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Publication Date

1962-05-01

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Contract No. W-7405-eng-48

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ABSTRACT

It is shown that for many-channel, arbitrary spin, potential-scattering problems, the previously defined continuation of the S matrix to nonphysical values of total angular momentum is such that one can make a generalized Sommerfeld-Watson transformation on the partial-wave expansions of the scattering amplitudes. An important symmetry of the partial-wave amplitudes is obtained, thus permitting an analogue of Mandelstam's modification of the Sommerfeld-Watson transform. It is shown that a pole of the S matrix at $J = \alpha$ is associated with a contribution to the scattering amplitude which behaves like z^α for large values of $z = \cos \theta$, θ being the scattering angle, and that such contributions dominate over the remaining integral terms. Exchange potentials and the generalization of J-parity are discussed, and some results on the positions of poles of the S matrix are derived.

ON COMPLEX ANGULAR MOMENTUM IN MANY-CHANNEL POTENTIAL-SCATTERING PROBLEMS, II¹

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I. INTRODUCTION

In a previous paper, (1) which we shall refer to as I, we introduced a continuation of a modified S matrix, $\tilde{S}^J$, as a function of J, away from the physical, integer or half-integer values of J. We showed that for a suitable potential this matrix is a meromorphic function of J in the entire finite J plane. In particular, we wrote the amplitude for scattering through center-of-mass angles θ , ϕ from channel c and helicities λ_1 and λ_2 to channel c' and helicities λ'_1 and λ'_2 as [see Eq. (I. 3. 19)]

$$\begin{aligned} \langle c' \lambda'_1 \lambda'_2 | f(\theta, \phi) | c \lambda_1 \lambda_2 \rangle &= (2i)^{-1} (k_c, k_c)^{-1/2} \exp [i(\lambda - \lambda')\phi] \\ &\times \sum_J \left\{ \sum_{tt' ss'} (2J + 2t' + 1)^{1/2} (2J + 2t + 1)^{1/2} C(J+t', s' J; 0 \lambda') \right. \\ &\times C(s'_1 s'_2 s'; \lambda'_1, -\lambda'_2) C(J+t, s J; 0 \lambda) C(s_1 s_2 s; \lambda_1, -\lambda_2) \\ &\times d_{\lambda \lambda'}^J(\theta) N(Jt' s') N(Jts)^{-1} i^{t-t'} \\ &\times \left. \langle k_c, JMc' J+t', s' | \tilde{S} - 1 | k_c JM c J+t, s \rangle \right\}, \end{aligned} \quad (1.1)$$

with

$$N(Jts) = \left[\frac{(2J + t + s + 1)!}{(2J + t - s)! (2J + 2t + 1)} \right]^{1/2} \quad (1.2)$$

We showed that the J summand of Eq. (1.1) was holomorphic apart from the poles in the matrix elements of $\tilde{S}^J$. We recall that $\lambda = \lambda_1 - \lambda_2$ and $\lambda' = \lambda'_1 - \lambda'_2$. Note that we considered only the case where all channels contained two particles.

We emphasize here that it was implicit in I that the Clebsch-Gordan coefficients were defined for unphysical J by means of Wigner's closed expression, (2)

$$C(J+t, sJ; mm_s) = \left[(2J+1) \frac{(2J+t-s)! (s-t)! (s+t)! (J+m+m_s)! (J-m-m_s)!}{(2J+t+s+1)! (J+t-m)! (J+t+m)! (s-m_s)! (s+m_s)!} \right]^{1/2} \\ \times \sum_v \frac{(-1)^{v+s+m_s}}{v!} \frac{(J+s+m-v)! (J+t-m+v)!}{(s-t-v)! (J+m+m_s-v)! (J+t-s-m-m_s+v)!} \quad (1.3)$$

It is important that all the Clebsch-Gordan coefficients involving J which occur can be written in this form. Note that certain of the familiar relations involving Clebsch-Gordan coefficients do not hold when J is unphysical.

In this paper, we first justify the choice of the particular continuation defined in I; this we do by showing that it allows a generalization of the Sommerfeld-Watson transformation. (3) As in the case of the one-channel problem, this makes the continuation unique. (4-6) In section 3 we show how, following a method of Mandelstam for the one-channel problem, (7) we can make a modified form of this transformation which exhibits clearly the fact that the rightmost pole of $\tilde{S}^J$ dominates the amplitude for large z , wherever that pole may lie.

The complications that arise from exchange potentials are dealt with in section 4, and some results on the positions of poles of S^J are obtained in section 5. Finally, in section 6, we briefly summarize the results of I and of this article.

II. THE SOMMERFELD-WATSON TRANSFORMATION

It is convenient to write Eq. (1.1) in the abbreviated form

$$2i(k'k)^{1/2} \exp[i(\lambda' - \lambda)\phi] \left\langle c' \lambda'_1 \lambda'_2 | f(\theta, \phi) | c \lambda_1 \lambda_2 \right\rangle$$

$$= \sum_J R(J) d_{\lambda\lambda'}^J(\theta), \quad (2.1)$$

where this defines $R(J)$, and where we have written k' and k for k_c and k_c respectively.

We consider, for example, the case where the physical values of J are integers. Then we have

$$\sum_J R(J) d_{\lambda\lambda'}^J(\theta) = -\frac{1}{2i} \int_C \frac{dJ R(J) d_{\lambda\lambda'}^J(\theta) (-1)^J}{\sin \pi J}, \quad (2.2)$$

where the contour C is given in Fig. 1 and encloses only the singularities of the integrand arising from the zeros of $\sin \pi J$. We may reasonably expect that both sides of Eq. (2.2) are defined in some domain of $\cos \theta$, analogous to the Lehmann ellipse, which contains the physical values $-1 < \cos \theta < +1$. Before distorting the contour, we remove the $(-1)^J$ factor by using, for integer J ,

$$d_{\lambda\lambda'}^J(\theta) (-1)^J = (-1)^\lambda d_{\lambda-\lambda'}^J(\pi-\theta), \quad (2.3)$$

which allows us to replace Eq. (2.2) by

$$\sum_J R(J) d_{\lambda\lambda'}^J(\theta) = -\frac{1}{2i} \int_C \frac{dJ R(J) d_{\lambda-\lambda'}^J(\pi-\theta) (-1)^\lambda}{\sin \pi J}. \quad (2.4)$$

We shall show below that the asymptotic behavior of the integrand in Eq. (2.4) for large J is such that we can deform the contour of integration to obtain

$$\sum_J (-1)^J R(J) d_{\lambda\lambda'}^J(\theta) = -\frac{1}{2i} \int_{C'} \frac{dJ R(J) d_{\lambda-\lambda'}^J(\pi-\theta)}{\sin \pi J} \\ - \pi \sum_a \frac{\beta(a) d_{\lambda-\lambda'}^a(\pi-\theta)}{\sin \pi a} \quad (2.5)$$

Here the contour C' (also given in Fig. 1) is a parallel to the imaginary axis between $\text{Re } J = s_{\text{max}} - 1/2$ and $\text{Re } J = s_{\text{max}}'$, and also includes a loop around $J = 0, 1, \dots, s_{\text{max}} - 1$. We have written s_{max} for the larger of s and s' . The terms in the sum arise from the poles of $R(J)$ which lie to the right of C' . The residue of $R(J)$ at such a pole at $J = a$ is denoted by $\beta(a)$.

