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**Increasing Returns in Infinite Horizon Economies**

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## Abstract

This paper shows that in a general equilibrium model with an infinite horizon in which production may exhibit increasing returns to scale or nonconvexities which are internal to the firm, marginal cost pricing equilibria exist and are essential, that is, stable with respect to small perturbations of the economy. As in models with a finite number of commodities, marginal cost pricing equilibria need not be Pareto Optimal, yet most systematic approaches to equilibrium analysis in infinite dimensional commodity spaces rely crucially on the First Welfare Theorem, which fails not only for marginal cost pricing equilibria in models with nonconvexities in production, but also for equilibria in many economies with distortions such as externalities or incomplete markets. This paper models equilibria in infinite dimensional commodity spaces as the solutions to a system of nonlinear equations and introduces Leray-Schauder degree theory, the extension of degree theory to Banach and locally convex spaces, as the natural methodology by which to demonstrate that solutions to this system of equations exist, and by which to analyze qualitative features of these solutions. These methods are used to show that marginal cost pricing equilibria exist and are essential in an infinite horizon model with a finite number of heterogeneous agents, and a single firm with nonconvexities in production.

# 1 Introduction

In his *Principles of Economic Analysis*, Alfred Marshall introduced the term “external economies” to describe economies of scale which were exogenous to individual firms but endogenous to the industry. As refined by Edgeworth (1905), Pigou (1912), and other proponents of this concept, it came to stand for the idea that increasing returns to scale might exist in an industry in which each individual firm operated under constant or decreasing returns to scale, but in which external effects between firms produced industry-wide increasing returns to scale. As such, proponents of the idea of external economies argued that increasing returns could be compatible with perfect competition, since each firm did not recognize the industry-wide externality, and would act like a classical price-taking firm producing subject to decreasing returns to scale. The concept of external economies, and in particular the conclusion that increasing returns to scale could be compatible with perfect competition, was rejected by many economists of the time, including Sraffa, Young, and Knight. Indeed, Knight objected to the entire concept of external economies, claiming “external economies in one business unit are internal economies in some other” (1924, p.597), and introduced the famous phrase “empty economic boxes” (1925) to describe external economies.

The same dichotomy can be found in contemporary work, but with the advent of general equilibrium theory, the claim that increasing returns to scale can be compatible with perfect competition was formalized, starting with the work of Chipman (1970) who considers a single period homogeneous agent economy with production characterized by such external nonconvexities. More recently, work on endogenous growth models, which can be viewed as dynamic generalizations of Chipman’s model, has begun to explore the implications of nonconvexities for the long run behavior of dynamic, infinite horizon economies. Most of this work posits this specialized type of nonconvexity in which individual firms face constant or decreasing returns to scale, but operate in an industry characterized by externalities between firms, due for instance to the effects of research and development or learning by doing, so that the economy as a whole exhibits increasing returns to scale (see, e.g., Romer, 1986, 1990; Stokey 1988). Suzuki (1990) has proven that competitive equilibria exist in a heterogeneous agent version of these models in which firms face what Chipman called “parameterized returns to scale.”

The general equilibrium literature on economies with nonconvexities has focused instead on nonconvexities which need not be external, but may be internal to the firm, so that the source of increasing returns to scale need not be some external force in the economy, but may arise from internal characteristics of the firm. Since competitive equilibria will not generally exist in such models, the competitive notion is replaced by notions such as marginal cost pricing equilibria or average cost pricing equilibria. Mantel (1979) shows that in the finite horizon heterogeneous agent version of Chipman’s model, the introduction of fundamental internal nonconvexities in the firm’s production does not preclude the existence of equilibria, albeit not competitive

equilibria but marginal cost pricing equilibria, and proves that for smooth nonconvexities, marginal cost pricing equilibria exist. Indeed, there are quite robust results on existence and determinacy of marginal cost pricing equilibria in finite horizon models with nonconvexities in production (see Beato (1976, 1982), Beato and Mas Colell (1985), Brown and Heal (1979), and for a survey, Brown (1990)). One of the most important and illuminating methods of studying equilibria in economies with nonconvexities in production, and indeed, in other models with distortions such as incomplete markets, has been degree theory, whereby one can at once establish both existence and local uniqueness of equilibria, and which also leads to computational algorithms for calculating equilibria (Brown and Heal, 1982; Kamiya, 1988, Brown, DeMarzo and Eaves, 1993).

Although a considerable amount of research has focused on optimal growth models with nonconvexities, in which a social planner or representative agent seeks to maximize an objective function over an infinite horizon consumption stream subject to production constraints which may be nonconvex (see, e.g., Chichilnisky, 1977, 1981; Majumdar and Mitra, 1982, 1983; Majumdar and Nermuth, 1982), nothing is known about the existence or nature of decentralized equilibria in models with an infinite horizon, many different households, and general nonconvexities in production, as would result in heterogeneous agent versions of the endogenous growth models if nonconvexities were not required to be external but rather were internalized by the firm. Moreover, it is certainly not clear from the existing literature on infinite dimensional models with convex production how such results might be obtained.<sup>1</sup> With the exception of Bewley's (1972) work, all of the systematic approaches to equilibrium analysis in infinite dimensional spaces rely crucially on the First Welfare Theorem, which, as the seminal work of Guesnerie (1975) shows, fails for marginal cost pricing equilibria, as well as for equilibria in other models with distortions such as incomplete markets models. Both the Negishi approach (MasColell, 1986), searching over the Pareto optima for prices and allocations at which each agent's budget constraint is satisfied, and the Edgeworth equilibrium approach (Aliprantis, Brown, and Burkinshaw, 1987a, 1987b, 1989), showing the existence of prices supporting allocations which are in the core of every replica economy, are of no use when the equilibrium need not be Pareto optimal or the core may be empty, as is the case with marginal cost pricing equilibria. The limiting argument of Bewley, which demonstrates the existence of an equilibrium as the limit of a sequence of equilibria of finite dimensional economies, becomes quite complicated as the economy in question grows more complex, and could never be used as the basis for a systematic approach to studying qualitative features of equilibria in infinite dimensional models, features such as local uniqueness, comparative statics,

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<sup>1</sup>Indeed, Mas-Colell (1992) argues, "Equilibrium theory, however, is not exhausted by the classical model. Its practical use has required the consideration of many departures and the incorporation of many forms of "imperfections" and market failures.... It would be comforting if we could assert that once the infinite-dimensional extension of the classical model is well understood the extension of the non-classical theory presents no particular difficulty. Unfortunately, this is not so."

stability, or for computation of equilibria.

This paper shows that marginal cost pricing equilibria exist in an infinite dimensional analogue of Mantel's (1979) model, a model with a finite number of heterogeneous agents, a single firm with nonconvex production, and an infinite horizon, and that the set of marginal cost pricing equilibria is essential or "stable" with respect to small perturbations in the economy. Marginal cost pricing equilibria are characterized as solutions to a system of equations, and Leray-Schauder degree theory, the extension of degree theory to Banach and locally convex spaces, is introduced as the natural and systematic means by which to demonstrate that solutions to this system of equations exist, and that the set of solutions is essential.

The paper proceeds as follows. Section 2 presents the assumptions of the model, characterizes marginal cost pricing equilibria as the solutions to a particular system of equations, and uses Leray-Schauder degree theory to show that marginal cost pricing equilibria exist and are essential. Examples are given in section 3, and concluding remarks appear in section 4. Finite dimensional degree theory is briefly reviewed in the first appendix, and the main definitions and results concerning Leray-Schauder degree theory are discussed there. Lengthy or tedious proofs are contained in the second appendix.

## 2 Marginal Cost Pricing Equilibria

The setting for this paper is a private ownership economy with a finite number of households in which production is carried out by a single firm. Increasing returns to scale are allowed by permitting the firm's production set to be nonconvex. To capture the idea that this is a generalization of discrete time, infinite horizon growth models, the commodity space is  $\ell_\infty$ , the space of all bounded real-valued sequences. Before precise assumptions about the nature of consumers' preferences or the firm's technology can be stated, some notation will be required. Let  $Y$  denote the production possibility set of the firm. Given the social endowment vector  $\omega$ , let  $\hat{Y} \equiv (Y + \omega) \cap \ell_{\infty+}$  denote the feasible production set, let  $\partial Y$  denote the boundary of the production set, and let  $\partial \hat{Y}$  denote the efficiency frontier. The following properties analogous to those assumed, for example, by Mantel (1979), Brown and Heal (1982), or Kamiya (1988) in the finite dimensional case, are assumed to hold for the production set  $Y$  of the economy.

### Assumption P.

1.  $0 \in Y$ ,  $Y - (\ell_{\infty+}) \subset Y$  (free disposal);
2.  $\partial Y$  is a smooth hypersurface in  $\ell_\infty$ , i.e., there exists a continuously Fréchet differentiable function  $f : \ell_\infty \rightarrow \mathcal{R}$  such that  $Y \equiv \{x \in \ell_\infty : f(x) \leq 0\}$  and  $\partial Y = \{y \in \ell_\infty : f(y) = 0\}$ ;

3.  $f : \ell_\infty \rightarrow \mathcal{R}$  is weak\* continuous<sup>2</sup> and bounded;
4.  $Df(y) \in \ell_{1++} = \{p \in \ell_1 : p_t > 0 \forall t\}$  for every  $y$ .

Assumption P1 is standard, with the exception that  $Y$  is not assumed to be convex. Assumption P2 requires that the production possibility frontier be smooth, so that while nonconvexities are allowed, they must be smooth ones. This assumption ensures that the marginal cost pricing rule  $y \mapsto Df(y)$  is a well-defined function. However, this assumption, and the restrictions it places on the type of nonconvexities that production can exhibit, is not necessary; the results of this paper carry over immediately to a more general model with a pricing rule  $\phi(y)$ , such as average cost pricing, so long as this rule satisfies the requisite assumptions made in the paper concerning the marginal cost pricing rule. Assumption P4 reflects the requirement that marginal cost always be strictly positive. The role that these assumptions play will become clearer after the equilibrium equations are defined. Examples satisfying all of these assumptions are discussed in section 3.

Each of the  $m$  households in this model has an endowment vector  $\omega_i$ , owns share  $\theta_i$  in the firm, and has utility function  $U_i$ . The following properties of households' characteristics are assumed to hold.

**Assumption H.** For  $i = 1, 2, \dots, m$ ,

1.  $U_i : \ell_{\infty+} \rightarrow \mathcal{R}$  is Mackey continuous,<sup>3</sup> strictly concave, strictly monotone, and  $U_i(0) = 0$ ;
2.  $\omega_i = \theta_i \omega$ , where  $\omega \in \ell_{\infty++} = \{x \in \ell_\infty : x_t > 0 \forall t\}$ ,  $\theta_i > 0$ , and  $\sum_{i=1}^m \theta_i = 1$ .

Again assumption H1 is relatively standard, with the exception that consumers' utility functions are assumed to be strictly concave, and serves among other purposes to guarantee that certain optimization problems are well-defined and have nonempty solution sets. The assumption that utility functions are Mackey continuous carries the behavioral assumption of impatience or generalized discounting on the part of the agents (see Brown and Lewis, 1981). Assumption H2 is the standard assumption of a fixed structure of revenues, or a fixed income distribution. Under this assumption, given prices  $p \in \ell_1$  and a production plan  $y \in Y$ , a consumer's income is simply  $\theta_i p \cdot (y + \omega)$ . As Brown (1990) points out, this expression for income should be interpreted as after-tax income, as the agents' shares carry unlimited liability, and unlike the

<sup>2</sup>The weak\* topology on  $\ell_\infty$  is the locally convex topology on  $\ell_\infty$  under which a net  $\{x^\alpha\}$  in  $\ell_\infty$  converges to an element  $x \in \ell_\infty$  if and only if  $p \cdot x^\alpha \rightarrow p \cdot x$  for every  $p \in \ell_1$ , and will be denoted  $\sigma(\ell_\infty, \ell_1)$ .

<sup>3</sup>The Mackey topology for the pairing  $(\ell_\infty, \ell_1)$ , which will be denoted  $\tau(\ell_\infty, \ell_1)$ , is the locally convex topology on  $\ell_\infty$  of uniform convergence on weak\* compact, convex, balanced sets. A set  $S$  is balanced if  $x \in S$  and  $|\lambda| \leq 1$  implies that  $\lambda x \in S$ . Unqualified topological statements will refer to the norm topology, and the standard topology on  $\mathcal{R}^n$  will be denoted  $\tau_{\mathcal{R}^n}$ .

convex case, by pricing at marginal cost with nonconvexities the firm may incur losses. When the firm makes losses, owning shares in the firm then amounts to paying a lump sum tax to cover those losses and provide a balanced budget for the economy as a whole. While simplifying the analysis to a certain extent and representing a plausible structure for income distribution, this assumption of a fixed structure of revenues could be weakened, replaced by an assumption like that used by Bonnisseau and Cornet (1988a), for example, of some exogenously specified income distribution  $\{r_i(y), i = 1, \dots, m\}$ , provided these income functions are bounded, nonnegative, weak\* continuous, and satisfy Walras's Law, so that  $p \cdot \sum_{i=1}^m r_i(y) = p \cdot (y + \omega)$  for all  $p$  and  $y$ .

In this framework, a *marginal cost pricing (MCP) quasiequilibrium* is a combination of consumption plans  $(x_1, \dots, x_m)$ , a production plan  $y$ , and prices  $p$  such that

- i.  $y + \omega \in \partial \hat{Y}$ ;
- ii.  $p = Df(y)$ ;
- iii.  $\sum_{i=1}^m x_i = y + \omega$ ;
- iv.  $U_i(\hat{x}) \geq U_i(x_i) \Rightarrow p \cdot \hat{x} \geq \theta_i p \cdot (y + \omega)$  for  $i = 1, \dots, m$ .

A marginal cost pricing quasiequilibrium requires that production be efficient, prices be set equal to the firm's marginal cost of producing the plan  $y$ , all markets clear, and that given prices  $p$ , agents minimize expenditures. If  $Y$  is assumed to be convex, this notion of quasiequilibrium coincides with the standard notion of competitive quasiequilibrium, thus competitive quasiequilibrium in a model with a single firm is a special case of marginal cost pricing quasiequilibrium in this model. If income is distributed according to a fixed structure of revenues, then in a quasiequilibrium with strictly positive prices, every consumer has positive income, which guarantees that any such consumption plans are also utility maximizing given the budget constraint. When there is a fixed structure of revenues, marginal cost pricing quasiequilibria are then equivalent to marginal cost pricing equilibria.

The assumptions made thus far will not be sufficient to guarantee that marginal cost pricing quasiequilibria will exist, however. In finite and infinite dimensional models, whether production is convex or nonconvex, in general one must ensure that the set of feasible allocations

$$\mathcal{A} = \{(x_1, \dots, x_m, y) \in \prod_{i=1}^m X_i \times Y : \sum_{i=1}^m x_i = y + \omega\}$$

is compact to guarantee the existence of quasiequilibria. Since the consumption sets  $X_i$  and the production set  $Y$  will be assumed to be closed, as is standard, the set of attainable allocations  $\mathcal{A}$  will at least be closed. In finite dimensional models,  $\mathcal{A}$

will then be compact provided it is bounded. The standing assumption in the most general and encompassing work on general equilibrium in models with nonconvex production, that of Bonnisseau and Cornet (1988a, 1988b), is precisely that: that the set of attainable allocations is bounded. As shown by Hurwicz and Reiter (1973), in a finite dimensional setting the boundedness of the set of attainable allocations can also be shown to be a consequence of more primitive assumptions on the production set, such as irreversibility.

In models with an infinite dimensional commodity space, the compactness of the set of attainable allocations will not follow from assumptions such as irreversibility, but rather more direct assumptions of compactness of the feasible production or consumption sets are required to guarantee compactness of the set of attainable allocations. As the following proposition, which is a straightforward consequence of a proposition of Aliprantis, Brown, and Burkinshaw (1989, Theorem 4.2.4), demonstrates, there are several assumptions which will guarantee that the set of attainable allocations is weak\* compact.

