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Results on Invariance-based Feedback Control for Hybrid Dynamical Systems

Jun Chai and Ricardo G. Sanfelice

***Abstract*— We present results for forward invariance-based control of hybrid dynamical systems via static feedback. Using recent results on forward invariance for hybrid systems without inputs, we present conditions on the state-feedback laws to induce forward invariance of a set for hybrid systems with inputs. In addition, we propose a notion of control Lyapunov function (CLF) that is suitable for the study of forward invariance of sublevel sets. Conditions that guarantee the existence of CLF-based feedback laws inducing forward invariance of sublevel sets are established. Examples are given to illustrate the results.**

I. INTRODUCTION

Forward invariance of a set is the property that every solution to a system starting in the set stays in the set. Also referred to as flow-invariance [1], positive invariance [2] and viability [3], such a property is useful when analyzing systems with complex dynamics for which locating the omega-limit of solutions is challenging; for example systems with possibly nonunique solutions and nonlinear set-valued dynamics, and those which potentially combine continuous and discrete behaviors. The work in [2] considers weak and strong notions of forward invariance as well as invariance of sublevel sets for single-valued purely continuous-time and discrete-time systems. Ensuring conditions on a family of Lyapunov like functions, [1] guarantees forward invariant properties of single-valued continuous-time systems with nonunique solutions. For nonlinear single-valued continuous-time systems, [4] proposes generalized Lyapunov functions to define the invariant set.

Relying on forward invariance results, invariance-based control design techniques are widely used tools for controller design for systems with inputs. For example, invariance-based control designs are provided in [5] for power inversion, in [6] for a genetic network, and in [7] for collision avoidance in automotive systems. While tools for the study of forward invariance and viability in hybrid systems can be found in [8], [9], [10], to the best of our knowledge, results enabling the systematic design of feedback controllers inducing forward invariance for hybrid systems are not available.

In this paper, we present results for the design of state-feedback laws inducing forward invariance for hybrid systems that are given in terms of hybrid inclusions. Building on our previous work introducing notions and sufficient conditions for analysis of forward invariance of a set [9], we present results for the existence and synthesis of static state-feedback laws inducing invariance of sets for hybrid systems with inputs $u = (u_c, u_d) \in \mathcal{U}_c \times \mathcal{U}_d \subset \mathbb{R}^{m_c} \times \mathbb{R}^{m_d}$ that are given by

$$\mathcal{H}_u \begin{cases} (x, u_c) \in C_u & \dot{x} \ \in F_u(x, u_c) \\ (x, u_d) \in D_u & x^+ \in G_u(x, u_d), \end{cases} \tag{1}$$

where C_u is the flow set, F_u is the flow map, D_u is the jump set, and G_u is the jump map. The state-feedback laws of interest are given by continuous pairs (κ_c, κ_d), when controlling $\mathcal{H}_u$, lead to the closed-loop hybrid system

$$\mathcal{H} \begin{cases} x \in C & \dot{x} \ \in F(x) := F_u(x, \kappa_c(x)) \\ x \in D & x^+ \in G(x) := G_u(x, \kappa_d(x)), \end{cases} \tag{2}$$

where F is the flow map governing the continuous evolution of the state on the flow set $C := \{x \in \mathbb{R}^n : (x, \kappa_c(x)) \in C_u\}$, and G is the jump map governing the discrete evolution from the jump set $D := \{x \in \mathbb{R}^n : (x, \kappa_d(x)) \in D_u\}$. To provide conditions that guarantee their existence, we employ control Lyapunov-like functions that are tailored to forward invariance. In particular, the proposed existence conditions guarantee invariance of sublevel sets of such functions by assuring the existence of continuous selections. The definitions and results are illustrated by a running example.

The remainder of the paper is organized as follows. Section II lists needed definitions and results for hybrid systems. Results for the design of invariance-inducing feedback laws are given in Section III. Conditions to guarantee existence of such laws using control Lyapunov functions are proposed in Section IV. The proofs of the results will be published elsewhere.

Notation: Given a closed set $S \subset \mathbb{R}^n \times \mathbb{R}^{m_\star}$ for some $\star \in \{c, d\}$, the projection of S onto $\mathbb{R}^n$ is denoted by $\Pi(S) := \{x : \exists u_\star \in \mathbb{R}^{m_\star} \text{ s.t. } (x, u_\star) \in S\}$; given $x \in \mathbb{R}^n$, the set of values $u_\star$ such that $(x, u_\star) \in S$ is denoted as $\widetilde{\Psi}_\star(x, S) := \{u_\star : (x, u_\star) \in S\}$. In addition, given a set $K \subset \mathbb{R}^n$, we define $\widetilde{\Upsilon}_\star(K, S) := \{(x, u_\star) \in S : x \in K, u_\star \in \widetilde{\Psi}_\star(x, S)\}$. The set-valued maps $\Psi_c : \mathbb{R}^n \rightrightarrows \mathcal{U}_c$ and $\Psi_d : \mathbb{R}^n \rightrightarrows \mathcal{U}_d$ are defined for each $x \in \mathbb{R}^n$ as $\Psi_c(x) := \widetilde{\Psi}_c(x, C_u)$ and $\Psi_d(x) := \widetilde{\Psi}_d(x, D_u)$, respectively. Given a set $K \subset \mathbb{R}^n$, we define $\Upsilon_c(K) := \widetilde{\Upsilon}_c(K, C_u)$ and $\Upsilon_d(K) := \widetilde{\Upsilon}_d(K, D_u)$. A closed unit ball around the origin in $\mathbb{R}^n$ is denoted by $\mathbb{B}$. Given a vector x, $|x|$ denotes the 2-norm of x. Given $r \in \mathbb{R}$, the r-sublevel set of a function $V : \mathbb{R}^n \to \mathbb{R}$ is $L_V(r) := \{x \in \mathbb{R}^n : V(x) \leq r\}$. Given a closed set K, we denote the tangent cone of the set K at a point $x \in K$ as $T_K(x)$. Given a set-valued mapping $M : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$, the range of M is denoted as $\operatorname{rge} M = \{y \in \mathbb{R}^n : \exists x \in \mathbb{R}^m \text{ s.t. } y \in M(x)\}$

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and its domain is denoted as $\operatorname{dom} M = \{x \in \mathbb{R}^m : M(x) \neq \emptyset\}$.