As in the one-channel case (3) the right-hand side of Eq. (2.5) in general exists in a much wider domain of θ than the original sum over J . General arguments (5, 6) show that there is no other meromorphic function of J , equal to $R(J)$ at integer J values, for which Eq. (2.5) is satisfied. Note that it follows from Eq. (I. 3. 18) that in general all elements of the S matrix have poles at the same value of J , and also that the residues at these poles are factorizable. (8-10)

A. Asymptotic Form for Large $|J|$

To determine the conditions on $\tilde{S}^J$ for which the Sommerfeld-Watson transform is permissible, we consider first the $d_{\lambda\lambda'}^J(\theta)$ factor which is given by (2)

$$d_{\lambda\lambda'}^J(\theta) = \left[\frac{(J-\lambda')! (J+\lambda)!}{(J+\lambda')! (J-\lambda)!} \right]^{1/2} \frac{(-1)^{\lambda-\lambda'}}{(\lambda-\lambda')!} \left(\frac{1-z}{1+z} \right)^{(\lambda-\lambda')/2} \\ \times \left(\frac{1+z}{2} \right)^J F\left(\lambda-J, -\lambda'-J; \lambda-\lambda'+1; \frac{z-1}{z+1} \right) \quad (2.6)$$

for $\lambda \geq \lambda'$. We transform the hypergeometric function using (11)

$$F(a, b; c; \zeta) = (1-\zeta)^{-a} F[a, c-b; c; \zeta(\zeta-1)^{-1}] \quad (2.7)$$

and the fact that $F(a, b; c; \zeta)$ is symmetrical in a and b , to obtain

$$d_{\lambda\lambda'}^J(\theta) = \left[\frac{(J-\lambda')! (J+\lambda)!}{(J+\lambda')! (J-\lambda)!} \right]^{1/2} \frac{(-1)^{\lambda-\lambda'}}{(\lambda-\lambda')!} \left(\frac{1-z}{2} \right)^{(\lambda-\lambda')/2} \left(\frac{1+z}{2} \right)^{(\lambda+\lambda')/2} \\ \times F(J+\lambda+1, \lambda-J; \lambda-\lambda'+1; \frac{1}{2}(1-z)). \quad (2.8)$$

Using the asymptotic form (11) of the hypergeometric function in Eq. (2.8)

for large J , we obtain

$$d_{\lambda\lambda'}^J(\theta) \Big|_{|J| \rightarrow \infty} \sim (-1)^{\lambda-\lambda'} 2^{\lambda} \pi^{-1/2} (1-z)^{(\lambda-\lambda')/2} (1+z)^{(\lambda+\lambda')/2} \\ \times \left(1 - e^{-i\theta} \right)^{\lambda' - \lambda - 1/2} \left(1 + e^{-i\theta} \right)^{-\lambda' - \lambda - 1/2} \\ \times J^{-1/2} \left[e^{i(J-\lambda)\theta} + e^{i\pi(\lambda-\lambda'+1/2)} e^{-i(J+\lambda+1)\theta} \right]. \quad (2.9)$$

For physical values of θ , i. e. real θ between 0 and π , this expression is valid for $\arg J \neq \pm \pi$, and is therefore adequate for our purpose.

In order to obtain an asymptotic expansion valid in $-\pi/2 \leq \arg J \leq \pi/2$ for nonphysical values of θ --in particular, for example, for large and real $\cos\theta$ --some further transformations on the hypergeometric function in Eq. (2.8) can be made. Alternatively, we use an expansion of the $d_{\lambda\lambda'}^J$ in terms of the Legendre functions, which will also be required later. This we obtain as a special case of the Clebsch-Gordan series, (2)² viz:

$$d_{\lambda\lambda'}^J(\theta) d_{\lambda\lambda'}^{\lambda}(\theta) = (-1)^{\lambda-\lambda'} \sum_{\tau=-\lambda}^{\lambda} C(J\lambda, J+\tau; \lambda, -\lambda) \times C(J\lambda, J+\tau; \lambda', -\lambda') P_{J+\tau}(z), \quad (2.10)$$

where we have taken, for example $\lambda \geq 0$ and $\lambda \geq \lambda'$. Other cases can be brought to this form by using simple symmetry properties of the $d_{\lambda\lambda'}^J$.

Although Eq. (2.10) is normally given only for integer or half-integer J values, it is in fact true for all J . An easy way to see this is to note that, for physical z , both sides of the equation are bounded by a power as $|J|$ approaches infinity in the half-plane $-\pi/2 \leq \arg J \leq \pi/2$. This follows for the left-hand side from Eq. (2.9), and for the right-hand side from, for example, the discussion by Squires. (5) Hence, as they are equal for integer J , they are equal for all J . Since both sides are analytic in z , this result can be extended to all z . Note that the presence of isolated branch points in J does not vitiate this argument, since they may be removed by multiplication with simple factors.

The Clebsch-Gordan coefficients in Eq. (2.10) tend to constants as $|J|$ approaches infinity. This can most easily be seen by using

Racah's formula (2) for them (the equality of this with Wigner's expression for all J follows in the usual way from the equality at integral J), together with Stirling's formula. It then follows from Eq. (2.10) that, apart from trivial constants, $d_{\lambda\lambda'}^J(\theta)$ has an asymptotic behavior for large J similar to that of $P_J(\cos\theta)$.

Returning to Eq. (1.1), and noting again that the Clebsch-Gordan coefficients tend to constants for large $|J|$, we see that a sufficient requirement on the matrix elements of $\underline{S}^J$ for the Sommerfeld-Watson transform to be possible is the same as for the one-channel zero-spin case, namely that $\underline{S}^J$ be bounded by a power of J as $|J|$ approaches infinity with $-\pi/2 < \arg J < \pi/2$, (5) and that $\underline{S}^J$ approach zero at least as fast as $J^{-3/2}$ as J approaches $\pm i\infty$. Of course, the existence of a domain of θ containing the physical values in which the original partial-wave expansion converges requires a stronger condition on the behavior as $\text{Re } J$ approaches infinity, namely that $\underline{S}^J$ be bounded by e^{-AJ} for some positive real constant A . The condition on the power as J approaches $\pm i\infty$ is only required if we wish to take $\text{Re } \theta = 0$, and is otherwise unnecessary.

B. Asymptotic Form of $\underline{S}^J$ in the Born Approximation

We shall now make plausible that $\underline{S}^J$ does satisfy the above conditions, at least for $\text{Re } J > 1/2 + s_{\text{max}}$, and in fact that

$$\underline{S}_{\mu\mu'}^J - \delta_{\mu\mu'} = O(|J|^{t-t'}|^{-3/2} e^{-Ja}) \quad (2.11)$$

as $|J|$ approaches infinity with $\text{Re } J > -(1+t+t')/2$, where $t = t_\mu$ and $t' = t_{\mu'}$, and a is a real positive number defined by Eq. (2.29). It will be seen that this expression gives the required behavior except possibly for the case of $\text{Re } \theta = 0$, when convergence of the integral along C' requires that the power of J be $-3/2$. We observe, however, that if, with a given potential and with the continuation we have chosen, the integral fails to converge for $\text{Re } \theta = 0$, then it will certainly not converge with any other continuation.

To make Eq. (2.11) plausible, we use the Born approximation. It is physically reasonable that when this gives an amplitude which tends to zero, the true amplitude tends to zero in a similar way.³ We shall assume this to be the case and demonstrate that the Born approximation $\underline{S}_B^J$ satisfies the bound of Eq. (2.11).

