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Negligible Cohomology

Author
Gherman, Matthew Michael

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Negligible Cohomology

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by

Matthew Michael Gherman

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ABSTRACT OF THE DISSERTATION

Negligible Cohomology

by

Matthew Michael Gherman
Doctor of Philosophy in Mathematics
University of California, Los Angeles, 2023
Professor Alexander Sergee Merkurjev, Chair

For a finite group G, a G-module M, and a field F, an element $u \in H^d(G, M)$ is negligible over F if for each field extension L/F and every continuous group homomorphism from $\mathrm{Gal}(L_{\mathrm{sep}}/L)$ to G, u is in the kernel of the induced homomorphism $H^d(G, M) \to H^d(L, M)$. We determine the group of negligible elements in $H^2(G, M)$ for every abelian group M with trivial G-action in Chapter 3.

For p a prime and a trivial G-action on the coefficients, the negligible elements in the cohomology ring $H^*(G, \mathbb{Z}/p\mathbb{Z})$ form an ideal. In Chapter 4, we show that when p is odd or $p = 2$ and either $|G|$ is odd or F is not formally real, the Krull dimension of the quotient of mod p cohomology by the negligible ideal is 0. However, when $p = 2$, $|G|$ is even, and F is formally real, the Krull dimension of the quotient of mod 2 cohomology of a finite 2-group by the negligible ideal is 1.

In Chapter 5, we compute generators of the negligible ideal in the mod p cohomology of elementary abelian p-groups. We also partially compute generators of the negligible ideal in the mod p cohomology of cyclic groups, finite abelian p-groups, dihedral groups, symmetric groups, and generalized quaternion groups under certain roots of unity assumptions.

The dissertation of Matthew Michael Gherman is approved.

Paul Balmer

Raphael Alexis Rouquier

Burt Totaro

Alexander Sergee Merkurjev, Committee Chair

University of California, Los Angeles

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Professor Merkurjev and I co-authored [GM22] in which he generalized and wrote the majority of the key results. Professor Merkurjev and I also submitted [GM23] for publication. He was instrumental in streamlining the proofs and editing the document. Both papers were supported by the National Science Foundation grant DMS #1801530 for which I am extremely grateful.

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VITA

2017	B.S. (Mathematics), Duke University, Durham, North Carolina.
2019	M.A. (Mathematics), University of California, Los Angeles, Los Angeles
2021	Candidate in Philosophy for Mathematics, University of California, Los Angeles
2022	Liggett Fellow Award, Department of Mathematics, University of California, Los Angeles

CHAPTER 1

Introduction

The notion of negligible cohomology was introduced by J.-P. Serre in [Ser13] (see also [GMS03, Part I, §26]). Let G be a finite group, M a G-module, and F a field. A continuous group homomorphism $j : \Gamma_L = \text{Gal}(L_{\text{sep}}/L) \to G$ from the absolute Galois group Γ_L of a field extension L of F to G yields a homomorphism $j^* : H^d(G,M) \to H^d(L,M)$ of cohomology groups for every $d \geq 0$. An element $u \in H^d(G,M)$ is called *negligible over* F if $u \in \ker(j^*)$ for all field extensions L/F and all j. All negligible over F elements form a subgroup $H^d(G,M)_{\text{neg}} = H^d(G,M)_{\text{neg},F} \subset H^d(G,M)$.

The following examples outline important connections between negligible classes and a wide range of algebraic concepts.

Example 1.1. (1) Negligible cohomology elements are related to the embedding problem. Let K/F be a finite Galois field extension with $G = \text{Gal}(K/F)$. Let

$$1 \longrightarrow M \longrightarrow G' \xrightarrow{f} G \longrightarrow 1 \tag{1.1}$$

be an exact sequence of finite groups with M abelian. The conjugation G'-action on M makes M a G-module. The embedding problem for the exact sequence (1.1) and field extension K/F is to find a Galois G'-algebra K' over F such that the restriction map $G' = \text{Gal}(K'/F) \to \text{Gal}(K/F) = G$ coincides with f. Equivalently, one needs to find a lifting $\Gamma_F \to G'$ of the homomorphism $\Gamma_F \to G$ corresponding to the extension K/F. Let $u \in H^2(G,M)$ be the class of the exact sequence (1.1) and let $j : \Gamma_L \to G$ be the group homomorphism given by a field extension L/F. Then j extends to a homomorphism $\Gamma_L \to G'$ if and only if the pull-back of the sequence

(1.1) under j is split. The latter is equivalent to the triviality of the image of u under $j^* : H^2(G, M) \to H^2(L, M)$. In other words, the class u is negligible if and only if all embedding problems for the exact sequence (1.1) and all G-Galois field extensions L'/L of fields containing F have solutions.

(2) Let M be an abelian group which we view as a module over any profinite group with trivial action. The cohomology group $H^d(F, M) = H^d(\Gamma_F, M)$ is the colimit of the groups $H^d(G, M)$ over all finite discrete factor groups G of Γ_F. The group $H^d(G, M)_{\text{neg}}$ is contained in the kernel of the natural homomorphism $H^d(G, M) \to H^d(F, M)$.

(3) Negligible cohomology elements of G are related to the invariants of G as follows. Let M be an abelian group with trivial group action. Write $\text{Inv}^d(G, M)$ for the group of degree d (normalized) invariants of G with values in M over a field F (for the definition of the invariant see [GMS03]). The homomorphism

$$\text{inv} : H^d(G, M) \to \text{Inv}^d(G, M),$$

takes $u \in H^d(G, M)$ to the invariant sending the class of a G-algebra N over a field extension L/F (that is a G-torsor over Spec$L$) to $j^*(u) \in H^d(L, M)$ for $j : \Gamma_L \to G$ the natural group homomorphism. By definition, $H^d(G, M)_{\text{neg}} = \ker(\text{inv})$.

1.1 History of Negligible Cohomology

The definition of negligible cohomology that Serre used in [GMS03, Part I, §26] is slightly more restrictive than the one proposed in this dissertation. We make reference to a ground field F whereas a negligible class for Serre needs to be in the kernel of the restriction map for every field. Serre described the mod 2 negligible classes over $\mathbb{Q}$ of an elementary abelian 2-group in [GMS03, Part I, Lemma 26.4] and the negligible classes over $\mathbb{Q}$ of symmetric groups in [GMS03, Part I, Theorem 26.3].

Saltman, in [Sal95], introduced our notion of negligible class referring to a fixed ground field. However, he only worked with fields F that are algebraically closed of characteristic

0, which we often generalize in the dissertation. [Sal95] is an early notable study of higher unramified cohomology, focusing on $H^3(\mathbb{C}(V)^G, \mathbb{Q}/\mathbb{Z})$ for V a faithful $\mathbb{C}$-representation of a finite group G. He proved that the third unramified cohomology is contained in the image of the inflation map from $H^3(G, \mathbb{Q}/\mathbb{Z})$ to $H^3(\mathbb{C}(V)^G, \mathbb{Q}/\mathbb{Z})$. The kernel of the inflation map corresponds to $H^3(G, \mathbb{Q}/\mathbb{Z})_{\text{neg},\mathbb{C}}$.

[Sal95] discusses an easy to describe subgroup of the degree three negligible classes known as permutation negligible classes,

$$H^3(G, \mathbb{Q}/\mathbb{Z})_{\text{per}} = \ker(H^3(G, \mathbb{Q}/\mathbb{Z}) \to H^3(G, \mathbb{C}(V)^\times)).$$

For a finite p-group G, Saltman used permutation negligible classes to find a surjection from degree three negligible classes to a kind of equivariant Chow group. The permutation negligible classes make up part of the kernel of the map.

Emmanuel Peyre constructed examples of non-trivial degree three negligible classes for groups that are central extensions of an F_p-vector space by another in [Pey98]. He assumed that the ground field F contains a fourth root of unity and that $\text{char}(F) \neq 2$. In a specific example, Peyre provided a negligible class that is not permutation negligible, extending Saltman's examples of such elements for only 2-groups G.

In a subsequent paper [Pey99], Emmanuel Peyre adapted an argument by Bruno Kahn to study the kernel and cokernel of a map from the Galois cohomology of a field F to unramified degree three Galois cohomology of the rational function field of a variety with certain properties. Applying his result to negligible classes, Peyre proved that the negligible cohomology of $H^3(G, \mathbb{Q}/\mathbb{Z}(2))$ is canonically isomorphic to the equivariant Chow group $\text{CH}^2_G(F)$ for an algebraically closed field F and a finite group G.

Peyre continued his study of negligible classes in [Pey08]. Over the ground field $\mathbb{C}$, he proved that the inflation map induces a surjection

$$H^3_{\text{nr}}(G, \mathbb{Q}/\mathbb{Z})/H^3_{\text{per}}(G, \mathbb{Q}/\mathbb{Z}) \to H^3_{\text{nr}}(\mathbb{C}(V)^G, \mathbb{Q}/\mathbb{Z})$$

from a quotient of degree three unramified cohomology to the unramified Galois cohomology of $\mathbb{C}(V)^G$. The kernel of the surjection is killed by a power of 2 so, when G is of odd order, Peyre proved that all degree three unramified negligible classes are permutation negligible. The inflation map does not induce this isomorphism in general.

Peyre proved that if G is a finite group, then the prime to 2 part of $H^3(G, \mathbb{Q}/\mathbb{Z})_{\text{neg},\mathbb{C}}$ is equal to the permutation negligible classes. As an application of this result, Peyre constructed a group G and a non-trivial degree three unramified class of $\mathbb{C}(V)^G$. The corresponding unramified Brauer group of $\mathbb{C}(V)^G$ is trivial. Thus the G-invariant function field is not rational over $\mathbb{C}$, but second cohomology cannot detect it.

1.2 Outline of Dissertation

Chapter 2 of the dissertation includes basic results about negligible cohomology. Chapter 3 includes methods for computing the degree two negligible classes of finite groups in any coefficients with a trivial action. Much of Chapters 2 and 3 were published by the author and Alexander Merkurjev in [GM22].

A fundamental and difficult problem in Galois theory is to characterize those profinite groups which are realizable as absolute Galois groups of fields. One of the most common approaches has been to find constraints on the cohomology of absolute Galois groups. For instance, the Bloch-Kato conjecture, proved by Rost and Voevodsky, provides a presentation of the cohomology of absolute Galois groups with generators in degree one and relations in degree two.

Quillen proved in [Qui71, Corollary 7.8] that the Krull dimension of $H^{\text{even}}(G, \mathbb{Z}/p\mathbb{Z})$ is equal to the maximum rank of an elementary abelian p-subgroup of G. With Quillen's result as inspiration, Chapter 4 is dedicated to the Krull dimension of the mod p cohomology ring of finite groups modulo the negligible cohomology ideal. We find that most classes in the cohomology of a finite group disappear when mapped to the cohomology of an absolute Galois group. We can interpret the size of negligible cohomology as a further restriction on profinite groups that are realizable as absolute Galois groups of fields.

In [QV72], Quillen and Venkov proved that nilpotent elements in the group cohomology of a finite group G are detected on the elementary abelian p-subgroups of G. Likewise, the elementary abelian p-subgroups of G provide an effective tool for detecting negligible classes in the cohomology of G. Section 5.1 contains the computation of generators of the mod p negligible cohomology ideal of elementary abelian p-groups. The material from Chapter 4 and Section 5.1 has been submitted for publication by the author and Alexander Merkurjev as [GM23].

The remainder of Chapter 5 is a collection of computations of generators of the negligible cohomology ideal of cyclic groups (Section 5.3), finite abelian p-groups (Section 5.4), dihedral groups (Section 5.5), symmetric groups (Section 5.6), and generalized quaternion groups (Section 5.7). There is special focus on Conjecture 5.6 that the mod 2 negligible classes of a finite group G can be detected on elementary abelian 2-subgroups of G.

In the case of finite abelian p-groups, dihedral groups, symmetric groups, and generalized quaternion groups, there is still work that needs to be done to find a complete description of the generators of the negligible cohomology ideal. Many of the computations focus on mod p coefficients and have restrictions on the roots of unity present in the base field. These are notable avenues for future projects on the negligible cohomology ideal.

Another promising future area of study is the negligible cohomology of profinite groups. If a profinite group Γ is the absolute Galois group of a field K, then an automorphism of Γ induces an isomorphism on cohomology in any coefficients with trivial Γ-action. Thus Γ has trivial negligible cohomology over F for any subfield F of K. Loosely, then, negligible cohomology of a profinite group could detect how far a profinite group is from being an absolute Galois group.

1.3 Notation and Facts

We use the following notations in the paper.

G is a finite group;

F is the base field, F_{sep} is a separable closure of F, $\Gamma_F = \text{Gal}(F_{\text{sep}}/F)$ is the absolute Galois group of F;

μ is the group of roots of unity in F_{sep} and μ_m is the group of m-th roots of unity in F_{sep}, $\mu_m(F) = \mu_m \cap F^\times$, fix a generator ξ_m of μ_m;

For an abelian group A write A_{tors} for the torsion part of A and set $A[q] := \ker(A \xrightarrow{q} A)$, where q is an integer;

$A[p^\infty] := \bigcup_{s>0} A[p^s]$, where p is a prime integer;

$H^d(F, M) := H^d(\Gamma_F, M)$ for a (discrete) Γ_F-module (Galois module) M.

Let K be a field extension of F. We will fix a primitive m-th root of unity $\xi_m \in F_{\text{sep}}$ throughout the dissertation. When $\mu_m \subset K$, we identify μ_m with $\mathbb{Z}/m\mathbb{Z}$ as Γ_K-modules. Then $H^1(K, \mathbb{Z}/m\mathbb{Z}) \simeq H^1(K, \mu_m) \simeq K^\times/(K^\times)^m$, and we write an element of $H^1(K, \mathbb{Z}/m\mathbb{Z})$ as a class (a) for $a(K^\times)^m \in K^\times/(K^\times)^m$. Let $(a_i) \in K^\times/(K^\times)^m$ for $1 \le i \le d$. We often write $(a_1, \dots, a_d)$ for the cup product $(a_1) \cup \cdots \cup (a_d)$ in $H^d(K, \mathbb{Z}/m\mathbb{Z})$. Note that $(a, a) = (a, -1)$ and $(a, b) + (b, a) = 0$ for all $(a), (b) \in H^1(K, \mathbb{Z}/m\mathbb{Z})$.

In order to discuss Krull dimension, we define the commutative ring

$$\mathcal{H}(G, \mathbb{Z}/p\mathbb{Z}) = \begin{cases} H^{\text{even}}(G, \mathbb{Z}/p\mathbb{Z}) & \text{if } p \neq 2 \\ H^*(G, \mathbb{Z}/2\mathbb{Z}) & \text{if } p = 2. \end{cases}$$

Since inflation maps are ring homomorphisms, the negligible elements of $\mathcal{H}(G, \mathbb{Z}/p\mathbb{Z})$ form an ideal, denoted $\mathcal{I}(G, \mathbb{Z}/p\mathbb{Z})$. We write $Q(G, \mathbb{Z}/p\mathbb{Z}) = \mathcal{H}(G, \mathbb{Z}/p\mathbb{Z})/\mathcal{I}(G, \mathbb{Z}/p\mathbb{Z})$ for the *negligible quotient*. In the possibly non-commutative ring $H^*(G, \mathbb{Z}/p\mathbb{Z})$, we denote the two-sided ideal of negligible elements $I(G, \mathbb{Z}/p\mathbb{Z})$. The radical of the negligible ideal $I(G, \mathbb{Z}/p\mathbb{Z})$ in $H^*(G, \mathbb{Z}/p\mathbb{Z})$ is the ideal of *eventually negligible* elements.

CHAPTER 2

Background and Preliminary Results

Let V be a faithful (finite dimensional) representation of the group G over F. The group G acts on the field $F(V)$ of rational functions on V over F making $F(V)/F(V)^G$ a Galois G-extension. The following proposition shows that in the definition of negligible elements it suffices to consider only surjective group homomorphisms j and, moreover, only one (generic) Galois field extension $F(V)/F(V)^G$.

Proposition 2.1. *Let G be a finite group, M a G-module, $u \in H^d(G, M)$, and F a field. Let V be a faithful representation of G. The following conditions are equivalent:*

(1) *u is negligible over F, i.e., $u \in H^d(G, M)_{neg}$;*

(2) *$j^*(u) = 0$ for all field extensions L/F and every surjective group homomorphism $j : \Gamma_L \to G$;*

(3) *If $K = F(V)^G$ and $j_K : \Gamma_K \to G$ is given by the Galois G-extension $F(V)/K$, then $j_K^*(u) = 0$ in $H^d(K, M)$.*

Proof. $(1) \Rightarrow (2)$ is trivial.

$(2) \Rightarrow (3)$ is clear since the map j_K in (3) is surjective.

$(3) \Rightarrow (1)$: Let N/L be a Galois G-algebra for a field extension L/F and $j : \Gamma_L \to G$ a group homomorphism. We need to show that $j^*(u) = 0$. As the natural homomorphism $H^d(L, M) \to H^d(L(t), M)$, where $L(t)$ is the rational function field over L, is injective, replacing F by $F(t)$ and L by $L(t)$ if necessary, we may assume that the field L is infinite.

The scheme Spec(K) is the limit of the family of varieties U/G, where $U \subset V$ is a nonempty open G-invariant subscheme such that the morphism $U \to U/G$ is a G-torsor. For every such U write $i_U : H^d(G, M) \to H^d_{\text{ét}}(U/G, M)$ for the edge homomorphism in the Hochschild-Serre spectral sequence [Mil80, Chapter III, Theorem 2.20],

$E_2^{p,q} = H^p(G, H^q_{\text{ét}}(U, M)) \Rightarrow H^{p+q}_{\text{ét}}(U/G, M)$. Since $j_K^*(u) = 0$ and étale cohomology takes limits of schemes to colimits of cohomology groups [Mil80, Chapter III, Lemma 1.16], there is U such that $i_U(u) = 0$. As L is infinite, by [GMS03, Part I, §5], there is a morphism $k : \text{Spec}(L) \to U/G$ such that $\text{Spec}(N) \to \text{Spec}(L)$ is the pull-back of $U \to U/G$ with respect to k. Then the composition

$$H^d(G, M) \xrightarrow{i_U} H^d_{\text{ét}}(U/G, M) \xrightarrow{k^*} H^d(K, M)$$

coincides with j^*. Since $i_U(u) = 0$ we have $j^*(u) = 0$. □

Corollary 2.2. *(cf., [Ser13] and [Sal95, Proposition 4.5])*

(1) *In the notation of the proposition,*

$$H^d(G, M)_{neg} = \ker(H^d(G, M) \xrightarrow{j^*} H^d(F(V)^G, M)).$$

(2) *The group $H^d(G, M)_{neg}$ is trivial if $d \leq 1$.*

Proof. (1) This follows immediately from Proposition 2.1.

