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General Relativity & Compact Stars

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General Relativity

&

COMPACT STARS

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Introduction

"In the deathless boredom of the sidereal calm we cry with regret for a lost sun ..."
Jean de la Ville de Mirmont, L'Horizon Chimérique.

Compact stars—broadly grouped as neutron stars and white dwarfs—are the ashes of luminous stars. One or the other is the fate that awaits the cores of most stars after a lifetime of tens to thousands of millions of years. Whichever of these objects is formed at the end of the life of a particular luminous star, the compact object will live in many respects unchanged from the state in which it was formed. Neutron stars themselves can take several forms—hyperon, hybrid, or strange quark star. Likewise white dwarfs take different forms though only in the dominant nuclear species. A black hole is probably the fate of the most massive stars, an inaccessible region of spacetime into which the entire star, ashes and all, falls at the end of the luminous phase.

Neutron stars are the smallest, densest stars known. Like all stars, neutron stars rotate—some as many as a few hundred times a second. A star rotating at such a rate will experience an enormous centrifugal force that must be balanced by gravity else it will be ripped apart. The balance of the two forces informs us of the lower limit on the stellar density. Neutron stars are 10^{14} times denser than Earth. Some neutron stars are in binary orbit with a companion. Application of orbital mechanics allows an assessment of masses in some cases. The mass of a neutron star is typically 1.5 solar masses. We can therefore infer their radii: about ten kilometers. Into such a small object, the entire mass of our sun and more, is compressed.

We infer the existence of neutron stars from the occurrence of supernova explosions (the release of the gravitational binding of the neutron star) and observe them in the periodic emission of pulsars. Just as neutron stars acquire high angular velocities through conservation of angular momentum, they acquire strong magnetic fields through conservation of magnetic flux during the collapse of normal stars. The two attributes, rotation and strong magnetic dipole field, are the principle means by which neutron stars can be detected—the beamed periodic signal of pulsars.

The extreme characteristics of neutron stars set them apart in the physical principles that are required for their understanding. All other stars can be described in Newtonian gravity with atomic and low-energy nuclear physics under conditions essentially known in the laboratory[1]. Neutron stars in their several forms push matter to such extremes of density that nuclear and particle physics—pushed to their extremes—are essential for their description. Further, the intense concentration of matter in neutron stars can be described only in General Relativity, Einstein's theory of gravity which alone describes the way the weakest force in nature arranges the distribution of the mass and constituents of the densest objects in the universe.

1.1 Compact Stars

Of what are compact stars made? The name "neutron star" is suggestive and at the same time misleading. No doubt neutron stars are made of baryons like nucleons and hyperons but also likely contain cores of quark matter in some cases. We use "neutron star" in a generic sense to refer to stars as compact as described above. How does a star become so compact as neutron stars and why is there little doubt that they are made of baryons or quarks? The notion of a neutron star made from the ashes of a luminous star at the end point of its evolution goes back to 1934 and the study of supernova explosions by Baade and Zwicky [1].

[1]Luminous stars evolve through thermonuclear reactions. These are nuclear reactions induced by high temperatures but involving collision energies that are small on the nuclear scale. In some cases the reaction cross-sections can be measured with nuclear accelerators, and in others, measured cross-sections must be extrapolated to lower energy.

During the luminous life of a star, part of the original hydrogen is converted in fusion reactions to heavier elements by the heat produced by gravitational compression. When sufficient iron—the end point of exothermic fusion—is made, the core containing this heaviest ingredient collapses and an enormous energy is released in the explosion of the star. Baade and Zwicky guessed that the source of such a magnitude as makes these stellar explosions visible in daylight and for weeks thereafter must be gravitational binding energy. This energy is released by the solar mass core as the star collapses to densities high enough to tear all nuclei apart into their constituents.

By a simple calculation one learns that the gravitational energy acquired by the collapsing core is more than enough to power such explosions as Baade and Zwicky were detecting. Their view as concerns the compactness of the residual star has since been supported by many detailed calculations, and most spectacularly by the supernova explosion of 1987 in the Large Magellanic Cloud, a nearby minor galaxy visible in the southern hemisphere. The pulse of neutrinos observed in several large detectors carried the evidence for an integrated energy release over 4π steradians of the expected magnitude.

The gravitational binding energy of a neutron star is about 10 percent of its mass. Compare this with the nuclear binding energy of 9 MeV per nucleon in iron which is one percent of the mass. We conclude that the release of gravitational binding energy at the death of a massive star is of the order ten times greater than the energy released by nuclear fusion reactions during the entire luminous life of the star. The evidence that the source of energy for a supernova is the binding energy of a compact star—a neutron star—is compelling. How else could a tenth of a solar mass of energy be generated and released in such a short time?

Neutron stars are more dense than was thought possible by physicists at the turn of the century. At that time astronomers were grappling with the thought of white dwarfs whose densities were inferred to be about a million times denser than the earth. It was only following the discovery of the quantum theory and Fermi-Dirac statistics that very dense *cold* matter—denser than could be imagined on the basis of atomic sizes—was conceivable.

Prior to the discovery of Fermi-Dirac statistics, the high density inferred for the white dwarf Sirius seemed to present a dilemma. For while the high density was understood as arising from the ionization of the atoms in the hot star making possible their compaction by gravity, what would become of this dense object when ultimately it had consumed its nuclear fuel? Cold matter was known only in the atomic form it is on earth with densities of a few grams per cubic centimeter. The great scientist Sir Arthur Eddington surmised for a time that the star had "got itself into an awkward fix"—that it must some how re-expand to matter of familiar densities as it cooled, but it had no remaining source of energy to do so.

The perplexing problem of how a hot dense body without a source of energy could cool persisted until R. H. Fowler "came to the rescue"[2] by showing that Fermi-Dirac degeneracy allowed the star to cool by remaining comfortably in a previously unknown state of cold matter, in this case a degenerate[3] electron state. A little later Baade and Zwicky conceived of a similar degenerate state as the final resting place for nucleons after the supernova explosion of a luminous star.

The constituents of neutron stars — leptons, baryons and quarks — are degenerate. They lie helplessly in the lowest energy states available to them. They must. Fusion reactions in the original star have reached the end point for energy release—the core has collapsed, and the immense gravitational energy converted to neutrinos has been carried away. The star has no remaining source of energy to excite the fermions. Only the Fermi pressure and the short-range repulsion of the nuclear force sustain the neutron star against further gravitational collapse—sometimes. At other times the mass is so concentrated that it falls into a black hole, a dynamical object whose existence and external properties can be understood in the Classical Theory of General Relativity.

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[2]Eddington in an address in 1936 at Harvard University.

[3]Nucleons and electrons obey the Pauli exclusion principle, according to which each particle must occupy a different quantum state from the others. A degenerate state refers to the complete occupation of the lowest available energy states. In that event, no reaction and therefore no energy generation is possible—hence the name —degenerate state.

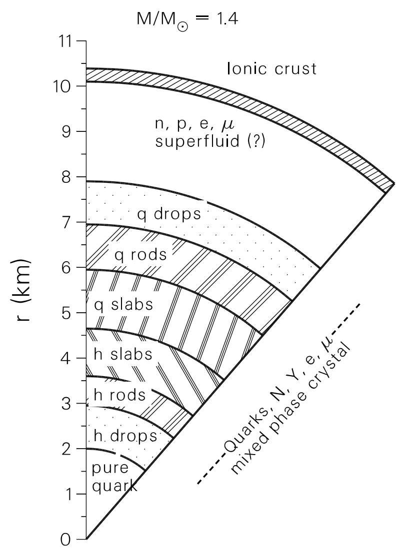

FIGURE 1.1. A section through a neutron star model that contains an inner sphere of pure quark matter surrounded by a crystalline region of mixed hadronic and quark matter. The mixed phase region consists of various geometrical objects of the rare phase immersed in the dominant one labeled by h(adronic) drops immersed in quark matter ... through to q(uark) drops immersed in hadronic matter. The particle composition of these regions is quarks, nucleons, hyperons, and leptons. A liquid of neutron star matter containing nucleons and leptons surrounds the mixed phase. A thin crust of heavy ions forms the stellar surface.[2]

1.2 Compact Stars and Relativistic Physics

Classical General Relativity is completely adequate for the description of neutron stars, white dwarfs, and for the most part, the exterior region of black holes as well as some aspects of the interior.[4] The first chapter is devoted to General Relativity. The goal is to rigorously arrive at the equations that describe the structure of relativistic stars—the Oppenheimer - Volkoff equations—the form that Einstein's equations take for spherical static stars. Two important facts emerge immediately. No

[4]The density at which quantum gravity would be relevant is 10^{78} higher than found in neutron stars.

form of matter whatsoever can support a relativistic star above a certain mass called the limiting mass. Its value depends on the nature of matter but the existence of the limit does not. The implied fate of stars more massive than the limit is that either mass is lost in great quantity during the evolution of the star or it collapses to form a black hole.

Black holes—the most mysterious objects of the universe—are treated at the classical level and only briefly. The peculiar difference between time as measured at a distant point and on an object falling into the hole is discussed. And it is shown that in black holes there is no statics. Everything at all times must approach the central singularity. Unlike neutron stars and white dwarfs, the question of their internal constitution does not arise at the classical level. They are enclosed within a horizon from which no information can be received. The ultimate fate of black holes is unknown.

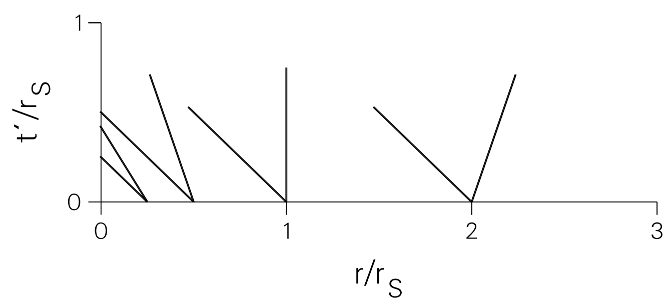

FIGURE 1.2. The possible futures of any event at the vertex of each cone, lies *within* the cone. Light propagates along the cone itself. On the scale of distance relative to the Schwarzschild radius of the black hole, the cones narrow and are tipped toward the black hole. At the critical radius, the outer edge of the cone is vertical; not even light can escape. Within the black hole, light can propagate only inward, as with anything else.

Luminous stars are known to rotate because of the Doppler broadening of spectral lines. Therefore their collapsed cores, spun up by conservation of angular momentum, may rotate very rapidly. Consequently, no account of compact stars would be complete without a discussion of rotation, its effects on the structure of the star and spacetime in the vicinity, the limits on rotation imposed by mass loss at the equator and by gravitational radiation, and the nature of compact stars that would be implied by very rapid rotation.

Rotating relativistic stars set local inertial frames into rotation with respect to the distant stars. An object falling from rest at great distance toward a rotating star would fall—not toward its center but would acquire an ever larger angular velocity as it approached. The effect of rotating stars on the fabric of spacetime acts back upon the structure of the stars and so is essential to our understanding.

1.3 Compact Stars and Dense-Matter Physics

The physics of dense matter is not as simple as the final resting place of stars imagined by Baade and Zwicky. The constitution of matter at the high densities attained in a neutron star—the particle types, abundances and their interactions—pose challenging problems in nuclear and particle physics. How should matter at supernuclear densities be described? In addition to nucleons, what exotic baryon species constitute it? Does a transition in phase from quarks confined in nucleons to the deconfined phase of quark matter occur in the density range of such stars? And how is the transition to be calculated? What new structure is introduced into the star? Do other phases like pion or kaon condensates play a role in their constitution?

In Fig. 1.1 we show a computation of the possible constitution and interior crystalline structure of a neutron star near the limiting mass of such stars. Only now are we beginning to appreciate the complex and marvelous structure of these objects. Surely the study of neutron stars and their astronomical realization in pulsars will serve as a guide in the search for a solution to some of the fundamental problems of dense many-body physics both at the level of nuclear physics—the physics of baryons and mesons—and ultimately at the level of their constituents—quarks and gluons. And neutron stars may be the only objects in which a Coulomb lattice structure (Fig. 1.1) formed from two phases of one and the same substance (hadronic matter) exists.

We do not know from experiment what the properties of superdense matter are. However we can be guided by certain general principles in our investigation of the possible forms that compact stars may take. Some of the possibilities lead to quite striking consequences that may in time be observable. The rate of discovery of new pulsars, X-ray neutron stars and other high-energy phenomena associated with neutron stars is astonishing, and was unforeseen a dozen years ago.

White dwarfs are the cores of stars whose demise is less spectacular than a supernova—a more quiescent thermal expansion of the envelope of a low mass star into a planetary nebula. White dwarf constituents are nuclei immersed in an electron gas and therefore arranged in a Coulomb lattice. White dwarfs are supported against collapse by Fermi pressure of degenerate electrons—while neutron stars— are supported by the Fermi pressure of degenerate nucleons. White dwarfs pose less severe and less fundamental problems than neutron stars. The nuclei will comprise varying proportions of helium, carbon, and oxygen, and in some cases heavier elements like magnesium, depending on how far in the chain of exothermic nuclear fusion reactions the precursor star burned before it was disrupted by instabilities leaving behind the dwarf. White dwarfs are barely relativistic.

Of a vastly different nature than neutron stars are *strange stars*. Like neutron stars they are, if they exist, very dense, of the same order as neutron stars. However their very existence hinges on a hypothesis that at first sight seems absurd. According to the hypothesis, sometimes referred to as the *strange-matter hypothesis*, quark matter—consisting of an approximately equal number of up, down and strange quarks—has an equilibrium energy per nucleon that is lower than the mass of the nucleon or the energy per nucleon of the most bound nucleus, iron. In other words, under the hypothesis, strange quark matter is the *absolute* ground state of the strong interaction.

We customarily find that systems, if not in their ground state, readily decay to it. Of course this is not always so. Even in well known objects like nuclei, there are certain excited states whose structure is such that the transition to the ground state is hindered. The first excited state of ^{180}Ta has a half-life of

10^{15} years, five orders of magnitude longer than the age of the universe! The strange-matter hypothesis is consistent with the present universe—a long-lived excited state—if strange matter is the ground state. The structure of strange stars is fascinating as are some of their properties.

2

General Relativity

> “Scarcely anyone who fully comprehends this theory can escape its magic.”
> *A. Einstein*
>
> “Beauty is truth, truth beauty–that is all
> Ye know on earth, and all ye need to know.”
> *J. Keats*

General Relativity—Einstein’s theory of gravity—is the most beautiful and elegant of physical theories. Not only that; it is the foundation for our understanding of compact stars. Neutron stars and black holes owe their very existence to gravity as formulated by Einstein [3, 4]. Dense objects like neutron stars could also exist in Newton’s theory, but they would be very different objects. Chandrasekhar found (in connection with white dwarfs) that all degenerate stars have a maximum possible mass. In Newton’s theory such a maximum mass is attained asymptotically when all fermions whose pressure supports the star are ultrarelativistic. Under such conditions stars populated by heavy quarks would exist. Such unphysical stars do not occur in Einstein’s theory.

Perhaps the beauty of Einstein’s theory can be attributed to the essentially simple but amazing answer it provides to a fundamental question: what meaning is attached to the absolute equality of inertial and gravitational masses? If all bodies move in gravitational fields in precisely the same way, no matter what their constitution or binding forces, then this means that their motion has nothing to do with their nature, but rather with the *nature of spacetime*. And if spacetime determines the motion of bodies, then according to the notion of action and reaction, this implies that spacetime in turn is *shaped by bodies and their motion*.

Beautiful or not, the predictions of theory have to be tested. The first three tests of General Relativity were proposed by Einstein, the gravitational redshift, the deflection of light by massive bodies and the perihelion shift of Mercury. The latter had already been measured. Einstein computed the anomalous part of the precession to be 43 arcseconds per century compared to the measurement of 42.98 ± 0.04. A fourth test was suggested by Shapiro in 1964—the time delay in the radar echo of a signal sent to a planet whose orbit is carrying it toward superior conjunction[1] with the sun. Eventually agreement to 0.1 percent with the prediction of Einstein's theory was achieved in these difficult and remarkable experiments. It should be remarked that all of the above tests involved weak gravitational fields.

The crowning achievement was the 20-year study by Taylor and his colleagues of the Hulse–Taylor pulsar binary discovered in 1974. Their work yielded a measurement of 4.22663 degrees *per year* for the periastron shift of the orbit of the neutron star binary and a measurement of the decay of the orbital period by $7.60 \pm 0.03 \times 10^{-7}$ seconds per year. This rate of decay agrees to less than 1% with careful calculations of the effect of energy loss through gravitational radiation as predicted by Einstein's theory [5]. A fuller discussion of these experiments and other intricacies involved in the tests of relativity can be found in the book by Will [6]. Since these early experiments, more accurate tests are being made by Dick Manchester and collaboraters at Parkes Obsevatory in Ausralia, who have discovered a closer binary pair of neutron stars — "We have verified GR to 0.1% already in two years" — ten times better than the early experiment." (Private communication: R. N. Manchester, 6/15/2005).

The goal of this chapter is to provide a rigorous derivation of the Oppenheimer–Volkoff equations that describe the structure of relativistic stars. We start by briefly outlining the Special Theory of Relativity for it is an essential ingredient of General Relativity. Then we formulate the General Theory of Relativ-

[1]Superior conjunction refers to the situation when the Earth and the planet are on opposite sides of the sun.

ity and derive all parts of the theory that are necessary to our goal.

2.1 Relativity

> "The views of space and time which I wish to lay before you have sprung from the soil of experimental physics, and therein lies their strength. They are radical. Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality." *H. Minkowski* [7]

The *principle of relativity* in physics goes back to Galileo who asserted that the laws of nature are the same in all *uniformly* moving laboratories. The relativity principle, stated in the narrow terms of reference frames in uniform motion, referred to as inertial frames, implies the existence of an absolute space. The notion of the absoluteness of time goes back to time immemorial. A *Galilean* transformation assumes the absoluteness of space and time:

$$x' = x - a - vt, \quad y' = y, \quad z' = z, \quad t' = t - b\,. \tag{2.1}$$

Newton's second law $F_x = m\,d^2x/dt^2$ is evidently invariant under this transformation if one assumes that force and mass are independent of the state of motion.

In contrast, Maxwell's equations do not take on the same form if subjected to a Galilean transformation whereas under a *Lorentz transformation* they do.[2] This fact led Einstein to the postulate that the speed of light is the same in all inertial systems and consequently that the principle of relativity should hold with respect to inertial frames connected by *Lorentz* transformations. That is the historical role that light speed played in the discovery of Special Relativity, and the reason for the undoubted influence that the Michelson–Morley experiment [9] had on the early acceptance of the theory.

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[2]See, for example, Ref. [8] for the Lorentz invariant form of Maxwell's equations.

However, the underlying physics is quite different from how it appears in the historical development of the Special Theory. The speed of light need not have been postulated as an invariant. Minkowski realized soon after Einstein's epochal discovery in 1905 that the spacetime manifold of our world is not Euclidean space in which events unfold in an absolute foliated time[3]. Spacetime is a 'Minkowski' manifold having such a nature that $d\tau^2 \equiv k^2 dt^2 - dx^2 - dy^2 - dz^2$ is invariant in the absence of gravity. The constant k is a conversion factor between length and time. Voigt observed in 1887 that $\Box\phi = 0$ preserved its form under a transformation that differed from the Lorentz transformation by only a scale factor [10]. In fact we will see shortly that the d'Alembertian $\Box$ is a Lorentz scalar. Consequently,

$$\Big(\frac{1}{k^2}\frac{\partial^2}{\partial t^2} - \nabla^2\Big)\phi = 0$$

informs us that a disturbance described by a wave equation for a massless particle in Minkowski spacetime propagates with velocity k in vacuum as viewed from this and any other reference frame connected to it by a Lorentz transformation. Hence, the constant k of the spacetime manifold is determined empirically by a measurement of the speed of light, c. In this way it is seen that the constancy of the speed of light is a *consequence* of the nature of the spacetime manifold in a gravity-free universe, or in a sufficiently small region of our gravity-filled universe. It is determined by the conversion factor between time and length of the manifold.