To derive the Born approximation, we first introduce solutions ϕ_0^J of the Schrödinger equation with $\underline{V} = 0$, analogous to the ϕ^J introduced in Eq. (1.2.6). Explicitly, these are

$$\phi_{0\mu p}^J(r) = (k_\mu r)^{1/2} J_{\ell_\mu + 1/2}(k_\mu r) \delta_{\mu p}, \quad (2.12)$$

and they satisfy

$$\phi_{0\mu p}^J(r) \underset{r \rightarrow \infty}{\sim} \frac{i}{2} \frac{i^{\ell_\mu}}{k_\mu^{1/2}} \left(f_{0\mu p}^J e^{-ik_\mu r} - e^{-i\pi \ell_\mu} f_{0\mu p}^J e^{ik_\mu r} \right) \delta_{\mu p}, \quad (2.13)$$

where the free Jost matrices f_0^J are given by

$$f_{0\mu p}^J = 2^{1/2} \pi^{-1/2} k_\mu^{1/2} \delta_{\mu p}. \quad (2.14)$$

From the Schrödinger equations satisfied by ϕ and ϕ_0 ,

$$\ddot{\phi} + \left[\underline{K}^2 - r^{-2} \underline{L}(\underline{L}+1) \right] \phi = \underline{V} \phi \quad (2.15)$$

and

$$\ddot{\phi}_0 + \left[\underline{K}^2 - r^{-2} \underline{L}(\underline{L}+1) \right] \phi_0 = 0, \quad (2.16)$$

where the matrices $\underline{K}$ and $\underline{L}$ are defined in Eqs. (I. 3. 12) and (I. 3. 26), we readily obtain

$$\int_0^\infty dr \phi_0^T \underline{V} \phi = \left[\phi_0^T \dot{\phi} - \dot{\phi}_0^T \phi \right]_0^\infty \quad (2.17)$$

For $\text{Re } J > -(1/2)(1 + t + t')$, the $(\mu\mu')$ element of the evaluation of the right-hand side at the lower limit is zero. The evaluation at the upper limit gives

$$\frac{i}{2} f_0^T \left[f(+) - f(-) \right] = \frac{i}{2} f_0^T (\underline{S} - 1) f(-), \quad (2.18)$$

where we have used Eq. (I. 2. 17). For $\text{Re } J$ sufficiently large, we have then

$$\frac{i}{2} f_0^T (\underline{S} - 1) f(-) = \int_0^\infty \phi_0^T \underline{V} \phi. \quad (2.19)$$

From this we obtain the Born approximation S_B^J by replacing ϕ by ϕ_0 and $f(-)$ by f_0 . Thus, for $\text{Re } J > -(1 + t + t')/2$, with $k_\mu = k$ and $k_{\mu'} = k'$, we have

$$(S_B^J)_{\mu\mu'} - \delta_{\mu\mu'} = -\pi i (kk')^{-1} \int_0^\infty dr r V_{\mu\mu'}^J(r) \\ \times J_{J+t+1/2}(kr) J_{J+t'+1/2}(k'r). \quad (2.20)$$

If the potential $\underline{V}$ can be written as a superposition of Yukawa functions, then we require (12)

$$\begin{aligned} \int_0^\infty dr e^{-mr} J_{J+t+1/2}(kr) J_{J+t'+1/2}(k'r) &= \left(\frac{k}{2}\right)^{J+t+1/2} \left(\frac{k'}{2}\right)^{J+t'+1/2} \\ &\times \frac{\Gamma(2J+t+t'+2)}{\Gamma(J+t+3/2)\Gamma(J+t'+3/2)} (m+ik+ik')^{-(2J+t+t'+2)} \\ &\times F_2(2J+t+t'+2; J+t+1, J+t'+1; 2J+2t+2, 2J+2t'+2; \frac{2ik}{m+ik+ik'}, \frac{2ik'}{m+ik+ik'}), \end{aligned} \quad (2.21)$$

where F_2 is one of Appel's generalized hypergeometric functions. (11)

Unfortunately the asymptotic form of these functions for $J \rightarrow \infty$ has not been discussed in the literature, and some effort on our part has not produced any way of obtaining the required result.⁴ We therefore use an alternative treatment of this question.

We first replace the potential $V_{\mu\mu'}^J(r)$ in Eq. (2.20) by

$$W_{\mu\mu'}^{J\sigma}(r) = V_{\mu\mu'}^J(r) \left[1 - e^{-\sigma r} \right]^P \quad (2.22)$$

with some positive integral value of P . Then we have

$$\begin{aligned} &\int_0^\infty dr r V_{\mu\mu'}^J(r) J_\nu(kr) J_{\nu'}(k'r) \\ &= \lim_{\sigma \rightarrow \infty} \int_0^\infty dr r W_{\mu\mu'}^{J\sigma}(r) J_\nu(kr) J_{\nu'}(k'r), \end{aligned} \quad (2.23)$$

provided $\text{Re } \nu, \text{Re } \nu'$ are sufficiently large that the integral on the left converges. Next we use the recurrence relation

$$J_{\nu+1}(kr) = \frac{1}{r} \left(\frac{\nu}{k} - \frac{\partial}{\partial k} \right) J_{\nu}(kr), \quad (2.24)$$

which gives

$$J_{J+t+1/2}(kr) = r^{-\Delta} \prod_{p=1}^{\Delta} \left(\frac{J+t+1/2-p}{k} - \frac{\partial}{\partial k} \right) J_{J+t'+1/2}(kr), \quad (2.25)$$

where, without loss of generality we have taken $t - t' = \Delta$ to be a non-negative integer. We substitute in Eq. (2.20), which becomes

$$\begin{aligned} (S_B^J)_{\mu\mu'} - \delta_{\mu\mu'} &= -\pi i (kk')^{-1} \prod_{p=1}^{\Delta} \left(\frac{J+t+1/2-p}{k} - \frac{\partial}{\partial k} \right) \\ &\times \lim_{\sigma \rightarrow \infty} \int_0^{\infty} dr r^{1-\Delta} W_{\mu\mu'}^{J\sigma}(r) J_{J+t'+1/2}(kr) J_{J+t'+1/2}(k'r). \end{aligned} \quad (2.26)$$

We now note that, since by our assumption $\underline{V}^J$ can be expressed as a superposition of Yukawa functions, the same will certainly be true of $r^{-\Delta} \underline{W}^{J\sigma}$, provided we have $P > \Delta$, i.e.,

$$r^{-\Delta} \underline{W}^{J\sigma}(r) = \frac{1}{\pi} \int_{m_0^2}^{\infty} dm^2 \frac{e^{-mr}}{mr} \underline{w}^{J\sigma}(m^2) m^{\Delta}. \quad (2.27)$$

If this is substituted in Eq. (2.26), the order of integrations may be interchanged and the r -integration performed explicitly. There results

$$\begin{aligned} (S_B^J)_{\mu\mu'} - \delta_{\mu\mu'} &= -\pi i (kk')^{-1} \prod_{p=1}^{\Delta} \left(\frac{J+t+1/2-p}{k} - \frac{\partial}{\partial k} \right) \\ &\times \frac{1}{\pi^2} \lim_{\sigma \rightarrow \infty} \int_{m_0^2}^{\infty} dm^2 m^{\Delta-1} w_{\mu\mu'}^{J\sigma}(m^2) (kk')^{-1/2} \\ &\times Q_{J+t} \left(\frac{m^2 + k^2 + k'^2}{2kk'} \right). \end{aligned} \quad (2.28)$$

We define

$$\cosh a = \frac{m_0^2 + k^2 + k'^2}{2kk'}, \quad (2.29)$$

and from the asymptotic form of $Q_{J\pm t}$ for large $|J|$, (11) obtain, at least for $\cosh a > 3/2\sqrt{2}$, the result given in Eq. (2.11). Here we must also use the fact that $V^J = O(1)$ as $|J|$ approaches ∞ , which can be seen from Eq. (I. 3. 5).