(2) As j is surjective, the inflation map j^* is injective if $d \leq 1$. □

In the following proposition we collect some functorial properties of negligible elements.

Proposition 2.3. *Let L/F be a field extension, G a finite group, M a G-module and $f : H \to G$ a homomorphism of finite groups. Then*

(1) *The map $f^* : H^d(G, M) \to H^d(H, M)$ takes $H^d(G, M)_{neg}$ into $H^d(H, M)_{neg}$;*
(2) *$H^d(G, M)_{neg} \subset H^d(G, M)_{neg,L}$;*
(3) *If L/F is finite, then $[L : F] \cdot H^d(G, M)_{neg,L} \subset H^d(G, M)_{neg}$;*
(4) *If $\alpha : M \to N$ is a G-module homomorphism, then the map $\alpha^* : H^d(G, M) \to H^d(G, N)$ takes $H^d(G, M)_{neg}$ into $H^d(G, N)_{neg}$.*

Proof. (1): Let $j : \Gamma_L \to H$ be a group homomorphism for a field extension L of F and $u \in H^d(G, M)_{\text{neg}}$. Then $j^*(f^*(u)) = (f \circ j)^*(u) = 0$, hence $f^*(u) \in H^d(H, M)_{\text{neg}}$.

(2): Let $K = F(V)^G$ as in Proposition 2.1(3) and set $KL := L(V)^G$. Let $u \in H^d(G, M)_{\text{neg}}$. By definition, $j^*(u) = 0$ in $H^d(K, M)$. It follows that $j^*_{KL}(u) = \text{res}_{KL/K} \circ j^*_K(u) = 0$ in $H^d(KL, M)$, hence $u \in H^d(G, M)_{\text{neg},L}$ by Corollary 2.2(1).

(3): If L/F is finite and $u \in H^d(G, M)_{\text{neg},L}$, then $\text{res}_{KL/K} \circ j^*_K(u) = j^*_{KL}(u) = 0$. Applying the corestriction homomorphism, we get

$$[L : F] \cdot j^*_K(u) = \text{cor}_{KL/K} \circ \text{res}_{KL/K} \circ j^*_K(u) = \text{cor}_{KL/K} \circ j^*_{KL}(u) = 0,$$

therefore, $[L : F] \cdot u \in H^d(G, M)_{\text{neg}}$.

(4) is clear. □

Corollary 2.4. *If p is a prime integer such that char$(F) \neq p$ and $p^s \cdot M = 0$ for some s, then*

$$H^d(G, M)_{neg} = H^d(G, M)_{neg, F(\xi_p)}.$$

Proof. Indeed, the degree $[F(\xi_p) : F]$ is prime to p. □

From now on assume that M is an abelian group with trivial G-action.

Lemma 2.5. *If M is a torsion free abelian group then $H^2(G, M)_{neg} = 0$.*

Proof. The exact sequence

$$0 \longrightarrow M \longrightarrow M \otimes \mathbb{Q} \longrightarrow M \otimes (\mathbb{Q}/\mathbb{Z}) \longrightarrow 0$$

yields the isomorphisms

$$H^2(G, M) \simeq H^1(G, M \otimes (\mathbb{Q}/\mathbb{Z})),\ H^2(L, M) \simeq H^1(L, M \otimes (\mathbb{Q}/\mathbb{Z}))$$

for every field L. Then $H^2(G, M)_{\text{neg}} \simeq H^1(G, M \otimes (\mathbb{Q}/\mathbb{Z}))_{\text{neg}} = 0$ by Corollary 2.2(2). □

The following proposition reduces the computation of negligible elements to the case when M is a torsion group.

Proposition 2.6. *Let M be an abelian group. Then the natural map*

$$H^2(G, M_{tors})_{neg} \to H^2(G, M)_{neg}$$

is an isomorphism.

Proof. If Γ is a profinite group and N is a torsion free abelian group, then $H^1(\Gamma, N) = \mathrm{Hom}(\Gamma, N) = 0$ since the image of every (continuous) homomorphism $\Gamma \to N$ is finite. The factor group M/M_{tors} is torsion free so the natural homomorphism $H^2(\Gamma, M_{\mathrm{tors}}) \to H^2(\Gamma, M)$ is injective. Therefore, both horizontal maps in the commutative diagram

$$\begin{array}{ccc} H^2(G, M_{\mathrm{tors}}) & \longrightarrow & H^2(G, M) \\ \downarrow{\scriptstyle j^*} & & \downarrow{\scriptstyle j^*} \\ H^2(L, M_{\mathrm{tors}}) & \longrightarrow & H^2(L, M) \end{array}$$

are injective for every field extension L/F and a group homomorphism $j : \Gamma_L \to G$.

Let $u \in H^2(G, M)_{\mathrm{neg}}$. By Lemma 2.5, the group $H^2(G, M/M_{\mathrm{tors}})_{\mathrm{neg}}$ is trivial, hence u comes from an element $w \in H^2(G, M_{\mathrm{tors}})$. The diagram chase shows $w \in H^2(G, M_{\mathrm{tors}})_{\mathrm{neg}}$, i.e., the map in the statement of the proposition is surjective. $\square$

Let $M = \mathrm{colim} M_i$ be a directed colimit of abelian groups M_i. By [Ser02, Chapter I, §2 Proposition 8], the cohomology of profinite groups commutes with directed colimits so $H^2(G, M)_{\mathrm{neg}} = \mathrm{colim} H^2(G, M_i)_{\mathrm{neg}}$. Since every torsion abelian group is the union of finite groups and every finite group is a direct sum of primary cyclic groups, Proposition 2.6 shows that in order to compute $H^2(G, M)_{\mathrm{neg}}$ for an arbitrary abelian group M, it suffices to determine the structure of $H^2(G, \mathbb{Z}/p^s\mathbb{Z})_{\mathrm{neg}}$ for all primes p and positive integers s.

If $\mathrm{char}(F) = p > 0$, then $H^d(G, \mathbb{Z}/p^s\mathbb{Z})_{\mathrm{neg}} = H^d(G, \mathbb{Z}/p^s\mathbb{Z})$ for $d \geq 2$ since $H^d(L, \mathbb{Z}/p^s\mathbb{Z})$ is trivial for $d \geq 2$ and every field extension L/F by [Ser02, Chapter II, §2.2 Proposition 3]. In what follows when computing the group $H^2(G, \mathbb{Z}/p^s\mathbb{Z})_{\mathrm{neg}}$ we will assume that $\mathrm{char}(F) \neq p$.

Lemma 2.7. *Let G be a finite group and L a field. For $d \geq 1$,*

$$H^d(G, \mathbb{Q}/\mathbb{Z}) \simeq H^{d+1}(G, \mathbb{Z})$$
$$H^d(L, \mathbb{Q}/\mathbb{Z}) \simeq H^{d+1}(L, \mathbb{Z})$$

and the isomorphisms respect negligible classes.

Proof. The short exact sequence $0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0$ induces a long exact sequence with the following portion.

$$H^d(-, \mathbb{Q}) \longrightarrow H^d(-, \mathbb{Q}/\mathbb{Z}) \longrightarrow H^{d+1}(-, \mathbb{Z}) \longrightarrow H^{d+1}(-, \mathbb{Q})$$

Since $\mathbb{Q}$ is uniquely divisible, $H^d(G, \mathbb{Q})$ and $H^d(L, \mathbb{Q})$ are trivial for $d \geq 1$. Let K be a field extension of F and $j : \Gamma_K \to G$ a continuous group homomorphism. The short exact sequence also induces the following commutative square for $d \geq 1$.

$$\begin{array}{ccc} H^d(G, \mathbb{Q}/\mathbb{Z}) & \hookrightarrow & H^{d+1}(G, \mathbb{Z}) \\ \downarrow{\scriptstyle j^*} & & \downarrow{\scriptstyle j^*} \\ H^d(K, \mathbb{Q}/\mathbb{Z}) & \hookrightarrow & H^{d+1}(K, \mathbb{Z}) \end{array}$$

Therefore, the connecting map induces an isomorphism on negligible classes. □

2.1 Cyclic Algebras

Let F be a field and $\Gamma_F = \text{Gal}(F_{\text{sep}}/F)$. Write $(\Gamma_F)^*$ for the group of (continuous) characters $\Gamma_F \to \mathbb{Q}/\mathbb{Z}$, i.e. $(\Gamma_F)^* = \text{Hom}(\Gamma_F, \mathbb{Q}/\mathbb{Z}) = H^1(F, \mathbb{Q}/\mathbb{Z}) = H^2(F, \mathbb{Z})$. For a character $x \in (\Gamma_F)^*$ and an element $a \in F^\times$ denote by (x, a) the class of the corresponding cyclic algebra in the Brauer group $\text{Br}(F)$ (see [GS17, §2.5]). By definition, $(x, a) = x \cup a$ with respect to the cup-product

$$(\Gamma)^* \otimes F^\times = H^2(F, \mathbb{Z}) \otimes H^0(F, F_{\text{sep}}^\times) \to H^2(F, F_{\text{sep}}^\times) = \text{Br}(F).$$

If $x \in (\Gamma_F)^*[2]$, i.e., $2x = 0$, then (x, a) is the class of a *quaternion algebra* split by the quadratic extension $F(a^{1/2})/F$. Conversely, every element in $\mathrm{Br}(F)$ that is split by $F(a^{1/2})/F$ is of the form (x, a) for some $x \in (\Gamma_F)^*[2]$.

Lemma 2.8. *If char$(F) \neq 2$, the kernel of the homomorphism $(\Gamma_F)^* \to$ Br(F) taking a character x to $(x, -1)$ coincides with $2(\Gamma_F)^*$.*

Proof. Let $x \in (\Gamma_F)^*$ and let m be the order of x. Consider the matrix $A \in \mathrm{GL}_m(F)$ defined by $(a_1, a_2, \dots, a_m) \cdot A = (a_2, a_3, \dots, a_m, -a_1)$ for all $a_i \in F$. Note that $A^m = -1$, hence we have a homomorphism $i : \mathbb{Z}/2m\mathbb{Z} \to \mathrm{GL}_m(F_{\text{sep}})$ defined by $i(r + 2m\mathbb{Z}) = A^r$. The upper row of the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \xrightarrow{\frac{1}{2}} & \mathbb{Q}/\mathbb{Z} & \xrightarrow{2} & \mathbb{Q}/\mathbb{Z} & \longrightarrow & 0 \\
 & & \Vert & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \xrightarrow{\frac{1}{2}} & \frac{1}{2m}\mathbb{Z}/\mathbb{Z} & \xrightarrow{2} & \frac{1}{m}\mathbb{Z}/\mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle i} & & \downarrow & & \\
1 & \longrightarrow & F_{\text{sep}}^{\times} & \longrightarrow & \mathrm{GL}_m(F_{\text{sep}}) & \xrightarrow{2} & \mathrm{PGL}_m(F_{\text{sep}}) & \longrightarrow & 1,
\end{array}$$

where $k(1 + 2\mathbb{Z}) = -1$ yields an exact sequence $(\Gamma_F)^* \xrightarrow{2} (\Gamma_F)^* \xrightarrow{\delta} H^2(F, \mathbb{Z}/2\mathbb{Z})$. Identifying $\mathbb{Z}/2\mathbb{Z}$ with μ_2 and $H^2(F, \mathbb{Z}/2\mathbb{Z})$ with the subgroup $H^2(F, \mu_2) = \mathrm{Br}(F)[2]$ of the Brauer group $H^2(F, F_{\text{sep}}^{\times}) = \mathrm{Br}(F)$ we see that it suffices to show that $\delta(x)$ is equal to the cyclic class $(x, -1)$.

It is shown in [GS17, §2.5] that the image of x under the composition

$$(\Gamma_F)^{\times} = H^1(F, \mathbb{Q}/\mathbb{Z}) \to H^1(F, \mathrm{PGL}(F_{\text{sep}})) \to H^2(F, F_{\text{sep}}^{\times}) = \mathrm{Br}(F)$$

given by the bottom row of the diagram coincides with $(x, -1)$. $\square$

2.2 The Negligible Ideal and Quotient

For a prime p, we will often take $M = \mathbb{Z}/p\mathbb{Z}$ with a trivial G-action. If char$(F) = p > 0$, then $H^d(K, \mathbb{Z}/p\mathbb{Z}) = 0$ for $d \geq 2$ by [Ser02, Chapter II, §2 Proposition 3] so $H^d(G, \mathbb{Z}/p\mathbb{Z})$ is entirely negligible for $d \geq 2$. We will, therefore, assume F is a field with char$(F) \neq p$ when computing the negligible classes of $H^*(G, \mathbb{Z}/p\mathbb{Z})$.

The Norm Residue Isomorphism Theorem (proved by Voevodsky and Rost) [HW19] reveals that the ideal $H^{>0}(K, \mathbb{Z}/p\mathbb{Z})$ is generated by elements of $H^1(K, \mathbb{Z}/p\mathbb{Z})$ when K contains a primitive p-th root of unity. Therefore, it is often sufficient to check properties on generators $(a) \in H^1(K, \mathbb{Z}/p\mathbb{Z})$ of $H^{>0}(K, \mathbb{Z}/p\mathbb{Z})$.

When $p = 2$, let $(a_i) \in H^1(K, \mathbb{Z}/2\mathbb{Z})$ for $1 \leq i \leq d$. Since $H^*(K, \mathbb{Z}/2\mathbb{Z})$ is a commutative ring, $(a_1, \ldots, a_d)^2 = (a_1, \ldots, a_d) \cup (-1)^d$. The squaring map and cup product by $(-1)^d$ are linear. Therefore, $\alpha^2 = \alpha \cup (-1)^d$ for any $\alpha \in H^d(K, \mathbb{Z}/2\mathbb{Z})$. An inductive argument reveals $\alpha^{k+1} = \alpha \cup (-1)^{dk}$.

The *level* of a field F, denoted $s(F)$, is the least number of squares that sum to -1 in F. We say that F is *formally real* if -1 cannot be written as a sum of squares. By [Lam05, Chapter VIII Theorem 1.10], F is formally real if and only if F has an ordering. Pfister's Level Theorem, [Lam05, Chapter XI Theorem 2.2], proves that when $s(F)$ is finite, $s(F)$ is a power of 2. If F is a field with $s(F) = 2^r$, we can, equivalently, say that the r-fold Pfister form $\langle\langle 1, \ldots, 1\rangle\rangle$ is anisotropic over F while the $(r+1)$-fold Pfister form $\langle\langle 1, \ldots, 1, 1\rangle\rangle$ is isotropic over F. By [EKM08, Section 16], the class $(-1)^{r+1} \in H^*(F, \mathbb{Z}/2\mathbb{Z})$ is trivial while $(-1)^r \in H^*(F, \mathbb{Z}/2\mathbb{Z})$ is not. For a proof of the result, see [OVV07, Theorem 4.1].

CHAPTER 3

Negligible Degree Two Cohomology of Finite Groups

3.1 Fields with many roots of unity

Proposition 3.1. *Let G be a finite group and F a field and let m be a positive integer such that char(F) does not divide m and $\mu_m \subset F^\times$. Then*

$$H^2(G, \mu_m)_{neg} = \ker(H^2(G, \mu_m) \to H^2(G, F^\times)),$$

where we view μ_m and $F^\times$ as trivial G-modules.

Proof. Let V be a finite dimensional faithful representation of G such that there is a G-invariant open subset $U \subset V$ with the property that $V \setminus U$ is of codimension at least 2 in V and there is a G-torsor $U \to X$ for a variety X over F. Such representations exist by [Tot99, Remark 1.4]. Since U is an open subscheme of an affine scheme, it is smooth over F. Note that $U \to X$ is étale and, hence, smooth. By [Sta23, Lemma 29.34.19], X is smooth over F so X is regular.

The Hochschild-Serre spectral sequence [Mil80, Chapter III, Theorem 2.20]

$$E_2^{p,q} = H^p(G, H^q_{\text{ét}}(U, \mathbb{G}_m)) \Rightarrow H^{p+q}_{\text{ét}}(X, \mathbb{G}_m)$$

yields an exact sequence

$$\text{Pic}(U)^G \to H^2(G, F[U]^\times) \to \text{Br}(X).$$

The group Pic(U) is trivial as U is an open subset of the affine space V. By the choice of

U every invertible regular function on U is constant, i.e., $F[U]^\times = F^\times$ and hence the map $H^2(G, F^\times) \to \mathrm{Br}(X)$ is injective.

By [Mil80, III, Example 2.22], the natural map $\mathrm{Br}(X) \to \mathrm{Br}(K)$, where $K = F(X)$, is injective. It follows that the bottom map of the commutative diagram

$$\begin{array}{ccc} H^2(G, \mu_m) & \longrightarrow & H^2(K, \mu_m) \\ \downarrow & & \downarrow \\ H^2(G, F^\times) & \longrightarrow & \mathrm{Br}(K) \end{array}$$

is injective. The right vertical morphism is also injective identifying $H^2(K, \mu_m)$ with $\mathrm{Br}(K)[m]$. Hence the other two homomorphisms in the diagram have equal kernels. Now the statement follows from Corollary 2.2(1). □

Remark. The proposition also follows from the isomorphism $\mathrm{Inv}^2(G, \mathbb{Q}/\mathbb{Z}) \simeq H^2(G, F^\times)$ established in [Bai17].

It follows from Proposition 3.1 that $H^2(G, \mu_m)_{\mathrm{neg}}$ coincides with the image of the connecting homomorphism

$$H^1(G, F^\times/\mu_m) \to H^2(G, \mu_m)$$

for the exact sequence $1 \to \mu_m \to F^\times \to F^\times/\mu_m \to 1$. An element of the group $H^1(G, F^\times/\mu_m)$ is a group homomorphism $G \to F^\times/\mu_m$. Its image is contained in $\mu(F)/\mu_m$. Consider the exact sequence

$$1 \to \mu_m \to \mu(F) \to \mu(F)/\mu_m \to 1. \tag{3.1}$$

We have proved the following result.

Corollary 3.2. *In the conditions of Proposition 3.1 the group $H^2(G, \mu_m)_{neg}$ coincides with the image of the connecting homomorphism $H^1(G, \mu(F)/\mu_m) \to H^2(G, \mu_m)$ for exact sequence (3.1).*

Exact sequence $0 \to \mathbb{Z}/m\mathbb{Z} \xrightarrow{\frac{1}{m}} \mathbb{Q}/\mathbb{Z} \xrightarrow{m} \mathbb{Q}/\mathbb{Z} \to 0$ for integer $m > 0$ yields an embedding

$$G^*/mG^* \hookrightarrow H^2(G, \mathbb{Z}/m\mathbb{Z}),$$

where $G^* := \mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z}) = H^1(G, \mathbb{Q}/\mathbb{Z})$ is the *character group* of G. We identify G^*/mG^* with a subgroup of $H^2(G, \mathbb{Z}/m\mathbb{Z})$.

3.2 Primary case

Let p be a prime integer and F a field such that $\mathrm{char}(F) \neq p$.