That the constancy of the speed of light is a consequence of the local spacetime manifold and not its determiner is most clearly illustrated by a thought experiment proposed by Swiatecki [11]. He shows that the invariance of the differential interval between spacetime events

$$d\tau^2 = k^2 dt^2 - dx^2 - dy^2 - dz^2$$

can be verified (at least in principle) *without* resort to propagation of light signals, but with only measuring rods and clocks.

[3]Foliated time refers to the time of events as being arranged as pages in a book, one following the other, there never being a question of which preceded another.

And if it were technically feasible to perform the experiment with sufficient accuracy, k would be measured and its value would be found to equal c.

Minkowski's fundamental discovery of the nature of spacetime in the absence of gravity was inspired by Einstein's postulate of the constancy of the speed of light. However, the constancy of the speed of light is a consequence of the spacetime manifold of our universe and its value (as for any massless particle) is equal to the conversion factor between space and time, as we have seen. The Minkowski invariant describes the nature of our spacetime (in a suitably limited region); the speed of light and that of any other massless particle is equal to the conversion factor k between time and length, as emphasized by W. Swiatecki[11]. In other words, Special Relativity is a consequence of the local spacetime manifold in which we live. The significance of the local restriction will become clear as we follow the development of the General Theory.

2.2 Lorentz Invariance

The Special Theory of Relativity, which holds in the absence of gravity, plays a central role in physics. Even in the strongest gravitational fields the laws of physics must conform to it in a sufficiently small locality of any spacetime event. That was a fundamental insight of Einstein. Consequently, the Special Theory plays a central role in the development of the General Theory of Relativity and its applications.

2.2.1 Lorentz transformations

The Lorentz transformation leaves invariant the *proper time* or *differential interval* in Minkowski spacetime

$$d\tau^2 = dt^2 - dx^2 - dy^2 - dz^2 \quad (\text{units } c = 1) \tag{2.2}$$

as measured by observers in frames moving with constant relative velocity (called inertial frames because they move freely under the action of no forces). The Minkowski manifold also implies an absolute spacetime in which spacetime events that can be connected by a Lorentz transformation lie within the cone defined by $d\tau = 0$. Absolute means unaffected by any physical conditions. This was the same criticism that Einstein made of Newton's space and time, and the one that powered his search for a new theory in which the expression of physical laws does not depend on the frame of reference, but, nevertheless, in which Lorentz invariance would remain a *local* property of spacetime. We will develop the core of the General Theory which extends the relativity principle to arbitrary frames and therefore to a gravity-filled universe, not just unaccelerated frames in relative uniform motion; but here we review briefly the Special Theory.

A *pure* Lorentz transformation is one without spatial rotation, while a general Lorentz transformation is the product of a rotation in space and a pure Lorentz transformation. We recall the pure transformation, sometimes also referred to as a *boost*. For convenience, define

$$x^0 = t, \quad x^1 = x, \quad x^2 = y, \quad x^3 = z\,. \tag{2.3}$$

(In spacetime a point such as that above is sometimes referred to as an *event*.) The linear homogeneous transformation connecting two reference frames can be written

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu\,. \tag{2.4}$$

(We shall use the convenient notation introduced by Einstein whereby repeated indices are summed—Greek over time and space, Roman over space.)

Any set of four quantities A^μ ($\mu = 0, 1, 2, 3$) that transforms under a change of reference frame in the same way as the coordinates is a *contravariant* Lorentz four-vector,

$$A'^\mu = \Lambda^\mu{}_\nu A^\nu\,. \tag{2.5}$$

The invariant interval (also variously called the proper time, the line element, or the separation formula) can be written

$$d\tau^2 = \eta_{\mu\nu}\, dx^\mu\, dx^\nu\,, \tag{2.6}$$

where $\eta_{\mu\nu}$ is the Minkowski metric which in rectilinear coordinates is

$$\eta_{\mu\nu} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\,. \tag{2.7}$$

The condition of the invariance of $d\tau^2$ is

$$\eta_{\alpha\beta}\, dx^\alpha\, dx^\beta = \eta_{\mu\nu}\, dx'^\mu\, dx'^\nu = \eta_{\mu\nu}\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta\, dx^\alpha\, dx^\beta\,. \qquad (2.8)$$

Since this holds for any dx^α, dx^β we conclude that the $\Lambda^\mu{}_\nu$ must satisfy the fundamental relationship assuring invariance of the proper time:

$$\eta_{\alpha\beta} = \eta_{\mu\nu}\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta\,. \qquad (2.9)$$

Transformations that leave $d\tau^2$ invariant leave the speed of light the same in all inertial systems, because if $d\tau = 0$ in one system, it is true in all, and the content of $d\tau = 0$ is that $d\mathbf{x}/dt = 1$.

Let us find the transformation matrix $\Lambda^\mu{}_\alpha$ for the special case of a boost along the x-axis. In this case it is clear that

$$x'^2 = x^2, \qquad x'^3 = x^3, \qquad (2.10)$$

and, moreover, that x'^0 and x'^1 cannot involve x^2 and x^3. So,

$$\begin{aligned} x'^0 &= \Lambda^0{}_0 x^0 + \Lambda^0{}_1 x^1 \\ x'^1 &= \Lambda^1{}_0 x^0 + \Lambda^1{}_1 x^1 \end{aligned} \qquad (2.11)$$

with the remaining Λ elements zero. So the above quadratic form in Λ yields the three equations,

$$\begin{aligned} 1 &= (\Lambda^0{}_0)^2 - (\Lambda^1{}_0)^2 \\ -1 &= (\Lambda^0{}_1)^2 - (\Lambda^1{}_1)^2 \\ 0 &= \Lambda^0{}_0\Lambda^0{}_1 - \Lambda^1{}_0\Lambda^1{}_1\,. \end{aligned} \qquad (2.12)$$

To get a fourth equation, suppose that the origins of the two frames in uniform motion coincide at $t = 0$ and the primed x-axis x'^1 is moving along x^1 with velocity v. That is, $x^1 = vt$ is the equation of the *primed* origin as it moves along the *unprimed* x-axis. The equation for the primed coordinate is

$$0 = x'^1 = \Lambda^1{}_0 x^0 + \Lambda^1{}_1 x^1 = (\Lambda^1{}_0 + \Lambda^1{}_1 v)t \qquad (2.13)$$

or

$$\Lambda^1{}_0 = -\Lambda^1{}_1 v\,. \qquad (2.14)$$

The four equations can now be solved with the result,

$$\begin{aligned} \Lambda^0{}_0 &= \Lambda^1{}_1 = \gamma \\ \Lambda^1{}_0 &= \Lambda^0{}_1 = -v\gamma \\ \Lambda^2{}_2 &= \Lambda^3{}_3 = 1\,, \end{aligned} \tag{2.15}$$

where

$$\gamma \equiv (1-v^2)^{-1/2} \equiv \cosh\theta, \qquad v\gamma \equiv \sinh\theta, \qquad v \equiv \tanh\theta \tag{2.16}$$

So

$$\begin{aligned} x'^0 &= x^0\cosh\theta - x^1\sinh\theta \\ x'^1 &= -x^0\sinh\theta + x^1\cosh\theta \\ x'^2 &= x^2, x'^3 = x^3\,. \end{aligned} \tag{2.17}$$

The combination of two boosts in the same direction, say v_1 and v_2, corresponds to $\theta = \theta_1 + \theta_2$. A boost in an arbitrary direction with the *primed* axis having velocity $\mathbf{v} = (v^1, v^2, v^3)$ relative to the *unprimed* is

$$\begin{aligned} \Lambda^0{}_0 &= \gamma \\ \Lambda^0{}_j &= \Lambda^j{}_0 = -v^j\gamma \\ \Lambda^j{}_k &= \Lambda^k{}_j = \delta^j_k + (\gamma-1)v^jv^k/\mathbf{v^2}\,. \end{aligned} \tag{2.18}$$

For a spatial rotation, say in the x-y plane, the transformation for a positive rotation about the common z-axis is

$$\begin{aligned} x'^1 &= x^1\cos\omega + x^2\sin\omega \\ x'^2 &= -x^1\sin\omega + x^2\cos\omega \\ x'^0 &= x^0, \qquad x'^3 = x^3\,. \end{aligned} \tag{2.19}$$

Transformation of vectors according to either of the above, or a product of them, preserves the invariance of the interval $d\tau^2$. For convenience they can be written in matrix form as

$$\Lambda \equiv \begin{pmatrix} \cosh\theta & -\sinh\theta & 0 & 0 \\ -\sinh\theta & \cosh\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.20}$$

$$R \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\omega & \sin\omega & 0 \\ 0 & -\sin\omega & \cos\omega & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\,. \tag{2.21}$$

2.2.2 Covariant vectors

Two contravariant Lorentz vectors such as

$$A^\mu \equiv (A^0, A^1, A^2, A^3) \tag{2.22}$$

and B^μ may be used to create a *scalar* product (Lorentz scalar)

$$A' \cdot B' \equiv \eta_{\mu\nu} A'^\mu B'^\nu = \eta_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta A^\alpha B^\beta = \eta_{\alpha\beta} A^\alpha B^\beta \equiv A \cdot B \tag{2.23}$$

Because of the minus signs in the Minkowski metric we have

$$A \cdot B = A^0 B^0 - \mathbf{A} \cdot \mathbf{B}\,, \tag{2.24}$$

and the *covariant* Lorentz vector is defined by

$$A_\mu \equiv (A^0, -A^1, -A^2, -A^3)\,. \tag{2.25}$$

A *covariant* Lorentz vector is obtained from its contravariant dual by the process of lowering indices with the metric tensor,

$$A_\mu = \eta_{\mu\nu} A^\nu\,. \tag{2.26}$$

Conversely, raising of indices is achieved by

$$A^\mu = \eta^{\mu\nu} A_\nu\,. \tag{2.27}$$

It is straightforward to show that

$$\eta^{\mu\alpha}\eta_{\alpha\nu} \equiv \eta^\mu_\nu = \delta^\mu_\nu\,, \tag{2.28}$$

where

$$\delta^\mu_\nu = \begin{cases} 1 & \text{if } \mu = \nu \\ 0 & \text{otherwise} \end{cases} \tag{2.29}$$

is the Kroneker delta. It follows that

$$\eta_{\mu\nu} = \eta^{\mu\nu}\,. \tag{2.30}$$

The Lorentz transformation for a covariant vector is written in analogy with that of a contravariant vector:

$$A'_\mu = \Lambda_\mu{}^\nu A_\nu\,. \tag{2.31}$$

To obtain the elements of $\Lambda_\mu^{\ \nu}$ we write the above in two different ways,

$$\eta_{\mu\beta}\Lambda^\beta_{\ \alpha}A^\alpha = \eta_{\mu\beta}A'^\beta = A'_\mu = \Lambda_\mu^{\ \nu}A_\nu = \Lambda_\mu^{\ \nu}\eta_{\nu\alpha}A^\alpha\,. \tag{2.32}$$

This holds for arbitrary A^μ so

$$\Lambda_\mu^{\ \nu} = \eta_{\mu\alpha}\Lambda^\alpha_{\ \beta}\eta^{\beta\nu}\,. \tag{2.33}$$

Using (2.28) in the above we get the inverse relationship

$$\Lambda^\mu_{\ \nu} = \eta^{\mu\alpha}\Lambda_\alpha^{\ \beta}\eta_{\beta\nu}\,. \tag{2.34}$$

Multiplying (2.33) by $A^\mu_{\ \sigma}$, summing on μ, and employing the fundamental condition of invariance of the proper time (2.9) we find

$$\Lambda^\mu_{\ \sigma}\Lambda_\mu^{\ \tau} = \delta_\sigma^\tau\,. \tag{2.35}$$

We can now invert (2.4) and find that $\Lambda_\mu^{\ \nu}$ is the inverse Lorentz transformation,

$$x^\mu = \Lambda_\nu^{\ \mu}x'^\nu\,. \tag{2.36}$$

The elements of the inverse transformation are given in terms of (2.15) or (2.18) by (2.33). We have

$$\begin{aligned}\Lambda_0^{\ 0} &= \Lambda_1^{\ 1} = \gamma\,,\\ \Lambda_1^{\ 0} &= \Lambda_0^{\ 1} = v\gamma\,,\\ \Lambda_2^{\ 2} &= \Lambda_3^{\ 3} = 1\,.\end{aligned} \tag{2.37}$$

A boost in an arbitrary direction with the primed axis having velocity $\mathbf{v} = (\mathbf{v^1}, \mathbf{v^2}, \mathbf{v^3})$ relative to the unprimed is

$$\begin{aligned}\Lambda_0^{\ 0} &= \gamma\,,\\ \Lambda_0^{\ j} &= \Lambda_j^{\ 0} = v^j\gamma\,,\\ \Lambda_j^{\ k} &= \Lambda_k^{\ j} = \delta_k^j + (\gamma-1)v^jv^k/\mathbf{v^2}\,.\end{aligned} \tag{2.38}$$

The *four-velocity* is a vector of particular interest and defined as

$$u^\mu = \frac{dx^\mu}{d\tau}\,. \tag{2.39}$$

Because $d\tau$ is an invariant scalar and dx^μ is a vector, u^μ is obviously a contravariant vector. From the expression for the invariant interval we have

$$d\tau = \sqrt{1-\mathbf{v^2}}\,dt, \qquad \mathbf{v} = \frac{d\mathbf{r}}{dt} \tag{2.40}$$

with $\mathbf{r} = (x^1, x^2, x^3)$; it therefore follows that

$$u^0 \equiv \frac{dt}{d\tau} = \gamma, \qquad u^i \equiv \frac{dx^i}{d\tau} = \frac{dx^i}{dt}\frac{dt}{d\tau} = v^i\gamma\,, \tag{2.41}$$

or

$$u^\mu = \gamma(1, v^1, v^2, v^3),\ \ u_\mu = \gamma(1, -v^1, -v^2, -v^3),\ \ u^\mu u_\mu = 1. \tag{2.42}$$

The transformation of a tensor under a Lorentz transformation follows from (2.5) and (2.31) according to the position of the indices; for example,

$$T'^{\mu\nu} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta T^{\alpha\beta}\,. \tag{2.43}$$

We note that according to (2.9), the Minkowski metric $\eta^{\mu\nu}$ is a tensor; moreover, it has the same constant form in every Lorentz frame.

2.2.3 Energy and Momentum

The relativistic analogue of Newton's law $F = ma$ is

$$F^\mu = m\frac{d^2x^\mu}{d\tau} \tag{2.44}$$

and the four-momentum is

$$p^\mu = m\frac{dx^\mu}{d\tau}\,. \tag{2.45}$$

Hence, from (2.39) and (2.40)

$$\begin{aligned} p^0 &\equiv E = m\gamma \\ \mathbf{p} &= \mathbf{m}\gamma\mathbf{v}\,. \end{aligned} \tag{2.46}$$

2.2.4 Energy-momentum tensor of a perfect fluid

A perfect fluid is a medium in which the pressure is isotropic in the rest frame of each fluid element, and shear stresses and heat transport are absent. If at a certain point the velocity of the fluid is $\mathbf{v}$, an observer with this velocity will observe the fluid in the neighborhood as isotropic with an energy density ϵ and pressure p. In this local frame the energy-momentum tensor is

$$T'^{\mu\nu} \equiv \begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} . \tag{2.47}$$

As viewed from an arbitrary frame, say the laboratory system, let this fluid element be observed to have velocity $\mathbf{v}$. According to (2.36) we obtain the transformation

$$T^{\mu\nu} = \Lambda_\alpha^{\ \mu} \Lambda_\beta^{\ \nu} T'^{\alpha\beta} . \tag{2.48}$$

The elements of the transformation are given by (2.37) in the case that the fluid element is moving with velocity v along the laboratory x-axis, or by (2.38) if it has the general velocity $\mathbf{v}$. It is easy to check that in the arbitrary frame

$$T^{\mu\nu} = -p\eta^{\mu\nu} + (p + \epsilon)u^\mu u^\nu \tag{2.49}$$

and reduces to the diagonal form above when $\mathbf{v} = \mathbf{0}$. We have used the four-velocity defined above by (2.41). Relative to the laboratory frame it is the four-velocity of the fluid element.

2.2.5 Light cone

For vanishing proper time intervals, $d\tau = 0$ given by (2.2) defines a cone (figure 1.2) in the four-dimensional space x^μ with the time axis as the axis of the cone. Events separated from the vertex event for which the proper time, (or invariant interval) vanishes ($d\tau = 0$), are said to have null separation. They can be connected to the event at the vertex by a light signal. Events separated from the vertex by a real interval $d\tau^2 > 0$ can be connected by a subluminal signal—a material particle can travel from one event to the other. An event for which $d\tau^2 < 0$ refers to an event outside the two cones; a light signal cannot join the vertex event to such an event. Therefore, events in the cone with t greater than that of the vertex of the cone lie in the future of the event at the vertex, while events in the other cone lie in its past. Events lying outside the cone are not causally connected to the vertex event.

2.3 Scalars, Vectors, and Tensors in Curvilinear Coordinates

In the last section we dealt with inertial frames of reference in flat spacetime. We now wish to allow for curvilinear coordinates. Our scalars, vectors, and tensors now refer to a point in spacetime. Their components refer to the reference frame at that point.

A scalar field $S(x)$ is a function of position, but its value does not depend on the coordinate system. An example is the temperature as registered on thermometers located in various rooms in a house. Each registered temperature may be different, and therefore is a function of position, but independent of the coordinates used to specify the locations:

$$S'(x') = S(x). \tag{2.50}$$

A vector is a quantity whose components change under a coordinate transformation. One important vector is the displacement vector between adjacent points. Near the point x^μ we consider another, $x^\mu + dx^\mu$. The four displacements dx^μ are the components of a vector. Choose units so that time and distance are measured in the same units ($c = 1$). In Cartesian coordinates we can write the *invariant interval* $d\tau$ of the Special Theory of Relativity, sometimes called the *proper time*, as

$$d\tau^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2. \tag{2.51}$$

Under a coordinate transformation from these rectilinear coordinates to arbitrary coordinates, $x^\mu \to x'^\mu$, we have (from the rules of partial differentiation)

$$dx'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} dx^\nu. \tag{2.52}$$

As before, repeated indices are summed. We can also write the inverse of the above equation and substitute for the spacetime differentials in the invariant interval to obtain an equation of the form

$$d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu \,, \tag{2.53}$$

where the $g_{\mu\nu}$ are defined in terms of products of the partial derivatives of the coordinate transformation.

Depending on the nature of the coordinate system, say rectilinear, oblique, or curvilinear, or on the presence of a gravitational field, the invariant interval may involve bilinear products of different dx^μ, and the $g_{\mu\nu}$ will be functions of position and time. The $g_{\mu\nu}$ are field quantities—the components of a tensor called the *metric tensor*. Because the $g_{\mu\nu}$ appear in a quadratic form (2.53), we may take them to be symmetric:

$$g_{\mu\nu} = g_{\nu\mu} \,. \tag{2.54}$$

In regions of spacetime for which the rectilinear system of the Special Theory of Relativity holds, the metric tensor $g_{\mu\nu}$ is equal to the Minkowski tensor (2.7). In fact, as we shall see, Special Relativity holds *locally* anywhere at any time. We shall refer to reference frames in which the metric is given by the Minkowski tensor as *Lorentz frames*.

The invariant interval or proper time $d\tau$ is real for a timelike interval and imaginary for a spacelike.[4] The notation *proper time* is seen to be appropriate because, when two events occur at the same *space* point, what remains of the invariant interval is dt.

Any four quantities A^μ that transform as dx^μ comprise a *contravariant vector*

$$A'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} A^\nu \,, \tag{2.55}$$

and

$$g_{\mu\nu} A^\mu A^\nu \equiv A^2 \tag{2.56}$$

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[4] The opposite convention $ds^2 = -d\tau^2$ could also be employed. The interval ds is often referred to as the line element.

is its invariant squared length. It is obviously invariant under the same transformations that leave (2.51) invariant because the four quantities A^μ form a four-vector like dx^μ.