C. Asymptotic Form for Large z

One of the principle uses of the study of complex angular momenta is to exhibit explicitly the form of the amplitude for large real $z (= \cos\theta)$. We can obtain this in our case simply from Eq. (2.5) if we first note that, for $\text{Re } J > -1/2$,

$$d_{\lambda\lambda'}^J(\pi - \theta) \underset{z \rightarrow \infty}{\sim} \text{const. } z^J. \quad (2.30)$$

This result follows from the expression for $d_{\lambda\lambda'}^J$ given in Eq. (2.6) and from the properties of $F(a, b; c; 1)$. It is then obvious that the right-most pole in Eq. (2.5) dominates the asymptotic behavior, provided it lies to the right of the contour C' . Similar restrictions were present in the use of the Sommerfeld-Watson transform for the one-channel zero-spin case by Regge et al., (3) and were removed by a device due to Mandelstam. (7) In the next section we shall show that a similar technique is available in our more general situation.

III. GENERALIZATION OF THE MANDELSTAM TRANSFORMATION

It is convenient to express the partial-wave expansion Eq. (2.1) in terms of the Legendre functions rather than the $d_{\lambda\lambda'}^J$. Therefore we use Eq. (2.10) and obtain, again for the case $\lambda \geq \lambda'$ and $\lambda \geq 0$, for example,

$$d_{\lambda\lambda'}^\lambda(\theta) 2i(k'k)^{1/2} \exp \left[i(\lambda' - \lambda)\phi \right] \left\langle c' \lambda'_1 \lambda'_2 \left| f(\theta\phi) \right| c \lambda_1 \lambda_2 \right\rangle$$

$$= \sum_J \sum_{\tau=-\lambda}^{\lambda} R(J, \tau) P_{J+\tau}(z) \quad (3.1)$$

Here the J summation extends to infinity by integral steps from either zero or $1/2$. The quantity $R(J, \tau)$ is given explicitly by

$$R(J, \tau) = \sum_{tt' ss'} C(J\lambda J+\tau; \lambda_1, -\lambda) C(J\lambda J+\tau; \lambda', -\lambda')$$

$$\times C(J+t, sJ; 0, \lambda) C(J+t', s'J; 0, \lambda')$$

$$\times C(s_1 s_2 s; \lambda_1, -\lambda_2) C(s'_1 s'_2 s'; \lambda'_1, -\lambda'_2)$$

$$\times (2J+2t'+1)^{1/2} (2J+2t+1)^{1/2} N(Jt's') N(Jts)^{-1}$$

$$\times i^{t-t'} \left\langle J+t' \left| \tilde{S}^J - 1 \right| J+t \right\rangle, \quad (3.2)$$

where we have dropped the irrelevant indices on the S -matrix element.

We now write

$$\begin{aligned} \sum_J \sum_{\tau=-\lambda}^{\lambda} R(J, \tau) P_{J+\tau}(z) &= \sum_J \sum_{\tau=-\lambda}^{\lambda} \left[R(J, \tau) P_{J+\tau}(z) \right. \\ &\quad \left. + \pi^{-1} (-1)^{J+\tau} R(J+1/2, \tau) Q_{J+\tau+1/2}(-z) \right] \\ &- \sum_J \sum_{\tau=-\lambda}^{\lambda} \pi^{-1} (-1)^{J+\tau} R(J+1/2, \tau) Q_{J+\tau+1/2}(-z), \end{aligned} \quad (3.3)$$

and replace the first sum by

$$- \frac{1}{2i} \sum_{\tau=-\lambda}^{\tau=+\lambda} \int_C dJ R(J, \tau) \left[\frac{P_{J+\tau}(-z)}{\sin \pi(J+\tau)} - \frac{1}{\pi} \frac{Q_{J+\tau}(-z)}{\cos \pi(J+\tau)} \right] \quad (3.4)$$

where the contour C is that given in Fig. 1 when physical values of J are integers, and is displaced to the right by a half unit when physical values are half-odd-integers. Following Mandelstam (7) we simplify the expression in the square bracket in Eq. (3.4) by using

$$\frac{P_\nu(z)}{\sin \pi \nu} - \frac{1}{\pi} \frac{Q_\nu(z)}{\cos \pi \nu} = - \frac{1}{\pi} \frac{Q_{-\nu-1}(z)}{\cos \pi \nu} \quad (3.5)$$

We then distort the contour to run parallel to the imaginary axis at

$$\operatorname{Re} J = -M - \epsilon, \quad (3.6)$$

where $0 < \epsilon < 1/2$ and $2M$ is a positive integer, even or odd according as physical values of $2J$ are even or odd. We denote the new contour, shown in Fig. 1, by C_M . Then, for $M > 2$ and for integral values of

physical J, we have

$$\begin{aligned}
 \sum_J \sum_{\tau=-\lambda}^{\lambda} R(J, \tau) P_{J+\tau}(z) &= \frac{1}{2\pi i} \sum_{\tau=-\lambda}^{\lambda} \int_{C_M} dJ R(J, \tau) \frac{Q_{-J-\tau-1}(-z)}{\cos \pi(J+\tau)} \\
 &+ \sum_{\tau=-\lambda}^{\lambda} \sum_{\substack{a \\ \operatorname{Re} a > -M-\epsilon}} \beta(a, \tau) \frac{Q_{-a-\tau-1}(-z)}{\cos \pi(a+\tau)} \\
 &- \sum_J \sum_{\tau=-\lambda}^{\lambda} \pi^{-1} (-1)^{J+\tau} R(J+1/2, \tau) Q_{J+\tau+1/2}(-z) \\
 &- \sum_J^{M-2} \sum_{\tau=-\lambda}^{\lambda} \pi^{-1} (-1)^{J+\tau} R(-J-3/2, -\tau) Q_{J+\tau+1/2}(-z) \\
 &+ \sum_{\tau=-\lambda}^{\lambda} \pi^{-1} (-1)^{\tau} R(-1/2, \tau) Q_{\tau-1/2}(-z). \tag{3.7}
 \end{aligned}$$

The last two sums in this equation have arisen from the zeros of $\cos \pi(J+\tau)$.

For the case where physical values of J are half-odd-integers, the last sum is to be replaced by

$$\sum_{\tau=-\lambda}^{\lambda} \pi^{-1} (-1)^{\tau+1/2} \left[R(0, \tau) + R(-1, -\tau) \right] Q_{\tau}(-z). \tag{3.8}$$

The term involving $\beta(a, \tau)$ in Eq. (3.7) arises from a pole of R , i. e. of $\tilde{S}$, at $J = a$, and $\beta(a, \tau)$ is the residue of $R(J, \tau)$ at this pole.