**Proposition 2.1.** *Assume that the firm's production set  $Y$  is weak\* closed.*

i. *If the attainable production set*

$$\hat{Y} = \{y \in Y : \exists(x_1, \dots, x_m; z) \in \mathcal{A} \text{ such that } z = y\}$$

*is order bounded, then it is weak\* compact.*

ii. *If  $\hat{Y}$  is weak\* compact, then the attainable consumption sets*

$$\tilde{X}_i = \{x \in X_i : \exists(x_1, \dots, x_m; y) \in \mathcal{A} \text{ such that } x_i = x\}$$

*are weak\* compact for each  $i$ . Moreover,  $\mathcal{A}$  is weak\* compact.*

Thus if the firm's attainable production set is bounded, it will be weak\* compact, and if the firm's attainable production set is weak\* compact, the set  $\mathcal{A}$  of attainable allocations is weak\* compact as well. The standard assumption in the literature on infinite dimensional models with convex production is then either that each attainable production set is weakly compact, or that the aggregate feasible production set  $\hat{Y}$  is weakly compact (see e.g., Aliprantis, Brown and Burkinshaw, 1987b, or Zame 1987). As the following lemma shows, when the commodity space is  $\ell_\infty$ , this assumption is actually no weaker than the assumption that each set is order bounded.

**Lemma 2.1.** *Let  $A \subset \ell_\infty$ . Then  $A$  is weak\* compact if and only if it is weak\* closed and order bounded.*

*Proof:* Since order intervals are weak\* compact by Alaoglu's Theorem, clearly if  $A$  is weak\* closed and order bounded, it is weak\* compact. The converse follows from Corollary 20.10 and Exercise 5 p. 163 of Aliprantis and Burkinshaw (1978). ■

This lemma also demonstrates that if the set of attainable allocations is to be weak\* compact, it must be order bounded. Moreover, in a model with a single firm, the boundedness of the set of attainable allocations has a clear connection with the boundedness of the feasible production and consumption sets, as shown by the following lemma, the proof of which should be clear.

**Lemma 2.2.** *If each consumer's feasible consumption set  $X_i$  is a subset of the positive cone  $\ell_{\infty+}$ , then the set of attainable allocations  $\mathcal{A}$  is order bounded if and only if either*

- i. *the firm's feasible production set  $\hat{Y} = (Y + \omega) \cap \ell_{\infty+}$  is order bounded above, or*
- ii. *each consumer's feasible consumption set  $X_i$  is order bounded above.*

Provided that the firm's production set is weak\* closed, which follows from assumption P, and that each consumer's consumption set  $X_i$  is weak\* closed, which will also be assumed, the set of attainable allocations will be weak\* closed, and thus by Lemma 2.1, it will be weak\* compact if and only if it is order bounded above. Then as in the literature on finite dimensional models with nonconvex production, the standing assumption here will be the following:

**Assumption B.** *The set  $\mathcal{A}$  of attainable allocations is order bounded.*

As argued above, this assumption will only hold if either the firm's feasible production set is bounded, or if each consumer's feasible consumption set is bounded. Accordingly, we will consider these two implications of the boundedness condition separately. The first part of this section considers the case in which the firm's feasible production set is bounded, and each consumer's feasible consumption set is the positive cone in  $\ell_{\infty}$ ; that is, the following is assumed.

**Assumption BF.**

1.  $\hat{Y} = (Y + \omega) \cap \ell_{\infty+}$  is order bounded.
2.  $X_i = \ell_{\infty+}$  for  $i = 1, \dots, m$ .

To show that equilibria exist in this model and to study qualitative features of the equilibrium set, the approach taken in this paper will be the same one which has been fundamental to studying questions of existence, determinacy, and computation of equilibria in finite-dimensional models with market distortions such as incomplete markets or nonconvexities. Equilibria will first be characterized as the solutions to a system of equations. Questions about the existence and nature of equilibria can then be rephrased as questions concerning the existence and nature of solutions to

this system of equations, questions which are perhaps most naturally and powerfully answered by degree theory and related methods.

In a standard finite dimensional setting, the degree of a function  $f : \mathcal{R}^n \rightarrow \mathcal{R}^n$ , calculated on a set  $A$  and at a point  $y$  in the range of the function, gives a rough “count” of the number of solutions to the equation  $f(x) = y$  which lie in the set  $A$ . This is easiest to see in the case when the function  $f$  is continuously differentiable and the point  $y$  is a regular value of  $f$ , i.e., a point such that for every preimage  $x \in f^{-1}(y)$ , the derivative  $Df(x)$  is nonsingular. In that case, the implicit function theorem implies that each solution is locally unique, and if in addition the set  $A$  is bounded and no solutions lie on the boundary of  $A$ , there can only be a finite number of solutions. In this case, the degree counts each solution with multiplicity  $+1$  or  $-1$ , corresponding to the sign of the determinant of the derivative at that solution. More precisely, in this case the degree, denoted  $d(f, A, y)$ , is given by the formula

$$d(f, \Omega, y) = \sum_{x \in f^{-1}(y)} \operatorname{sgn}(\det Df(x)).$$

However, even if the function  $f$  fails to be continuously differentiable, the degree still conveys information about the existence and qualitative features of the set of solutions. If  $d(f, A, y) \neq 0$ , then  $f^{-1}(y) \neq \emptyset$ , i.e., if the degree is nonzero, then there exist solutions in  $A$  to the equation  $f(x) = y$ . Moreover, even if the function  $f$  is only continuous or if  $y$  is not a regular value of  $f$ , the degree still contains information about the qualitative features of solutions to the equation  $f(x) = y$ , and in particular, the degree may yield information about the stability of solutions with respect to perturbations in the model. A subset  $K$  of  $f^{-1}(y)$  is called *stable* or *essential* if for every neighborhood  $U$  of  $K$ , there exists a neighborhood  $V$  of the graph of  $f$  such that if  $g$  is any continuous function whose graph lies in  $V$ , the equation  $g(x) = y$  must have a solution in  $U$ . Essential solutions or sets of solutions cannot be removed by arbitrarily small perturbations of the underlying equations, as sufficiently small perturbations of the original equations must have nearby solutions. In contrast, if a solution or set of solutions is inessential, arbitrarily small perturbations of the system of equations need not have any solutions close to the original solutions, so that these small perturbations in effect remove those inessential solutions. For example, if  $f : \mathcal{R} \rightarrow \mathcal{R}$  is given by  $f(x) = x^2$ , then  $f(0) = 0$ , but this is an inessential solution, since arbitrarily small perturbations of the function  $f$  have no zeros at all. This concept of stability, introduced by Fort (1950) and discussed by Dierker (1974) and Geanakoplos and Shafer (1990), is closely related to degree. Indeed, if  $d(f, \Omega, y) \neq 0$ , then  $f^{-1}(y)$  is an essential set of solutions by homotopy invariance (see Geanakoplos and Shafer (1990)). Even in the continuous case then, the degree contains information not only about existence of solutions but also about the qualitative behavior of solutions.

Homotopy invariance is the key to most economic applications of degree theory to the problem of demonstrating that the solution set of the equilibrium equations

is nonempty and finite. This is the method applied by both Dierker (1972) and Kamiya (1988), and more recently by Brown, DeMarzo, and Eaves (1993) to compute equilibria in incomplete markets models. Even if  $f$  is continuously differentiable and  $y$  is a regular value of  $f$ , the degree is usually impossible to calculate from scratch, since without a great deal of information about the derivative  $Df$ , one would have to be able to compute all of the solutions to the equation  $f(x) = y$  for  $x \in \Omega$  in order to compute the degree. These are the solutions we are searching for; being able to compute them would obviate the need to know the degree. A more useful way to calculate the degree is to try to construct a well-behaved homotopy between the function  $f$  and some function  $g$ , where the degree of  $g$  is easily computable and nonzero, and then to use homotopy invariance to conclude that  $d(f, \Omega, y) = d(g, \Omega, y)$ . Such an argument also yields in theory a method for computing equilibria by constructing a path between solutions of the equation  $g(x) = y$ , which should ideally be easily computable, and solutions to the equation  $f(x) = y$ . Indeed, this will be the approach employed in this paper.

One encounters a number of difficulties in attempting to extend the notion of degree to infinite dimensional spaces. The degree should still provide a useful and meaningful answer to questions concerning the existence and number of solutions to a system of equations. In infinite dimensional spaces however, degree cannot be defined for all continuous functions and still satisfy desirable properties like homotopy invariance; see e.g., Krasnosel'skii and Zabreiko (1980) and Appendix I for a more complete discussion of this point. As is often the case in infinite dimensional spaces, what is lacking is some form of compactness. The key idea which leads to an appropriate class of functions for which a meaningful notion of degree can be defined is that of a compact operator. If  $X$  is a Banach space,  $\Omega \subset X$ , and  $C : \Omega \rightarrow X$ , then  $C$  is a compact operator if  $C$  is continuous, and if for every bounded set  $B \subset \Omega$ ,  $\overline{C(B)}$  is compact. If  $X$  and  $Y$  are finite dimensional and  $\Omega$  is bounded, then any continuous function which is defined and finite on  $\bar{\Omega}$  is compact. Of course this is no longer true in infinite dimensional spaces. More generally, if  $(X, \tau_1)$  and  $(Y, \tau_2)$  are locally convex linear topological vector spaces,  $\Omega \subset X$  is open and  $C : \Omega \rightarrow Y$ , then  $C$  is called a  $\tau_1 - \tau_2$  compact operator if  $C$  is  $\tau_1 - \tau_2$  continuous, i.e., continuous as a map  $C : (X, \tau_1) \rightarrow (Y, \tau_2)$ , and if  $\overline{C(\Omega)}$  is  $\tau_2$  compact. If  $(X, \tau_1) = (Y, \tau_2)$ ,  $C$  will simply be called a  $\tau_1$  compact operator, and references to the topology may sometimes be omitted altogether if the relevant topology is clear. For a more detailed discussion of the properties of compact operators see Appendix I.

The appropriate extension of Brouwer's degree to infinite dimensional spaces, called *Leray-Schauder degree*, can then be defined for the class of compact perturbations of the identity, which are maps of the form  $I - C$  where  $C : \Omega \rightarrow X$  is a compact operator. As in the finite dimensional case, the distinction remains between the interpretation of the degree for compact perturbations of the identity which are in addition continuously differentiable and for which  $y$  is a regular value, and those which are merely continuous. If  $C$  is continuously Fréchet differentiable on  $\Omega$  and

$y$  is a regular value of  $I - C$ , that is, a value for which  $D(I - C)(x)$  is surjective for every point  $x$  satisfying the equation  $(I - C)(x) = y$ , then the implicit function theorem again implies that  $(I - C)^{-1}(y)$  is a finite set. Moreover, as in the finite dimensional case, in this case the degree also counts the solutions with multiplicities  $+1$  or  $-1$  depending on the derivative at each solution. If  $C$  is only continuous or if  $y$  is not a regular value of  $I - C$ , one can still ask as in the finite dimensional case whether a solution or set of solutions is essential. Again by homotopy invariance, if  $D(I - C, \Omega, y) \neq 0$ , the set of solutions to the equation  $(I - C)(x) = y$  is essential, and hence given any neighborhood  $U$  of the set of solutions, arbitrarily small perturbations of the system of equations  $I - C$  must still have solutions in this neighborhood  $U$ .

Then to apply the tools of the previous section to demonstrate the existence and essentiality of MCP quasiequilibria, we need to characterize the quasiequilibria as the zeros to a system of nonlinear equations of the form  $I - C$  on a suitable domain  $\Omega \subset X$  containing all of the marginal cost pricing equilibria, such that  $C : \Omega \rightarrow X$  is a compact operator in the relevant topology, and such that  $I - C$  has no zeros on the boundary of the domain  $\Omega$ . Often the most difficult part of this process is finding a suitable domain  $\Omega$  which contains all of the solutions of interest, and for which the homotopy has no solutions on the boundary  $\partial\Omega$ , and knowledge about a priori bounds on the possible solutions of the equation  $(I - C)(x) = y$  is often necessary to determine this domain  $\Omega$ .

There are natural a priori bounds on the possible MCP quasiequilibria. Since  $\hat{Y}$  is bounded, there exists some  $b \in \ell_{\infty+}$  such that  $\hat{Y} \subset [0, b] \equiv \{x \in \ell_{\infty} : 0 \leq x \leq b\}$ .<sup>4</sup> Any MCP equilibrium allocation  $(x_1, \dots, x_m)$  must be feasible, thus for any MCP equilibrium, both total output  $y + \omega$  and individual consumption bundles  $x_i$  must lie in the interval  $[0, b]$ . Moreover,  $U_i(x_i) \leq U_i(b)$  for every  $i = 1, \dots, m$ . In that respect, the problem of finding MCP equilibria is well suited to the methods of Leray-Schauder degree theory.

However, most finite-dimensional equilibrium existence proofs rely on degree-theoretic or fixed point arguments built around maps from the price space to the commodity space, such as the excess demand map, or between the commodity space and the price space, such as the first order conditions. Of course when there are a finite number of commodities, the price space and the commodity space are the same, or at least isomorphic; both are simply isomorphic to  $\mathcal{R}^n$  for some  $n$ , and using Brouwer's degree is not a problem. This certainly won't be true in general when the commodity space is infinite dimensional, as the price space will typically be some subspace of the dual space of the commodity space. For example, in this model, the price space is the separable space  $\ell_1$ , whereas the commodity space is the nonseparable space  $\ell_{\infty}$ .<sup>5</sup>

<sup>4</sup>The notational conventions here are that  $x \leq y \Rightarrow x_n \leq y_n$  for every  $n$ ,  $x < y \Rightarrow x_n < y_n$  for every  $n$  and  $x \neq y$ , and that  $x \ll y \Rightarrow x_n < y_n$  for every  $n$ .

<sup>5</sup>Even though  $\ell_1$  is a subspace of  $\ell_{\infty}$ , it is a proper subspace, and degree can be utilized only

Consequently, quasiequilibria will be characterized in terms of Hicksian demand functions. As before, let  $b \in \ell_{\infty++}$  be such that  $\hat{Y} \subset [0, b]$  and  $y + \omega \ll b$  for every  $y + \omega \in \hat{Y}$ . Choose  $\bar{b} \in \text{int } \ell_{\infty++}$  such that  $\bar{b} \gg b$  (see Figure 1). Define for  $i = 1, \dots, m$ , for  $p \in \ell_1$ , and for  $U \in [0, U_i(\bar{b})]$ , the expenditure function for consumer  $i$

$$E_i(p, U) = \min_{\substack{U_i(x) \geq U \\ 0 \leq x \leq \bar{b}}} p \cdot x$$

and consumer  $i$ 's Hicksian demand

$$x_i(p, U) = \arg \min_{\substack{U_i(x) \geq U \\ 0 \leq x \leq \bar{b}}} p \cdot x.$$

**Theorem 2.1.** *For  $p \in \ell_1$  and  $U \in [0, U_i(\bar{b})]$ ,  $E_i(p, U)$  and  $x_i(p, U)$  are well-defined. For  $p \in \ell_{1++}$ ,  $x_i(p, U)$  is single-valued.*

*Proof:* By Alaoglu's theorem, the order interval  $[0, \bar{b}]$  is weak\* compact, and by assumption H1,  $\{x \in \ell_{\infty+} : U_i(x) \geq U\}$  is Mackey closed and convex for every  $U \geq 0$ , and hence is weak\* closed as well.