II. Preliminaries

In this paper, we follow the hybrid systems framework in [11], in which a closed-loop hybrid system $\mathcal{H}$ is given as in (2). A solution to the hybrid system $\mathcal{H}$ is parameterized by the ordinary time variable $t \in \mathbb{R}_{\geq 0} := [0, \infty)$ and by the discrete counter $j \in \mathbb{N} := \{0, 1, 2, ...\}$, and defined on a hybrid time domain $E \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$; see [11, Definition 2.3]. The set E is a hybrid time domain if, for each $(T, J) \in E$, $E \cap ([0, T] \times \{0, 1, ..., J\})$ can be written as $\cup_{j=0}^{J-1} ([t_j, t_{j+1}], j)$ for some finite sequence of times $0 = t_0 \leq t_1 \leq t_2 \; ... \leq t_J$. A hybrid arc ϕ is a function on a hybrid time domain if, for each $j \in \mathbb{N}$, $t \mapsto \phi(t, j)$ is absolutely continuous on the interval $\{t : (t, j) \in \operatorname{dom} \phi \}$. A solution to $\mathcal{H}$ is a hybrid arc $\phi : \operatorname{dom} \phi \to \mathbb{R}^n$ that satisfies the dynamics of $\mathcal{H}$, where $\operatorname{dom} \phi$ is a hybrid time domain E; see [11, Definition 2.6]. A solution ϕ to $\mathcal{H}$ is said to be *complete* if $\operatorname{dom} \phi$ is unbounded and *maximal* if there does not exist another solution ϕ' such that ϕ is a truncation of ϕ' to some proper subset of $\operatorname{dom} \phi'$. Given a set K, $\mathcal{S}_{\mathcal{H}}(K)$ represents a set including all maximal solutions to system $\mathcal{H}$ that are initialized from set K.

The following regularity conditions on the system data for the closed-loop hybrid system $\mathcal{H}$ will be needed. In addition, they guarantee robustness of stability with respect to perturbations, see [11, Chapter 6] for details.

Definition 2.1: ([11, Assumption 6.5]) A hybrid system $\mathcal{H}$ with state $x \in \mathbb{R}^n$ is said to satisfy the hybrid basic conditions if its data (C, F, D, G) is such that

(A1) C and D are closed sets;

(A2) $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is outer semicontinuous and locally bounded, and $F(x)$ is a nonempty and convex for all $x \in C$;

(A3) $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is outer semicontinuous and locally bounded, and $G(x)$ is a nonempty subset of $\mathbb{R}^n$ for all $x \in D$. □

We recall the following lemma from [12]. This result states conditions on the system data of a hybrid system with inputs $\mathcal{H}_u$ as in (1) and on the state-feedback pair (κ_c, κ_d) that lead to the closed-loop system $\mathcal{H}$ in (2) satisfying (A1)-(A3) in Definition 2.1.

Lemma 2.2: ([12, Lemma 3.2]) Suppose $\kappa_c : \Pi(C_u) \to \mathcal{U}_c$ and $\kappa_d : \Pi(D_u) \to \mathcal{U}_d$ are continuous and $\mathcal{H}_u = (C_u, F_u, D_u, G_u)$ is such that

(A1') C_u and D_u are closed subsets of $\mathbb{R}^n \times \mathcal{U}_c$ and $\mathbb{R}^n \times \mathcal{U}_d$ respectively;

(A2') $F_u : \mathbb{R}^n \times \mathbb{R}^{m_c} \rightrightarrows \mathbb{R}^n$ is outer semicontinuous relative to C_u and locally bounded, and for all $(x, u_c) \in C_u$, $F(x, u_c)$ is nonempty and convex;

(A3') $G_u : \mathbb{R}^n \times \mathbb{R}^{m_d} \rightrightarrows \mathbb{R}^n$ is outer semicontinuous relative to D_u and locally bounded, and for all $(x, u_d) \in D_u$, $G(x, u_d)$ is nonempty.

Then, $\mathcal{H}$ satisfies conditions (A1)-(A3) in Definition 2.1.

In this paper, we focus on the forward invariance notions defined in [9, Definition 2.3 - 2.6].

Definition 2.3: (forward invariance) The set $K \subset \mathbb{R}^n$ is *weakly forward pre-invariant* for $\mathcal{H}$ if for every $x \in K$ there exists at least one maximal solution ϕ with $\operatorname{rge} \phi \subset K$. The set $K \subset \mathbb{R}^n$ is *weakly forward invariant* for $\mathcal{H}$ if for every $x \in K$ there exists at least one complete solution $\phi \in \mathcal{S}_{\mathcal{H}}(x)$ with $\operatorname{rge} \phi \subset K$. The set $K \subset \mathbb{R}^n$ is *forward pre-invariant* for $\mathcal{H}$ if for every $x \in K$ there exists at least one solution, and every maximal solution from K satisfies $\operatorname{rge} \phi \subset K$. The set $K \subset \mathbb{R}^n$ is *forward invariant* for $\mathcal{H}$ if K is forward pre-invariant for $\mathcal{H}$ and every maximal solution $\phi \in \mathcal{S}_{\mathcal{H}}(K)$ is complete. □

Remark 2.4: According to Definition 2.3, forward invariance implies weak forward invariance. In the literature, the weak forward invariance notion in Definition 2.3 is usually associated with the term "viability ," while the forward invariance notion therein is referred to as an "invariance" property; see [3]. In the special case when the system has unique maximal solutions from the set of interest, the two notions are equivalent.