A *covariant* vector can be obtained through the process of *lowering* indices with the metric tensor:

$$A_\mu = g_{\mu\nu} A^\nu \,. \tag{2.57}$$

In terms of this vector, the magnitude equation (2.56) can be written as

$$A_\mu A^\mu = A^2 \,. \tag{2.58}$$

Let A^μ and B^μ be distinct contravariant vectors. Then so is $A^\mu + \lambda B^\mu$ for all finite λ. The quantity

$$g_{\mu\nu}(A^\mu + \lambda B^\mu)(A^\nu + \lambda B^\nu)$$

is the invariant squared length. Because this is true for all λ, the coefficient of each power of λ is also an invariant; for the linear term we find

$$g_{\mu\nu}(A^\mu B^\nu + B^\mu A^\nu) = 2 g_{\mu\nu} A^\mu B^\nu \,, \tag{2.59}$$

where we have used the symmetry of $g_{\mu\nu}$. Thus, we obtain the invariant scalar product of two vectors:

$$g_{\mu\nu} A^\mu B^\nu = A_\nu B^\nu = A \cdot B \,. \tag{2.60}$$

To derive the transformation law for a covariant vector use the fact, just proven, that $A_\mu B^\mu$ is a scalar. Then using the law of transformation of a contravariant vector (2.55), we have

$$A'_\mu B'^\mu = A_\alpha B^\alpha = A_\alpha \frac{\partial x^\alpha}{\partial x'^\mu} B'^\mu \,, \tag{2.61}$$

where A'_μ is the same vector as A_μ but referred to the primed reference frame. From the above equation it follows that

$$\Big(A'_\mu - \frac{\partial x^\alpha}{\partial x'^\mu} A_\alpha\Big) B'^\mu = 0 \,. \tag{2.62}$$

Because B^μ is any vector, the quantity in brackets must vanish; thus we have the law of transformation of a covariant vector,

$$A'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} A_\nu \,. \tag{2.63}$$

Compare this transformation law with that of (2.55).

Let the determinant of $g_{\mu\nu}$ be g,

$$g = \det|g_{\mu\nu}| \,. \tag{2.64}$$

As long as g does not vanish, the equations (2.57) can be inverted. Let the coefficients of the inverse be called $g^{\mu\nu}$. Then find

$$A^{\nu} = g^{\nu\mu} A_{\mu} \,. \tag{2.65}$$

Multiply (2.57) by $g^{\alpha\mu}$ and sum on μ with the result

$$A^{\alpha} = g^{\alpha\mu} A_{\mu} = g^{\alpha\mu} g_{\mu\nu} A^{\nu} \,, \tag{2.66}$$

or

$$(g^{\alpha\mu} g_{\mu\nu} - \delta^{\alpha}_{\nu}) A^{\nu} = 0 \,, \tag{2.67}$$

where δ^{α}_{ν} is the Kroneker delta. Because this equation holds for any vector, we have

$$g^{\alpha}_{\beta} \equiv g^{\alpha\mu} g_{\mu\beta} = \delta^{\alpha}_{\beta} \,. \tag{2.68}$$

The two g's, one with subscripts, the other with superscripts, are inverses. In the same way as $g_{\mu\nu}$ can be used to lower an index, $g^{\mu\nu}$ can be used to raise one. Both are symmetric;

$$g_{\mu\nu} = g_{\nu\mu}, \qquad g^{\mu\nu} = g^{\nu\mu} \,. \tag{2.69}$$

The derivative of a scalar field $S(x) = S'(x')$ with respect to the components of a contravariant position vector yields a covariant vector field and, vice versa:

$$\frac{\partial S}{\partial x'^{\mu}} = \frac{\partial x^{\nu}}{\partial x'^{\mu}} \frac{\partial S}{\partial x^{\nu}} \,. \tag{2.70}$$

Accordingly, we shall sometimes use the abbreviations

$$\partial_{\mu} \equiv \frac{\partial}{\partial x^{\mu}} \quad \text{and} \quad \partial^{\mu} \equiv \frac{\partial}{\partial x_{\mu}} \,, \tag{2.71}$$

especially in writing Lagrangians of fields. In relativity it is also useful to have an even more compact notation for the coordinate derivative—the "comma subscript":

$$S_{,\mu} \equiv \frac{\partial S}{\partial x^{\mu}} \,. \tag{2.72}$$

The d'Alembertian,

$$\Box = \partial_\mu \partial^\mu \,, \tag{2.73}$$

is manifestly a scalar.

Tensors are similar to vectors, but with more than one index. A simple tensor is one formed from the product of the components of two vectors, $A^\mu B^\nu$. But this is special because of the relationships between its components. A general tensor of the second rank can be formed by a sum of such products:

$$T^{\mu\nu} = A^\mu B^\nu + C^\mu D^\nu + \cdots . \tag{2.74}$$

The superscripts can be lowered as with a vector, either one index, or both,

$$T_\mu{}^\nu = g_{\mu\alpha} T^{\alpha\nu} \,, \quad T^\nu{}_\mu = T^{\nu\alpha} g_{\alpha\mu} \,, \quad T_{\mu\nu} = g_{\mu\alpha} g_{\nu\beta} T^{\alpha\beta} \,. \tag{2.75}$$

Similarly, we may have tensors of higher rank, either contravariant with respect to all indices, or covariant, or mixed. The position of the indices on the mixed tensor (the lower to the left or right of the upper) refers to the position of the index that was lowered. If $T^{\mu\nu}$ is symmetric, then $T^\mu{}_\nu = T_\nu{}^\mu$ and it is unimportant to keep track of the position of the index that has been lowered (or raised). But if $T^{\mu\nu}$ is antisymmetric, then the two orderings differ by a sign.

If two of the indices on a tensor, one a superscript the other a subscript, are set equal and summed, the rank is reduced by two. This process is called *contraction*. If it is done on a second-rank mixed tensor, the result is a scalar,

$$S = T^\mu{}_\mu = T_\mu{}^\mu . \tag{2.76}$$

When $T^{\mu\nu}$ is antisymmetric, the contractions $T_\mu{}^\mu$ and $T^\mu{}_\mu$ are identically zero.

The test of tensor character is whether the object in question transforms under a coordinate transformation in the obvious generalization of a vector. For example,

$$T'^\mu{}_\nu = \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x^\beta}{\partial x'^\nu} T^\alpha{}_\beta \tag{2.77}$$

is a tensor.

In general, we deal with curved spacetime in General Relativity. We must therefore deal with curvilinear coordinates. Vectors and tensors at a point in such a spacetime have components referring to the axis at that point. The components will change according to the above laws, depending on the way the axes change at that point. Therefore, the metric tensors $g_{\mu\nu}, g^{\mu\nu}$ cannot be constants. They are field quantities which vary from point to point. As we shall see, they can be referred to collectively as the gravitational field. Because the formalism of this section is expressed by local equations, it holds in curved spacetime, for curved spacetime is flat in a sufficiently small locality.

Because the derivative of a scalar field is a vector (2.70), one might have thought that the derivative of a vector field is a tensor. However, by checking the transformation properties one finds that this supposition is not true.

We have referred invariably to the $g_{\mu\nu}$ as tensors. Now we show that this is so. Let A^μ, B^ν be arbitrary vector fields, and consider two coordinate systems such that the same point P has the coordinates x^μ and x'^μ when referred to the two systems, respectively. Then we have

$$g'_{\alpha\beta} A'^\alpha A'^\beta = g_{\mu\nu} A^\mu A^\nu = g_{\mu\nu} \frac{\partial x^\mu}{\partial x'^\alpha} \frac{\partial x^\nu}{\partial x'^\beta} A'^\alpha A'^\beta. \tag{2.78}$$

Because this holds for arbitrary vectors, we find

$$g'_{\alpha\beta} = g_{\mu\nu} \frac{\partial x^\mu}{\partial x'^\alpha} \frac{\partial x^\nu}{\partial x'^\beta} , \tag{2.79}$$

which, by comparison with (2.63), shows that $g_{\mu\nu}$ is a covariant tensor. Similarly $g^{\mu\nu}$ is a contravariant tensor:

$$g'^{\alpha\beta} = g^{\mu\nu} \frac{\partial x'^\alpha}{\partial x^\mu} \frac{\partial x'^\beta}{\partial x^\nu} . \tag{2.80}$$

These are called the *fundamental tensors*. Of course, the above tensor character of the metric is precisely what is required to make the square of the interval $d\tau^2$ of (2.53) an invariant, as is trivially verified.

Mixed tensors of arbitrary rank transform, for each index, according to the transformation laws (2.55, 2.63) depending on whether the index is a superscript or a subscript, as can be derived in obvious analogy to the above manipulations.

Tensors and tensor algebra are very powerful techniques for carrying the consequences discovered in one frame to another. That the linear combination of tensors of the same rank and arrangement of upper and lower indices is also a tensor; that the direct product of two tensors of the same or different rank and arrangement of indices, $A^{\mu\cdots}_{\alpha\cdots}B^{\nu\cdots}_{\cdots} = T^{\mu\cdots\nu\cdots}_{\alpha\cdots}$ is also a tensor; and that contraction (defined above) of a pair of indices, one upper, one lower produces a tensor of rank reduced by two—are all easy theorems that we do not need to prove, but only note in passing. Of particular note, if the difference of two tensors of the same transformation rule vanishes in one frame, then it vanishes in all (i.e., the two tensors are equal in all frames).

2.4 Principle of Equivalence of Inertia and Gravitation

> "The possibility of explaining the numerical equality of inertia and gravitation by the unity of their nature gives to the general theory of relativity, according to my conviction, such a superiority over the conceptions of classical mechanics, that all the difficulties encountered in development must be considered as small in comparison." *A. Einstein* [4]

Eötvös established that all bodies have the same ratio of inertial to gravitational mass with high precision [12]. With an appropriate choice of units, the two masses are equal for all bodies to the accuracy established for the ratio. One might have expected such conceptually different properties, one having to do with inertia to motion (m_I), the other with "charge" (m_G), in an expression of mutual attraction between bodies, to be entirely different. The relation between the force exerted by the gravitational attraction of a body of mass M at a distance R upon the object, and the acceleration imparted to it are expressed by Newton's equation, valid for weak fields and small material velocities:

$$m_I a = G\frac{m_G M}{R^2}\,. \tag{2.81}$$

Einstein reasoned that the near equality of two such different properties must be more than mere coincidence and that inertial and gravitational masses must be exactly equal: $m_I = m_G = m$. The mass drops out! In that case all bodies experience precisely the same acceleration in a gravitational field, as was presaged by Galileo's experiments centuries earlier. For all other forces that we know, the acceleration is inverse to the mass.

The equivalence of inertial and gravitational mass is established to high accuracy for atomic and nuclear binding energies.[5] Moreover, as a result of very careful lunar laser-ranging experiments, the earth and moon are found to fall with equal acceleration toward the sun to a precision of almost 1 part in 10^{13}, better than the most accurate Eötvös-type experiments on laboratory bodies. This exceedingly important test involving bodies of different gravitational binding was conceived by Nordtvedt [13]. The essentially null result establishes the so-called *strong* statement of equivalence of inertial and gravitational mass: Free bodies—no matter their nature or constituents, nor how much or little those constituents are bound, nor by what force—all move in the spacetime of an arbitrary gravitational field as if they were identical test particles! *Because their motion has nothing to do with their nature, it evidently has to do with the nature of spacetime.*

Einstein felt certain that a deep meaning was attached to the equivalence; "The experimentally known matter independence of the acceleration of fall is .... a powerful argument for the fact that the relativity postulate has to be extended to coordinate systems which, relative to each other, are in non-uniform motion" [14]. This conviction led him to the formulation of the *equivalence principle*. The equivalence principle provides the link between the physical laws as we discern them in our laboratories and their form under any circumstance in the universe—more

[5] Eötvös' experiments on such diverse media as wood, platinum, copper, glass, and other materials involve different molecular, atomic, and nuclear binding energies and different ratios of neutrons and protons.

precisely, in arbitrarily strong and varying gravitational fields. It also provides a tool for the development of the theory of gravitation itself, as we shall see throughout the sequel.

The universe is populated by massive objects moving relative to one another. The gravitational field may be arbitrarily changing in time and space. However, the presence of gravity cannot be detected in a sufficiently small reference frame falling freely with a particle under no influence other than gravity. The particle will remain at rest in such a frame. It is a *local inertial frame*. A local inertial frame and a *local Lorentz frame* are synonymous. The laws of Special Relativity hold in inertial frames and therefore in the neighborhood of a *freely falling frame*. In this way the relativity principle is extended to arbitrary gravitational fields.

Associated with a given spacetime event there are an infinity of locally inertial frames related by Lorentz transformations. All are equivalent for the description of physical phenomena in a sufficiently small region of spacetime. So we arrive at a statement of the equivalence principle: *At every spacetime point in an arbitrary gravitational field (meaning anytime and anywhere in the universe), a local inertial (Lorentz) frame can be chosen so that the laws of physics take on the form they have in Special Relativity.* This is the meaning of the equality of inertial and gravitational masses that Einstein sought. The restricted validity of inertial frames to small localities of any event suggested the very fruitful analogy with local flatness on a curved surface.

Einstein went further than the above statement of the equivalence principle. He spoke of the laws of nature rather than just the laws of physics. It seems entirely plausible that the extension is true, but we deal here only with physics.

The equivalence principle has great power. It is the instrument by which all the special relativistic laws of physics—valid in a gravity-free universe—can be generalized to a gravity-filled universe. We shall see how Einstein was able to give dynamic meaning to the spacetime continuum as an integral part of the physical world quite unlike the conception of an absolute spacetime in which the rest of physical processes take place.

2.4.1 Photon in a gravitational field

Employing the conservation of energy and Newtonian physics, Einstein reasoned that the gravitational field acts on photons. Let a photon be emitted from z_1 vertically to z_2, and only for simplicity, let the field be uniform. A device located at z_2 converts its energy on arrival to a particle of mass m with perfect efficiency. The particle drops to z_1 where its energy is now $m + mgh$, where g is the acceleration due to the uniform field. A device at z_1 converts it into a photon of the same energy as possessed by the particle. The photon again is directed to z_2. If the original (and each succeeding photon) does not lose energy $(h\nu)gh$ in climbing the gravitational field equal to the energy gained by the particle in dropping in the field, we would have a device that creates energy. By the law of conservation of energy Einstein discovered the gravitational redshift, commonly designated by the factor z and equal in this case to gh. The shift in energy of a photon by falling (in this case blue-shifted) in the earth's gravitational field has been directly confirmed in an experiment performed by Pound and Rebka [15].

In the above discussion the equivalence principle entered when the photon's inertial mass $(h\nu)$ was used also as its gravitational mass in computing the gravitational work. One can also see the role of the equivalence principle by considering a pulse of light emitted over a distance h along the axis of a spaceship in uniform acceleration g in outer space. The time taken for the light to reach the detector is $t = h$ (we use units $G = c = 1$). The difference in velocity of the detector acquired during the light travel time is $v = gt = gh$, the Doppler shift z in the detected light. This experiment, carried out in the gravity-free environment of a spaceship whose rockets produce an acceleration g, must yield the same result for the energy shift of the photon in a uniform gravitational field g according to the equivalence principle. The Pound–Rebka experiment can therefore be regarded as an experimental proof of the equivalence principle.

We may regard a radiating atom as a clock, with each wave crest regarded as a tick of the clock. Imagine two identical atoms situated one at some height above the other in the gravitational field of the earth. Since, by dropping in the gravitational field, the light is blue-shifted when compared to the radiation of an identical atom (clock) at the bottom, the clock at the top is seen to be running faster than the one at the bottom. Therefore, identical clocks, stationary with respect to the earth, run at different rates according to their different heights above the earth. Time flows at different rates in different gravitational fields.

The trajectory of photons is also bent by the gravitational field. Imagine a freely falling elevator in a constant gravitational field. Its walls constitute an inertial frame as guaranteed by the equivalence principle. Therefore, a photon (as for a free particle) directed from one wall to the opposite along a path parallel to the floor will arrive at the other wall at the same height from which it started. But relative to the earth, the elevator has fallen during the traversal time. Therefore the photon has been deflected toward the earth and follows a curved path as observed from a frame fixed on the earth.

2.4.2 Tidal gravity

Einstein predicted that a clock near a massive body would run more slowly than an identical distant clock. In doing so he arrived at a hint of the deep connection of the structure of spacetime and gravity. Two parallel straight lines never meet in the gravity-free, flat spacetime of Minkowski. A single inertial frame would suffice to describe all of spacetime. In formulating the equivalence principle (knowing that gravitational fields are not uniform and constant but depend on the motion of gravitating bodies and the position where gravitational effects are experienced), Einstein understood that only in a suitably small locality of spacetime do the laws of Special Relativity hold. Gravitational effects will be observed on a larger scale. *Tidal gravity* refers to the deviation from uniformity of the gravitational field at nearby points.

These considerations led Einstein to the notion of spacetime curvature. Whatever the motion of a free body in an arbitrary gravitational field, it will follow a straight-line trajectory over any small locality as guaranteed by the equivalence principle. And in a gravity-endowed universe, free particles whose trajectories are parallel in a local inertial frame, will not remain parallel over a large region of spacetime. This has a striking analogy with the surface of a sphere on which two straight lines that are parallel over a small region *do* meet and cross. What if in fact the particles are freely falling in curved spacetime? In this way of thinking, the law that free particles move in straight lines remains true in an arbitrary gravitational field, thus obeying the principle of relativity in a larger sense. Any sufficiently small region of curved spacetime is locally flat. The paths in curved spacetime that have the property of being locally straight are called geodesics.

2.4.3 Curvature of spacetime

Let us now consider a thought experiment. Two nearby bodies released from rest above the earth follow parallel trajectories over a small region of their trajectories, as we know from the equivalence principle. But if holes were drilled in the earth through which the bodies could fall, the bodies would meet and cross at the earth's center. So there is clearly no single Minkowski spacetime that covers a large region or the whole region containing a massive body.

Einstein's view was that spacetime curvature caused the bodies to cross, bodies that in this curved spacetime were following straight line paths in every small locality, just as they would have done in the whole of Minkowski (flat) spacetime in the absence of gravitational bodies. The presence of gravitating bodies denies the existence of a global inertial frame. Spacetime can be flat everywhere only if there exists such a global frame. Hence, spacetime is curved by massive bodies. In their presence a test particle follows a geodesic path, one that is always locally straight. The concept of a "gravitational force" has been replaced by the curvature of spacetime, and the natural free motions of particles in it are defined by geodesics.

2.4.4 Energy conservation and curvature

Interestingly, the conservation of energy can also be used to inform us that spacetime is curved. Consider a static gravitational field. Let us conjecture that spacetime is flat so that the Minkowski metric holds; we will arrive at a contradiction.

Imagine the following experiment performed by observers and their apparatus at rest with respect to the gravitational field and their chosen Lorentz frame in the supposed flat spacetime of Minkowski. At a height z_1 in the field, let a monochromatic light signal be emitted upward a height h to $z_2 = z_1 + h$. Let the pulse be emitted for a specific time dt_1 during which N wavelengths (or photons) are emitted. Let the time during which they are received at z_2 be measured as dt_2. (Because the spacetime is assumed to be described by the Minkowski metric and the source and receiver are at rest in the chosen frame, the proper times and coordinate times are equal.)

Because the field in the above experiment is static, the path in the z-t plane will have the same shape for both the beginning and ending of the pulse (as for each photon) as they trace their path in the Minkowski space we postulate to hold. The trajectories will not be lines at 45 degrees because of the field, but the curved paths will be congruent; a translation in time will make the paths lie one upon the other. Therefore $d\tau_2 = dt_2 = dt_1 = d\tau_1$ will be measured at the stationary detector if spacetime is Minkowskian. In this case, the frequency (and hence the energy received at z_2) is the same as that sent from z_1. But this cannot be. The photons comprising the signal must lose energy in climbing the gravitational field (see Section 2.4.1).