Since the function  $x \mapsto p \cdot x$  is certainly weak\* continuous, and  $\{x \in \ell_{\infty+} : U_i(x) \geq U, 0 \leq x \leq \bar{b}\}$  is weak\* compact, both  $E_i(p, U)$  and  $x_i(p, U)$  are well-defined. Since  $x \mapsto p \cdot x$  is convex,  $x_i(p, U)$  is convex-valued. If  $p \in \ell_{1++}$ , fix  $x_1, x_2 \in x_i(p, U)$ . If  $U = 0$ ,  $p \in \ell_{1++} \Rightarrow x_1 = x_2 = 0$ , since  $U_i(0) = 0$  and  $x_i(p, U) \subset [0, \bar{b}]$ . Similarly, for  $U = U_i(\bar{b})$ , by strict convexity  $x_1 = x_2 = \bar{b}$ . Suppose  $U \in (0, U_i(\bar{b}))$ . Then  $x_1, x_2 \neq 0$ , so  $p \cdot x_1 > 0$ . But then

$$x_i(p, U) = \arg \max_{\substack{p \cdot x \leq p \cdot x_1 \\ 0 \leq x \leq \bar{b}}} U_i(x).$$

If  $x_1 \neq x_2$ , then for  $\alpha \in (0, 1)$ ,  $\alpha x_1 + (1 - \alpha)x_2 \in x_i(p, U)$ , and  $U_i(x)$  strictly concave  $\Rightarrow U_i(\alpha x_1 + (1 - \alpha)x_2) > \alpha U_i(x_1) + (1 - \alpha)U_i(x_2) = U_i(x_1) = U_i(\alpha x_1 + (1 - \alpha)x_2)$  since  $\alpha x_1 + (1 - \alpha)x_2, x_1 \in x_i(p, U)$ . This is a contradiction, hence  $x_1 = x_2$ . Therefore, for  $p \in \ell_{1++}$ ,  $x_i(p, U)$  is single-valued. ■

By definition, a MCP quasiequilibrium requires that in equilibrium, output is chosen to equate marginal cost to price, so that  $p = Df(y)$ . Using this observation and the expenditure and Hicksian demand functions, MCP quasiequilibria can be characterized as the tuples  $(y, U_1, \dots, U_m) \in \ell_{\infty} \times \mathcal{R}^m$  such that

$$\begin{aligned} \sum_{i=1}^m x_i(Df(y), U_i) - (y + \omega) &= 0, \\ f(y) &= 0, \\ Df(y) \cdot (x_2(Df(y), U_2) - \theta_2(y + \omega)) &= 0, \end{aligned}$$

indirectly for mappings into a proper subspace, as any such mapping must have degree 0 at every point (Deimling p.92).

$$\begin{aligned} & \vdots \\ Df(y) \cdot (x_m(Df(y), U_m) - \theta_m(y + \omega)) &= 0. \end{aligned}$$

As usual, Walras' law is invoked to eliminate the redundant budget equation of the first agent. Note that the budget equation for agent  $i$  can be rewritten as

$$E_i(Df(y), U_i) - \theta_i Df(y) \cdot (y + \omega) = 0.$$

Furthermore, if we define

$$F(y, U_1, \dots, U_m) = \begin{pmatrix} \sum_{i=1}^m x_i(Df(y), U_i) - (y + \omega) \\ f(y) \\ Df(y) \cdot [x_2(Df(y), U_2) - \theta_2(y + \omega)] \\ \vdots \\ Df(y) \cdot [x_m(Df(y), U_m) - \theta_m(y + \omega)] \end{pmatrix},$$

then  $F$  is a map from  $\ell_\infty \times \mathcal{R}^m$  to  $\ell_\infty \times \mathcal{R}^m$ , and  $F$  can be written in the form  $F = I - C$ , where  $C$  is also a map from  $\ell_\infty \times \mathcal{R}^m$  to  $\ell_\infty \times \mathcal{R}^m$ . To see this, note that  $F$  can be rewritten as

$$F(y, U_1, \dots, U_m) = \begin{pmatrix} y - \left( \sum_{i=1}^m x_i(Df(y), U_i) - \omega \right) \\ U_1 - (U_1 - f(y)) \\ U_2 - (U_2 - Df(y) \cdot [x_2(Df(y), U_2) - \theta_2(y + \omega)]) \\ \vdots \\ U_m - (U_m - Df(y) \cdot [x_m(Df(y), U_m) - \theta_m(y + \omega)]) \end{pmatrix}.$$

Also, a tuple  $(\bar{y}, \bar{U}_1, \dots, \bar{U}_m) \in \ell_\infty \times \mathcal{R}^m$  will satisfy the equation  $F(\bar{y}, \bar{U}_1, \dots, \bar{U}_m) = 0$  if and only if the tuple  $(\bar{x}_1, \dots, \bar{x}_m, \bar{y}, \bar{p})$  is a MCP quasiequilibrium, where  $\bar{x}_i = x_i(Df(\bar{y}), \bar{U}_i)$  and  $\bar{p} = Df(\bar{y})$ .

The properties of the expenditure function and the Hicksian demand function in this setting will then be crucial to establishing that  $F$  is a compact perturbation of the identity, which is the first step in showing that MCP quasiequilibria exist, and in examining the qualitative features of the set of MCP quasiequilibria. Berge's Theorem (Berge, p. 115) will be used to establish that the expenditure and Hicksian demand functions are continuous.<sup>6</sup>

**Theorem 2.2.** *Assume that H1 and H2 hold. For  $p \in \ell_1$  and  $U_i \in [0, U_i(\bar{b})]$ ,  $E_i(p, U_i)$  is norm continuous, and  $x_i(p, U_i)$  is norm - weak\* u.s.c. for  $i = 1, \dots, m$ .*

*Proof:* Since  $(p, x) \mapsto p \cdot x$  is norm  $\times$  weak\* continuous, by Berge's theorem it suffices to show that  $\beta_i(U) = \{x \in \ell_\infty : U_i(x) \geq U, 0 \leq x \leq \bar{b}\}$  is a continuous

<sup>6</sup>To be precise here, I should say that  $E_i(p, U_i)$  is norm  $\times \tau_{\mathcal{R}} - \tau_{\mathcal{R}}$  continuous, and  $x_i(p, U_i)$  is norm  $\times \tau_{\mathcal{R}} - \sigma(\ell_\infty, \ell_1)$  u.s.c.; however, this notation becomes cumbersome, and thus I will omit reference to the topology on the finite-dimensional factors, as the standard Euclidian topology will always be imposed on these factors.

correspondence, as by Alaoglu's theorem and assumption A1,  $\beta_i(U)$  is convex- and weak\* compact-valued. Since  $\ell_1$  is separable,  $\sigma(\ell_\infty, \ell_1)$  is metrizable on bounded sets (Aliprantis and Burkinshaw (1985), Theorem 10.7), so in particular  $\sigma(\ell_\infty, \ell_1)$  is metrizable on  $[0, \bar{b}]$ . To show  $\beta_i(U)$  is u.s.c., suppress the dependence on  $i$ , and suppose  $U_n \rightarrow U$  and  $x_n \xrightarrow{\sigma(\ell_\infty, \ell_1)} x$ , where  $x_n \in \beta(U_n)$ . We must show that  $x \in \beta(U)$ . Clearly  $x \in [0, \bar{b}]$ , and  $x_n \in \beta(U_n) \Rightarrow U(x_n) \geq U_n$  for every  $n$ . But  $x_n \xrightarrow{\sigma(\ell_\infty, \ell_1)} x \Rightarrow x_n \xrightarrow{\tau(\ell_\infty, \ell_1)} x$  since  $\sigma(\ell_\infty, \ell_1)$  and  $\tau(\ell_\infty, \ell_1)$  agree on norm-bounded sets (Rubel and Ryff 1970), hence the Mackey continuity of  $U_i(\cdot)$  implies that

$$\lim_n U(x_n) = U(x) \geq U = \lim_n U_n.$$

That is,  $x \in \beta(U)$ .

To show that  $\beta$  is l.h.c., let  $U_n \rightarrow U$ , and  $x \in \beta(U)$ . We must show that there exists a sequence  $\{x_n\}$  such that  $x_n \in \beta(U_n)$  and  $x_n \xrightarrow{\sigma(\ell_\infty, \ell_1)} x$ . First, suppose  $U(x) > U$ . Then  $\exists \epsilon > 0$  such that  $U(x) - U > \epsilon$ , but  $\exists N$  such that for  $n \geq N$ ,  $|U - U_n| < \epsilon/2$ ,  $\Rightarrow U(x) > U_n$  for  $n \geq N \Rightarrow x \in \{\hat{x} | U(\hat{x}) \geq U_n\}$  for  $n \geq N$ . So the sequence  $x_n = \bar{b}$  for  $n < N$ , and  $x_n = x$  for  $n \geq N$  will suffice.

Suppose  $U(x) = U$ . Without loss of generality,  $x \neq \bar{b}$ , else setting  $x_n = x$  for every  $n$  produces the required sequence. Since  $U_n \rightarrow U$ ,  $\exists N$  such that for  $n \geq N$ ,  $|U_n - U| < U(\bar{b}) - U$ ; i.e., such that  $U_n < U(\bar{b})$ . Set  $\beta(U) = \bar{b}$  for  $U > U(\bar{b})$  and set  $x_n = \bar{b}$  for  $n < N$ . For  $n \geq N$ , if  $U_n \leq U$ , set  $x_n = x$ . Since  $x \neq \bar{b}$  and  $x \in [0, \bar{b}]$ , there exists  $j$  such that  $x_j < \bar{b}_j$ . For  $n$  such that  $U_n > U$ , consider  $\{(1 - \alpha)x + \alpha\bar{b} : \alpha \in [0, 1]\}$ ; for every such  $n$ ,  $\exists 1 \geq \tilde{\alpha}_n > 0$  such that  $U((1 - \tilde{\alpha}_n)x + \tilde{\alpha}_n\bar{b}) > U_n$ . Using the intermediate value theorem, choose  $\alpha_n$  such that  $U((1 - \alpha_n)x + \alpha_n\bar{b}) = U_n$ , so  $(1 - \alpha_n)x + \alpha_n\bar{b} \in \beta(U_n)$ . Then  $(1 - \alpha_n)x + \alpha_n\bar{b} \xrightarrow{\sigma(\ell_\infty, \ell_1)} x$ , since if not, by the  $\sigma(\ell_\infty, \ell_1)$  compactness of  $[0, \bar{b}]$ , there exists a convergent subsequence  $(1 - \alpha_{n_k})x + \alpha_{n_k}\bar{b} \xrightarrow{\sigma(\ell_\infty, \ell_1)} \bar{x} \neq x$ . Hence  $\bar{x} = (1 - \bar{\alpha})x + \bar{\alpha}\bar{b}$  for some  $\bar{\alpha} > 0$ . But  $U((1 - \alpha_{n_k})x + \alpha_{n_k}\bar{b}) = U_{n_k} \rightarrow U$  and  $U(\cdot)$  is  $\sigma(\ell_\infty, \ell_1)$  continuous on  $[0, \bar{b}]$ , so  $U((1 - \alpha_{n_k})x + \alpha_{n_k}\bar{b}) \rightarrow U(\bar{x})$ , i.e.,  $U(\bar{x}) = U = U(x)$ . This is a contradiction, since  $\bar{x} = (1 - \bar{\alpha})x + \bar{\alpha}\bar{b} > x$  for  $\bar{\alpha} > 0$  and  $U(\cdot)$  is strictly monotone. Thus  $(1 - \alpha_n)x + \alpha_n\bar{b} \xrightarrow{\sigma(\ell_\infty, \ell_1)} x$ , and the sequence

$$x_n = \begin{cases} \bar{b}, & \text{if } n < N; \\ x & \text{if } U_n \leq U; \\ (1 - \alpha_n)x + \alpha_n\bar{b} & \text{if } U_n > U \end{cases}$$

satisfies the requirement. Thus  $\beta(U)$  is l.h.c. ■

Since  $x_i(p, U)$  is a function for  $p \in \ell_{1++}$  by Theorem 2.1, this theorem shows that  $x_i(p, U)$  is a norm - weak\* continuous function on  $\ell_{1++}$ .<sup>7</sup> To prove the existence of MCP quasiequilibria, and to determine whether the set of MCP quasiequilibria is

<sup>7</sup>This is a weaker conclusion than one might like; if  $F$  is continuously Fréchet differentiable and has 0 as a regular value, both existence and local uniqueness of MCP quasiequilibria can be

essential, we can utilize Theorem 2.2 together with the extension of Leray-Schauder degree theory to locally convex spaces. The difficulty presented by using the extension of Leray-Schauder degree theory to locally convex spaces lies in the choice of domain  $\Omega$ . All of the equilibrium quantities lie in the interval  $[0, b]$ : equilibrium production plans  $y + \omega$  must lie both in  $[0, b]$ , and on the hypersurface  $\{y + \omega : f(y) = 0\}$ ; not only must agents' individual consumption bundles lie in  $[0, b]$ , but the sum of their consumption bundles in equilibrium must lie in  $[0, b]$  as well. However,  $[0, b]$  is not a weak\* open set, in fact it has empty weak\* interior, whereas the domain must be an open set in order to define degree in general locally convex spaces. Hence we choose to work with production plans in a weak\* open set containing  $[0, b]$ . Let  $\Omega_y = \{y : f(y) < f(\hat{b} - \omega)\}$ , where  $\hat{b} \geq m\bar{b}$  (see Figures 1 and 2). Since  $f(\cdot)$  is weak\* continuous,  $\Omega_y$  is weak\* open. Moreover, since  $Df(y) > 0$  for all  $y$ ,  $\hat{Y} \subset [0, b] \subset (\Omega_y + \omega)$ .

In a finite dimensional model, assumptions guaranteeing compactness of the set of attainable allocations, together with continuity of preferences or demand functions, would essentially be sufficient to guarantee existence of equilibria. However, in infinite dimensional models with production, compactness of feasible production sets is not sufficient in general to guarantee existence of equilibria, and the work on infinite dimensional economies with production has stressed the need for extra restrictions on production sets such as uniform properness (see MasColell (1986) or Aliprantis, Brown and Burkinshaw (1987b)) or bounded marginal efficiencies of production (Zame (1987)). Both of these assumptions essentially place bounds on the marginal rates of transformation across all goods in the model. The analogous assumption required to guarantee the existence of marginal cost pricing equilibria in this model is the following.

**Assumption K.**  $Df : \Omega_y \rightarrow \ell_{1++}$  is a weak\* - norm compact operator; that is  $Df : \Omega_y \rightarrow \ell_1$  is weak\* - norm continuous and  $\overline{Df(\Omega_y)}$  is norm compact in  $\ell_1$ .

Since the firm's production set is described by the transformation function  $f$ , the derivative  $Df(y)$  gives the sequence of marginal rates of transformation, and this assumption requires that all such sequences of marginal rates of transformation lie in some norm compact subset of  $\ell_1$ . In particular, as is discussed in more detail in section 3, order intervals in  $\ell_1$  are norm compact, so a sufficient condition to guarantee

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established by showing that  $D(F, \Omega, 0) \neq 0$ . In order for  $F$  to be Fréchet differentiable, it must of course be norm continuous, and to use Leray-Schauder degree theory it would have to be a norm compact perturbation of the identity as well. Because the constraint correspondence  $\beta(U)$  is not in general norm compact, the argument of Theorem 2.2 cannot be used to show that  $x_i(\cdot)$  is norm - norm continuous. Furthermore, although versions of the Implicit Function Theorem hold for Banach spaces, they are of no use here when trying to show that  $E_i(p, U)$  or  $x_i(p, U)$  are Fréchet differentiable, since to apply those theorems, one must show that the agent's bordered Hessian is a continuous bijection, which would in this case be a continuous bijection between the nonseparable space  $\ell_\infty$  and the separable space  $\ell_1$  which is impossible. I will return to a discussion of this point in the concluding section.

that this restriction on marginal rates of transformation is satisfied is the existence of upper and lower bounds  $\underline{m}$  and  $\bar{m} \in \ell_1$  such that  $\underline{m} \leq Df(y) \leq \bar{m}$  for each  $y$  in  $\Omega_y$ .

It is also important to note the fundamental role played by the assumption that  $Df : \Omega_y \rightarrow \ell_1$  is a weak\* - norm continuous operator. The structure of MCP equilibrium in this model, which requires that in equilibrium  $p = Df(y)$ , together with the assumption that  $Df$  is weak\* - norm continuous, mean that the map  $y \mapsto Df(y) \cdot y$  is always weak\* continuous, and thus that consumers' incomes will be weak\* continuous as a function of the production plan  $y$ . Characterizing quasiequilibria in this way then avoids the problem that the evaluation functional is not jointly  $\sigma(\ell_1, \ell_\infty) \times \sigma(\ell_\infty, \ell_1)$  continuous. Examples of production functions satisfying this compactness assumption, together with all of the conditions in assumption P, are given in section 3, together with sufficient conditions to guarantee that these various assumptions are satisfied. Before continuing, the reader may wish to turn to section 3 for a discussion of these examples.

