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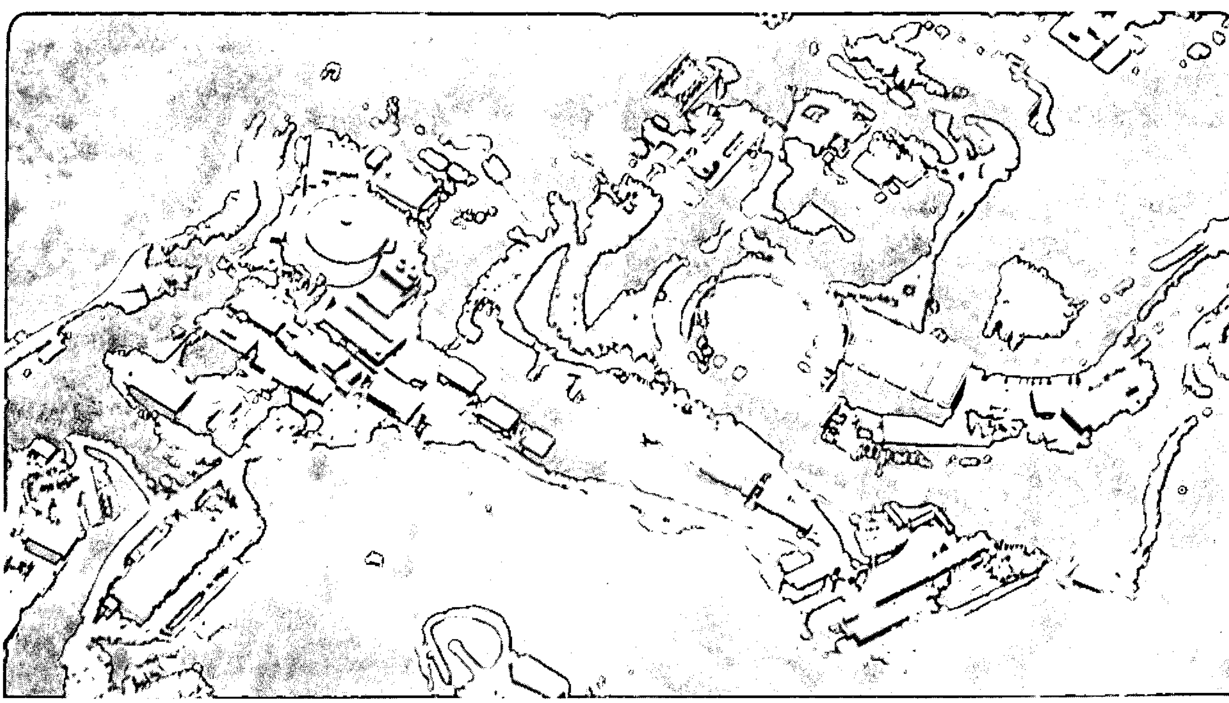

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Properties of Quantum 2×2 Matrices†

S. VOKOS (*) - B. ZUMINO (*) - J. WESS (**)

1. - Introduction

The theory of quantum groups has attracted great interest recently. In mathematics quantum groups are related to Hopf algebras, non commutative geometry and the theory of knots and links. In physics they are relevant for the theory of integrable systems, certain problems in statistical physics and the study of conformal field theories in two dimensions.

One approach to the study of quantum groups, followed especially by Faddeev and collaborators, defines them in terms of their basic representation by matrices. Thus the quantum version of $SL(2,C)$, denoted by $SL_q(2,C)$, is defined by giving quantization relations for the elements of the 2×2 $SL(2,C)$ matrix, as described briefly in Section 2. Similarly, for other Lie groups, one starts from the basic representation and quantizes the matrix elements of the classical matrix. Higher representations of the same quantum group can be obtained by multiplying and reducing quantum representations.