The conjecture that spacetime in the presence of a gravitational field is Minkowskian must therefore be false. We conclude that the presence of the gravitational field has caused spacetime to be curved. Such a line of reasoning was first conceived by Schild [16, 17, 18].

2.5 Gravity

> "I was sitting in a chair at the patent office at Bern when all of a sudden a thought occurred to me: 'If a person falls freely he will not feel his own weight.' I was startled. This simple thought had a deep impression on me. It impelled me toward a theory of gravitation." *A. Einstein* [19]

Massive bodies generate curvature. Galaxies, stars, and other bodies are in motion; therefore the curvature of spacetime is everywhere changing. For this reason there is no "prior geometry". There are no immutable reference frames to which events in spacetime can be referred. Indeed, the changing geometry of spacetime and of the motion and arrangement of mass-energy in spacetime are inseparable parts of the description of physical processes. This is a very different idea of space and time from that of Newton and even of the Special Theory of Relativity. We now take up the unified discussion of gravitating matter and motion.

The power of the equivalence principle in informing us so simply that spacetime must be curved by the presence of massive bodies in the universe suggests a fruitful way of beginning. Following Weinberg [20], or indeed, following the notion expressed by Einstein in the quotation above, we seek the connection between an arbitrary reference frame and a reference frame that is freely falling with a particle that is moving only under the influence of an arbitrary gravitational field. In this freely falling and therefore locally inertial frame, the particle moves in a straight line. Denote the coordinates by ξ^α. The equations of motion are

$$\frac{d^2\xi^\alpha}{d\tau^2} = 0 \ , \tag{2.82}$$

and the invariant interval (or proper time) between two neighboring spacetime events expressed in this frame, from (2.6), is

$$d\tau^2 = \eta_{\alpha\beta} d\xi^\alpha d\xi^\beta . \tag{2.83}$$

The freely falling coordinates may be regarded as functions of the coordinates x^μ of any arbitrary reference frame—curvilinear, accelerated, or rotating. We seek the connection between the equations of motion in the freely falling frame and the arbitrary one which, for example, might be the laboratory frame. From the chain rule for differentiation we can rewrite (2.82) as

$$\begin{aligned} 0 &= \frac{d}{d\tau}\Big(\frac{\partial \xi^\alpha}{\partial x^\mu}\frac{dx^\mu}{d\tau}\Big) \\ &= \frac{\partial \xi^\alpha}{\partial x^\mu}\frac{d^2 x^\mu}{d\tau^2} + \frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\,. \end{aligned}$$

Multiply by $\partial x^\lambda/\partial \xi^\alpha$, and use the chain rule again to obtain

$$\frac{dx^\lambda}{dx^\mu} = \frac{\partial x^\lambda}{\partial \xi^\alpha}\frac{\partial \xi^\alpha}{\partial x^\mu} = \delta^\lambda_\mu\,. \tag{2.84}$$

The equation of motion of the particle in an arbitrary frame when the particle is moving in an arbitrary gravitational field therefore is

$$\frac{d^2 x^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0\,. \tag{2.85}$$

Here $\Gamma^\lambda_{\mu\nu}$, defined by

$$\Gamma^\lambda_{\mu\nu} \equiv \frac{\partial x^\lambda}{\partial \xi^\alpha}\frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu}\,, \tag{2.86}$$

is called the *affine connection*. The affine connection is symmetric in its lower indices.

The path defined by equation (2.85) is called a *geodesic*, the extremal path in the spacetime of an arbitrary gravitational field. We do not see here that it is an extremal, but this is hinted at inasmuch as it defines the same path of (2.82), the straight-line path of a free particle as observed from its freely falling frame. In the next section we will see that a geodesic path is locally a straight line.

The invariant interval (2.83) can also be expressed in the arbitrary frame by writing $d\xi^\alpha = (\partial \xi^\alpha/\partial x^\mu)dx^\mu$ so that

$$d\tau^2 = g_{\mu\nu}dx^\mu dx^\nu \tag{2.87}$$

with

$$g_{\mu\nu} = \frac{\partial \xi^\alpha}{\partial x^\mu}\frac{\partial \xi^\beta}{\partial x^\nu}\eta_{\alpha\beta}\,. \tag{2.88}$$

In the new and arbitrary reference frame, the second term of (2.85) causes a deviation from a straight-line motion of the particle in this frame. Therefore, the second term represents the effect of the gravitational field. (To be sure, the connection coefficients also represent any other noninertial effects that may have been introduced by the choice of reference frame, such as rotation.)

The affine connection (2.86) appearing in the geodesic equation clearly plays an important role in gravity, and we study it further. We first show that the affine connection is a nontensor, and then show how it can be expressed in terms of the metric tensor and its derivatives. In this sense the metric behaves as the gravitational potential and the affine connection as the force. Write $\Gamma^{\lambda}_{\mu\nu}$ expressed in (2.86) in another coordinate system x'^{μ} and use the chain rule several times to rewrite it:

$$\begin{aligned}\Gamma'^{\lambda}_{\mu\nu} &= \frac{\partial x'^{\lambda}}{\partial x^{\rho}}\frac{\partial x^{\rho}}{\partial \xi^{\alpha}}\frac{\partial}{\partial x'^{\mu}}\Big(\frac{\partial x^{\sigma}}{\partial x'^{\nu}}\frac{\partial \xi^{\alpha}}{\partial x^{\sigma}}\Big) \\ &= \frac{\partial x'^{\lambda}}{\partial x^{\rho}}\frac{\partial x^{\rho}}{\partial \xi^{\alpha}}\Big[\frac{\partial x^{\sigma}}{\partial x'^{\nu}}\frac{\partial x^{\tau}}{\partial x'^{\mu}}\frac{\partial^2 \xi^{\alpha}}{\partial x^{\tau}\partial x^{\sigma}} + \frac{\partial^2 x^{\sigma}}{\partial x'^{\mu}\partial x'^{\nu}}\frac{\partial \xi^{\alpha}}{\partial x^{\sigma}}\Big] \\ &= \frac{\partial x'^{\lambda}}{\partial x^{\rho}}\frac{\partial x^{\tau}}{\partial x'^{\mu}}\frac{\partial x^{\sigma}}{\partial x'^{\nu}}\Gamma^{\rho}_{\tau\sigma} + \frac{\partial x'^{\lambda}}{\partial x^{\rho}}\frac{\partial^2 x^{\rho}}{\partial x'^{\mu}\partial x'^{\nu}} \ . \end{aligned} \tag{2.89}$$

According to the transformation laws of tensors developed in Section 2.3, the second term on the right spoils the transformation law of the affine connection. It is therefore a nontensor.

Let us now obtain the expression of the affine connection in terms of the derivatives of the metric tensor. Form the derivative of (2.79):

$$\frac{\partial}{\partial x'^{\kappa}}g'_{\mu\nu} = \frac{\partial}{\partial x'^{\kappa}}\Big(g_{\rho\sigma}\frac{\partial x^{\rho}}{\partial x'^{\mu}}\frac{\partial x^{\sigma}}{\partial x'^{\nu}}\Big) \ .$$

Take the derivatives and form the following combination and find that it is equal to the above derivative:

$$\begin{aligned}\frac{\partial g'_{\kappa\nu}}{\partial x'^{\mu}} + \frac{\partial g'_{\kappa\mu}}{\partial x'^{\nu}} - \frac{\partial g'_{\mu\nu}}{\partial x'^{\kappa}} = \frac{\partial x^{\tau}}{\partial x'^{\kappa}}\frac{\partial x^{\rho}}{\partial x'^{\mu}}\frac{\partial x^{\sigma}}{\partial x'^{\nu}}\Big(\frac{\partial g_{\sigma\tau}}{\partial x^{\rho}} + \frac{\partial g_{\rho\tau}}{\partial x^{\sigma}} - \frac{\partial g_{\rho\sigma}}{\partial x^{\tau}}\Big) \\ +2g_{\rho\sigma}\frac{\partial x^{\sigma}}{\partial x'^{\kappa}}\frac{\partial^2 x^{\rho}}{\partial x'^{\mu}\partial x'^{\nu}} \ .\end{aligned}$$

Multiply this equation by $\frac{1}{2}$ and then multiply the left and right sides by the left and right sides, respectively, of the law of transformation (2.80), namely,

$$g'^{\lambda\kappa} = g^{\alpha\beta}\frac{\partial x'^{\lambda}}{\partial x^{\alpha}}\frac{\partial x'^{\kappa}}{\partial x^{\beta}} \,.$$

Use the chain rule and rename several dummy indices to obtain

$$\begin{Bmatrix} \lambda \\ \mu\nu \end{Bmatrix}' = \frac{\partial x'^{\lambda}}{\partial x^{\rho}}\frac{\partial x^{\tau}}{\partial x'^{\mu}}\frac{\partial x^{\sigma}}{\partial x'^{\nu}}\begin{Bmatrix} \rho \\ \tau\sigma \end{Bmatrix} + \frac{\partial x'^{\lambda}}{\partial x^{\rho}}\frac{\partial^2 x^{\rho}}{\partial x'^{\mu}\partial x'^{\nu}} \,, \tag{2.90}$$

where the prime on $\{\}$ means that it is evaluated in the x'^{μ} frame and the symbol stands for

$$\begin{Bmatrix} \lambda \\ \mu\nu \end{Bmatrix} = \tfrac{1}{2}g^{\lambda\kappa}\left[\frac{\partial g_{\kappa\nu}}{\partial x^{\mu}} + \frac{\partial g_{\kappa\mu}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\kappa}}\right]. \tag{2.91}$$

This is called a *Christoffel symbol of the second kind*. It is seen to transform in exactly the same way as the affine connection (2.89). Subtract the two to obtain

$$\left[\Gamma^{\lambda}_{\mu\nu} - \begin{Bmatrix} \lambda \\ \mu\nu \end{Bmatrix}\right]' = \frac{\partial x'^{\lambda}}{\partial x^{\rho}}\frac{\partial x^{\tau}}{\partial x'^{\mu}}\frac{\partial x^{\sigma}}{\partial x'^{\nu}}\left[\Gamma^{\rho}_{\tau\sigma} - \begin{Bmatrix} \rho \\ \tau\sigma \end{Bmatrix}\right]. \tag{2.92}$$

This shows that the difference is a tensor. According to the equivalence principle, at anyplace and anytime there is a local inertial frame ξ^{α} in which the effects of gravitation are absent, the metric is given by (2.7), and $\Gamma^{\lambda}_{\mu\nu}$ vanishes (compare (2.82) and (2.85)). Because the first derivatives of the metric tensor vanish in such a local inertial system, the Christoffel symbol also vanishes. Because the difference of the affine connection and the Christoffel symbol is a tensor which vanishes in this frame, the difference vanishes in all reference frames. So everywhere we find

$$\Gamma^{\lambda}_{\mu\nu} = \begin{Bmatrix} \lambda \\ \mu\nu \end{Bmatrix} = \tfrac{1}{2}g^{\lambda\kappa}(g_{\kappa\nu,\mu} + g_{\kappa\mu,\nu} - g_{\mu\nu,\kappa}) \,. \tag{2.93}$$

We use the "comma subscript" notation introduced earlier to denote differentiation (2.72).

Sometimes it is useful to have the superscript lowered on the affine connection

$$\Gamma_{\kappa\mu\nu} = g_{\kappa\lambda}\Gamma^{\lambda}_{\mu\nu} \,. \tag{2.94}$$

It is equal to the Christoffel symbol of the first kind

$$\Gamma_{\kappa\mu\nu} = \begin{bmatrix} \kappa \\ \mu\nu \end{bmatrix} = \tfrac{1}{2}(g_{\kappa\nu,\mu} + g_{\kappa\mu,\nu} - g_{\mu\nu,\kappa}) \; . \tag{2.95}$$

The above formulas provide a means of computing the affine connection from the derivatives of the metric tensor and will prove very useful. It is trivial from the above to prove that

$$\Gamma_{\kappa\mu\nu} + \Gamma_{\mu\kappa\nu} = g_{\mu\kappa,\nu} \, . \tag{2.96}$$

2.5.1 Mathematical definition of local Lorentz frames

Spacetime is curved globally by the massive bodies in the universe. Therefore, we need to define mathematically the meaning of "local Lorentz frame". In a rectilinear Lorentz frame the metric tensor is $\eta_{\mu\nu}$ (2.7). Therefore, in the local region around an event P (a point in the four-dimensional spacetime continuum), the metric tensor, its coordinate derivatives, and the affine connection have the following values:

$$g_{\mu\nu}(P) = \eta_{\mu\nu}, \qquad g_{\mu\nu,\alpha}(P) = 0, \qquad \Gamma^{\lambda}_{\mu\nu}(P) = 0 \, . \tag{2.97}$$

The third of these equations follows from the second and from (2.93). All local effects of gravitation disappear in such a frame. The geodesic equation (2.85) defining the path followed by a free particle in an arbitrary gravitational field becomes locally the equation of a uniform straight line, in accord with the equivalence principle.

Of course, physical measurements are always subject to the precision of the measuring devices. The extent of the local region around P, in which the above equations will hold and in which spacetime is said to be flat, will depend on the accuracy of the devices and therefore their ability to detect deviations from the above conditions as one measures further from P.

2.5.2 Geodesics

In the Special Theory of Relativity a free particle remains at rest or moves with constant velocity in a straight line. A straight line is the shortest distance between two points in Euclidean three-dimensional space. In Minkowski spacetime a straight line is the longest interval between two events, as we shall shortly see. Both situations are covered by saying that a straight line is an extremal path between two points. We shall show that in an arbitrary gravitational field, a particle moving under the influence of only gravity, follows a path that is, in the sense that we shall define, the straightest line possible in curved spacetime.

We first show that a straight-line path between two events in Minkowski spacetime maximizes the proper time. This is easily proved. Orient the axis so that the two events marking the ends of the path, A and B, lie on the t-axis with coordinates $(0,0,0,0)$ and $(T,0,0,0)$, and consider an alternate path in the t-x plane that consists of two straight-line segments that pass from A to B through $(T/2, R/2, 0, 0)$. The proper time as measured on the second path is

$$\tau = 2\sqrt{(T/2)^2 - (R/2)^2} = \sqrt{T^2 - R^2}\,. \tag{2.98}$$

For any finite R, τ is smaller than the proper time along the straight-line path from A to B, namely, T. Therefore, a straight-line path is a maximum in proper time.

We have referred to the equation of motion of a particle moving freely in an arbitrary gravitational field (2.85) as a geodesic equation. In general, a geodesic that is not null (a null geodesic, as is the case for a light particle, has $d\tau = 0$), is the extremal path of

$$\int_A^B d\tau \tag{2.99}$$

where A and B refer to spacetime events on the geodesic. To prove this result, let $x^\mu(\tau)$ denote the coordinates along the geodesic path, parameterized by the proper time, and let $x^\mu(\tau) + \delta x^\mu(\tau)$ denote a neighboring path with the same end points, A to B. From

$$d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu\,, \tag{2.100}$$

we have to first order in the variation,

$$\begin{aligned} 2d\tau\, \delta(d\tau) &= \delta g_{\mu\nu} dx^\mu dx^\nu + 2 g_{\mu\nu} dx^\mu \delta(dx^\nu) \\ &= dx^\mu dx^\nu g_{\mu\nu,\lambda} \delta x^\lambda + 2 g_{\mu\nu} dx^\mu d(\delta x^\nu)\,. \end{aligned} \tag{2.101}$$

Recalling the four-velocity, $u^\mu = dx^\mu/d\tau$, we have

$$\delta(d\tau) = \Big(\tfrac{1}{2}u^\mu u^\nu g_{\mu\nu,\lambda}\delta x^\lambda + g_{\mu\lambda}u^\mu \frac{d}{d\tau}\delta x^\lambda\Big)d\tau\,. \tag{2.102}$$

Thus

$$\delta\int_A^B d\tau = \int_A^B \Big[\tfrac{1}{2}u^\mu u^\nu g_{\mu\nu,\lambda} - \frac{d}{d\tau}\Big(g_{\mu\lambda}u^\mu\Big)\Big]\delta x^\lambda\, d\tau \tag{2.103}$$

where an integration by parts in the second term was performed. Because the variation of the path δx^λ is arbitrary save for its end points being zero, we obtain as the extremal condition,

$$\frac{d}{d\tau}\Big(g_{\mu\lambda}u^\mu\Big) - \tfrac{1}{2}u^\mu u^\nu g_{\mu\nu,\lambda} = 0\,. \tag{2.104}$$

The first and second terms can be rewritten:

$$\begin{aligned}\frac{d}{d\tau}\Big(g_{\mu\lambda}u^\mu\Big) &= g_{\mu\lambda}\frac{du^\mu}{d\tau} + g_{\mu\lambda,\nu}u^\mu u^\nu\,,\\ g_{\mu\lambda,\nu}u^\mu u^\nu &= \tfrac{1}{2}(g_{\mu\lambda,\nu} + g_{\lambda\nu,\mu})u^\mu u^\nu\,.\end{aligned} \tag{2.105}$$

Now using the relationship (2.95), we find

$$g_{\mu\lambda}\frac{du^\mu}{d\tau} + \Gamma_{\lambda\mu\nu}u^\mu u^\nu = 0\,. \tag{2.106}$$

Multiplying by $g^{\sigma\lambda}$ and summing on λ, we obtain the geodesic equation (2.85):

$$\frac{du^\sigma}{d\tau} + \Gamma^\sigma_{\mu\nu}u^\mu u^\nu = 0\,. \tag{2.107}$$

This completes the proof that the path defined by the geodesic equation, the equation of motion of a particle in a purely gravitational field, extremizes the proper time between any two events on the path.

The straight-line path between two events in Minkowski spacetime maximizes the interval between the events. We proved that a geodesic path, in the general case that a gravitational field is present, will be an extremum, but if the spacetime separation of the ends of the path is large, there may be two geodesic paths, one of minimum and one of maximum length. The geodesic path of a particle in spacetime is frequently referred to as its *world line*. A world line is a continuous sequence of points in spacetime; it represent the history of a particle or photon.

In a region of spacetime sufficiently small that the Minkowski metric holds (the existence of which locality is guaranteed by the equivalence principle), we see that the geodesic equation reduces to that for uniform straight-line motion,

$$\frac{du^\mu}{d\tau} = 0 . \tag{2.108}$$

Therefore, the path of a particle moving under the influence of a general gravitational field will be locally straight. But we know that no global Lorentz frame exists in the presence of gravitating bodies; therefore, geodesic paths will in general be curved. However, in the above sense they will be as straight as possible in curved spacetime.

2.5.3 Comparison with Newton's gravity

We confirm the assertion made earlier that the metric tensor $g_{\mu\nu}$ takes the place in General Relativity that the Newtonian potential occupies in Newton's theory. Of course this must be done in a weak field situation for it is only there that Newton's theory applies. For this reason, of the ten independent $g_{\mu\nu}$'s, only one can be involved in the correspondence.