We shall show below that

$$R(J + 1/2, \tau) = -R(-J - 3/2, -\tau) \quad (3.9)$$

for physical values of $\pm J$. An immediate consequence is that Eq. (3.7) simplifies in all cases to

$$\begin{aligned} \sum_J \sum_{\tau=-\lambda}^{\lambda} R(J, \tau) P_{J+\tau}(z) &= \frac{1}{2\pi i} \sum_{\tau=-\lambda}^{\lambda} \int_{C_M} dJ \frac{R(J, \tau) Q_{-J-\tau-1}(z)}{\cos \pi(J+\tau)} \\ &+ \sum_{\tau=-\lambda}^{\lambda} \sum_{\substack{a \\ \operatorname{Re} a > -M-\epsilon}} \beta(a, \tau) \frac{Q_{-a-\tau-1}(z)}{\cos \pi(a+\tau)} \\ &- \sum_{\substack{J \\ J > M-2}} \pi^{-1} (-1)^{J+\tau} R(J+1/2, \tau) Q_{J+\tau+1/2}(z). \end{aligned} \quad (3.10)$$

The utility of this way of writing the partial wave sum is now immediately apparent, since for large z , the three sums on the right-hand side of Eq. (3.10) behave as $z^{-M-\epsilon+\lambda}$, $z^{a+\lambda}$, and $z^{-M-1/2+\lambda}$, respectively. Hence, since we have $\operatorname{Re} a > -M-\epsilon > -M-1/2$, the expression is dominated by the contributions arising from the poles of $\tilde{S}^J$.

Note that, in deriving Eq. (3.10), we have neglected the contribution to the integrals from the arcs at infinity, introduced in deforming the integration contour from C to C_M . The asymptotic behavior discussed in section 2 justifies this for $\operatorname{Re} J > -1/2 + s_{\max}$, but we

know of no proof, even in the one-channel, zero-spin case, which covers the region to the left of $\text{Re } J = -1/2 + s_{\text{max}}$.

We now turn to the proof of the symmetry relation Eq. (3.9). An essential ingredient in our proof is the symmetry

$$C(J+t, s, J; m m_s) = (-1)^{s+m_s} C(-J-t-1, s, -J-1; m m_s), \quad (3.11)$$

which is valid for all J and m , and for $t = -s, -s+1, \dots, s$ and $m_s = -s, -s+1, \dots, s$, with $2s$ a nonnegative integer. This is proved in the Appendix. We also use the symmetry

$$N(J t s) = (-1)^{-s} N(-J-1, -t s), \quad (3.12)$$

which can readily be obtained from the definition of $N(Jts)$ given in Eq. (1.2). Then, using Eqs. (3.11) and (3.12) in Eq. (3.2), we see that Eq. (3.9) follows immediately, provided that $\tilde{S}^J$ has the symmetry

$$i^{t-t'} \langle J+t' | \tilde{S}^J | J+t \rangle = i^{t'-t} \langle -J-t'-1 | \tilde{S}^{-J-1} | -J-t-1 \rangle, \quad (3.13)$$

for $J = 1/2$ physical, i.e., for $2J+2s$ an odd integer $\geq s$. We shall now prove this relation.

First we note that from Eqs. (3.11) and (3.12), together with the definition of the potential given in Eqs. (I.3.5) and (I.3.7), there follows the symmetry of $\tilde{V}^J$,

$$\langle J+t', s' | \tilde{V}^J(r) | J+t, s \rangle = \langle -J-t'-1, s' | \tilde{V}^{-J-1}(r) | -J-t-1, s \rangle. \quad (3.14)$$

It follows that the Schrödinger equation,

$$\tilde{\psi}^J(r) + \left[K^2 - \frac{L(L+1)}{r^2} \right] \tilde{\psi}^J(r) - \tilde{V}^J(r) \tilde{\psi}^J(r) = 0 \quad (3.15)$$

is unchanged by the reflection $J \rightarrow -J-1, t \rightarrow -t$.

From the boundary condition for the solution $\tilde{\chi}_{\mu}^J$,

$$\lim_{r \rightarrow \infty} e^{\pm i k_{\mu} r} \tilde{\chi}_{\mu p}^J(\pm) = k_{\mu}^{-1/2} (\pm i)^{\ell_{\mu}} \delta_{\mu p}, \quad (3.16)$$

it is clear that under this reflection we have

$$\tilde{\chi}_{\mu p}^J(\pm) \rightarrow \tilde{\chi}_{\bar{\mu} \bar{p}}^{-J-1}(\pm) = (\pm i)^{-2\ell_{\mu}-1} \tilde{\chi}_{\mu p}^J(\pm). \quad (3.17)$$

Here, if μ is the integer representing the channel labels c , ℓ and s , then $\bar{\mu}$ represents c , $-\ell-1$, and s ; also, the relation of $\bar{p}$ to p is the same as the relation of $\bar{\mu}$ to μ .

The situation for the $\tilde{\phi}_{\mu}^J$ solutions is less simple, since their defining boundary condition depends on J in a nontrivial way. We recall that in Eq. (I. 3. 22) we could write

$$\tilde{\phi}_{\mu}^J(r) = r^{J-s_{\max}+1} \sum_{n=0}^{\infty} \sum_{q=0}^{2s_{\max}+1} \underline{a}_{\mu}^J(n, q) r^n (r \ln r)^q, \quad (3.18)$$

where

$$\begin{aligned} & \left[(J-s_{\max}+n+q+1)(J-s_{\max}+n+q) - \underline{L}(\underline{L}+1) \right] \underline{a}_{\mu}^J(n, q) \\ &= \sum_m \underline{a}_{\mu}^J(m) \underline{a}_{\mu}^J(n-m-1, q) \\ & - (2J-2s_{\max}+2n+2q+1)(q+1) \underline{a}_{\mu}^J(n-1, q+1) \\ & - (q+2)(q+1) \underline{a}_{\mu}^J(n-2, q+2). \end{aligned} \quad (3.19)$$

Here the quantities $\underline{a}_{\mu}^J(n, q)$ are defined by

$$(r K^2 - r \tilde{V}^J) = \sum_{n=0}^{\infty} \underline{a}^J(n) r^n. \quad (3.20)$$

The arbitrary parameters in Eq. (3.19) are $\underline{a}_{\mu p}^J(t_{\mu} + s_{\max}, 0)$, and it is convenient to choose these as

$$\underline{a}_{\mu p}^J(t_{\mu} + s_{\max}, 0) = \left[\Gamma(2J - 2s_{\max} + 1) \right]^{-1} \bar{a}_{\mu p}, \quad (3.21)$$

the $\bar{a}_{\mu p}$ being N^2 arbitrary parameters independent of J .

It turns out that $\tilde{\phi}^J$ does not have any simple symmetry under the above reflection except at integer or half-integer values. We therefore consider $\tilde{\phi}^J$ for $2J = -N_0 - 1 + 2s_{\max}$ and compare it with $\tilde{\phi}^J$ for $2J = N_0 - 1 - 2s_{\max}$, where N_0 is a nonnegative integer. At the former values of $2J$, the recursion formula, Eq. (3.19), becomes ambiguous, and we therefore consider

$$\lim_{2J \rightarrow -N_0 - 1 + 2s_{\max}} \underline{a}^J(n, q) \equiv \underline{a}^{-(N_0+1)/2 + s_{\max}}(n, q). \quad (3.22)$$

From Eqs. (3.19) and (3.21), we have

$$\underline{a}_{\mu p}^{-(N_0+1)/2 + s_{\max}}(n, q) = 0 \quad (3.23)$$

for $n + q < N_0 - t_{\mu} - s_{\max}$. The first nonzero value of

$$\underline{a}_{\mu p}^{-(N_0+1)/2 + s_{\max}}(n, 0) \text{ occurs for}$$

$$n = N_0 - t_{\mu} - s_{\max}. \quad (3.24)$$

The N^2 quantities, $\underline{a}_{\mu p}^{-(N_0+1)/2 + s_{\max}}(t_{\mu} - s_{\max} + N_0, 0)$, are given directly in terms of the arbitrary parameters $\bar{a}$, and the other nonzero

values of $a^{-(N_0+1)/2 + s_{\max}}$ are obtained from these by the recursion formula, Eq. (3.19).