It can happen that the basic representation of a quantum group possesses special interesting properties. In this paper we describe the special properties we have found for the 2×2 representation of $SL_q(2,C)$. Although the results can be stated very simply, the proofs are usually somewhat lengthy and involved. We shall only sketch the basic ideas of the proofs here and will describe the details in a longer paper.

The literature on quantum groups is very extensive. At the end we list only a few papers where numerous other mathematical and physical references can be found.

We are deeply indebted to Ludwig Faddeev and to Vaughan Jones for introducing us to the theory of quantum groups.

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It is a pleasure to dedicate this paper to Luigi Radicati on the occasion of his 70th birthday.

2. - Quantum $SL(2, C)$

We review the usual definition of quantum $SL(2, C)$, i.e. $SL_q(2, C)$, in terms of its two dimensional representation. We consider first the general linear group in two dimensions. A matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{2.1}$$

is said to belong to the quantum linear group $GL_q(2, C)$ if its matrix elements, instead of being complex numbers, are non commuting quantities (which can be realized as operators in a Hilbert space) satisfying the commutation relations

$$\left.\begin{array}{l} ab = qba \\ ac = qca, \quad bc = cb \\ bd = qdb, \quad ad - da = \left(q - \dfrac{1}{q}\right) bc \\ cd = qdc \end{array}\right\}. \tag{2.2}$$

Here q is a complex number, the quantum parameter. Matrices like A have the following remarkable property which can be taken as the definition of a quantum group. Let

$$A' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \tag{2.3}$$

be a matrix of the same type, i.e. let its elements satisfy commutation relations similar to (2.2)

$$\left.\begin{array}{l} a'b' = qb'a' \\ a'c' = qc'a' \\ \text{etc.} \end{array}\right\}. \tag{2.4}$$

Let also a', b', c', d' commute with a, b, c, d. Then the matrix $A'' = AA'$

$$A'' = \begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix} = \begin{pmatrix} aa' + bc' & ab' + bd' \\ ca' + dc' & cb' + dd' \end{pmatrix} \tag{2.5}$$

is also of the same type, i.e.

$$\left.\begin{aligned} a''b'' &= qb''a'' \\ a''c'' &= qc''a'' \\ &\text{etc.} \end{aligned}\right\}. \tag{2.6}$$

With reference to matrices like A, A' and A'' one talks of a quantum group, the corresponding classical group being obtained in the limit $q \to 1$, when the matrix elements commute. A quantum group is not a group; a better epression would be quantized group.

The quantum determinant of the matrix A is defined as

$$D_q = \det_q A = ad - qbc = da - \frac{1}{q} bc. \tag{2.7}$$

It reduces to the usual determinant for $q = 1$. Using (2.2) it is easy to verify that D_q is central, i.e. it commutes with $a, b, c,$ and d. Using the quantum determinant one obtains the (both right and left) inverse matrix

$$A^{-1} = \frac{1}{D_q} \begin{pmatrix} d & -\frac{1}{q} b \\ -qc & a \end{pmatrix}. \tag{2.8}$$

Notice that A^{-1} is a quantum matrix which corresponds to the quantum parameter q^{-1}. Indeed, from (2.2),

$$d\left(-\frac{1}{q} b\right) = \frac{1}{q} \left(-\frac{1}{q} b\right) d \quad \text{etc.} \tag{2.9}$$

Similarly the matrix

$$A^2 = \begin{pmatrix} a^2 + bc & ab + bd \\ ca + dc & cb + d^2 \end{pmatrix} \tag{2.10}$$

corresponds to the quantum parameter q^2, a fact which can also be easily verified using the commutation relations (2.2). In general one can show that the matrix A^n is a quantum matrix corresponding to the quantum parameter q^n, as we discuss in Sections 3 and 4. This fact was also noticed and proven by Corrigan and Tunstall [7].