We consider a particle moving slowly in a weak static gravitational field. From the Special Theory of Relativity we have

$$d\tau = (dt^2 - d\mathbf{r}^2)^{1/2} = (1 - \mathbf{v}^2)^{1/2} dt, \qquad \mathbf{v} = \frac{d\mathbf{r}}{dt} , \tag{2.109}$$

where boldface symbols denote three-vectors. The slowly moving assumption is

$$\frac{d\mathbf{r}}{dt} << \frac{dt}{d\tau} \approx 1 . \tag{2.110}$$

So the geodesic equation (2.85) can be written with the neglect of the velocity terms as

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{00} \Big(\frac{dt}{d\tau}\Big)^2 = 0 . \tag{2.111}$$

Because the field is static, the time derivatives of $g_{\mu\nu}$ vanish. Consequently,

$$\Gamma^{\mu}_{00} = \tfrac{1}{2} g^{\mu\nu}(2g_{\nu 0,0} - g_{00,\nu}) = -\tfrac{1}{2} g^{\mu\nu} g_{00,\nu}(1 - \delta^{\nu}_{0}) \,. \qquad (2.112)$$

Because the field is weak we may take

$$g_{00} = (1+\delta)\eta_{00} \ , \qquad (2.113)$$

where $\delta << 1$ and similarly for the other $g_{\mu\nu}$. To first order in the small quantities, we have

$$\Gamma^{\mu}_{00} = -\tfrac{1}{2}\eta^{\mu\nu}\eta_{00}\frac{d\delta}{dx^{\nu}}(1 - \delta^{\nu}_{0}) \,. \qquad (2.114)$$

Thus the geodesic equations become

$$\frac{d^2\mathbf{r}}{d\tau^2} = -\tfrac{1}{2}\Big(\frac{dt}{d\tau}\Big)^2 \nabla\delta, \qquad \frac{d^2 t}{d\tau^2} = 0 \,. \qquad (2.115)$$

The second of these tells us that $\tau = at + b$. So we may write the first as

$$\frac{d^2\mathbf{r}}{dt^2} = -\tfrac{1}{2}\nabla\delta \,. \qquad (2.116)$$

Newton's equation is

$$\frac{d^2\mathbf{r}}{dt^2} = -\nabla V \ , \qquad (2.117)$$

where V is the gravitational potential. Comparing, we have

$$g_{00} = 1 + 2V \,. \qquad (2.118)$$

In particular, if the gravitational field is produced by a body of mass M,

$$V = -\frac{GM}{r} \quad \Longrightarrow \quad g_{00} = 1 - \frac{2GM}{r} \ , \qquad (2.119)$$

where G is Newton's constant. Thus we see for weak fields how the metric is related to the Newtonian potential.

2.6 Covariance

2.6.1 Principle of general covariance

Physical laws in their form ought to be independent of the frame in which they are expressed and of the location in the universe, that is, independent of the gravitational field. The principle of general covariance states that a law of physics holds in a general gravitational field if it holds in the absence of gravity and its form is invariant to any coordinate transformation. Physical laws frequently involve space-time derivatives of scalars, vectors, or tensors. We have seen that the derivative of a scalar is a vector but that the ordinary derivative of a vector or a tensor is not a tensor (page 28). Therefore, we need a type of derivative—a covariant derivative—that reduces to ordinary differentiation in the absence of gravity and which retains its form under any coordinate transformation, that is, in any gravitational field.

2.6.2 Covariant differentiation

Take the derivative of the expression of the covariant vector transformation law (2.63),

$$\frac{dA'_\mu}{dx'^\rho} = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\rho}\frac{\partial A_\nu}{\partial x^\sigma} + \frac{\partial^2 x^\nu}{\partial x'^\rho \partial x'^\mu} A_\nu \ .$$

If only the first term were present we would have the correct transformation law for a covariant tensor. Now multiply the left and right sides of (2.89) by the left and right sides of (2.63), respectively, and rearrange to find

$$\Gamma'^\lambda_{\mu\nu} A'_\lambda = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}\Gamma^\kappa_{\alpha\beta} A_\kappa + \frac{\partial^2 x^\kappa}{\partial x'^\mu \partial x'^\nu} A_\kappa \ .$$

Subtracting the above two equations after renaming dummy indices of summation, we get

$$\left(\frac{dA'_\mu}{dx'^\nu} - \Gamma'^\lambda_{\mu\nu} A'_\lambda\right) = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}\left(\frac{dA_\alpha}{dx^\beta} - \Gamma^\lambda_{\alpha\beta} A_\lambda\right) , \qquad (2.120)$$

which proves the tensor character of the quantity in brackets. This we call the *covariant derivative* of a covariant vector. We denote it by

$$A_{\mu;\nu} \equiv \frac{dA_\mu}{dx^\nu} - \Gamma^\lambda_{\mu\nu} A_\lambda \ , \tag{2.121}$$

and the "semicolon subscript" shall denote the covariant derivative, and imply the operations shown on the right. The covariant derivative of a covariant vector is a second-rank covariant tensor which reduces to ordinary differentiation in inertial frames—and therefore locally in any gravitational field.

Through similar manipulations we find the covariant derivative of a contravariant vector,

$$A^\mu{}_{;\nu} \equiv \frac{dA^\mu}{dx^\nu} + \Gamma^\mu_{\sigma\nu} A^\sigma \ . \tag{2.122}$$

This is a second-rank mixed tensor because its transformation law is

$$\left(\frac{dA'^\mu}{dx'^\nu} + \Gamma'^\mu_{\lambda\nu} A'^\lambda\right) = \frac{\partial x'^\mu}{\partial x^\alpha}\frac{\partial x^\beta}{\partial x'^\nu}\left(\frac{dA^\alpha}{dx^\beta} + \Gamma^\alpha_{\kappa\beta} A^\kappa\right) \ . \tag{2.123}$$

The covariant derivative of a mixed tensor of arbitrary order can be obtained by successive application of the above two rules to each index; there is one ordinary derivative of the tensor and an affine connection for each index with sign as indicated by the above.

In particular, the covariant derivative of the metric tensor is

$$g_{\mu\nu;\lambda} = \frac{\partial g_{\mu\nu}}{\partial x^\lambda} - \Gamma^\rho_{\mu\lambda} g_{\rho\nu} - \Gamma^\rho_{\nu\lambda} g_{\rho\mu} \ . \tag{2.124}$$

In a local inertial frame, where the affine connection and the derivative of the metric tensor vanish, we see that the covariant derivative of the metric tensor vanishes in that frame. But because this itself is a tensor, it must vanish in all frames. Similarly, for the covariant derivative of $g^{\mu\nu}$,

$$g_{\mu\nu;\lambda} = 0 = g^{\mu\nu}{}_{;\lambda} \, . \tag{2.125}$$

2.6.3 Geodesic equation from covariance principle

As an important example of the application of the covariant derivative, consider the four-velocity of a free particle in a Lorentz frame in the absence of gravity. We denote the four-velocity by $u^\mu = dx^\mu/d\tau$ and its equation of motion is $du^\mu/d\tau = 0$, or equivalently in differential form,

$$du^\mu = 0\,. \tag{2.126}$$

The covariant derivative (2.123) was introduced to preserve the vector or tensor character so that a law expressed in such form is preserved in form for all coordinate transformations in accord with the principle of relativity. The equation expressing the law is said to be covariant if its form is preserved. Therefore the law of free motion (2.126) in a Lorentz frame in the absence of gravity is generalized to frames in arbitrary gravitational fields by requiring that the covariant differential of the four-velocity vanish:

$$\begin{aligned} 0 = u^\mu{}_{;\nu} dx^\nu &= \frac{du^\mu}{dx^\nu} dx^\nu + \Gamma^\mu_{\sigma\nu} u^\sigma dx^\nu \\ &= du^\mu + \Gamma^\mu_{\sigma\nu} u^\sigma dx^\nu\,. \end{aligned} \tag{2.127}$$

Dividing the above equation by $d\tau$ yields the expected result—the geodesic equation (2.85)—the equation of motion derived previously for a free particle in an arbitrary gravitational field:

$$\frac{d^2 x^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu} \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0\,. \tag{2.128}$$

This is an example of the application of the principle of general covariance and it is seen to rest on the equivalence principle, which assures us that a Lorentz frame can be erected locally.

To restate the principle briefly, *any law that holds in the special theory of relativity and in the absence of gravity can be generalized by replacing the metric* $\eta_{\mu\nu}$ by $g_{\mu\nu}$ *and replacing ordinary derivatives by covariant derivatives.*

We obtain an additional result that we need later, namely, the equations of motion for the covariant components of the four-velocity. The law of motion of a free particle in the special theory, expressed in differential form as in (2.126), implies at once that $du_\mu = g_{\mu\nu} du^\nu = 0$. The covariant translation of this fact is

$$0 = u_{\mu;\nu} dx^\nu = \frac{du_\mu}{dx^\nu} dx^\nu - \Gamma^\lambda_{\mu\nu} u_\lambda dx^\nu \tag{2.129}$$

or

$$\frac{d^2x_\mu}{d\tau^2} - \Gamma^\lambda_{\mu\nu}\frac{dx_\lambda}{d\tau}\frac{dx^\nu}{d\tau} = 0\,. \qquad (2.130)$$

This is the equation corresponding to (2.128) for the covariant acceleration. We carry the analysis a step further. Examine the second term on the left.

$$\begin{aligned}\Gamma^\lambda_{\mu\nu}u_\lambda u^\nu &= \frac{1}{2}g^{\lambda\kappa}(g_{\kappa\nu,\mu} + g_{\kappa\mu,\nu} - g_{\mu\nu,\kappa})u_\lambda u^\nu \\ &= \frac{1}{2}(g_{\kappa\nu,\mu} + g_{\kappa\mu,\nu} - g_{\mu\nu,\kappa})u^\kappa u^\nu\,. \qquad (2.131)\end{aligned}$$

Because of the symmetry of the product $u^\kappa u^\nu$, the last two terms in the bracket cancel. We are left with

$$\frac{du_\mu}{d\tau} = \frac{1}{2}g_{\kappa\nu,\mu}u^\kappa u^\nu\,. \qquad (2.132)$$

This proves that if all the $g_{\alpha\beta}$ are independent of some coordinate component, say x^μ, then the covariant velocity u_μ is a constant along the particle's trajectory. We will use this result in a much later chapter during a discussion of the phenomenon of dragging of local inertial frames by a rotating star (according to which a body dropped freely from a great distance falls, not toward the star's center, but is dragged ever more strongly in the sense of the star's rotation).

2.6.4 Covariant divergence and conserved quantities

The element of four-volume transforms under coordinate change as

$$dx'^0dx'^1dx'^2dx'^3 = J\,dx^0dx^1dx^2dx^3\,, \qquad (2.133)$$

where J is the Jacobian of the transformation,

$$J = \det\left|\frac{\partial x'^\rho}{\partial x^\mu}\right|\,. \qquad (2.134)$$

For brevity the four-volume element is often written d^4x.

The transformation law for the metric tensor is

$$g_{\mu\nu} = \frac{\partial x'^\alpha}{\partial x^\mu} g'_{\alpha\beta} \frac{\partial x'^\beta}{\partial x^\nu} \,. \tag{2.135}$$

We may regard this as an element in the product of three matrices. The corresponding determinant equation is

$$g = J g' J = J^2 g' \tag{2.136}$$

where $g = \det|g_{\mu\nu}|$ and is a negative quantity as can be verified by looking at the Minkowski metric. Thus, we may write

$$\sqrt{-g} = J\sqrt{-g'} \,. \tag{2.137}$$

If $S = S'$ is a scalar field, then

$$\int_{V_4} S\sqrt{-g}\, d^4x = \int_{V_4} S\sqrt{-g'} J\, d^4x = \int_{V_4} S'\sqrt{-g'}\, d^4x' \tag{2.138}$$

is an invariant where V_4 is a prescribed four-volume. The quantity

$$\mathcal{S} \equiv S\sqrt{-g} \tag{2.139}$$

is called a *scalar density*, and its integral over a region of spacetime is invariant to a coordinate transformation. Also, and very important to us, $\sqrt{-g}\, d^4x$ is the invariant volume element.

The covariant derivative of a vector A^μ is given by (2.122). If we contract indices, according to (2.76) we have a scalar. This is the *covariant divergence* of A^μ:

$$A^\mu{}_{;\mu} \equiv A^\mu{}_{,\mu} + \Gamma^\mu_{\nu\mu} A^\nu \,. \tag{2.140}$$

From (2.93) we find

$$\Gamma^\nu_{\mu\nu} = \tfrac{1}{2} g^{\nu\kappa}(g_{\kappa\nu,\mu} + g_{\kappa\mu,\nu} - g_{\mu\nu,\kappa}) \,. \tag{2.141}$$

Interchange the names of the dummy summation indices in the second term on the right to see that it cancels the third. Thus

$$\Gamma^\nu_{\mu\nu} = \tfrac{1}{2} g^{\nu\kappa} g_{\kappa\nu,\mu} \,. \tag{2.142}$$

We need still another result. Denote the cofactor of the element $g_{\alpha\beta}$ by $C^{\alpha\beta}$. The determinant $g = \det|g_{\alpha\beta}|$ can be expanded in any of the set of minors (i.e., any $\alpha = 0, 1, 2,$ or 3) in the equation

$$g = g_{(\alpha)\beta} C^{(\alpha)\beta} \quad \text{(no sum on } \alpha). \tag{2.143}$$

Because the cofactor contains no elements $g_{(\alpha)\beta}$, we find

$$\frac{\partial g}{\partial g_{\alpha\nu}} = \frac{\partial (g_{(\alpha)\beta} C^{(\alpha)\beta})}{\partial g_{\alpha\nu}} = \frac{\partial g_{\alpha\mu}}{\partial g_{\alpha\nu}} C^{\alpha\mu} = \delta^{\nu}_{\mu} C^{\alpha\mu} = C^{\alpha\nu} . \quad (2.144)$$

Therefore,

$$g_{,\alpha} = \frac{\partial g}{\partial g_{\mu\nu}} g_{\mu\nu,\alpha} = C^{\mu\nu} g_{\mu\nu,\alpha} . \quad (2.145)$$

We need the expression

$$C^{\mu\nu} = g g^{\mu\nu} , \quad (2.146)$$

which can be proved by multiplying by $g_{\mu\nu}$ and summing only over ν,

$$g_{(\mu)\nu} C^{(\mu)\nu} = g_{(\mu)\nu} g^{(\mu)\nu} g = g . \quad (2.147)$$

This is the determinant expansion in minors (2.143). Thus, we have derived the result

$$g_{,\alpha} = g g^{\mu\nu} g_{\mu\nu,\alpha} . \quad (2.148)$$

Hence,

$$\Gamma^{\nu}_{\mu\nu} = \tfrac{1}{2} g^{-1} g_{,\mu} = \tfrac{1}{2} (\ln(-g))_{,\mu} = \frac{1}{\sqrt{-g}} (\sqrt{-g})_{,\mu} . \quad (2.149)$$

We can use this to rewrite the covariant divergence of A^{μ} as

$$A^{\mu}{}_{;\mu} = \frac{1}{\sqrt{-g}} (\sqrt{-g} A^{\mu})_{,\mu} \quad (2.150)$$

With (2.149) in (2.140), we obtain the important result for the covariant divergence,

$$\sqrt{-g} A^{\mu}{}_{;\mu} = (\sqrt{-g} A^{\mu})_{,\mu} . \quad (2.151)$$

The left side is a scalar density. From the invariance of the integral of a scalar density over a prescribed four-volume, we have the invariant

$$\int_{V_4} \sqrt{-g} A^{\mu}{}_{;\mu} \, d^4x = \int_{V_4} (\sqrt{-g} A^{\mu})_{,\mu} \, d^4x . \quad (2.152)$$

The right side can be converted to a surface integral over a three-volume at a definite time x^0 by Gauss' theorem.

If the covariant divergence vanishes, we get a conservation law as follows:

$$A^{\mu}{}_{;\mu} = 0 \quad \Longrightarrow \quad (\sqrt{-g}A^{\mu})_{,\mu} = 0\,. \tag{2.153}$$

As a result, we obtain

$$(\sqrt{-g}A^0)_{,0} = -(\sqrt{-g}A^m)_{,m} \quad \text{(summed over } m = 1-3)\tag{2.154}$$

Integrate the above expression over a three-volume at definite time x^0 to find

$$\frac{\partial}{\partial x^0}\int_V \sqrt{-g}A^0\, d^3x = -\int_V (\sqrt{-g}A^m)_{,m} d^3x \tag{2.155}$$

$$= -\int_S \sqrt{-g}\underline{\mathrm{A}} \cdot d\mathbf{S}\,. \tag{2.156}$$

If there is no three-current $\sqrt{-g}\mathbf{A}$ crossing the surface, then the quantity of density $\sqrt{-g}A^0$ contained within V is constant,

$$\int_V \sqrt{-g}A^0\, d^3x = \text{ constant} \tag{2.157}$$

This quantity is frequently referred to as the total charge of whatever A^{μ} represents.

We can apply precisely the same reasoning to the covariant divergence of an antisymmetric tensor:

$$\text{If} \quad A^{\mu\nu} = -A^{\nu\mu}, \quad \text{then} \quad \sqrt{-g}A^{\mu\nu}{}_{;\nu} = (\sqrt{-g}A^{\mu\nu})_{,\nu}\,, \tag{2.158}$$

where the quantity on the left is a *vector density* according to the previous section. Similarly we can derive conservation laws for the three-volume integral of the four densities $\sqrt{-g}A^{\mu 0}$ if the covariant divergence vanishes and there is no three-flux through the surface of the volume. However, if the tensor is not antisymmetric, the above theorem does not generally apply in curved spacetime to a tensor of more than one index.

2.7 Riemann Curvature Tensor

The order of ordinary differentiation in flat spacetime does not matter. The order of covariant differentiation does matter in curved spacetime. From an investigation of this fact we arrive at a measure of curvature.

2.7.1 Second covariant derivative of scalars and vectors

If we take the covariant derivative of a scalar twice and then invert the order, the answer is easily verified to be the same:

$$S_{;\mu;\nu} = S_{;\mu,\nu} - \Gamma^{\alpha}_{\mu\nu} S_{;\alpha} = S_{,\mu,\nu} - \Gamma^{\alpha}_{\mu\nu} S_{;\alpha}\,, \tag{2.159}$$

where we use the fact in the second equality that the covariant derivative of a scalar is the ordinary derivative $S_{;\mu} = S_{,\mu}$. The above result is symmetrical in μ, ν.

However for vectors and tensors, a changed order of differentiation in general produces a different result. The operations involved, all defined above, are many but straightforward. The result for the vector A_σ is

$$A_{\sigma;\mu;\nu} - A_{\sigma;\nu;\mu} = A_\rho R^{\rho}_{\sigma\mu\nu}\,, \tag{2.160}$$

where

$$R^{\rho}_{\sigma\mu\nu} \equiv \Gamma^{\rho}_{\sigma\nu,\mu} - \Gamma^{\rho}_{\sigma\mu,\nu} + \Gamma^{\alpha}_{\sigma\nu}\Gamma^{\rho}_{\alpha\mu} - \Gamma^{\alpha}_{\sigma\mu}\Gamma^{\rho}_{\alpha\nu} \tag{2.161}$$

is the *Riemann–Christoffel curvature tensor*. We know that it is a tensor because the left side of (2.160) is a tensor and A_ν is any vector. Riemann is the only tensor that can be constructed from the metric tensor and its first and second derivatives (cf. Ref. [20], p. 133).

2.7.2 Symmetries of the Riemann tensor

Riemann has a number of symmetry properties that can be easily derived from the above expression:

$$\begin{gathered} R^{\mu}_{\nu\rho\sigma} = -R^{\mu}_{\nu\sigma\rho}\,, \\ R^{\alpha}_{\nu\rho\sigma} + R^{\alpha}_{\sigma\nu\rho} + R^{\alpha}_{\rho\sigma\nu} = 0\,. \end{gathered} \tag{2.162}$$

Lowering the index on the Riemann tensor, we get

$$R_{\rho\sigma\mu\nu} = g_{\rho\alpha} R^{\alpha}_{\sigma\mu\nu}\,. \tag{2.163}$$

The additional symmetries follow:

$$\begin{aligned} R_{\mu\nu\rho\sigma} &= -R_{\nu\mu\rho\sigma} = -R_{\mu\nu\sigma\rho}\,, \\ R_{\mu\nu\rho\sigma} &= R_{\rho\sigma\mu\nu} = R_{\sigma\rho\nu\mu}\,. \end{aligned} \tag{2.164}$$

As a consequence of the symmetries only 20 of the $4^4 = 256$ components of Riemann are independent. In two dimensions there are 15 such symmetry relationships. Consequently, there are $2^4 - 15 = 1$ independent components of the Riemann tensor, namely, the Gaussian curvature. (See Ref. [21] p. 60 and appendix B for a discussion of curvature in two dimensions.)

