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Elastic Dynamic Response of Axisymmetric Structures

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ELASTIC DYNAMIC RESPONSE OF AXISYMMETRIC STRUCTURES

by
EDWARD L. WILSON

REPORT TO
WATERWAYS EXPERIMENT STATION
U.S. ARMY CORPS OF ENGINEERS

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STRUCTURAL ENGINEERING LABORATORY
COLLEGE OF ENGINEERING
UNIVERSITY OF CALIFORNIA
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INTRODUCTION

Previous application of the finite element idealization in the blast analysis of axisymmetric solids has been conducted utilizing the SLAM code [1]. The SLAM code is a large capacity program utilizing lower-order finite elements. Because of its many complex options the program is not readily used by engineers in the dynamic analysis of complex structures.

In reference [2] the finite element method coupled with a stable step-by-step integration procedure was used for the elastic dynamic analysis of two-dimensional plane strain solids. The computer program developed was machine independent and was designed to be used in the dynamic analysis of day-to-day engineering problems. The purpose of this investigation is to extend the step-by-step approach to the analysis of axisymmetric structures. In addition, new coding techniques have been introduced in order to increase the capacity of the program. Also, a higher order axisymmetric finite element has been developed which has significantly increased the accuracy of the procedure for the same numerical effort. Since the axisymmetric formulation reduces the plane strain case for a large radius, the present axisymmetric computer program replaces the program given in reference [2].

DYNAMIC EQUILIBRIUM EQUATIONS

The force equilibrium of a system of structural elements is expressed by the following matrix equation:

$$\underline{M}\,\ddot{\underline{U}}_t + \underline{C}\,\dot{\underline{U}}_t + \underline{K}\,\underline{U}_t = \underline{P}_t \qquad (1)$$

where $\underline{U}$, $\dot{\underline{U}}$, and $\ddot{\underline{U}}$ are vectors of nodal point displacements, velocities, and accelerations at time "t". The formation of the stiffness matrix $\underline{K}$ for an axisymmetric finite element system is discussed in the Appendix A.

A formal mathematical development of the mass matrix $\underline{M}$ is possible. Such an approach would result in a mass matrix with the same coupling properties as the stiffness matrix. However, if the physical lumped mass approximation is made the mass matrix will be diagonal. The lumped mass approximation results in a small reduction in accuracy and a considerable saving in computer storage and time. In this investigation one-fourth the mass of each quadrilateral is assumed to be concentrated at each of the four nodal points.

For most structures the exact form of the damping matrix $\underline{C}$ is unknown. In the solution procedure the damping matrix may be completely arbitrary; however, there is little experimental justification for selecting specific damping coefficients. A form of viscous damping, which is sufficiently general for most structures, is given by the following matrix equation:

$$\underline{C} = \alpha\underline{M} + \beta\underline{K} \qquad (2)$$

By assuming the damping matrix is proportional to the mass and stiffness matrices the effect of viscous damping is included without requiring additional storage within the computer program.

STEP-BY-STEP INTEGRATION OF EQUILIBRIUM EQUATIONS

The dynamic equilibrium of the finite element system is given by Eq. 1. The solution of this set of second order differential equations is accomplished by a step-by-step procedure [3]. The only approximation which is made is that the acceleration of each point in the system varies linearly within a small time interval, Δt. This assumption leads to a parabolic variation of velocity and a cubic variation of displacement within the time interval $t - \Delta t$ and t.

A direct integration over the interval gives the following equations for acceleration and velocity at the end of the time interval:

$$\ddot{\underline{U}}_t = \frac{6}{\Delta t^2} \underline{U}_t - \frac{6}{\Delta t^2} \underline{U}_{t-\Delta t} - \frac{6}{\Delta t} \dot{U}_{t-\Delta t} - 2\, \ddot{\underline{U}}_{t-\Delta t} \tag{3}$$

$$\dot{\underline{U}}_t = \frac{3}{\Delta t} \underline{U}_t - \frac{3}{\Delta t} \underline{U}_{t-\Delta t} - 2\, \dot{\underline{U}}_{t-\Delta t} - \frac{t}{2} \ddot{\underline{U}}_{t-\Delta t} \tag{4}$$

The substitution of Eqs. (2), (3) and (4) into the equilibrium relationship, Eq. (1), results in a set of linear equations in terms of the unknown vector $\underline{U}_t$. A solution of this set of equations yields the displacements of the system at time t. The acceleration and velocities may then be found from Eqs. (3) and (4). This procedure may then be repeated for subsequent time steps.

STABILITY OF THE STEP-BY-STEP METHOD

The previously described step-by-step integration technique is accurate if the time step is small compared to the shortest period of the finite element system. If the time step is long compared to the shortest period, the method will become unstable and fail to produce realistic results. Newmark [4] has studied this instability and has suggested a constant acceleration method. Newmark's procedure was found to be stable when applied to finite element systems; however, spurious finite oscillations associated with the high frequencies of the system were still present in the results. Several other stable step-by-step methods were investigated with respect to finite element systems; the method found to be completely stable was a modification of the previously described linear acceleration method.

The instability in the linear acceleration method is first initiated by an oscillation of the displacements about the true solution. In the early stages of instability it is apparent that the displacements at the center of the time interval are a good approximation of the true solution. Therefore, if this mid-point solution is utilized, the tendency for oscillations to develop is eliminated.

In order to modify the previous step-by-step equations to reflect this approach a time increment of 2 Δt is introduced and the acceleration, $\ddot{U}_{t + \Delta t}$, at the end of the time interval is calculated. The midpoint acceleration is calculated as

$$\ddot{\underline{U}}_t = \frac{1}{2}[\ddot{\underline{U}}_{t-\Delta t} + \ddot{\underline{U}}_{t+\Delta t}] \qquad (5)$$

The velocities and displacements at time "t" are calculated from

$$\dot{\underline{U}}_t = \dot{\underline{U}}_{t-\Delta t} + \frac{\Delta t}{2} \ddot{\underline{U}}_{t-\Delta t} + \frac{\Delta t}{2} \ddot{\underline{U}}_t \tag{6}$$

$$\underline{U}_t = \underline{U}_{t-\Delta t} + \Delta t \, \dot{\underline{U}}_{t-\Delta t} + \frac{\Delta t^2}{3} \underline{U}_{t-\Delta t} + \frac{\Delta t^2}{6} \ddot{\underline{U}}_t \tag{7}$$

This modification eliminates all stability problems from the linear acceleration method. However, the new procedure tends to introduce damping in the higher frequencies of the system. Fortunately, this partial truncation of the higher modes is justified in many dynamic analyses. The selection of the time step and the finite element idealization for a particular problem will depend on the experience of the user with similar problems.

The step-by-step procedure, which is presented in a form which minimizes computer storage and execution time, is summarized in Table 1. The "effective" stiffness matrix is normally banded and its triangularized form is also banded; therefore, a large amount of computer storage is not required. Also, the time for the solution of the equations for each time step is not large since the matrix was initially triangularized.

Table 1 Summary of Step-by-Step Procedure

I. INITIAL CALCULATION

a. Form Stiffness Matrix $\underline{K}$ and Diagonal Mass Matrix $\underline{M}$

b. Calculate the following constants:

$$\tau = 2\Delta t \qquad a_5 = \frac{3\beta}{\tau} a_4 - \frac{3}{\tau^2}$$

$$a_0 = \frac{6+3\alpha\tau}{\tau^2+3\beta\tau} \qquad a_6 = 2\beta a_4 - \frac{3}{\tau}$$

$$a_1 = \frac{6}{\tau^2} + \frac{3}{\tau}(\alpha-\beta a_0) \qquad a_7 = \frac{\beta\tau a_4 - 1}{2}$$

$$a_2 = \frac{6}{\tau} + 2(\alpha-\beta a_0) \qquad a_8 = \frac{\Delta t}{2}$$

$$a_3 = 2 + \frac{\tau}{2}(\alpha-\beta a_0) \qquad a_9 = \frac{\Delta t^2}{3}$$

$$a_4 = \frac{3}{3\beta\tau+\tau^2} \qquad a_{10} = \frac{\Delta t^2}{6}$$

c. Form Effective Stiffness Matrix $\overline{\underline{K}} = \underline{K} + \underline{a_0}\,\underline{M}$

d. Triangularize $\overline{\underline{K}}$

II. FOR EACH TIME INCREMENT

a. Form Effective Load

$$\overline{\underline{K}} = \underline{P}_{t-\Delta t} + \underline{M}\,[a_1 \underline{U}_{t-\Delta t} + a_2 \dot{\underline{U}}_{t-\Delta t} + a_3 \ddot{\underline{U}}_{t-\Delta t}]$$

b. Solve for Effective Displacement Vector $\overline{\underline{U}}_t$

$$\overline{\underline{K}}\ \overline{\underline{U}}_t = \overline{\underline{P}}_t$$

c. Calculate Accelerations, Velocities and Displacements at time t.

$$\ddot{\underline{U}}_t = a_4 \overline{\underline{U}}_t + a_5 \underline{U}_{t-\Delta t} + a_6 \dot{\underline{U}}_{t-\Delta t} + a_7 \ddot{\underline{U}}_{t-\Delta t}$$

$$\dot{\underline{U}}_t = \dot{\underline{U}}_{t-\Delta t} + a_8 [\ddot{\underline{U}}_{t-\Delta t} + \ddot{\underline{U}}_t]$$

$$\underline{U}_t = \underline{U}_{t-\Delta t} + \Delta t\, \dot{\underline{U}}_{t-\Delta t} + a_9 \ddot{\underline{U}}_{t-\Delta t} + a_{10} \ddot{\underline{U}}_t$$

d. Calculate Element Stresses If Desired

e. Repeat For Next Time Increment

APPLICATION

The validity of the finite element method as applied to the dynamic analysis of axisymmetric systems has been demonstrated in reference [1]. Therefore, the purpose of this section is to illustrate the application of this particular computer program to a complex structure and to compare the results with an experimental study.

The method of analysis is compared with an experimental study conducted by the Ralph M. Parsons Company. A steel encased concrete closure model mounted at the end of a detonation tube is shown in figure (1). The model was subjected to a blast load as shown and strains, displacements and accelerations were measured at various points within the model. The same model was idealized by a system of finite elements as shown in figure (2). A listing of the input data for this structure is given in Appendix D. A comparison of the strains in the steel at the axis of symmetry is illustrated by figure (3). Since in this case the model was not loaded beyond the elastic range, good agreement is obtained.

In another analysis the results of an elastic finite element analysis were compared with an experimental study of a structure buried in a soil material. This experimental study was conducted in the Blast Load Simulator at Vicksburg, Mississippi. In figure (4) the displacements at a point in the soil are plotted. In this case the need for a nonlinear analysis is clear -- the experimental results indicate a permanent set in the material; whereas, the displacements from the elastic analysis return to zero.

COMPUTER PROGRAM

A FORTRAN IV listing of the computer program for the dynamic elastic analysis of axisymmetric structures is given in Appendix C. The program utilizes axisymmetric elements with a triangular, quadrilateral or one-dimensional cross-section. The capacity of the program will depend on the storage of the computer used.