It is easily seen that the recursion formula for $a_{\mu p}^{-(N_0+1)/2 + s_{\max}}(n + N_0 - 2s_{\max}, q)$ is identical with that for $a_{\mu p}^{(N_0+1)/2 - s_{\max}}(n, q)$, from which it follows that

$$\tilde{\phi}_{\mu p}^{-J-1}(r) = \sum_{p'} \tilde{\phi}_{\mu p'}^J(r) C_{p' p}^J. \quad (3.25)$$

The constant matrix $C_{p' p}^J$ differs from the unit matrix because the N^2 quantities which are the starting values in the recursion formula are in general different in the two cases.

From Eqs. (3.17) and (3.25) it follows that the Jost matrices, defined by

$$\tilde{f}^J(\pm) = \lim_{r \rightarrow \infty} W \left[\tilde{\chi}^J(\pm), \tilde{\psi}^J \right], \quad (3.26)$$

satisfy

$$\tilde{f}_{\bar{p} p'}^{-J-1}(\pm) = (\pm i)^{-2\ell} p'^{-1} \sum_{p''} f_{p p''}^J(\pm) C_{p'' p'}^J. \quad (3.27)$$

The S matrix is given by

$$\tilde{S}^J = \left[\tilde{f}_{(-)}^J T \right]^{-1} \tilde{f}_{(+)}^J T; \quad (3.28)$$

hence we have

$$\tilde{S}_{\bar{p} p'}^{-J-1} = (-1)^{-1-2J-t-t'} \tilde{S}_{p p'}^J \quad (3.29)$$

for $J = \text{integer or half-integer}$. We note that $1 + 2J + 2t$ is even for values of J halfway between the physical values (i.e. for $2J + 2s$ an odd integer), so for these values we can write

$$\tilde{S}_{\bar{p} p'}^{-J-1} = (-1)^{t-t'} \tilde{S}_{p p'}^J \quad (3.30)$$

which is Eq. (3.13) in a different notation.

IV. EXCHANGE POTENTIALS

So far we have considered only direct potentials, and in this section we show how the treatment can be generalized to include exchange potentials. These are given in their most general form by replacing the $\delta(\underline{r} - \underline{r}')$ factor in Eq. (I. 3. 4) by $\delta(\underline{r} + \underline{r}')$. This means that for physical J values the potential in the angular-momentum representation can be written as

$$(\cos \pi J + \sin \pi J) \langle J + t', s' | V_e^J(r) | J + t, s \rangle,$$

where the last factor is of the form of the right-hand side of Eq. (I. 3. 5) and, in particular, as a function of J has the same properties as the matrix element of the direct potential. The $(\cos \pi J + \sin \pi J)$ factor has an unsuitable asymptotic behavior for large $\text{Im } J$, so we consider two separate potential problems, with potentials $V_d \pm V_e$, where V_d denotes the direct potential. These give rise to two modified S matrices, which we denote by $\tilde{S}^{J\pm}$. Corresponding to these we have quantities $R^\pm(J)$, analogous to the $R(J)$ introduced in Eq. (2. 1).

It is clear that we have

$$\begin{aligned} R(J) = & 1/2 [1 + \cos \pi J + \sin \pi J] R^+(J) \\ & + 1/2 [1 - \cos \pi J - \sin \pi J] R^-(J), \end{aligned} \quad (4. 1)$$

for physical values of J . Thus we can write

$$\begin{aligned} 2i(kk')^{1/2} [\exp i(\lambda' - \lambda)\phi] \langle c' \lambda'_1 \lambda'_2 | f(\theta, \phi) | c \lambda_1 \lambda_2 \rangle \\ = \sum_J R(J) d_{\lambda\lambda'}^J(\theta) \\ = \frac{1}{2} \sum_J \left\{ R^+(J) [d_{\lambda\lambda'}^J(\theta) + (\cos \pi \lambda - \sin \pi \lambda) d_{\lambda-\lambda'}^J(\pi-\theta)] \right. \\ \left. + R^-(J) [d_{\lambda\lambda'}^J(\theta) - (\cos \pi \lambda - \sin \pi \lambda) d_{\lambda-\lambda'}^J(\pi-\theta)] \right\}. \end{aligned} \quad (4. 2)$$

Here we have used Eq. (2.3) and thereby eliminated the troublesome $(\cos \pi J + \sin \pi J)$ factors. The Sommerfeld-Watson transformation, or the Mandelstam transformation, can now be made on the J sum as before.

Since the quantities $\tilde{S}^{J\pm}$ arise from solutions of Schrödinger equations with different potentials, we expect that they will have poles at different values of J . Thus, as in the one-channel problem, (13) we can characterize any given pole by its "J-parity", or "signature", which is + or - according to whether the pole occurs in $\tilde{S}^{J+}$ or $\tilde{S}^{J-}$.

V. POLES OF $\tilde{S}^J$

Regge has shown that in the one-channel, zero-spin problem there are no poles of the S matrix in the region $\text{Re } J > -1/2$, $\text{Im } k > 0$, and $\text{Re } k \cdot \text{Im } J < 0$. (3) For $\text{Re } k = 0$, $\text{Im } k > 0$ and $\text{Re } J > -1/2$, he has also obtained an equation for the derivative of the position of a pole with respect to E , which shows in particular that, in this region, this derivative is always positive. Here we shall show that the first of these results has a simple generalization, and find an analogous expression for the derivative, which, however, in general is not now positive below threshold.

First we consider the Schrödinger equation, which we write in matrix form as

$$\ddot{\underline{\psi}} + \left[\underline{K}^2 - \underline{L}(\underline{L} + 1) r^{-2} \right] \underline{\psi} - \underline{V}^J \underline{\psi} = 0, \quad (5.1)$$

and its Hermitian conjugate

$$\ddot{\underline{\psi}}^\dagger + \underline{\psi}^\dagger \left[\underline{K}^{2\dagger} - \underline{L}^\dagger(\underline{L}^\dagger + 1) r^{-2} \right] - \underline{\psi}^\dagger \underline{V} = 0. \quad (5.2)$$

Here we have used the fact that V is Hermitian. From these equations it follows that

$$\frac{d}{dr} \left[\underline{\psi}^\dagger \dot{\underline{\psi}} - \dot{\underline{\psi}}^\dagger \underline{\psi} \right] + \underline{\psi}^\dagger \left[\underline{K}^2 - \underline{K}^{2\dagger} \right] \underline{\psi} = \underline{\psi}^\dagger \left[\underline{L}(\underline{L} + 1) - \underline{L}^\dagger(\underline{L}^\dagger + 1) \right] \underline{\psi} r^{-2}. \quad (5.3)$$

We consider this equation at a value of $J = \alpha(E)$, at which $\underline{S}^J$ has a pole. At this value there exists a regular solution $\phi_\mu^{\alpha(E)}(r)$ which behaves like $e^{ik_\mu r}$ as r approaches infinity, and like $r^{\alpha(E)+t_\mu+1}$ as r approaches zero.