Under this assumption then, consumer  $i$ 's income  $\theta_i Df(y) \cdot (y + \omega)$  is a weak\* continuous function. However, as the equilibrium equations must be defined over not only the order interval  $[0, \bar{b}]$  which contains all of the potential equilibria, but also over the entire unbounded set  $\Omega_y$ , this assumption will not guarantee that  $\theta_i Df(y) \cdot (y + \omega)$  is a weak\* compact operator, so a slight adjustment to the budget equations will be necessary. Since  $[0, m\bar{b}]$  is weak\* compact,  $\bar{r} = \max_{y+\omega \in [0, m\bar{b}]} Df(y) \cdot (y + \omega)$  is defined and finite. Define

$$r_i(y + \omega) = \theta_i |Df(y) \cdot (y + \omega)| \wedge \bar{r} \equiv \theta_i \min(|Df(y) \cdot (y + \omega)|, \bar{r}).$$

By definition, for  $y + \omega \in [0, \bar{b}]$ ,  $r_i(y + \omega) \equiv \theta_i Df(y) \cdot (y + \omega)$ , the consumer's actual income, and as will become clearer below, we will essentially never need to consider production plans  $y$  such that  $y + \omega \notin [0, \bar{b}]$ . Moreover,  $r_i(y + \omega)$  is both bounded and weak\* continuous. In order to ensure that quasiequilibria are defined by a weak\* compact perturbation of the identity, redefine  $F : \bar{\Omega} \rightarrow \ell_\infty \times \mathcal{R}^m$  as follows:

$$F(y, U_1, \dots, U_m) = \begin{pmatrix} \sum_{i=1}^m x_i(Df(y), U_i) - (y + \omega) \\ f(y) \\ Df(y) \cdot x_2(Df(y), U_2) - r_2(y + \omega) \\ \vdots \\ Df(y) \cdot x_m(Df(y), U_m) - r_m(y + \omega) \end{pmatrix}.$$

**Theorem 2.3.** *Let  $\Omega \equiv \Omega_y \times \prod_{i=1}^m (0, U_i(\bar{b}))$ . Under assumptions P, BP, K, and H, the equilibrium map  $F$  can be written as  $F = I - C$ , where  $C : \Omega \rightarrow \ell_\infty \times \mathcal{R}^m$  is a weak\* compact operator.*

*Proof:*  $F$  can be written in the form  $I - C$  as shown above. As the sum of a finite number of compact operators is a compact operator (see Appendix I), it suffices to prove

- i.  $x_i(Df(y), U_i) : \Omega_y \times [0, U_i(\bar{b})] \rightarrow [0, \bar{b}]$  is weak\*-weak\* compact operator;
  - ii.  $E_i(Df(y), U_i) : \Omega_y \times [0, U_i(\bar{b})] \rightarrow \mathcal{R}$  is a weak\* compact operator;
- and
- iii.  $r_i(y + \omega) : \Omega_y \rightarrow \mathcal{R}$  is a weak\* compact operator.

Statements (i) and (ii) follow from Proposition A3 (Appendix I) and Theorem 2.2, together with assumption K, which assumes that  $Df$  is a weak\* - norm compact operator, since by Theorem 2.2,  $E_i(p, U_i)$  is a weak\* continuous function and  $x_i(p, U_i)$  is a norm - weak\* continuous function for  $p \in \ell_{1++}$ . By assumption K,  $Df$  is a weak\* - norm compact operator, and by assumption P5,  $Df(y) \in \ell_{1++}$  for every  $y$ , so that  $E_i(Df(y), U_i)$  and  $x_i(Df(y), U_i)$  are each the composition of a continuous function with a compact operator, which by Proposition A3 (Appendix I) is compact. Since  $Df$  is weak\* - norm compact, and the function  $(p, y) \mapsto p \cdot y$  is norm  $\times$  weak\* continuous,  $Df(y) \cdot (y + \omega)$  is weak\* continuous, so  $r_i(y + \omega)$  is a weak\* compact operator as argued above. ■

The equilibrium map  $F$  then constitutes a compact perturbation of the identity. By the homotopy invariance principle, in order to show that MCP equilibria exist and are essential, it suffices to find a compact perturbation of the identity  $G$  which is homotopic to  $F$ , which has nonzero degree at 0, and for which the resulting homotopy has no zeros on  $\partial\Omega$ . Such an approach is the main idea of the proof, which is contained in Appendix II.

**Theorem 2.4.** *Under assumptions P, BP, K and H, marginal cost pricing quasiequilibria exist. Moreover, the set of marginal cost pricing quasiequilibria is essential.*

*Proof:* See Appendix II. ■

Since the fixed structure of revenues assumption guarantees that each household has positive income at the quasiequilibrium prices, the quasiequilibria are actually equilibria, i.e., an immediate corollary of this theorem is the following.

**Corollary 2.5.** *Under assumptions P, BP, K and H, marginal cost pricing equilibria exist and are essential.*

As discussed above, if the set of attainable allocations is to be bounded, either the feasible production set must be bounded, the assumption just considered, or consumers' feasible consumption sets must be bounded, the case considered next. In addition, to ensure that any equilibria exist, we must guarantee that the set of attainable allocations at which the firm is producing an efficient production bundle is nonempty. A sufficient condition to guarantee both that the boundedness condition holds and that there are attainable allocations at which production is efficient is the following:

**Assumption BC.** Each consumer's consumption set  $X_i \subset \ell_{\infty+}$  is a weak\* closed, convex set such that  $0 \in X_i$  and such that  $X_i$  has a maximal element  $\bar{b} \in \text{int } \ell_{\infty++}$  satisfying  $x \leq \bar{b}$  for every  $x \in X_i$  and  $\bar{b} \notin \hat{Y}$ , that is,  $\bar{b}$  is not feasible for the firm.

This assumption requires that consumers' consumption sets contain an upper bound which is sufficiently large, that is, which lies outside of the firm's feasible production set  $\hat{Y}$ . Under this assumption,  $X_i \subset [0, \bar{b}]$ , and hence  $X_i$  is weak\* compact. Since equilibrium requires that  $y + \omega = \sum_{i=1}^m x_i$ , the equilibrium production plans  $y + \omega$  must lie in the order interval  $[0, m\bar{b}]$ . Moreover, Theorems 2.1, 2.2, and 2.3 hold after trivial modifications accounting for the fact that each consumer's constrained consumption set is  $X_i$  rather than  $[0, \bar{b}]$ . The marginal cost pricing equilibria can then be characterized as above as vectors  $(y, U_1, \dots, U_m) \in \ell_{\infty} \times \mathcal{R}^m$  such that

$$F(y, U_1, \dots, U_m) = \begin{pmatrix} (y + \omega) - \sum_{i=1}^m x_i(Df(y), U_i) \\ f(y) \\ Df(y) \cdot x_2(Df(y), U_2) - r_2(y + \omega) \\ \vdots \\ Df(y) \cdot x_m(Df(y), U_m) - r_m(y + \omega) \end{pmatrix} = 0.$$

Moreover, an identical argument using homotopy invariance and path following techniques can be used to establish existence and essentiality of marginal cost pricing equilibria in this model.

**Theorem 2.6.** *Under assumptions P, H, BC, and K, marginal cost pricing quasiequilibria exist and are essential.*

*Proof:* The proof is exactly the same as the proof of Theorem 2.4. ■

Approaching equilibria in infinite horizon economies using this methodology reveals an interesting feature of production sets in infinite dimensional models. In a finite dimensional model, most "reasonable" specifications of technology will give rise to a transformation function and corresponding production set for which the feasible production set  $\hat{Y}$  is bounded. In fact, as long as the marginal rates of transformation are bounded, the feasible production set will be bounded. In an infinite dimensional setting, this need not be true, an anomaly which arises not as a result of increasing returns or nonconvexities, but rather as a result of the infinite dimensionality of the model. For example, suppose the production technology exhibits constant returns to scale, so that in a finite dimensional setting the transformation function is  $f(y) = a \cdot y$ , where  $a \in \mathcal{R}_{++}^n$  is a fixed vector. Under this specification, the feasible production set  $\hat{Y}$  is convex and also bounded. The natural generalization of constant returns to scale to the setting of this paper is the technology which is given by the transformation function  $f(y) = a \cdot y$ , where  $a \in \ell_{1++}$ . In

this infinite dimensional setting with constant returns to scale, the feasible production set  $\hat{Y}$  will certainly still be convex, but will no longer be bounded in  $\ell_\infty$ , as  $\hat{Y} = \{y + \omega : a \cdot y = 0, y + \omega \geq 0\} = \{z \in \ell_{\infty+} : a \cdot z = a \cdot \omega\}$ .<sup>8</sup> Moreover, given this production set, it is straightforward to construct economies in which no MCP equilibria exist, even with a representative agent.

In such cases in which the feasible production set  $\hat{Y}$  is not bounded, the previous result shows that for all sufficiently large bounds on consumption, equilibria exist as long as consumers are constrained to consume less than this upper bound. If the set of equilibria is bounded, then there exists an upper bound large enough so that this constraint on consumption is never binding for any consumer in equilibrium, and unless the set of equilibria is bounded, these artificial upper bounds will always bind some consumer and will hence create new, artificial distortions in the economy. This restriction is analogous to short sales restrictions in financial markets models such as Radner (1968), in which the possibility of unlimited short sales means that consumers' consumption sets are unbounded below. In order to guarantee the existence of equilibria in such a model, one must either impose an artificial short sales restriction on consumers as in Radner (1968), or rule out arbitrage opportunities, that is, impose joint restrictions across consumers which essentially rule out the possibility that there are prices at which some agent wants to go arbitrarily short and some other agent arbitrarily long in the same asset (see, e.g., Werner (1987) or Brown-Werner (1991)).

Similarly, to weaken the assumption of an arbitrary bound on consumption in this model, it will be necessary to impose joint restrictions on consumers' preferences and production that rule out similar "arbitrage opportunities" and guarantee that the set of possible equilibrium allocations is bounded. Note first that by assumption K, for each date  $T$  there exists a number  $M_T$  such that for each feasible production plan  $y$ ,  $|y_t| \leq M_T$  for each  $t \leq T$ . Thus the feasible production set is only unbounded if there exist production plans  $\{y^n\}$  in  $\hat{Y}$  such that  $y_t^n \rightarrow \infty$  as  $T, n \rightarrow \infty$ . Consumption of an unbounded amount is possible, but consumers will have to wait arbitrarily long to consume it. If consumers are sufficiently impatient, they will not be willing to forgo the amount of current consumption necessary to obtain arbitrarily large consumption arbitrarily far in the future. To formalize these ideas, we will need an additional assumption and some additional notation. Assume that each consumer's utility function satisfies the partial differentiability condition that  $D_t U_i(x) = \partial U_i / \partial x_t(x)$  exists for each period  $t$  and for each nonzero  $x \in \ell_{\infty+}$ . For each  $t$ , let  $\tilde{y}_t = (0, \dots, 0, y_t, y_{t+1}, \dots)$ , and let  $\hat{y}_t = (y_1, \dots, y_t)$ . Then for each consumer and each  $t$ , define

$$\beta(t) = \sup_{\substack{x^n \in \ell_{\infty+} \\ \|\tilde{x}_t^n\| \rightarrow \infty}} \frac{D_t U(x^n)}{D_1 U(x^n)}.$$

<sup>8</sup>For every  $n$ ,  $\frac{a \cdot \omega}{a_n} e_n \in \hat{Y}$ , where  $e_n$  is the  $n^{\text{th}}$  standard basis vector, and since  $a \in \ell_1$ ,  $a_n \rightarrow 0$ , hence  $\{\frac{a \cdot \omega}{a_n} e_n\}$  is an unbounded subset of  $\hat{Y}$ .

The vector  $(\beta(1), \beta(2), \dots)$  is the consumer's vector of *asymptotic discount factors*. Similarly, define the firm's vector of *asymptotic transformation factors* by

$$\gamma(t) = \inf_{\substack{y^n \in \partial Y \\ \|\tilde{y}_n^n\| \rightarrow \infty}} \frac{D_t f(y^n)}{D_1 f(y^n)}.$$

Then we will say that the consumer is *myopic with respect to feasible production* if there exists a  $T$  such that for  $t \geq T$ ,

$$\frac{\beta(t)}{\gamma(t)} < 1.$$

If each consumer is myopic with respect to feasible production, then no one is willing to give up the consumption necessary in current periods in order to purchase arbitrarily large amounts of consumption arbitrarily far in the future when prices are equal to the marginal costs of the firm, and hence the set of equilibrium allocations will be bounded.

**Theorem 2.7.** *Under assumptions P, H, and K, if each consumer is myopic with respect to feasible production, then the set of equilibrium allocations is bounded. Moreover, marginal cost pricing equilibria exist and are essential.*

*Proof:* As argued above, it suffices to establish the first claim. Suppose by way of contradiction that the set of equilibrium allocations is unbounded. Then there exists a sequence of equilibrium allocations  $(x_1^n, x_2^n, \dots, x_m^n, y^n)$  such that  $\|\tilde{y}_n^n\| \rightarrow \infty$  and  $\|\tilde{x}_{j,n}^n\| \rightarrow \infty$  for some  $j$ . Since  $(x_1^n, x_2^n, \dots, x_m^n, y^n)$  is an equilibrium allocation for every  $n$ , the prices  $p^n = Df(y^n)$  must support the bundle  $x_i^n$  for each  $i$  and each  $n$ . However, there exists  $T$  such that if  $t \geq T$ , then for all  $i$ ,

$$\frac{D_t U_i(x_i^n)}{D_1 U_i(x_i^n)} \leq \beta(t) < \gamma(t) \leq \frac{D_t f(y^n)}{D_1 f(y^n)}$$

for all  $n$  sufficiently large. Moreover, since  $\|\tilde{y}_n^n\| \rightarrow \infty$  and  $\|\tilde{x}_{j,n}^n\| \rightarrow \infty$  for some  $j$ , for all  $n$  there exists  $t \geq n$  such that  $x_{j,t}^n > 0$ . Thus there exists  $N$  such that  $p^N$  does not support  $x_j^N$ . This contradiction implies that the set of equilibrium allocations is bounded by some number  $M$ . The existence and essentiality of marginal cost pricing equilibria then follows from Theorem 2.6 by taking the bound on consumption  $b$  to be some vector greater than  $(M, M, M, \dots)$ . ■

### 3 Compact Technologies

The results of the previous section made use of several main assumptions about the nature of the firm's production technology. This section details sufficient conditions to ensure that these assumptions are satisfied in turn, and discusses a class of examples which satisfy all of these assumptions.

There are several different ways to describe the production possibility set of a firm. By using a transformation function  $f$  to characterize the firm's production set  $Y$  as  $\{y : f(y) \leq 0\}$  and the firm's efficiency frontier as  $\{y : f(y) = 0\}$ , the equilibrium conditions that prices be set equal to marginal cost and that the equilibrium production plan be efficient can be easily translated into the equations  $p = Df(y)$  and  $f(y) = 0$ . In infinite horizon models, the production possibility set of the firm is often expressed in terms of production functions  $g_t$  in each time period rather than in terms of a transformation function. There is a natural and straightforward way to convert production functions into a transformation function in finite dimensional fixed factor supply models which can be adapted to an infinite dimensional setting. Suppose that there is a single input good, which is inelastically supplied at a fixed level  $\bar{x}$ . For a finite dimensional model, given  $(y_2, \dots, y_n)$  define

$$\begin{aligned} y_1(y_2, \dots, y_n) &= \max_{x_1} g_1(x_1) \\ &\text{s.t. } g_t(x_t) \geq y_t \quad t = 2, \dots, n; \\ &\quad \sum_{t=1}^n x_t \leq \bar{x}. \end{aligned}$$

The transformation function  $f(y)$  is then given by  $f(y_1, \dots, y_n) = y_1 - y_1(y_2, \dots, y_n)$ . Similarly, one can define the transformation function for an infinite horizon model by  $f(y) = y_1 - y_1(y_2, y_3, \dots)$  for  $y \in \ell_\infty$ , where  $y_1(y_2, y_3, \dots)$  is the optimal value function of the analogous infinite horizon optimization problem. If each  $g_t$  is strictly increasing and continuous, it is easy to see what  $y_1(\cdot)$  must be. If each  $g_t$  is increasing, the constraints become

$$\begin{aligned} g_t(x_t) &= y_t \quad t = 2, 3, \dots \\ \sum_{t=1}^{\infty} x_t &= \bar{x}. \end{aligned}$$

Since each  $g_t : \mathcal{R} \rightarrow \mathcal{R}$  is continuous and strictly increasing, each function  $g_t$  is invertible. The constraints then require that  $x_t = g_t^{-1}(y_t)$  for  $t = 2, 3, \dots$ , and  $x_1 = \bar{x} - \sum_{t=2}^{\infty} g_t^{-1}(y_t)$ , provided this sum exists. Then  $y_1(y_2, y_3, \dots) = g_1(\bar{x} - \sum_{t=2}^{\infty} g_t^{-1}(y_t))$ . For example, suppose that  $g : \mathcal{R} \rightarrow \mathcal{R}$  is continuous and strictly increasing,  $g^{-1}$  is continuous,  $0 < \beta < 1$ , and that  $g_t(x_t) = g(\frac{x_t}{\beta^t})$ . Then  $x_t = \beta^t g^{-1}(y_t)$  for each  $t = 2, 3, \dots$ , and  $f(y) = y_1 - g_1(\bar{x} - \sum_{t=2}^{\infty} \beta^t g^{-1}(y_t))$ . Since  $y \in \ell_\infty$  and  $g^{-1}$  is continuous,  $\{g^{-1}(y_t)\} \in \ell_\infty$ , and thus under this specification of  $g_t$ , the sum is well-defined.