Given a hybrid system $\mathcal{H}$ as in (2) and a set $K \subset \mathbb{R}^n$, the following mild assumptions are imposed in some of our results.

Assumption 2.5: The sets K, C, and D are such that $K \subset \overline{C} \cup D$ and that $K \cap C$ is closed. The map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is outer semicontinuous, locally bounded relative to $K \cap C$, and $F(x)$ is convex for every $x \in K \cap C$.

To derive conditions inducing forward invariance properties, we employ the following lower semicontinuous and Lipschitz properties of set-valued maps.

Definition 2.6: (lower semicontinuous set-valued maps) A set-valued map $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is lower semicontinuous if for every $x \in \mathbb{R}^n$, one has that $\liminf_{x_i \to x} S(x_i) \supset S(x)$, where

$$\liminf_{x_i \to x} S(x_i) := \{z : \forall x_i \to x, \exists z_i \to z \text{ s.t. } z_i \in S(x_i)\}$$

is the inner limit of S (see [13, Chapter 5.B]). □

Definition 2.7: (locally Lipschitz set-valued maps) A set-valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is locally Lipschitz on a set $K \subset \mathbb{R}^n$ if for every $x \in K$, there exist a neighborhood U of x and a constant $\lambda \geq 0$ such that

$$F(x) \subset F(\xi) + \lambda |x - \xi| \mathbb{B} \qquad \forall \xi \in U \cap \operatorname{dom} F.$$

Furthermore, F is locally Lipschitz when it is locally Lipschitz on $\operatorname{dom} F$ (see [14, Chapter 1, Definition 4]). □

III. Invariance-based Control of Hybrid Systems via Static State-Feedback Laws

In this section, we present results on forward invariance properties of a set for hybrid system $\mathcal{H}_u$ given as in (1) under the effect of the state-feedback pair ($\kappa_c : \mathbb{R}^n \to \mathbb{R}^{m_c}, \kappa_d : \mathbb{R}^n \to \mathbb{R}^{m_d}$). More precisely, inspired by [9], we provide conditions on the static state-feedback pair (κ_c, κ_d) that ensure forward invariance of a set for the closed-loop system.

Our first result is for weak forward invariance of a set. By establishing weak forward invariance property of a set, we know the set is "viable," in the sense that from every initial condition in the set, at least one complete solution never leaves it.

Proposition 3.1: (weak invariance-based control) The set $K \subset \mathbb{R}^n$ is weakly forward invariant for the closed-loop hybrid system $\mathcal{H} = (C, F, D, G)$ in (2)*, obtained from controlling the hybrid system $\mathcal{H}_u$ in* (1) *by a continuous static state-feedback pair (κ_c, κ_d), if the following conditions hold:*

1.1) $K, C, F,$ and D satisfy Assumption 2.5;
1.2) For every $x \in K \cap D, G_u(x, \kappa_d(x)) \cap K \neq \emptyset$;
1.3) For every $x \in K \setminus D, F_u(x, \kappa_c(x)) \cap T_{K \cap C}(x) \neq \emptyset$;
1.4) $K \cap C$ is compact or $F(x)$ is bounded on $K \cap C$.

The following conditions guarantee forward invariance of a set for a given hybrid system with inputs and state-feedback pair. When a state-feedback pair renders a set forward invariant for the closed-loop system, all maximal solutions that start from such set are complete and stay within it. This is a desired property for many control problems.

Proposition 3.2: (invariance-based control) The set $K \subset \mathbb{R}^n$ is forward invariant for the closed-loop hybrid system $\mathcal{H} = (C, F, D, G)$ in (2)*, obtained from controlling the hybrid system $\mathcal{H}_u$ in* (1) *by a continuous static state-feedback pair (κ_c, κ_d), if the following conditions hold:*

2.1) $K, C, F,$ and D satisfy Assumption 2.5;
2.2) For every $x \in K \cap D, G_u(x, \kappa_d(x)) \subset K$;
2.3) For every $x \in K \cap C, F_u(x, \kappa_c(x)) \subset T_{K \cap C}(x)$;
2.4) $K \cap C$ is compact or $F(x)$ is bounded on $K \cap C$;
2.5) $x \mapsto F(x)$ is locally Lipschitz on $K \cap C$.

We illustrate Proposition 3.2 with the following example.

Example 3.3: (nonlinear planar system with jumps) Consider a hybrid system with inputs $\mathcal{H}_u$ given as in (1) with the following data:[1]

$$
\begin{aligned}
&F_u(x, u_c) := \left\{ \begin{bmatrix} x_1^2 - \gamma \\ x_1 x_2 \end{bmatrix} u_c : \gamma \in [3, 4] \right\}, \\
&G_u(x, u_d) := \{-R(u_d)x, R(u_d)x\}, \\
&C_u := \{(x, u_c) \in \mathbb{R}^2 \times \mathbb{R} : |x| \geq 1, |x_1| \geq |u_c|, \\
&\qquad\qquad (|x|^2 - 2)x_1^2 \leq u_c x_1 \leq (|x|^2 - 1)x_1^2\}, \\
&D_u := \left\{(x, u_d) \in \mathbb{R}^2 \times \mathbb{R} : x_1 = 0, |x| \geq 1, u_d \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right]\right\}.
\end{aligned}
$$

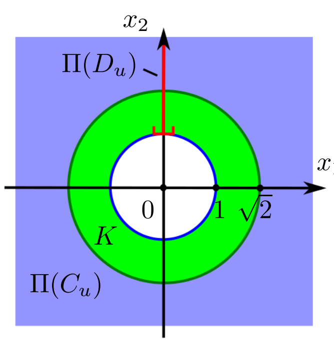

Fig. 1: Sets in Example 3.3.