Lemma 3.3. *Let $\mu_{p^\infty}(F(\xi_p)) = \mu_{p^t}$ for some t with $1 \leq t \leq \infty$. Assume that $t \geq 2$ if $p = 2$. Then $\mu_{p^\infty}(F(\xi_{p^r})) = \mu_{p^r}$ for every $r \geq t$.*

Proof. The image of the injective homomorphism $\chi : \Gamma = \mathrm{Gal}(F(\mu_{p^\infty})/F(\xi_p)) \to \mathbb{Z}_p^\times$ taking an automorphism σ to the unique p-adic unit a such that $\sigma(\xi) = \xi^a$ for all $\xi \in \mu_{p^\infty}$ is contained in $U_t = \{a \in \mathbb{Z}_p^\times | a \equiv 1 \text{ mod } p^t\}$. Choose an element $\sigma \in \Gamma$ such that $\chi(\sigma) \notin U_{t+1}$. By assumption, U_t is a topological cyclic group generated by σ. It follows that $\mathrm{im}(\chi) = U_t$ and $F(\xi_{p^r})$ for all $r \geq t$ are all intermediate fields between $F(\xi_p)$ and $F(\mu_{p^\infty})$ corresponding to all closed subgroups $U_r \subset U_t$. □

Theorem 3.4. *Let G be a finite group, p a prime integer and s a positive integer. Let F be a field such that char$(F) \neq p$ and $\mu_{p^\infty}(F(\xi_p)) = \mu_{p^t}$ for some t with $1 \leq t \leq \infty$.*

(1) *If $t \geq s$, then*

$$H^2(G, \mathbb{Z}/p^s\mathbb{Z})_{neg} = (G^*[p^{t-s}] + p^sG^*)/p^sG^* \subset G^*/p^sG^* \subset H^2(G, \mathbb{Z}/p^s\mathbb{Z}).$$

(2) *If $t < s$ and $t \geq 2$ in the case $p = 2$, then $H^2(G, \mathbb{Z}/p^s\mathbb{Z})_{neg} = 0$.*

Proof. (1): Since $t \geq s$, by Corollary 2.4, we may assume that $\mu_{p^s} \subset F^\times$, hence $\mathbb{Z}/p^s\mathbb{Z} \simeq \mu_{p^s}$ as Galois modules. The p-primary component of the exact sequence (3.1) is isomorphic to the upper row of the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}/p^s\mathbb{Z} & \xrightarrow{p^{-s}} & p^{-t}\mathbb{Z}/\mathbb{Z} & \xrightarrow{p^s} & p^{s-t}\mathbb{Z}/\mathbb{Z} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{Z}/p^s\mathbb{Z} & \xrightarrow{p^{-s}} & \mathbb{Q}/\mathbb{Z} & \xrightarrow{p^s} & \mathbb{Q}/\mathbb{Z} & \longrightarrow & 0 \end{array}$$

Applying cohomology groups to the diagram and using Corollary 3.2 we see that the group $H^2(G, \mathbb{Z}/p^s\mathbb{Z})_{\text{neg}}$ coincides with the image of the composition

$$G^*[p^{t-s}] = H^1(G, p^{s-t}\mathbb{Z}/\mathbb{Z}) \to H^1(G, \mathbb{Q}/\mathbb{Z}) = G^* \to G^*/p^sG^* \subset H^2(G, \mathbb{Z}/p^s\mathbb{Z}),$$

whence the result.

(2): Let $L = F(\mu_{p^s})$. By Lemma 3.3, we have $\mu_{p^\infty}(L) = \mu_{p^s}$. The first part of the theorem applied to the field L show that $H^2(G, \mathbb{Z}/p^s\mathbb{Z})_{\text{neg},L} = 0$. It follows from Proposition 2.3(2) that $H^2(G, \mathbb{Z}/p^s\mathbb{Z})_{\text{neg}} = 0$. □

3.3 The case $p = 2$ and $t = 1$

It remains to consider the case $p = 2$ and $t = 1$ and F is a field of characteristic different from 2. The condition $t = 1$ means that -1 is not a square in F.

Proposition 3.5. *Let $b \geq a$ be positive integers, L a field such that $\xi_{2^b} \in L(\sqrt{-1})$ and $\Gamma = \Gamma_L$. Then $\Gamma^*[2^{b-a}] \cap 2\Gamma^* \subset 2^a\Gamma^*$.*

Proof. We prove the statement by induction on a. The case $a = 1$ is obvious.

$a = 2$: Let $x \in \Gamma^*[2^{b-2}] \cap 2\Gamma^*$. Write $x = 2y$ for $y \in \Gamma^*[2^{b-1}]$. Consider the cyclic class $(y, -1) \in \text{Br}(L)$. As $-1 = (\xi_{2^b})^{2^{b-1}}$ in $L' := L(\sqrt{-1})$, we have

$$(y, -1) \otimes_L L' = (y_{L'}, -1) = 2^{b-1} \cdot (y_{L'}, \xi_{2^b}) = (2^{b-1}y_{L'}, \xi_{2^b}) = 0$$

in the Brauer group $\text{Br}(L')$ since $2^{b-1}y = 0$. We proved that $(y, -1)$ is split by the extension

$L(\sqrt{-1})$ of L, hence $(y,-1)$ is the class of the quaternion algebra $(z,-1)$ for some $z \in \Gamma^*[2]$. It follows that $(y-z,-1)=0$, hence $y-z \in 2\Gamma^*$ by Lemma 2.8 and therefore,

$$x = 2y = 2(y-z) \in 4\Gamma^*.$$

$a-1 \Rightarrow a$: Let $x \in \Gamma^*[2^{b-a}] \cap 2\Gamma^*$. By the induction hypothesis, $x = 2^{a-1}y$ for some $y \in \Gamma^*[2^{b-1}]$. Then $2y \in \Gamma^*[2^{b-2}] \cap 2\Gamma^*$ and hence $2y \in 4\Gamma^*$ by the first part of the proof. Finally, $x = 2^{a-2} \cdot 2y \in 2^{a-2} \cdot 4\Gamma^* = 2^a\Gamma^*$. □

Theorem 3.6. *Let G be a finite group and s a positive integer. Let F be a field such that char$(F) \neq 2$ and $-1 \notin (F^\times)^2$. Write $\mu_{2^\infty}(F(\sqrt{-1})) = \mu_{2^{t'}}$ for some t' with $1 \leq t' \leq \infty$.*

(1) *If $t' \geq s$, then*

$$H^2(G,\mathbb{Z}/2^s\mathbb{Z})_{neg} = ((G^*[2^{t'-s}] \cap 2G^*) + 2^sG^*)/2^sG^* \subset G^*/2^sG^* \subset H^2(G,\mathbb{Z}/2^s\mathbb{Z}).$$

(2) *If $t' < s$, then $H^2(G,\mathbb{Z}/2^s\mathbb{Z})_{neg} = 0$.*

Proof. (1): It follows from Theorem 3.4(1) applied to the field $F' := F(\sqrt{-1})$ and Proposition 2.3(2) that

$$H^2(G,\mathbb{Z}/2^s\mathbb{Z})_{\text{neg}} \subset H^2(G,\mathbb{Z}/2^s\mathbb{Z})_{\text{neg},F'} = (G^*[2^{t'-s}] + 2^sG^*)/2^sG^*.$$

Applying Corollary 3.2 in the case $m=2$ we see that $H^2(G,\mathbb{Z}/2\mathbb{Z})_{\text{neg}} = 0$ since $t=1$. The commutativity of the diagram

$$\begin{array}{ccc} G^*/2^sG^* & \longrightarrow & G^*/2G^* \\ \downarrow & & \downarrow \\ H^2(G,\mathbb{Z}/2^s\mathbb{Z}) & \longrightarrow & H^2(G,\mathbb{Z}/2\mathbb{Z}) \end{array}$$

shows that $H^2(G,\mathbb{Z}/2^s\mathbb{Z})_{\text{neg}} \subset 2G^*/2^sG^*$. It follows that

$$H^2(G,\mathbb{Z}/2^s\mathbb{Z})_{\text{neg}} \subset ((G^*[2^{t'-s}] \cap 2G^*) + 2^sG^*)/2^sG^*.$$

Conversely, let $x \in G^*[2^{t'-s}] \cap 2G^*$. We show that the corresponding element in $G^*/2^sG^* \subset H^2(G, \mathbb{Z}/2^s\mathbb{Z})$ is negligible. Let L/F be a field extension and $j : \Gamma_L \to G$ a group homomorphism. Consider the following commutative diagram

$$\begin{array}{ccc} G^*/2^sG^* & \xrightarrow{j^*} & (\Gamma_L)^*/2^s(\Gamma_L)^* \\ \downarrow & & \downarrow \\ H^2(G, \mathbb{Z}/2^s\mathbb{Z}) & \longrightarrow & H^2(L, \mathbb{Z}/2\mathbb{Z}) \end{array}$$

By Proposition 3.5 applied to $a = s$ and $b = t'$ we see that the image of x in $(\Gamma_L)^*/2^s(\Gamma_L)^*$ is trivial and hence the image of x in $H^2(L, \mathbb{Z}/2^s\mathbb{Z})$ is also trivial, i.e., x is negligible.

(2): Let $L = F(\mu_{2^s}) = F'(\mu_{2^s})$. By Lemma 3.3 applied to F', we have $\mu_{2^\infty}(L) = \mu_{2^s}$. The first part of the theorem applied to the field L shows that $H^2(G, \mathbb{Z}/2^s\mathbb{Z})_{\text{neg},L} = 0$. It follows from Proposition 2.3(2) that $H^2(G, \mathbb{Z}/2^s\mathbb{Z})_{\text{neg}} = 0$. □

3.4 $\mathbb{Q}/\mathbb{Z}$ coefficients in characteristic zero

Proposition 3.7. *Assume that F is a field such that char$(F) = 0$ and $\mu \subset F$. The negligible cohomology of $H^2(G, \mathbb{Q}/\mathbb{Z})$ over F is trivial.*

Proof. Identify $\mu \simeq \mathbb{Q}/\mathbb{Z}$. Let V be a faithful F-representation of G. In [Bai17, Theorem 3.1], the normalized elements of $\text{Inv}^2(G, \mathbb{Q}/\mathbb{Z}(1)) = \text{Inv}^2(G, \mathbb{Q}/\mathbb{Z})$ are identified with $H^2(G, F^\times)$. The short exact sequence

$$1 \longrightarrow \mu \longrightarrow F^\times \longrightarrow F^\times/\mu \longrightarrow 1.$$

induces a long exact sequence in cohomology with the portion

$$H^1(G, F^\times/\mu) \longrightarrow H^2(G, \mu) \longrightarrow H^2(G, F^\times).$$

Since the G-action on $F^\times/\mu$ is taken to be trivial, we have $H^1(G, F^\times/\mu) = \text{Hom}(G, F^\times/\mu)$. The group $F^\times/\mu$ is torsion-free so $\text{Hom}(G, F^\times/\mu)$ is trivial for G finite. The induced map $H^2(G,\mu) \to H^2(G,F^\times)$ is injective. Therefore, inv : $H^2(G,\mathbb{Q}/\mathbb{Z}) \to \mathrm{Inv}^2(G,\mathbb{Q}/\mathbb{Z})$ is injective. We conclude $H^2(G,\mathbb{Q}/\mathbb{Z})_{\text{neg}}$ is trivial as in Example 1.1(3). □

Corollary 3.8. *Assume that F is a field such that char$(F) = 0$. The negligible cohomology of $H^2(G,\mathbb{Q}/\mathbb{Z})$ over F is trivial.*

Proof. Proposition 3.7 implies that $H^2(G,\mathbb{Q}/\mathbb{Z})_{\text{neg},F(\mu)} = 0$. Thus $H^2(G,\mathbb{Q}/\mathbb{Z})_{\text{neg}} = 0$ by Proposition 2.3(2). □

CHAPTER 4

Krull Dimension of the Negligible Quotient

4.1 Krull dimension of the negligible quotient over fields that are not formally real

In all cases except when $p = 2$, F is formally real, and G has even order, we prove that the mod p cohomology of a finite group G becomes entirely negligible after some degree. We begin with a more general result about the nilpotence of elements in Galois cohomology.

Lemma 4.1. *Let p be a prime. If p is odd, assume $\mu_p \subset K$. If $p = 2$, assume that the field K is not formally real. Then every element of $H^{>0}(K, \mathbb{Z}/p\mathbb{Z})$ is nilpotent.*

Proof. In the graded ring $H^*(K, \mathbb{Z}/p\mathbb{Z})$, the sum of nilpotent elements is nilpotent and the p-th power map is linear. It is thus sufficient to check nilpotence on homogeneous generators $(a) \in H^1(K, \mathbb{Z}/p\mathbb{Z})$ of $H^{>0}(K, \mathbb{Z}/p\mathbb{Z})$. When p is odd, we have $(a)^2 = (a, -1) = 0$. When $p = 2$ and $s(K) = 2^r$, $(-1)^{r+1}$ is trivial in $H^{r+1}(K, \mathbb{Z}/2\mathbb{Z})$. Let m be a power of 2 such that $r + 1 \leq m - 1$. Then $(a)^m = (a) \cup (-1)^{m-1} = 0$. $\square$

Corollary 4.2. *Assume that K is a field such that char$(K) \neq 2$ and $s(K) = 1$. The square of any element of $H^{>0}(K, \mathbb{Z}/2\mathbb{Z})$ is trivial.*

Theorem 4.3. *Let p be a prime, G a finite group, and F a field. If $p = 2$, assume that F is not formally real or G has odd order. Then the negligible quotient $Q(G, \mathbb{Z}/p\mathbb{Z})$ is finite. In particular, $Q(G, \mathbb{Z}/p\mathbb{Z})$ has Krull dimension 0.*

Proof. If $p = 2$ and $|G|$ is odd, $H^{>0}(G, \mathbb{Z}/2\mathbb{Z}) = 0$. Hence, we may assume that F is not formally real when $p = 2$. By Corollary 2.4, for negligible cohomology computations we may assume that $\mu_p \subset F$. The ring $\mathcal{H}(G, \mathbb{Z}/p\mathbb{Z})$ is finitely generated by [Eve91, Corollary 7.4.6]. Let K be a field extension of F and $j : \Gamma_K \to G$ a continuous group homomorphism. The image of each generator via j^* will be nilpotent in $H^{>0}(K, \mathbb{Z}/p\mathbb{Z})$ by Lemma 4.1. Therefore, each generator of $\mathcal{H}(G, \mathbb{Z}/p\mathbb{Z})$ is in the radical of $\mathcal{I}(G, \mathbb{Z}/p\mathbb{Z})$ and, hence, $Q(G, \mathbb{Z}/p\mathbb{Z})$ is finite. We conclude that $Q(G, \mathbb{Z}/p\mathbb{Z})$ is a ring of Krull dimension 0. □

4.2 Krull dimension of the negligible quotient over formally real fields

The final case to consider is when $p = 2$, F is formally real, and G has even order. With these assumptions, we prove the Krull dimension of the negligible quotient is always 1.

Lemma 4.4. *Let G be a finite group of even order. Assume that F is formally real. Then the Krull dimension of $Q(G, \mathbb{Z}/2\mathbb{Z})$ is positive.*

Proof. Let H be an order 2 cyclic subgroup of G. By Proposition 2.3(1), the restriction $\mathrm{res} : \mathcal{H}(G, \mathbb{Z}/2\mathbb{Z}) \to \mathcal{H}(H, \mathbb{Z}/2\mathbb{Z})$ factors as $f : Q(G, \mathbb{Z}/2\mathbb{Z}) \to Q(H, \mathbb{Z}/2\mathbb{Z})$. By [Eve61, Theorem 7.1], $\mathcal{H}(H, \mathbb{Z}/2\mathbb{Z})$ is a finite algebra over the subring $\mathrm{im}(\mathrm{res})$ so $Q(H, \mathbb{Z}/2\mathbb{Z})$ is a finite algebra over the subring $\mathrm{im}(f)$. [AM16, Corollary 5.9] shows $\dim(Q(G, \mathbb{Z}/2\mathbb{Z})) \geq \dim(Q(H, \mathbb{Z}/2\mathbb{Z}))$. By Theorem 5.1, $Q(H, \mathbb{Z}/2\mathbb{Z}) = \mathcal{H}(H, \mathbb{Z}/2\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}[x]$ has Krull dimension 1. □

Lemma 4.5. *Let G be a finite group. The Krull dimension of $Q(G, \mathbb{Z}/2\mathbb{Z})$ is at most 1.*

Proof. Let u and v be homogeneous elements of $H^*(G, \mathbb{Z}/2\mathbb{Z})$. Denote $k = \deg(u)$ and $\ell = \deg(v)$. We will show that $uv(u^\ell + v^k)$ is negligible. Let K be a field extension of F and $j : \Gamma_K \to G$ a continuous group homomorphism. Let $\alpha = j^*(u)$ and $\beta = j^*(v)$. Then

$$\begin{aligned} j^*(u^{\ell+1}v + uv^{k+1}) &= \alpha^{\ell+1} \cup \beta + \alpha \cup \beta^{k+1} \\ &= \alpha \cup (-1)^{k\ell} \cup \beta + \alpha \cup \beta \cup (-1)^{k\ell} \\ &= (\alpha \cup \beta + \alpha \cup \beta) \cup (-1)^{k\ell} \\ &= 0. \end{aligned}$$

We conclude that elements of the form $uv(u^\ell + v^k)$ are negligible.

For a set of generators $\{u_1, \ldots, u_m\}$ of $R = H^*(G, \mathbb{Z}/2\mathbb{Z})$ with $d_i = \deg(u_i)$, define the ideal $I = \langle u_i u_j (u_i^{d_j} + u_j^{d_i}) : 1 \leq i < j \leq m \rangle$. We showed above that $I \subset I(G, \mathbb{Z}/2\mathbb{Z})$. Let P be a prime ideal of R that contains $I(G, \mathbb{Z}/2\mathbb{Z})$ and, thus, I. It suffices to show that $\dim(R/P) \leq 1$ since $\dim(Q(G, \mathbb{Z}/2\mathbb{Z})) = \max_{P \supset I(G,\mathbb{Z}/2\mathbb{Z})} \dim(R/P)$. If $u_i \in P$ for all $1 \leq i \leq m$, then $R/P = \mathbb{Z}/2\mathbb{Z}$ and $\dim(R/P) = 0$. We may assume that $u_i \notin P$ for some $1 \leq i \leq m$. Since $u_i u_j (u_i^{d_j} + u_j^{d_i}) \in P$ and P is prime, $u_j \in P$ or $u_i^{d_j} + u_j^{d_i} \in P$ for every j. For the ring homomorphism $\varphi : \mathbb{Z}/2\mathbb{Z}[t] \to R/P$ defined as $\varphi(t) = u_i$, u_j is integral over $\operatorname{im}(\varphi)$ in either case. Thus R/P is a finite $\mathbb{Z}/2\mathbb{Z}[t]$-algebra so $\dim(R/P) \leq \dim(\mathbb{Z}/2\mathbb{Z}[t]) = 1$. $\square$

Theorem 4.6. *Let G be a finite group of even order and F a formally real field. Then the negligible quotient $Q(G, \mathbb{Z}/2\mathbb{Z})$ has Krull dimension 1.*

CHAPTER 5

Negligible Cohomology Ideal Computations

5.1 Elementary Abelian p-groups

In this section, G is an elementary abelian p-group of rank n or $G \simeq (\mathbb{Z}/p\mathbb{Z})^n$ for p a prime. We wish to compute generators of the negligible cohomology ideal of the mod p cohomology of G. We will first study the $p = 2$ case, which is a generalization of Serre's computation of negligible classes over $\mathbb{Q}$ for elementary abelian 2-groups found in [GMS03, Part I, Lemma 26.4]. By [CTV03, Proposition 4.5.4], the mod 2 cohomology of a rank n elementary abelian 2-group G is a polynomial ring in n variables,

$$H^*(G, \mathbb{Z}/2\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}[x_1, \ldots, x_n]$$

where $\{x_1, \ldots, x_n\}$ is a basis for $H^1(G, \mathbb{Z}/2\mathbb{Z})$ as a $\mathbb{Z}/2\mathbb{Z}$-vector space.