As the results of the rest of this section will show, a class of examples which satisfy the main assumptions of the previous section can be developed by considering transformation functions of the form  $f(y) = \sum_{t=1}^{\infty} \beta^t g(y_t)$ , where  $\beta \in (0, 1)$ ,  $g : \mathcal{R} \rightarrow \mathcal{R}$

is  $C^1$ ,  $g$  and  $g'$  are bounded, and  $g'(r) > 0$  for every  $r \in \mathcal{R}$ .<sup>9</sup> That such functions do indeed satisfy these assumptions will be demonstrated below, as each assumption is discussed in turn.

The four main assumptions concerning the transformation function  $f$  describing the production set required to show that MCP equilibria exist and are essential are the following:

1.  $f$  is weak\* continuous;
2.  $f$  is continuously Fréchet differentiable and  $Df : \ell_\infty \rightarrow \ell_{1++}$ ;
3.  $Df(\Omega_y)$  is relatively norm compact;
4.  $Df(\cdot)$  is weak\*-norm continuous.

Assumptions 3 and 4 together comprise assumption K, that  $Df(\cdot)$  is a weak\*-norm compact operator.

The first main assumption, that  $f$  is weak\* continuous, will simply have to be verified for any given example, as it is for this class of examples at the end of this section. The second main assumption employed here concerning the transformation function  $f$  is that  $Df : \ell_\infty \rightarrow \ell_{1++}$ . The heart of this assumption is that  $Df(y) \in \ell_1$  for every  $y \in \ell_\infty$ . Given that  $f$  is Fréchet differentiable,  $Df(y)$  in general would be an element of  $ba$ , the norm dual of  $\ell_\infty$ . The requirement that  $Df(y) \in \ell_1 \subset ba$  reflects the standard requirement that prices be economically meaningful, that is, be representable as a vector of prices for each good. If  $f$  is strictly concave, so that the technology exhibits generalized increasing returns to scale, this will essentially follow from weak\* continuity, as shown by the following theorem.

**Theorem 3.1.** *If  $f : \ell_\infty \rightarrow \mathcal{R}$  is strictly concave, strictly monotone, continuously Fréchet differentiable, and weak\* continuous, then  $Df(y) \in \ell_1$  for every  $y \in \ell_\infty$ .*

*Proof:* See Appendix II. ■

If the transformation function  $f$  is not strictly concave, it may be possible to verify directly that  $Df(y) \in \ell_1$  for every  $y$ , as is the case with the class of functions  $f(y) = \sum_{i=1}^{\infty} \beta^i g(y_i)$ , as the following result demonstrates.

**Theorem 3.2.** *Suppose  $f(y) = \sum_{i=0}^{\infty} \beta^i g(y_i)$ , where  $g : \mathcal{R} \rightarrow \mathcal{R}$  is  $C^1$ ,  $|g'(r)| \leq M$  for some  $M < \infty$ , and  $0 < \beta < 1$ . Then  $f$  is Fréchet differentiable and  $Df(y) = \{\beta^i g'(y_i)\} \in \ell_1$  for every  $y \in \ell_\infty$ .*

*Proof:* See Appendix II.

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<sup>9</sup>An example of such a function is the principal branch of the function  $g(r) = \arctan r$ . For this function,  $|g(r)| \leq \pi/2$  for every  $r \in \mathcal{R}$ , and  $g'(r) = 1/(1+r^2)$ , which is both bounded and always positive.

To understand the third main assumption, that  $Df(\Omega_y)$  is relatively norm compact, it is important to know which subsets of  $\ell_1$  are norm compact. The space  $\ell_1$  has many nice properties which are useful in this regard; the most important for our purposes is the Schur property (see, e.g., Diestel, 1984, p.85).

**Proposition 3.1** (Schur). *In  $\ell_1$ ,  $\sigma(\ell_1, \ell_\infty)$  and norm convergence of sequences coincide; i.e.,  $x_n \xrightarrow{\sigma(\ell_1, \ell_\infty)} x \Leftrightarrow x_n \xrightarrow{n} x$ .*

The Schur property is fundamental to identifying norm compact subsets of  $\ell_1$ , as the following lemma demonstrates.

**Lemma 3.1.** *A subset  $A$  of  $\ell_1$  is norm compact if and only if it is  $\sigma(\ell_1, \ell_\infty)$  compact. In particular, if  $a_1, a_2 \in \ell_1$  and  $a_1 \leq a_2$ , then  $[a_1, a_2]$  is norm compact.*

*Proof:* By the Eberlein-Smulian theorem, a subset  $A$  of  $\ell_1$  is  $\sigma(\ell_1, \ell_\infty)$  compact if and only if every sequence  $\{x_n\} \subset A$  has a  $\sigma(\ell_1, \ell_\infty)$  convergent subsequence  $\{x_{n_k}\}$ . Schur's theorem implies that  $\{x_{n_k}\}$  is also norm convergent, i.e., every sequence in  $A$  has a norm convergent subsequence, so  $A$  is norm compact. The converse is immediate. That the order interval  $[a_1, a_2]$  is norm compact follows immediately from the fact that  $\ell_1$  is Dedekind complete and  $\sigma(\ell_1, \ell_\infty)$  is order continuous, so order intervals in  $\ell_1$  are  $\sigma(\ell_1, \ell_\infty)$  compact (Aliprantis and Burkinshaw (1985), p. 168). ■

In order to show that  $\overline{Df(\Omega_y)}$  is norm compact, it then suffices to show that it is  $\sigma(\ell_1, \ell_\infty)$  compact, or that  $Df(\Omega_y) \subset [a_1, a_2]$  for some  $a_1, a_2 \in \ell_1$ . For example, if  $f(y) = \sum_{t=1}^{\infty} \beta^t g(y_t)$ , as in Theorem 3.2,  $Df(y) = \{\beta^t g'(y_t)\}$ , and by assumption for every  $r \in \mathcal{R}$ ,  $g'(r) \in [0, M]$  for some  $M > 0$ . Hence  $Df(\Omega_y) \subset [0, a]$ , where  $a = \{M\beta^t\}$ .

The fourth main assumption on technology employed in the previous section is that  $Df : \Omega_y \rightarrow \ell_1$  is weak\* - norm continuous. If  $f$  is continuously Fréchet differentiable,  $Df$  will be norm - norm continuous by definition. Since the weak\* topology is coarser than the norm topology on  $\ell_\infty$ , satisfying this assumption requires more than just continuous differentiability; that is, the coarser is the topology on the domain, the more restrictive is the requirement that the function be continuous in that topology. The following theorem shows that the class of examples we are considering do have weak\* - norm continuous differential.

**Theorem 3.3.** *Suppose  $f(y) = \sum_{t=0}^{\infty} \beta^t g(y_t)$ , where  $g : \mathcal{R} \rightarrow \mathcal{R}$  is  $C^1$ ,  $|g'(r)| \leq M$  for some  $M < \infty$ , and  $0 < \beta < 1$ . Then  $Df : \ell_\infty \rightarrow \ell_1$  is weak\* -norm continuous.*

*Proof:* Recall from Theorem 4.7 that  $Df(y) = \{\beta^t g'(y_t)\}$ . To show  $Df$  is weak\* - norm continuous, suppose  $x^\alpha \xrightarrow{\sigma(\ell_\infty, \ell_1)} x$  for some net  $\{x^\alpha : \alpha \in \mathcal{A}\}$ . We must show  $\sum_{t=0}^{\infty} \beta^t |g'(x_t^\alpha) - g'(x_t)| \rightarrow 0$  in  $\mathcal{R}$ . Let  $\epsilon > 0$  be given. Choose  $T$  such that  $\sum_T^{\infty} \beta^t < \epsilon/2M$ , and using the continuity of  $g'$ , choose  $\delta_t > 0, t = 1, \dots, T$  such that for  $|y - x_t| < \delta_t$ ,

$$|g'(y) - g'(x_t)| < \epsilon/(1 - \beta).$$

For every  $t$ ,  $x_t^\alpha \rightarrow x_t$  in  $\mathcal{R}$ , so for  $t = 1, \dots, T$ , by the definition of convergence of nets,  $\exists \alpha_{o_t}$  such that for  $\alpha \geq \alpha_{o_t}$ ,  $|x_t^\alpha - x_t| < \delta_t$ . Let  $\alpha_o \geq \max\{\alpha_{o_t} : t = 1, \dots, T\}$ , which exists as  $T$  is finite and  $\mathcal{A}$  is a directed set. Then for  $\alpha \geq \alpha_o$ ,

$$\begin{aligned} \sum_{i=0}^{\infty} \beta^i |g'(x_i^\alpha) - g'(x_i)| &= \sum_{i=0}^{T-1} \beta^i |g'(x_i^\alpha) - g'(x_i)| + \sum_T^{\infty} \beta^i |g'(x_i^\alpha) - g'(x_i)| \\ &< \frac{\epsilon}{1-\beta} \sum_{i=0}^{T-1} \beta^i + 2M \cdot \frac{\epsilon}{2M} < 2\epsilon. \end{aligned}$$

Thus  $Df$  is weak\*-norm continuous. ■

The same argument can be used to show that if  $g : \mathcal{R} \rightarrow \mathcal{R}$  is also bounded, then  $f(y) = \sum_{i=0}^{\infty} \beta^i g(y_i)$  is weak\* continuous. The class of examples  $f(y) = \sum_{i=0}^{\infty} \beta^i g(y_i)$ , where  $g : \mathcal{R} \rightarrow \mathcal{R}$  is  $C^1$ ,  $g, g'$  are bounded, and  $g'(r) > 0$  for every  $r \in \mathcal{R}$  then satisfies all of the main assumptions of the previous section. It should be noted that this class of examples will not in general satisfy the assumption that  $\hat{Y} = (Y + \omega) \cap \ell_{\infty+}$  is bounded. However, if consumers are myopic with respect to feasible production, then marginal cost pricing equilibria exist and are essential. For example, if each consumer's utility function is additively separable with discount factor  $\beta_i < \beta$ , and if marginal utilities and rates of transformation are bounded, so that there exists  $b, B > 0$  and  $k, K > 0$  such that  $b \leq u'_i(c) \leq B$  for each  $i$  and for all  $c \geq 0$  and  $k \leq g'(r) \leq K$  for all  $r$ , then each consumer will be myopic with respect to feasible production.

## 4 Discussion

The results of the previous sections show that marginal cost pricing equilibria exist and are essential in a broad class of infinite horizon economies with nonconvexities in production. These results indicate that, in contrast with most prior approaches to equilibrium analysis in economies with an infinite dimensional commodity space, no added difficulties arise in a degree-theoretic approach to existence when the underlying technology is not convex or more generally when equilibria are not Pareto optimal. Moreover, unlike Bewley-type limiting arguments, this approach to equilibrium analysis in infinite economies based on homotopy or path-following techniques immediately yields conclusions regarding qualitative properties of equilibria, the conclusion that equilibria are essential and thus stable with respect to sufficiently small perturbations in the economy, as well as providing a framework for studying issues like determinacy, and for the computation of equilibria, at least in theory.

Although a detailed exploration of these ideas is beyond the scope of this paper and is left for future work, the basic framework for studying marginal cost pricing equilibria developed in this paper could easily be adapted to study issues of determinacy or computation of equilibria in infinite horizon economies, provided one can

show that the equilibrium equations satisfy stronger continuity or differentiability conditions. For example, note that if each consumer's Hicksian demand function  $x(p, U)$  is norm continuous rather than just weak\* continuous and if the marginal cost pricing function  $Df(y)$  is a norm compact operator,<sup>10</sup> then the composition  $x(y, U) \equiv x(Df(y), U)$  will be a norm compact operator, and minor modifications of the arguments of section 2 show that  $D(F, \Omega, 0) = 1$ , so that marginal cost pricing equilibria exist and are norm essential in such economies. Furthermore, note that for this result to hold, the Hicksian demand  $x(p, U)$  need be norm continuous only on the set  $\mathcal{P} = \{p \in \ell_1 : p = Df(y) \text{ for some } y \in \hat{Y} - \omega\}$  rather than for all possible prices  $p$ .

If in addition the composite Hicksian demand function  $x(y, U)$  is  $C^1$  on  $\hat{Y}$ , then questions about local uniqueness and determinacy can be meaningfully discussed using the same methodology developed in this paper. Of course the difficulty in applying such results is determining reasonable conditions on the primitives of the economy, preferences and technologies, which guarantee that the Hicksian demand will have these properties (for example, see Araujo (1987) and the discussion in footnote 7). As a simple example, consider an economy in which each consumer's utility function is of the form  $U_i(x) = \inf_t \alpha_t^i x_t$  for some sequence  $\alpha^i \in \text{int } \ell_{\infty++}$ , a generalization of Leontief preferences. Hicksian demand functions will then be independent of prices and linear in utility levels, and thus trivially  $C^1$  on the feasible production set  $\hat{Y}$  regardless of the firm's technology. Finding more general and less trivial conditions on preferences and technology under which these continuity and smoothness conditions hold would thus be the heart of a study of determinacy in such models, and is left for future work. However, as soon as such results are developed, the methodology and results of this paper should also provide a framework for establishing the generic determinacy of equilibria in such infinite horizon economies with increasing returns to scale.

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<sup>10</sup>Note that this latter assumption is weaker than the corresponding assumption K used in the paper, since by assumption  $Df(y)$  is norm continuous, and thus will be a norm compact operator if for each norm bounded set  $B$ , the image  $Df(B)$  has norm compact closure. Sufficient conditions for this were given in section 3.

## 5 Appendix I-Finite and Infinite Dimensional Degree Theory

This appendix discusses the basic properties of compact operators and Leray-Schauder degree theory which are used in the paper, after first giving a brief review of finite-dimensional degree theory.

