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Willmore flow of complete surfaces in Euclidean space

DISSERTATION

submitted in partial satisfaction of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

in Mathematics

by

Long-Sin Li

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DEDICATION

To

my parents and my brother,

who, in spite of their careers in IT or IT-related fields and understanding of how pursuing Ph.D. in pure math leads to financial risks, always support my decisions wholeheartedly.

To

Dr. Streets, Dr. Schoen, Dr. Tsui, Dr. Wang, Dr. Chi, Ms. Chen, and Mr. Sun,

my teachers and mentors

who have shown me and helped me explore the beauty of the mathematical world, in which, ironically, ugly objects are almost everywhere.

Also to

the friends I have met in Irvine, California.

\hspace{8cm}\it\tiny\rotatebox[origin=c]{5}{Yes, I'm talking about you.}

An apology

in case there are any typos or grammatical mistakes in this piece of work which I would be glad to know while reluctant to fix.

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VITA

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ABSTRACT OF THE DISSERTATION

Willmore flow of complete surfaces in Euclidean space

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Professor Jeffrey D. Streets, Chair

In this dissertation, we discuss the behavior of Willmore flow, a fourth-order geometric flow, for complete, properly immersed surfaces in Euclidean space.

We develop a-priori estimates for weighted Willmore flows. The estimates are later used to generalize Kuwert and Schätzle's short-time existence theorem in [10] for complete surfaces with bounded geometry, to find a condition for uniqueness of Willmore flows on complete surfaces, and show gap phenomena of Willmore energy. We also discuss blow-ups of Willmore flow, as constructed by Kuwert and Schätzle in [13].

We also discuss the Fredholm property of Laplacian operator on the space of normal vector fields, in view of weighted Sobolev spaces as defined by Lockhart in [17]. We give a few conjectures regarding the Fredholm property of linearization of Willmore tensor, Łojasiewicz–Simon inequality, and stability of minimal surfaces with finite energy as Willmore surfaces.

Chapter 1

Introduction

1.1 Background of the study

One of the most basic and important problems in differential geometry is to find a "canonical representative" for each "shape". For example, the uniformization theorem for closed surfaces, which shows that any closed surface admits a Riemannian metric such that the Gaussian curvature is constant. In view of Gauss' Theorema Egregium and Gauss–Bonnet theorem, this essentially concluded the study of intrinsic geometry of closed surfaces.

To generalize the idea, instead of treating the "canonical representatives" as solutions to a PDE (partial differential equation), we think of them as the equilibria of a function that we refer to as an energy function, or simply as an energy. We can hence study the gradient flows of the energy, which are called geometric flows, and study if a geometric flow converges to an equilibrium. Famous examples of geometric flows include Ricci flow (which in 2-dimensional case rediscovers the uniformization theorem, and in 3-dimensional case is used to solve the famous Poincaré conjecture), mean-curvature flow, curve-shortening flow, etc.

The Willmore energy of an immersed surface $f : \Sigma^2 \looparrowright \mathbb{R}^n$ is defined as

$$\mathcal{W}(f) := \frac{1}{2}\int_\Sigma |A|^2 \, \mathrm{d}\mu \,,$$

where A denotes the second fundamental form. By Gauss–Bonnet theorem, if f is a closed surface,

$$\mathcal{W}(f) = \int_\Sigma |A^0|^2 \, \mathrm{d}\mu + 2\pi\chi(\Sigma) = \frac{1}{2}\int_\Sigma |H|^2 \, \mathrm{d}\mu - 2\pi\chi(\Sigma),$$

where A^0 denotes the trace-free part of A. Therefore, all the aforementioned expressions are used as the definition in different literature for different needs, while essentially they are all the same. (cf. [11], [37], etc.)

An equilibrium of the Willmore energy is called a Willmore surface, while the gradient flow for the Willmore energy is called a Willmore flow. It is worth noting that there are compact Willmore surfaces, such as spheres and Clifford torus, as well as non-compact Willmore surfaces, such as planes and catenoids. In fact, all minimal surfaces are Willmore surfaces.

The first variation of $\mathcal{W}$ is given by the Willmore tensor (cf. [11], etc.):

$$\mathbf{W}(f) = \triangle H + Q(A^0)H,$$

where Q is defined by

$$Q(\eta)\phi = g^{ik}g^{j\ell}\eta_{ij}\langle \eta_{k\ell}, \phi\rangle_{N_\Sigma}$$

for tensors $\eta \in \Gamma(N_\Sigma \otimes \mathrm{Sym}^2(T^*\Sigma))$ and $\phi \in \Gamma(N_\Sigma)$. Note that the first variation formula holds whether Σ is compact or non-compact. Therefore, given initial data $f_0 : \Sigma \looparrowright \mathbb{R}^n$, we can consider the Willmore flow equation

$$\begin{cases} \partial_t f = -\mathbf{W}(f), \\ f\big|_{t=0} = f_0. \end{cases} \tag{1.1}$$

Some fundamental studies regarding Willmore surfaces and convergence of Willmore flows to Willmore surfaces can be found in [7], [12], [13], [14], [20], [32] etc. There are also Willmore flows that develop singularities (cf. [2], [22]), and there are open questions regarding such singularities for compact surfaces, including existence of finite-time singularities and classification of singularity types, etc. Related results include: in [10, Theorem 1.2], where the authors showed that finite-time singularities require energy concentration, [4, Theorem 1.1], where the authors showed that blow-ups are not compact, and [24, Theorem 1.4], where the authors showed an upper bound for the existence time of locally constrained Willmore flows, while the upper bound increases to infinity as the PDE converges to the classical Willmore flow equation, etc. It could hence be interesting to approximate Willmore blow-ups by complete surfaces.

Willmore surfaces and Willmore flows are also studied in other works, including [9], [28], [29], etc. Similar frameworks are also used to study 4-th order parabolic PDEs that are related to the Willmore tensor and in fact sharing the leading order terms, in [23], [25], [30], [31], [34], [35], [36], etc. In addition, regarding developments of studies on Willmore flow, see, e.g., [11] and [21]; regarding general strategies for parabolic geometric PDEs, e.g., [19]; regarding Łojasiewicz inequalities, which are discussed in chapter 3 and section 4.2, see, e.g., [3] and [5].

1.2 Results

This dissertation is devoted to extend the study of Willmore flow of closed surfaces to complete, properly immersed surfaces in $\mathbb{R}^n$.

In the first half of chapter 2, we generalize Kuwert and Schätzle's short-time existence theorem [10, Theorem 1.2] for complete surfaces, and find a similar lower bound for the existence time with a similar upper bound for concentration concentration of curvature, which only depend on concentration of curvature on the initial surface:

Theorem 2.4.6. *Let $f_0 : \Sigma \to \mathbb{R}^n$ be a smooth, complete, properly immersed surface in $\mathbb{R}^n$. Then there exist $\varepsilon_1 > 0$ and $c_1 > 0$, both depending only on n, such that whenever the initial energy concentration condition*

$$\int_{\Sigma_0 \cap B_\varrho(x)} |A_0|^2 \,\mathrm{d}\mu_0 \le e_0 \le \varepsilon_1, \quad \forall x \in \mathbb{R}^n$$

holds for some $\varrho > 0$ and $e_0 > 0$, there exists a solution $f : \Sigma \times [0, T) \to \mathbb{R}^n$ to the Willmore flow equation (1.1) such that $T \ge c_1^{-1} \varrho^4$. Moreover, f satisfies the following estimate for the growth of energy concentration:

$$\int_{\Sigma_t \cap B_\varrho(x)} |A_t|^2 \,\mathrm{d}\mu_t \le a_n e_0 (1 + c_1 \varrho^{-4} t), \quad \forall x \in \mathbb{R}^n \text{ and } 0 \le t \le c_1^{-1} \varrho^4.$$

To prove our theorem, we consider solutions to a weighted PDE

$$\begin{cases} \partial_t f = -\theta^r \mathbf{W}(f), \\ f\big|_{t=0} = f_0, \end{cases} \tag{1.2}$$

where $0 \le \theta \le 1$ is a smooth function defined on the ambient space, i.e., $\theta = \widehat{\theta} \circ f$ for some $\widehat{\theta} : \mathbb{R}^n \to \mathbb{R}$, and r is a sufficiently large integer that we will specify later. First, we recover that the a-priori estimates in [10] hold for the weighted PDE. From the estimates, we can show that if there were a Willmore flow with $T < c_1^{-1}\varrho^4$, f would converge as $t \to T$ and hence can be extended to $[0, T]$. However, well-posedness of (1.2), which allows f to extended over T, generally only holds when Σ is compact and $\theta > 0$.