We shall encounter two additional objects that are obtained from the Riemann tensor, the *Ricci tensor*,

$$R_{\mu\nu} = R^{\rho}_{\mu\nu\rho}\,, \tag{2.165}$$

and the *scalar curvature*,

$$R = g^{\mu\nu} R_{\mu\nu}\,. \tag{2.166}$$

Multiply the left and right side of (2.164) by $g^{\mu\sigma}$ and then rename indices to find

$$R_{\mu\nu} = R_{\nu\mu}\,. \tag{2.167}$$

Because of this symmetry, when we raise an index on the Ricci tensor, it is unnecessary to preserve the location,

$$R^{\mu}{}_{\nu} = R_{\nu}{}^{\mu} = R^{\mu}_{\nu}\,. \tag{2.168}$$

From the definition of the Ricci tensor in terms of the Riemann tensor, we have the following explicit expression:

$$R_{\mu\nu} = \Gamma^{\alpha}_{\mu\alpha,\nu} - \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\nu}\Gamma^{\beta}_{\alpha\beta} + \Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha}\,. \tag{2.169}$$

The first term might appear to contradict the assertion that $R_{\mu\nu}$ is symmetric in μ, ν. However the result (2.149) proves that the Ricci tensor is symmetric.

2.7.3 Test for flatness

If spacetime is flat, then we may choose a rectilinear coordinate system in which case the metric tensor is a constant throughout spacetime. Then according to (2.93) the nontensor $\Gamma^{\lambda}_{\mu\nu}$ vanishes in this frame in all spacetime. So also do the derivatives of $\Gamma^{\lambda}_{\mu\nu}$. Therefore the Riemann tensor (2.161) vanishes everywhere at all times in flat spacetime. Because this is a statement about a tensor, it is true in any coordinate system, rectilinear or not. The converse is true but more difficult to prove: If Riemann vanishes, spacetime is flat. We prove this later in Section 2.9.3.

2.7.4 Second covariant derivative of tensors

An arbitrary second-rank tensor can be expressed as the sum of products $A_\mu B_\nu$. It is simpler to start by examining the second covariant derivative of such a product:

$$\begin{aligned}(A_\mu B_\nu)_{;\rho;\sigma} &= (A_{\mu;\rho}B_\nu + A_\mu B_{\nu;\rho})_{;\sigma} \\ &= A_{\mu;\rho;\sigma}B_\nu + A_\mu B_{\nu;\rho;\sigma} + A_{\mu;\rho}B_{\nu;\sigma} + A_{\mu;\sigma}B_{\nu;\rho}\,.\end{aligned}$$

Interchange ρ, σ, and subtract to find

$$\begin{aligned}(A_\mu B_\nu)_{;\rho;\sigma} &= (A_\mu B_\nu)_{;\sigma;\rho} \\ &= A_\mu(B_{\nu;\rho;\sigma} - B_{\nu;\sigma;\rho}) + (A_{\mu;\rho;\sigma} - A_{\mu;\sigma;\rho})B_\nu \\ &= A_\mu B_\alpha R^{\alpha}_{\nu\rho\sigma} + A_\alpha R^{\alpha}_{\mu\rho\sigma} B_\nu\,.\end{aligned}$$

We can form an arbitrary linear combination of such products of first-rank tensors to obtain the result for a general tensor,

$$T_{\mu\nu;\rho;\sigma} - T_{\mu\nu;\sigma;\rho} = T_{\mu\alpha}R^{\alpha}_{\nu\rho\sigma} + T_{\alpha\nu}R^{\alpha}_{\mu\rho\sigma}\,. \qquad (2.170)$$

2.7.5 Bianchi identities

The Bianchi identities are extremely important for the further development of the theory of gravity, allowing us to prove that the Einstein tensor, which we come to next, has vanishing divergence.

Apply the above result to the particular case that the second-rank tensor is the covariant derivative of a vector $T_{\mu\nu} = A_{\mu;\nu}$,

$$A_{\mu;\nu;\rho;\sigma} - A_{\mu;\nu;\sigma;\rho} = A_{\mu;\alpha}R^{\alpha}_{\nu\rho\sigma} + A_{\alpha;\nu}R^{\alpha}_{\mu\rho\sigma}\,. \qquad (2.171)$$

Now write down the additional two equations obtained from this by cyclic permutation of the indices $(\nu\rho\sigma)$, and add the three equations. First study the left side of the sum. Use (2.160) to get

$$\begin{aligned} \text{LHS} &= (A_{\mu;\nu;\rho} - A_{\mu;\rho;\nu})_{;\sigma} + (A_{\mu;\sigma;\nu} - A_{\mu;\nu;\sigma})_{;\rho} \\ +(A_{\mu;\rho;\sigma} - A_{\mu;\sigma;\rho})_{;\nu} &= (A_\alpha R^\alpha_{\mu\nu\rho})_{;\sigma} + (A_\alpha R^\alpha_{\mu\sigma\nu})_{;\rho} + (A_\alpha R^\alpha_{\mu\rho\sigma})_{;\nu} \,. \end{aligned}$$

Using (2.162) in the sum of the right-hand sides of the cyclic permutation, we are left with

$$\text{RHS} = A_{\alpha;\nu} R^\alpha_{\mu\rho\sigma} + A_{\alpha;\sigma} R^\alpha_{\mu\nu\rho} + A_{\alpha;\rho} R^\alpha_{\mu\sigma\nu} \,.$$

Equating left and right sides and cancelling common terms, we find

$$A_\alpha (R^\alpha_{\mu\nu\rho;\sigma} + R^\alpha_{\mu\sigma\nu;\rho} + R^\alpha_{\mu\rho\sigma;\nu}) = 0 \,. \tag{2.172}$$

Because A_α is any vector,

$$R^\alpha_{\mu\nu\rho;\sigma} + R^\alpha_{\mu\sigma\nu;\rho} + R^\alpha_{\mu\rho\sigma;\nu} = 0 \,. \tag{2.173}$$

In addition to the symmetry relationships derived earlier, the Riemann tensor satisfies the differential equations above known as the *Bianchi identities*.

2.7.6 Einstein tensor

Let us multiply the differential equations for the Bianchi identities (2.173) by $g^{\mu\nu}$, contract σ with α, and use the fact, already established, that the covariant derivatives of the metric tensor vanish:

$$\begin{aligned} 0 &= g^{\mu\nu} (R^\alpha_{\mu\nu\rho;\sigma} + R^\alpha_{\mu\sigma\nu;\rho} + R^\alpha_{\mu\rho\sigma;\nu}) \\ &= (g^{\mu\nu} R^\alpha_{\mu\nu\rho})_{;\alpha} + (g^{\mu\nu} R^\alpha_{\mu\alpha\nu})_{;\rho} + (g^{\mu\nu} R^\alpha_{\mu\rho\alpha})_{;\nu} \,. \end{aligned} \tag{2.174}$$

Examine each term in brackets using the Riemann tensor symmetries. The first term is

$$\begin{aligned} g^{\mu\nu} R^\alpha_{\mu\nu\rho} = g^{\mu\nu} g^{\alpha\beta} R_{\beta\mu\nu\rho} = g^{\mu\nu} g^{\alpha\beta} R_{\mu\beta\rho\nu} = g^{\alpha\beta} R^\nu_{\beta\rho\nu} = g^{\alpha\beta} R_{\beta\rho} \\ = R^\alpha_\rho \,. \end{aligned}$$

The second term is

$$g^{\mu\nu} R^\alpha_{\mu\alpha\nu} = -g^{\mu\nu} R^\alpha_{\mu\nu\alpha} = -g^{\mu\nu} R_{\mu\nu} = -R \,.$$

The third term is

$$g^{\mu\nu} R^{\alpha}_{\mu\rho\alpha} = g^{\mu\nu} R_{\mu\rho} = R^{\nu}_{\rho} \,.$$

Now put these results back into their brackets with the covariant derivatives as indicated in (2.174) to obtain

$$0 = R^{\alpha}_{\rho;\alpha} - R_{;\rho} + R^{\nu}_{\rho;\nu} = 2R^{\alpha}_{\rho;\alpha} - R_{;\rho} \,.$$

Multiply by $g^{\mu\rho}$, and note that

$$g^{\mu\rho} R^{\alpha}_{\rho;\alpha} = (g^{\mu\rho} R^{\alpha}_{\rho})_{;\alpha} = R^{\mu\alpha}{}_{;\alpha} = R^{\mu\nu}{}_{;\nu}$$
$$g^{\mu\rho} R_{;\rho} = g^{\mu\nu} R_{;\nu}$$

to arrive immediately at the vanishing divergence

$$(R^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu} R)_{;\nu} = 0 \,. \tag{2.175}$$

The object in the brackets is called the *Einstein curvature tensor*,

$$G^{\mu\nu} \equiv R^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu} R \,. \tag{2.176}$$

The Einstein tensor is constructed from the Riemann curvature tensor and has an identically vanishing covariant divergence. It is symmetric and of second rank. Einstein was motivated to seek a tensor that contained no differentials of the $g^{\mu\nu}$ higher than the second—a tensor which was a linear homogeneous combination of terms linear in the second derivative or quadratic in the first (in analogy with Poisson's equation for the gravitational potential in Newton's theory:

$$\nabla^2 V = 4\pi\rho \,, \tag{2.177}$$

where ρ is the mass density generating the field). For the expression of energy and momentum conservation, it is important that the divergence vanish. The energy-momentum tensor of matter accomplishes this and is of second rank.

2.8 Einstein's Field Equations

> "The geometry of spacetime is not given; it is determined by matter and its motion."[6]
> *W. Pauli, 1919*

We know that other bodies will experience gravity in the vicinity of massive bodies. So mass is a source of gravity, and from the Special Theory of Relativity we must say in general that mass and energy are sources. We have just seen that a construction from the Riemann curvature tensor, namely, Einstein's tensor, has vanishing covariant divergence. We have three possibilities,

$$G^{\mu\nu} = 0\,, \tag{2.178}$$

or

$$G^{\mu\nu} = kT^{\mu\nu}\,, \tag{2.179}$$

where $T^{\mu\nu}$ is a symmetric divergenceless tensor constructed from the mass-energy properties of the material medium, or

$$G^{\mu\nu} = kT^{\mu\nu} + \Lambda g^{\mu\nu}\,. \tag{2.180}$$

The constant Λ is the so-called *cosmological constant*. It was not present in the original theory and was added to obtain a static cosmology before it was known that the universe is expanding. Einstein regarded its numerical value as a matter to be settled by experiment—"The curvature constant [Λ] is, however, essentially determinable, and an increase in the precision of data derived from observations will enable us in the future to fix its sign and determine its value" [22].

It is apparent that the cosmological constant corresponds to a constant energy density $\Lambda/(8\pi)$ and a constant pressure of the same numerical value but of opposite sign. The cosmological constant is sometimes referred to as the vacuum energy density. In any case it is small; its value has been recently measures by perlmutter. Its effect is indeed cosmological; stellar structure is unaffected by it. We need not consider the cosmological term further.

[6] Very importantly, the converse is also true.

The first set of differential equations (2.178) are those that must be satisfied by the metric in empty space outside material bodies and energy concentrations. An example is the gravitational fields outside a star.

The second set of differential equations (2.179) determine the gravitational fields $g^{\mu\nu}$ inside a spacetime region of mass-energy and in addition determine how the mass-energy is arranged by gravity. With appropriate $T^{\mu\nu}$ it would provide the equations of stellar structure. We have yet to fix the constant k. This can be done by looking to the weak field limit where the General Theory of Relativity should agree with Newton's well-tested, weak-field theory.

There are several remarkable notes we can make at this point. Einstein's field equations tell spacetime how to curve and mass-energy how to configure itself and how to move. Spacetime acts upon matter and in turn is acted upon by matter. This was Einstein's intuition and motivation in seeking a theory that placed spacetime and matter as co-determiners in nature. He was displeased with the Special Theory of Relativity as anything but a local theory, for it gave spacetime an absolute status.

Second, the Einstein field equations are nonlinear in the fields $g^{\mu\nu}$. (This can be verified by tracing back through the objects from which the Einstein tensor is constructed.) Nonlinearity means that the gravitational field interacts with itself. This is because the field carries energy, and mass-energy in any form is a source of gravity. The nonlinearity of the Einstein equations accounts for some of the extraordinary phenomena encountered in strong gravity, including black holes [23] and the reversal of the centrifugal force in their vicinity [24].

We have seen in (2.175) that the Einstein tensor has identically vanishing covariant divergence. Hence (2.179) requires of the matter tensor that

$$T^{\mu\nu}{}_{;\nu} = 0\,. \qquad (2.181)$$

The corresponding equation in flat space is

$$T^{\mu\nu}{}_{,\nu} = 0\,. \qquad (2.182)$$

Vanishing of the ordinary divergence of the energy-momentum tensor in the Special Theory of Relativity corresponds to the conservation of energy and momentum. However, (2.181) does not assure us of the constancy of any quantity in time. In fact (2.179) ensures that matter and the gravitational fields exchange energy, or in other words do work on each other, for it is the divergence of $G^{\mu\nu} - kT^{\mu\nu}$ that vanishes. So neither matter nor the gravitational field can by itself conserve energy in any sense. No contradiction exists with laboratory experiments performed on earth. Over the dimensions of a typical laboratory, spacetime is essentially flat, and nothing that could be done in a laboratory could possibly disturb this flatness in any perceptible way.

This brings us back to the comparison of the weak-field limit between Newton's theory and Einstein's. The inverse-square law of the force between massive objects is not required by the inner structure of Newton's theory. He could have postulated an inverse α law, that force $F \sim Mm/r^{\alpha}$, and then attempted to fit α to the astronomical data of the solar system. Depending on what weight was given to the precession of planets, one would have found a value of α close to two.

Einstein's theory does not possess the flexibility of Newton's in this regard. We saw in (2.119) that Einstein predicts precisely the inverse square law. In this sense, he could claim as his own all the successes of the Newtonian theory in explaining the motion of planets in the solar system. They were computed with the inverse-square law, there being no flexibility in the choice of the power in his theory.

Concerning the precession of planets, an isolated planet in orbit about the sun under an inverse square law is an ellipse whose orientation is fixed in space. However the total precession of the orbit of Mercury is observed to be about 5600 seconds/century. Most of this is caused by the fact that an earthbound observer is not in an inertial frame far from the sun. For example, suppose that Mercury did not orbit about the sun, but instead held a fixed position. Nevertheless, from the earth it would be appear to move, sometimes to the left of the sun, sometimes to the right, and alternately passing in front of and in back of the sun. Taking account of this correction to the apparent motion of Mercury due to the earth's own motion, the precession of Mercury is about 574 seconds/century. This value is about 43 seconds/century larger than the precession computed by Newtonian physics as due to the perturbation of the orbit by other planets, a small but disturbing discrepancy. An early triumph of Einstein was that he calculated, within the observational errors, the precise value of the excess precession. In Newton's theory only mass contributes to gravity, whereas in Einstein's theory the kinetic energy of the motion of the planets contributes as well.

2.9 Relativistic Stars

Einstein's field equations are completely general and simple in appearance. However, they are exceedingly complicated because of their nonlinear character and because spacetime and matter act upon each other. As already remarked, there is no prior geometry of spacetime. There are a few cases in which solutions can be found in closed form. One of the most important closed-form solutions is the Schwarzschild metric outside a static spherical star. Another is the Kerr metric outside a rotating black hole. Einstein's equations can also be solved numerically as the coupled differential equations for the interior structure of a spherical static star, which are called the Oppenheimer–Volkoff equations for stellar structure.

In this section we take up the important problem of deriving the equations that govern spacetime and the arrangement of matter in the case of relativistic spherical static stars. They are the basic equations that underlie the development of neutron star models. They also demonstrate the mathematical existence of Schwarzschild black holes. They can also be used to develop white dwarf models, though Newtonian gravity is a good approximation for these stars.

2.9.1 Metric in static isotropic spacetime

We seek solutions to Einstein's field equations in static isotropic regions of spacetime such as would be encountered in the interior and exterior regions of static stars. Under these conditions the $g_{\mu\nu}$ are independent of time ($x^0 \equiv t$) and $g^{0m} = 0$. We choose spatial coordinates $x^1 = r$, $x^2 = \theta$, and $x^3 = \phi$. The most general form of the line element is then

$$d\tau^2 = U(r)\,dt^2 - V(r)\,dr^2 - W(r)r^2(\,d\theta^2 + \sin^2\theta\,d\phi^2)\,. \quad (2.183)$$

We may replace r by any function of r without disturbing the spherical symmetry. We do so in such a way that $W(r) \equiv 1$. Then we may write

$$d\tau^2 = e^{2\nu(r)}\,dt^2 - e^{2\lambda(r)}\,dr^2 - r^2\,d\theta^2 - r^2\sin^2\theta\,d\phi^2\ , \quad (2.184)$$

where λ, ν are functions only of r. Comparing with

$$d\tau^2 = g_{\mu\nu}\,dx^\mu\,dx^\nu$$

, we read off [7]

$$g_{00} = e^{2\nu(r)}, \quad g_{11} = -e^{2\lambda(r)}, \quad g_{22} = -r^2, \quad g_{33} = -r^2\sin^2\theta,$$
$$g_{\mu\nu} = g^{\mu\nu} = 0 \quad (\mu \neq \nu)\,. \quad (2.185)$$

Hence, from $g_{\mu\nu}g^{\nu\rho} = \delta^\rho_\mu$, we have in this special case

$$g_{\mu\mu} = 1/g^{\mu\mu} \qquad (\text{not summed})\,. \quad (2.186)$$

According to its definition as a contraction of the Riemann tensor, the Ricci tensor can be written

$$R_{\mu\nu} = \Gamma^\alpha_{\mu\alpha,\nu} - \Gamma^\alpha_{\mu\nu,\alpha} - \Gamma^\alpha_{\mu\nu}\Gamma^\beta_{\alpha\beta} + \Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\nu\alpha}\,. \quad (2.187)$$

We can derive the nonvanishing affine connections (2.93), which are symmetric in their lower indices, from the metric tensor whose general form for static isotropic regions was derived above:

$$\begin{array}{ll} \Gamma^1_{00} = \nu'\,e^{2(\nu-\lambda)}\ , & \Gamma^0_{10} = \nu'\ , \\ \Gamma^1_{11} = \lambda'\ , & \Gamma^2_{12} = \Gamma^3_{13} = 1/r\ , \\ \Gamma^1_{22} = -re^{-2\lambda}\ , & \Gamma^3_{23} = \cot\theta\ , \\ \Gamma^1_{33} = -r\sin^2\theta\,e^{-2\lambda}\ , & \Gamma^2_{33} = -\sin\theta\cos\theta\,. \end{array} \quad (2.188)$$

The primes denote differentiation with respect to r. Hence, for static isotropic spacetime

$$R_{00} \;=\; \Big(-\nu'' + \lambda'\nu' - \nu'^2 - \frac{2\nu'}{r}\Big)e^{2(\nu-\lambda)}\ ,$$

[7]Note that in (2.184) some authors use the opposite signs for time and space components, and some use the functions ν, λ but without the factor 2, or use different notation altogether for the metric. Great care has to be exercised in using results from different sources.

$$\begin{aligned} R_{11} &= \nu'' - \lambda'\nu' + \nu'^2 - \frac{2\lambda'}{r} \,, \\ R_{22} &= (1 + r\nu' - r\lambda')e^{-2\lambda} - 1 \,, \\ R_{33} &= R_{22}\sin^2\theta \,. \end{aligned} \tag{2.189}$$

2.9.2 The Schwarzschild solution

In the empty space outside a static star Einstein's equation is $G_{\mu\nu} = 0$, or equivalently

$$R_{\mu\nu} = \tfrac{1}{2} g_{\mu\nu} R \,. \tag{2.190}$$

Multiply by $g^{\alpha\mu}$, and sum on the dummy index to find

$$R^{\alpha}_{\nu} = \tfrac{1}{2} \delta^{\alpha}_{\nu} R \,. \tag{2.191}$$

Contract by setting $\alpha = \nu$, and sum to get

$$R = 2R \qquad \Longrightarrow \qquad R = 0 \,. \tag{2.192}$$

Hence, the vanishing of Einstein's tensor implies

$$G_{\mu\nu} = 0 \qquad \Longrightarrow \qquad R = 0, \quad R_{\mu\nu} = 0 \,. \tag{2.193}$$

In empty space, Einstein's equation is equivalent to the vanishing of the Ricci tensor or, equivalently, the scalar curvature. Its form for static isotropic spacetime was worked out in the previous section.