As noted in the paper, the basic idea behind the degree of a function  $f$  on a domain  $\Omega$  at a point  $y \in f(\Omega)$  is to “count” the number of solutions to the equation  $f(x) = y$  in the domain  $\Omega$ . As such it can be used to answer questions concerning the existence, local uniqueness, or indeterminacy of solutions to such a system of equations. There are several minimal properties which are required of any reasonable definition of degree, if it is to provide a meaningful answer to these questions. If  $f \equiv I : \mathcal{R}^n \rightarrow \mathcal{R}^n$ , the identity map, the equation  $f(x) = y$  has the unique solution  $y \in \Omega$ , so the first axiom of degree serves as a normalization.

$$(d1). \text{ (normalization) } d(I, \Omega, y) = 1 \text{ if } y \in \Omega$$

Furthermore, if degree is to express some idea of the number and location of solutions to the equation  $f(x) = y$  in  $\Omega$ , it should essentially be additive across domains.

$$(d2). \text{ (domain decomposition) } d(f, \Omega, y) = d(f, \Omega_1, y) + d(f, \Omega_2, y) \text{ for } \Omega_1, \Omega_2 \text{ disjoint open sets such that } y \notin f(\bar{\Omega} \setminus (\Omega_1 \cup \Omega_2))$$

Finally, degree should be a homotopy invariant, so that if a function  $f$  can be continuously deformed into another function  $g$  in such a manner so as to avoid solutions on the boundary at every step, then the degree of  $f$  and  $g$  should be the same at  $y$ .

$$(d3). \text{ (homotopy invariance) } d(h(t, \cdot), \Omega, y(t)) \text{ is independent of } t \in [0, 1] \text{ if } h : [0, 1] \times \bar{\Omega} \rightarrow \mathcal{R}^n \text{ is continuous, } y : [0, 1] \rightarrow \mathcal{R}^n \text{ is continuous, and } y(t) \notin h(t, \partial\Omega) \text{ for every } t \in [0, 1].$$

There is a unique function

$$d : \{(f, \Omega, y) : \Omega \subset \mathcal{R}^n \text{ open, bounded, } f : \bar{\Omega} \rightarrow \mathcal{R}^n \text{ is continuous, } y \notin f(\partial\Omega)\} \rightarrow \mathcal{Z}$$

satisfying these three conditions, which is called *Brouwer's degree*, and is denoted  $d(f, \Omega, y)$ . For a construction of this function and a more detailed discussion of Brouwer's degree, see Deimling (1985) or Lloyd (1978). It is important to notice that Brouwer's degree is defined in theory for all continuous functions  $f : \Omega \rightarrow \mathcal{R}^n$  and at all points  $y \notin f(\partial\Omega)$ , and not just for continuously differentiable functions for which  $y$  is a regular value, that is, a value such that for every preimage  $x \in f^{-1}(y)$ , the derivative  $Df(x)$  is surjective. However, if  $f$  is continuously differentiable on  $\bar{\Omega}$ , and  $y \notin f(\partial\Omega)$  is a regular value of  $f$ , then  $d(f, \Omega, y)$  can be computed by the convenient formula

$$d(f, \Omega, y) = \sum_{x \in f^{-1}(y)} \text{sgn det } Df(x).$$

The axioms of degree can be used to show that the degree has a number of useful properties: in particular, the degree does convey information about the existence of solutions to the equation  $f(x) = y$ , as

$$(d4). \quad d(f, \Omega, y) \neq 0 \text{ implies that } f^{-1}(y) \neq \emptyset.$$

Moreover, if  $f$  is actually continuously differentiable on  $\bar{\Omega}$  and  $y$  is a regular value of  $f$ , much stronger conclusions can be drawn from the result that  $d(f, \Omega, y) \neq 0$ . Indeed, if  $d(f, \Omega, y) \neq 0$ , then not only is  $f^{-1}(y)$  nonempty as it is in the case when  $f$  is merely continuous, but  $f^{-1}(y)$  is then a finite nonempty set of points. Knowing that the degree at a particular point  $y$  is nonzero will then lead directly to the determinacy conclusion that there are a finite nonzero number of solutions to the equation  $f(x) = y$  in the case that  $f \in C^1(\bar{\Omega})$  and  $y$  is a regular value of  $f$ .

Also as discussed in section 2, extending the notion of degree to infinite dimensional spaces is not trivial. The degree should still provide a useful and meaningful answer to questions concerning the existence and number of solutions to a system of equations, so it should still satisfy the degree axioms (d1)-(d3). In infinite dimensional spaces however, degree cannot be defined for all continuous functions and still satisfy the axioms (d1)-(d3). There are many counterexamples, see e.g., Krasnosel'skii and Zabreiko (1980), but perhaps the clearest way to see the necessity of working with a more restrictive class of maps than continuous maps is to recall that Brouwer's fixed point theorem, which states that a continuous function mapping a closed, bounded, convex subset of  $\mathcal{R}^n$  into itself must have a fixed point, can be proven using only the axioms (d1)-(d3). If it were possible to define degree for all continuous functions in an infinite dimensional space which still satisfied (d1)-(d3), the same proof of Brouwer's theorem could be used to show that every continuous function from a closed bounded convex subset of some infinite dimensional space into itself must have a fixed point, which clearly is not true. For example, consider the domain  $c_0$ , the space of sequences which converge to 0, where for  $x \in c_0$ ,  $\|x\| = \sup_n |x_n|$ , and let  $F : c_0 \rightarrow c_0$  be defined by  $F(x) = ((1 + \|x\|)/2, x_1, x_2, \dots)$ . Then  $F : B_1(0) \rightarrow B_1(0)$  but  $F$  has no fixed points (see Deimling p. 37). As is often the case in infinite dimensional spaces, what is lacking is compactness, since arbitrary closed bounded sets certainly need not be compact in an infinite dimensional space.

As noted in the paper, the key idea which leads to an appropriate class of functions for which degree satisfying axioms (d1)-(d3) can be defined is that of a compact operator. Compact operators have a number of important and useful properties, perhaps the most important of which is that compact operators can always be approximated by maps with finite dimensional range. With the above notation,  $F$  is called *finite dimensional* if  $F(\Omega) \subset X^\alpha$  where  $X^\alpha$  is a finite dimensional subspace of  $X$ . The following theorem is central to the development of degree for Banach spaces, and can be found in Deimling (1985, Proposition 8.1). Recall that a function  $f : X \rightarrow Y$  is *proper* if for every compact set  $K \subset Y$ ,  $f^{-1}(K)$  is compact.

**Proposition A1.** *Let  $X$  be a Banach space,  $\Omega \subset X$  closed and bounded, and let*

$F : \Omega \rightarrow X$  be a compact operator. For every  $\epsilon > 0$  there exists a finite-dimensional operator  $F_\epsilon : \Omega \rightarrow X$  such that

$$\sup_{\Omega} \|F(x) - F_\epsilon(x)\| < \epsilon.$$

Moreover,  $I - F$  is proper.

An analogous version of Proposition A1 holds for compact operators in locally convex spaces, with the appropriate changes in notation, and can again be found in Deimling (1985, Proposition 10.1).

**Proposition A2.** *Let  $X$  be locally convex,  $\Omega \subset X$ , and  $F : \Omega \rightarrow X$  compact, i.e., continuous and such that  $\overline{F(\Omega)}$  is compact. Then we have*

- i. *For  $U \in \mathcal{U}(0)$ , the set of neighborhoods of 0, there exists a finite dimensional map  $F_U$  such that  $F_U(x) - F(x) \in U$  for  $x \in \Omega$ .*
- ii.  *$I - F$  maps closed subsets of  $\Omega$  onto closed sets.*

Furthermore, linear combinations of compact operators are compact, as are compositions of continuous functions with compact operators (Krasnosel'skii and Zabreiko, p.73).

**Proposition A3.** *Let  $(X, \tau_1)$  and  $(Y, \tau_2)$  be locally convex linear topological vector spaces, and  $\Omega \subset X$ . If  $F, G : \Omega \rightarrow Y$  are  $\tau_1 - \tau_2$  compact operators, so is  $F + G$ . If  $H : Y \rightarrow X$  is  $\tau_2 - \tau_1$  continuous,  $H \circ F : \Omega \rightarrow X$  is a  $\tau_1$  compact operator.*

Using these properties of compact operators, the appropriate extension of Brouwer's degree to infinite dimensional spaces, called *Leray-Schauder degree*, can be defined for the class of *compact perturbations of the identity*, which are maps of the form  $I - F$  where  $F : \Omega \rightarrow X$  is a compact operator. That is, there exists a unique function

$$D : \{(I - F, \Omega, y) : \Omega \subset X \text{ open, bounded, } F : \bar{\Omega} \rightarrow X \text{ a compact operator, } y \notin (I - F)(\partial\Omega)\} \rightarrow \mathcal{Z}$$

satisfying (D1)- (D3):

(D1). (normalization)  $D(I, \Omega, y) = 1$  if  $y \in \Omega$ ;

(D2). (domain decomposition)  $D(I - F, \Omega, y) = D(I - F, \Omega_1, y) + D(I - F, \Omega_2, y)$  for  $\Omega_1, \Omega_2$  disjoint open sets such that  $y \notin (I - F)(\bar{\Omega} \setminus (\Omega_1 \cup \Omega_2))$ ;

(D3). (homotopy invariance)  $D(I - H(t, \cdot), \Omega, y(t))$  is independent of  $t \in [0, 1]$  if  $H : [0, 1] \times \bar{\Omega} \rightarrow X$  is compact,  $y : [0, 1] \rightarrow X$  is continuous, and  $y(t) \notin (I - H(t, \cdot))(\partial\Omega)$  for every  $t \in [0, 1]$ .

As in the finite dimensional case, it is a relatively straightforward consequence of (D1)-(D3) that

$$(D4). \quad D(I - F, \Omega, y) \neq 0 \text{ implies that } (I - F)^{-1}(y) \neq \emptyset.$$

Also as in the finite dimensional case, the distinction remains between the interpretation of the degree for compact perturbations of the identity which are in addition continuously differentiable and for which  $y$  is a regular value, and those which are merely continuous. If  $F$  is continuously Fréchet differentiable on  $\Omega$  and  $y$  is a regular value of  $I - F$ , that is, a value for which  $D(I - F)(x)$  is surjective for every point  $x$  satisfying the equation  $(I - F)(x) = y$ , then  $(I - F)^{-1}(y)$  is a finite set. Moreover, as with nonsingular  $n \times n$  matrices, there are exactly two homotopy classes  $H^+$  and  $H^-$  of linear isomorphisms of a Banach space, and  $D(I - F, \Omega, y)$  can be computed by the formula:

$$D(I - F, \Omega, y) = \sum_{x \in (I - F)^{-1}(y)} j(D(I - F)(x))$$

where

$$j(D(I - F)(x)) = \begin{cases} -1, & \text{if } D(I - F)(x) \in H^-; \\ 1, & \text{if } D(I - F)(x) \in H^+. \end{cases}$$

The function  $j(\cdot)$  is the natural generalization of  $\text{sgn det}(\cdot)$ , since for  $n \times n$  matrices,  $H^-$  is the set of matrices with negative determinant and  $H^+$  is the set of matrices with positive determinant. Again if  $y$  is a regular value of the continuously Fréchet differentiable function  $I - F$  and  $D(I - F, \Omega, y) \neq 0$ , one can conclude directly that the equation  $(I - F)(x) = y$  has a finite nonzero number of solutions in the set  $\Omega$ . Moreover, under these regularity conditions, if  $D(I - F, \Omega, y) > 1$ , then the equation  $(I - F)(x) = y$  has multiple solutions, an important implication for many economic applications such as macroeconomic models of coordination failure.

If  $F$  is only continuous or if  $y$  is not a regular value of  $I - F$ , one can still ask as in the finite dimensional case whether a solution or set of solutions is essential. Again by homotopy invariance, if  $D(I - F, \Omega, y) \neq 0$ , the set of solutions to the equation  $(I - F)(x) = y$  is essential, and hence given any neighborhood  $U$  of the set of solutions, arbitrarily small perturbations of the system of equations  $I - F$  must still have solutions in this neighborhood  $U$ .

Leray-Schauder degree can be extended to locally convex spaces, again with the requisite changes in notation and terminology. The major change from the Banach space formulation is that in locally convex spaces open sets are not generally bounded, and the degree is defined for triples  $(I - F, \Omega, y)$  where  $\Omega$  is open,  $\overline{F(\Omega)}$  is compact and  $F$  is continuous with respect to the given topology. For a more detailed discussion of Leray-Schauder degree theory, see Deimling (1985), Lloyd (1978), or Krasnosel'skii and Zabreiko (1980).

## 6 Appendix II

The first part of this appendix is concerned with proving the main result of the paper, that MCP equilibria exist and are essential; the remainder of this section is devoted to several other lengthy proofs.

In order to show that MCP equilibria exist and are essential, the idea of the proof is to apply a homotopy to reduce the problem to a simple starting problem whose solutions are known. To define the starting equations, choose  $(\bar{y}, \bar{U}_1, \dots, \bar{U}_m) \in \Omega$  such that  $\bar{y} + \omega \gg 0$ ,  $f(\bar{y}) < 0$  (see Figure 2), and such that  $\bar{U}_i \in (0, U_i(\omega))$  for each  $i$ . Set

$$G(y, U_1, \dots, U_m) = (y - \bar{y}, U_1 - \bar{U}_1, \dots, U_m - \bar{U}_m).$$

Certainly  $G$  is a compact perturbation of the identity, and by (D1),  $D(G, \Omega, 0) = 1$ . The map  $G$  will serve as the candidate starting equations.

**Theorem 2.4.** *Under assumptions P, BP, K and H, marginal cost pricing quasiequilibria exist. Moreover, the set of marginal cost pricing quasiequilibria is essential.*

*Proof:* Recall that  $\Omega = \Omega_y \times \prod_{i=1}^m (0, U_i(m\bar{b}))$ , where  $\Omega_y = \{y : f(y) < f(\bar{b} - \omega)\}$ , where  $\bar{b} \geq m\bar{b}$ . First, make a modification to  $F$ . For each  $i = 1, \dots, m$ , choose  $\tilde{U}_i$  such that  $U_i(\bar{b}) < \tilde{U}_i < U_i(m\bar{b})$ , and let  $\chi_{U_i} : [0, U_i(m\bar{b})] \rightarrow [0, 1]$  be a smooth function such that

$$\begin{aligned} \chi_{U_i}([0, U_i(\bar{b})]) &\equiv 1; \\ \chi_{U_i}([\tilde{U}_i, U_i(m\bar{b})]) &\equiv 0. \end{aligned}$$

Define

$$\hat{x}_i(Df(y), U_i) = \chi_{U_i}(U_i) x_i(Df(y), U_i \wedge U_i(\bar{b})) + (1 - \chi_{U_i}(U_i)) m\bar{b}.$$

Then  $\hat{x}_i(\cdot)$  is a  $\sigma(\ell_\infty, \ell_1)$  compact operator, and note that for all values  $U_i$  such that  $U_i \leq U_i(\bar{b})$ ,  $\hat{x}_i(Df(y), U_i) \equiv x_i(Df(y), U_i)$ . Now define

$$\tilde{F}(y, U_1, \dots, U_m) = \begin{pmatrix} (y + \omega) - \sum_{i=1}^m x_i(Df(y), U_i \wedge U_i(\bar{b})) \\ f(y) \\ Df(y) \cdot \hat{x}_2(Df(y), U_2) - r_2(y + \omega) \\ \vdots \\ Df(y) \cdot \hat{x}_m(Df(y), U_m) - r_m(y + \omega) \end{pmatrix}.$$

Then  $\tilde{F}$  is still a  $\sigma(\ell_\infty, \ell_1)$  compact perturbation of the identity. Moreover,  $\tilde{F}^{-1}(0) = F^{-1}(0)$ , as if  $U_i > U_i(\bar{b})$  for some  $i$ ,  $x_i(Df(y), U_i \wedge U_i(\bar{b})) \equiv x_i(Df(y), U_i(\bar{b})) \equiv \bar{b}$ , hence  $\sum_{i=1}^m x_i(Df(y), U_i \wedge U_i(\bar{b})) \geq \bar{b}$ . By choice of  $\bar{b}$ , if  $f(y) = 0$ ,  $y + \omega \not\geq \bar{b}$ , so  $y + \omega \neq \sum_{i=1}^m x_i(Df(y), U_i \wedge U_i(\bar{b}))$ . That is,  $\tilde{F}(\cdot) \neq 0$  if  $U_i > U_i(\bar{b})$  for any  $i$ , and

hence  $\tilde{F}^{-1}(0) = F^{-1}(0)$ . Thus the equation  $\tilde{F}(\cdot) = 0$  defines the MCP quasiequilibria. Then let

$$H(y, U_1, \dots, U_m, t) = (1-t)G(y, U_1, \dots, U_m) + t\tilde{F}(y, U_1, \dots, U_m).$$

So  $H = I - C(y, U_1, \dots, U_m, t)$ , where

$$C(y, U_1, \dots, U_m, t) = t \begin{pmatrix} \sum_{i=1}^m x_i(Df(y), U_i \wedge U_i(\bar{b})) - \omega \\ U_1 - f(y) \\ U_2 - [Df(y) \cdot x_2(Df(y), U_2) - r_2(y + \omega)] \\ \vdots \\ U_m - [Df(y) \cdot x_m(Df(y), U_m) - r_m(y + \omega)] \end{pmatrix} + (1-t) \begin{pmatrix} \bar{y} \\ \bar{U}_1 \\ \vdots \\ \bar{U}_m \end{pmatrix}$$

and  $C : \Omega \times [0, 1] \rightarrow \ell_\infty \times \mathcal{R}^m$  is a  $\sigma(\ell_\infty, \ell_1) \times \tau_{\mathcal{R}^m} \times \tau_{\mathcal{R}} - \sigma(\ell_\infty, \ell_1) \times \tau_{\mathcal{R}^m}$  compact operator. Using homotopy invariance, it suffices to show that  $H(\partial\Omega, t) \neq 0$  for all  $t \in [0, 1]$ .