[1] $R(s) = \begin{bmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{bmatrix}$ represents a rotation matrix.

We consider the set $K = \{x \in \mathbb{R}^2 : 1 \leq |x| \leq \sqrt{2}\}$, and a continuous state-feedback pair (κ_c, κ_d) given by

$$\kappa_c(x) = \left(|x|^2 - \frac{3}{2}\right) x_1 \text{ and } \kappa_d(x) = \frac{\pi}{3}.$$

Conditions *2.1)* and *2.4)* in Proposition 3.2 hold by construction of $\mathcal{H}$ and K. By definition of F_u and κ_c, we have

$$F(x) := \left\{ \begin{bmatrix} x_1^2 - \gamma \\ x_1 x_2 \end{bmatrix} \left(|x|^2 - \frac{3}{2}\right) x_1 : \gamma \in [3, 4] \right\},$$

which is Lipschitz on the set $C \cap K$. Since $C \cap K = K$ is closed, by definition of tangent cone, for each $x \in K$, we have

$$T_K(x) = \begin{cases} \mathbb{R}^2 & \text{if } x \in \text{int } K, \\ \{\omega \in \mathbb{R}^2 : \langle \nabla V(x), \omega \rangle \geq 0\} \\ \qquad \text{if } x \in K_1 := \{x \in \mathbb{R}^2 : |x| = 1\}, \\ \{\omega \in \mathbb{R}^2 : \langle \nabla V(x), \omega \rangle \leq 0\} \\ \qquad \text{if } x \in K_2 := \{x \in \mathbb{R}^2 : |x| = \sqrt{2}\}, \end{cases}$$

where $V : \mathbb{R}^2 \to \mathbb{R}$ is the continuously differentiable function $V(x) := x_1^2 + x_2^2$ for every $x \in \mathbb{R}^2$. For every $x \in \text{int } K$, $F(x) \subset \mathbb{R}^2$; for every $x \in K_1$ and $\xi \in F(x)$, we have

$$\langle \nabla V(x), \xi \rangle = 2x_1\xi_1 + 2x_2\xi_2 = (\gamma - 1)x_1^2,$$

which is nonnegative since $\gamma \in [3, 4]$; and for every $x \in K_2$ and $\xi \in F(x)$, we have

$$\langle \nabla V(x), \xi \rangle = 2x_1\xi_1 + 2x_2\xi_2 = (2 - \gamma)x_1^2 \leq 0.$$

Therefore, condition *2.3)* holds. Condition *2.2)* holds because the rotation matrix R only changes the direction of x, while its magnitude remains the same after each jump. Thus, by an application of Proposition 3.2, the given state-feedback pair (κ_c, κ_d) renders the set K forward invariant for the closed-loop system $\mathcal{H}$. △

IV. Invariance-based Control for Hybrid Systems via State-Feedback Laws Using CLFs

In this section, using control Lyapunov functions (CLFs), we present results on the existence of invariance-based control laws for hybrid systems $\mathcal{H}_u$ as in (1). The definition of a CLF for forward invariance for $\mathcal{H}_u$ is given as follows.

Definition 4.1: (CLFs for forward invariance) Given sets $\mathcal{U}_c \subset \mathbb{R}^{m_c}$, $\mathcal{U}_d \subset \mathbb{R}^{m_d}$, a constant $r \in \mathbb{R}$ and a continuous function $V : \mathbb{R}^n \to \mathbb{R}$ that is continuously differentiable on an open set containing $\overline{\Pi(C_u)}$, if

$$\inf_{u_c \in \Psi_c(x)} \sup_{\xi \in F_u(x, u_c)} \langle \nabla V(x), \xi \rangle \leq 0 \qquad \forall x \in L_V(r) \cap \Pi(C_u), \tag{3}$$

$$\inf_{u_d \in \Psi_d(x)} \sup_{\xi \in G_u(x, u_d)} V(\xi) - V(x) \leq 0 \qquad \forall x \in L_V(r) \cap \Pi(D_u), \tag{4}$$

then, the pair (V, r) defines a control Lyapunov function for forward invariance of the r-sublevel set of V with $\mathcal{U} = \mathcal{U}_c \times \mathcal{U}_d$ controls for $\mathcal{H}_u = (C_u, F_u, D_u, G_u)$. □

Next, we use a variation of Example 3.3 to illustrate this definition.

Example 4.2: (nonlinear planar system with jumps revised) Consider a hybrid system with inputs $\mathcal{H}_u$ given as in (1) with the following data:

$$
\begin{aligned}
&F_u(x,u_c) := \left\{ \begin{bmatrix} x_1^2-1 \\ \alpha x_1 x_2 \end{bmatrix} u_c : \alpha \in [1,2] \right\}, \\
&G_u(x,u_d) := \{\beta R(u_d)x : \beta \in [0,1]\} \\
&C_u := \{(x,u_c) \in \mathbb{R}^2 \times \mathbb{R} : |x| \geq 1, |x_1| \geq |u_c|, \\
&\qquad\qquad (|x|^2-3)x_1^2 \leq u_c x_1 \leq (|x|^2-2)x_1^2\}, \\
&D_u := \left\{ (x,u_d) \in \mathbb{R}^2 \times \mathbb{R} : x_1 = 0, |x| \geq 1, u_d \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right] \right\}.
\end{aligned}
$$