To solve (1.1), we approximate solutions to (1.1) with solutions to (1.2) with compactly supported θ. There are two main obstacles when having θ. One of them is extra efforts to balance powers of θ in the a-priori estimates, for the correct exponents for θ don't always coincide those for γ, where γ is the same cutoff function as in [10]. The other obstacle is that traditional short-time existence results fail when θ is not globally positive. We can view $\Sigma \cap [\theta > 0]$ as a subset of a closed surface, and hence we can modify θ to be positive everywhere on the closed surface, and then finally approximate solutions for (1.2).

For compact surfaces, the vector field $\mathbf{W}(f)$ is the gradient of the energy $\mathcal{W}(f)$ and hence we have energy identity for any family of surfaces:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{W}(f_t) = \int_{\Sigma_t} \langle \mathbf{W}(f_t), \partial_t f_t \rangle \, \mathrm{d}\mu_t \,.$$

In particular, energy decreases along a negative gradient flow. For complete surfaces, energy may escape into infinity and hence decreases even faster:

Corollary 2.4.7. *If $\mathcal{W}(f_0) < \infty$ and f is the Willmore flow constructed in Theorem 2.4.6, then we have*

$$\int_{\Sigma_t} |A_t|^2 \, \mathrm{d}\mu_t + \int_0^t \int_{\Sigma_{t'}} |\mathbf{W}(f_{t'})|^2 \, \mathrm{d}\mu_{t'} \, \mathrm{d}t' \leq \int_{\Sigma_0} |A_0|^2 \, \mathrm{d}\mu_0 \,.$$

In section 2.5, in view of the Sobolev inequalities, initial non-concentration conditions for $A, \dots, \nabla^5 A$ implies uniform bounds for $A, \dots, \nabla^3 A$, which give us sufficient flatness to obtain the following uniqueness result for the fourth-order PDE.

Theorem 2.5.9. *Assume that* $f_0 : \Sigma \to \mathbb{R}^n$ *is a smooth, complete, properly immersed surface in* $\mathbb{R}^n$ *such that*

$$\liminf_{R\to\infty} R^{-4}\mu_0\big(B_R(0)\big) = 0, \text{ and}$$

for some $\varrho > 0$ *and* $M > 0$,

$$\begin{cases} \displaystyle\int_{\Sigma_0\cap B_\varrho(x)} |A_0|^2 \,\mathrm{d}\mu_0 \le \varepsilon_1, & \forall x \in \mathbb{R}^n, \\ \displaystyle\int_{\Sigma_0\cap B_\varrho(x)} |\nabla^k A_0|^2 \,\mathrm{d}\mu_0 \le M, & \forall x \in \mathbb{R}^n \text{ and } k = 1,\dots,5, \end{cases}$$

where ε_1 *is as given in Theorem 2.4.6. Let* $f = f_i : \Sigma \times [0,T) \to \mathbb{R}^n$, *where* $i = 1,2$, *be two solutions to the Willmore flow equation (1.1), then there exists* $t_3 > 0$, *only depending on* n, ϱ, *and* M, *such that* $f_1 = f_2$ *for all* $0 \le t < \widehat{T} = \min(t_3, T)$.

In section 2.6, given a Willmore flow $f : \Sigma \times [0,T) \to \mathbb{R}^n$ with maximal existence time $T \in (0,\infty)$, we consider the first time when there exists an ambient ball of radius r where the curvature exceeds a given number $e > 0$. More precisely,

$$t(r,e) = \inf\left\{ t \in [0,T) : \sup_{x\in\mathbb{R}^n} \int_{\Sigma_t\cap B_r(x)} |A_t|^2 \,\mathrm{d}\mu_t > e \right\}.$$

A general strategy to study the singularity at $t = T$ is to blow-up the Willmore flow, that is, choose $r_j > 0$, $t_j \in [0,T)$, and $x_j \in \mathbb{R}^n$ so that the rescaled Willmore flows

$$f_j(p,\tau) = r_j^{-1}\big(f(p, t_j + r_j^4\tau) - x_j\big)$$

have a chance to converge smoothly as $j \to \infty$. Ideally, we take $r_j \to 0$ and let $t_j = t(r_j, e)$ for some fixed $e > 0$. Existence of blow-ups has been discussed by Kuwert and Schätzle in [13, Section 4]. We can then characterize the singularity by the behavior of the maximal existence time $r_j^{-4}(T-t_j)$. In particular, type-I singularities are defined as following:

Definition 2.6.2. *Given $0 < e \le \varepsilon_1$, we say f has a type-I singularity with respect to energy threshold e if*

$$\begin{cases} t(r,e) < T \text{ for all } r > 0, \text{ and} \\ \limsup_{r\to 0^+} \left[r^{-1}\big(T - t(r,e)\big)^{1/4}\right] < \infty, \end{cases}$$

which in particular implies $T < \infty$.

However, when blowing up the singularity, we see that:

Theorem 2.6.5. *For all $e < \varepsilon_1$, a Willmore flow f of closed surfaces cannot have a type-I singularity with respect to energy threshold e.*

In chapter 3, we adopt Lockhart's definition for weighted Sobolev spaces in [17] for complete manifolds with finitely many ends, where the metric on each end is diffeomorphic to a cylinder and is conformal to an asymptotically translation-invariant metric. For those surfaces that are also Willmore surfaces, we conjecture that

Conjecture 3.2.8. (4) *Let $f_W : \Sigma \to \mathbb{R}^n$ be a Willmore immersion that is complete and proper. Assume that for some $\beta \ge 0$,*

$$\begin{cases} \rho_0 := \inf_{\Sigma} \rho > -\infty, \text{ and} \\ \sup_{\Sigma}(e^{(t+1)(1-\beta)\rho}|\nabla^t_{(g)}A|_g) < \infty, \text{ for some } \beta \ge 0 \text{ and } \forall t = 0, 1. \end{cases}$$

Then for a.e. $\delta \in \mathbb{R}^L$, the Willmore energy $\mathcal{W}$ satisfies the Łojasiewicz–Simon inequality, namely, there exists $\theta \in (0, \frac{1}{2}]$ such that for all sufficiently small $\eta \in W^{4,2}_{\delta,-4+3\beta}(N\Sigma, g)$,

$$|\mathcal{W}(f_W + \eta) - \mathcal{W}(f_W)|^{1-\theta} \le C\|\mathbf{W}(f_W + \eta)\|_{W^{0,2}_{\delta,0}}.$$

We list a few examples in section 3.3.

In chapter 4, we derive gap phenomena, namely convergence of Willmore flow to planes or minimal surfaces given lower Willmore energy. The following result is also explained by more general theorems such as [13, Theorem 2.7] and [36, Theorem 1, (2)].