Within the program a method of dynamic storage allocation is used; therefore, for a given problem all required data is compressed into the smallest possible storage area. This also allows the capacity of the program to be increased or decreased by only changing one number within the program.

The operation of the program may be summarized by the following steps:

First:

Control information, material properties and nodal point geometry data is read (or generated) by the computer.

Second:

Element data is read (or generated) a single element at a time. For each element, an 8 x 8 stiffness matrix and a 4 x 8 stress-displacement are formed. These matrices, the element's mass and the element nodal point numbers are placed on tape for temporary storage. Therefore, there is practically no limit on the number of elements which can be used.

Third:

At this point in the execution of the program the nodal point data is no longer required; therefore, this storage is available to be used by the complete stiffness matrix for the structure. The element stiffnesses are then read from tape and added into the banded, symmetric stiffness matrix. Also, the diagonal mass matrix is formed at the same time.

Fourth:

The step-by-step solution technique, as summarized in Table 1, is used to evaluate the displacements as a function of time. At specified time points the element stresses can be calculated by reading the element stress-displacement matrices from tape. These 4 x 8 matrices are read in groups in order to minimize computer time.

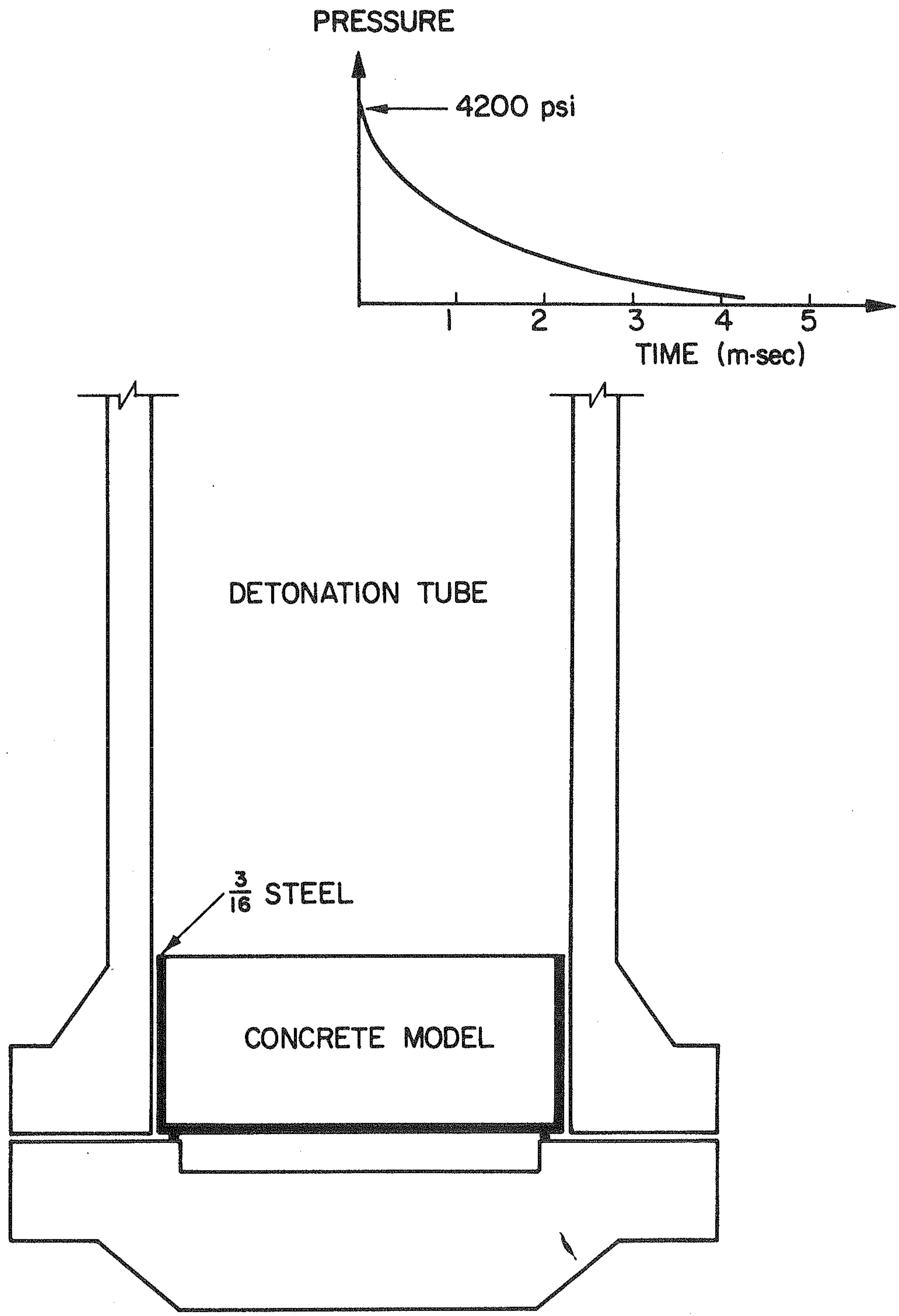

FIG. 1 SECTION OF CLOSURE MODEL

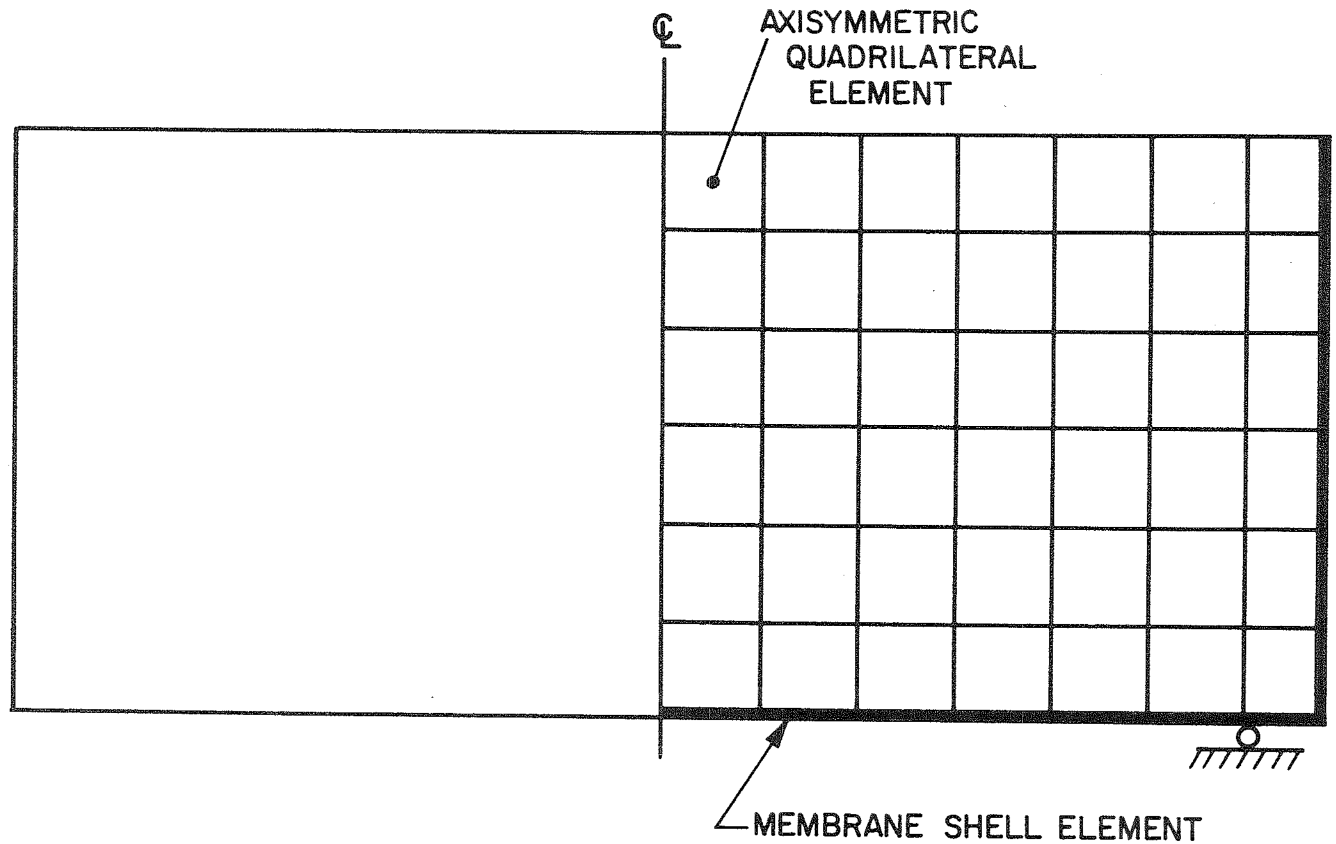

FIG. 2 FINITE ELEMENT REPRESENTATION OF CLOSURE

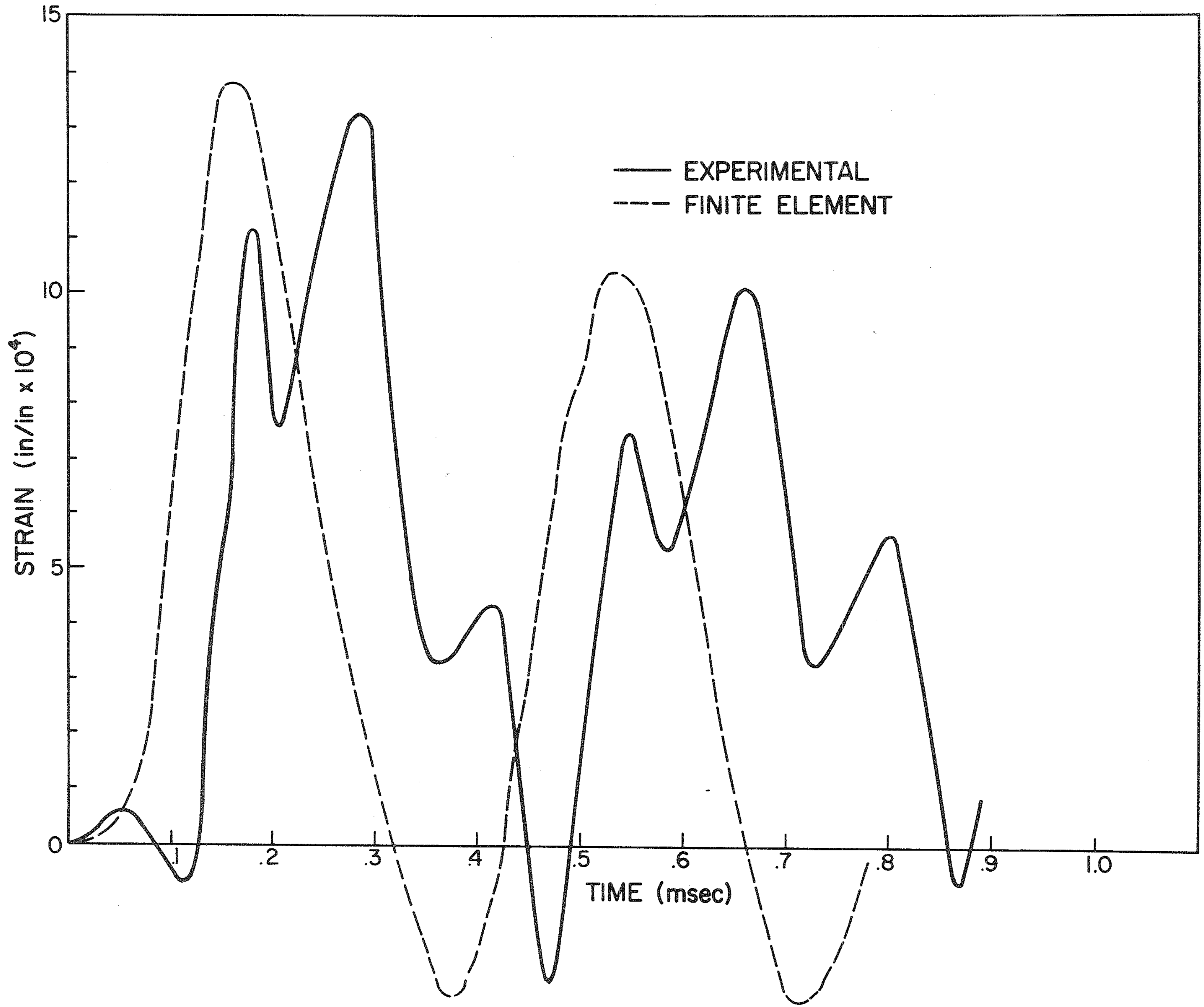

FIG. 3 STRAIN vs. TIME IN STEEL SHELL

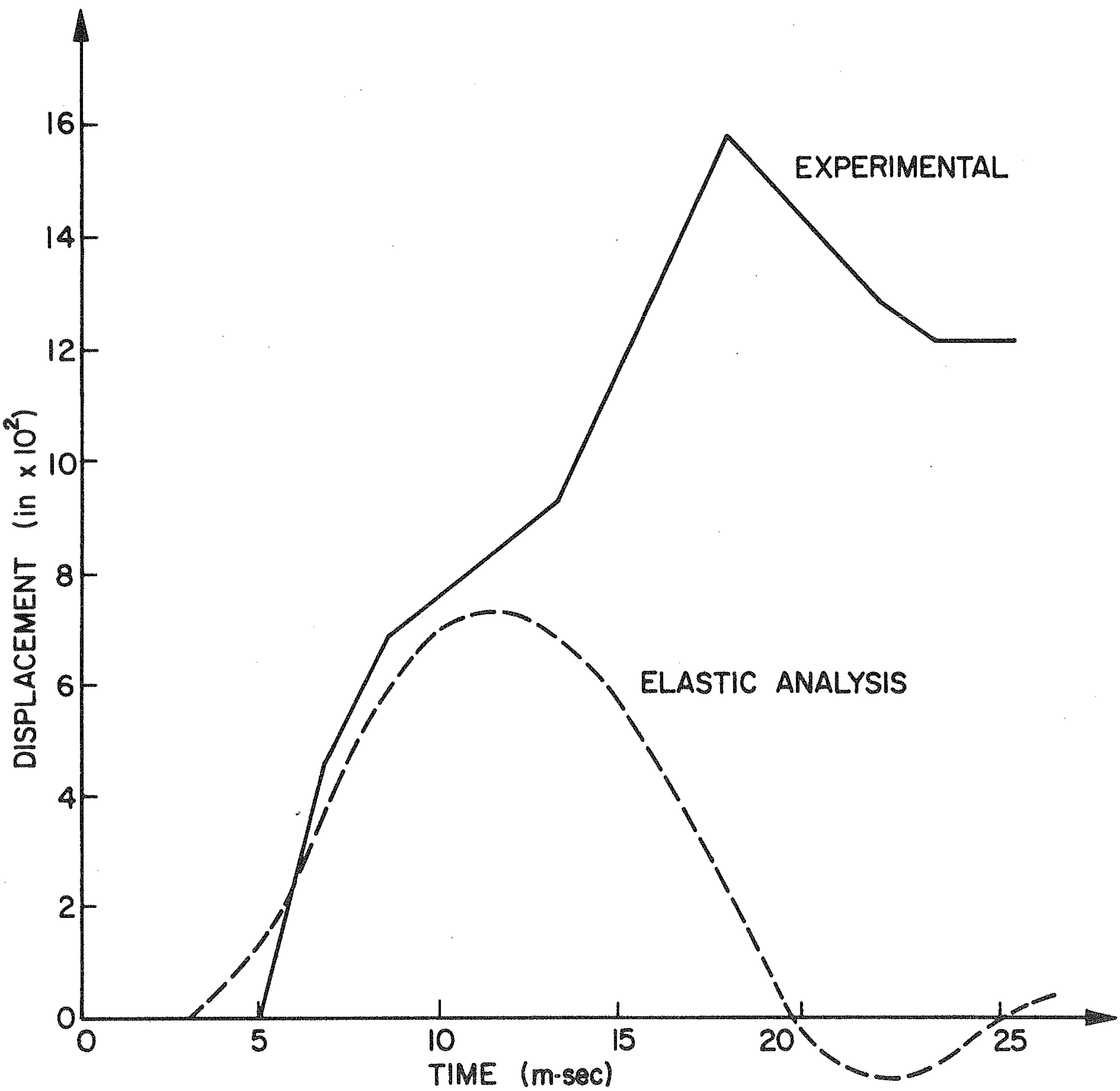

FIG. 4 DISPLACEMENTS vs. TIME IN SOIL MATERIAL

EXTENSION TO NONLINEAR MATERIALS

The method of analysis and the computer program presented in this report can be used in the elastic analysis of complex plane or axi-symmetric structures subjected to dynamic loads. The next step in the development program is the extension to the dynamic analysis of structures with nonlinear material properties.

Initially, the nonlinear behavior of soils will be incorporated into the computer program. In reference [5] the stress-strain behavior of soils subjected to dynamic loads is discussed. It appears that one of the most important nonlinear parameters is the volumetric strain. A typical test of a soil is illustrated in figure (5). If unloading occurs the material tends to have a different behavior (modulus) than if the loads are increased monotonically.