One can impose the condition

$$D_q = 1 \tag{2.11}$$

which restricts $L_q(2, C)$ to $SL_q(2, C)$. In addition one can impose reality conditions on the matrix A. One choice is that it be unitary

$$\bar{a} = d \quad \bar{b} = -qc \quad \bar{c} = -\frac{1}{q} b. \tag{2.12}$$

These relations restrict $SL_q(2,C)$ to $SU_q(2)$. They require for consistency that q be real. Another choice is that A be real

(2.13) $$\bar{a}=a \quad \bar{b}=b \quad \bar{c}=c \quad \bar{d}=d.$$

This gives $SL_q(2,R)$. For consistency with (2.2) it must now be $|q|=1$.

The commutation relations (2.2) can be interpreted as quantum symplectic conditions on A. Define the quantum epsilon matrix

(2.14) $$\varepsilon_q = \begin{pmatrix} 0 & \frac{1}{\sqrt{q}} \\ -\sqrt{q} & 0 \end{pmatrix}$$

which satisfies

(2.15) $$\varepsilon_q^2 = -1.$$

One has

(2.16) $$\varepsilon_q A^T \varepsilon_q^{-1} = \begin{pmatrix} d & -\frac{b}{q} \\ -qc & a \end{pmatrix} = D_q A^{-1},$$

where A^T is the transposed of the matrix A. (2.16) can be written as

(2.17) $$A^T \varepsilon_q A = A \varepsilon_q A^T = D_q \varepsilon_q.$$

For $D_q = 1$ this the quantum analogue of the usual conditions for a matrix to be symplectic. The two conditions (2.17) are equivalent to (2.2) plus (2.7).

3. - Properties of 2×2 quantum matrices

As we mentioned in Section 2, the $n-th$ power A^n of the matrix A is a quantum matrix corresponding to the quantum parameter q^n. In this section we sketch a proof of this fact in the case when $n \gtreqless 0$ is an integer, $n \in \mathbb{Z}$. As we shall see in the next section the result is valid for continuous values of n. We are dealing here with special properties of 2×2 quantum matrices. It would be interesting to see if and how they generalize to higher dimensional representations of $GL_q(2,C)$ or to other quantum groups.

Let us call a_n, b_n, c_n, d_n the matrix elements of the $n-th$ power of the matrix A in (1)

(3.1) $$A^n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$$

and let

(3.2) $$D_n = \det_{q^n} A^n = a_n d_n - q^n b_n c_n.$$

The following relations are valid

$$
\left.\begin{aligned}
&a_n b_m - q^m b_n a_m = -q^m D_m b_{n-m}\\
&a_n c_m - q^n c_n a_m = -q^{n-m} D_m c_{n-m}\\
&a_n d_m - q^{n+m} d_n a_m = D_m a_{n-m} - q^{n+m} D_m d_{n-m}\\
&b_n c_m - q^{n-m} c_n b_m = 0\\
&b_n d_m - q^n d_n b_m = q^m D_m b_{n-m}\\
&c_n d_m - q^m d_n c_m = D_m c_{n-m}\\
&D_m = D^m
\end{aligned}\right\}. \tag{3.3}
$$

Notice that (3.3) are not *commutation* relations except for $n = m$, in which case they imply that A^n is a quantum matrix of quantum parameter q^n (see below). We have proven (3.3) by double induction. First for $m = 1$ by induction in n, then for fixed n, by induction in m. This induction proof shows that (3.3) are valid for $n, m \in \mathbb{Z}$. In the course of the induction proof we also show that

$$a_n d_m - q^m b_n c_m = D_m a_{n-m}. \tag{3.4}$$

Using (3.3), (3.4) can be rewritten as

$$a_n d_m - q^n c_n b_m = D_m a_{n-m}, \tag{3.5}$$

or as

$$d_n a_m - q^{-n} b_n c_m = d_n a_m - q^{-m} c_n b_m = D_m d_{n-m}. \tag{3.6}$$

We shall not reproduce here the induction proof which is rather lengthy and tedious, although relatively straightforward. As mentioned in the introduction, we intend to give it in a longer paper together with the detailed proof of the statements of the next section.