Theorem 4.1.2. *If $f : \Sigma \to \mathbb{R}^n$ is a complete, smooth, properly immersed Willmore surface with $\mathcal{W}(f) \le \frac{1}{2}\varepsilon_0$, then Σ is a plane, where $\varepsilon_0 > 0$ only depends on n.*

As a result, we can prove that Willmore flows with small initial energy converge to planes:

Corollary 4.1.4. *Let $f : \Sigma \times [0, \infty) \to \mathbb{R}^n$ be a solution to (1.1). Assume that $\mathcal{W}(f_0) \le \frac{1}{2}\varepsilon_0$ and that*

$$\sup_{t \ge 0} \mu_t\big(B_R(0)\big) < \infty \text{ for all } R > 0.$$

Then as $t \to \infty$, any subsequence has a further subsequence such that Σ_t converges locally smoothly, up to diffeomorphisms, to a plane $L : \mathbb{R}^2 \to \mathbb{R}^n$ in the sense as in Definition 2.6.3.

In the statement, we assume space-time bounds to guarantee convergence and to avoid having a sum of planes as the limit. Alternatively, we have the same conclusion if we assume an Euclidean area growth rate for the initial surface:

Corollary 4.1.5. *Let $f : \Sigma \times [0, \infty) \to \mathbb{R}^n$ be a solution to (1.1). Assume that $\mathcal{W}(f_0) \le \frac{1}{2}\varepsilon_0$ and that*

$$\liminf_{R \to \infty} R^{-2} \mu_0\big(B_R(0)\big) < \infty.$$

Then as $t \to \infty$, any subsequence has a further subsequence such that Σ_t converges to a plane $L : \mathbb{R}^2 \to \mathbb{R}^n$ in the sense as in Definition 2.6.3.

In addition, using the Łojasiewicz inequality, we conjecture a stability result:

Conjecture 4.2.1. *Let $f_W : \Sigma \to \mathbb{R}^n$ be a Willmore immersion that is complete and proper. Assume that the induced metric $g = e^{2\rho}h$ is admissible, as in Definition 3.1.2. Assume condition (3.1) for some $\beta \geq 0$ and $s_0 = 1$.*

If $f : \Sigma \times [0, T)$ is a a Willmore flow, where:

- *T is the maximal existence time,*
- *$\mathcal{W}(f_t) \geq \mathcal{W}(f_W)$ whenever $\|K(f_t \circ \Phi - f_W)\|_{C^k(\Sigma,h)} \leq \eta$ up to some diffeomorphism $\Phi \in \mathrm{Aut}(\Sigma)$, and*
- *$\|f_0 - f_W\|_{W^{2,2}_{\delta,a} \cap C^1} < \varepsilon$, where $\varepsilon = \varepsilon(n, k, \eta)$,*

then $T = \infty$, and as $t \to \infty$, f_t converges locally smoothly up to diffeomorphisms to a Willmore surface f_∞ that satisfies $\mathcal{W}(f_\infty) = \mathcal{W}(f_W)$.

In the appendix, we list and prove various interpolation inequalities and Sobolev inequalities that are used in the main article.

1.3 Conventions

First, we list some notations that are used throughout the article.

- Σ is a smooth surface.
- f either denotes:
  - A smooth immersion $f : \Sigma \looparrowright \mathbb{R}^n$, where we identify Σ and $f(\Sigma)$, or

– A family of smooth immersions $f : \Sigma \times [0,T) \to \mathbb{R}^n$, where we denote $f_t(x) := f(x,t)$ and $\Sigma_t = f_t(\Sigma)$.

We should always assume the immersed surface $(\Sigma, f^* g_{\mathbb{R}^n})$ or $(\Sigma_t, f_t^* g_{\mathbb{R}^n})$ is complete. We denote the induced Levi-Civita connection as ∇ (where we don't specify t for Σ_t), and the 2-dimensional Hausdorff measure as μ or μ_t, correspondingly.

- A, A_t denote the second fundamental form of Σ and Σ_t, correspondingly. Similarly, H and H_t denote the mean curvature, while A^0 and A_t^0 denote the trace-free part of the second fundamental form.

- ϕ: A tensor of class $\Gamma((T^*\Sigma)^{\otimes r_\phi} \otimes N_\Sigma)$, where r_ϕ is a non-negative integer and N_Σ is the normal bundle on Σ.

- $\triangle = -\nabla^*\nabla$, where ∇^* is the formal adjoint of ∇.

- $s, r \geq 2$: sufficiently large positive integers.

- $P_k^m = \sum_{i_1+\cdots+i_r=m} \nabla^{i_1} A * \cdots * \nabla^{i_k} A$ with unspecified coefficients that are bounded by some $c(n,s,r)$. The "star product" notation denotes an unspecified universal multilinear form. See, for example, [10, Section 2] for more explanation. Here we don't specify t for Σ_t.

- $\varkappa(r,t) = \sup_{x\in\mathbb{R}^n} \int_{\Sigma_t \cap B_r(x)} |A_t|^2 \, \mathrm{d}\mu_t$ measures the concentration of curvature. (adopted from [13, Theorem 4.2].)

- $c = c(\ldots)$ denotes scalars that only depend on the arguments. All the c's can denote different numbers, even in the same line.

Next, we pick a smooth function χ defined on $\mathbb{R}$ such that

$$\begin{cases} \chi \text{ is decreasing,} \\ \chi(x) = 1 \text{ for all } x \leq 0, \text{ and} \\ \chi(x) = 0 \text{ for all } x \geq 1. \end{cases}$$

We will fix this choice so that $\sup |D^k \chi|$ only depends on k. Next, we construct functions $\widehat{\gamma}, \widehat{\theta}$ on $\mathbb{R}^n$ such that for some given $K > 0$,

$$\begin{cases} \widehat{\gamma} \text{ and } \widehat{\theta} \text{ are smooth,} \\ 0 \leq \widehat{\gamma}, \widehat{\theta} \leq 1 \text{ while also both are not identically } 0, \\ \widehat{\gamma}\widehat{\theta} \text{ has compact support, and} \\ \forall k \geq 1,\ |D^k \widehat{\gamma}| \leq K^k \sup |D^k \chi| \text{ and } |D^k \widehat{\theta}| \leq K^k \sup |D^k \chi|. \end{cases} \tag{1.3}$$

Lemma 1.3.1. *Given any $x_1, x_2 \in \mathbb{R}^n$, $R_1, R_2 > 0$, and $0 < K_1, K_2 \leq K$, we can let*

$$\widehat{\gamma}(x) = \chi\big(K_1(|x - x_1| - R_1)\big) \quad \textit{and} \quad \widehat{\theta}(x) = \chi\big(K_2(|x - x_2| - R_2)\big)$$

so that they satisfy (1.3).

Let $\gamma = \widehat{\gamma}\big|_\Sigma$ and $\theta = \widehat{\theta}\big|_\Sigma$. We derive estimates for the covariant derivatives of $\gamma^s \theta^r$:

Lemma 1.3.2. *For all $k \geq 1$,*

$$|\nabla^k (\gamma^s \theta^r)| \leq c \Bigg(\gamma^{\max(s-k,0)} \theta^{\max(r-k,0)} K^k + \sum_{\substack{1 \leq i_0 < k \\ i_1 \cdots, i_\ell \geq 0 \\ i_0 + \cdots + i_\ell + \ell = k}} \gamma^{\max(s-i_0,0)} \theta^{\max(s-i_0,0)} K^{i_0} \prod_{j=1}^{\ell} |\nabla^{i_\ell} A| \Bigg),$$

where $c = c(s, r, k)$. The cases when $k = 1, 2$ are especially frequently used:

$$|\nabla(\gamma^s\theta^r)| \leq c\,\gamma^{s-1}\theta^{r-1}K,$$

and

$$|\nabla^2(\gamma^s\theta^r)| \leq c\left(\gamma^{s-2}\theta^{r-2}K^2 + \gamma^{s-1}\theta^{r-1}K|A|\right).$$

Proof. The proof is clear by induction, while we only show the cases $k = 1, 2$. Let (u, v) be a normal coordinate at $p \in \Sigma$ and $e_1 = \partial_u$, $e_2 = \partial_v$. We have

$$\nabla(\gamma^s\theta^r)(e_i) = D\big(\widehat{\gamma}^s\widehat{\theta}^r\big)(e_i),$$

and

$$\nabla^2(\gamma^s\theta^r)(e_i, e_j) = D^2\big(\widehat{\gamma}^s\widehat{\theta}^r\big)(e_i, e_j) + D\big(\widehat{\gamma}^s\widehat{\theta}^r\big)\big(A(e_i, e_j)\big).$$

□

Chapter 2

Short-time existence and uniqueness

We consider the Willmore flow equation for complete, properly immersed surfaces in $\mathbb{R}^n$. Given bounded geometry on the initial surface, we extend the result in [10] with respect to a similar energy concentration condition.