Throughout this section, we will denote $\{1, 2, \ldots, n\}$ by $[1, n]$.

Theorem 5.1. *Let G be an elementary abelian 2-group of rank n and F a field with char$(F) \neq 2$. Denote by $s(F)$ the level of F.*

(1) *If F is formally real, then $I(G, \mathbb{Z}/2\mathbb{Z})$ over F is generated by*

$$\{x_i x_j^2 + x_j x_i^2 : 1 \leq i < j \leq n\}.$$

(2) *If $s(F) = 2^r > 1$, then $I(G, \mathbb{Z}/2\mathbb{Z})$ over F is generated by*

$$\{x_i x_j^2 + x_j x_i^2 : 1 \leq i < j \leq n\} \cup \{x_i^{r+2} : 1 \leq i \leq n\}.$$

(3) *If $s(F) = 1$, then $I(G, \mathbb{Z}/2\mathbb{Z})$ over F is generated by*

$$\{x_i^2 : 1 \leq i \leq n\}.$$

Proof. See Section 2.2 for an overview of $s(F)$, the level of F. Let I be the ideal generated by the elements in the proposition statement for an elementary abelian 2-group of rank n. We will first prove that $I \subset I(G, \mathbb{Z}/2\mathbb{Z})$. Let K be a field extension of F and $j : \Gamma_K \to G$ be a continuous group homomorphism. Denote $(a_i) = j^*(x_i) \in H^1(K, \mathbb{Z}/2\mathbb{Z}) \simeq K^\times/(K^\times)^2$.

$$j^*(x_i x_j^2 + x_j x_i^2) = (a_i, a_j, a_j) + (a_j, a_i, a_i) = (a_i, a_j, -1) + (a_j, a_i, -1) = 0$$

If $s(F) = 2^r$, then $(-1)^{r+1}$ is trivial and

$$j^*(x_i^{r+2}) = (a_i)^{r+2} = (a_i) \cup (-1)^{r+1} = 0.$$

We will now show that $I(G, \mathbb{Z}/2\mathbb{Z}) \subset I$. Define the iterated Laurent series field $E = F((a_1))((a_2)) \cdots ((a_n))$ with indeterminates a_i. For $S \subset [1, n]$, denote $x_S = \prod_{i \in S} x_i$ and $(a_S) = \prod_{i \in S}(a_i)$ in $H^{|S|}(K, \mathbb{Z}/2\mathbb{Z})$. Then $H^*(E, \mathbb{Z}/2\mathbb{Z})$ is a free $H^*(F, \mathbb{Z}/2\mathbb{Z})$-module with basis $\{(a_S) : S \subset [1, n]\}$ by [Kat06, Theorem 3]. The field extension $E(\sqrt{a_1}, \ldots, \sqrt{a_n})$ over E is Galois with Galois group G acting by $g \cdot \sqrt{a_i} = (-1)^{x_i(g)}\sqrt{a_i}$ for $g \in G$. As a result, there is a continuous group homomorphism $j_E : \Gamma_E \to G$, which induces a ring homomorphism $j_E^* : H^*(G, \mathbb{Z}/2\mathbb{Z}) \to H^*(E, \mathbb{Z}/2\mathbb{Z})$.

Define the subset $T = \{x_S x_j^i : S \subset [1, n], j \in S \text{ maximal}, 0 \leq i\}$ of $H^*(G, \mathbb{Z}/2\mathbb{Z})$ if F is formally real or $T = \{x_S x_j^i : S \subset [1, n], j \in S \text{ maximal}, 0 \leq i \leq r + 2\}$ if F is not formally real and $s(F) = 2^r$. Denote by W the subspace of $H^*(G, \mathbb{Z}/2\mathbb{Z})$ generated by T. Note that, modulo I, every element of $H^*(G, \mathbb{Z}/2\mathbb{Z})$ may be reduced to an element of W. Further, for all $x_S x_j^i \in T$,

$$j_E^*(x_S x_j^i) = (a_S) \cup (a_j)^i = (a_S) \cup (-1)^i.$$

Since $\{(a_S) : S \subset [1, n]\}$ is linearly independent in $H^*(E, \mathbb{Z}/2\mathbb{Z})$ as a $H^*(F, \mathbb{Z}/2\mathbb{Z})$-module, the restriction of j_E^* to W is injective. We build the following commutative square.

$$\begin{array}{ccc} W & \xhookrightarrow{j_E^*} & H^*(E,\mathbb{Z}/2\mathbb{Z}) \\ \downarrow\!\!\downarrow & & \uparrow{\scriptstyle j_E^*} \\ H^*(G,\mathbb{Z}/2\mathbb{Z})/I & \xrightarrow{f} & H^*(G,\mathbb{Z}/2\mathbb{Z})/I(G,\mathbb{Z}/2\mathbb{Z}) \end{array}$$

A diagram chase implies that f is injective and $I(G,\mathbb{Z}/2\mathbb{Z}) \subset I$. □

We will now focus on the case when p is an odd prime. By [CTV03, Proposition 4.5.4], the mod p cohomology of an elementary abelian p-group G is a polynomial ring over the exterior algebra of the character group G^* of G,

$$H^*(G,\mathbb{Z}/p\mathbb{Z}) \simeq \Lambda(G^*)[y_1,\ldots,y_n]$$

where $\deg(y_j) = 2$ and n is the rank of G as an elementary abelian p-group. Let $\{x_1,\ldots,x_n\}$ be a basis for $H^1(G,\mathbb{Z}/2\mathbb{Z})$ as a $\mathbb{Z}/2\mathbb{Z}$-vector space. For each $1 \le i \le n$, we can choose $y_i = B(x_i)$ for $B : H^1(G,\mathbb{Z}/p\mathbb{Z}) \to H^2(G,\mathbb{Z}/p\mathbb{Z})$ the Bockstein homomorphism.

The following result can be found in [EM11, proof of Proposition 3.2].

Lemma 5.2. *Let K be a field that contains a primitive p-th root of unity ξ_p. Then $B(\alpha) = \alpha \cup (\xi_p)$ for $\alpha \in H^1(K,\mathbb{Z}/p\mathbb{Z})$.*

Proof. Let $\tilde{B} : H^1(K,\mathbb{Z}/p\mathbb{Z}) \to H^2(K,\mathbb{Z})$ denote the integral Bockstein homomorphism. The homomorphism $f : \mathbb{Z} \to K_{\text{sep}}^\times$ satisfying $f(1) = \xi_p$ factors through $\mathbb{Z}/p\mathbb{Z}$. We can build the following commutative diagram.

$$\begin{array}{ccccc} H^1(K,\mathbb{Z}/p\mathbb{Z}) & \xrightarrow{\tilde{B}} & H^2(K,\mathbb{Z}) & \xrightarrow{f_*} & H^2(K,K_{\text{sep}}^\times) \simeq \text{Br}(K) \\ & {\scriptstyle B}\searrow & \downarrow & \nearrow & \\ & & H^2(K,\mathbb{Z}/p\mathbb{Z}) & & \end{array}$$

By [GS17, Proposition 4.7.3, Corollary 2.5.5, and Proposition 4.7.1], $f^*(\tilde{B}(\alpha))$ is $\alpha \cup (\xi_p)$ in $H^2(K,\mathbb{Z}/p\mathbb{Z})$. Therefore, $B(\alpha) = \alpha \cup (\xi_p)$ by commutativity. □

Let K be a field extension of F and $j : \Gamma_K \to G$ be a continuous group homomorphism. The Bockstein commutes with inflation so, for $x \in H^1(G, \mathbb{Z}/p\mathbb{Z})$,

$$j^*(B(x)) = B(j^*(x)) = j^*(x) \cup (\xi_p) \tag{5.1}$$

when $\mu_p \subset K$ by Lemma 5.2.

Theorem 5.3. *Let p be an odd prime. Let G be an elementary abelian p-group of rank n.*

(1) *If F does not contain a primitive p^2 root of unity, then $I(G, \mathbb{Z}/p\mathbb{Z})$ over F is generated by*

$$\{x_i y_j + x_j y_i : 1 \leq i \leq j \leq n\} \cup \{y_i y_j : 1 \leq i \leq j \leq n\}.$$

(2) *If F contains a primitive p^2 root of unity, then $I(G, \mathbb{Z}/p\mathbb{Z})$ over F is generated by*

$$\{y_i : 1 \leq i \leq n\}.$$

Proof. By Corollary 2.4, we may assume that $\mu_p \subset F$ for negligible cohomology computations. Let I be the ideal generated by the elements in the proposition statement for an elementary abelian p-group of rank n. We will first prove that $I \subset I(G, \mathbb{Z}/p\mathbb{Z})$. Let K be a field extension of F and $j : \Gamma_K \to G$ be a continuous group homomorphism. Denote $(a_i) = j^*(x_i) \in H^1(K, \mathbb{Z}/p\mathbb{Z}) \simeq K^\times/(K^\times)^p$ so

$$j^*(y_i) = j^*(B(x_i)) = B(j^*(x_i)) = B(a_i) = (a_i, \xi_p)$$

by equation (5.1). We have

$$\begin{aligned} j^*(x_i y_j + x_j y_i) &= (a_i, a_j, \xi_p) + (a_j, a_i, \xi_p) = 0 \\ j^*(y_i y_j) &= (a_i, \xi_p, a_j, \xi_p) = -(a_i, a_j, \xi_p, \xi_p) = -(a_i, a_j, \xi_p, -1) = 0. \end{aligned}$$

If F contains a primitive p^2 root of unity ξ_{p^2}, we obtain

$$j^*(y_i) = (a_i, \xi_p) = (a_i, \xi_{p^2}^p) = p(a_i, \xi_{p^2}) = 0.$$

We will now show that $I(G, \mathbb{Z}/p\mathbb{Z}) \subset I$. Define the field extension E of F as in the proof of Theorem 5.1. Once again, by [Kat06, Theorem 3], $H^*(E, \mathbb{Z}/p\mathbb{Z})$ is a free $H^*(F, \mathbb{Z}/p\mathbb{Z})$-module with basis $\{(a_S) : S \subset [1, n]\}$. As before, we have $j_E^* : H^*(G, \mathbb{Z}/p\mathbb{Z}) \to H^*(E, \mathbb{Z}/p\mathbb{Z})$.

Define the subsets

$$T_1 = \{x_S : S \subset [1, n]\}$$
$$T_2 = \{x_S y_j : S \subset [1, n], i < j \text{ for each } i \in S\}$$

of $H^*(G, \mathbb{Z}/p\mathbb{Z})$. If F does not contain a p^2 root of unity, let $T = T_1 \cup T_2$. If F does contain a p^2 root of unity, let $T = T_1$. Denote by W the subspace of $H^*(G, \mathbb{Z}/p\mathbb{Z})$ generated by T. Note that, modulo I, every element of $H^*(G, \mathbb{Z}/p\mathbb{Z})$ may be reduced to an element of W. Further,

$$j_E^*(x_S) = (a_S)$$
$$j_E^*(x_S y_j) = (a_S) \cup (a_j, \xi_p) = (a_{S \cup \{j\}}) \cup (\xi_p).$$

Since $\{(a_S) : S \subset [1, n]\}$ is linearly independent in $H^*(E, \mathbb{Z}/p\mathbb{Z})$ as a $H^*(F, \mathbb{Z}/p\mathbb{Z})$-module, the restriction of j_E^* to W is injective. We build the following commutative square.

$$\begin{array}{ccc} W & \xhookrightarrow{\quad j_E^* \quad} & H^*(E, \mathbb{Z}/p\mathbb{Z}) \\ \big\downarrow & & \big\uparrow {\scriptstyle j_E^*} \\ H^*(G, \mathbb{Z}/p\mathbb{Z})/I & \xrightarrow{\quad f \quad} & H^*(G, \mathbb{Z}/2\mathbb{Z})/I(G, \mathbb{Z}/p\mathbb{Z}) \end{array}$$

A diagram chase implies that f is injective and $I(G, \mathbb{Z}/p\mathbb{Z}) \subset I$. □

5.1.1 Open conjecture

In [QV72], Quillen and Venkov proved that nilpotent elements in the group cohomology of a finite group G are detected on the elementary abelian p-subgroups of G. We could hope that there is a similar result about negligible classes.

Recall from Section 2.2 that a class $u \in H^*(G, \mathbb{Z}/p\mathbb{Z})$ is eventually negligible if u^k is negligible for some $k \geq 1$.

Conjecture 5.4. *Let G be a finite group and F a field such that char$(F) \neq p$. An element of $H^*(G, \mathbb{Z}/p\mathbb{Z})$ is eventually negligible if and only if it restricts to an eventually negligible element in $H^*(H, \mathbb{Z}/p\mathbb{Z})$ for each elementary abelian p-subgroup $H \subset G$.*

If p is odd, every element of $H^{>0}(G, \mathbb{Z}/p\mathbb{Z})$ and $H^{>0}(H, \mathbb{Z}/p\mathbb{Z})$ is eventually negligible by Theorem 4.3. [QV72] handles the degree zero classes so the conjecture holds when p is odd. The conjecture holds for the same reasons if $p = 2$ and $|G|$ is odd or F is not a formally real field.

Lemma 5.5. *Let H be an elementary abelian 2-group of rank n and F a formally real field such that char$(F) \neq 2$. Then $I(H, \mathbb{Z}/2\mathbb{Z})$ is radical.*

Proof. Identify $H^*(H, \mathbb{Z}/2\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}[x_1, \ldots, x_n]$ as a polynomial ring in n variables. Then [GMS03, Part I, Lemma 26.4] proves an homogeneous polynomial $f \in \mathbb{Z}/2\mathbb{Z}[x_1, \ldots, x_n]$ is contained in $I(H, \mathbb{Z}/2\mathbb{Z})$ if and only if f vanishes when each x_i is evaluated on values of $\mathbb{Z}/2\mathbb{Z}$. We note that f^k vanishing implies f vanishes so $f^k \in I(H, \mathbb{Z}/2\mathbb{Z})$ implies f is in $I(H, \mathbb{Z}/2\mathbb{Z})$. For a possibly inhomogeneous $g \in H^*(H, \mathbb{Z}/2\mathbb{Z})$, an inductive argument will prove that $g^k \in I(H, \mathbb{Z}/2\mathbb{Z})$ implies $g \in I(H, \mathbb{Z}/2\mathbb{Z})$. □

Lemma 5.5 states that the eventually negligible elements of $H^*(H, \mathbb{Z}/2\mathbb{Z})$ are negligible for H an elementary abelian 2-group. We can rephrase the conjecture as follows.

Conjecture 5.6. *Let G be a finite group and F a formally real field. An element of $H^*(G, \mathbb{Z}/2\mathbb{Z})$ is eventually negligible if and only if it restricts to a negligible element in $H^*(H, \mathbb{Z}/2\mathbb{Z})$ for each elementary abelian 2-subgroup $H \subset G$.*

Let $v \in H^*(G, \mathbb{Z}/2\mathbb{Z})$ be an element that does not restrict to a negligible element in $H^*(H, \mathbb{Z}/2\mathbb{Z})$ for some elementary abelian 2-subgroup $H \subset G$. Since $I(H, \mathbb{Z}/2\mathbb{Z})$ is radical, no power of v restricts to a negligible element in $H^*(H, \mathbb{Z}/2\mathbb{Z})$. By Proposition 2.3(1), v is not eventually negligible in $H^*(G, \mathbb{Z}/2\mathbb{Z})$. Therefore, eventually negligible elements of

$H^*(G, \mathbb{Z}/2\mathbb{Z})$ restrict to negligible elements of $H^*(H, \mathbb{Z}/2\mathbb{Z})$. However, we have yet to prove that we can detect eventually negligible elements of $H^*(G, \mathbb{Z}/2\mathbb{Z})$ on the cohomology of elementary abelian 2-subgroups of G.

5.2 Negligible Ideal Computational Tools

When G is not an elementary abelian p-group, the following results will help compute the negligible cohomology ideal in some cases.

Lemma 5.7. *Let $H \subset G$ be a subgroup of a finite group G. Denote by*

$$\textit{cor}^d : H^d(H, M) \to H^d(G, M)$$

the degree d corestriction (or transfer) map. If $u \in H^d(H, M)$ is negligible over F, then $\textit{cor}^d(u)$ is negligible over F in $H^d(G, M)$.

Proof. Let $K = F(V)^G$ for V a faithful F-representation of G. Since H is a subgroup of G, V is likewise a faithful F-representation of H. Define $K_H = F(V)^H$. We build the following commutative square.

$$\begin{array}{ccc} H^d(H, M) & \xrightarrow{\text{cor}^d} & H^d(G, M) \\ \downarrow{\scriptstyle \text{inf}_H} & & \downarrow{\scriptstyle \text{inf}} \\ H^d(K_H, M) & \xrightarrow{\text{cor}^d} & H^d(K, M) \end{array}$$

Proposition 2.1 proves that the kernel of these inflation maps are the negligible classes of $H^d(H, M)$ and $H^d(G, M)$ respectively. Then $\text{inf}_H(u) = 0$ implies that $\text{inf}(\text{cor}^d(u)) = 0$. We conclude that $\text{cor}^d(u)$ is negligible in $H^d(G, M)$. □

Lemma 5.8. *Let G be a finite group and p be a prime integer. Let $H \subset G$ be a subgroup for which $\gcd([G : H], p) = 1$. Then restriction is an injection of rings $H^*(G, \mathbb{Z}/p\mathbb{Z}) \to H^*(H, \mathbb{Z}/p\mathbb{Z})$. Further, $I(G, \mathbb{Z}/p\mathbb{Z}) = H^*(G, \mathbb{Z}/p\mathbb{Z}) \cap I(H, \mathbb{Z}/p\mathbb{Z})$ when $H^*(G, \mathbb{Z}/p\mathbb{Z})$ is viewed as a subring of $H^*(H, \mathbb{Z}/p\mathbb{Z})$.*

Proof. The composition of the corestriction and restriction maps is multiplication by $[G : H]$

on each $H^d(G, \mathbb{Z}/p\mathbb{Z})$. Since $\gcd([G : H], p) = 1$, the composition is an isomorphism on each $H^d(G, \mathbb{Z}/p\mathbb{Z})$. We conclude that res : $H^*(G, \mathbb{Z}/p\mathbb{Z}) \to H^*(H, \mathbb{Z}/p\mathbb{Z})$ is an injective ring map and each $\mathrm{cor}^d : H^d(H, \mathbb{Z}/p\mathbb{Z}) \to H^d(G, \mathbb{Z}/p\mathbb{Z})$ is a surjective group homomorphism.