Thus, the set-valued maps Ψ_c and Ψ_d are given as follows:

$$
\Psi_c(x) = \begin{cases} [(|x|^2-3)x_1, (|x|^2-2)x_1] \\ \qquad\qquad \text{if } x \in \Pi(C_u), x_1 > 0, \\ [(|x|^2-2)x_1, (|x|^2-3)x_1] \\ \qquad\qquad \text{if } x \in \Pi(C_u), x_1 < 0, \\ \{0\} \qquad \text{if } x \in \Pi(C_u), x_1 = 0, \\ \emptyset \qquad \text{otherwise,} \end{cases} \tag{5}
$$

$$
\Psi_d(x) = \begin{cases} \left[\frac{\pi}{4}, \frac{\pi}{2}\right] & \text{if } x \in \Pi(D_u), \\ \emptyset & \text{otherwise.} \end{cases} \tag{6}
$$

Consider the candidate pair (V, r) given by a function $V : \mathbb{R}^2 \to \mathbb{R}$, where $V(x) = x_1^2 + x_2^2$ for each $x \in \mathbb{R}^2$, and $r = 2$. Note that V is continuously differentiable. We have $L_V(2) = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 2\}$, and $\Pi(C_u) = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 \geq 1\}$. Then, for every $x \in L_V(r) \cap \Pi(C_u) = \{x \in \mathbb{R}^2 : 1 \leq x_1^2 + x_2^2 \leq 2\}$ and each $u_c \in \Psi_c(x)$, we have

$$
\begin{aligned}
\sup_{\xi \in F_u(x,u_c)} \langle \nabla V(x), \xi \rangle &= \sup_{\xi \in F_u(x,u_c)} (2x_1\xi_1 + 2x_2\xi_2) \\
&= \max_{\alpha \in [1,2]} 2(|x|^2 - 1 + (\alpha-1)x_2^2)x_1 u_c.
\end{aligned} \tag{7}
$$

By (5), for every $x \in L_V(r) \cap \Pi(C_u)$, it is the case that $u_c x_1 \leq 0$. Hence, for every $x \in L_V(r) \cap \Pi(C_u)$ and every $u_c \in \Psi_c(x)$, the expression in (7) is maximum when $\alpha = 1$, and

$$
2x_1 u_c \leq \sup_{\xi \in F_u(x,u_c)} \langle \nabla V(x), \xi \rangle \leq 0.
$$

Thus, (3) holds. With given jump dynamics, for each $x \in L_V(r) \cap \Pi(D_u) = \{x \in \mathbb{R}^2 : 1 \leq x_2 \leq \sqrt{2}, x_1 = 0\}$, since $\beta \in [0,1]$, we have

$$
\begin{aligned}
\sup_{\xi \in G_u(x,u_d)} V(\xi) - V(x) &= (\xi_1^2 + \xi_2^2) - (x_1^2 + x_2^2) \\
&= (\beta(\cos(u_d)x_1 + \sin(u_d)x_2))^2 \\
&\quad + (\beta(-\sin(u_d)x_1 + \cos(u_d)x_2))^2 - (x_1^2 + x_2^2) \\
&= (\beta^2 - 1)(x_1^2 + x_2^2) \leq 0,
\end{aligned}
$$

which is independent of the choice for u_d, i.e., (4) holds. △

As stated in Section II, to have a well-posed closed-loop system $\mathcal{H}$ that satisfies conditions (A1)-(A3) in Definition 2.1, we require a continuous feedback-pair (κ_c, κ_d). Hence, given a pair (V, r) for $\mathcal{H}_u$ that satisfies conditions (A1')-(A3') in Lemma 2.2, we study the existence of a continuous state-feedback pair (κ_c, κ_d) inducing forward invariance properties of

$$
\mathcal{M}_r := L_V(r) \cap (\Pi(C_u) \cup \Pi(D_u)) \tag{8}
$$

for the resulting closed-loop system $\mathcal{H}$. For each $(x, u_c) \in \mathbb{R}^n \times \mathbb{R}^{m_c}$, we define the function

$$
\Gamma_c(x,u_c) := \begin{cases} \sup_{\xi \in F_u(x,u_c)} \langle \nabla V(x), \xi \rangle \\ \qquad\qquad \text{if } (x,u_c) \in C_u \cap (\mathcal{M}_r \times \mathbb{R}^{m_c}), \\ -\infty \qquad \text{otherwise} \end{cases}
$$

and for each $(x, u_d) \in \mathbb{R}^n \times \mathbb{R}^{m_d}$, we define the function

$$
\Gamma_d(x,u_d) := \begin{cases} \sup_{\xi \in G_u(x,u_d)} V(\xi) - V(x) \\ \qquad\qquad \text{if } (x,u_d) \in D_u \cap (\mathcal{M}_r \times \mathbb{R}^{m_d}), \\ -\infty \qquad \text{otherwise.} \end{cases}
$$

A. Existence of a State-feedback Pair for Weak Forward Invariance

This section pertains to weak forward invariance of $\mathcal{M}_r$ for $\mathcal{H}_u$ under the effect of a continuous state-feedback pair (κ_c, κ_d). For every $x \in \mathcal{M}_r$, we consider sets that include all inputs that keep "some solutions" to the system within $\mathcal{M}_r$ (see Section IV-B for the case of "all solutions"). More precisely, with $\Pi(C_u)$ closed, for every $x \in \Pi(C_u)$ we define

$$
\Theta_c(x) := \{u_c \in \Psi_c(x) : F_u(x,u_c) \cap T_{\Pi(C_u)}(x) \neq \emptyset\}; \tag{9}
$$

and for every $x \in \Pi(D_u)$, we define

$$
\begin{aligned}
\Theta_d(x) := \{&u_d \in \Psi_d(x) : \\
&G_u(x,u_d) \cap (\Pi(C_u) \cup \Pi(D_u)) \neq \emptyset\}.
\end{aligned} \tag{10}
$$

The following result presents conditions for the existence of a continuous feedback pair (κ_c, κ_d) inducing weak forward invariance of $\mathcal{M}_r$ for $\mathcal{H}_u$.