Set $n = m$ in (3.3). Since $a_0 = d_0 = 1$, $b_0 = c_0 = 0$, we find

$$
\left.\begin{aligned}
&a_n b_n - q^n b_n a_n = 0\\
&a_n c_n - q^n c_n a_n = 0\\
&a_n d_n - q^{2n} d_n a_n = (1 - q^{2n}) D_n\\
&b_n c_n - c_n b_n = 0\\
&b_n d_n - q^n d_n b_n = 0\\
&c_n d_n - q^n d_n c_n = 0
\end{aligned}\right\}. \tag{3.7}
$$

On the other hand, setting $n = m$ in (3.4), (3.5) and (3.6) we have

$$a_n d_n - q^n b_n c_n = D_n \tag{3.8}$$

and

$$d_n a_n - q^{-n} b_n c_n = D_n. \tag{3.9}$$

The third equation in (3.7) is a consequence of (3.8) and (3.9). Subtracting (3.9) from (3.8) we obtain

$$a_n d_n - d_n a_n = (q^n - q^{-n}) b_n c_n. \tag{3.10}$$

We have now all relations stating that A^n is a quantum matrix corresponding to q^n.

4. - Exponential description

The fact that A^n corresponds to the quantum parameter q^n suggest the ansatz

$$A = e^{hM}, \quad q = e^h \tag{4.1}$$

where the matrix elements of the 2×2 matrix M should satisfy commutation relations independent of h. The commutation relations (2.2) for the elements of A should be a consequence of those for the elements of M. If this is the case, the matrix $A_1 = e^{h_1 M}$ would have quantum parameter $q_1 = e^{h_1}$ and the matrix $AA_1 = e^{(h+h_1)M}$ would have quantum parameter $qq_1 = e^{h+h_1}$. In particular this would imply that A^n has quantum parameter $q^n = e^{nh}$, not only for integer n but also for continuous values of n, as long as e^{nhM} has a meaning. All this is actually true, at least formally, and furthermore the properties of M are extremely simple. It turns out that for our quantum matrices the usual relation between determinant and trace is valid for the quantum determinant

$$D_q = \det_q A = \exp\ \mathrm{tr}\ (hM), \tag{4.2}$$

where tr denotes the *ordinary* trace. Therefore we can limit ourselves at first to the case $D_q = 1$ when M is traceless

$$M = \begin{pmatrix} \lambda & \mu \\ \nu & -\lambda \end{pmatrix} \tag{4.3}$$

and introduce later a non trivial central trace to account for a determinant different from one. The correct commutation relations are simply

$$\left.\begin{array}{c} \lambda\mu - \mu\lambda = \mu, \ \ \lambda\nu - \nu\lambda = \nu \\ \mu\nu - \nu\mu = 0 \end{array}\right\}. \tag{4.4}$$

For the matrix elements of A, given by (4.1), the commutation relations (4.4), together with (4.3), imply (2.2). More precisely they imply that

$$\varepsilon_q A^T \varepsilon_q^{-1} = A^{-1}. \tag{4.5}$$

Comparing with (2.16) we see that also $D_q = 1$. We sketch now the proof of these statements.

The condition (4.5), or

$$A^T = \varepsilon_q^{-1} A^{-1} \varepsilon_q, \tag{4.6}$$

can be written more explicitly, using (4.1). Since

$$\varepsilon_q^{-1} M \varepsilon_q = \begin{pmatrix} -\lambda & -\frac{\nu}{q} \\ -q\mu & \lambda \end{pmatrix}, \tag{4.7}$$

(4.6) becomes

$$\left(\exp\left[h\begin{pmatrix} \lambda & \mu \\ \nu & -\lambda \end{pmatrix}\right]\right)^T = \exp\left[h\begin{pmatrix} \lambda & e^{-h}\nu \\ e^{h}\mu & -\lambda \end{pmatrix}\right]. \tag{4.8}$$

We have verified that this equation is correct to all orders in h by expanding the exponentials. In spite of the great simplicity of the commutation relations (4.4) the proof is not trivial. The crucial point for the validity of (4.8) is of course that, for quantum matrices like M, it is not true that $(M^n)^T = (M^T)^n$. Instead, left and right hand side are related exactly in such a way as to account for the extra factors e^h and e^{-h} occuring in the right hand side of (4.8).