For a nonlinear analysis, it is necessary to form an incremental stiffness of the system; therefore, an incremental stress-strain relationship is required. This incremental relationship will be assumed to be of the following form:

$$\Delta 6_{ij} = K^* \Delta e \delta_{ij} + 2G^* \Delta \varepsilon_{ij}^{1}$$

where

$$K^* = I^* (P, Pmax, e, \dot{e})$$

$$G^* = G(P)$$

The values of the incremental bulk modulus K^* and the incremental shear modulus G^* must be determined from experimental tests. As indicated by

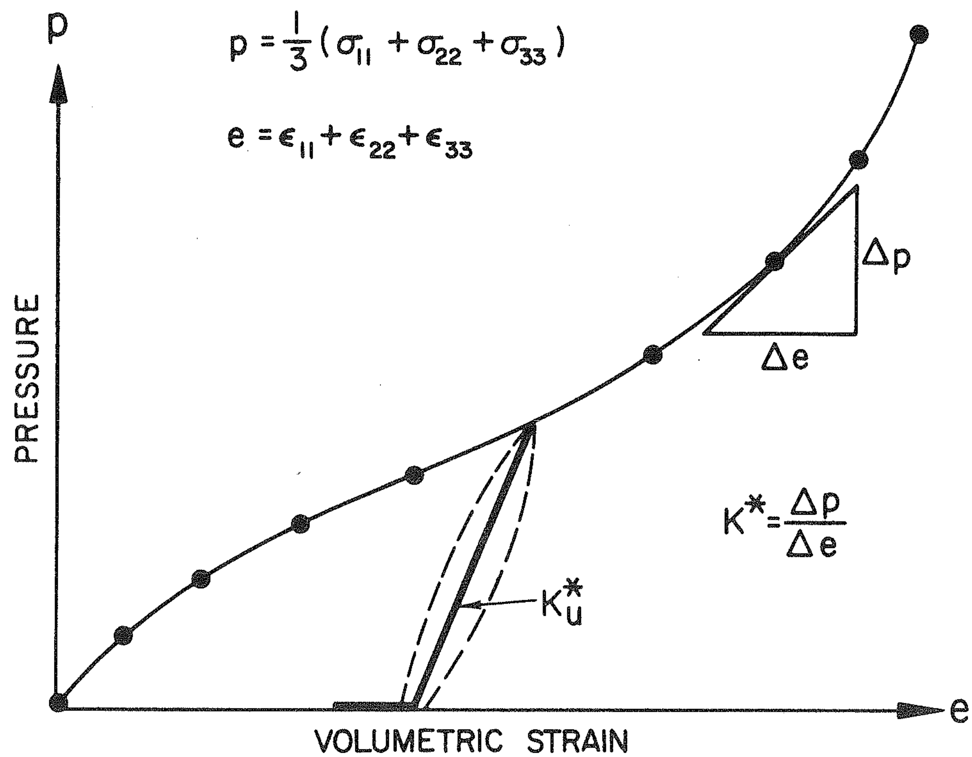

FIG. 5 NONLINEAR MATERIAL PROPERTIES

figure (5) the loading and unloading values of the bulk modulus will be different.

It is not necessary to express the material properties in mathematical form for the purpose of a numerical analysis; a sequence of points which describe the stress-strain behavior may be used. Therefore, a series of points with the following information will constitute the input to the computer program:

1. Pressure
2. Strain
3. Unloading Bulk Modulus
4. Shear Modulus

This type of material behavior is currently being incorporated into the computer program.

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APPENDIX A

STIFFNESS MATRIX FOR QUADRILATERAL ELEMENT

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STIFFNESS MATRIX FOR QUADRILATERAL ELEMENT

Introduction

The purpose of this section is to present the development of a five-point quadrilateral axisymmetric element. The general form of the stiffness matrix [K] for any finite element is

$$[K] = \int_{Vol.} [B]^T [D][B]\, dV \qquad \text{(A-1)}$$

where [B] is the strain-displacement relationship and [D] is the stress-strain law; i.e.

$$[\varepsilon] = [B]\,[d] \qquad\qquad [\sigma] = [D]\,[\varepsilon]$$

For a four-point axisymmetric solid the strains $[\varepsilon]$, stress $[\sigma]$ and nodal point displacements [d] are given by

$$[\sigma] = \begin{bmatrix} \sigma_r \\ \sigma_z \\ \sigma_\theta \\ \tau_{rz} \end{bmatrix} \qquad [\varepsilon] = \begin{bmatrix} \varepsilon_r \\ \varepsilon_z \\ \varepsilon_\theta \\ \gamma_{rz} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial r} \\ \frac{\partial v}{\partial z} \\ \frac{u}{r} \\ \frac{\partial u}{\partial z} + \frac{\partial v}{\partial r} \end{bmatrix}$$

$$[d]^T = [\ u_1 v_1 u_2 v_2 u_3 v_3 u_4 v_4]$$

The derivation of the five-point stiffness matrix is more readily demonstrated by starting with the four-point element.