By choice of  $(\bar{y}, \bar{U}_1, \dots, \bar{U}_m)$ , this is true for  $t = 0$ , since

$$\begin{aligned} \partial\Omega &= \{(y, U_1, \dots, U_m) : f(y) = f(\bar{b} - \omega)\} \\ &\cup \left( \bigcup_{i=1}^m \{(y, U_1, \dots, U_m) : U_i \in \{0, U_i(m\bar{b})\}\} \right). \end{aligned}$$

This is also true for  $t = 1$ . By feasibility, there can be no equilibria with  $y \in \partial\Omega_y$  since  $f(y) \neq 0$  for such  $y$  by definition, or such that  $y + \omega \not\geq 0$  since  $\sum_{i=1}^m x_i(Df(y), U_i \wedge U_i(\bar{b})) \geq 0$  for all  $y$ , and quasiequilibrium requires that  $\sum_{i=1}^m x_i(Df(y), U_i \wedge U_i(\bar{b})) = y + \omega$ . Since quasiequilibrium requires that  $\sum_{i=1}^m x_i(Df(y), U_i \wedge U_i(\bar{b})) = y + \omega$ ,  $y + \omega \in [0, m\bar{b}]$ , and hence  $r_i(y + \omega) \equiv \theta_i Df(y) \cdot (y + \omega)$  for every  $i$ . If  $U_i = 0$  for some  $i$ , then by definition,  $x_i(Df(y), 0) = 0$  for all  $y$ , and  $Df(y) \cdot (x_i(Df(y), 0) - \theta_i(y + \omega)) = -\theta_i Df(y) \cdot (y + \omega) < 0$  for all  $y$  such that  $f(y) = 0$  and  $y + \omega \geq 0$ . If  $U_i = U_i(m\bar{b})$ , again by definition  $x_i(Df(y), U_i(m\bar{b}) \wedge U_i(\bar{b})) = \bar{b}$ , which would violate feasibility.

Now for  $0 < t < 1$ , note that  $H(y, U_1, \dots, U_m, t) = 0 \Rightarrow$

$$(1). \quad t[\sum_{i=1}^m x_i(Df(y), U_i \wedge U_i(\bar{b})) - (y + \omega)] = (1-t)(y - \bar{y}) = (1-t)[(y + \omega) - (\bar{y} + \omega)].$$

Then first note that (1) will not hold if  $y + \omega \not\geq 0$ , since if for some component  $n$   $(y + \omega)_n < 0$ , then for the corresponding component of equation (1), the left hand side is nonnegative, while the right hand side is strictly negative, so equality in equation (1) cannot hold unless  $y + \omega \geq 0$ . Then assume  $y + \omega \geq 0$ . Equation (1) is also violated if  $y \in \partial\Omega_y$ , since then there is some component  $n$  such that  $(y + \omega)_n \geq (m\bar{b})_n$ , and for the corresponding component of equation (1), the left hand side is nonpositive while the right hand side is strictly positive by choice of  $\bar{y}$ . So if (1) is to hold,  $y + \omega \in [0, m\bar{b}]$ . For the remainder of the proof, assume  $y + \omega \in [0, m\bar{b}]$ . Then note that  $H(y, U_1, \dots, U_m, t) = 0 \Leftrightarrow$

- (1).  $t[\sum_{i=1}^m x_i(Df(y), U_i \wedge U_i(\bar{b})) - (y + \omega)] = (1 - t)(y - \bar{y})$ ;  
(2).  $-tDf(y) \cdot (\hat{x}_i(Df(y), U_i) - \theta_i(y + \omega)) = (1 - t)(U_i - \bar{U}_i)$  for  $i = 2, \dots, m$ ;

and

$$(3). \quad tf(y) = (1 - t)(U_1 - \bar{U}_1)$$

Moreover,  $\hat{x}_i(Df(y), 0) = x_i(Df(y), 0) = 0$  for all  $y$ , so if  $U_i = 0$  for some  $i = 2, \dots, m$ , for  $y + \omega \geq 0$  equation (2) becomes

$$0 \leq t\theta_i Df(y) \cdot (y + \omega) = -(1 - t)\bar{U}_i < 0,$$

which is a contradiction. Similarly,  $\hat{x}_i(Df(y), U_i(m\bar{b})) = m\bar{b}$  for all  $y$ , so for  $y + \omega \in [0, m\bar{b}]$ , equation (2) will be violated if  $U_i = U_i(m\bar{b})$  for some  $i = 2, \dots, m$ , as it becomes

$$0 \geq -tDf(y) \cdot (m\bar{b} - \theta_i(y + \omega)) = (1 - t)(U_i(m\bar{b}) - \bar{U}_i) > 0,$$

which is again a contradiction.

Now consider equation (3). The strategy for the remainder of the proof is to show that there exist values  $t'$  and  $t''$  such that if  $t \leq t'$  or  $t \geq t''$ ,  $H(\cdot, t) \neq 0$  if  $U_1 = 0$  or  $U_1 = U_1(m\bar{b})$ , and then to slightly change equation (3) so that the resulting homotopy will have no zeros for  $t \in [t', t'']$  as well if  $U_1 \in \{0, U_1(m\bar{b})\}$ .

Since  $f$  is bounded, there exists  $M > 0$  such that  $|f(y)| \leq M$  for all  $y + \omega \in [0, m\bar{b}]$ . Choose  $\gamma > 0$  such that  $\gamma < \min(\bar{U}_1, U_1(m\bar{b}) - \bar{U}_1)$ . For  $t < 1$ , the function  $h(t) = \frac{t}{1-t}M$  is continuous, and  $h(0) = 0$ , so there exists  $t' > 0$  such that if  $t \leq t'$ ,  $\frac{t}{1-t}M < \gamma$ . Then if  $t \leq t'$ , (3) fails if either  $U_1 = 0$  or  $U_1 = U_1(m\bar{b})$ , as (3) requires

$$-\frac{t}{1-t}f(y) = U_1 - \bar{U}_1$$

or

$$\frac{t}{1-t}|f(y)| = |U_1 - \bar{U}_1|.$$

If  $U_1 = 0$ , this becomes

$$\frac{t}{1-t}|f(y)| = \bar{U}_1$$

but by the choice of  $t'$  and the definition of  $M$  and  $\gamma$ ,

$$\gamma > \frac{t}{1-t}M \geq \frac{t}{1-t}|f(y)| = \bar{U}_1 > \gamma$$

which is a contradiction. Similarly, if  $U_1 = U_1(m\bar{b})$ , (3) requires

$$\frac{t}{1-t}|f(y)| = U_1(m\bar{b}) - \bar{U}_1$$

which by choice of  $t'$  and definition of  $M$  and  $\gamma$  implies that

$$\gamma > \frac{t}{1-t}M \geq \frac{t}{1-t}|f(y)| = U_1(m\bar{b}) - \bar{U}_1 > \gamma,$$

which is again a contradiction. So for  $t \leq t'$ , (3) fails if  $U_1 = 0$  or  $U_1 = U_1(m\bar{b})$ .

Now we want to show that there exists a  $t'' < 1$  such that if  $t \geq t''$ , equations (1), (2) and (3) cannot all hold if either  $U_1 = 0$  or  $U_1 = U_1(m\bar{b})$ . For simplicity, define  $\tilde{f}(z) = f(z - \omega)$ , so that  $\tilde{f}(y + \omega) = f(y)$ . Since  $\tilde{f}(\bar{b}) = f(\bar{b} - \omega) > 0$ , let  $\eta > 0$  be such that  $\eta < \tilde{f}(\bar{b})/4$ . Since  $\tilde{f}$  is  $\sigma(\ell_\infty, \ell_1)$  continuous it is norm continuous, so there exists  $\delta > 0$  such that if  $\|z - \bar{b}\| \leq \delta$ ,  $|\tilde{f}(z) - \tilde{f}(\bar{b})| < \eta$ . Choose  $\epsilon > 0$  such that  $\epsilon < \tilde{f}(\bar{b}) - \eta$ . Then if  $\tilde{f}(y + \omega) = f(y) < \epsilon$ ,  $\tilde{f}(\bar{b}) - \tilde{f}(y + \omega) > \eta$ , so  $\|(y + \omega) - \bar{b}\| > \delta$ . Also, as  $\tilde{f}$  is strictly monotone, if  $\tilde{f}(y + \omega) = f(y) < \epsilon$ ,  $\|(y + \omega) - s\| > \delta$  for all  $s \geq \bar{b}$ ; that is (see Figure 3)

$$\{r : \|r - s\| \leq \delta \text{ for some } s \geq \bar{b}\} \cap \{r : \tilde{f}(r) < \epsilon\} = \emptyset.$$

Now there exists  $t_1 < 1$  such that if  $t \geq t_1$ ,

$$\frac{1-t}{t}|U_1 - \bar{U}_1| < \epsilon \quad \text{and} \quad \frac{1-t}{t}\|y - \bar{y}\| < \delta/2$$

for all  $y$  such that  $y + \omega \in [0, m\bar{b}]$  and for all  $U_1 \in [0, U_1(m\bar{b})]$ . Then in particular, for  $t \geq t_1$ , if (3) is to hold we must have

$$|f(y)| = \frac{1-t}{t}|U_1 - \bar{U}_1| < \epsilon.$$

Then if  $t \geq t_1$ ,  $f(y) < \epsilon$ , and (1) holds, we have

$$\left\| \sum_{i=1}^m x_i (Df(y), U_i \wedge U_i(\bar{b})) - (y + \omega) \right\| = \frac{1-t}{t}\|y - \bar{y}\| < \delta/2. (*)$$

By definition of  $\delta$ , if  $t \geq t_1$  and  $f(y) < \epsilon$ , (1) will then fail unless  $U_i \leq U_i(\bar{b})$  for  $i = 1, \dots, m$ , since if  $U_i > U_i(\bar{b})$  for any  $i$ ,  $\sum_{i=1}^m x_i (Df(y), U_i \wedge U_i(\bar{b})) \geq \bar{b}$ , so  $(*) \Rightarrow$

$$\|(y + \omega) - s\| < \delta/2 \quad \text{where } s \geq \bar{b}.$$

But this is a contradiction, since  $f(y) < \epsilon$ . Then in particular, for  $t \geq t_1$ , (1) and (3) cannot both hold if  $U_1 = U_1(m\bar{b})$ .

Now suppose  $U_1 = 0$ . Without loss of generality assume  $\bar{U}_i$  have been chosen such that  $U_i(\bar{b}) - \bar{U}_i > \sum_{j \neq i} \bar{U}_j$  for each  $i$ , so that if  $U_i \geq U_i(\bar{b})$  for some  $i$ ,  $\sum_{i=1}^m (U_i - \bar{U}_i) > 0$ .

Since  $\bar{b} \in \text{int } \ell_{\infty+}$ , we can choose  $\delta$  such that  $\bar{b} - \delta e \equiv (\bar{b}_1 - \delta, \bar{b}_2 - \delta, \dots) \in \text{int } \ell_{\infty+}$ . Let  $\beta \in (0, 1)$ . Then if  $\|z - (y + \omega)\| < \delta$  and  $z \geq \bar{b}$ ,  $y + \omega \geq \beta(\bar{b} - \delta e)$  (see figure 3). Moreover, without loss of generality we can assume  $\bar{y}$  has been chosen so that  $\bar{y} + \omega \ll \beta(\bar{b} - \delta e)$ ,<sup>11</sup> so that for every  $y$ ,  $Df(y) \cdot (\bar{y} + \omega) < Df(y) \cdot \beta(\bar{b} - \delta e)$ , or  $Df(y) \cdot (\beta(\bar{b} - \delta e) - (\bar{y} + \omega)) > 0$ .

<sup>11</sup>Pick  $\bar{y} = -\omega + \alpha(\bar{b} - \delta e)$ , where using the Intermediate Value Theorem, we choose  $\alpha > 0$  such that  $\alpha < \beta$  and  $\tilde{f}(\alpha(\bar{b} - \delta e)) < 0$ , noting that at  $\alpha = 1$ ,  $\tilde{f}(\bar{b} - \delta e) > 0$  and at  $\alpha = 0$ ,  $\tilde{f}(0) = f(-\omega) < 0$ .

Now we claim that if  $U_1 = 0$  and  $t \geq t_1$ , (1) and (2) cannot both hold if  $U_i \geq U_i(\bar{b})$  for some  $i = 2, \dots, m$ , regardless of  $f(y)$ . To see this, note that  $x_1(Df(y), 0) = 0$ , so (1) becomes

$$t \left[ \sum_{i=2}^m x_i(Df(y), U_i \wedge U_i(\bar{b})) - (y + \omega) \right] = (1-t)(y - \bar{y})$$

and by summing (2) over  $i = 2, \dots, m$ ,

$$Df(y) \cdot \left[ \sum_{i=2}^m \tilde{x}_i(Df(y), U_i) - (1 - \theta_1)(y + \omega) \right] = -\frac{1-t}{t} \sum_{i=2}^m (U_i - \bar{U}_i).$$

Suppose some  $U_i \geq U_i(\bar{b})$ . Then  $\sum_{i=2}^m x_i(Df(y), U_i \wedge U_i(\bar{b})) \geq \bar{b}$  and  $\sum_{i=2}^m \tilde{x}_i(Df(y), U_i) \geq \sum_{i=2}^m x_i(Df(y), U_i \wedge U_i(\bar{b}))$ . Moreover,  $-\frac{1-t}{t} \sum_{i=2}^m (U_i - \bar{U}_i) < 0$ . Since  $\theta_1 > 0$ , we have

$$\sum_{i=2}^m x_i(Df(y), U_i \wedge U_i(\bar{b})) - (y + \omega) \leq \sum_{i=2}^m \tilde{x}_i(Df(y), U_i) - (1 - \theta_1)(y + \omega),$$

and thus applying  $Df(y)$  to both sides yields

$$\begin{aligned} Df(y) \left[ \sum_{i=2}^m x_i(Df(y), U_i \wedge U_i(\bar{b})) - (y + \omega) \right] &\leq Df(y) \left[ \sum_{i=2}^m \tilde{x}_i(Df(y), U_i) - (1 - \theta_1)(y + \omega) \right] \\ &= -\frac{1-t}{t} \sum_{i=2}^m (U_i - \bar{U}_i) < 0, \end{aligned}$$

i.e.,

$$Df(y) \cdot \left[ \sum_{i=2}^m x_i(Df(y), U_i \wedge U_i(\bar{b})) - (y + \omega) \right] < 0.$$

However, since  $t \geq t_1$ , by (1)

$$\left\| \sum_{i=2}^m x_i(Df(y), U_i \wedge U_i(\bar{b})) - (y + \omega) \right\| < \delta/2.$$

Applying  $Df(y)$  to both sides of (1) implies that

$$\begin{aligned} t Df(y) \cdot \left[ \sum_{i=2}^m x_i(Df(y), U_i \wedge U_i(\bar{b})) - (y + \omega) \right] &= (1-t) Df(y) \cdot (y - \bar{y}) \\ &= (1-t) Df(y) \cdot [(y + \omega) - (\bar{y} + \omega)]. \end{aligned}$$

Since  $\sum_{i=2}^m x_i(Df(y), U_i \wedge U_i(\bar{b})) \geq \bar{b}$ , by choice of  $\delta$ ,  $y + \omega \geq \beta(\bar{b} - \delta e)$ , and then by choice of  $\bar{y}$ ,

$$Df(y) \cdot [(y + \omega) - (\bar{y} + \omega)] \geq Df(y) \cdot [\beta(\bar{b} - \delta e) - (\bar{y} + \omega)] > 0,$$

that is

$$Df(y) \cdot \left[ \sum_{i=2}^m x_i(Df(y), U_i \wedge U_i(\bar{b})) - (y + \omega) \right] > 0,$$

which is a contradiction.