View $H^*(G, \mathbb{Z}/p\mathbb{Z})$ as a subring of $H^*(H, \mathbb{Z}/p\mathbb{Z})$ via restriction. If $u \in I(G, \mathbb{Z}/p\mathbb{Z})$, then $\mathrm{res}(u) \in I(H, \mathbb{Z}/p\mathbb{Z})$ by Proposition 2.3(1). Thus

$$I(G, \mathbb{Z}/p\mathbb{Z}) \subset H^*(G, \mathbb{Z}/p\mathbb{Z}) \cap I(H, \mathbb{Z}/p\mathbb{Z}).$$

If $v \in H^d(G, \mathbb{Z}/p\mathbb{Z}) \cap H^d(H, \mathbb{Z}/p\mathbb{Z})_{\text{neg}}$, then $v = \mathrm{res}(u)$ for some $u \in H^d(G, \mathbb{Z}/p\mathbb{Z})$. We have $\mathrm{cor}^d(v) = \mathrm{cor}^d(\mathrm{res}^d(u)) = [G : H]u$ is an element of $H^d(G, \mathbb{Z}/p\mathbb{Z})_{\text{neg}}$ by Lemma 5.7. We conclude that u is an element of $H^d(G, \mathbb{Z}/p\mathbb{Z})_{\text{neg}}$ since $[G : H]$ is invertible in $H^d(G, \mathbb{Z}/p\mathbb{Z})$. For an inhomogeneous $w \in H^*(G, \mathbb{Z}/p\mathbb{Z}) \cap I(H, \mathbb{Z}/p\mathbb{Z})$, perform the above procedure on each homogeneous piece. Therefore, $I(G, \mathbb{Z}/p\mathbb{Z}) \supset H^*(G, \mathbb{Z}/p\mathbb{Z}) \cap I(H, \mathbb{Z}/p\mathbb{Z})$. □

We will make frequent use of the following short exact sequence.

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\ n\ } \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0 \tag{5.2}$$

Lemma 5.9. *Assume that $\mu \subset F$. Let L be a field extension of F. Then the connecting map $H^d(L, \mathbb{Z}/n\mathbb{Z}) \to H^{d+1}(L, \mathbb{Z})$ in the long exact sequence induced by (5.2) is injective for $d \geq 2$.*

Proof. Since F contains all roots of unity, we can identify $\mu(d) \simeq \mathbb{Q}/\mathbb{Z}$ for any d. Taking a direct limit over all integers m of the Norm Residue Isomorphism [HW19] yields $H^{d-1}(L, \mathbb{Q}/\mathbb{Z}) \simeq K^M_{d-1}(L) \otimes \mathbb{Q}/\mathbb{Z}$ for $K^M_{d-1}(L)$ the $(d-1)$st Milnor K-group of the field L. Therefore, $H^{d-1}(L, \mathbb{Q}/\mathbb{Z})$ is n-divisible for $d \geq 2$. Lemma 2.7 proves that $H^{d-1}(L, \mathbb{Q}/\mathbb{Z})$ is isomorphic to $H^d(L, \mathbb{Z})$ so $H^d(L, \mathbb{Z})$ is also n-divisible. The connecting map of the long exact sequence induced by (5.2) is injective for $d \geq 2$. □

In the rest of Chapter 5, we will often assume that $\mu \subset F$. The next result proves that, from a negligible cohomology perspective, the same computations can be obtained by assuming F contains only finitely many roots of unity. However, the result does not indicate the exact roots of unity required.

Proposition 5.10. *Let L over F be an algebraic field extension. For each positive integer d, there is some finite intermediate field extension L_0/F for which $H^d(G, M)_{neg,L_0} = H^d(G, M)_{neg,L}$. Further, the negligible ideal $I(G, \mathbb{Z}/p\mathbb{Z})$ over some finite intermediate field extension L_0/F is the same as that over L.*

Proof. Let $u \in H^d(G, M)_{\text{neg},L}$. We can write L as the colimit of finite algebraic extension L_i/F. Let V be a faithful representation of G over L. Then $L(V)^G = \cup_i L_i(V)^G$, and we obtain the following commutative diagram for each i.

$$\begin{array}{ccc} H^d(L_i(V)^G, M) & \xrightarrow{\text{res}_i} & H^d(L(V)^G, M) \\ & {\nwarrow}^{\text{inf}_i} \quad {\nearrow}^{\text{inf}} & \\ & H^d(G, M) & \end{array}$$

Since inf$(u) = 0$, we have res$_i$(inf$_i(u)) = 0$ or inf$_i(u) \in \ker($res$_i)$ for each i. The universal map f : colim$_i H^d(L_i(V)^G, M) \to H^d(L(V)^G, M)$ induced by the restrictions is an isomorphism via [Ser02, Chapter I, §2 Proposition 8]. Therefore, inf$_i(u) = 0$ for some i and $u \in H^d(G, M)_{\text{neg},L_i}$ by Proposition 2.1.

By [Eve91, Corollary 7.4.6], the Noetherian ring $H^*(G, \mathbb{Z}/p\mathbb{Z})$ is finitely generated. Thus the negligible ideal $I(G, \mathbb{Z}/p\mathbb{Z})$ over L is finitely generated. We may assume that the generators are homogeneous. To construct L_0, take the compositum of the finite field extensions corresponding to each generator of the negligible ideal. □

5.3 Cyclic Groups

Unfortunately, negligible cohomology computations for groups more complicated than elementary abelian p-groups become difficult. Section 5.3.1 culminates with Proposition 5.15 in which we show degree three and degree four cohomology of cyclic groups is entirely negligible under certain roots of unity assumptions. In Section 5.3.2, we use the result to find generators of the mod p negligible cohomology ideal of cyclic p-groups in Propositions 5.17 and 5.19 under certain roots of unity assumptions. Proposition 5.17 describes the $p = 2$ case with a limitation on the roots of unity. Section 5.3.3 provides a slight relaxation of the roots of unity requirement of Proposition 5.15.

5.3.1 Degree Three and Degree Four Negligible Cohomology of Cyclic Groups

In order to compute the degree three and degree four negligible classes of cyclic groups, we will need preliminary results about connecting homomorphisms.

Let K be a field extension of F and $j : \Gamma_K \to G$ a continuous group homomorphism. We will assume throughout this section that $\mu_m \subset F$ so μ_m is a trivial Γ_K-module. Endow $\mathrm{Ext}^1_{\Gamma_K}(\mu_m, \mu_m)$ with a group structure via the Baer sum. The group homomorphism

$$\Phi_k : \mathrm{Ext}^1_{\Gamma_K}(\mu_m, \mu_m) \to \mathrm{Hom}(H^k(K, \mu_m), H^{k+1}(K, \mu_m))$$

identifies the class of an extension with a connecting homomorphism in the induced long exact sequence on cohomology.

Let C_i in $\mathrm{Ext}^1_{\Gamma_K}(\mu_m, \mu_m)$ denote the class of the extension

$$1 \longrightarrow \mu_m \longrightarrow \mu_{m^2}^{\otimes i} \longrightarrow \mu_m \longrightarrow 1.$$

Let ξ_{m^2} be a primitive m^2 root of unity. The cyclic extension $K(\xi_{m^2})/K$ provides a group homomorphism $\phi : \Gamma_K \to \mathrm{Gal}(K(\xi_{m^2})/K)$. The action on $\mu_{m^2}^{\otimes i}$ is given by $\sigma \cdot \xi_m^\ell = \phi(\sigma)^i(\xi_m^\ell)$.

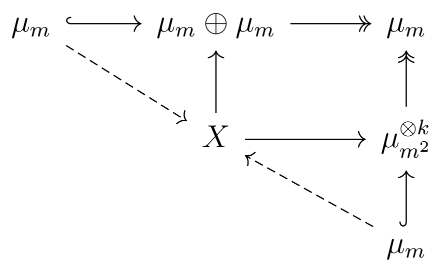
For a cyclic degree m field extension E of K, we know that $E = K(\sqrt[m]{x})$ for some $x \in E^\times$ since $\mu_m \subset K$. Denote by $\chi : \Gamma_K \to \mathrm{Gal}(E/K) \simeq \mathbb{Z}/m\mathbb{Z}$ the surjective group homomorphism induced by the field extension. Let $D(x)$ in $\mathrm{Ext}^1_{\Gamma_K}(\mu_m, \mu_m)$ be the class of the extension

$$1 \longrightarrow \mu_m \longrightarrow \mu_m \oplus \mu_m \longrightarrow \mu_m \longrightarrow 1.$$

with action $\sigma \cdot (\xi_1, \xi_2) = (\xi_1 \chi(\sigma)(\xi_2), \xi_2)$ for $\sigma \in \Gamma_K$.

Lemma 5.11. *Let K be a field and ξ_m a primitive m-th root of unity. Assume $\mu_m \subset K$. Then the Baer sum $kD(\xi_m) + C_0$ is equal to C_k in $Ext^1_{\Gamma_K}(\mu_m, \mu_m)$.*

Proof. We will proceed by induction on k. The base case $k = 0$ is trivial. Assume that $kD(\xi_m) + C_0 = C_k$. We will show that the Baer sum $D(\xi_m) + C_k = C_{k+1}$. Let X be the pullback in the following diagram.



Then $X = \{((\xi, \xi'), \xi'') : \xi, \xi' \in \mu_m, \xi'' \in \mu_{m^2}, \xi' = (\xi'')^m\}$. Define Y to be the quotient of X via the image of the skew-diagonal embedding of μ_m in X. The image of the skew-diagonal embedding of μ_m in X is $\{((\xi^{-1}, 0), \xi) : \xi \in \mu_m\}$ so each class of Y has a representative of the form $((0, \xi'), \xi'')$ for $\xi' \in \mu_m$, $\xi'' \in \mu_{m^2}$, and $\xi' = (\xi'')^m$. The Baer sum $D(\xi_m) + C_k$ is the class of the extension $0 \to \mu_m \to Y \to \mu_m \to 0$.

Both $D(\xi_m)$ and C_k refer to the cyclic field extension $K(\xi_{m^2})$ of K. Denote the group homomorphism corresponding to the extension by $\chi : \Gamma_K \to \text{Gal}(K(\xi_{m^2})/K)$. For $\sigma \in \Gamma_K$ such that $\chi(\sigma)$ generates $\text{Gal}(K(\xi_{m^2})/K)$ and $\xi'' \in \mu_{m^2}$, we have $\chi(\sigma)(\xi'') = (\xi'')^{m+1} = (\xi'')^m \xi''$. In other words, the action of σ is multiplication by $(\xi'')^m$. Then

$$\begin{aligned}
\sigma \cdot ((0, \xi'), \xi'') &= ((\chi(\sigma)(\xi'), \xi'), \chi(\sigma)^k(\xi'')) \\
&= ((\xi', \xi'), \chi(\sigma)^k(\xi'')) \\
&= ((1, \xi'), \chi(\sigma)^k(\xi'')(\xi')) \\
&= ((1, \xi'), \chi(\sigma)^{k+1}(\xi''))
\end{aligned}$$

since $\xi' = (\xi'')^m$. We conclude $Y \simeq \mu_{m^2}^{\otimes(k+1)}$ as Γ_K-modules and $D(\xi_m) + C_k = C_{k+1}$ in $\text{Ext}^1_{\Gamma_K}(\mu_m, \mu_m)$. $\square$

Lemma 5.12. *Let K be a field with $\mu_m \subset K$. Then $\Phi_k(C_k)$ is trivial and $\Phi_k(C_0) = -k\Phi_k(D(\xi_m))$ in Hom$(H^k(K, \mu_m), H^{k+1}(K, \mu_m))$.*

Proof. It is sufficient to show $\Phi_k(C_k)$ is trivial in $\mathrm{Hom}(H^k(K, \mu_m), H^{k+1}(K, \mu_m))$ by Lemma 5.11. Since $\mu_m \subset K$, we can identify $\mu_m \simeq \mu_m^{\otimes i}$ as Γ_K-modules for all $i \in \mathbb{Z}$. The extension

$$1 \longrightarrow \mu_m \longrightarrow \mu_{m^2}^{\otimes d} \longrightarrow \mu_m \longrightarrow 1$$

induces a long exact sequence in cohomology with portion

$$H^d(K, \mu_{m^2}^{\otimes d}) \longrightarrow H^d(K, \mu_m) \xrightarrow{\partial^d} H^{d+1}(K, \mu_m).$$

By the Norm Residue Isomorphism [HW19], the following commutative square

$$\begin{array}{ccc} H^d(K, \mu_{m^2}^{\otimes d}) & \longrightarrow & H^d(K, \mu_m^{\otimes d}) \\ \simeq\uparrow & & \simeq\uparrow \\ K_d(K)/m^2 K_d(K) & \twoheadrightarrow & K_d(K)/mK_d(K) \end{array}$$

proves that the top map is surjective. Therefore, $\Phi_k(C_k) = \partial^k$ is trivial. □

Lemma 5.13. *Let K be a field that contains a primitive m-th root of unity ξ_m. Then*

$$\Phi_k(D(\xi_m))(\gamma) = (\xi_m) \cup \gamma$$

for $\gamma \in H^k(K, \mu_m)$.

Proof. Let ξ_{m^2} be a primitive m^2 root of unity. For the cyclic field extension $K(\xi_{m^2})$ of K, there is a corresponding group homomorphism $\chi : \Gamma_K \to \mathrm{Gal}(E/K) \simeq \mathbb{Z}/m\mathbb{Z}$ in $H^1(K, \mathbb{Z}/m\mathbb{Z})$. The pairing $\mathbb{Z}/m\mathbb{Z} \otimes \mu_m \to \mu_m$ induces a cup product

$$\cup : H^1(K, \mathbb{Z}/m\mathbb{Z}) \otimes H^k(K, \mu_m) \to H^{k+1}(K, \mu_m)$$

for each $k \geq 0$. Since $\mu_m \subset K$, we can identify $D(\xi_m)$ with the short exact sequence

$$0 \longrightarrow \mathbb{Z}/m\mathbb{Z} \longrightarrow \mathbb{Z}/m\mathbb{Z} \oplus \mathbb{Z}/m\mathbb{Z} \longrightarrow \mathbb{Z}/m\mathbb{Z} \longrightarrow 0$$

with Γ_K-action on $\mathbb{Z}/m\mathbb{Z} \oplus \mathbb{Z}/m\mathbb{Z}$ given by $\sigma \cdot (x, y) = (x + \chi(\sigma)(y), y)$ for $\sigma \in \Gamma_K$. Then $D(\xi_m)$ induces the commutative square

$$\begin{array}{ccc} H^0(K, \mathbb{Z}/m\mathbb{Z}) \otimes H^k(K, \mu_m) & \xrightarrow{\cup} & H^k(K, \mu_m) \\ \downarrow{\scriptstyle \partial^0 \otimes \mathrm{id}} & & \downarrow{\scriptstyle \partial^k} \\ H^1(K, \mathbb{Z}/m\mathbb{Z}) \otimes H^k(K, \mu_m) & \xrightarrow{\cup} & H^{k+1}(K, \mu_m) \end{array}$$

for $\partial^k : H^k(K, \mu_m) \to H^{k+1}(K, \mu_m)$ a connecting map. We identify $H^0(K, \mathbb{Z}/m\mathbb{Z}) \simeq \mathbb{Z}/m\mathbb{Z}$ so ∂^0 maps $1 \in \mathbb{Z}/m\mathbb{Z}$ to $\chi \in H^1(K, \mathbb{Z}/m\mathbb{Z})$. By commutativity,

$$\Phi_k(D(\xi_m))(\gamma) = \partial^k(\gamma) = \chi \cup \gamma$$

for $\gamma \in H^k(K, \mu_m)$. In the notation we adopt, χ is written as $(\xi_m) \in K^\times/(K^\times)^m$. $\square$

Lemma 5.14. *Let K be a field that contains a primitive m-th root of unity ξ_m. The mod m surjection $q : \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ induces a map $q_* : H^2(K, \mathbb{Z}) \to H^2(K, \mathbb{Z}/m\mathbb{Z})$ on cohomology. A character $\chi \in H^2(K, \mathbb{Z})$ corresponds to a cyclic extension $K(\sqrt[m]{a})$ of K defined by $q \circ \chi : \Gamma_K \to \mathbb{Z}/m\mathbb{Z}$. Then $q_*(\chi) = (a) \cup (\xi_m)$.*

Proof. The following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Q} & \longrightarrow & \mathbb{Q}/\mathbb{Z} & \longrightarrow & 0 \\ & & \| & & \uparrow{\scriptstyle \frac{1}{m}} & & \uparrow & & \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z}/m\mathbb{Z} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle q} & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathbb{Z}/m\mathbb{Z} & \longrightarrow & \mathbb{Z}/m^2\mathbb{Z} & \longrightarrow & \mathbb{Z}/m\mathbb{Z} & \longrightarrow & 0 \end{array}$$

induces the commutative square

$$\begin{array}{ccc} H^1(K,\mathbb{Q}/\mathbb{Z}) & \xrightarrow{\simeq} & H^2(K,\mathbb{Z}) \\ \uparrow & & \downarrow{\scriptstyle q_*} \\ H^1(K,\mathbb{Z}/m\mathbb{Z}) & \xrightarrow{\partial^1} & H^2(K,\mathbb{Z}/m\mathbb{Z}). \end{array}$$

The image of (a) in $H^2(K,\mathbb{Z})$ is χ. By Lemma 5.12,

$$\partial^1(a) = -\Phi_1(C_0) = -(\xi_m) \cup (a) = (a) \cup (\xi_m).$$

The result follows from commutativity. □

Proposition 5.15. *Let G be a cyclic group of order m. If $m \not\equiv 2$ (mod 4), assume $\mu_m \subset F$. If $m \equiv 2$ (mod 4), assume $\mu_{2m} \subset F$. Then $H^d(G,\mathbb{Z}/m\mathbb{Z})_{neg} = H^d(G,\mathbb{Z}/m\mathbb{Z})$ for $d \in \{3,4\}$.*

Proof. Let K be a field extension of F and $j : \Gamma_K \to G$ a continuous group homomorphism. Let $x \in H^1(G,\mathbb{Z}/m\mathbb{Z})$ be a generator. Then $j^*(x) \in H^1(K,\mathbb{Z}/m\mathbb{Z})$ corresponds to a field extensions E/K for which Gal$(E/K) \simeq G \simeq \mathbb{Z}/m\mathbb{Z}$. Since $\mu_m \subset F$, we note $E = K(\sqrt[m]{a})$ for some $(a) \in K^\times/(K^\times)^m$. Then $j^*(x)$ corresponds to (a) in $H^1(K,\mathbb{Z}/m\mathbb{Z}) \simeq K^\times/(K^\times)^m$. By including $\mathbb{Z}/m\mathbb{Z}$ into $\mathbb{Q}/\mathbb{Z}$, the field extension induces a character $\chi_G : G \to \mathbb{Q}/\mathbb{Z}$ in $H^1(G,\mathbb{Q}/\mathbb{Z}) \simeq H^2(K,\mathbb{Z})$.