We take this solution as the first column of $\underline{\psi}$ and consider the (1, 1) element of Eq. (5.3). Then for

$$\text{Re } \alpha(E) > s_{\max} - 1/2 \quad (5.4)$$

and

$$\text{Im } k_\mu > 0, \quad (5.5)$$

we can integrate from $r = 0$ to $r = \infty$ and obtain

$$\begin{aligned} & \sum_{\mu} \operatorname{Re} k_{\mu} \operatorname{Im} k_{\mu} \int_0^{\infty} dr \left| \phi_{\mu}^a \right|^2 \\ &= \operatorname{Im} a \sum_{\mu} (\operatorname{Re} a + t_{\mu} + 1/2) \int_0^{\infty} dr r^{-2} \left| \phi_{\mu}^a \right|^2 \end{aligned} \quad (5.6)$$

It follows that for all $\operatorname{Re} k_{\mu} > 0$, we have

$$\operatorname{Im} a > 0, \quad (5.7)$$

and for $\operatorname{Re} k_{\mu} = 0$, i. e. for $E < E_{\mu}$ for all μ ,

$$\operatorname{Im} a = 0. \quad (5.8)$$

Thus when all channels are closed, poles to the right of $s_{\max}^{-1/2}$ are confined to the real axis, and when any channel is open, poles in this region are in the upper half-plane.

These considerations apply to poles of S^J , but, from Eqs. (I. 3. 8) and (I. 3. 17) it is easy to see that for $\operatorname{Re} a(E) < s_{\max}^{-1/2}$ the singularities (poles) of S^J and of $\tilde{S}^J$ are common.

To obtain the equation for the derivative of $a(E)$ with respect to E we differentiate Eq. (5. 1) with respect to a , regarding E as a function of a , and readily derive

$$\frac{d}{dr} \left(\psi^T \frac{d}{da} \psi - \dot{\psi}^T \frac{d}{da} \psi \right) + \psi^T \left[\frac{dE}{da} \frac{1}{r} - (2L + 1)r^{-2} \right] \psi - \psi^T \left(\frac{d}{da} V^a \right) \psi = 0. \quad (5.9)$$

Taking, as before, the (1, 1) element of this equation, and integrating from $r = 0$ to $r = \infty$, we obtain

$$\begin{aligned} \frac{dE}{da} \sum_{\mu} \int_0^{\infty} dr \phi_{\mu}^2 &= \sum_{\mu} (2a + 2t_{\mu} + 1) \int_0^{\infty} dr r^{-2} \phi_{\mu}^2 \\ &+ \sum_{\mu\mu'} \int_0^{\infty} dr \phi_{\mu} \phi_{\mu'} \frac{d}{da} V_{\mu\mu'}^a, \end{aligned} \quad (5.10)$$

provided that $E < E_\mu$ for all μ and $a > s_{\max} - 1/2$. Under these conditions we can take the solution ϕ_μ to be real. It follows from Eq. (5.10) that for $d/da \, \bar{V}^a = 0$, which is true for the zero-spin case if the potential is spherically symmetric and expressible in the form of Eq. (I.3.4), dE/da is positive. In the general case, however, this does not appear to be necessarily true.

VI. SUMMARY

We have shown how the essential results of the discussions of complex angular momentum, initiated by Regge, (3) in the one-channel zero-spin potential-scattering problem can be generalized to problems with many two-body channels and arbitrary spins. Specifically, we defined in paper I a modified S matrix, $\tilde{S}^J$, which we showed to be meromorphic as a function of J in the entire J plane. We expressed the J summand of the usual partial-wave expansion of the scattering amplitude in terms of $\tilde{S}^J$, and showed that it also is meromorphic, having only the poles of $\tilde{S}^J$. A further consequence of the results of paper I is that the residues of the poles of $\tilde{S}^J$ factorize. (8-10)

In the present paper, we considered first the asymptotic behavior of $\tilde{S}^J$, and made it plausible that the partial-wave sum can be reexpressed by a generalization of the Sommerfeld-Watson transformation. We then obtained an analogue of the Mandelstam symmetry, (7) and hence an analogue of Mandelstam's modification to the Sommerfeld-Watson transformation. This means that, for large $z = \cos\theta$, the Regge-pole terms always dominate over the integral term, and that a pole at $J = \alpha$ gives a contribution to the amplitude which behaves like z^α . As in the one-channel case, it is very plausible that for some region of z (perhaps outside of the Lehmann ellipse) the remaining contour can be pushed indefinitely to the left, leaving just an infinite, convergent sum of pole contributions. The complications due to exchange potentials were considered in section 4; these required the introduction of J parity as in the one-channel case. Finally, in section 5, we obtained some simple restrictions on the positions of poles in $\tilde{S}^J$, and derived an equation for the derivative of the position, α , of such a pole with respect to the energy, E.

The continuation to complex angular momentum in many-channel problems with zero spin has also been considered by Favella and Reineri (14) and by Jaffe and Kim. (15) These authors obtain analyticity properties in energy at fixed J , and their results can obviously be applied to arbitrary spin. Thus our work compliments theirs, and enables us to explore the domain of holomorphy of $\tilde{S}^J(E)$ as a function of J and E .

APPENDIX

SYMMETRY OF THE CLEBSCH-GORDAN COEFFICIENTS

In this appendix we prove the symmetry used in section 3,

$$C(J+t, s, J; m m_s) = (-1)^{s+m_s} C(-J-t-1, s, -J-1; m m_s) \quad (A.1)$$

for all J and m , and for $t = -s, -s+1, \dots, s$ and $m_s = -s, -s+1, \dots, s$

with $2s$ a nonnegative integer.

We prove the result first for $t = -s, m_s = -s$; i.e., we show that

$$C(J-s, s, J; m, -s) = C(-J+s-1, s, -J-1; m, -s). \quad (A.2)$$

To do this we need first to state and prove a lemma. We assert that for all a , and for all nonnegative integers or half-odd-integers s ,

$$\sum_v \frac{(-1)^v}{v!(2s-v)!} \frac{(a+s-v)!}{(a-s-v)!} = 1. \quad (A.3)$$

To prove this, consider the functions

$$g_r(s, a) = \sum_v \frac{(-1)^v}{v!(2s-v)!} \frac{(a+s-v-r)!}{(a-s-v)!}, \quad (A.4)$$

where r is a nonnegative integer. We can derive a recurrence relation on r , viz.,

$$g_r(s, a) - g_r(s-1/2, a-1/2) = \sum_v \frac{(-1)^v}{v!(2s-v)!} \times \frac{(a+s-v-r)!}{(a-s-v)!} \left[1 - \frac{2s-v}{a+s-v-r} \right] \quad (A.5)$$

$$= (a-s-r) g_{r+1}(s, a). \quad (A.6)$$

Now we have $g_{2s}(s, a) \equiv \sum_v \frac{(-1)^v}{v!(2s-v)!}$

$$= 0, \quad (A.7)$$

for all $s \geq 0$, independent of a . On this basis, we may construct an induction proof that

$$g_{2s-n}(s, a) \equiv 0 \quad (\text{A. 8})$$

for all $s \geq 0$, and all n with $0 \leq n < 2s$. Suppose Eq. (A. 8) is true for some $n = n_0$ and for all s with $2s > n_0$. Then we also have

$$g_{2s-n-1}(s - 1/2, a - 1/2) \equiv 0 \quad (\text{A. 9})$$

for all s with $2s > n_0 + 1$. From Eq. (A. 6) it now follows that Eq. (A. 8) is true for $n = n_0 + 1$ and for all s with $2s > n_0 + 1$. Thus, in particular, we have

$$g_1(s, a) \equiv 0 \quad (\text{A. 10})$$

for all s . Using Eq. (A. 6) once more, we have

$$g_0(s, a) = g_0(s - 1/2, a - 1/2). \quad (\text{A. 11})$$

Repeated application of this formula leads to

$$\begin{aligned} g_0(s, a) &= g_0(0, a-s) \\ &= \sum_v \frac{(-1)^v}{v! (-v)!} \\ &= 1, \end{aligned} \quad (\text{A. 12})$$

identically in a and for all nonnegative integers or half-odd-integers s .