Coordinate Systems

The coordinates (r,z) are cartesian while the natural coordinates (s,t) may be skewed and are defined such that s and t vary from -1 to 1,

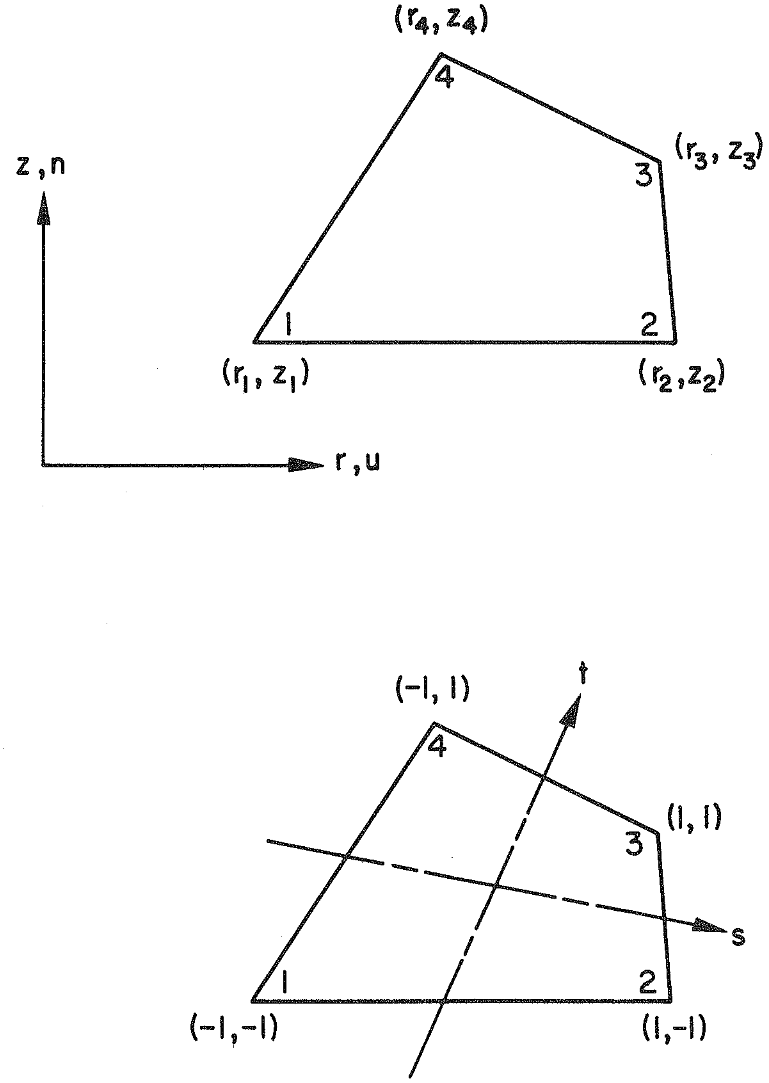

FIG. A1 THE NATURAL COORDINATE SYSTEM

as shown in figure A-1. The (r,z) coordinates are given in terms of (s,t) via the interpolating functions:

$$r(s,t) = \sum_{i=1}^{4} h_i r_i \qquad z(s,t) = \sum_{i=1}^{4} h_i z_i \qquad \text{(A-2)}$$

$$h_1 = (1-s)(1-t)/4 \qquad h_3 = (1+s)(1+t)/4$$
$$h_2 = (1+s)(1-t)/4 \qquad h_4 = (1-s)(1+t)/4 \qquad \text{(A-3)}$$

Since strains are defined by derivatives with respect to (r,z) and the displacement expansions are given in the (s,t) system, the chain rule for differentiation must be used to calculate

$$\frac{\partial s}{\partial r},\ \frac{\partial s}{\partial z},\ \frac{\partial t}{\partial r} \text{ and } \frac{\partial t}{\partial z}$$

Inverting the chain rule;

$$\begin{Bmatrix} \frac{\partial}{\partial s} \\ \frac{\partial}{\partial t} \end{Bmatrix} = \begin{bmatrix} \frac{\partial r}{\partial s} & \frac{\partial z}{\partial s} \\ \frac{\partial r}{\partial t} & \frac{\partial z}{\partial t} \end{bmatrix} \begin{Bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial z} \end{Bmatrix} \qquad \text{(A-4)}$$

gives

$$\begin{Bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial z} \end{Bmatrix} = \frac{1}{J} \begin{bmatrix} \frac{\partial z}{\partial t} & -\frac{\partial z}{\partial s} \\ -\frac{\partial r}{\partial t} & \frac{\partial r}{\partial s} \end{bmatrix} \begin{Bmatrix} \frac{\partial}{\partial s} \\ \frac{\partial}{\partial t} \end{Bmatrix} \qquad \text{(A-5)}$$

where

$$J = J(s,t) = \frac{\partial r}{\partial s}\frac{\partial z}{\partial t} - \frac{\partial r}{\partial t}\frac{\partial z}{\partial s}$$

Expressions used in evaluating J are useful in obtaining the strain-displacement relationships. Substitution of (A-2) and (A-3) into (A-5) gives

$$J = \sum_{i=1}^{4} \sum_{j=1}^{4} r_i \left(\frac{hi}{s} \frac{hj}{t} - \frac{hi}{t} \frac{hj}{s} \right) zj$$

$$= \sum_{i=1}^{4} \sum_{j=1}^{4} r_i P_{ij} z_i = [\bar{r}]^T [P] [\bar{z}] \qquad \text{(A-6)}$$

where

$$[\bar{r}]^T = [\ r_1 r_2 r_3 r_4]$$

$$[\bar{z}]^T = [\ z_1 z_2 z_3 z_4]$$

$$[P] = \frac{1}{8}\begin{bmatrix} 0 & 1-t & -s+t & -1+s \\ & 0 & 1+s & -s-t \\ & & 0 & t+1 \\ \text{skew} \\ \text{-symmetric} & & & 0 \end{bmatrix} \qquad \text{(A-7)}$$

Therefore,

$$[P][\bar{z}] = \frac{1}{8}\begin{bmatrix} z_{24} - z_{34}s - z_{23}t \\ -z_{13} + z_{34}s + z_{14}t \\ -z_{24} + z_{12}s - z_{14}t \\ z_{13} - z_{12}s + z_{23}t \end{bmatrix} \qquad \text{(A-8)}$$

and

$$J = [\bar{r}]^T[P][\bar{z}] = \frac{1}{8}[(r_{13}z_{24}-r_{24}z_{13}) + s(r_{34}z_{12}-r_{12}z_{34}) + t(r_{23}z_{14}-r_{14}z_{23})] \qquad \text{(A-9)}$$

where

$$r_{ij} = r_i - r_j$$

$$z_{ij} = z_i - z_j$$

Strain Displacement Transformation [B]

Let nodal point values of the displacements u and v be given by

$$[\bar{u}]^T = [u_1 u_2 u_3 u_4]$$

$$[v]^T = [v_1 v_2 v_3 v_4] \qquad \text{(A-10)}$$

The assumed displacement expansion uses the same interpolation functions as appeared in (A-2); i.e.

$$u_{(s,t)} = \sum_{i=1}^{4} h_i u_i \qquad v_{(s,t)} = \sum_{i=1}^{4} h_i v_i \qquad \text{(A-11)}$$

The ε_θ strain is given immediately by

$$\varepsilon_\theta = \sum_{i=1}^{4} \frac{h_i u_i}{r_i} = \sum_{i=1}^{4} G_i u_i$$

ε_r must be obtained by differentiation

$$\varepsilon_r = \frac{\partial u}{\partial r} = \frac{\partial u}{\partial s}\frac{\partial s}{\partial r} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial r}$$

$$= \frac{1}{J}\sum_{i=1}^{4}\sum_{j=1}^{4} u_i\left(\frac{\partial hi}{\partial s}\frac{\partial hj}{\partial t} - \frac{\partial hi}{\partial t}\frac{\partial hj}{\partial s}\right)z_i$$

$$= \frac{[\bar{u}]^T[P][\bar{z}]}{J} \qquad \text{(A-12)}$$

where [P] is given by (A-8)

Similarly

$$\varepsilon_z = \frac{\partial v}{\partial z} = -\frac{[\bar{v}]^T[P][\bar{r}]}{J}$$

$$\gamma_{rz} = \frac{\partial u}{\partial z} + \frac{\partial v}{\partial r} = -\frac{[\bar{u}]^T[P][\bar{r}]}{J} + \frac{[\bar{v}]^T[P][\bar{z}]}{J}$$

Let $\frac{[P][\bar{z}]}{J} = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{bmatrix} = [Y]$ and $\frac{-[P][\bar{r}]}{J} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = [X]$

The Xi and Yi are evaluated directly from (A-8)

$$X_1 = -\frac{1}{8}(r_{24}-r_{34}s - r_{23}t)/J$$

$$Y_1 = \frac{1}{8}(z_{24}-z_{34}s - z_{23}t)/J \text{, and so on.}$$

With this definition the strains are given by

$$\varepsilon_\theta = \sum_{i=1}^{4} G_i u_i \text{ , } \varepsilon_r = \sum_{i=1}^{4} Y_i u_i \text{ , } \varepsilon_z = \sum_{i=1}^{4} X_i v_i$$

$$\gamma_{rz} = \sum_{i=1}^{4} (X_i u_i + Y_i v_i)$$

In matrix form

$$\begin{bmatrix} \varepsilon_r \\ \varepsilon_z \\ \varepsilon_\theta \\ \gamma_{rz} \end{bmatrix} = \begin{bmatrix} Y_1 & 0 & Y_2 & 0 & Y_3 & 0 & Y_4 & 0 \\ 0 & X_1 & 0 & X_2 & 0 & X_3 & 0 & X_4 \\ G_1 & 0 & G_2 & 0 & G_3 & 0 & G_4 & 0 \\ X_1 & Y_1 & X_2 & Y_2 & X_3 & Y_3 & X_4 & Y_4 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{bmatrix} \qquad \text{(A-13)}$$

or $[\varepsilon] = [B][d]$

[B] is the strain-displacement relationship required in (A-1) to evaluate the stiffness matrix of the element.

Strain - Displacement Transformation

A modified quadrilateral element stiffness is obtained by assuming that the shear strain is constant over the element (this assumption improves the bending behavior of the element significantly). The value of the constant shear strain is given by the evaluation of the shear strain at (t,s) = (o,o).