If the matrix M is not traceless

$$M = \begin{pmatrix} \alpha & \mu \\ \nu & \beta \end{pmatrix} \tag{4.9}$$

the commutation relations (4.4) must be replaced by the slightly more general relations

$$\left.\begin{array}{c} \alpha\mu - \mu\alpha = \mu, \ \ \alpha\nu - \nu\alpha = \nu \\ \mu\nu - \nu\mu = 0 \\ \beta\mu - \mu\beta = -\mu, \ \ \beta\nu - \nu\beta = -\nu \\ \alpha\beta - \beta\alpha = 0 \end{array}\right\}. \tag{4.10}$$

It is clear from these relations that

$$\operatorname{tr} M = \alpha + \beta \tag{4.11}$$

is central, i.e. it commutes with α, β, μ and ν. From (4.2) we see that

$$D_q = \det_q A = \exp\left[h(\alpha+\beta)\right]. \tag{4.12}$$

It is now also obvious that

$$\det_{q^n} A^n = (\det_q A)^n. \tag{4.13}$$

5. - Concluding remarks

We restrict ourselves to the case $D_q = 1$. Since A, given by (4.1), (4.3) and (4.4), is an $SL_q(2, C)$ matrix, it behaves as described in Section 2 under multiplication. This means that, if

$$A' = e^{hM'}, \quad M' = \begin{pmatrix} \lambda' & \mu' \\ \nu' & -\lambda' \end{pmatrix} \tag{5.1}$$

is a matrix of the same type as A

$$\left. \begin{matrix} \lambda'\mu' - \mu'\lambda' = \mu', \ \lambda'\nu' - \nu'\lambda' = \nu' \\ \mu'\nu' - \nu'\mu' = 0 \end{matrix} \right\} \tag{5.2}$$

and furthermore λ, μ, ν commute with λ', μ', ν', then

$$A'' = AA' = e^{hM''}, \quad M'' = \begin{pmatrix} \lambda'' & \mu'' \\ \nu'' & -\lambda'' \end{pmatrix} \tag{5.3}$$

is also an $SL_q(2, C)$ matrix, i.e.

$$\left. \begin{matrix} \lambda''\mu'' - \mu''\lambda'' = \mu'', \ \lambda''\nu'' - \nu''\lambda'' = \nu'' \\ \mu''\nu'' - \nu''\mu'' = 0 \end{matrix} \right\}. \tag{5.4}$$

One can find explicit expressions for λ'', μ'' and ν'' by means of the Baker-Campbell-Hausdorff formula, which gives

$$hM'' = hM + hM' + \frac{1}{2}h^2[M, M'] + \frac{1}{12}h^3[M, [M, M']] - \frac{1}{12}h^3[M', [M, M']] + \dots \tag{5.5}$$

To the order indicated one finds

$$
\begin{aligned}
\lambda'' &= \lambda + \lambda' + \frac{h}{2}(\mu\nu' - \nu\mu') + \\
&+ \frac{h^2}{6}\,[2\lambda\mu'\nu' + 2\mu\nu\lambda' - \mu\nu'\lambda' - \nu\lambda\mu' - \nu\mu'\lambda' - \mu\lambda\nu'] + \dots \\
\mu'' &= \mu + \mu' + h\,(\lambda\mu' - \mu\lambda') + \\
&+ \frac{h^2}{6}\,[\mu\mu'\nu' + \mu\nu\mu' - \nu\mu'^2 - \mu^2\nu' + 2\mu\lambda'^2 + 2\lambda^2\mu' \\
&- \lambda\mu'\lambda' - \mu\lambda\lambda' - \lambda\lambda'\mu' - \lambda\mu\lambda'] + \dots \\
\nu'' &= \nu + \nu' + h\,(\nu\lambda' - \lambda\nu') + \\
&+ \frac{h^2}{6}\,[\nu\mu'\nu' + \mu\nu\nu' - \mu\nu'^2 - \nu^2\mu' + 2\nu\lambda'^2 + 2\lambda^2\nu' \\
&- \lambda\nu'\lambda' - \nu\lambda\lambda' - \lambda\lambda'\nu' - \lambda\nu\lambda'] + \dots
\end{aligned} \tag{5.6}
$$