But this is just the identity, Eq. (A. 3), Q. E. D.

We are now in a position to write down explicit expressions for the left- and right-hand sides of Eq. (A. 2), and so to prove the symmetry by inspection. For from the Wigner expression Eq. (1. 3) for the Clebsch-Gordan coefficients, we have

$$C(J-s, s, J; m, -s) = \left[\frac{(2J-2s)!}{(2J)!} \frac{(J-m+s)!}{(J-m-s)!} \right]^{1/2} \\ \times \sum_v \frac{(-1)^v}{v!(2s-v)!} \frac{(J+m+s-v)!}{(J+m-s-v)!},$$

or using Eq. (A. 3),

$$C(J-s, s, J; m, -s) = \left[\frac{(2J-2s)!}{(2J)!} \frac{(J-m+s)!}{(J-m-s)!} \right]^{1/2}. \quad (A. 13)$$

For the right-hand side, we have

$$C(-J+s-1, s, -J-1; m, -s) = - \left[\frac{(2J+1)(-2J-2)!(-J-1+s+m)!}{(-2J-1+2s)!(-J-1-s+m)!} \right]^{1/2} \\ = \left[\frac{(2J-2s)!}{(2J)!} \frac{(J-m+s)!}{(J-m-s)!} \right]^{1/2} \quad (A. 14)$$

and Eq. (A. 2) follows.

We now prove the symmetry Eq. (A. 1) for $m_s = -s$ and $t = -s+1, -s+2, \dots, s$. We use the relationship between contiguous Clebsch-Gordan coefficients (2) in the form

$$(2J+2t+1) \left\{ 2s(J+t)(J+t+1) - m \left[s(s+1) - J(J+1) + (J+t)(J+t+1) \right] \right\} \\ \times C(J+t, s, J; m, -s) \\ = (J+t+1) \left\{ \left[(J+t)^2 - m^2 \right] (2J+t-s)(2J+t+s+1)(s+t)(s-t+1) \right\}^{1/2} \\ \times C(J+t-1, s, J; m, -s) \\ + (J+t) \left\{ \left[(J+t+1)^2 - m^2 \right] (2J+t-s+1)(2J+t+s+2)(s-t)(s+t+1) \right\}^{1/2} \\ \times C(J+t+1, s, J; m, -s). \quad (A. 15)$$

Suppose that for some integer r , with $0 < r < 2s$, Eq. (A. 1) is true for $m_s = -s$ and $t = -s, -s+1, \dots, -s+r$. Then if we write Eq. (A. 15) for $t = -s+r$, we have an equation of the form

$$\begin{aligned} \alpha C(J - s + r, s, J; m, -s) &= \beta C(J - s + r - 1, s, J; m, -s) \\ &+ \gamma C(J - s + r + 1, s, J; m, -s). \end{aligned} \quad (A. 16)$$

We substitute $J \rightarrow -J - 1$, $J - s + r \rightarrow -J - 1 + s - r$, and notice that

$\alpha \rightarrow -\alpha$, $\beta \rightarrow -\gamma$, $\gamma \rightarrow -\beta$. Thus we have

$$\begin{aligned} \alpha C(-J - 1 + s - r, s, -J - 1; m, -s) &= \beta C(-J + s - r, s, -J - 1; m, -s) \\ &+ \gamma C(-J + s - r - 2, s, -J - 1; m, -s), \end{aligned}$$

or, using the inductive hypothesis,

$$\begin{aligned} \alpha C(J - s + r, s, J; m, -s) &= \beta C(J - s + r - 1, s, J; m, -s) \\ &+ \gamma C(-J + s - r - 2, s, -J - 1; m, -s). \end{aligned} \quad (A. 17)$$

Comparing this with Eq. (A. 16), we deduce that Eq. (A. 1) is true also for

$m_s = -s$, $t = -s + r + 1$. To complete the induction proof, we must show

that Eq. (A. 1) is true for $m_s = -s$, $t = -s$, and $t = -s + 1$. The former

case-- $m_s = -s$, $t = -s$ --has been dealt with directly above. For the latter, we

note that the coefficient β of $C(J - s - 1, s, J; m, -s)$ in Eq. (A. 16)

written for $r = 0$ vanishes, so that this case follows from the former one.

This, together with the induction proof then gives the result, Eq. (A. 1),

for $m_s = -s$ and $t = -s, -s + 1, \dots s$.

We now use this as the basis for another induction which will enable us to increase m_s . For this we use the relationship (2)

$$\begin{aligned} &\left[(s + m_s + 1)(s - m_s) \right]^{1/2} C(J + t, s, J; m, m_s + 1) \\ &= \left[(J + m + m_s + 1)(J - m - m_s) \right]^{1/2} C(J + t, s, J; m, m_s) \\ &\quad - \left[(J + m + t + 1)(J - m + t) \right]^{1/2} C(J + t, s, J; m + 1, m_s). \end{aligned} \quad (A. 18)$$

We suppose Eq. (A. 1) to be true for all $t = -s, -s + 1, \dots s$ and for

$m_s = -s + r$, with $0 \leq r < 2s$, for all J and all m . Then if we change

$J \rightarrow -J - 1$, $t \rightarrow -t$ in Eq. (A. 18) written for $m_s = -s + r$, the right-hand side of Eq. (A. 18) acquires a factor $(-1)^{s+m_s+1}$ while the square-root factor on the left is invariant and nonzero. It follows that Eq. (A. 1) is true also for $m_s = -s + r + 1$, and so, by induction and on the basis of the proved result for $r = 0$, true for $m_s = -s, -s + 1, \dots, s$.

This completes the proof of the symmetry Eq. (A. 1) under the stated conditions.

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FOOTNOTES

¹This work was done under the auspices of the U. S. Atomic Energy Commission.

²We are grateful to Mr. I. Muzinich, Lawrence Radiation Laboratory, Berkeley, for suggesting this method.

³This has been proved for the one-channel zero-spin case by Dr. C. Zemach, University of California, Berkeley (private communication), and we anticipate that a similar proof holds in the present case.

⁴We are grateful to Professor A. Erdélyi (California Institute of Technology) for correspondence on this problem.

FIGURE LEGEND

Fig. 1. Integration contours C , C' , and C_M for the Sommerfeld-Watson and Mandelstam transformations. We have taken $s_{\max} = 2$.

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