Hence when  $t \geq t_1$ , if  $f(y) < \epsilon$  or  $U_1 = 0$ , equations (1) and (2) cannot hold unless  $U_i \leq U_i(\bar{b})$  for all  $i = 1, \dots, m$ . Then assuming  $U_i \leq U_i(\bar{b})$  for all  $i$ , equations (1), (2) and (3) become

$$(1). \quad t \left( \sum_{i=1}^m x_i(Df(y), U_i) - (y + \omega) \right) = (1-t)(y - \bar{y});$$

$$(2). \quad -t Df(y) \cdot (x_i(Df(y), U_i) - \theta_i(y + \omega)) = (1-t)(U_i - \bar{U}_i) \text{ for } i = 2, \dots, m;$$

and

$$(3). \quad t f(y) = (1-t)(U_1 - \bar{U}_1).$$

By applying  $Df(y)$  to both sides of (1) we get

$$t Df(y) \cdot \left[ \sum_{i=1}^m x_i(Df(y), U_i) - (y + \omega) \right] = (1-t) Df(y) \cdot (y - \bar{y}),$$

so that after summing (2) over  $i = 2, \dots, m$  and subtracting we have

$$(4). \quad t Df(y) \cdot (x_1(Df(y), U_1) - \theta_1(y + \omega)) = (1-t) [Df(y) \cdot (y - \bar{y}) + \sum_{i=2}^m (U_i - \bar{U}_i)].$$

Now since  $\tilde{f}(0) = f(-\omega) < 0$ , without loss of generality choose  $\epsilon > 0$  so that  $\epsilon < |f(-\omega)|/2$  as well. Then by choice of  $\epsilon$ ,  $D = \{y : f(y) \geq -\epsilon, y + \omega \in [0, m\bar{b}]\}$  is a  $\sigma(\ell_\infty, \ell_1)$  compact set, and  $-\omega \notin D$ , so that  $Df(y) \cdot (y + \omega) > 0$  for all  $y \in D$ . Since  $Df(\cdot)$  is  $\sigma(\ell_\infty, \ell_1)$ -norm continuous,  $Df(y) \cdot (y + \omega)$  is  $\sigma(\ell_\infty, \ell_1)$  continuous, and thus  $C = \min_{y \in D} Df(y) \cdot (y + \omega)$  is attained and hence positive. Then there exists  $t'' < 1$  such that  $t'' \geq t_1$  and such that if  $t \geq t''$ ,

$$\frac{1-t}{t} |Df(y) \cdot (y - \bar{y}) + \sum_{i=2}^m (U_i - \bar{U}_i)| < \theta_1 C.$$

Hence if  $t \geq t''$  and  $f(y) \geq -\epsilon$ , (4) will fail if  $U_1 = 0$ , as it becomes

$$-\theta_1 Df(y) \cdot (y + \omega) = \frac{1-t}{t} [Df(y) \cdot (y - \bar{y}) + \sum_{i=2}^m (U_i - \bar{U}_i)]$$

which implies that

$$\theta_1 C \leq |\theta_1 Df(y) \cdot (y + \omega)| = \frac{1-t}{t} |Df(y) \cdot (y - \bar{y}) + \sum_{i=2}^m (U_i - \bar{U}_i)| < \theta_1 C$$

by choice of  $t''$ , which is a contradiction. So if  $t \geq t''$ , (1), (2), and (3) cannot all hold if either  $U_1 = 0$  or or  $U_1 = U_1(m\bar{b})$ .

Now choose  $t_*, t^*$  such that  $0 < t_* < t' \leq t'' < t^* < 1$  and let  $\chi_t : [0, 1] \rightarrow [0, 1]$  be a smooth function such that

$$\begin{aligned}\chi_t([t', t'']) &\equiv 1; \\ \chi_t([0, t_*]) &\equiv 0; \\ \chi_t([t^*, 1]) &\equiv 0.\end{aligned}$$

Also, choose  $U_{1*}, U_1^*$  such that  $0 < U_{1*} < U_1^* < U_1(m\bar{b})$  and let  $\chi_{U_1} : [0, U_1(m\bar{b})] \rightarrow [0, 1]$  be a smooth function such that

$$\begin{aligned}\chi_{U_1}([0, U_{1*}]) &\equiv 1; \\ \chi_{U_1}([U_1^*, U_1(m\bar{b})]) &\equiv 0.\end{aligned}$$

Note that  $l \equiv f(-\omega) < 0$  and  $L \equiv f(\bar{b} - \omega) > 0$ . Define

$$g(y, t, U_1) = (1 - \chi_t(t))y + \chi_t(t)[(1 - \chi_{U_1}(U_1))(\bar{b} - \omega) + \chi_{U_1}(U_1)(-\omega)],$$

and replace (3) by

$$(3') \quad -tf(g(t, y, U_1)) = (1 - t)(U_1 - \bar{U}_1).$$

Now we claim that we have constructed equation(3') such that (1), (2), and (3') cannot all hold if either  $U_1 = 0$  or  $U_1 = U_1(m\bar{b})$  for any  $t \in [0, 1]$ . To see this, consider the cases. If  $t \leq t'$ , (3') will fail by construction if  $U_1 = 0$  or  $U_1 = U_1(m\bar{b})$ , as by definition of  $t'$ , for  $t \leq t'$ ,  $\frac{t}{1-t}|f(g(y, t, U_1))| < \gamma < \min(\bar{U}_1, U_1(m\bar{b}) - \bar{U}_1)$ . So if  $U_1 \in \{0, U_1(m\bar{b})\}$ , (3') requires

$$\gamma > \frac{t}{1-t}|f(g(y, t, U_1))| = |U_1 - \bar{U}_1| \geq \gamma$$

which is a contradiction as argued before.

If  $t \in [t', t'']$ , (3') also fails for  $U_1 \in \{0, U_1(m\bar{b})\}$  by construction. If  $U_1 = 0$  and  $t \in [t', t'']$ ,  $g(y, t, 0) = -\omega$ , so (3') becomes

$$0 < -tf(-\omega) = (1 - t)(-\bar{U}_1) < 0,$$

a contradiction. Similarly, if  $U_1 = U_1(m\bar{b})$  and  $t \in [t', t'']$ ,  $g(y, t, U_1(m\bar{b})) = \bar{b} - \omega$ , so (3') becomes

$$0 > -tf(\bar{b} - \omega) = -tL = (1 - t)(U_1(m\bar{b}) - \bar{U}_1) > 0$$

also a contradiction.

If  $t \geq t^*$ , then  $g(y, t, U_1) \equiv y$ , so equation (3') is equivalent to equation (3), and since  $t^* \geq t''$ , if  $t \geq t^*$ , then by choice of  $t''$  equations (1), (2), and (3) cannot all hold if  $U_1 = 0$  or  $U_1 = U_1(m\bar{b})$ .

Finally, suppose  $t \in [t'', t^*]$ . Since  $t \geq t''$ , if (3') holds then

$$|f(g(y, t, U_1))| = \frac{1-t}{t}|U_1 - \bar{U}_1| < \epsilon.$$

Note that for  $t \in [t'', t^*]$ , if  $U_1 = 0$ ,  $g(y, t, 0) \leq y$ , so  $f(g(y, t, 0)) \leq f(y)$ , and hence since  $f(g(y, t, 0)) > -\epsilon$ ,  $f(y) > -\epsilon$ . We chose  $t''$  so that if  $t \geq t''$ ,  $U_1 = 0$  and  $f(y) \geq -\epsilon$ , equations (1) and (2) cannot both hold. Similarly, if  $U_1 = U_1(m\bar{b})$  and  $t \in [t'', t^*]$ ,  $g(y, t, U_1(m\bar{b})) \geq y$ , so  $f(g(y, t, U_1(m\bar{b}))) \geq f(y)$ , and hence since  $f(g(y, t, U_1(m\bar{b}))) < \epsilon$ ,  $f(y) < \epsilon$ . But we chose  $t''$  such that if  $t \geq t''$  and  $f(y) < \epsilon$ , (1) and (2) cannot both hold if  $U_1 = U_1(m\bar{b})$ .

Replace the original homotopy  $H$  by  $H'$ , where  $H'$  is identical to  $H$  but has (3') in place of (3). Then  $H'$  is a  $\sigma(\ell_\infty, \ell_1)$  compact perturbation of the identity,  $H'(\cdot, 0) \equiv G(\cdot)$ ;  $H'(\cdot, 1) \equiv \tilde{F}(\cdot)$ , and by construction  $0 \notin H'(\partial\Omega, t)$  for every  $t \in [0, 1]$ . Then by homotopy invariance,  $D(\tilde{F}, \Omega, y) = D(G, \Omega, y) = 1$ . Since  $D(\tilde{F}, \Omega, 0) \neq 0$ ,  $\tilde{F}^{-1}(0) \neq \emptyset$ , so that marginal cost pricing quasiequilibria exist. Moreover, as discussed in section 2, the set of marginal cost pricing quasiequilibria must also be essential. ■

**Theorem 3.1.** *If  $f : \ell_\infty \rightarrow \mathcal{R}$  is strictly concave, strictly monotone, continuously Fréchet differentiable, and weak\* continuous, then  $Df(y) \in \ell_1$  for every  $y \in \ell_\infty$ .*

*Proof:* Since  $f(y)$  is  $\sigma(\ell_\infty, \ell_1)$  continuous, the hypograph of  $f$ , which is the set

$$\text{hypo } f = \{(y, z) : z \leq f(y)\}$$

has nonempty  $\sigma(\ell_\infty, \ell_1)$  interior. To see this, let  $\epsilon > 0$  and consider  $(y, f(y) - \epsilon)$ . Since  $f$  is  $\sigma(\ell_\infty, \ell_1)$  continuous, there exists a neighborhood  $U$  of  $y$  such that for  $y' \in U$ ,  $f(y') \in (f(y) - \epsilon/2, f(y) + \epsilon/2)$ . So for instance  $U \times (f(y) - 5\epsilon/4, f(y) - 3\epsilon/4)$  is a neighborhood of  $(y, f(y) - \epsilon)$  which is contained in  $\text{hypo } f$ , since if  $(y', z) \in U \times (f(y) - 5\epsilon/4, f(y) - 3\epsilon/4)$ ,  $z \leq f(y) - 3\epsilon/4 < f(y) - \epsilon/2 < f(y')$ ; i.e.,  $(y', z) \in \text{hypo } f$ . Hence  $\{(y, z) : z < f(y)\}$  is a  $\sigma(\ell_\infty, \ell_1)$  open, convex set which is disjoint from  $\text{graph } f$ . For every  $(\bar{y}, f(\bar{y})) \in \text{graph } f$ , by the Hahn-Banach theorem for locally convex spaces (see, e.g., Aliprantis and Burkinshaw (1985), Theorem 9.10), there exists  $(p, r) \in \ell_1 \times \mathcal{R}$  such that  $(p, r) \neq 0$  and such that for every  $(y, z) \in \{(y, z) : z < f(y)\}$ ,

$$(p, r) \cdot (\bar{y}, f(\bar{y})) \leq (p, r) \cdot (y, z). \quad (1)$$

Consider  $(\bar{y}, z) \in \text{hypo } f$  such that  $z < f(\bar{y})$ , or  $0 < f(\bar{y}) - z$ . Then (1) implies that  $0 \leq -r(f(\bar{y}) - z)$ , or  $r \leq 0$ . Furthermore, (1) implies that  $(p, r) \cdot (\bar{y}, f(\bar{y})) \leq (p, r) \cdot (y, f(y) - \epsilon)$  for every  $\epsilon > 0$ . Letting  $\epsilon \rightarrow 0$  yields

$$(p, r) \cdot (\bar{y}, f(\bar{y})) \leq (p, r) \cdot (y, f(y))$$

for every  $y \in \ell_\infty$ . So

$$p \cdot (\bar{y} - y) \leq -r(f(\bar{y}) - f(y)) \quad \forall y \in \ell_\infty. \quad (2)$$

Now  $r \neq 0$ . To see this, suppose not, that is, suppose  $r = 0$ . By (2),  $p \cdot \bar{y} \leq p \cdot y$  for every  $y \in \ell_\infty$ . Since  $(p, r) \neq 0$ ,  $\exists j$  such that  $p_j \neq 0$ . Let  $e_j$  be the  $j^{\text{th}}$  standard basis vector in  $\ell_\infty$  and set

$$x = \begin{cases} -\frac{2|p \cdot \bar{y}|}{p_j} e_j & \text{if } p \cdot \bar{y} \neq 0, \\ -p_j e_j & \text{if } p \cdot \bar{y} = 0. \end{cases}$$

Certainly  $x \in \ell_\infty$ , and  $p \cdot x < p \cdot \bar{y}$ , a contradiction. Thus  $r \neq 0$ . Let  $\bar{p} = -\frac{1}{r}p$ . By (2),

$$\bar{p} \cdot (y - \bar{y}) \geq f(y) - f(\bar{y}) \quad \forall y \in \ell_\infty,$$

or  $\bar{p} \in \partial f(\bar{y})$ , the set of subgradients of  $f$  at  $\bar{y}$ . As  $f$  is continuously Fréchet differentiable,  $\partial f(\bar{y}) = \{Df(\bar{y})\}$ , so  $Df(\bar{y}) = \bar{p} \in \ell_1$ . ■

**Theorem 3.2.** Suppose  $f(y) = \sum_{t=0}^{\infty} \beta^t g(y_t)$ , where  $g : \mathcal{R} \rightarrow \mathcal{R}$  is  $C^1$ ,  $|g'(r)| \leq M$  for some  $M < \infty$ , and  $0 < \beta < 1$ . Then  $f$  is Fréchet differentiable and  $Df(y) = \{\beta^t g'(y_t)\} \in \ell_1$  for every  $y \in \ell_\infty$ .

*Proof:* We claim that  $f$  is Fréchet differentiable and that  $Df(y) = \{\beta^t g'(y_t)\}$ . Clearly  $\{\beta^t g'(y_t)\} \in \ell_1$  for every  $y \in \ell_\infty$  as  $g'$  is continuous on  $\mathcal{R}$ , so it suffices to prove that this is actually the derivative of  $f$  at  $y$ . This can be proven using the definition, since it suffices to show that

$$\lim_{\|h\| \rightarrow 0} \frac{|\sum_{t=0}^{\infty} \beta^t g(x_t + h_t) - \beta^t g(x_t) - \beta^t g'(x_t) h_t|}{\|h\|} = 0.$$

But

$$\begin{aligned} \frac{|\sum_{t=0}^{\infty} \beta^t g(x_t + h_t) - \beta^t g(x_t) - \beta^t g'(x_t) h_t|}{\|h\|} &\leq \frac{\sum_{t=0}^{\infty} \beta^t |g(x_t + h_t) - \beta^t g(x_t) - \beta^t g'(x_t) h_t|}{\|h\|} \\ &\leq \sum_{t=0}^{\infty} \beta^t \left| \frac{g(x_t + h_t) - g(x_t) - g'(x_t) h_t}{h_t} \right|, \end{aligned}$$

by definition of  $\|h\|$ . Then given  $\epsilon > 0$ , choose  $T$  such that  $\sum_T^{\infty} \beta^t < \epsilon/2M$ . Since for every  $t$ ,

$$\left| \frac{g(x_t + h_t) - g(x_t) - g'(x_t) h_t}{h_t} \right| \rightarrow 0,$$

for  $t = 1, \dots, T$  there exists  $\delta_t > 0$  such that for  $|h_t| < \delta_t$ ,

$$\left| \frac{g(x_t + h_t) - g(x_t) - g'(x_t) h_t}{h_t} \right| < \frac{\epsilon}{1 - \beta}.$$

Set  $\delta = \min\{\delta_t : t = 1, \dots, T\} > 0$ . For  $\|h\| < \delta$ ,

$$\begin{aligned} \frac{\sum_{t=0}^{\infty} |\beta^t g(x_t + h_t) - \beta^t g(x_t) - \beta^t g'(x_t) h_t|}{\|h\|} &\leq \sum_{t=0}^{\infty} \beta^t \left| \frac{g(x_t + h_t) - g(x_t) - g'(x_t) h_t}{h_t} \right| \\ &\leq \sum_{t=0}^{T-1} \beta^t \left| \frac{g(x_t + h_t) - g(x_t) - g'(x_t) h_t}{h_t} \right| \end{aligned}$$

$$\begin{aligned}
& + \sum_{t=T}^{\infty} \beta^t \left| \frac{g(x_t + h_t) - g(x_t) - g'(x_t)h_t}{h_t} \right| \\
\leq & \sum_{t=0}^{T-1} \beta^t \left| \frac{g(x_t + h_t) - g(x_t) - g'(x_t)h_t}{h_t} \right| \\
& + \sum_{t=T}^{\infty} \beta^t \left| \frac{g'(r_t)|h_t| - g'(x_t)h_t}{h_t} \right|,
\end{aligned}$$

for some  $r_t \in (x_t, x_t + h_t)$ ,

$$< \epsilon + 2M \cdot \frac{\epsilon}{2M} = 2\epsilon.$$

Thus  $Df(y) = \{\beta^t g'(y_t)\}$ . ■

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