Theorem 4.3: (existence of state-feedback pair for weak forward invariance) Given a hybrid system $\mathcal{H}_u$ as in (1) *satisfying conditions (A1')-(A3') in Lemma 2.2, suppose there exists a pair (V, r) for forward invariance of r-sublevel sets with $\mathcal{U}$ controls for $\mathcal{H}_u$. Furthermore, suppose the following conditions hold:*

- *R1) The set-valued maps Ψ_c, Ψ_d are lower semicontinuous, and $\operatorname{gph}\Theta_c, \operatorname{gph}\Theta_d$ in* (9) *and* (10) *are open relative to $\operatorname{gph}\Psi_c, \operatorname{gph}\Psi_d$, respectively;*
- *R2) For every $x \in \mathcal{M}_r \cap \Pi(C_u), \Theta_c(x)$ is nonempty and convex, and for every $x \in \mathcal{M}_r \cap \Pi(D_u), \Theta_d(x)$ is nonempty and convex;*
- *R3) For every $x \in \mathcal{M}_r \cap \Pi(C_u)$, the map $u_c \mapsto \Gamma_c(x,u_c)$ is convex on $\Psi_c(x)$ and, for every $x \in \mathcal{M}_r \cap \Pi(D_u)$, the map $u_d \mapsto \Gamma_d(x,u_d)$ is convex on $\Psi_d(x)$.*

Then, there exists a continuous state-feedback pair (κ_c, κ_d) rendering the set $\mathcal{M}_r$ in (8) *weakly forward pre-invariant for $\mathcal{H} = (C, F, D, G)$ as in* (2)*. Furthermore, if either F is bounded on $\mathcal{M}_r \cap C$ or $\mathcal{M}_r \cap C$ is compact, $\mathcal{M}_r$ is weakly forward invariant for $\mathcal{H}$.*

Remark 4.4: Note that Theorem 4.3 provides sufficient conditions for the existence of continuous state feedback pair (κ_c, κ_d) to render the set $\mathcal{M}_r$ weakly forward invariant for $\mathcal{H}$. Thus, some solutions to $\mathcal{H}$ starting in $\mathcal{M}_r$ might leave this set. In such a case, while the Lyapunov inequalities in Definition 4.1 guarantee that solutions stay within $L_V(r)$, they may leave the set $\mathcal{M}_r$ by either jumping or flowing out of $C \cup D$.

Next, we illustrate Theorem 4.3 in an example.

Example 4.5: (nonlinear planar system with jumps revised revisited) Consider the hybrid system and the pair (V, r) in Example 4.2. It is easy to verify that $\mathcal{H}_u$ satisfies (A1')-(A3') in Lemma 2.2. The (trivial) extension to $\mathbb{R}^n$ of the set-valued maps Ψ_c and Ψ_d in (5) and (6) are lower semicontinuous by construction. By the definition of Θ_c given in (9), we verify that $\Theta_c = \Psi_c$, i.e., for every $x \in \Pi(C_u)$ and every $u_d \in \Psi_c(x)$, there exist $\xi \in F_u(x, u_c)$, such that $F_u(x, u_c) \cap T_{\Pi(C_u)}(x) \neq \emptyset$. Since $\Pi(C_u)$ is closed, for each $x \in \Pi(C_u) = \{x \in \mathbb{R}^2 : |x| \geq 1\}$, the tangent cone is given by

$$T_{\Pi(C_u)}(x) = \begin{cases} \mathbb{R}^2 & \text{if } x \in \text{int } \Pi(C_u), \\ \{\omega \in \mathbb{R}^2 : \langle \nabla V(x), \omega \rangle \geq 0\} & \\ & \text{if } x \in \partial \Pi(C_u), \end{cases}$$

where $V(x) = x_1^2 + x_2^2$ for each $x \in \mathbb{R}^2$. For every $x \in \text{int } \Pi(C_u)$, trivially, $F_u(x, u_c) \subset \mathbb{R}^2$. For every $x \in \Pi(C_u)$ and $\xi \in F_u(x, u_c)$, we have

$$\begin{aligned} \langle \nabla V(x), \xi \rangle &= 2x_1\xi_1 + 2x_2\xi_2 \\ &= 2(|x|^2 - 1 + (\alpha - 1)x_2^2)x_1 u_c. \end{aligned}$$

Then, when $u_c \in \Psi_c(x)$, by definition of Ψ_c as in (5), every $x \in \{x \in \mathbb{R}^2 : |x| = 1\}$ satisfies the inequality

$$-4(\alpha - 1)x_2^2 x_1^2 \leq \langle \nabla V(x), \xi \rangle \leq -(\alpha - 1)x_2^2 x_1^2.$$