2.1 Geometry with low energy concentration

In this section, we derive general inequalities regarding low energy concentration. These inequalities are later applied in the context of Willmore flows.

First, for convenience, we rewrite [10, Lemma 4.2], replacing γ^4 with $\gamma^s\theta^r$:

Lemma 2.1.1. *If $s, r \geq 4$, then*

$$\int_\Sigma \gamma^s\theta^r \left(|\nabla A|^2|A|^2 + |A|^6\right) \mathrm{d}\mu$$
$$\leq c \int_{[\gamma\theta>0]} |A|^2 \,\mathrm{d}\mu \int_\Sigma \gamma^s\theta^r \left(|\nabla^2 A|^2 + |A|^6\right) \mathrm{d}\mu + c\, K^4 \left(\int_{[\gamma\theta>0]} |A|^2 \,\mathrm{d}\mu\right)^2,$$

where $c = c(n, s, r)$. Moreover, there exists $\varepsilon_0 > 0$, only depending on n, s, and r, such that whenever

$$\int_{[\gamma\theta>0]} |A|^2 \,\mathrm{d}\mu \le \varepsilon_0, \tag{2.1}$$

we have

$$\int_\Sigma \gamma^s\theta^r \left(|\nabla A|^2|A|^2 + |A|^6\right) \mathrm{d}\mu \le \int_\Sigma \gamma^s\theta^r |\nabla^2 A|^2 + c\, K^4 \left(\int_{[\gamma\theta>0]} |A|^2 \,\mathrm{d}\mu\right)^2 .$$

Lemma 2.1.2. *If $s \ge 6$ and $r \ge 8$, then we can choose ε_0 so that assuming (2.1), we have*

$$\int_\Sigma \gamma^s\theta^{r-2} K^2 |A|^8 \,\mathrm{d}\mu \le \int_\Sigma \gamma^s\theta^r |\nabla A|^4|A|^2 + K^8 \int_{[\gamma\theta>0]} |A|^2 \,\mathrm{d}\mu\,.$$

Proof. By Theorem A.2.1,

$$\begin{aligned}
&\int_\Sigma \gamma^s\theta^{r-2}K^2|A|^8\,\mathrm{d}\mu \\
&\le c\bigg(\int_\Sigma \gamma^{s/2}\theta^{r/2-1}K|\nabla A|\,|A|^3\,\mathrm{d}\mu + \int_\Sigma \gamma^{s/2-1}\theta^{r/2-2}K^2|A|^4\,\mathrm{d}\mu \\
&\qquad\quad + \int_\Sigma \gamma^{s/2}\theta^{r/2-1}K|A|^5\,\mathrm{d}\mu\bigg)^2 \\
&\le c\bigg(\int_\Sigma \gamma^{s/2}\theta^{r/2}|\nabla A|^2|A|^2\,\mathrm{d}\mu + \int_\Sigma \gamma^{s/2-1}\theta^{r/2-2}K^2|A|^4\,\mathrm{d}\mu + \int_\Sigma \gamma^{s/2}\theta^{r/2-1}K|A|^5\,\mathrm{d}\mu\bigg)^2 \\
&\le c\bigg(\int_\Sigma \gamma^{s/2}\theta^{r/2}|\nabla A|^2|A|^2\,\mathrm{d}\mu + \int_\Sigma \gamma^{s/2-3}\theta^{r/2-4}K^4|A|^2\,\mathrm{d}\mu + \int_\Sigma \gamma^{s/2}\theta^{r/2-1}K|A|^5\,\mathrm{d}\mu\bigg)^2 \\
&\le c\,\varepsilon_0\bigg(\int_\Sigma \gamma^s\theta^r|\nabla A|^4|A|^2\,\mathrm{d}\mu + \int_\Sigma \gamma^s\theta^{r-2}K^2|A|^8\,\mathrm{d}\mu + K^8\int_{[\gamma\theta>0]}|A|^2\,\mathrm{d}\mu\bigg).
\end{aligned}$$

We require $c\,\varepsilon_0 \le \frac{1}{2}$ to obtain the claimed statement. $\square$

Proposition 2.1.3. *If $s \geq 2$ and $r \geq 4$, then*

$$\int_\Sigma \gamma^s \theta^r |\nabla A|^4 \,\mathrm{d}\mu$$
$$\leq c\,\|\theta^{r/4} A\|^2_{\infty,[\gamma>0]} \left(\int_\Sigma \gamma^s \theta^{r/2} |\nabla^2 A|^2 \,\mathrm{d}\mu + K^2 \int_\Sigma \gamma^{s-2} \theta^{r/2-2} |\nabla A|^2 \,\mathrm{d}\mu \right),$$

where $c = c(n, s, r)$.

Proof. Using integration by parts,

$$\int_\Sigma \gamma^s \theta^r |\nabla A|^4 \,\mathrm{d}\mu$$
$$\leq c \int_\Sigma \big(\gamma^s \theta^r |\nabla^2 A|\,|\nabla A|^2 + \gamma^{s-1}\theta^{r-1} K |\nabla A|^3\big) |A| \,\mathrm{d}\mu$$
$$\leq c\,\|\theta^{r/4} A\|_{\infty,[\gamma>0]} \left(\int_\Sigma \gamma^s \theta^r |\nabla A|^4 \,\mathrm{d}\mu \right)^{1/2}$$
$$\cdot \left[\left(\int_\Sigma \gamma^s \theta^{r/2} |\nabla^2 A|^2 \,\mathrm{d}\mu \right)^{1/2} + \left(\int_\Sigma \gamma^{s-2} \theta^{r/2-2} K^2 |\nabla A|^2 \,\mathrm{d}\mu \right)^{1/2} \right]$$
$$\leq \frac{1}{2} \int_\Sigma \gamma^s \theta^r |\nabla A|^4 \,\mathrm{d}\mu$$
$$+ c\,\|\theta^{r/4} A\|^2_{\infty,[\gamma>0]} \left(\int_\Sigma \gamma^s \theta^{r/2} |\nabla^2 A|^2 \,\mathrm{d}\mu + K^2 \int_\Sigma \gamma^{s-2} \theta^{r/2-2} |\nabla A|^2 \,\mathrm{d}\mu \right),$$

and hence we can obtain the stated inequality. □

Proposition 2.1.4. *If $s \geq 6$ and $r \geq 20$, then we can choose ε_0 so that assuming (2.1), we have*

$$\int_\Sigma \gamma^s \theta^r |\nabla A|^4 |A|^2 \,\mathrm{d}\mu$$
$$\leq c\,K^2 \int_\Sigma \gamma^{s-2} \theta^{r-2} |\nabla A|^3 \,\mathrm{d}\mu + \|\theta^{r/4} A\|^4_{\infty,[\gamma>0]} \int_{[\theta>0]} \gamma^s |\nabla^2 A|^2 \,\mathrm{d}\mu$$
$$+ c\left(K^5 \|\theta^{r/4} A\|^3_{\infty,[\gamma>0]} + K^8\right) \int_{[\gamma\theta>0]} |A|^2 \,\mathrm{d}\mu,$$

where $c = c(n, s, r)$.