Let $q : \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ be a surjection with induced map $q_* : H^2(-,\mathbb{Z}) \to H^2(-,\mathbb{Z}/m\mathbb{Z})$ on cohomology. Denote $\chi = j^*(\chi_G) \in H^2(K,\mathbb{Z})$ and generator $y = q_*(\chi_G) \in H^2(G,\mathbb{Z}/m\mathbb{Z})$. By Lemma 5.14, the following commutative square proves $j^*(y) = (a) \cup (\xi_m) \in H^2(K,\mathbb{Z}/m\mathbb{Z})$.

$$\begin{array}{ccc} H^2(G,\mathbb{Z}) & \xrightarrow{q_*} & H^2(G,\mathbb{Z}/m\mathbb{Z}) \\ \downarrow{\scriptstyle j^*} & & \downarrow{\scriptstyle j^*} \\ H^2(K,\mathbb{Z}) & \xrightarrow{q_*} & H^2(K,\mathbb{Z}/m\mathbb{Z}) \end{array}$$

Cup product by $\chi_G \in H^2(G,\mathbb{Z})$ gives isomorphisms $H^d(G,\mathbb{Z}/m\mathbb{Z}) \to H^{d+2}(G,\mathbb{Z}/m\mathbb{Z})$ in each degree $d \geq 1$. Thus $x \cup y$ is a generator of $H^3(G,\mathbb{Z}/m\mathbb{Z})$ and $y \cup y$ is a generator of

$H^4(G,\mathbb{Z}/m\mathbb{Z})$. We have

$$j^*(x \cup y) = (a) \cup (a,\xi_m) = (a,-1,\xi_m)$$
$$j^*(y \cup y) = (a,\xi_m) \cup (a,\xi_m) = -(a,-1,\xi_m,-1).$$

When $m \not\equiv 2 \pmod 4$ and $\mu_m \subset F$, we note (-1) is trivial. When $m \equiv 2 \pmod 4$, we have

$$j^*(x \cup y) = (a,-1,\xi_m) = (a,-1,\xi_{2m}^2) = 2(a,-1,\xi_{2m}) = 0$$
$$j^*(y \cup y) = -(a,-1,\xi_m,-1) = -(a,-1,\xi_{2m}^2,-1) = -2(a,-1,\xi_{2m},-1) = 0.$$

Therefore, $H^3(G,\mathbb{Z}/m\mathbb{Z})$ and $H^4(G,\mathbb{Z}/m\mathbb{Z})$ are entirely negligible. □

Corollary 5.16. *Let G be cyclic of order n. If $n \not\equiv 2$ (mod 4), assume $\mu_n \subset F$. If $n \equiv 2$ (mod 4), assume $\mu_{2n} \subset F$. Then $H^d(G,\mathbb{Z})_{neg} = H^d(G,\mathbb{Z})$ for $d \in \{3,4\}$.*

Proof. Since multiplication by n on $H^d(G,\mathbb{Z})$ is trivial in each degree, short exact sequence (5.2) induces a surjective connecting map on cohomology $H^d(G,\mathbb{Z}/n\mathbb{Z}) \to H^{d+1}(G,\mathbb{Z})$ for each $d \geq 1$. The connecting map respects negligible cohomology since short exact sequence (5.2) produces an analogous long exact sequence in Galois cohomology. Proposition 5.15 completes the argument in degrees 3 and 4. □

5.3.2 Negligible Cohomology Ideal of Cyclic Groups

If we restrict our view to the coefficients $M = \mathbb{Z}/p\mathbb{Z}$ with a trivial G-action for prime p, then, from a negligible cohomology standpoint, Lemma 5.8 proves that it is sufficient to only consider cyclic groups of order p^k. Let C_{p^k} be a cyclic group of order p^k. The case $k = 1$ is taken care of in Section 5.1. Hence, we will assume $k > 1$ throughout this section.

When $p = 2$ and $k > 1$, the cohomology ring of a cyclic 2-group is

$$H^*(C_{2^k},\mathbb{Z}/2\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}[x,y]/\langle x^2\rangle$$

such that $\deg(x) = 1$ and $\deg(y) = 2$ by [Eve91, Section 3.2].

When p is an odd prime and $k > 1$, the cohomology ring of a cyclic p-group is

$$H^*(C_{p^k}, \mathbb{Z}/p\mathbb{Z}) \simeq \Lambda_{\mathbb{Z}/p\mathbb{Z}}(C_{p^k}^*)[y]$$

for $C_{p^k}^*$ the characters of C_{p^k} and $\deg(y) = 2$ by [Eve91, Section 3.2].

Proposition 5.17. *Assume that F is a field such that char$(F) \neq 2$ and $\mu_4 \subset F$.*

(1) *If $\mu_{2^{k+1}} \subset F$, then the negligible cohomology ideal of $H^*(C_{2^k}, \mathbb{Z}/2\mathbb{Z})$ is generated by $\{y\}$.*

(2) *If $\mu_{2^{k+1}} \not\subset F$, then the negligible cohomology ideal of $H^*(C_{2^k}, \mathbb{Z}/2\mathbb{Z})$ is generated by $\{xy, y^2\}$.*

Proof. By Corollary 2.2(2), there are no negligible classes in $H^1(C_{2^k}, \mathbb{Z}/2\mathbb{Z})$. Corollary 3.2 proves that $H^2(C_{2^k}, \mathbb{Z}/2\mathbb{Z})$ is entirely negligible if and only if $\mu_{2^{k+1}} \subset F$.

Since $\mu_4 \subset F$, Proposition 5.15 implies that $H^3(C_4, \mathbb{Z}/4\mathbb{Z})$ is entirely negligible. The quotient $\mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ induces a surjection $H^3(C_4, \mathbb{Z}/4\mathbb{Z}) \to H^3(C_4, \mathbb{Z}/2\mathbb{Z})$ that respects negligible classes by Proposition 2.3(4). Thus $H^3(C_4, \mathbb{Z}/2\mathbb{Z})$ is entirely negligible. Let $C_{2^{k-1}}$ denote the unique cyclic subgroup of C_{2^k} of order 2^{k-1}. The corestriction map $H^3(C_{2^{k-1}}, \mathbb{Z}/2\mathbb{Z}) \to H^3(C_{2^k}, \mathbb{Z}/2\mathbb{Z})$ is an isomorphism for $k \geq 2$ and respects negligible classes by Lemma 5.7. Via induction on k, $H^3(C_{2^k}, \mathbb{Z}/2\mathbb{Z})$ is entirely negligible for all $k \geq 2$.

Let K be a field extension of F and $j : \Gamma_K \to C_{2^k}$ a continuous group homomorphism. Then $j^*(y^2) = j^*(y)^2 = j^*(y) \cup (-1)^2$ is trivial since $\mu_4 \subset F$. We conclude that $H^4(C_{2^k}, \mathbb{Z}/2\mathbb{Z})$ is entirely negligible. □

Remark. Assume that $\mu_4 \not\subset F$ or, equivalently, $s(F) > 1$. We have not yet developed techniques to determine when classes of $H^{2d-1}(C_{2^k}, \mathbb{Z}/2\mathbb{Z})$ are negligible. However, the next result, Proposition 5.18, implies that the unique elementary abelian 2-subgroup of C_{2^k} can detect the negligible classes of $H^{2d}(C_{2^k}, \mathbb{Z}/2\mathbb{Z})$. The result aligns with Conjecture 5.6 although, when r is even, there is one choice for ℓ missing.

Proposition 5.18. *Assume that F is a field such that char$(F) \neq 2$.*

(1) *If* $s(F) = 2^r$*, then* y^ℓ *is not negligible in* $H^{2\ell}(C_{2^k}, \mathbb{Z}/2\mathbb{Z})$ *if* $\ell \leq \lfloor \frac{r+1}{2} \rfloor$*. Further,* y^ℓ *is negligible if* $\ell \geq \lceil \frac{r+3}{2} \rceil$*.*

(2) *If* F *is formally real, then* y^ℓ *is not negligible in* $H^{2\ell}(C_{2^k}, \mathbb{Z}/2\mathbb{Z})$ *for any* $1 \leq \ell$*.*

Proof. The group C_{2^k} has a unique elementary abelian 2-subgroup H, which is cyclic of order 2. The restriction map $\mathrm{res}^d : H^d(C_{2^k}, \mathbb{Z}/2\mathbb{Z}) \to H^d(H, \mathbb{Z}/2\mathbb{Z})$ is an isomorphism in even degrees. Theorem 5.1 proves that $\mathrm{res}^{2\ell}(y^\ell)$ is not negligible in $H^{2\ell}(H, \mathbb{Z}/2\mathbb{Z})$ when $\ell \leq \lfloor \frac{r+1}{2} \rfloor$. By Proposition 2.3(1), y^ℓ is not negligible when $\ell \leq \lfloor \frac{r+1}{2} \rfloor$. If F is formally real, y^ℓ is not negligible for any $1 \leq \ell$.

We will now assume $\ell > \lceil \frac{r+3}{2} \rceil$. Let K be a field extension of F and $j : \Gamma_K \to C_{2^k}$ a continuous group homomorphism. Then

$$j^*(y^\ell) = j^*(y)^\ell = j^*(y) \cup (-1)^{2\ell-2}.$$

Since $2\ell - 2 \geq r + 1$, we find $j^*(y^\ell)$ is trivial when $s(F) = 2^r$. $\square$

Proposition 5.19. *Let* p *be an odd prime. Assume* F *is a field such that char*$(F) \neq p$*.*

(1) y^2 *is negligible in* $H^4(C_{p^k}, \mathbb{Z}/p\mathbb{Z})$*.*

(2) *If* $\mu_{p^{k+1}} \subset F$*, then the negligible cohomology ideal of* $H^*(C_{p^k}, \mathbb{Z}/p\mathbb{Z})$ *is generated by* $\{y\}$*.*

(3) *If* $\mu_{p^k} \subset F$ *but* $\mu_{p^{k+1}} \not\subset F$*, then the negligible cohomology ideal of* $H^*(C_{p^k}, \mathbb{Z}/p\mathbb{Z})$ *is generated by* $\{xy, y^2\}$*.*

Proof. Let K be a field extension of F and $j : \Gamma_K \to C_{p^k}$ a continuous group homomorphism. Then $j^*(y^2) = j^*(y)^2 = j^*(y) \cup (-1)^2$ is trivial.

By Corollary 2.2(2), there are no negligible classes in $H^1(C_{p^k}, \mathbb{Z}/p\mathbb{Z})$. Corollary 3.2 proves that $H^2(C_{p^k}, \mathbb{Z}/p\mathbb{Z})$ is entirely negligible if and only if $\mu_{p^{k+1}} \subset F$.

If $\mu_{p^k} \subset F$, Proposition 5.15 proves that $H^3(C_{p^k}, \mathbb{Z}/p^k\mathbb{Z})$ and $H^4(C_{p^k}, \mathbb{Z}/p^k\mathbb{Z})$ are entirely negligible. The quotient $\mathbb{Z}/p^k\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ induces a surjection

$$H^d(C_{p^k}, \mathbb{Z}/p^k\mathbb{Z}) \to H^d(C_{p^k}, \mathbb{Z}/p\mathbb{Z})$$

in each degree $d \geq 0$ that respects negligible classes by Proposition 2.3(4). Therefore, $H^3(C_{p^k}, \mathbb{Z}/p\mathbb{Z})$ and $H^4(C_{p^k}, \mathbb{Z}/p\mathbb{Z})$ are entirely negligible. □

5.3.3 Relaxation of Roots of Unity for Cyclic Groups of Odd Prime Power Order

In order to compute the negligible classes of $H^3(C_{p^k}, \mathbb{Z}/p^k\mathbb{Z})$, Proposition 5.15 requires $\mu_{p^k} \subset F$. In this subsection, we prove a slight relaxation of the requirement $\mu_{p^k} \subset F$. For an odd prime p, we will show that $H^3(C_{p^{2k}}, \mathbb{Z}/p^{2k}\mathbb{Z})$ and $H^4(C_{p^{2k}}, \mathbb{Z}/p^{2k}\mathbb{Z})$ are entirely negligible over F if $\mu_{p^k} \subset F$.

Proposition 5.20. *Let p be an odd prime. Assume $\mu_{p^\ell} \subset F$. Then $H^3(C_{p^{2\ell}}, \mathbb{Z}/p^{2\ell}\mathbb{Z})$ is entirely negligible over F.*

Proof. If $\mu_{p^{2\ell}} \subset F$, we obtain the result by Proposition 5.15. Assume that $\mu_{p^{2\ell}} \not\subset F$. Let V be a faithful F-representation of $C_{p^{2\ell}}$ and $K = F(V)^{C_{p^{2\ell}}}$. Denote by

$$\mathrm{inf}_{p^k} : H^3(C_{p^{2\ell}}, \mathbb{Z}/p^k\mathbb{Z}) \to H^3(K, \mathbb{Z}/p^k\mathbb{Z})$$

an inflation map. By Proposition 2.1, the kernel of inf_{p^ℓ} is the negligible cohomology of $H^3(C_{p^{2\ell}}, \mathbb{Z}/p^\ell\mathbb{Z})$.

Let m be the largest integer for which $\mu_{p^m} \subset F$ so $\ell \leq m < 2\ell$. Let $L = K(\xi_{p^{m+\ell}})$ be the separable degree p^ℓ field extension of K. Note that $2\ell \leq m + \ell$ so $\mu_{p^{2\ell}} \subset L$. Denote by $\mathrm{res}_{p^k} : H^*(K, \mathbb{Z}/p^k\mathbb{Z}) \to H^*(L, \mathbb{Z}/p^k\mathbb{Z})$ a restriction map. Over $F(\xi_{p^{m+\ell}})$, $V \otimes_F F(\xi_{p^{m+\ell}})$ is a faithful representation of $C_{p^{2\ell}}$. Then $L = F(\xi_{p^{m+\ell}})(V \otimes_F F(\xi_{p^{m+\ell}}))^{C_{p^{2\ell}}}$, and inflation $\mathrm{inf}_{L,p^\ell} : H^3(C_{p^k}, \mathbb{Z}/p^\ell\mathbb{Z}) \to H^3(L, \mathbb{Z}/p^\ell\mathbb{Z})$ factors as $\mathrm{inf}_{L,p^\ell} = \mathrm{res}_{p^\ell} \circ \mathrm{inf}_{p^\ell}$.

The short exact sequence

$$0 \longrightarrow \mathbb{Z}/p^\ell\mathbb{Z} \longrightarrow \mathbb{Z}/p^{2\ell}\mathbb{Z} \longrightarrow \mathbb{Z}/p^\ell\mathbb{Z} \longrightarrow 0$$

induces the following commutative diagram with exact rows.

$$\begin{array}{ccccccc}
 & & & & H^3(C_{p^{2\ell}}, \mathbb{Z}/p^{2\ell}\mathbb{Z}) & \xrightarrow{g_*} & H^3(C_{p^{2\ell}}, \mathbb{Z}/p^{\ell}\mathbb{Z}) \\
 & & & & \downarrow{\scriptstyle \inf_{p^{2\ell}}} & & \downarrow{\scriptstyle \inf_{p^{\ell}}} \\
H^2(K, \mathbb{Z}/p^{\ell}\mathbb{Z}) & \xrightarrow{\delta^2} & H^3(K, \mathbb{Z}/p^{\ell}\mathbb{Z}) & \longrightarrow & H^3(K, \mathbb{Z}/p^{2\ell}\mathbb{Z}) & \longrightarrow & H^3(K, \mathbb{Z}/p^{\ell}\mathbb{Z}) \\
\downarrow & & \downarrow{\scriptstyle \mathrm{res}_{p^{\ell}}} & & \downarrow{\scriptstyle \mathrm{res}_{p^{2\ell}}} & & \downarrow{\scriptstyle \mathrm{res}_{p^{\ell}}} \\
H^2(L, \mathbb{Z}/p^{\ell}\mathbb{Z}) & \xrightarrow{\partial^2} & H^3(L, \mathbb{Z}/p^{\ell}\mathbb{Z}) & \longrightarrow & H^3(L, \mathbb{Z}/p^{2\ell}\mathbb{Z}) & \longrightarrow & H^3(L, \mathbb{Z}/p^{\ell}\mathbb{Z})
\end{array}$$

Let u be a generator of $H^3(C_{p^{2\ell}}, \mathbb{Z}/p^{2\ell}\mathbb{Z})$. Since $\mu_{p^\ell} \subset F$, $\inf_{p^\ell}(g_*(u)) = 0$ in $H^3(K, \mathbb{Z}/p^\ell\mathbb{Z})$ by Proposition 5.15. Equivalently, $\inf_{p^{2\ell}}(u)$ lifts to an element $\alpha \in H^3(K, \mathbb{Z}/p^\ell\mathbb{Z})$. Then u is negligible in $H^3(C_{p^{2\ell}}, \mathbb{Z}/p^{2\ell}\mathbb{Z})$ if and only if α is in the image of the connecting map δ^2.