Hence, for every $(x, u_c) \in \Upsilon_c(\partial \Pi(C_u))$,$\langle \nabla V(x), \xi \rangle = 0$ when $\alpha = 1$. Thus, $\Theta_c(x) = \Psi_c(x)$ for every $x \in \Pi(C_u)$. Similarly, for every $x \in \Pi(D_u)$, $\Theta_d(x) = \Psi_d(x)$. This can be checked by noticing that the rotation matrix $R(u_d)$ does not effect the 2-norm of x at jumps. Then, since $\Theta_\star = \Psi_\star$, condition *R1)* in Theorem 4.3 holds. Moreover, with $\mathcal{M}_r \cap \Pi(C_u) = \{x \in \mathbb{R}^2 : 1 \leq x_1^2 + x_2^2 \leq 2\}$ and $\mathcal{M}_r \cap \Pi(D_u) = \{x \in \mathbb{R}^2 : 1 \leq x_2 \leq \sqrt{2}, x_1 = 0\}$, *R2)* holds. In addition, condition *R3)* holds since F_u and G_u are convex functions of u_c and u_d, respectively. Then, by Theorem 4.3, there exist a state-feedback pair (κ_c, κ_d) that renders the set $\mathcal{M}_r = \{x \in \mathbb{R}^2 : 1 \leq x_1^2 + x_2^2 \leq 2\}$ weakly forward invariant for the closed-loop system. One such continuous state-feedback pair is defined by $\kappa_c(x) = (x_1^2 + x_2^2 - \frac{5}{2})x_1$ for every $x \in \Pi(C_u)$, and $\kappa_d(x) = \frac{\pi}{3}$, for every $x \in \Pi(D_u)$. Note that some of the maximal solutions from $\mathcal{M}_r$ with such state-feedback pair leave $\mathcal{M}_r$. △

B. Existence of a State-feedback Pair for Forward Invariance

In addition to the weak forward invariance result, to get the stronger forward invariance property in Definition 2.3, we further assume that the flow map is locally Lipschitz. To get Lipschitz continuity of F in (2), we consider a locally Lipschitz set-valued map $F_u(x, u)$ under the effect of a locally Lipschitz state-feedback κ_c. The following result is immediate.

Lemma 4.6: (Lipschitzness of F with state-feedback κ_c) Suppose $F_u : S_1 \times S_2 \rightrightarrows S_1$ is locally Lipschitz (as a set-valued map) and $\kappa_c : S_1 \to S_2$ is locally Lipschitz (as a function). Then, $F := F_u(x, \kappa_c)$ is locally Lipschitz on S_1 (as a set-valued map).

Since forward invariance requires that every solution to $\mathcal{H}$ stays in $\mathcal{M}_r$, we define the following two set-valued maps. With $\Pi(C_u)$ closed, for each $x \in \Pi(C_u)$, we define

$$\widetilde{\Theta}_c(x) := \{u_c \in \Psi_c(x) : F_u(x, u_c) \subset T_{\Pi(C_u)}(x)\},$$

and for each $x \in \Pi(D_u)$,

$$\widetilde{\Theta}_d(x) := \{u_d \in \Psi_d(x) : G_u(x, u_d) \subset (\Pi(C_u) \cup \Pi(D_u))\}.$$

Then, the following proposition establishes conditions that guarantee the existence of a continuous state-feedback pair (κ_c, κ_d) for $\mathcal{H}_u$ to render the set $\mathcal{M}_r$ forward invariant.

Theorem 4.7: (existence of state-feedback pair for forward invariance) Given a hybrid system $\mathcal{H}_u$ as in (1) satisfying conditions (A1')-(A3') in Lemma 2.2, suppose there exists a pair (V, r) for forward invariance of r-sublevel sets with $\mathcal{U}$ controls for $\mathcal{H}_u$. Furthermore, suppose the following conditions hold:

- *R1') The set-valued map $\widetilde{\Theta}_c$ is locally Lipschitz on $\Pi(C_u)$; the set-valued map Ψ_d is lower semicontinuous, and $\text{gph}\, \widetilde{\Theta}_d$ is open relative to $\text{gph}\, \Psi_d$;*
- *R2') For every $x \in \mathcal{M}_r \cap \Pi(C_u)$, $\widetilde{\Theta}_c(x)$ is nonempty, compact and convex; and for every $x \in \mathcal{M}_r \cap \Pi(D_u)$, $\widetilde{\Theta}_d(x)$ is nonempty and convex;*
- *R3') For every $x \in \mathcal{M}_r \cap \Pi(C_u)$, the function $u_c \mapsto \Gamma_c(x, u_c)$ is convex and locally Lipschitz on $\Psi_c(x)$, for every $x \in \Pi(D_u) \cap \mathcal{M}_r$, the function $u_d \mapsto \Gamma_d(x, u_d)$ is convex on $\Psi_d(x)$;*
- *R4) The flow map F_u is locally Lipschitz on C_u.*

Then, there exists a continuous state-feedback pair (κ_c, κ_d) rendering the set $\mathcal{M}_r$ in (8) forward pre-invariant for the closed-loop system $\mathcal{H} = (C, F, D, G)$. Furthermore, if either F is bounded on $\mathcal{M}_r \cap C$ or $\mathcal{M}_r \cap C$ is compact, $\mathcal{M}_r$ is forward invariant for $\mathcal{H}$.

The following example illustrates Theorem 4.7.

Example 4.8: (nonlinear planar system with jumps revisited) Consider the hybrid system with inputs $\mathcal{H}_u$ in Example 3.3. Note that $\mathcal{H}_u$ satisfies (A1')-(A3') in Lemma 2.2. Consider the pair (V, r) in Example 4.2. The set-valued maps Ψ_c and Ψ_d are given as follows:

$$\Psi_c(x) = \begin{cases} [(|x|^2 - 2)x_1, (|x|^2 - 1)x_1] & \\ & \text{if } x \in \Pi(C_u), x_1 > 0, \\ [(|x|^2 - 1)x_1, (|x|^2 - 2)x_1] & \\ & \text{if } x \in \Pi(C_u), x_1 < 0, \\ \{0\} & \text{if } x \in \Pi(C_u), x_1 = 0, \\ \emptyset & \text{otherwise,} \end{cases} \tag{11}$$

$$\Psi_d(x) = \begin{cases} \left[\frac{\pi}{4}, \frac{\pi}{2}\right] & \text{if } x \in \Pi(D_u), \\ \emptyset & \text{otherwise.} \end{cases}$$