Let
$$\left.\begin{aligned} X_{i+4} &= X_i(r,z) \\ Y_{i+4} &= Y_i(r,z) \end{aligned}\right\} \quad t = s = 0 \qquad i = 1, 2, 3, 4$$

The [B] matrix for the element is simply:

$$[B] = \begin{bmatrix} Y_1 & 0 & Y_2 & 0 & Y_3 & 0 & Y_4 & 0 \\ 0 & X_1 & 0 & X_2 & 0 & X_3 & 0 & X_4 \\ G_1 & 0 & G_2 & 0 & G_3 & 0 & G_4 & 0 \\ X_5 & Y_5 & X_6 & Y_6 & X_7 & Y_7 & X_8 & Y_8 \end{bmatrix} \qquad \text{(A-14)}$$

Extension to the Modified Five-Point Quadrilateral

A five-point quadrilateral with an internal degree of freedom associated with the displacement function $(1 - s^2)(1 - t^2)$ can be added to the above formulation. Letting the new degrees of freedom be Uc and Vc the displacement expansion becomes

$$U = \sum_{i=1}^{4} h_i U_i + (1 - s^2)(1 - t^2)\, Uc$$

$$V = \sum_{i=1}^{4} h_i V_i + (1 - s^2)(1 - t^2)Vc$$

Let

$$Hc = (1 - s^2)(1 - t^2)$$

$$Gc = (1 - s^2)(1 - t^2)/r$$

$$Xc = \frac{\partial Hc}{\partial z} = -\frac{2}{J}\left((1 - s^2)t\frac{\partial r}{\partial s} - (1 - t^2)s\frac{\partial r}{\partial t}\right)$$

$$Yc = \frac{\partial Hc}{\partial r} = \frac{2}{J}\left((1 - s^2)t\frac{\partial z}{\partial s} - (1 - t^2)s\frac{\partial z}{\partial t}\right)$$

The [B] matrix for the five-point element is

$$[B] = \begin{bmatrix} Y_1 & 0 & Y_2 & 0 & Y_3 & 0 & Y_4 & 0 & Yc & 0 \\ 0 & X_1 & 0 & X_2 & 0 & X_3 & 0 & X_4 & 0 & Xc \\ G_1 & 0 & G_2 & 0 & G_3 & 0 & G_4 & 0 & Gc & 0 \\ X_1 & Y_1 & X_2 & Y_2 & X_3 & Y_3 & X_4 & Y_4 & Xc & Yc \end{bmatrix} \qquad \text{(A-15)}$$

The [B] matrix for the modified five-point element is obtained by assuming the shear to be constant within the element. Or

$$\begin{bmatrix} Y_1 & 0 & Y_2 & 0 & Y_3 & 0 & Y_4 & 0 & Yc & 0 \\ 0 & X_1 & 0 & X_2 & 0 & X_3 & 0 & X_4 & 0 & Xc \\ G_1 & 0 & G_2 & 0 & G_3 & 0 & G_4 & 0 & Gc & 0 \\ X_5 & Y_5 & X_6 & Y_6 & X_7 & Y_7 & X_8 & Y_8 & 0 & 0 \end{bmatrix} \qquad \text{(A-16)}$$

Numerical Integration

The stiffness matrix is given by the integral (A-1)

$$[K] = \int_z \int_r \int_\theta [B]^T[D][B] \, r \, d\theta \, dr \, dz$$

For a one radian segment

$$[K] = \int_r \int_z [B]^T[D][B] \, r \, dr \, dz$$

The derived [B] matrices are functions of (s,t). The variables of integration are changed to (s,t) by means of the Jacobian determinant J (A-5)

$$[K] = \int_{-1}^{1} \int_{-1}^{1} [B]^T[D][B] \, r \, J \, ds \, dt$$

Using and N point numerical integration scheme to evaluate [K] gives

$$[K] = \sum_{i=1}^{N} [B(s_i, t_i)]^T[D][B(s_i, t_i)] \, r_i \, J(s_i, t_i) \, W_i$$

where W_i is a weighting factor.

Triangular elements are obtained by letting two adjacent nodes of a quadrilateral coincide. This would not be possible if closed formed integration were used to evaluate [K] because of singular derivatives at the double node. A numerical formula with integration points internal to the element does not "see" the singularity.

The above procedure yields the 10 x 10 element stiffness matrix. The standard "static condensation" technique may be applied to develop the 8 x 8 quadrilateral stiffness matrix.

APPENDIX B

DESCRIPTION OF INPUT DATA FOR COMPUTER PROGRAM

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DESCRIPTION OF INPUT DATA FOR COMPUTER PROGRAM

The purpose of this computer program is to determine time-dependent displacements and stresses within elastic axisymmetric structures of arbitrary shape and materials. In order to define the computer input a two-dimensional cross-section of the axisymmetric structure must be idealized by a system of finite elements. Quadrilateral, triangular and one-dimensional membrane elements can be used. Elements in the system are identified by a sequence of numbers starting with one. Also, all nodal points are identified by a separate numbering sequence. The reference coordinate system to be used and a simple finite element representation of a structure is shown in Figure B-1.

The following sequence of punched cards numerically define the axisymmetric structure to be analyzed.

A. IDENTIFICATION CARD. (72 H)

Columns 1 to 72 contain information to be printed with results.

B. CONTROL CARD. (7I5, 4F10.0)

Columns	1 - 5	Number of nodal points (n)
	6 - 10	Number of elements (no limit)
	11 - 15	Number of different materials (m)
	16 - 20	Number of time steps
	21 - 25	Number of time increments between the print displacements and stresses
	26 - 30	Number of load cards (ℓ)

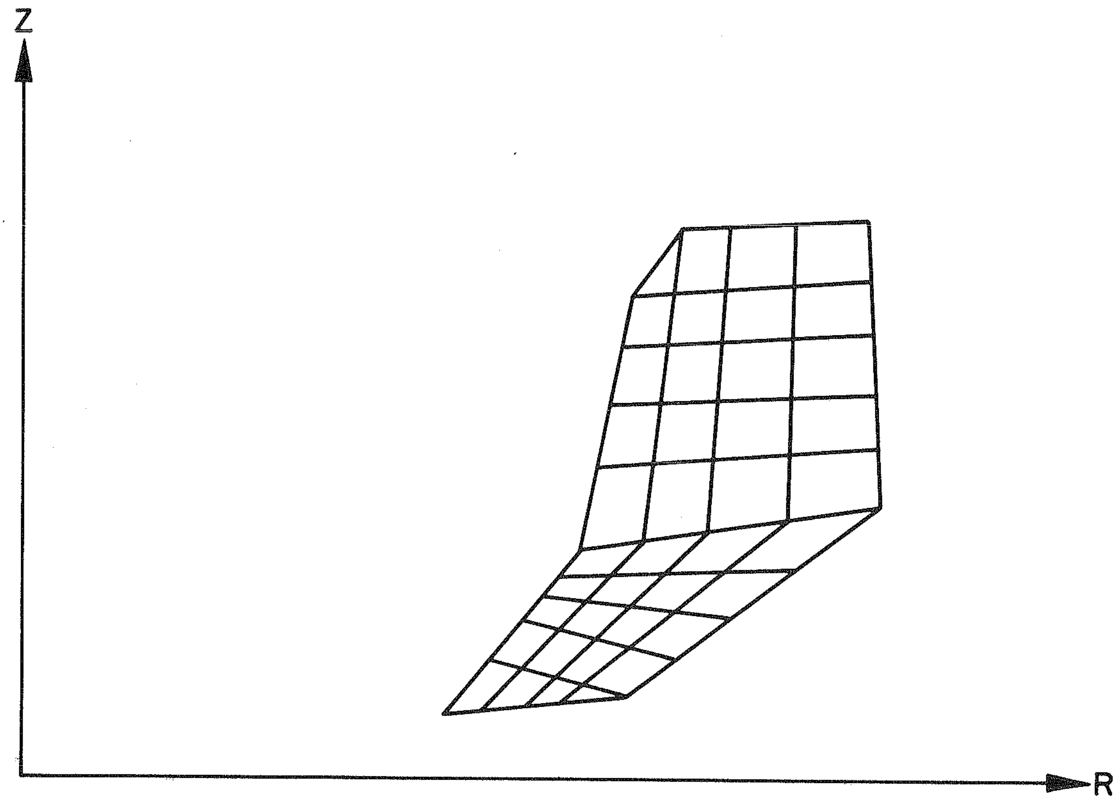

FIG. B-1 REFERENCE COORDINATE SYSTEM

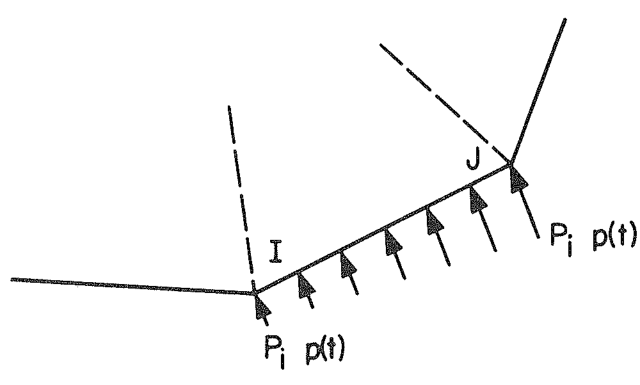

FIG. B-2 PRESSURE BOUNDARY CONDITIONS

31 - 35	Number of boundary pressure cards (p)
36 - 45	Damping coefficient α
46 - 55	Damping coefficient β
56 - 65	Time increment Δt
66 - 75	Reference number to be added to all R ordinates

C. MATERIAL PROPERTY INFORMATION.

The following card must be supplied for each different material (I5, 4F10.0)

Columns

1 - 5	Material identification number
6 - 15	Modulus of elasticity
16 - 25	Poisson's ratio
26 - 35	Mass density of material
36 - 45	Thickness (for membrane shell elements)

D. NODAL POINT CARDS, (I5, F5.0, 2F.10.0)

One card is required for each nodal point with the following information:

Columns	1 - 5	Nodal point number
	6 - 10	Boundary condition code "k"
	11 - 20	R-ordinate
	21 - 30	Z-ordinate

Specifications for code "k". If

k = 0	load in the R-direction
	load in the Z-direction

k = 1	zero displacement in the R-direction
	load in the Z-direction
k = 2	load in the R-direction
	zero displacement in the Z-direction
k = 3	zero displacement in the R-direction
	zero displacement in the Z-direction

Nodal point cards must be in numerical sequence. If cards are omitted, the omitted nodal points are generated at equal intervals along a straight line between the defined nodal points. The boundary condition code is set equal to zero.

E. ELEMENT CARDS. (6I5)

Columns		
1 - 5	Element number	
6 - 10	Nodal point I	The maximum difference "b" between these numbers is an indication of the band width. The execution time for the program will be proportional to this number squared.
11 - 15	Nodal point J	
16 - 20	Nodal point K	
21 - 25	Nodal point L	
26 - 30	Material Identification	

For a right hand coordinate system the nodal point numbers I, J, K and L must be in sequence in a counter-clockwise direction around the element. Element cards must be in element number sequence. If element cards are omitted the program automatically generates the omitted information by incrementing by one the preceeding I, J, K and L. The material identification for the generated cards is set equal to the corresponding value on the last card. The last element card must always be supplied. Triangular elements are also permissible; they are identified by repeating the last nodal point number (i.e. I, J. K, K). One dimensional membrane elements are identified by a nodal point numbering sequence of the form I, J, J, I.

F. PRESSURE CARDS (2I5, 3F10.0)

One card for each boundary element which is subjected to a normal pressure.

Columns	1 - 5	Nodal point I
	6 - 10	Nodal point J
	11 - 20	Pressure multiplier P_i
	21 - 30	Pressure multiplier P_j
	31 - 40	Arrival time of pressure at the center of the surface element

As shown in Figure B-2 the boundary element must be on the left as one progresses from I to J. Surface tensile force is input as a negative pressure.