Since $\mu_{p^{2\ell}} \subset L$, Proposition 5.15 implies that $\inf_{L,p^{2\ell}}(u) = \mathrm{res}_{p^{2\ell}}(\inf_{p^{2\ell}}(u)) = 0$. By commutativity and row exactness, $\mathrm{res}_{p^\ell}(\alpha) \in \mathrm{im}(\partial^2)$. Lemma 5.12 proves that $\partial^2 = 0$ so $\alpha \in \ker(\mathrm{res}_{p^\ell})$. By [MS82, Corollary 15.3], $\ker(\mathrm{res}_{p^\ell}) = \chi \cdot H^2(K, \mathbb{Z}/p^\ell\mathbb{Z})$ for χ the image of a generator of $H^1(C_{p^{2\ell}}, \mathbb{Z}/p^\ell\mathbb{Z})$ via inflation. The element χ is some power of (ξ_{p^ℓ}). Lemma 5.12 proves $\delta^2 = -2\Phi_2(D(\xi_{p^\ell}))$ so $\delta^2(\gamma) = -2((\xi_{p^\ell}) \cup \gamma)$ for $\gamma \in H^2(K, \mathbb{Z}/p^\ell\mathbb{Z})$. Therefore, $\alpha \in \mathrm{im}(\delta^2)$ as long as p is odd. We conclude u is negligible over F. □

Corollary 5.21. *Let $\mu_{p^\ell} \subset F$. Then $H^3(C_{p^{2\ell}}, \mathbb{Z}/p^m\mathbb{Z})$ is entirely negligible over F for any $m \geq 0$.*

Proof. By Proposition 5.20, $H^3(\mathbb{Z}/p^{2\ell}\mathbb{Z}, \mathbb{Z}/p^{2\ell}\mathbb{Z})$ is entirely negligible over F.

For now, assume $m \leq 2\ell$. The surjection $\mathbb{Z}/p^{2\ell}\mathbb{Z} \to \mathbb{Z}/p^m\mathbb{Z}$ induces a surjective homomorphism $H^3(C_{p^{2\ell}}, \mathbb{Z}/p^k\mathbb{Z}) \to H^3(C_{p^{2\ell}}, \mathbb{Z}/p^m\mathbb{Z})$ by inspecting the corresponding long exact sequence in cohomology. Proposition 2.3(4) shows that the induced map respects negligible classes. Thus $H^3(C_{p^{2\ell}}, \mathbb{Z}/p^m\mathbb{Z})$ is entirely negligible for all $m \leq 2\ell$.

Assume $m > 2\ell$. The inclusion $\mathbb{Z}/p^{2\ell}\mathbb{Z} \to \mathbb{Z}/p^m\mathbb{Z}$ induces an isomorphism in cohomology $H^3(C_{p^{2\ell}}, \mathbb{Z}/p^{2\ell}\mathbb{Z}) \to H^3(C_{p^{2\ell}}, \mathbb{Z}/p^m\mathbb{Z})$ by inspecting the corresponding long exact sequence in cohomology. Proposition 2.3(4) shows that the induced map respects negligible classes. Thus $H^3(C_{p^{2\ell}}, \mathbb{Z}/p^m\mathbb{Z})$ is entirely negligible for all $m > 2\ell$. □

The next result extends the relaxed degree three negligible computation of Proposition

5.20 to degree four when the order of the cyclic group and the order of the coefficients coincide.

Lemma 5.22. *Let p be prime. If $H^3(C_{p^k}, \mathbb{Z}/p^k\mathbb{Z})$ is entirely negligible over F, then $H^4(C_{p^k}, \mathbb{Z}/p^k\mathbb{Z})$ is entirely negligible over F.*

Proof. Let K be a field extension of F and $j : \Gamma_K \to C_{p^k}$ a continuous group homomorphism. The short exact sequence

$$0 \longrightarrow \mathbb{Z}/p^k\mathbb{Z} \longrightarrow \mathbb{Z}/p^{2k}\mathbb{Z} \longrightarrow \mathbb{Z}/p^k\mathbb{Z} \longrightarrow 0$$

induces the following commutative square.

$$\begin{array}{ccc} H^3(G, \mathbb{Z}/p^k\mathbb{Z}) & \xrightarrow{\simeq} & H^4(G, \mathbb{Z}/p^k\mathbb{Z}) \\ \downarrow{\scriptstyle j^*} & & \downarrow{\scriptstyle j^*} \\ H^3(K, \mathbb{Z}/p^k\mathbb{Z}) & \longrightarrow & H^4(K, \mathbb{Z}/p^k\mathbb{Z}) \end{array}$$

By assumption, j^* is the zero map in degree three so $H^4(G, \mathbb{Z}/p^k\mathbb{Z})$ is contained in the kernel of j^* in degree four. □

5.4 Finite Abelian Groups

5.4.1 Finite Abelian 2-groups

Let G be a finite abelian 2-group. Assume that G has n cyclic direct summands in elementary divisor form. Denote by G_i the ith cyclic direct summand of G so $G \simeq \bigoplus_{i=1}^n G_i$. Denote by m the first index for which $|G_i| > 2$. [Eve91, Sections 3.2 and 3.5] prove the cohomology ring of G is

$$H^*(G, \mathbb{Z}/2\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}[x_1, \ldots, x_n, y_m, \ldots, y_n]/\langle x_m^2, \ldots, x_n^2\rangle$$

where $\deg(x_i) = 1$ for $1 \leq i \leq n$ and $\deg(y_j) = 2$ for $m \leq j \leq n$. We can pick a basis $\{x_1, \ldots, x_n\}$ for $H^1(G, \mathbb{Z}/2\mathbb{Z})$ as a $\mathbb{Z}/2\mathbb{Z}$-vector space such that x_i is the inflation of a generator of $H^1(G_i, \mathbb{Z}/2\mathbb{Z})$. Further, we can pick y_i so that it is the inflation of a generator of $H^2(G_i, \mathbb{Z}/2\mathbb{Z})$.

Proposition 5.23. *Let G be a finite abelian 2-group with n direct summands when written in elementary divisor form. Denote by m the index of the first direct summand of order greater than 2. The following classes are negligible over a field F with char$(F) \neq 2$.*

(1) $\{x_i x_j (x_i + x_j) : 1 \leq i < j \leq m\}$
(2) $\{x_i y_j (x_i^2 + y_j) : 1 \leq i < m \leq j \leq n\}$
(3) $\{y_i y_j (y_i + y_j) : m \leq i < j \leq n\}$
(4) *If* $s(F) = 2^r$, $\{x_i^{r+2} : 1 \leq i < m\}$
(5) *If* $s(F) = 2^r$, $\{y_j^\ell : m \leq j \leq n, \lceil \frac{r+3}{2} \rceil \leq \ell\}$.

Proof. The proof of Lemma 4.5 reveals that classes of the form (1), (2), (3) are negligible.

We will now assume that $s(F) = 2^r$. Let K be a field extension of F and $j : \Gamma_K \to G$ a continuous group homomorphism. Then $j^*(x_i) = (a_i) \in K^\times/(K^\times)^2$ for some square free $a_i \in K^\times$. We have

$$j^*(x_i^{r+2}) = (a_i, \ldots, a_i) = (a_i) \cup (-1)^{r+1} = 0$$

so classes of the form (4) are negligible. Assume that $\lceil \frac{r+3}{2} \rceil \leq \ell$. Therefore,

$$j^*(y_j^\ell) = j^*(y_j)^\ell = j^*(y_j) \cup (-1)^{2(\ell-1)} = 0$$

so classes of the form (5) are negligible. □

Remark. In order to detect that many classes are not negligible, we restrict to the cohomology of a maximal elementary abelian 2-subgroup. Combine the results of Theorem 5.1 and Proposition 2.3(1). There are, however, classes like $x_j x_i^2$ or $x_j y_k$ for $1 \leq i < m$ and $m \leq j < k \leq n$ that restrict to 0 in the cohomology of all elementary abelian 2-subgroups. We do not yet have a way of detecting whether these classes are negligible.

Remark. By [QV72], classes that restrict to 0 in the cohomology of all elementary abelian 2-subgroups of G are nilpotent and, thus, eventually negligible in $H^*(G, \mathbb{Z}/2\mathbb{Z})$. Proposition

5.23 reveals that the classes of $H^*(G, \mathbb{Z}/2\mathbb{Z})$ that restrict to non-zero negligible classes in the cohomology of elementary abelian 2-subgroups of G are negligible. Conjecture 5.6 holds for finite abelian 2-groups.

5.4.2 Finite Abelian p-Groups for Odd p

Let p be an odd prime. Assume now that G is a finite abelian p-group with n cyclic direct summands in elementary divisor form. Denote by G_i the ith cyclic direct summand of G so $G \simeq \bigoplus_{i=1}^n G_i$. [Eve91, Sections 3.2 and 3.5] proves

$$H^*(G, \mathbb{Z}/p\mathbb{Z}) \simeq \Lambda_{\mathbb{Z}/p\mathbb{Z}}(G^*)[y_1, \ldots, y_n]$$

where G^* is the characters of G and $\deg(y_i) = 2$ for $1 \leq i \leq n$. We can pick a basis $\{x_1, \ldots, x_n\}$ for $G^* = H^1(G, \mathbb{Z}/p\mathbb{Z})$ as a $\mathbb{Z}/p\mathbb{Z}$-vector space such that x_i is the inflation of a generator of $H^1(G_i, \mathbb{Z}/p\mathbb{Z})$. Further, we can pick y_i so that it is the inflation of a generator of $H^2(G_i, \mathbb{Z}/p\mathbb{Z})$.

Proposition 5.24. *Let p be an odd prime and G be a finite abelian p-group with cyclic direct summands G_i for $1 \leq i \leq n$ in elementary divisor form. The following are negligible over any field F with char$(F) \neq p$.*

(*1*) $\{x_i y_i : 1 \leq i \leq n\}$

(*2*) $\{y_i^2 : 1 \leq i \leq n\}$

Proof. Let x be a generator of $H^1(G_i, \mathbb{Z}/p\mathbb{Z})$ such that inf$(x) = x_i$ and y a generator of $H^2(G_i, \mathbb{Z}/p\mathbb{Z})$ such that inf$(y) = y_i$. Let K be a field extension of F and $j : \Gamma_K \to G$ a continuous group homomorphism. Let $j_i : \Gamma_K \to G_i$ be the composition of j with the surjection $G \to G_i$. We have the following commutative diagram.

$$\begin{array}{ccc} H^*(G_i, \mathbb{Z}/p\mathbb{Z}) & \xhookrightarrow{\text{inf}} & H^*(G, \mathbb{Z}/p\mathbb{Z}) \\ & \searrow^{j_i^*} & \downarrow j^* \\ & & H^*(K, \mathbb{Z}/p\mathbb{Z}) \end{array}$$

By Proposition 5.19, $j_i^*(xy) = 0$ and $j_i^*(y^2) = 0$. Therefore, $j^*(x_i y_i) = 0$ and $j^*(y_i^2) = 0$ so $x_i y_i$ and y_i^2 are negligible in $H^*(G, \mathbb{Z}/p\mathbb{Z})$. $\square$

The following integral negligible cohomology result will be useful for the negligible cohomology computations of non-abelian groups.

Lemma 5.25. *Let G be a group of order n. If $n \not\equiv 2 \ (mod\ 4)$, assume $\mu_n \subset F$. If $n \equiv 2 \ (mod\ 4)$, assume $\mu_{2n} \subset F$. The square of any element of $H^2(G, \mathbb{Z})$ is negligible in $H^4(G, \mathbb{Z})$ over F.*

Proof. Denote by $[G, G]$ the commutator subgroup of G so $G/[G, G]$ is the abelianization of G. The group of characters $H^2(G, \mathbb{Z})$ is isomorphic to $H^2(G/[G, G], \mathbb{Z})$ via inflation. Since $G/[G, G]$ is a finite abelian group, we can write $G/[G, G]$ in elementary divisor form with cyclic direct summands G_i. We can choose a generating set $\{x_1, \ldots, x_\ell\}$ of $H^2(G/[G, G], \mathbb{Z})$ in which each x_i is the inflation of a generator of $H^2(G_i, \mathbb{Z})$.

Let K be a field extension of F and $j : \Gamma_K \to G$ a continuous group homomorphism. Let $\overline{j} : \Gamma_K \to G/[G, G]$ be the composition of j with the projection $G \to G/[G, G]$ and $j_i : \Gamma_K \to G_i$ the composition of $\overline{j}$ with the surjection $G/[G, G] \to G_i$. We build the following commutative diagram for each $d \geq 1$.

$$\begin{array}{ccccc} H^d(G_i, \mathbb{Z}) & \xrightarrow{\text{inf}} & H^d(G/[G,G], \mathbb{Z}) & \xrightarrow{\text{inf}} & H^d(G, \mathbb{Z}) \\ & \searrow^{j_i^*} & & \searrow^{\overline{j}^*} & \downarrow{\scriptstyle j^*} \\ & & & & H^d(K, \mathbb{Z}) \end{array}$$

By Corollary 5.16, $H^4(G_i, \mathbb{Z})$ is entirely negligible over F. Thus x_i^2 in $H^4(G/[G, G], \mathbb{Z})$ is negligible over F for each $1 \leq i \leq \ell$. We conclude that the square of each character in $H^4(G, \mathbb{Z})$ is negligible over F. $\square$

5.5 Dihedral Groups

Let D_{2n} be the dihedral group of order $2n$. Dihedral groups are examples of Coxeter groups and, in some cases, of Weyl groups (e.g. D_6, D_8, and D_{12}). Under the assumption that the characteristic of the base field is coprime to that of the group, [Hir20, Part II] provides a detailed description of the mod 2 ring of cohomological invariants of a Weyl group. Although there are noted issues with the paper, [Duc11, §3.2 Theorem 7] claims that Conjecture 5.6 holds for Weyl groups over $\mathbb{Q}$. Further, [Duc11, Theorem 2] combined with [Duc11, §1 Proposition 1] would show that Conjecture 5.6 holds for Coxeter groups over some finite field extension of $\mathbb{Q}$.

5.5.1 Odd Order Subgroup of Rotations

We will assume that the subgroup of rotations of D_{2n} has odd order. In other words, n is odd. [Han93, Theorem 5.6] shows $H^*(D_{2n}, \mathbb{Z}/2\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}[v_1]$ for $\deg(v_1) = 1$.

Proposition 5.26. *Let n be an odd natural number and F a field with char$(F) \neq 2$.*

(1) *If $s(F) = 2^r$, the negligible cohomology ideal of $H^*(D_{2n}, \mathbb{Z}/2\mathbb{Z})$ over F is generated by $\{v_1^{r+2}\}$.*

(2) *If F is formally real, the negligible cohomology ideal of $H^*(D_{2n}, \mathbb{Z}/2\mathbb{Z})$ over F is trivial.*

Proof. Let $s \in D_{2n}$ represent a reflection and $H = \langle s \rangle$ be the order 2 cyclic subgroup of D_{2n} generated by s. Since n is odd, the abelianization of D_{2n} is cyclic of order 2 generated by the class of s. Therefore, res : $H^*(D_{2n}, \mathbb{Z}/2\mathbb{Z}) \to H^*(H, \mathbb{Z}/2\mathbb{Z})$ is a ring isomorphism. Combining Theorem 5.1 and Proposition 2.3(1), we find v_1^k is not negligible for $1 \leq k < r+2$ when $s(F) = 2^r$ and v_1^k is not negligible for any $1 \leq k$ when F is formally real.

Let K be a field extension of F and $j : \Gamma_K \to G$ a continuous group homomorphism. We have $j^*(v_1) = (a) \in K^\times/(K^\times)^2$ for a square-free element $a \in K^\times$. Then

$$j^*(v_1^k) = (a)^k = (a) \cup (-1)^{k-1}.$$

If $s(F) = 2^r$, then v_1^k is negligible for $k \geq r + 2$. □

Remark. Conjecture 5.6 is supported by the result of Proposition 5.26.

5.5.2 Even Order Subgroup of Rotations

We will now assume that the subgroup of rotations of D_{2n} has even order. In other words, n is even. Then [Han93, Theorem 5.5] shows

$$H^*(D_{2n}, \mathbb{Z}/2\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}[u_1, v_1, w_2]/\langle u_1^2 + u_1v_1 + (n/2)w_2\rangle$$

for $\deg(u_1) = \deg(v_1) = 1$ and $\deg(w_2) = 2$.

Lemma 5.27. *Let $k \geq 2$. Assume that F is a field such that char$(F) \neq 2$ and $\mu_{2^k} \subset F$. The 2-torsion of $H^4(D_{2^{k+1}}, \mathbb{Z})$ is negligible.*

Proof. Let $D_{2^{k+1}}$ be the dihedral group of order 2^{k+1} for $k \geq 2$. We note $2k - 2 \geq k$ so 2^{2k-2} is a multiple of 2^k. Thus the cohomology ring of $D_{2^{k+1}}$ is

$$H^*(D_{2^{k+1}}, \mathbb{Z}) \simeq \mathbb{Z}[a_2, b_2, c_3, d_4]/\langle 2a_2, 2b_2, 2c_3, 2^kd_4, b_2^2 + a_2b_2, c_3^2 + a_2d_4\rangle$$

where $\deg(a_2) = \deg(b_2) = 2$, $\deg(c_3) = 3$, and $\deg(d_4) = 4$ by [Han93, Theorem 5.2]. The 2-torsion of $H^4(D_{2^{k+1}}, \mathbb{Z})$ is generated by $\{a_2^2, b_2^2, 2^{k-1}d_4\}$.

The unique cyclic subgroup of $D_{2^{k+1}}$ of order 2^k is $H = \langle r\rangle$, the subgroup of rotations. The integral cohomology ring of H implies $H^4(H, \mathbb{Z}) \simeq \mathbb{Z}/2^k\mathbb{Z}$ generated by x^2 for generator $x \in H^2(H, \mathbb{Z})$. Let $\mathcal{N}_H^{D_{2^{k+1}}} : H^d(H, \mathbb{Z}) \to H^{2d}(D_{2^{k+1}}, \mathbb{Z})$ be the norm map defined in [Eve91, Section 6.1]. By [Eve91, Theorem 6.1.1 (N4)],

$$\text{res}^H_{D_{2^{k+1}}}(\mathcal{N}_H^{D_{2^{k+1}}}(x)) = \prod_{\sigma \in D_{2^{k+1}}/H} \sigma \cdot x = -x^2.$$

Since $\text{res}^H_{D_{2^{k+1}}}(\mathcal{N}_H^{D_{2^{k+1}}}(x))$ is a generator of $H^4(H, \mathbb{Z})$, the order of $\mathcal{N}_H^{D_{2^{k+1}}}(x)$ is at least 2^k. Thus d_4 is a linear combination of $\{a_2^2, b_2^2, \mathcal{N}_H^{D_{2^{k+1}}}(x)\}$.

By Lemma 5.25, $\{a_2^2, b_2^2\}$ is negligible. By Corollary 5.16, $H^4(H, \mathbb{Z})$ is entirely negligible since $\mu_{|H|} \subset F$. Lemma 5.7 implies that $\text{cor}^G_H(\text{res}^H_G(\mathcal{N}^G_H(x))) = 2\mathcal{N}^G_H(x)$ is negligible. Therefore, $2^{k-1}d_4$ is negligible. $\square$

Proposition 5.28. *Assume F is a field such that char$(F) \neq 2$ and $\mu \subset F$. The negligible cohomology ideal of $H^*(D_{2^{k+1}}, \mathbb{Z}/2\mathbb{Z})$ over F is generated by $\{u_1^2, v_1^2, u_1w_2, v_1w_2, w_2^2\}$. In particular, $H^d(D_{2^{k+1}}, \mathbb{Z}/2\mathbb{Z})$ is entirely negligible for $d \geq 3$.*

Proof. By Corollary 2.2(2), there are no negligible classes in $H^1(D_{2n}, \mathbb{Z}/2\mathbb{Z})$. Corollary 3.2 proves that $H^2(D_{2n}, \mathbb{Z}/2\mathbb{Z})_{\text{neg},F}$ is generated by $\{u_1^2, v_1^2\}$ when $\mu_4 \subset F$.