We check condition (3). For every $x \in L_V(r) \cap \Pi(C_u) = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 = 2\}$, we have

$$\begin{aligned} \sup_{\xi \in F_u(x,u_c)} \langle \nabla V(x), \xi \rangle &= \sup_{\xi \in F_u(x,u_c)} 2x_1\xi_1 + 2x_2\xi_2 \\ &= \max_{\gamma \in [3,4]} 2x_1 u_c(|x|^2 - \gamma). \end{aligned} \tag{12}$$

We have the following two cases:

- For every $x \in L_V(r) \cap \Pi(C_u), u_c \in \Psi_c(x)$ such that $u_c x_1 \geq 0$, the expression in (12) is maximum when $\gamma = 3$, and
$$\sup_{\xi \in F_u(x,u_c)} \langle \nabla V(x), \xi \rangle \leq -2x_1 u_c \leq 0;$$
- For every $x \in L_V(r) \cap \Pi(C_u), u_c \in \Psi_c(x)$ such that $u_c x_1 < 0$, the expression in (12) is maximum when $\gamma = 4$, and
$$\sup_{\xi \in F_u(x,u_c)} \langle \nabla V(x), \xi \rangle \geq -6x_1 u_c > 0.$$

By definition of Ψ_c in (5), we have for every $x \in L_V(r) \cap \Pi(C_u)$,

$$\inf_{u_c \in \Psi_c(x)} \sup_{\xi \in F_u(x,u_c)} \langle \nabla V(x), \xi \rangle = \min_{u_c \in \Psi_c(x)} -2x_1 u_c,$$

which is nonpositive for all x since for each such x, there exists u_c such that $u_c x_1 \geq 0$. Then, (3) holds.

Then, as in Example 4.5, we find that $\widetilde{\Theta}_\star = \Psi_\star$. More precisely, for every $x \in \operatorname{int} \Pi(C_u), T_{\Pi(C_u)}(x) = \mathbb{R}^2$ and $F_u(x, u_c) \subset \mathbb{R}^2$; for every $x \in \partial\Pi(C_u)$, and $\xi \in F_u(x, u_c)$, we have

$$\langle \nabla V(x), \xi \rangle = 2x_1\xi_1 + 2x_2\xi_2 = 2x_1 u_c(1 - \gamma).$$

Then, when $u_c \in \Psi_c(x)$, by definition of $\Psi_c(x)$ as in (11), every $x \in \{x \in \mathbb{R}^2 : |x| = 1\}$ satisfies the inequality

$$2(\gamma - 1)x_1^2 \geq \langle \nabla V(x), \xi \rangle \geq 0.$$

Therefore, $\widetilde{\Theta}_c = \Psi_c$ for every $x \in \Pi(C_u)$, and $\widetilde{\Theta}_c$ is locally Lipschitz on $\Pi(C_u)$. Similarly for the jump dynamics, Ψ_d is lower semicontinuous by construction, we have $\widetilde{\Theta}_d = \Psi_c$ for every $x \in \Pi(D_u)$, thus, $\operatorname{gph} \widetilde{\Theta}_d$ is open relative to $\operatorname{gph} \Psi_d$. Hence, *R1')* and *R2')* in Theorem 4.7 hold. Then, condition *R3')* holds, because F_u and G_u are convex functions of u_c and u_d, respectively. Moreover, *R4)* holds by construction of F_u. Hence, by Theorem 4.7, there exists a state-feedback pair (κ_c, κ_d) that renders the set $\mathcal{M}_r = \{x \in \mathbb{R}^2 : 1 \leq x_1^2 + x_2^2 \leq 2\}$ forward invariant for the closed-loop system. One such particular state-feedback pair is given in Example 3.3, which confirm the findings by Proposition 3.2. △

Condition *R1')* in Theorem 4.7 is somewhat restrictive in the sense that it requires a locally Lipschitz property of $\widetilde{\Theta}_c$ rather than of the general input projection Ψ_c. This is due to the fact that, in general, intersections of locally Lipschitz maps are not Lipschitz. However, as the following lemma suggests, it is possible to relax that condition.

Lemma 4.9: In Theorem 4.7, when either

1) for each $x \in \Pi(C_u), F_u(x, u_c) \subset T_{\Pi(C_u)}(x)$ for each $u_c \in \Psi_c(x)$; or

2) there exist Lipschitz functions $\gamma : \Pi(C_u) \to \mathbb{R}_{>0}$ and $\varepsilon : \Pi(C_u) \to (0, 1)$ such that $\Psi_c(x) \cap \varepsilon(x)r(x)\mathbb{B} \neq \emptyset$, and for every $x \in \Pi(C_u), \widetilde{\Theta}_c(x) = \Psi_c(x) \cap r(x)\mathbb{B}$, condition R1') in Theorem 4.7 can be replaced by

R1) The set-valued map Ψ_d is lower semicontinuous, $\operatorname{gph} \widetilde{\Theta}_d$ is open relative to $\operatorname{gph} \Psi_d$, and the set-valued map Ψ_c is locally Lipschitz.*

V. CONCLUSION

In this paper, building from previous work on forward invariance properties of hybrid systems without inputs, we presented conditions for the design of invariance-based static state-feedback controllers for hybrid dynamical systems. Using a novel concept of control Lyapunov functions for forward invariance, regulation maps were built to ensure the existence of a (Lipschitz) continuous state-feedback law that leads to forward invariance properties of sublevel sets. This work is part of ongoing research on the construction of invariance-based control laws using selection theorems that guarantee optimality.

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