From the vanishing of R_{00}, R_{11} we find that

$$\lambda' + \nu' = 0 \,. \tag{2.194}$$

(Do not confuse ν and λ when used to denote indices and when used to denote the metric functions as in the above equation.) For large r, space must be unaffected by the star and therefore flat so that λ and ν tend to zero; therefore

$$\lambda + \nu = 0 \,. \tag{2.195}$$

Using these results in $R_{22} = 0$, we find that

$$(1 + 2r\nu')e^{2\nu} = 1 \,. \tag{2.196}$$

This condition integrates to

$$g_{00} \equiv e^{2\nu} = 1 - \frac{2GM}{r} \qquad (r > R) \;, \tag{2.197}$$

where M is the constant of integration, and we introduced Newton's constant. Having studied the Newtonian approximation, one identifies M with the mass of the star. From the foregoing results,

$$g_{11} = -e^{2\lambda} = -e^{-2\nu} = -\Big(1 - \frac{2GM}{r}\Big)^{-1} \qquad (r > R). \tag{2.198}$$

This completes the derivation of the *Schwarzschild solution* of 1916 of Einstein's equations outside a spherical static star. It was the first exact solution found for Einstein's equations. The proper time is

$$d\tau^2 \;=\; \Big(1 - \frac{2GM}{r}\Big)dt^2 - \Big(1 - \frac{2GM}{r}\Big)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\phi^2 \qquad (r > R)\,, \tag{2.199}$$

where R, in this context, denotes the radius of the star.

Let us summarize the Schwarzschild solution found above:

$$\begin{aligned} g_{00}(r) &= e^{2\nu(r)} = \Big(1 - \frac{2GM}{r}\Big), \qquad r > R\;, \\ g_{11}(r) &= -e^{2\lambda(r)} = -\Big(1 - \frac{2GM}{r}\Big)^{-1}, \qquad r > R\;, \\ g_{22}(r) &= -r^2, \qquad g_{33}(r,\theta) = -r^2 \sin^2\theta\,. \end{aligned} \tag{2.200}$$

Notice that the Schwarzschild metric is singular at the radius $r = r_S \equiv 2GM$. This does not mean that spacetime itself is singular at that radius, but only that this particular metric is. Other nonsingular metrics have been found, in particular, the Kruskal–Szerkeres metric [25, 26]. However, further analysis shows that if r_S lies outside the star where the Schwarzschild solution holds, then it is a black hole—no particle or even light can leave the region $r < r_S$. This radius r_S is called the *Schwarzschild radius* or *singularity* or *horizon*. But because the above metric holds only outside the star, r_S has no special significance if it is smaller than the radius of the star. For then a different metric holds inside the star which does not possess a singularity. We come to this solution shortly.

2.9.3 Riemann tensor outside a Schwarzschild star

If spacetime is flat, then the Riemann curvature tensor vanishes (Section 2.7.3). We are now prepared to address the converse (albeit not rigorously): if spacetime is curved, some components of the Riemann tensor are finite (which components, of course, will depend upon how convoluted spacetime is).

The metric tensor and, indeed, the affine connection for the empty space outside a massive body were computed in the preceding section. We have seen in Section 2.4.3 that massive bodies curve spacetime. So we know that the Schwarzschild metric tensor refers to curved spacetime. Referring to the definition of the Riemann tensor (2.161) and the specific form that the affine connection takes for a static spherical star (2.188), we can compute

$$R^1_{010} = (\nu'' + 2\nu'^2 - \nu'\lambda')e^{2(\nu-\lambda)}\,. \qquad (2.201)$$

Thus we exhibit at least one nonvanishing component of the Riemann tensor in the curved spacetime outside a Schwarzschild star. This suggests that Riemann is not identically zero in curved spacetime. An actual proof that if Riemann is finite then spacetime is curved requires the formulation of parallel transport, which we do not take up here. We declare, without rigorous proof, that the Riemann tensor vanishes if and only if spacetime is flat. Notice that, far from an isolated star where spacetime approaches flatness, Riemann approaches zero as it should.

2.9.4 Energy-Momentum tensor of matter

From the success of Newtonian physics in describing celestial mechanics and other weak gravitational field phenomena, we know that mass is a source of gravity. From the experimental verifications of the Special Theory of Relativity, we know that all forms of energy are equivalent and must contribute equally as sources of gravity. Normally, of course, it is mass that dominates, and the average mass density in the solar system and in the universe is very small; that is why Newtonian physics is so accurate under the typical conditions mentioned above.

An essential aspect of Einstein's curvature tensor is that it automatically has vanishing covariant divergence (2.175). It is also a symmetric second-rank tensor. Accordingly, mass-energy—the source of the gravitational field—must be incorporated into a divergenceless, symmetric, second- rank tensor in flat space. As a tensor, it can be transcribed immediately to its form in an arbitrary spacetime frame by the general covariance principle. Such a tensor is the energy-momentum tensor.

In other parts of this book we shall be interested in specific theories of dense matter from which we will be able to explicitly construct the energy-momentum tensor of the theory. Here we are interested in the general form such a tensor takes. Frequently, matter may be regarded as a perfect fluid. The fluid velocity is assumed to vary continuously from point to point. The perfect fluid energy-momentum tensor in the Special Theory of Relativity can be expressed in terms of the local values of the pressure p and energy density ϵ as in (2.49). The General Relativistic energy-momentum tensor can be written immediately using the Principle of General Covariance spelled out on page 47:

$$T^{\mu\nu} = -pg^{\mu\nu} + (p+\epsilon)u^{\mu}u^{\nu}\,,$$
$$g_{\mu\nu}u^{\mu}u^{\nu} = 1\,.$$

In the above equations, u^{μ} is the local fluid four-velocity

$$u^{\mu} = \frac{dx^{\mu}}{d\tau} \tag{2.202}$$

and satisfies (2.202) because of (2.53).

The pressure and total energy density (including mass) are related by the equation of state of matter, frequently written in either form

$$p = p(\epsilon) \qquad \text{or} \qquad \epsilon = \epsilon(p) \tag{2.203}$$

where p and ϵ are the pressure and energy density (including mass) in the local rest-frame of the fluid. In the next section we shall see how the equations for stellar structure involve these quantities and this relationship.

2.9.5 The Oppenheimer–Volkoff equations

We are now prepared to derive the differential equations for the structure of a static, spherically symmetric, relativistic star. For the region outside a star, we found that the vanishing of the Einstein tensor was equivalent to the vanishing of the Ricci tensor or the scalar curvature. This is not the case for the interior of the star. We need both the Ricci tensor and scalar curvature to construct the Einstein tensor. The general form of the metric for a static isotropic spacetime was obtained in (2.185). From Section 2.9.1 we find the scalar curvature,

$$\begin{aligned} R &= g^{\mu\nu}R_{\mu\nu} = e^{-2\nu}R_{00} - e^{-2\lambda}R_{11} - \frac{2}{r^2}R_{22} \\ &= e^{-2\lambda}\Big\{-2\nu'' + 2\lambda'\nu' - 2\nu'^2 - \frac{2}{r^2} + 4\frac{\lambda'}{r} - 4\frac{\nu'}{r}\Big\} \\ &+ \frac{2}{r^2}\,. \end{aligned} \tag{2.204}$$

It is more convenient to work with mixed tensors. For example,

$$G_0^{\ 0} = R_0^{\ 0} - \tfrac{1}{2}R \tag{2.205}$$

is obtained with the results of Section 2.9.1 for a static isotropic field, namely,

$$g_0^0 = g_{0\nu}g^{0\nu} = g_{00}g^{00} = 1\,. \tag{2.206}$$

So using results obtained earlier in this section we can find that the components of the Einstein tensor are

$$\begin{aligned} r^2G_0^{\ 0} &\equiv e^{-2\lambda}(1-2r\lambda') - 1 = -\frac{d}{dr}[r(1-e^{-2\lambda})]\,, \\ r^2G_1^{\ 1} &\equiv e^{-2\lambda}(1+2r\nu') - 1\,, \\ G_2^{\ 2} &\equiv e^{-2\lambda}\Big(\nu'' + \nu'^2 - \lambda'\nu' + \frac{\nu'-\lambda'}{r}\Big)\,, \\ G_3^{\ 3} &= G_2^{\ 2}. \end{aligned} \tag{2.207}$$

Because of the assumption that the star is static, the three-velocity of every fluid element is zero, so

$$u^\mu = 0 \quad (\mu \neq 0), \qquad u^0 = 1/\sqrt{g_{00}}\,, \tag{2.208}$$

according to (2.202). The energy-momentum tensor expressed as a mixed tensor, we have the nonzero components in the present metric,

$$T_0{}^0 = \epsilon, \qquad T_\mu{}^\mu = -p \qquad (\mu \neq 0). \tag{2.209}$$

So the (00) component of the Einstein equations gives

$$r^2 G_0{}^0 = -\frac{d}{dr}\{r(1 - e^{-2\lambda(r)})\} = kr^2 T_0{}^0 = kr^2 \epsilon(r)\,. \tag{2.210}$$

This can be integrated immediately to yield

$$e^{-2\lambda(r)} = 1 + \frac{k}{r}\int_0^r \epsilon(r) r^2 dr\,. \tag{2.211}$$

Let us define

$$M(r) \equiv 4\pi \int_0^r \epsilon(r) r^2 dr\ , \tag{2.212}$$

and let R denote the radius of the star, the radial coordinate exterior to which the pressure vanishes. Zero pressure defines the edge of the star because zero pressure can support no material against the gravitational attraction from within. Denote the corresponding value of $M(R)$ by

$$M \equiv M(R)\,. \tag{2.213}$$

Now comparing (2.119, 2.197, 2.198) we see that, to obtain agreement with the Newtonian limit, we must choose

$$k = -8\pi G \tag{2.214}$$

and interpret M as the *gravitational mass* of the star. Therefore, $M(r)$ is referred to as the *included mass* within the coordinate r. So Einstein's field equations can now be written

$$G^{\mu\nu} = -8\pi G T^{\mu\nu}. \tag{2.215}$$

From the above, we have found so far that

$$g_{11}(r) = -e^{2\lambda(r)} = -\Big(1 - \frac{2GM(r)}{r}\Big)^{-1}\ , \tag{2.216}$$

which agrees with (2.198), but now we see that $g_{11}(r)$ has the same form inside and outside the star although it is the included mass $M(r)$, not the total mass, that appears in the interior solution.

Having learned the constant of proportionality in Einstein's equations (2.215), let us now write out the field equations for a spherically symmetric static star, including the one we have already solved. In passing we note that our solution gives a relationship between the included mass $M(r)$ at any radial coordinate and the metric function $g_{11}(r)$ or $\lambda(r)$, but we have yet to learn how to compute one or the other. The differential equations from (2.207) are

$$G_0{}^0 \equiv e^{-2\lambda}\Big(\frac{1}{r^2}-\frac{2\lambda'}{r}\Big)-\frac{1}{r^2} = -8\pi G\epsilon(r)\,, \tag{2.217}$$

$$G_1{}^1 \equiv e^{-2\lambda}\Big(\frac{1}{r^2}+\frac{2\nu'}{r}\Big)-\frac{1}{r^2} = 8\pi Gp(r)\,, \tag{2.218}$$

$$G_2{}^2 \equiv e^{-2\lambda}\Big(\nu''+\nu'^2-\lambda'\nu'+\frac{\nu'-\lambda'}{r}\Big) = 8\pi Gp(r)\,, \tag{2.219}$$

$$G_3{}^3 = G_2{}^2 = 8\pi Gp(r)\,. \tag{2.220}$$

The last equation contains no information additional to that provided by those preceding it.

To simplify notation, choose units so that $G = c = 1$. Solve (2.217) to find

$$-2r\lambda' = (1-8\pi r^2\epsilon)e^{2\lambda}-1 \tag{2.221}$$

and (2.218) to find

$$2r\nu' = (1+8\pi r^2 p)e^{2\lambda}-1\,. \tag{2.222}$$

Take the derivative of (2.222) and then multiply by r:

$$2r\nu' + 2r^2\nu'' = \Big[2r\lambda'(1+8\pi r^2 p) + (16\pi r^2 p + 8\pi r^3 p')\Big]e^{2\lambda}\,.$$

Solve for ν'' using (2.222, 2.221):

$$2r^2\nu'' = 1 + (16\pi r^2 p + 8\pi r^3 p')e^{2\lambda} \tag{2.223}$$

$$-(1+8\pi r^2 p)(1-8\pi r^2\epsilon)e^{4\lambda}\,. \tag{2.224}$$

Square (2.222) to obtain the result

$$2r^2\nu'^2 = \tfrac{1}{2}(1+8\pi r^2 p)^2 e^{4\lambda} - (1+8\pi r^2 p)e^{2\lambda} + \tfrac{1}{2}\,. \tag{2.225}$$

The last four numbered equations provide expressions for λ', ν', ν'', and ν'^2 in terms of p, p', ϵ, and $e^{2\lambda}$ the latter of which, according to (2.216), can be expressed in terms of the included mass. Therefore the metric can be eliminated altogether by substitution of the above results into the remaining field equation (2.219). After a number of cancellations, we emerge with the result

$$\frac{dp}{dr} = -\frac{[p(r) + \epsilon(r)][M(r) + 4\pi r^3 p(r)]}{r[r - 2M(r)]}\,. \tag{2.226}$$

This and equation (2.212) represent the reduction of Einstein's equations for the interior of a spherical, static, relativistic star. These equations are frequently referred to as the Oppenheimer–Volkoff equations. The stars they describe—static and spherically symmetric—are sometimes referred to as Schwarzschild stars.

Given an equation of state (2.203), the stellar structure equations (2.212) and (2.226) can be solved simultaneously for the radial distribution of pressure, $p(r)$, and hence for the distribution of mass-energy density $\epsilon(r)$. Moreover, in any detailed theory of dense matter, the baryon and lepton populations are obtained as a function of density; hence the distribution of particle populations in a star can be found coincident with a solution of the Oppenheimer–Volkoff equations.

It may seem curious that the expression (2.212) for mass has precisely the same form as one would write in nonrelativistic physics for the mass whose distribution is given by $\epsilon(r)$. How can this be, inasmuch as we know that spacetime is curved by mass and mass in turn is moved and arranged by spacetime in accord with Einstein's equations? The answer is that (2.212) is not a prescription for computing the total mass of an arbitrary distribution $\epsilon(r)$. There are no arbitrary distributions in gravity; rather $\epsilon(r)$ is precisely prescribed by another of Einstein's equations (2.226). As such, M comprises the mass of the star and its gravitational field. Because of the mutual interaction of mass-energy and spacetime, there is no meaning to the question "What is the mass of the star?" in isolation from the field energy. That is why we refer to M as the gravitational mass or the mass-energy of the star. It is the only type of mass that enters Einstein's theory and is the only stellar mass to which we will refer in this book. Therefore, we shall generally refer to a star's mass as simply the mass without the adjective "gravitational". Sometimes a so-called proper mass is defined. It appears nowhere in Einstein's equations and is an artifact.

It does make sense to inquire about the mass of the totality of nucleons in a star if they were dispersed to infinity. This mass is referred to as the baryon mass. The difference between gravitational mass and baryon mass, if negative, is the gravitational binding of the star. As we shall find, the gravitational binding is of the order of 100 MeV per nucleon in stars near the mass limit as compared to 10 MeV binding by the strong force in nuclei.

Notice that, according to (2.226), the pressure is a monotonic decreasing function from the inside of the star to its edge because all the factors in (2.226) are positive, leaving the explicit negative sign. This makes sense. Any region is weighted down by all that lies above. We have assumed that the denominator in (2.226) is positive. Overall this is true of the earth, the sun, and a neutron star. In fact, $2M/R < 8/9$ for any static stable star. It can also be shown that $2M(r)/r < 1$ for all regions of a stable star [27]; so indeed we are justified in taking the last factor in (2.226) as positive.

In (2.199) we saw a singularity in the Schwarzschild solution if a star lies within $r = 2M$. Such stars are highly relativistic objects called black holes. No light or particle can escape from within their Schwarzschild radius. A luminous star is highly nonrelativistic. A neutron star is relativistic. Newtonian gravity would not produce the same results as General Relativity. This fact is clear, given that $2M$ can be as large as $\frac{8}{9}R$ for a neutron star, which makes the denominator of (2.226) a large correction (as much as 9 instead of 1).

We already have an expression for the radial metric function both inside and outside a star. It is sometimes useful to have the time metric function g_{00}. No general expression for the solution can be obtained, as for g_{11}, (2.216). However using the latter in (2.222) we obtain a differential equation,

$$\frac{d\nu}{dr} = \frac{M(r) + 4\pi r^3 p(r)}{r[r - 2M(r)]}\,. \tag{2.227}$$

The solution must match the exterior solution (2.198). This is easily accomplished. If $\nu(r)$ is a solution, we can add any constant to it and still have a solution. We obtain the correct condition at R if we make the change

$$\nu(r) \longrightarrow \nu(r) - \nu(R) + \tfrac{1}{2}\ln\Bigl(1 - \frac{2M}{R}\Bigr), \qquad r \leq R. \quad (2.228)$$

We can start the integration at $r = 0$ with any convenient value of $\nu(0)$, say zero.

Alternately, once the OV equations have been solved so that $p(r)$ and hence $\epsilon(r)$ are known, one can find $\nu(r)$ by integration of

$$\frac{d\nu}{dr} = -\frac{1}{p+\epsilon}\frac{dp}{dr}\,, \quad (2.229)$$

namely,

$$\nu(r) = -\int_0^r \frac{1}{p+\epsilon}\frac{dp}{dr} + \text{constant}, \qquad \nu(\infty) = 0\,. \quad (2.230)$$

The Oppenheimer–Volkoff equations can be integrated from the origin with the initial conditions $M(0) = 0$ and an arbitrary value for the central energy density $\epsilon(0)$, until the pressure $p(r)$ becomes zero at, say R. Because zero pressure can support no overlying matter against the gravitational attraction, R defines the gravitational radius of the star and $M(R)$ its gravitational mass. For the given equation of state, there is a unique relationship between the mass and central density $\epsilon(0)$. So for each possible equation of state, there is a unique family of stars, parameterized by, say, the central density or the central pressure. Such a family is often referred to as the *single parameter sequence* of stars corresponding to the given equation of state.

2.9.6 Gravitational collapse and limiting mass

In Newtonian physics mass alone generates gravity. In the Special Theory of Relativity mass is equivalent to energy, so in the general theory all forms of energy contribute to gravity. It is surprising that pressure also plays a most consequential role in the structure of relativistic stars beyond the role it plays in

Newtonian gravity. Pressure supports stars against gravity, but surprisingly, it ultimately assures the gravitational collapse of relativistic stars whose mass lies above a certain limit.

Pressure appears together with energy density in determining the monotonic decrease of pressure (2.226) in a relativistic star. Gravity acts to compress the material of the star. As it does so, the pressure of the material is increased toward the center. But inasmuch as pressure appears on the right side of the equation, this increase serves to further enhance the grasp of gravity on the material. Therefore, for stars of increasing mass, for which the supporting pressure must correspondingly increase, the pressure gradient (which is negative) is increased in magnitude, making the radius of the star smaller because its edge necessarily occurs at $p = 0$. As a consequence, if the mass of a relativistic star exceeds a critical value, there is no escape from gravitational collapse to a black hole [28]. Whatever the equation of state, the one-parameter sequence of stable configurations belonging to that equation of state is terminated by a maximum-mass compact star. The mass of this star is referred to as the *mass limit* or *limiting mass* of the sequence.