Short exact sequence (5.2) for $n = 2$ induces the following commutative square.

$$\begin{array}{ccc} H^3(D_{2^{k+1}}, \mathbb{Z}/2\mathbb{Z}) & \longrightarrow & H^4(D_{2^{k+1}}, \mathbb{Z}) \\ \downarrow \text{inf}^3 & & \downarrow \text{inf}^4 \\ H^3(K, \mathbb{Z}/2\mathbb{Z}) & \hookrightarrow & H^4(K, \mathbb{Z}) \end{array}$$

Lemma 5.9 proves that the bottom map is injective when F contains all roots of unity. When $\mu_{2^k} \subset F$, Lemma 5.27 shows that the 2-torsion of $H^4(D_{2^{k+1}}, \mathbb{Z})$ is negligible. Thus $H^3(D_{2^{k+1}}, \mathbb{Z}/2\mathbb{Z})$ is entirely negligible.

In degree 4, $\{u_1^4, v_1^4, u_1^2w_2, v_1^2w_2\}$ are negligible since u_1^2 and v_1^2 are negligible. Corollary 4.2 proves that w_2^2 is negligible over F. The cohomology in higher degrees is generated by products of negligible classes. □

Corollary 5.29. *Assume F is a field such that char$(F) \neq 2$ and $\mu \subset F$. Let n be an even natural number. The negligible cohomology ideal of $H^*(D_{2n}, \mathbb{Z}/2\mathbb{Z})$ over F is generated by $\{u_1^2, v_1^2, u_1w_2, v_1w_2, w_2^2\}$. In particular, $H^d(D_{2n}, \mathbb{Z}/2\mathbb{Z})$ is entirely negligible for $d \geq 3$.*

Proof. Let H be a Sylow 2-subgroup of D_{2n}. By Lemma 5.8, restriction induces an injection $H^*(D_{2n}, \mathbb{Z}/2\mathbb{Z}) \to H^*(H, \mathbb{Z}/2\mathbb{Z})$ for which $I(D_{2n}, \mathbb{Z}/2\mathbb{Z}) = H^*(D_{2n}, \mathbb{Z}/2\mathbb{Z}) \cap I(H, \mathbb{Z}/2\mathbb{Z})$.

If 4 does not divide n, $H \simeq (\mathbb{Z}/2\mathbb{Z})^2$. Apply Theorem 5.1.

If 4 divides n, $H \simeq D_{2^{k+1}}$ for some $k \geq 2$. Apply Proposition 5.28. □

Remark. Corollary 5.29 requires $\mu \subset F$ so, in particular, $s(F) = 1$. For an elementary abelian 2-subgroup H of D_{2n}, Theorem 5.1 proves that the negligible cohomology ideal $H^*(H, \mathbb{Z}/2\mathbb{Z})$ is generated by the square of characters in $H^1(H, \mathbb{Z}/2\mathbb{Z})$. Corollary 5.29 is not immediately a counterexample to Conjecture 5.6 although we have yet to study how w_2 restricts to the cohomology of H. We also have yet to compute the negligible cohomology of a dihedral group D_{2n} when n is even over a field that does not contain all roots of unity.

5.6 Symmetric Groups

Serre classifies the negligible classes of symmetric groups over $\mathbb{Q}$ for finite coefficients with a trivial action in [GMS03, Theorem 26.3]. He compares group cohomology to the computation of the cohomological invariants of a symmetric group found in [GMS03, Theorem 25.13]. Serre finds that the mod 2 negligible classes over $\mathbb{Q}$ are those that restrict to negligible classes in the cohomology of elementary abelian 2-subgroups by comparing the result with [GMS03, Lemma 26.4]. Serre's results confirm Conjecture 5.6 for symmetric groups over $\mathbb{Q}$.

Proposition 5.30. *Let p be an odd prime. Assume that F is a field such that char$(F) \neq p$.*

(1) *If $p \leq n < 2p$, then $H^d(S_n, \mathbb{Z}/p\mathbb{Z})$ is entirely negligible over F in non-zero degrees.*

(2) *If $2p \leq n < 3p$, then $H^d(S_n, \mathbb{Z}/p\mathbb{Z})$ is entirely negligible over F for non-zero $d \neq 3$. If $\mu_{p^2} \subset F$, then $H^d(S_n, \mathbb{Z}/p\mathbb{Z})$ is entirely negligible over F in all non-zero degrees.*

Proof. The abelianization of S_n is $\mathbb{Z}/2\mathbb{Z}$ for $n \geq 3$. Thus $H^1(S_n, \mathbb{Z}/p\mathbb{Z}) = 0$ for p an odd prime. By [Hup13, Theorem 25.12], $H^2(S_n, \mathbb{Z}/p\mathbb{Z}) = 0$.

(1) By assumption, p divides $n!$ but p^2 does not divide $n!$. Thus a Sylow p-subgroup C_p of S_n is cyclic of order p. The composition of corestriction and restriction with respect to C_p is multiplication by $\frac{n!}{p}$, an isomorphism on each $H^d(S_n, \mathbb{Z}/p\mathbb{Z})$. Therefore,

$$\text{cor} : H^d(C_p, \mathbb{Z}/p\mathbb{Z}) \to H^d(S_n, \mathbb{Z}/p\mathbb{Z})$$

is surjective for all $d \geq 0$ and respects negligible classes by Lemma 5.7. We may assume, without loss of generality, that $\mu_p \subset F$ by Corollary 2.4. Then Proposition 5.15 implies that $H^d(C_p, \mathbb{Z}/p\mathbb{Z})$ is entirely negligible for $d \geq 3$. We conclude that

$H^d(S_n, \mathbb{Z}/p\mathbb{Z})$ is entirely negligible for all $d \geq 3$.

(2) By assumption, p^2 divides $n!$ but p^3 does not divide $n!$. Since there is no element of order p^2 in S_n, the Sylow p-subgroups H of S_n are elementary abelian of order p^2. The composition of corestriction and restriction with respect to H is multiplication by $\frac{n!}{p^2}$, an isomorphism on each $H^d(S_n, \mathbb{Z}/p\mathbb{Z})$. Therefore,

$$\text{cor} : H^d(H, \mathbb{Z}/p\mathbb{Z}) \to H^d(S_n, \mathbb{Z}/p\mathbb{Z})$$

is surjective for all $d \geq 0$ and respects negligible classes by Lemma 5.7. When $\mu_{p^2} \not\subset F$, Theorem 5.3 proves $H^d(H, \mathbb{Z}/p\mathbb{Z})$ is entirely negligible for $d \geq 4$. We conclude that $H^d(S_n, \mathbb{Z}/p\mathbb{Z})$ is entirely negligible for all $d \geq 4$. When $\mu_{p^2} \subset F$, Theorem 5.3 proves $H^d(H, \mathbb{Z}/p\mathbb{Z})$ is entirely negligible for $d \geq 2$. We conclude that $H^d(S_n, \mathbb{Z}/p\mathbb{Z})$ is entirely negligible for all $d \geq 3$. □

Since $S_3 \simeq D_6$, the mod 2 negligible cohomology ring of S_3 is computed in Section 5.5.1. We will compute generators of the mod p negligible cohomology ideal for S_4 and S_5.

Proposition 5.30(1) handles the negligible cohomology ideal of $H^*(S_4, \mathbb{Z}/3\mathbb{Z})$. [Nak62, Theorem 4.1] proves that

$$H^*(S_4, \mathbb{Z}/2\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}[u_1, v_2, w_3]/\langle u_1 w_3 \rangle$$

for $\deg(u_1) = 1$, $\deg(v_2) = 2$, and $\deg(w_3) = 3$.

Proposition 5.30(1) handles the negligible ideals of $H^*(S_5, \mathbb{Z}/3\mathbb{Z})$ and $H^*(S_5, \mathbb{Z}/5\mathbb{Z})$. [KG23] proves that

$$H^*(S_5, \mathbb{Z}/2\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}[u_1, v_2, w_3]/\langle u_1 w_3 \rangle$$

for $\deg(u_1) = 1$, $\deg(v_2) = 2$, and $\deg(w_3) = 3$.

Proposition 5.31. *Assume that F is a field such that char$(F) \neq 2$ and $\mu \subset F$. Then the negligible cohomology ideals of $H^*(S_4, \mathbb{Z}/2\mathbb{Z})$ over F and $H^*(S_5, \mathbb{Z}/2\mathbb{Z})$ over F are generated by $\{u_1^2, u_1 v_2, w_3, v_2^2\}$. In particular, $H^d(S_4, \mathbb{Z}/2\mathbb{Z})$ and $H^d(S_5, \mathbb{Z}/2\mathbb{Z})$ are entirely negligible for $d \geq 3$.*

Proof. Let $n \in \{4,5\}$. By Corollary 2.2(2), the negligible cohomology of $H^1(S_n, \mathbb{Z}/2\mathbb{Z})$ over F is trivial. Corollary 3.2 proves $H^2(S_n, \mathbb{Z}/2\mathbb{Z})_{\text{neg}}$ is generated by $\{u_1^2\}$ when $\mu_4 \subset F$.

A Sylow 2-subgroup H of S_n is isomorphic to D_8 and of index 3 if $n = 4$ or index 15 if $n = 5$. Thus $\text{cor}^d : H^d(H, \mathbb{Z}/2\mathbb{Z}) \to H^d(S_n, \mathbb{Z}/2\mathbb{Z})$ is surjective for all $d \geq 0$. Proposition 5.28 implies $H^d(H, \mathbb{Z}/2\mathbb{Z})$ is entirely negligible for $d \geq 3$ when $\mu \subset F$. By Lemma 5.7, $H^d(S_n, \mathbb{Z}/2\mathbb{Z})$ is entirely negligible for $d \geq 3$. □

Remark. Proposition 5.31 requires $\mu \subset F$ so, in particular, $s(F) = 1$. Once again, Proposition 5.31 is not immediately a counterexample to Conjecture 5.6 although we have yet to study the restriction to elementary abelian 2-subgroups or generalize the computations to an arbitrary base field.

5.7 Generalized Quaternion Groups

Let Q_{2^k} denote the generalized quaternion group of order 2^k for $k \geq 3$ with presentation

$$Q_{2^k} = \langle g, h : g^{2^{k-1}} = h^4 = 1, g^{2^{k-2}} = h^2, h^{-1}gh = g^{-1} \rangle.$$

[MP91, Theorem 1] proves that for $n \geq 4$

$$H^*(Q_8, \mathbb{Z}/2\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}[u_1, v_1, w_4]/\langle u_1^2 + u_1v_1 + v_1^2, u_1^2v_1 + u_1v_1^2 \rangle$$
$$H^*(Q_{2^n}, \mathbb{Z}/2\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}[u_1, v_1, w_4]/\langle u_1v_1, u_1^3 + v_1^3 \rangle$$

where $\deg(u_1) = \deg(v_1) = 1$ and $\deg(w_4) = 4$.

Proposition 5.32. *Assume char$(F) \neq 2$.*

(1) *If $s(F) = 1$, then $\{u_1^2, v_1^2\}$ is negligible in $H^2(Q_8, \mathbb{Z}/2\mathbb{Z})$ over F.*

(2) *Assume $s(F) = 2^r > 1$. There are no negligible classes in $H^2(Q_8, \mathbb{Z}/2\mathbb{Z})$ over F.*

(3) *Assume $s(F) = 2^r$. If $\ell > \lceil \frac{r}{4} \rceil + 1$, then w_4^ℓ is negligible in $H^*(Q_8, \mathbb{Z}/2\mathbb{Z})$ over F. If $\ell \leq \lfloor \frac{r+1}{4} \rfloor$, then w_4^ℓ is not negligible in $H^*(Q_8, \mathbb{Z}/2\mathbb{Z})$ over F.*

(4) *If F is formally real, then there are no negligible classes in $H^2(Q_8, \mathbb{Z}/2\mathbb{Z})$ over F. Further,*

w_4 is not eventually negligible in $H^*(Q_8, \mathbb{Z}/2\mathbb{Z})$ over F.

Proof. By Corollary 2.2(2), the negligible cohomology of $H^1(Q_8, \mathbb{Z}/2\mathbb{Z})$ over F is trivial.

As groups, $H^2(Q_8, \mathbb{Z}/2\mathbb{Z})$ is generated by $\{u_1^2, v_1^2\}$ and $H^3(Q_8, \mathbb{Z}/2\mathbb{Z})$ is generated by $\{u_1^2 v_1 = u_1 v_1^2\}$. Let K be a field extension of F and $j : \Gamma_K \to Q_8$ be a continuous group homomorphism. Then $j^*(u_1) = (a_1) \in K^\times/(K^\times)^2$ and $j^*(v_1) = (a_2) \in K^\times/(K^\times)^2$ for $a_1, a_2 \in K^\times$ square-free. We have

$$j^*(u_1^2) = (a_1, a_1) = (a_1, -1)$$
$$j^*(v_1^2) = (a_2, a_2) = (a_2, -1)$$
$$j^*(u_1^2 v_1) = j^*(u_1^2) \cup j^*(v_1) = (a_1, a_1, a_2) = (a_1, a_2, -1)$$

If $s(F) = 1$, then $(a_1, -1) = (a_2, -1) = 0$. If $s(F) > 1$, then Corollary 3.2 proves $H^2(Q_8, \mathbb{Z}/2\mathbb{Z})_{\text{neg}}$ is trivial.

The order 2 center Z of the generalized quaternion group Q_8 is its unique elementary abelian 2-subgroup. Since w_4 is not nilpotent in $H^*(Q_8, \mathbb{Z}/2\mathbb{Z})$, [QV72] implies that w_4 does not restrict to 0 on $H^*(Z, \mathbb{Z}/2\mathbb{Z})$. When $s(F) = 2^r$ and $\ell \leq \lfloor \frac{r+1}{4} \rfloor$ or F is formally real for any $\ell \geq 1$, the restriction of w_4^ℓ to $H^{4\ell}(Z, \mathbb{Z}/2\mathbb{Z})$ is not negligible. Theorem 5.1 and Proposition 2.3(1) provide the relevant results. When $s(F) = 2^r$ and $\ell > \lceil \frac{r}{4} \rceil + 1$,

$$j^*(w_4^\ell) = j^*(w_4) \cup (-1)^{4(\ell-1)} = 0$$

so w_4^ℓ is negligible. □

Proposition 5.33. *Assume char$(F) \neq 2$. Let $k \geq 4$.*

(1) *If $s(F) = 1$, then $\{u_1^2, v_1^2\}$ in $H^2(Q_{2^k}, \mathbb{Z}/2\mathbb{Z})$ is negligible over F.*
(2) *If $s(F) = 2$, then $\{u_1^3 = v_1^3\}$ in $H^3(Q_{2^k}, \mathbb{Z}/2\mathbb{Z})$ is negligible over F.*
(3) *If $s(F) = 2^r > 1$, then there are no negligible classes in $H^2(Q_{2^k}, \mathbb{Z}/2\mathbb{Z})$ over F.*
(4) *Assume $s(F) = 2^r$. If $\ell > \lceil \frac{r}{4} \rceil + 1$, then w_4^ℓ is negligible in $H^*(Q_{2^k}, \mathbb{Z}/2\mathbb{Z})$ over F. If $\ell \leq \lfloor \frac{r+1}{4} \rfloor$, then w_4^ℓ is not negligible in $H^*(Q_{2^k}, \mathbb{Z}/2\mathbb{Z})$ over F.*

(5) *If F is formally real, then there are no negligible classes in $H^2(Q_{2^k}, \mathbb{Z}/2\mathbb{Z})$ and $H^3(Q_{2^k}, \mathbb{Z}/2\mathbb{Z})$ over F. Further, w_4 is not eventually negligible in $H^*(Q_{2^k}, \mathbb{Z}/2\mathbb{Z})$ over F.*

Proof. By Corollary 2.2(2), the negligible cohomology of $H^1(Q_{2^k}, \mathbb{Z}/2\mathbb{Z})$ over F is trivial.

As groups, $H^2(Q_{2^k}, \mathbb{Z}/2\mathbb{Z})$ is generated by $\{u_1^2, v_1^2\}$ and $H^3(Q_{2^k}, \mathbb{Z}/2\mathbb{Z})$ is generated by $\{u_1^3 = v_1^3\}$. Let K be a field extension of F and $j : \Gamma_K \to Q_{2^k}$ be a continuous group homomorphism. Then $j^*(u_1) = (a_1) \in K^\times/(K^\times)^2$ and $j^*(v_1) = (a_2) \in K^\times/(K^\times)^2$ for $a_1, a_2 \in K^\times$ square-free. We have

$$j^*(u_1^2) = (a_1, a_1) = (a_1, -1)$$
$$j^*(v_1^2) = (a_2, a_2) = (a_2, -1)$$
$$j^*(u_1^3) = (a_1, a_1, a_1) = (a_1, -1, -1).$$

If $s(F) = 1$, then $(a_1, -1) = (a_2, -1) = 0$. If $s(F) \geq 2$, then Corollary 3.2 proves $H^2(Q_{2^k}, \mathbb{Z}/2\mathbb{Z})_{\text{neg}}$ is trivial. If $s(F) = 2$, then $(a_1, -1, -1) = 0$.

The order 2 center Z of the generalized quaternion group Q_{2^k} is its unique elementary abelian 2-subgroup. Since w_4 is not nilpotent in $H^*(Q_{2^k}, \mathbb{Z}/2\mathbb{Z})$, [QV72] implies that w_4 does not restrict to 0 on $H^*(Z, \mathbb{Z}/2\mathbb{Z})$. When $s(F) = 2^r$ and $\ell \leq \lfloor \frac{r+1}{4} \rfloor$ or F is formally real for any $\ell \geq 1$, the restriction of w_4^ℓ to $H^{4\ell}(Z, \mathbb{Z}/2\mathbb{Z})$ is not negligible. Theorem 5.1 and Proposition 2.3(1) provide the relevant results. When $s(F) = 2^r$ and $\ell > \lceil \frac{r}{4} \rceil + 1$,

$$j^*(w_4^\ell) = j^*(w_4) \cup (-1)^{4(\ell-1)} = 0$$

so w_4^ℓ is negligible. □

Remark. Conjecture 5.6 has not been confirmed in the generalized quaternion case. If $s(F) = 2^r > 1$, there are sometimes choices for ℓ between $\lfloor \frac{r+1}{4} \rfloor$ and $\lceil \frac{r}{4} \rceil + 1$. We have yet to determine whether w_4^ℓ is negligible over F in these cases.

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