Notice that, when the matrix elements of two matrices don't commute, the trace of the commutator does not vanish in general. However M'' is traceless, because the traces of terms of a given order in h cancel. For instance in (5.5) the two terms in h^3 separately have non zero trace, but the sum of the traces is zero.

In (5.6) the ordering of non commuting quantities is important. Formulas (5.6) give a realization of the quantum group in terms of exponential quantum group parameters. However, in spite of the great simplicity of the commutation relations (4.4) the group composition laws (5.6) are relatively complicated because of the ordering. We emphasize that, to obtain (5.6), we have used explicitly the 2×2 representation. The reason is that we are dealing here with linear combinations with non commuting coefficients of the generators of a Lie algebra: in general the commutator of two such quantities is not an object of the same kind unless additional algebraic relations (such as nilpotency, square equal to the unit matrix, etc.) are imposed on the Lie algebra generators.

Finally, we consider the limit $q \to 1$, $h \to 0$. In this limit the quantities a, b, c, d of Section 2 commute. With the usual relation between Poisson brackets (for which we use round brackets) and commutators we find

$$
(a, b) = \lim_{h \to 0} \frac{[a, b]}{h} = ab \tag{5.7}
$$

and similarly

$$
\left.
\begin{aligned}
&(a, c) = ac, \quad (b, c) = 0 \\
&(b, d) = bd, \quad (a, d) = 2bc \\
&(c, d) = cd
\end{aligned}
\right\}. \tag{5.8}
$$

If a', b', c', d' have similar Poisson brackets

$$\left.\begin{aligned} (a', b') &= a'b' \\ (a', c') &= a'c' \\ \text{etc.} & \end{aligned}\right\}, \tag{5.9}$$

then a'', b'', c'', d'' given by (2.5) also do, i.e.

$$\left.\begin{aligned} (a'', b'') &= a''b'' \\ (a'', c'') &= a''c'' \\ \text{etc.} & \end{aligned}\right\}. \tag{5.10}$$

In this case one speaks of a Poisson group.

For the exponential description we must first rescale the variables and introduce new quantities $\hat{\lambda}, \hat{\mu}, \hat{\nu}$

$$\hat{\lambda} = h\lambda, \quad \hat{\mu} = h\mu, \quad \hat{\nu} = h\nu, \tag{5.11}$$

so that

$$A = \exp \begin{pmatrix} \hat{\lambda} & \hat{\mu} \\ \hat{\nu} & -\hat{\lambda} \end{pmatrix}. \tag{5.12}$$

In the Poisson limit $\hat{\lambda}, \hat{\mu}$ and $\hat{\nu}$ commute and their Poisson brackets are

$$\left.\begin{aligned} (\hat{\lambda}, \hat{\mu}) = \hat{\mu}, \quad (\hat{\lambda}, \hat{\nu}) = \hat{\nu} \\ (\hat{\mu}, \hat{\nu}) = 0 \end{aligned}\right\}. \tag{5.13}$$

In the limit the composition laws (5.6) become the usual composition laws for the (commuting) exponential parameters of the Lie group and depend now only on the Lie algebra, not on additional properties of the particular representation. In terms of the rescaled variables the composition laws do not contain h. They have, of course, the usual non linear structure typical of exponential parameters but preserve exactly the very simple Poisson relations (5.13).

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