Proof. First,

$$\int_\Sigma \gamma^s\theta^r|\nabla A|^4|A|^2\,\mathrm{d}\mu \le \|\theta^{r/4}A\|^2_{\infty,[\gamma>0]}\int_\Sigma \gamma^s\theta^{r/2}|\nabla A|^4\,\mathrm{d}\mu\,.$$

Next, using integration by parts,

$$\begin{aligned}
&\int_\Sigma \gamma^s\theta^{r/2}|\nabla A|^4\,\mathrm{d}\mu\\
&\le c\int_\Sigma \big(\gamma^s\theta^{r/2}|\nabla^2 A|\,|\nabla A|^2 + \gamma^{s-1}\theta^{r/2-1}K|\nabla A|^3\big)|A|\,\mathrm{d}\mu\\
&\le c\,\|\theta^{r/4}A\|_{\infty,[\gamma>0]}\left(\int_\Sigma \gamma^s\theta^{r/2}|\nabla A|^4\,\mathrm{d}\mu\right)^{1/2}\left(\int_{[\theta>0]}\gamma^s|\nabla^2 A|^2\,\mathrm{d}\mu\right)^{1/2}\\
&\qquad + c\,K\left(\int_\Sigma \gamma^s\theta^{r/2}|\nabla A|^4\,\mathrm{d}\mu\right)^{3/4}\left(\int_\Sigma \gamma^{s-4}\theta^{r/2-4}|A|^4\,\mathrm{d}\mu\right)^{1/4}\\
&\le \frac{1}{2}\int_\Sigma \gamma^s\theta^{r/2}|\nabla A|^4\,\mathrm{d}\mu + c\,\|\theta^{r/4}A\|^2_{\infty,[\gamma>0]}\int_{[\theta>0]}\gamma^s|\nabla^2 A|^2\,\mathrm{d}\mu\\
&\qquad + c\,K^4\int_\Sigma \gamma^{s-4}\theta^{r/2-4}|A|^4\,\mathrm{d}\mu\,,
\end{aligned}$$

so that we have

$$\begin{aligned}
&\int_\Sigma \gamma^s\theta^r|\nabla A|^4|A|^2\,\mathrm{d}\mu\\
&\le c\,\|\theta^{r/4}A\|^4_{\infty,[\gamma>0]}\int_{[\theta>0]}\gamma^s|\nabla^2 A|^2\,\mathrm{d}\mu + c\,K^4\|\theta^{r/4}A\|^2_{\infty,[\gamma>0]}\int_\Sigma \gamma^{s-4}\theta^{r/2-4}|A|^4\,\mathrm{d}\mu\,.
\end{aligned}$$

Next, by Theorem A.2.1,

$$\begin{aligned}
&\int_\Sigma \gamma^{s-4}\theta^{r/2-4}|A|^4\,\mathrm{d}\mu\\
&\le c\bigg(\int_\Sigma \gamma^{s/2-2}\theta^{r/4-2}|\nabla A|\,|A|\,\mathrm{d}\mu + K\int_\Sigma \gamma^{s/2-3}\theta^{r/4-3}|A|^2\,\mathrm{d}\mu\\
&\qquad + \int_\Sigma \gamma^{s/2-2}\theta^{r/4-2}|A|^3\,\mathrm{d}\mu\bigg)^2\\
&\le c\,\varepsilon_0\left(\int_\Sigma \gamma^{s-4}\theta^{r/2-4}|\nabla A|^2\,\mathrm{d}\mu + \int_\Sigma \gamma^{s-4}\theta^{r/2-4}|A|^4\,\mathrm{d}\mu\right) + c\,\varepsilon_0 K^2\int_{[\gamma\theta>0]}|A|^2\,\mathrm{d}\mu\,,
\end{aligned}$$

and hence we can require $c\,\varepsilon_0 \le \frac{1}{2}$ so that we have

$$\begin{aligned}
&\int_\Sigma \gamma^s\theta^r|\nabla A|^4|A|^2\,\mathrm{d}\mu\\
&\le c\,\|\theta^{r/4}A\|^4_{\infty,[\gamma>0]}\int_{[\theta>0]}\gamma^s|\nabla^2 A|^2\,\mathrm{d}\mu + c\,K^4\|\theta^{r/4}A\|^2_{\infty,[\gamma>0]}\int_\Sigma \gamma^{s-4}\theta^{r/2-4}|\nabla A|^2\,\mathrm{d}\mu\\
&\qquad + c\,K^6\|\theta^{r/4}A\|^2_{\infty,[\gamma>0]}\int_{[\gamma\theta>0]}|A|^2\,\mathrm{d}\mu\,.
\end{aligned}$$

Next, by Proposition A.1.4 with $\alpha = K^{1/2}\|\theta^{r/4}A\|^{1/2}_{\infty,[\gamma>0]}$, we have

$$\begin{aligned}
&K^2\|\theta^{r/4}A\|^2_{\infty,[\gamma>0]}\int_\Sigma \gamma^{s-4}\theta^{r/2-4}|\nabla A|^2\,\mathrm{d}\mu\\
&\le K^2\|\theta^{r/4}A\|^2_{\infty,[\gamma>0]}\int_\Sigma \gamma^{s-4}\theta^{(r-2)/3}|\nabla A|^2\,\mathrm{d}\mu \qquad (r\ge 20)\\
&\le \int_\Sigma \gamma^{s-2}\theta^{r-2}|\nabla^3 A|^2\,\mathrm{d}\mu + c\left(K^3\|\theta^{r/4}A\|^3_{\infty,[\gamma>0]} + K^6\right)\int_{[\gamma\theta>0]}|A|^2\,\mathrm{d}\mu\,.
\end{aligned}$$

In summary,

$$\begin{aligned}
&\int_\Sigma \gamma^s\theta^r|\nabla A|^4|A|^2\,\mathrm{d}\mu\\
&\le c\,K^2\int_\Sigma \gamma^{s-2}\theta^{r-2}|\nabla A|^3\,\mathrm{d}\mu + \|\theta^{r/4}A\|^4_{\infty,[\gamma>0]}\int_{[\theta>0]}\gamma^s|\nabla^2 A|^2\,\mathrm{d}\mu\\
&\qquad + c\left(K^5\|\theta^{r/4}A\|^3_{\infty,[\gamma>0]} + K^8\right)\int_{[\gamma\theta>0]}|A|^2\,\mathrm{d}\mu\,.
\end{aligned}$$

□

Lemma 2.1.5 ([10, Lemma 4.3]). *(i) We have*

$$\|\phi\|^4_{\infty,[\gamma=1]} \le c\,\|\phi\|^2_{2,[\gamma>0]}\big(\|\nabla^2\phi\|^2_{2,[\gamma>0]} + \|\phi\|^2_{2,[\gamma>0]} + \|\,|A|^4|\phi|^2\|_{1,[\gamma>0]}\big),$$

where $c = c(n, r_\phi, K)$.

(ii) Moreover, assuming (2.1), we have

$$\|A\|^4_{\infty,[\gamma=1]} \le c\,\|A\|^2_{2,[\gamma>0]}\big(\|\nabla^2 A\|^2_{2,[\gamma>0]} + \|A\|^2_{2,[\gamma>0]}\big).$$

The following corollary refines both the previous lemma and [13, Lemma 2.8].

Corollary 2.1.6. *If $r \ge 6$, assuming (2.1), we have*

$$\|\theta^{r/4} A\|^4_{\infty,[\gamma=1]} \le c\,\|A\|^2_{2,[\gamma\theta>0]}\big(\|\theta^{r/2}\nabla^2 A\|^2_{2,[\gamma>0]} + \|A\|^2_{2,[\gamma\theta>0]}\big)$$

where $c = c(n, r, K)$.