G. LOAD CARDS (2F10.0)

These cards specify the normal pressure as a function of time in the form of straight line segments. One card is required for each point with the following information:

Columns	1 - 10	Time t
	11 - 20	Normal pressure p (t)

OUTPUT INFORMATION

The following information is developed and printed by the program:

1. Reprint of input data
2. Pressure boundary conditions
3. Nodal point displacements, velocities and accelerations as a function of time
4. Stresses at the center of each element as a function of time

PROGRAM LIMITATIONS

The capacity of the program is limited by the dimension "d" of the "A" array in program DYNS.

$$[9n + 2\ell + 7p + 4n\,(b + 1)] \text{ must not be greater than } d.$$

The symbols n, ℓ, p and b have been defined previously and their values will depend on the particular structure to be analyzed. The maximum size which d can have will depend on the particular computer being utilized. For a computer with 32K storage the maximum value for d will be approximately 20000.

APPENDIX C

FORTRAN IV LISTING OF COMPUTER PROGRAM

```
      PROGRAM DYNS(INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT,TAPE1)
      COMMON NUMNP,MBAND,NT,NPRINT,NP,NUMPC,DELT,TT,NUMEL,ALFA,BETA,RA
     1,HED(12),A(33200)
C***********************************************************************
C     READ AND PRINT OF CONTROL INFORMATION
C***********************************************************************
   50 READ  (5,1000) HED,NUMNP,NUMEL,NUMMAT,NT,NPRINT,NP,NUMPC,ALFA,
     1BETA,DELT,RA
      WRITE (6,2000) HED,NUMNP,NUMEL,NUMMAT,NT,NPRINT,NP,NUMPC,ALFA,
     1BETA,DELT
C***********************************************************************
C     READ DATA AND FORM ELEMEMT STIFFNESSES
C***********************************************************************
      NEQ=2*NUMNP
      N2=1+NUMPC
      N3=N2+NUMPC
      N4=N3+NUMPC
      N5=N4+NUMPC
      N6=N5+NUMPC
      N7=N6+NUMPC
      N8=N7+NUMPC
      N9=N8+NEQ
      N10=N9+NEQ
      N11=N10+NUMNP
      N12=N11+NUMNP
      N13=N12+NUMNP
      N14=N13+NUMMAT
      N15=N14+NUMMAT
      N16=N15+NUMMAT
      IF (N16+NUMMAT.LE.33200) GO TO 100
      WRITE (6,1100)
      STOP
  100 CALL STIFF(A(1),A(N2),A(N3),A(N4),A(N5),A(N6),A(N7),A(N8),A(N9)
     1 ,A(N10),A(N11),A(N12),A(N13),A(N14),A(N15),A(N16),NUMMAT)
C***********************************************************************
C     FORM TOTAL STIFFNESS AND MASS MATRICES AND SOLVE STEP-BY-STEP
C***********************************************************************
      N11=N10+NEQ
      N12=N11+2*NP
      N13=N12+NUMNP
      N14=N13+NEQ
      IF (N14+NEQ*MBAND.LE.33200) GO TO 200
      WRITE (6,1100)
      STOP
  200 CALL SOLVE(A(1),A(N2),A(N3),A(N4),A(N5),A(N6),A(N7),A(N8),A(N9),
     1 A(N10),A(N11),A(N12),A(N13),A(N14),NEQ)
C***********************************************************************
      GO TO 50
C***********************************************************************
 1000 FORMAT (12A6/7I5,4F10.0)
 1100 FORMAT (25H0 DIMENSION OF A EXCEEDED)
 2000 FORMAT (1H1 12A6/
     1 30H0 NUMBER OF NODAL POINTS------ I4 /
     2 30H0 NUMBER OF ELEMENTS---------- I4 /
     3 30H0 NUMBER OF DIFF. MATERIALS--- I4 /
```

```
4 30H0 NUMBER OF TIME INCREMENTS--- I4 /
5 30H0 PRINT INTERVAL-------------- I4 /
6 30H0 NUMBER OF LOAD POINTS------- I4 /
7 30H0 NUMBER OF PRESSURE CARDS---- I4 /
8 30H0 DAMPING COEFFICIENT ALFA---- F10.5 /
9 30H0 DAMPING COEFFICIENT BETA---- F10.5 /
0 30H0 TIME INCREMENT-------------- F10.5)
 END
```

```
      SUBROUTINE STIFF (T,INI,JNJ,HI,HJ,VI,VJ,X0,X1,CODE,R,Z,YMOD,ENU,
     1 RO,H,NUMMAT)
      COMMON NUMNP,MBAND,NT,NPRINT,NP,NUMPC,DELT,TT,NUMEL,ALFA,BETA,RA
      COMMON /LS4ARG/ LM(8),SS(4,8),XC,YC,ELMASS,S(8,8),C(4,4)
      DIMENSION T(1),INI(1),JNJ(1),HI(1),HJ(1),VI(1),VJ(1),X0(1),X1(1),
     1 CODE(1),R(1),Z(1),IX(4),IE(5),YMOD(1),ENU(1),RO(1),H(1)
C***********************************************************************
   56 DO 59 M=1,NUMMAT
      READ (5,1001) MTYPE,YMOD(MTYPE),ENU(MTYPE),RO(MTYPE),H(MTYPE)
   59 WRITE (6,2011) MTYPE,YMOD(MTYPE),ENU(MTYPE),RO(MTYPE),H(MTYPE)
C***********************************************************************
C     READ AND PRINT OF NODAL POINT DATA
C***********************************************************************
      WRITE (6,2004)
      L=0
   60 READ (5,1002) N,CODE(N),R(N),Z(N)
      R(N)=R(N)+RA
      IF (L.EQ.0) GO TO 85
      ZX=N-L
      DR=(R(N)-R(L))/ZX
      DZ=(Z(N)-Z(L))/ZX
   85 NL=L+1
   70 L=L+1
      IF(N-L) 100,90,80
   80 CODE(L)=0.0
      R(L)=R(L-1)+DR
      Z(L)=Z(L-1)+DZ
      GO TO 70
   90 WRITE (6,2002) (K,CODE(K),R(K),Z(K),K=NL,N)
      IF(NUMNP-N) 100,110,60
  100 WRITE (6,2009) N
      STOP
  110 CONTINUE
C***********************************************************************
C        READ AND PRINT OF ELEMENT PROPERTIES
C***********************************************************************
      WRITE (6,2001)
      N=0
      MBAND=0
  130 READ (5,1003) M,(IE(I),I=1,5)
  140 N=N+1
      IF(M.EQ.N) GO TO 145
      DO 142 I=1,4
  142 IX(I)=IX(I)+1
      GO TO 150
  145 DO 148 I=1,4
  148 IX(I)=IE(I)
      MTYPE=IE(5)
C-----DETERMINE BAND WIDTH--------
  150 MB=0
      DO 160 I=1,4
      DO 160 J=I,4
      MM=IABS(IX(I)-IX(J))
      IF(MM.GT.MB) MB=MM
```

```
  160 CONTINUE
      MB=2*MB+2
      IF(MB.GT.MBAND) MBAND=MB
      WRITE (6,2003) N,(IX(I),I=1,4),MTYPE,MB
C*************************************************************************
C     FORM ELEMENT STIFFNESS MATRICES
C*************************************************************************
      IF(IX(2).NE.IX(3)) GO TO 420
C*************************************************************************
C     FORM STIFFNESS FOR ONE-D ELEMENT
C*************************************************************************
      CALL ONED (R,Z,YMOD,ENU,H,IX,MTYPE,VOL)
      GO TO 430
C*************************************************************************
C           FORM QUADRILATERAL STIFFNESS MATRIX
C*************************************************************************
  420 F=YMOD(MTYPE)/(1.0+ENU(MTYPE))/(1.-2.*ENU(MTYPE))
      C(1,1)=F*(1.-ENU(MTYPE))
      C(1,2)=F*ENU(MTYPE)
      C(1,3)=C(1,2)
      C(1,4)=0.0
      C(2,1)=C(1,2)
      C(2,2)=C(1,1)
      C(2,3)=C(1,2)
      C(2,4)=0.0
      C(3,1)=C(1,3)
      C(3,2)=C(2,3)
      C(3,3)=C(1,1)
      C(3,4)=0.0
      C(4,1)=0.0
      C(4,2)=0.0
      C(4,3)=C(3,4)
      C(4,4)=0.5*F*(1.-2.*ENU(MTYPE))
C
      I=IX(1)
      J=IX(2)
      K=IX(3)
      L=IX(4)
      CALL QUAD(R(I),R(J),R(K),R(L),Z(I),Z(J),Z(K),Z(L),XC,YC,VOL,C,
     1 S,SS)
  430 ELMASS=VOL*RO(MTYPE)/4.
C*************************************************************************
C     MODIFY FOR ZERO DISPLACEMENTS
C*************************************************************************
      DO 600 I=1,4
      II=IX(I)
      LM(2*I)=2*II
      LM(2*I-1)=2*II-1
      IF (CODE(II).EQ.0.0) GO TO 600
      IF (CODE(II).EQ.1.0) GO TO 580
      DO 570 J=1,8
      S(2*I,J)=0.0
  570 S(J,2*I)=0.0
  580 IF (CODE(II).EQ.2.0) GO TO 600
      DO 590 J=1,8
```

```
      S(2*I-1,J)=0.0
  590 S(J,2*I-1)=0.0
  600 CONTINUE
C
      CALL WRITEB(LM,107,N,NUMEL)
      IF(N.EQ.NUMEL) GO TO 700
      IF(N.EQ.M) GO TO 130
      GO TO 140
C*************************************************************************
C     PRESSURE BOUNDARY CONDITIONS
C*************************************************************************
  700 WRITE (6,2010)
      DO 330 K=1,NUMPC
      READ (5,1007) INI(K),JNJ(K),A,B,T(K)
      I=INI(K)
      J=JNJ(K)
      DZ=(Z(I)-Z(J))/12.0
      DR=(R(J)-R(I))/12.0
      RX=A*(3.0*R(I)+R(J))+B*(R(I)+R(J))
      ZX=A*(R(I)+R(J))+B*(R(I)+3.0*R(J))
      HI(K)=RX*DZ
      HJ(K)=ZX*DZ
      VI(K)=RX*DR
      VJ(K)=ZX*DR
  330 WRITE (6,2013) I,J,A,B,HI(K),VI(K),HJ(K),VJ(K),T(K)
C*************************************************************************
      RETURN
 1001 FORMAT (I5,4F10.0)
 1002 FORMAT  (I5,F5.0,2F10.0)
 1003 FORMAT (6I5)
 1007 FORMAT (2I5,3F10.0)
 2001 FORMAT (49H1ELEMENT NO.      I      J      K      L     MATERIAL  )
 2002 FORMAT (I7, F10.2,2F10.3)
 2003 FORMAT (1I13,4I6,2I12)
 2004 FORMAT (37H1NODAL POINT TYPE     X-ORD      Y-ORD )
 2009 FORMAT (26H0NODAL POINT CARD ERROR N= I5)
 2010 FORMAT (29H1PRESSURE BOUNDARY CONDITIONS/
     15X,1HI,5X,1HJ,7X,4HPI/P,8X,4HPJ/P,8X,2HHI,10X,2HVI,10X,2HHJ,10X,
     2 2HVJ,11X,1HT)
 2011 FORMAT (16H0MATERIAL NUMBER I3/
     1 4H0 E= F16.6/
     2 4H0NU= F16.6/
     3 4H0RO= F16.6/
     4 4H0 H= F16.6)
 2013 FORMAT (2I6,7F12.3)
      END
```

```
      SUBROUTINE SOLVE (T,INI,JNJ,HI,HJ,VI,VJ,X0,X1,X2,P,MASS,B,A,NEQ)
      COMMON NUMNP,MBAND,NT,NPRINT,NP,NUMPC,DELT,TT,NUMEL,ALFA,BETA
      COMMON /LS4ARG/ LM(8),SS(4,8),XC,YC,ELMASS,S(8,8),C(4,4)
      DIMENSION T(1),INI(1),JNJ(1),HI(1),HJ(1),VI(1),VJ(1),X0(1),X1(1),
     1 X2(1),P(2,1),MASS(1),B(1),A(NEQ,1)
      REAL MASS
C*****************************************************************************
C     FORM TOTAL MASS AND STIFFNESS MATRICES
C*****************************************************************************
      DO 100 I=1,NEQ
      X0(I)=0.0
      X1(I)=0.0
      X2(I)=0.0
      B(I)=0.0
      MASS(I)=0.0
      DO 100 J=1,MBAND
  100 A(I,J)=0.0
      DO 375 N=1,NUMEL
C
      CALL READB (LM,107,N,NUMEL)
C
      DO 300 I=1,8
      II=LM(I)
      DO 300 J=1,8
      JJ=LM(J)-II+1
      IF (JJ.LT.1) GO TO 300
      A(II,JJ)=A(II,JJ)+S(I,J)
  300 CONTINUE
      DO 350 I=1,4
      II=LM(2*I)/2
  350 MASS(II)=MASS(II)+ELMASS
  375 CONTINUE
C*****************************************************************************
C        READ AND PRINT OF LOAD DATA
C*****************************************************************************
      WRITE (6,2007)
      DO 380 M=1,NP
  380 READ(5,1004) (P(K,M),K=1,2)
      WRITE (6,2005) ((P(K,M),K=1,2),M=1,NP)
C*****************************************************************************
C     CONSTANTS FOR THE STEP-BY-STEP SOLUTION
C*****************************************************************************
      DELT1=2.0*DELT
      DELT2=DELT1**2
      A0=(3.0*ALFA*DELT1+6.0)/(DELT2+3.0*BETA*DELT1)
      B0=ALFA-BETA*A0
      A1=6.0/DELT2+3.0*B0/DELT1
      A2=6.0/DELT1+2.0*B0
      A3=2.0+B0*DELT
      A4=3.0/(3.0*BETA*DELT1+DELT2)
      B1=BETA*A4
      A5=3.0*B1/DELT1-3.0/DELT2
      A6=2.0*B1-3.0/DELT1
      A7=0.5*B1*DELT1-0.5
```

```