2.10 Action Principle in Gravity

We arrived at Einstein's equations by noting the vanishing divergence of the Einstein curvature tensor and equating it to the energy-momentum tensor of matter as the source of the gravitational field. We did not comment on how the energy-momentum tensor might be obtained. In general, this tensor is not given but must be calculated from a theory of matter. In what frame should the theory be solved? Evidently in the general frame of the gravitational field. But this is an entirely different problem than is normally solved in many-body theory.

We are accustomed to solving problems in nuclear and particle theory in flat spacetime (or even flat space) in which the constant Minkowski metric $\eta_{\mu\nu}$ appears, not a general and as yet unspecified field $g_{\mu\nu}(x)$. A tacit assumption is made in passing from the energy-momentum tensor in a Lorentz frame (2.49) to its form (2.202) in a general frame by means of the principle of general covariance, as was done in deriving the Oppenheimer–Volkoff equations of stellar structure. The local region over which Lorentz frames extend is assumed to be sufficiently large that the equations of motion of the matter fields can be solved in a Lorentz frame, that is, in the absence of gravity, and the corresponding energy-momentum tensor constructed from the solution for such a region.

As we shall see in the next chapter, the local inertial frames in the gravitational field of neutron stars (and therefore for the less dense white dwarfs and all other stars) are actually sufficiently extensive that the matter from which they are constituted can be described by theories in flat Minkowski spacetime. We shall refer to such a situation as a partial decoupling of matter from gravity. In other words, the equations of motion for the matter and radiation fields can be solved in Minkowski spacetime. The solutions will provide the means of calculating the energy density and pressure of matter ϵ and p throughout the star. But the general metric functions of gravity $g_{\mu\nu}(x)$ reappear on the right side of Einstein's field equations in the energy-momentum tensor, (2.202), when referred to a general frame in accord with the principle of general covariance. Therefore the gravitational fields $g_{\mu\nu}(x)$ still appear on both sides of Einstein's field equations, and matter in bulk shapes spacetime just as spacetime shapes and moves matter in bulk. However, the local structure of matter is determined only by the equations of motion in Minkowski spacetime.

There are conceivable situations where the partial decoupling just described may not hold. In that case the equations of motion themselves contain, not the Minkowski metric tensor (a diagonal tensor with constant elements), but the general, spacetime-dependent, metric tensor. This is the fully coupled problem and obviously would be enormously difficult to solve. While we do not encounter this situation in this book (see as an example where strong coupling is used, Refs. [29, 30]), nonetheless it is worth seeing in symbolic form what the fully coupled problem looks like. The expectation that the stress-energy tensor should be obtained in general from a theory of matter by solving the field equations of the theory in a general gravitational field will be verified. It is also interesting to see Einstein's equations emerge from a variational principle.

We employ the gravitational action principle. As in all cases, the Lagrangian of gravity ought to be a scalar. We have encountered the Ricci scalar curvature $R = g^{\mu\nu}R_{\mu\nu}$, and from it, as Hilbert did, the Lagrangian density can be formed (with a prefactor that can be known only in hindsight):

$$\mathcal{L}_g = -\frac{1}{16\pi G} R\sqrt{-g}\,. \tag{2.231}$$

Here G is Newton's constant and g is the determinant of the metric $g_{\mu\nu}$, which is negative for our choice of signature for the metric. (Recall, for example, the Minkowski metric, or the Schwarzschild metric.) We also define the Lagrangian density

$$\mathcal{L}_m = L_m\sqrt{-g} \tag{2.232}$$

from the Lagrangian L_m of the matter and radiation fields ϕ. The total action is

$$I = \int (\mathcal{L}_g + \mathcal{L}_m)\, d^4x\,. \tag{2.233}$$

The coupled field equations for the matter and metric functions emerge as the conditions that yield vanishing variation of the action with respect to all the fields—the gravitational fields described by $g_{\mu\nu}$ and matter fields described by ϕ. The manipulations are quite tedious and are relegated to the next section. The field equations obtained are

$$\frac{\partial L_m}{\partial \phi} - \partial_\mu \frac{\partial L_m}{\partial(\partial_\mu \phi)} = 0\,, \tag{2.234}$$

$$G^{\mu\nu} = -8\pi G T^{\mu\nu}\,, \tag{2.235}$$

where $G^{\mu\nu} \equiv R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R$ is the Einstein tensor. The first of the field equations reduces to the familiar Euler–Lagrange equations in the limit of weak gravitational fields ($g_{\mu\nu} \to \eta_{\mu\nu}$). We shall encounter the Euler–Lagrange equations in studying theories of dense nuclear matter. The second are Einstein's field equations (2.215).

The matter-radiation energy-momentum tensor that emerges from the variational principle is given by

$$T^{\mu\nu} \equiv -g^{\mu\nu}L_m + 2\frac{\partial L_m}{\partial g_{\mu\nu}}\,. \tag{2.236}$$

The second term is

$$2\frac{\partial L_m}{\partial g_{\mu\nu}} = 2\frac{\partial L_m}{\partial(\partial_\alpha \phi)}\frac{1}{2}\frac{\partial(g_{\alpha\beta}\partial^\beta \phi)}{\partial g_{\mu\nu}} = \frac{\partial L_m}{\partial(\partial_\mu \phi)}\partial^\nu \phi\,.$$

Combining these results yields the canonical form of the energy-momentum tensor in field theory (for example, see Ref. [8]) except that the Minkowski tensor is replaced by the general metric of gravity. Thus we have

$$T^{\mu\nu} = -g^{\mu\nu}L_m + \sum_\phi \frac{\partial L_m}{\partial(\partial_\mu \phi)} g^{\nu\alpha}\partial_\alpha \phi\ , \qquad (2.237)$$

where the sum is over the various fields ϕ in L_m. In this way we see how the equations couple all matter and gravitational fields, $\phi(x), ..., g^{\mu\nu}(x)$ in the general case.

2.10.1 Derivations

We write down most of the steps in deriving the Einstein field and matter-radiation equations from the variation of the action. The gravitational and matter fields will be subjected to arbitrary variations except the values and first derivatives will be kept constant on the boundaries. We concentrate on the gravitational part because that is the hardest. First, from (2.68) and (2.96) we readily obtain by differentiation,

$$g^{\alpha\mu}{}_{,\sigma}\, g_{\mu\nu} + g^{\alpha\mu}(\Gamma_{\nu\mu\sigma} + \Gamma_{\mu\nu\sigma}) = 0$$

Multiply by $g^{\beta\nu}$ and sum on ν to find

$$g^{\alpha\beta}{}_{,\sigma} = -g^{\alpha\mu}\Gamma^\beta_{\mu\sigma} - g^{\beta\nu}\Gamma^\alpha_{\nu\sigma}\,. \qquad (2.238)$$

Next, evaluate

$$\begin{aligned}(g^{\mu\nu}\sqrt{-g})_{,\sigma} &= g^{\mu\nu}{}_{,\sigma}\sqrt{-g} + g^{\mu\nu}(\sqrt{-g})_{,\sigma} \\ &= -g^{\mu\rho}\Gamma^\nu_{\rho\sigma}\sqrt{-g} - g^{\nu\rho}\Gamma^\mu_{\rho\sigma}\sqrt{-g} + g^{\mu\nu}(\sqrt{-g})_{,\sigma}\,.\end{aligned}$$

Use (2.149) to find

$$(g^{\mu\nu}\sqrt{-g})_{,\sigma} = \left(-g^{\mu\rho}\Gamma^\nu_{\rho\sigma} - g^{\nu\rho}\Gamma^\mu_{\rho\sigma} + g^{\mu\nu}\Gamma^\rho_{\sigma\rho}\right)\sqrt{-g}\,. \quad (2.239)$$

Now set $\sigma = \nu$ and contract. After a cancelation of two terms, find,

$$(g^{\mu\nu}\sqrt{-g})_{,\nu} = -g^{\nu\alpha}\Gamma^{\mu}_{\alpha\nu}\sqrt{-g}\,. \tag{2.240}$$

The above results can now be employed to rewrite the gravitational action (where, as usual, we set $G = c = 1$ whenever convenient);

$$16\pi I_g = \int R\sqrt{-g}\, d^4x\,. \tag{2.241}$$

From (2.169) define

$$\begin{aligned} L &\equiv g^{\mu\nu}\left(\Gamma^{\sigma}_{\mu\nu}\Gamma^{\rho}_{\sigma\rho} - \Gamma^{\rho}_{\mu\sigma}\Gamma^{\sigma}_{\nu\rho}\right) \\ M &\equiv g^{\mu\nu}\left(\Gamma^{\sigma}_{\mu\sigma,\nu} - \Gamma^{\sigma}_{\mu\nu,\sigma}\right)\,. \end{aligned} \tag{2.242}$$

With these definitions the scalar curvature becomes

$$R = M - L\,. \tag{2.243}$$

Use (2.239) and (2.240) to find

$$\begin{aligned} M\sqrt{-g} &= (g^{\mu\nu}\Gamma^{\sigma}_{\mu\sigma}\sqrt{-g})_{,\nu} - (g^{\mu\nu}\Gamma^{\sigma}_{\mu\nu}\sqrt{-g})_{,\sigma} \\ &\quad -(g^{\mu\nu}\sqrt{-g})_{,\nu}\Gamma^{\sigma}_{\mu\sigma} + (g^{\mu\nu}\sqrt{-g})_{,\sigma}\Gamma^{\sigma}_{\mu\nu}\,. \end{aligned} \tag{2.244}$$

The first two terms are perfect differentials and so contribute nothing under the integral because of the vanishing of the fields and their derivatives on the boundaries. After some manipulation, the remaining two terms are found to be $2L\sqrt{-g}$. Consequently,

$$\begin{aligned} 16\pi I_g &= \int L\sqrt{-g}\, d^4x \\ &= \int g^{\mu\nu}\left(\Gamma^{\sigma}_{\mu\nu}\Gamma^{\rho}_{\sigma\rho} - \Gamma^{\rho}_{\mu\sigma}\Gamma^{\sigma}_{\nu\rho}\right)\sqrt{-g}\, d^4x\,. \end{aligned} \tag{2.245}$$

Now evaluate the variation, examining separately the two terms in the integrand. Use (2.149) and (2.240) to rewrite the variation of the first term;

$$\begin{aligned} &\delta\left(g^{\mu\nu}\Gamma^{\sigma}_{\mu\nu}\Gamma^{\rho}_{\sigma\rho}\sqrt{-g}\right) = \\ &\quad \Gamma^{\alpha}_{\mu\nu}\delta(g^{\mu\nu}(\sqrt{-g})_{,\alpha}) - \Gamma^{\beta}_{\alpha\beta}\delta(g^{\alpha\nu}\sqrt{-g})_{,\nu} - \Gamma^{\beta}_{\alpha\beta}\Gamma^{\alpha}_{\mu\nu}\delta(g^{\mu\nu}\sqrt{-g})\,. \end{aligned}$$

Use (2.238) to develop the variation of the second term in (2.245) and find

$$\delta(g^{\mu\nu}\Gamma^{\rho}_{\mu\sigma}\Gamma^{\sigma}_{\nu\rho}\sqrt{-g}) = -\Gamma^{\alpha}_{\nu\beta}\delta(g^{\nu\beta}{}_{,\sigma}\sqrt{-g}) - \Gamma^{\beta}_{\mu\alpha}\Gamma^{\alpha}_{\nu\beta}\delta(g^{\mu\nu}\sqrt{-g})\,.$$

Assemble the above two results and introduce the perfect differentials with compensating terms to form the variation of the integrand of (2.245);

$$\begin{aligned}\delta\mathcal{L}_g \;=\; & [\Gamma^{\alpha}_{\mu\nu}\delta(g^{\mu\nu}\sqrt{-g})]_{,\alpha} - [\Gamma^{\beta}_{\alpha\beta}\delta(g^{\alpha\nu}\sqrt{-g})]_{,\nu} \\ & +(-\Gamma^{\alpha}_{\mu\nu,\alpha} + \Gamma^{\beta}_{\mu\beta,\nu} + \Gamma^{\beta}_{\mu\alpha}\Gamma^{\alpha}_{\nu\beta} - \Gamma^{\beta}_{\alpha\beta}\Gamma^{\alpha}_{\mu\nu})\delta(g^{\mu\nu}\sqrt{-g})\,.\end{aligned}$$

The first two terms are perfect differentials and yield zero because the variations vanish on the boundary. The remaining bracket is the Ricci tensor (2.169). Therefore, we have for the variation of the gravitational action

$$\delta I_g = \frac{1}{16\pi}\int R_{\mu\nu}\delta(g^{\mu\nu}\sqrt{-g})\,d^4x\,. \tag{2.246}$$

Next, again from (2.68) deduce that

$$\delta g^{\alpha\nu} = -g^{\mu\alpha}g^{\nu\beta}\delta g_{\alpha\beta}\,. \tag{2.247}$$

Also, from (2.148) find

$$2\sqrt{-g}(\sqrt{-g})_{,\alpha} = (\sqrt{-g}\sqrt{-g})_{,\alpha} = g_{,\alpha} = g g^{\mu\nu}g_{\mu\nu,\alpha}\,.$$

Consequently,

$$(\sqrt{-g})_{,\alpha} = \tfrac{1}{2}\sqrt{-g}g^{\mu\nu}g_{\mu\nu,\alpha}\,. \tag{2.248}$$

With this result we now have

$$\delta(g^{\mu\nu}\sqrt{-g}) = -\left(g^{\mu\alpha}g^{\nu\beta} - \tfrac{1}{2}g^{\mu\nu}g^{\alpha\beta}\right)\sqrt{-g}\,\delta g_{\alpha\beta}\,. \tag{2.249}$$

Recalling the definition of Einstein's curvature tensor, we have obtained

$$\delta I_g = -\frac{1}{16\pi}\int G^{\alpha\beta}\sqrt{-g}\,\delta g_{\alpha\beta}\,d^4x\,. \tag{2.250}$$

Thus we derive Einstein's field equation in empty space from the vanishing of the variation of the gravitational action.

If we add the action of matter and radiation fields to the gravitational action, we get the total action. Under arbitrary variations of the gravitational and other fields we insist that the total action vanish. This leads to the Euler–Lagrange equations (2.234) for the matter-radiation fields and to the Einstein equations (2.235) for the gravitational fields. Note, however, that the equations of motion for the matter-radiation fields contain the gravitational fields $g^{\mu\nu}$ and they reduce to the usual form in Minkowski spacetime only when the $g^{\mu\nu}$ can be replaced by the Minkowski tensor.

Return now to the total action (2.233) and vary all fields. Also use the result (2.250) for the variation of $\mathcal{L}_g$

$$\delta I = \int \Big[\Big\{ -\frac{1}{16\pi} G^{\alpha\beta}\sqrt{-g} + \Big(\frac{\partial \mathcal{L}_m}{\partial g_{\alpha\beta}} - \partial_\nu \frac{\partial \mathcal{L}_m}{\partial g_{\alpha\beta,\nu}} \Big) \Big\} \delta g_{\alpha\beta} + \Big\{ \frac{\partial \mathcal{L}_m}{\partial \phi} - \partial_\nu \frac{\partial \mathcal{L}_m}{\partial(\partial_\nu \phi)} \Big\} \delta\phi \Big] \, d^4x \,. \quad (2.251)$$

The last term in each curly brackets was obtained by an integration by parts, as follows:

$$\frac{\partial \mathcal{L}}{\partial f_\mu} \partial f_{,\mu} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial f_{,\mu}} \delta f \right) - \partial_\mu \frac{\partial \mathcal{L}}{\partial f_{,\mu}} \delta f \,,$$

where f stands for either $g_{\mu\nu}$ or ϕ. The integral over the first term on the right vanishes because the f are not varied on the boundary. Because the variations are otherwise arbitrary, the vanishing of the action implies the vanishing of the coefficients of the varied fields. The variation of the matter fields yields (2.234) (where we have removed $\sqrt{-g}$ because it is unaffected by the ϕ variation). The variation of the gravitational fields yields

$$G^{\alpha\beta}\sqrt{-g} = 16\pi \left(\frac{\partial \mathcal{L}_m}{\partial g_{\alpha\beta}} - \partial_\nu \frac{\partial \mathcal{L}_m}{\partial g_{\alpha\beta,\nu}} \right) . \quad (2.252)$$

This equation shows how the gravitational and matter fields are coupled. We use it to derive (2.235) by showing that the right side is the energy-momentum tensor in the form (2.236). The familiar form (2.237) then follows. Evaluate the first term on the right side of 2.252:

$$\frac{\partial \mathcal{L}_m}{\partial g_{\alpha\beta}} = L_m \frac{\partial \sqrt{-g}}{\partial g_{\alpha\beta}} \,. \quad (2.253)$$

Because

$$\frac{\partial g}{\partial g_{\mu\nu}} = g g^{\mu\nu}\,, \tag{2.254}$$

which follows by differentiating the identity $g = g_{\mu\nu}g^{\mu\nu}$, we obtain

$$\frac{\partial \mathcal{L}_m}{\partial g_{\alpha\beta}} = \left(-\tfrac{1}{2}g^{\alpha\beta}L_m + \frac{\partial L_m}{\partial g_{\alpha\beta}}\right)\sqrt{-g}\,. \tag{2.255}$$

The matter Lagrangian will not usually depend on derivatives of the metric, but only on the metric itself. Thus, with the above equation we have derived (2.235) with the energy momentum tensor given by (2.236).

2.11 Problems for Chapter 2

1. Derive and solve the equations (2.12) for the Lorentz transformation (2.15).
2. Derive the transformation for an arbitrary boost, (2.18).
3. Derive the four-velocity components (2.42).
4. Check that the energy–momentum tensor takes the form (2.49).
5. Derive the coordinate expression for $g_{\mu\nu}$ in (2.53). Review the motion of a free particle in an arbitrary gravitational field, and derive the geodesic equation of motion, (2.85).
6. Derive the transformation property of the Christoffel symbol (2.90).
7. Prove the expression of the affine connection (2.93).
8. Prove the expression involving the Christoffel symbols (2.96).
9. Obtain the geodesic equation (2.107) as the extremal of the propertime.
10. Follow all the steps in the derivation of the g_{00} in (2.119).

11. Derive the expression (2.122) for the covariant derivative of a contravariant vector.

12. Two esoteric-looking results that are used in the variational principle for the derivation of Einstein's equations are (2.148) and (2.149). Derive them in detail.

13. Review the details of the derivation of the conservation of total charge (2.157).

14. Derive (2.160) and with it the expression for the Riemann curvature tensor (2.161).

15. Derive the expression for the Ricci tensor (a contraction of the Riemann tensor) given by (2.169). Show that it is symmetric, though not manifestly so.

16. Prove the Bianchi identities (2.173), thus paving the way to the proof that the Einstein curvature tensor has vanishing covariant divergence (2.175). Prove the latter also. Einstein was unaware of the Bianchi identities and this delayed his discovery of General Relativity.

17. From the above, understand the three possibilities of Section 2.8.

18. Derive at least three of the expressions for the affine connections in static spherical isotropic spacetime (2.188).

19. Derive the Schwarzschild solution for relativistic static stars (2.200).

20. Derive the relationship of the components of the Einstein tensor to the metric functions of static spherical spacetime (2.207).

21. Derive all intermediate steps including the identification of k with the Newton constant G (2.214).

22. Derive the explicit Einstein equations for a static spherical star, (2.217) to (2.220).

23. Go through the details of manipulation of the above equations that are outlined in (2.221) to (2.225).

24. Hence, derive the Oppenheimer–Volkoff equation (2.226).

25. Follow the principle steps in the derivation of the coupled matter and gravitational fields as expressed by (2.234) and (2.235) as given in Section 2.10.1.

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