Proof. First, by Lemma A.2.4 with $m = 2$, $p = 4$, etc.,

$$\|\gamma^2\theta^{r/4} A\|_\infty \le c\,\|A\|^{1/3}_{2,[\gamma\theta>0]}\big(\|\gamma^3\theta^{3r/8}\nabla A\|_4 + \|\gamma^2\theta^{3r/8-1}A\|_4 + \|\gamma^3\theta^{3r/8}|A|^2\|_4\big)^{2/3}.$$

Next, by Lemma A.2.3 with $\phi = A$, $p = 2$, etc.,

$$\|\gamma^3\theta^{3r/8}\nabla A\|^2_4 \le c\,\big(\|\gamma^4\theta^{r/2}\nabla^2 A\|_2\|\gamma^2\theta^{r/4}A\|_\infty + \|\gamma^3\theta^{r/2-1}\nabla A\|_2\|\gamma^2\theta^{r/4}A\|_\infty\big).$$

Moreover, we have

$$\|\gamma^3\theta^{r/2-1}\nabla A\|_2 \le c\,\big(\|\gamma^4\theta^{r/2}\nabla^2 A\|_2 + \|A\|_{2,[\gamma\theta>0]}\big), \qquad \text{(Lemma A.1.2)}$$
$$\|\gamma^2\theta^{3r/8-1}A\|^4_4 \le \|\gamma^2\theta^{r/4}A\|^2_\infty\|A\|^2_{2,[\gamma\theta>0]},$$
$$\|\gamma^3\theta^{3r/8}|A|^2\|^4_4 \le \|\gamma^2\theta^{r/4}A\|^2_\infty\|\gamma^8\theta^r|A|^6\|_1, \text{ and}$$
$$\|\gamma^8\theta^r|A|^6\|_1 \le c\,\big(\|\gamma^4\theta^{r/2}\nabla^2 A\|^2_2 + \|A\|^2_{2,[\gamma\theta>0]}\big). \qquad \text{(Lemma 2.1.1)}$$

Combining all the inequalities above,

$$\|\gamma^2\theta^{r/4}A\|_\infty \le c\,\|A\|_{2,[\gamma\theta>0]}^{1/3}\|\gamma^2\theta^{r/4}A\|_\infty^{1/3}\big(\|\gamma^4\theta^{r/2}\nabla^2 A\|_2^{1/3} + \|A\|_{2,[\gamma\theta>0]}^{1/3}\big),$$

and hence

$$\|\gamma^2\theta^{r/4}A\|_\infty^4 \le c\,\|A\|_{2,[\gamma\theta>0]}^2\big(\|\gamma^4\theta^{r/2}\nabla^2 A\|_2^2 + \|A\|_{2,[\gamma\theta>0]}^2\big),$$

which leads to the result we need to prove. □

2.2 Evolution equations

In this section, we derive the evolution of tensors along Willmore flows. In particular, those of $\nabla^m A$. First, as stated in section 2 of [10], we have the following lemmas.

Lemma 2.2.1. *Let $\phi \in \Gamma((T^*\Sigma)^{\otimes(\ell-1)} \otimes N_\Sigma)$, then*

$$(\nabla\nabla^* - \nabla^*\nabla)\phi = A * A * \phi - (\nabla^* T),$$

where

$$\begin{aligned} T(X_0,\dots,X_\ell) &= (\nabla_{X_0}\phi)(X_1,X_2,\dots,X_\ell) - (\nabla_{X_1}\phi)(X_0,X_2,\dots,X_\ell) \\ &= (\mathrm{R}^{\ell-1}(X_0,X_1)\phi)(X_2,\dots,X_\ell) \\ &= A * A * \phi. \qquad \text{(Gauss–Codazzi equation)} \end{aligned}$$

Corollary 2.2.2.

$$(\triangle\nabla - \nabla\triangle)\phi = (\nabla\nabla^* - \nabla^*\nabla)(\nabla\phi) = A * A * \nabla\phi + A * \nabla A * \phi,$$

and hence

$$(\triangle\nabla^m - \nabla^m\triangle)\phi = P_2^m(A) * \phi + P_2^{m-1}(A) * \nabla\phi + \cdots + P_2^0(A) * \nabla^m\phi$$
$$= \nabla^m(\phi * P_2^0).$$

Lemma 2.2.3 (Simons' identity)**.**

$$\triangle A_{ij} = \nabla_{ij}^2 H + g^{k\ell}g^{pq}(\langle A_{ik}, A_{jp}\rangle A_{q\ell} - \langle A_{qk}, A_{jp}\rangle A_{i\ell}).$$

In particular,

(a) $\triangle A = \nabla^2 H + A * A * A$, *and*

(b) $\triangle A^0 = S^0(\nabla^2 H) + \frac{1}{2}|H|^2 A^0 + A^0 * A^0 * A^0$, *where* $S^0(\nabla^2 H)_{ij} = \nabla_{ij}^2 H - \frac{1}{2}H g_{ij} - \frac{1}{2}(R^\perp H)_{ij}$ *denotes the symmetric, trace-free part of* $\nabla^2 H$.

Lemma 2.2.4. *Letting* $V = \partial_t f$ *be a normal vector field on* Σ*, we have*

(a) $\partial_t^\perp \nabla_X\phi - \nabla_X\partial_t^\perp\phi = A(X, e_i)\langle\nabla_{e_i}V, \phi\rangle + \nabla_{e_i}V\langle A(X, e_i), \phi\rangle = A * \nabla V * \phi$,

(b) $\partial_t(\nabla_X Y) = \big[- \langle(\nabla_{e_i}A)(X, Y), V\rangle + \langle A(X, Y), \nabla_{e_i}V\rangle - \langle A(X, e_i), \nabla_Y V\rangle - \langle A(Y, e_i), \nabla_X V\rangle\big] e_i$, *and*

(c) $\partial_t^\perp A(X, Y) = \nabla_{X,Y}^2 V - A(e_i, X)\langle A(e_i, Y), V\rangle$, *i.e.,* $\partial_t^\perp A = \nabla^2 V - (A \llcorner e_i) \otimes \langle (A \llcorner e_i), V\rangle$.

In this article, we will let $V = -\theta^r \mathbf{W}(f)$, where θ is the cutoff function described in section 1.3. The following statements are some consequences:

Lemma 2.2.5.

$$(\partial_t^\perp + \theta^r\triangle^2)A = \theta^r(P_3^2 + P_5^0) + \nabla\big(\nabla(\theta^r) * (P_1^2 + P_3^0)\big).$$

Proof. By (c) of Lemma 2.2.4,

$$\begin{aligned}
\partial_t^{\perp} A &= \nabla^2 V + A * A * V \\
&= -\nabla^2(\theta^r \triangle H + \theta^r P_3^0) + \theta^r P_2^0 * (P_1^2 + P_3^0) \\
&= \theta^r(-\nabla^2 \triangle H + P_3^2 + P_5^0) + \nabla(\theta^r) * (P_1^3 + P_3^1) + \nabla^2(\theta^r) * (P_1^2 + P_3^0).
\end{aligned}$$

Also, by Lemma 2.2.2 and (a) of Lemma 2.2.3,

$$\triangle^2 A = \triangle(\nabla^2 H + P_3^0) = \triangle \nabla^2 H + P_3^2 = \nabla^2 \triangle H + P_3^2.$$

Therefore,

$$\begin{aligned}
(\partial_t^{\perp} + \theta^r \triangle^2) A &= \theta^r(P_3^2 + P_5^0) + \nabla(\theta^r) * (P_1^3 + P_3^1) + \nabla^2(\theta^r) * (P_1^2 + P_3^0) \\
&= \theta^r(P_3^2 + P_5^0) + \nabla\big(\nabla(\theta^r) * (P_1^2 + P_3^0)\big).
\end{aligned}$$

□

Lemma 2.2.6 (Cf. [10, Lemma 2.3])**.** *If* $(\partial_t^{\perp} + \theta^r \triangle^2)\phi = Y$ *and* $\psi = \nabla\phi$, *then*

$$(\partial_t^{\perp} + \theta^r \triangle^2)\psi = \nabla Y + \nabla(\theta^r) * ((P_2^2 + P_4^0) * \phi + \triangle^2 \phi) + \theta^r \nabla^3(P_2^0 * \phi).$$