      A8=0.5*DFLT
      A9=DFLT**2/3.0
      A10=0.5*A9
C***********************************************************************
C     FORM EFFECTIVE STIFFNESS MATRIX
C***********************************************************************
      TT=P(1,1)
      IK=1
      CALL LOAD (T,P,B,INI,JNJ,HI,HJ,VI,VJ,IK)
      DO 400 I=1,NEQ
      II=(I+1)/2
      X2(I)=B(I)/MASS(II)
      IF(A(I,1).NE.0.0) A(I,1)=A(I,1)+A0*MASS(II)
  400 CONTINUE
C***********************************************************************
C     TRIANGULARIZE STIFFNESS MATRIX
C***********************************************************************
      CALL TRIA (NEQ,MBAND,A)
C***********************************************************************
C     STEP-BY-STEP SOLUTION
C***********************************************************************
      LL=0
      DO 500 NNN=1,NT
      TT=TT+DFLT
C
C     EFFECTIVE LOAD CALCULATION
C
      CALL LOAD (T,P,B,INI,JNJ,HI,HJ,VI,VJ,IK)
      DO 460 I=1,NEQ
      II=(I+1)/2
      B(I)=B(I)+MASS(II)*(A1*X0(I)+A2*X1(I)+A3*X2(I))
      IF(A(I,1).EQ.0.0) B(I)=0.0
  460 CONTINUE
C
C     SOLUTION AT END OF TIME STEP
C
      CALL BACKS(NEQ,MBAND,A,B)
      DO 480 I=1,NEQ
      ACC=A4*B(I)+A5*X0(I)+A6*X1(I)+A7*X2(I)
      X0(I)=X0(I)+DFLT*X1(I)+A9*X2(I)+A10*ACC
      X1(I)=X1(I)+A8*(X2(I)+ACC)
  480 X2(I)=ACC
C
C     PRINT DISPLACEMENTS AND STRESSES
C
      LL=LL+1
      IF(LL.NE.NPRINT) GO TO 500
      LL=0.0
      WRITE (6,2006) TT
      WRITE (6,2008) (N,X0(2*N-1),X0(2*N),X1(2*N-1),X1(2*N),X2(2*N-1),
     1 X2(2*N),N,N=1,NUMNP)
C
C     COMPUTE STRESSES
C
      CALL STRESS(X0)
```

```
  500 CONTINUE
      RETURN
C***********************************************************************
 1004 FORMAT (2F10.0)
 2005 FORMAT (2F15.7)
 2006 FORMAT (8H1TIME T=F10.6/118 H0NODAL POINT X-DISPLACEMENT  Y-DISPLA
     1CEMENT    X-VELOCITY      Y-VELOCITY  X-ACCELERATION  Y-ACCELERATI
     2ON  NODAL POINT )
 2007 FORMAT  (27H1       TIME       PRESSURE P)
 2008 FORMAT (I9,6F16.4,I9)
      END
```

```
      SUBROUTINE LOAD (T,P,B,INI,JNJ,HI,HJ,VI,VJ,IK)
      COMMON NUMNP,MBAND,NT,NPRINT,NP,NUMPC,DELT,TT,NUMEL,ALFA,BETA
      DIMENSION T(1),P(2,1),B(1),INI(1),JNJ(1),HI(1),HJ(1),VI(1),VJ(1)
C
      DO 600 I=1,NUMNP
      B(2*I-1)=0.0
  600 B(2*I)=0.0
      N=1
  100 TAU=TT-T(N)
      IF(TAU) 500,200,200
  200 IF(TAU.GE.P(1,IK).AND.TAU.LE.P(1,IK+1)) GO TO 300
      IF (TAU.GT.P(1,IK+1)) IK=IK+1
      IF (TAU.LT.P(1,IK)) IK=IK-1
      GO TO 200
  300 D=P(1,IK+1)-P(1,IK)
      DH=P(2,IK+1)-P(2,IK)
      IF (TT.EQ.P(1,1)) TAU=-DELT
      DT=TAU-P(1,IK)+DELT
      F=P(2,IK)+DT*DH/D
  400 I=INI(N)
      J=JNJ(N)
      B(2*I-1)=B(2*I-1)+F*HI(N)
      B(2*I)=B(2*I)+F*VI(N)
      B(2*J-1)=B(2*J-1)+F*HJ(N)
      B(2*J)=B(2*J)+F*VJ(N)
  500 N=N+1
      IF (N.GT.NUMPC) RETURN
      IF(T(N).EQ.T(N-1)) GO TO 400
      GO TO 100
      END
```

```
      SUBROUTINE STRESS(XO)
      COMMON NUMNP,MBAND,NT,NPRINT,NP,NUMPC,DELT,TT,NUMEL,ALFA,BETA
      DIMENSION XO(1),SIG(7)
      COMMON /LS4ARG/ LM(8),SS(4,8),XC,YC,ELMASS,S(8,8),C(4,4)
C***********************************************************************
C     COMPUTE ELEMENT STRESSES
C***********************************************************************
      MPRINT=0
C
      REWIND 1
      DO 300 N=1,NUMEL
      CALL READB (LM,107,N,NUMEL)
C
      DO 180 I=1,4
      SIG(I)=0.0
      DO 180 J=1,8
      JJ=LM(J)
  180 SIG(I)=SIG(I)+SS(I,J)*XO(JJ)
C
C
C     CALCULATE PRINCIPAL STRESSES
C
      CC=(SIG(1)+SIG(2))/2.0
      BB=(SIG(1)-SIG(2))/2.
      CR=SQRT(BB**2+SIG(4)**2)
      SIG(5)=CC+CR
      SIG(6)=CC-CR
      IF ((BB.EQ.0.0).AND.(SIG(4).EQ.0.0)) GO TO 255
      SIG(7)=28.648*ATAN2(SIG(4),BB)
C
  255 IF (MPRINT) 110,105,110
  105 WRITE (6,2000)
      MPRINT=50
  110 MPRINT=MPRINT-1
C
  305 WRITE (6,2001)  N,XC,YC,(SIG(I),I=1,7)
  300 CONTINUE
C
  320 RETURN
C
 2000 FORMAT (6H1EL.NO  7X  1HR  7X  1HZ  7X  5HSIG-R  7X  5HSIG-Z  7X
     1 5HSIG-T  6X  6HTAU-RZ  5X  7HSIG-MAX  5X  7HSIG-MIN  7H  ANGLE )
 2001 FORMAT (I5,1X,2F8.2,6E12.4,F6.2)
C
      END
```

```
      SUBROUTINE QUAD (R1,R2,R3,R4,Z1,Z2,Z3,Z4,RM,ZM,VOL,D,QK,QS)
C
C     FORMS STIFFNESS MATRIX QK, CENTROIDAL STRESS MATRIX QS
C              FOR A FIVE POINT AXISYMMETRIC IRON'S QUADRILATERAL USING
C     A FOUR POINT INTEGRATION FORMULA.
C     CONSTANT SHEAR STRAIN INTRODUCES INCOMPATIBILITY
      DIMENSION QK(8,8),QS(4,8),D(4,4),TT(4),QC(4,10),SS(4),QQ(10,10)
      DATA SS/ -1.,1.,1.,-1. / , TT /-1.,-1.,1.,1. /
C
      DO 6 I=1,100
    6 QQ(I)=0.0
      R12=R1-R2
      R13=R1-R3
      R14=R1-R4
      R23=R2-R3
      R24=R2-R4
      R34=R3-R4
      Z12=Z1-Z2
      Z13=Z1-Z3
      Z14=Z1-Z4
      Z23=Z2-Z3
      Z24=Z2-Z4
      Z34=Z3-Z4
      VOL=R13*Z24-R24*Z13
      RM=(R1+R2+R3+R4)/4.0
      ZM=(Z1+Z2+Z3+Z4)/4.0
      Y5=Z24/VOL
      X6=R13/VOL
      X7=R24/VOL
      Y8=Z13/VOL
      X5=-X7
      Y6=-Y8
      Y7=-Y5
      X8=-X6
      DO 30 II=1,4
      S=SS(II)*0.577350269189626
      T=TT(II)*0.577350269189626
      XJ =VOL+S*(R34*Z12-R12*Z34)+T*(R23*Z14-R14*Z23)
      XJAC=XJ/8.0
      SM=1.0-S
      SP=1.0+S
      TM=1.0-T
      TP=1.0+T
      H1=0.25*SM*TM
      H2=0.25*SP*TM
      H3=0.25*SP*TP
      H4=0.25*SM*TP
      R=H1*R1+H2*R2+H3*R3+H4*R4
      G1=H1/R
      G2=H2/R
      G3=H3/R
      G4=H4/R
      GC=SM*SP*TM*TP/R
      X1=(-R24+R34*S+R23*T)/XJ
```

```
      X2=( R13-R34*S-R14*T)/XJ
      X3=( R24-R12*S+R14*T)/XJ
      X4=(-R13+R12*S-R23*T)/XJ
      Y1=( Z24-Z34*S-Z23*T)/XJ
      Y2=(-Z13+Z34*S+Z14*T)/XJ
      Y3=(-Z24+Z12*S-Z14*T)/XJ
      Y4=( Z13-Z12*S+Z23*T)/XJ
      RS=0.25*(-TM*R1+TM*R2+TP*R3-TP*R4)
      ZS=0.25*(-TM*Z1+TM*Z2+TP*Z3-TP*Z4)
      RT=0.25*(-SM*R1-SP*R2+SP*R3+SM*R4)
      ZT=0.25*(-SM*Z1-SP*Z2+SP*Z3+SM*Z4)
      XC=-2.0*(T*SM*SP*RS-S*TM*TP*RT)/XJAC
      YC= 2.0*(T*SM*SP*ZS-S*TM*TP*ZT)/XJAC
      FAC=XJAC*R
C
C     FORM STIFFNESS QK
C
      DO 10 I=1,4
      D1=D(I,1)*FAC
      D2=D(I,2)*FAC
      D3=D(I,3)*FAC
      D4=D(I,4)*FAC
      QC(I,1)=          D1*Y1+D4*X5+D3*G1
      QC(I,3)=          D1*Y2+D4*X6+D3*G2
      QC(I,5)=          D1*Y3+D4*X7+D3*G3
      QC(I,7)=          D1*Y4+D4*X8+D3*G4
      QC(I,9)=          D1*YC      +D3*GC
      QC(I,2)=          D2*X1+D4*Y5
      QC(I,4)=          D2*X2+D4*Y6
      QC(I,6)=          D2*X3+D4*Y7
      QC(I,8)=          D2*X4+D4*Y8
      QC(I,10)=         D2*XC
   10 CONTINUE
      DO 20 I=1,10
      D1=QC(1,I)
      D2=QC(2,I)
      D3=QC(3,I)
      D4=QC(4,I)
      QQ(1,I)=QQ(1,I)+D1*Y1+D4*X5+D3*G1
      QQ(3,I)=QQ(3,I)+D1*Y2+D4*X6+D3*G2
      QQ(5,I)=QQ(5,I)+D1*Y3+D4*X7+D3*G3
      QQ(7,I)=QQ(7,I)+D1*Y4+D4*X8+D3*G4
      QQ(9,I)=QQ(9,I)+D1*YC      +D3*GC
      QQ(2,I)=QQ(2,I)+D2*X1+D4*Y5
      QQ(4,I)=QQ(4,I)+D2*X2+D4*Y6
      QQ(6,I)=QQ(6,I)+D2*X3+D4*Y7
      QQ(8,I)=QQ(8,I)+D2*X4+D4*Y8
      QQ(10,I)=QQ(10,I)+D2*XC
   20 CONTINUE
   30 CONTINUE
C
C     FORM STRESS MATRIX QS AT CENTROID (RM,ZM) OF ELEMENT
C
      DO 40 I=1,4
      D1=D(I,1)
```

```
      D2=D(I,2)
      D3=D(I,3)/(4.0*RM)
      D4=D(I,4)
      T1=( D1*Z24-D4*R24)/VOL
      T2=(-D1*Z13+D4*R13)/VOL
      T3=(-D2*R24+D4*Z24)/VOL
      T4=( D2*R13-D4*Z13)/VOL
      QC(I,1)=D3+T1
      QC(I,3)=D3+T2
      QC(I,5)=D3-T1
      QC(I,7)=D3-T2
      QC(I,9)=4.0*D3
      QC(I,2)= T3
      QC(I,4)= T4
      QC(I,6)=-T3
      QC(I,8)=-T4
      QC(I,10)=0.0
   40 CONTINUE
C
C     ELIMINATE CENTRE NODE
C
      DO 50 N=1,2
      L=10-N
      M=L+1
      DO 45 I=1,4
      E=QC(I,M)/QQ(M,M)
      DO 45 J=1,L
   45 QC(I,J)=QC(I,J)-E*QQ(M,J)
      DO 50 I=1,L
      C=QQ(I,M)/QQ(M,M)
      DO 50 J=1,L
   50 QQ(I,J)=QQ(I,J)-C*QQ(M,J)
C
C     RELOCATE STRESS, STIFFNESS AND LOAD MATRICES
C
      DO 70 J=1,8
      DO 70 I=1,4
      QS(I,J)=QC(I,J)
      QK(I,J)=QQ(I,J)
   70 QK(I+4,J)=QQ(I+4,J)
      VOL=VOL*RM/2.
      RETURN
      END
```

```
      SUBROUTINE ONED (R,Z,YMOD,FNU,H,IX,MTYPE,VOL)
      COMMON /LS4ARG/ LM(8),SS(4,8),XC,YC,ELMASS,S(8,8),C(4,4)
      DIMENSION R(1),Z(1),YMOD(1),FNU(1),H(1),IX(4),ST(4,8)
      DO 410 I=1,8
      DO 405 J=1,4
  405 ST(J,I)=0.0
      DO 410 J=1,8
  410 S(I,J)=0.0
      I=IX(1)
      J=IX(2)
      XC=(R(I)+R(J))/2.0
      YC=(Z(I)+Z(J))/2.0
      DX=R(J)-R(I)
      DY=Z(J)-Z(I)
      XL=SQRT(DX**2+DY**2)
      VOL=H(MTYPE)*XL*XC
C
      E1=YMOD(MTYPE)/(1.0-FNU(MTYPE)**2)
      C(1,1)=E1
      C(2,2)=E1
      C(1,2)=FNU(MTYPE)*E1
      C(2,1)=C(1,2)
C
      ST(1,1)=-DX/XL**2
      ST(1,2)=-DY/XL**2
      ST(1,3)=-ST(1,1)
      ST(1,4)=-ST(1,2)
      ST(2,1)=.5/XC
      ST(2,3)=ST(2,1)
C
      DO 411 I=1,4
      DO 411 J=1,8
  411 SS(I,J)=0.0
C
      DO 412 I=1,2
      DO 412 J=1,4
      DO 412 K=1,2
  412 SS(I,J)=SS(I,J)+C(I,K)*ST(K,J)
C
      DO 414 J=1,4
      DO 414 I=1,4
      DO 414 K=1,2
  414 S(I,J)=S(I,J)+ST(K,I)*SS(K,J)*VOL
      RETURN
      END
```

```
      SUBROUTINE TRIA (NN,MM,A)
      DIMENSION A(NN,1)
 1000 N=0
  100 N=N+1
      IF(N.EQ.NN) RETURN
      IF(A(N,1).EQ.0.0) GO TO 100
      I=N
      MB=MINO(MM,NN-N+1)
      DO 260 L=2,MB
      I=I+1
      C=A(N,L)/A(N,1)
      IF(C.EQ.0.0) GO TO 260
      J=0
      DO 250 K=L,MB
      J=J+1
  250 A(I,J)=A(I,J)-C*A(N,K)
      A(N,L)=C
  260 CONTINUE
      GO TO 100
      END
```

```
      SUBROUTINE BACKS(NN,MM,A,B)
C
      DIMENSION A(1),B(1)
C
      MMM=MM-1
      N=0
  270 N=N+1
      C=B(N)
      IF(A(N).NE.0.0) B(N)=B(N)/A(N)
      IF(N.EQ.NN) GO TO 300
      IL=N+1
      IH=MINO(NN,N+MMM)
      M=N
      DO 285 I=IL,IH
      M=M+NN
  285 B(I)=B(I)-A(M)*C
      GO TO 270
C
  300 IL=N
      N=N-1
      IF(N.EQ.0) RETURN
      IH=MINO(NN,N+MMM)
      M=N
      DO 400 I=IL,IH
      M=M+NN
  400 B(N)=B(N)-A(M)*B(I)
      GO TO 300
C
      END
```

```
      SUBROUTINE WRITEB (A,LA,N,NUMEL)
      DIMENSION A(LA)
      COMMON /BUF/ B(2140)
      LB=2140
      IF ( N .NE. 1 ) GO TO 100
       REWIND 1
      M=0
  100 MM=M+LA
      DO 200 I=1,LA
      II=I+M
  200 B(II)=A(I)
      M=MM
      IF ( N .EQ. NUMEL ) GO TO 300
      IF ( (M+LA) .LE. LB ) GO TO 400
  300 WRITE (1) B
      M=0
  400 RETURN
      END
```

```
      SUBROUTINE READB (A,LA,N,NUMEL)
      DIMENSION A(LA)
      COMMON /BUF/ B(2140)
      LB=2140
      IF ( N .NE. 1 ) GO TO 100
       REWIND 1
      M=0
      READ (1) B
  100 MM=M+LA
      DO 200 I=1,LA
      II=I+M
  200 A(I)=B(II)
      M=MM
      IF ( N .EQ. NUMEL ) GO TO 400
      IF ( (M+LA) .LE. LB ) GO TO 400
  300 READ (1) B
      M=0
  400 RETURN
      END
```