Proof. Let $X_1, \ldots, X_\ell$ be time-independent. WLOG, assume $\nabla_{X_k} X_h$ vanishes at a given position and time so that we have

$$\begin{aligned}
&(\partial_t^{\perp}\psi)(X_1, \ldots, X_\ell) \\
&= \partial_t^{\perp}\left[(\nabla_{X_1}\phi)(X_2, \ldots, X_\ell) - \sum_{k=2}^{\ell} \phi(X_2, \ldots, \nabla_{X_1} X_k, \ldots, X_\ell)\right] \\
&= (\partial_t^{\perp}\nabla_{X_1}\phi)(X_2, \ldots, X_\ell) - \sum_{k=2}^{\ell} \phi(X_2, \ldots, \partial_t(\nabla_{X_1} X_k), \ldots, X_\ell).
\end{aligned}$$

By (a) of Lemma 2.2.4,

$$(\partial_t^\perp \nabla_{X_1}\phi)(X_2,\ldots,X_\ell) = (\nabla_{X_1}\partial_t^\perp \phi)(X_2,\ldots,X_\ell) + A * \nabla V * \phi.$$

Also, by (b) of Lemma 2.2.4,

$$\partial_t \nabla_{X_1} X_k = \nabla(A * V) * (X_1 \otimes X_k).$$

As a result,

$$\partial_t^\perp \psi = \nabla \partial_t^\perp \phi + \nabla(A * V) * \phi.$$

Next, by Lemma 2.2.2,

$$\begin{aligned}
\triangle^2 \psi &= \triangle^2 \nabla \phi \\
&= \triangle \nabla \triangle \phi + \triangle \nabla (P_2^0 * \phi) \\
&= \nabla \triangle^2 \phi + \nabla (P_2^0 * \triangle \phi) + \triangle \nabla (P_2^0 * \phi) \\
&= \nabla \triangle^2 \phi + \nabla^3 (P_2^0 * \phi).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&(\partial_t^\perp + \theta^r \triangle^2)\psi \\
&= \nabla \partial_t^\perp \phi + \nabla(A * V) * \phi + \theta^r \nabla \triangle^2 \phi + \theta^r \nabla^3 (P_2^0 * \phi) \\
&= \nabla \partial_t^\perp \phi + \nabla(\theta^r \triangle^2 \phi) + \nabla(\theta^r) * \triangle^2 \phi + \nabla(A * V) * \phi + \theta^r \nabla^3 (P_2^0 * \phi) \\
&= \nabla Y + \nabla(\theta^r) * \triangle^2 \phi + \nabla(A * V) * \phi + \theta^r \nabla^3 (P_2^0 * \phi).
\end{aligned}$$

Since $V = -\theta^r \mathbf{W}(f) = \theta^r(P_1^2 + P_3^0)$,

$$\begin{aligned}
&\nabla(\theta^r) * \triangle^2\phi + \nabla(A * V) * \phi + \theta^r \nabla^3(P_2^0 * \phi) \\
&= \nabla(\theta^r) * \triangle^2\phi + \nabla\big(\theta^r(P_2^2 + P_4^0)\big) * \phi + \theta^r \nabla^3(P_2^0 * \phi) \\
&= \nabla(\theta^r) * \big((P_2^2 + P_4^0) * \phi + \triangle^2\phi\big) + \theta^r \nabla^3(P_2^0 * \phi).
\end{aligned}$$

□

Proposition 2.2.7.

$$(\partial_t^\perp + \theta^r \triangle^2)(\nabla^m A) = \nabla^m\big(\theta^r(P_3^2 + P_5^0)\big) + \nabla^{m+1}\big(\nabla(\theta^r) * (P_1^2 + P_3^0)\big).$$

Proof. We have proved the case $m = 0$ in Lemma 2.2.5. Inductively, assume that we have the conclusion for $m - 1$. Let $\phi = \nabla^{m-1} A$ in Lemma 2.2.6 so that

$$\begin{aligned}
&(\partial_t^\perp + \theta^r \triangle^2)(\nabla^m A) \\
&= \nabla\big((\partial_t^\perp + \theta^r \triangle^2)(\nabla^{m-1} A)\big) + \nabla(\theta^r) * \big((P_2^2 + P_4^0) * P_1^{m-1} + P_1^{m+3}\big) \\
&\quad + \theta^r \nabla^3(P_2^0 * P_1^{m-1}).
\end{aligned}$$

For the first term:

$$\begin{aligned}
&\nabla\big((\partial_t^\perp + \theta^r \triangle^2)(\nabla^{m-1} A)\big) \\
&= \nabla\big[\nabla^{m-1}\big(\theta^r(P_3^2 + P_5^0)\big) + \nabla^m\big(\nabla(\theta^r) * (P_1^2 + P_3^0)\big)\big] \\
&= \nabla^m\big(\theta^r(P_3^2 + P_5^0)\big) + \nabla^{m+1}\big(\nabla(\theta^r) * (P_1^2 + P_3^0)\big);
\end{aligned}$$

for the second term:

$$\nabla(\theta^r) * \big((P_2^2 + P_4^0) * P_1^{m-1} + P_1^{m+3}\big) = \nabla^{m+1}\big(\nabla(\theta^r) * (P_1^2 + P_3^0)\big);$$

and for the third term:

$$\theta^r \nabla^3 (P_2^0 * P_1^{m-1}) = \nabla^m \big(\theta^r (P_3^2 + P_5^0)\big).$$

We can derive

$$(\partial_t^\perp + \theta^r \triangle^2)(\nabla^m A) = \nabla^m \big(\theta^r (P_3^2 + P_5^0)\big) + \nabla^{m+1}\big(\nabla(\theta^r) * (P_1^2 + P_3^0)\big),$$

so the claim holds by mathematical induction. □

2.3 Energy estimates

In this section, we estimate the evolution of L^2 norms of tensors.

Lemma 2.3.1. *Let $Y = (\partial_t^\perp + \theta^r \triangle^2)\phi$. We have*

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Sigma_t} \frac{1}{2}\gamma^s |\phi|^2 \,\mathrm{d}\mu_t + \int_{\Sigma_t} \langle \triangle\phi, \triangle(\gamma^s \theta^r \phi)\rangle - \langle Y, \gamma^s \phi\rangle \,\mathrm{d}\mu_t \\
&= \frac{1}{2}\int_{\Sigma_t} \big(\partial_t(\gamma^s) - \gamma^s \langle H_t, V\rangle\big)|\phi|^2 \,\mathrm{d}\mu_t \\
&\quad - \int_{\Sigma_t} \gamma^s \langle V, A_t(e_{i_k}, e_j)\rangle \langle \phi(e_{i_1}, \dots, e_{i_{r_\phi}}), \phi(e_{i_1}, \dots, e_{i_{k-1}}, e_j, e_{i_{k+1}}, \dots, e_{i_{r_\phi}})\rangle \,\mathrm{d}\mu_t\,,
\end{aligned}$$

where $\{e_i\}$ denotes an orthonormal frame of Σ_t, and we use Einstein's conventions on the indices $i_1, \dots, i_{r_\phi}, j \in \{1, 2\}$ and $k \in \{1, \dots, r_\phi\}$.