APPENDIX D

LISTING OF INPUT DATA FOR SAMPLE PROBLEM

```
AXISYMMETRIC SOLID SUBJECTED TO BLAST LODDING
 56    55     2    40     1    12     7                                  .00002
  12900    .       .3      .0007           .1875
  2 300    .       .17    .000225
  1     1                 .1875
  2     1                 .9879
  3     1                1.7883
  4     1                2.5887
  5     1                3.3891
  6     1                4.1895
  7     1                  4.99
  8          .8308       .1875
 14          .8308      4.99
 15         1.6616       .1875
 21         1.6616      4.99
 22         2.4924       .1875
 28         2.4924      4.99
 29         3.3232       .1875
 35         3.3232      4.99
 36         4.1540       .1875
 42         4.1540      4.99
 43    2.   4.9850       .1875
 49         4.9850      4.99
 50  2.0    5.5635      .1875
 56         5.5635      4.99
  1     1     8     8     1     1
  2     2     1     8     9     2
  7     7     6    13    14     2
  8     8    15    15     8     1
  9     9     8    15    16     2
 14    14    13    20    21     2
 15    15    22    22    15     1
 16    16    15    22    23     2
 21    21    20    27    28     2
 22    22    29    29    22     1
 23    23    22    29    30     2
 28    28    27    34    35     2
 29    29    36    36    29     1
 30     3    29    36    37     2
 35    35    34    41    42     2
 36    36    43    43    36     1
 37    37    36    43    44     2
 42    42    41    48    49     2
 43    43    50    50    43     1
 44    44    43    50    51     2
 49    49    48    55    56     2
 50    51    50    50    51     1
 55    56    55    55    56     1
 14     7    1.0         1.0
 21    14    1.0         1.0
 28    21    1.0         1.0
```