Proof. (i) Abusing the notations, let $\{e_i\}$ denote the coordinate tangent vectors of a coordinate system on Σ in a neighborhood of $x \in \Sigma$ that is orthonormal with respect to

$f^*g_{\mathbb{R}^n}$ at x for some particular t. Then we have

$$\begin{aligned}
&\partial_t^\perp\left[\phi(e_{i_1},\ldots,e_{i_\ell})\right]\\
&=(Y-\theta^r\triangle^2\phi)(e_{i_1},\ldots,e_{i_{r_\phi}})+\sum_{k=1}^{\ell}\phi(e_{i_1},\ldots,e_{i_{k-1}},\partial_t^\top e_{i_k},e_{i_{k+1}},\ldots,e_{i_{r_\phi}}),
\end{aligned}$$

where

$$\begin{aligned}
g(\partial_t^\top e_i,e_j)&=\langle\partial_t e_i,e_j\rangle=\langle D_V e_j,e_j\rangle=\langle D_{e_i}V,e_j\rangle=-\langle V,D_{e_i}e_j\rangle\\
&=-\langle V,A_t(e_i,e_j)\rangle.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\partial_t\left(\frac{1}{2}\gamma^s|\phi(e_{i_1},\ldots,e_{i_{r_\phi}})|^2\right)\\
&=\gamma^s\langle\phi(e_{i_1},\ldots,e_{i_{r_\phi}}),(Y-\theta^r\triangle^2\phi)(e_{i_1},\ldots,e_{i_{r_\phi}})\rangle\\
&\quad-\gamma^s\langle V,A_t(e_{i_k},e_j)\rangle\langle\phi(e_{i_1},\ldots,e_{i_{r_\phi}}),\phi(e_{i_1},\ldots,e_{i_{k-1}},e_j,e_{i_{k+1}},\ldots,e_{i_{r_\phi}})\rangle\\
&\quad+\frac{1}{2}\partial_t(\gamma^s)|\phi(e_{i_1},\ldots,e_{i_{r_\phi}})|^2.
\end{aligned}$$

That is,

$$\begin{aligned}
&\partial_t\left(\frac{1}{2}\gamma^s|\phi|^2\right)\\
&=\langle Y,\gamma^s\phi\rangle-\theta^r\langle\triangle^2\phi,\gamma^s\phi\rangle+\frac{1}{2}\partial_t(\gamma^s)|\phi|^2\\
&\qquad-\gamma^s\langle V,A_t(e_{i_k},e_j)\rangle\langle\phi(e_{i_1},\ldots,e_{i_{r_\phi}}),\phi(e_{i_1},\ldots,e_{i_{k-1}},e_j,e_{i_{k+1}},\ldots,e_{i_{r_\phi}})\rangle.
\end{aligned}$$

In summary,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Sigma_t}\frac{1}{2}\gamma^s|\phi|^2\,\mathrm{d}\mu_t$$

$$
\begin{aligned}
&= \int_{\Sigma_t} \bigg(\langle Y, \gamma^s \phi \rangle - \theta^r \langle \triangle^2 \phi, \gamma^s \phi \rangle + \frac{1}{2} \partial_t(\gamma^s) |\phi|^2 - \frac{1}{2} \gamma^s \langle H_t, V \rangle |\phi|^2 \\
&\quad - \gamma^s \langle V, A_t(e_{i_k}, e_j) \rangle \langle \phi(e_{i_1}, \ldots, e_{i_{r_\phi}}), \phi(e_{i_1}, \ldots, e_{i_{k-1}}, e_j, e_{i_{k+1}}, \ldots, e_{i_{r_\phi}}) \rangle \bigg) \, \mathrm{d}\mu_t \\
&= \int_{\Sigma_t} \bigg(\langle Y, \gamma^s \phi \rangle - \langle \triangle \phi, \triangle(\gamma^s \theta^r \phi) \rangle + \frac{1}{2} \partial_t(\gamma^s) |\phi|^2 - \frac{1}{2} \gamma^s \langle H_t, V \rangle |\phi|^2 \\
&\quad - \gamma^s \langle V, A_t(e_{i_k}, e_j) \rangle \langle \phi(e_{i_1}, \ldots, e_{i_{r_\phi}}), \phi(e_{i_1}, \ldots, e_{i_{k-1}}, e_j, e_{i_{k+1}}, \ldots, e_{i_{r_\phi}}) \rangle \bigg) \, \mathrm{d}\mu_t \, .
\end{aligned}
$$

□

Lemma 2.3.2 (Cf. [10, Lemma 3.2]). *Again let $Y = (\partial_t^\perp + \theta^r \triangle^2)\phi$. If $s \geq 4$, $r \geq 4$, we have*

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Sigma_t} \gamma^s |\phi|^2 \, \mathrm{d}\mu_t + \frac{7}{8} \int_{\Sigma_t} \gamma^s \theta^r |\nabla^2 \phi|^2 \, \mathrm{d}\mu_t \\
&\leq c \int_{\Sigma_t} \gamma^s \phi * \big(Y + A_t * \phi * V + \theta^r |\nabla A_t|^2 \phi + \theta^r |A_t|^4 \phi \big) \, \mathrm{d}\mu_t \\
&\quad + c \, K^4 \int_{\Sigma_t} \gamma^{s-4} \theta^{r-4} |\phi|^2 \, \mathrm{d}\mu_t \, ,
\end{aligned}
$$

where $c = c(s, r, r_\phi)$.

Proof. (i) By Lemma 2.3.1, we have

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Sigma_t} \gamma^s |\phi|^2 \, \mathrm{d}\mu_t + 2 \int_{\Sigma_t} \langle \triangle \phi, \triangle(\gamma^s \theta^r \phi) \rangle - \gamma^s \langle Y, \phi \rangle \, \mathrm{d}\mu_t \\
&= \int_{\Sigma_t} \bigg(s \gamma^{s-1} \big(\partial_t \gamma - \gamma \langle H_t, V \rangle \big) |\phi|^2 \\
&\quad - 2\gamma^s \sum_{k=1}^{\ell} \langle V, A_t(e_{i_k}, e_j) \rangle \langle \phi(e_{i_1}, \ldots, e_{i_\ell}), \phi(e_{i_1}, \ldots, e_{i_{k-1}}, e_j, e_{i_{k+1}}, \ldots, e_{i_\ell}) \rangle \bigg) \, \mathrm{d}\mu_t \\
&\leq c \int_{\Sigma_t} \gamma^{s-1} \partial_t \gamma |\phi|^2 \, \mathrm{d}\mu_t + c \int_{\Sigma_t} \gamma^s A_t * \phi * \phi * V \, \mathrm{d}\mu_t \, ,
\end{aligned}
$$

so that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Sigma_t}\gamma^s|\phi|^2\,\mathrm{d}\mu_t\\
&= -2\int_{\Sigma_t}\langle\triangle\phi,\triangle(\gamma^s\theta^r\phi)\rangle\,\mathrm{d}\mu_t + c\int_{\Sigma_t}\gamma^{s-1}\partial_t\gamma|\phi|^2\,\mathrm{d}\mu_t\\
&\quad + c\int_{\Sigma_t}\gamma^s\phi*(Y+A_t*\phi*V)\,\mathrm{d}\mu_t\,.
\end{aligned}
$$

(ii) Using integration by parts and the Cauchy-Schwarz inequality,

$$
\begin{aligned}
&-\int_{\Sigma_t}\langle\triangle\phi,\triangle(\gamma^s\theta^r\phi)\rangle\,\mathrm{d}\mu_t\\
&= \int_{\Sigma_t}\langle\nabla\triangle\phi,\nabla(\gamma^s\theta^r\phi)\rangle\,\mathrm{d}\mu_t\\
&= \int_{\Sigma_t}\langle(\triangle\nabla\phi + A_t*A_t*\nabla\phi + A_t*\nabla A_t*\phi),\nabla(\gamma^s\theta^r\phi)\rangle\,\mathrm{d}\mu_t\\
&\le -\int_{\Sigma_t}\langle\nabla^2\phi,\nabla^2(\gamma^s\theta^r\phi)\rangle\,\mathrm{d}\mu_t\\
&\quad + c\int_{\Sigma_t}\big(|A_t|^2|\nabla\phi| + |A_t|\,|\nabla A_t|\,|\phi|\big)\big(\gamma^{s-1}\theta^{r-1}K|\phi| + \gamma^s\theta^r|\nabla\phi|\big)\,\mathrm{d}\mu_t\\
&\le -\int_{\Sigma_t}\langle\nabla^2\phi,\gamma^s\theta^r\nabla^2\phi\rangle\,\mathrm{d}\mu_t\\
&\quad + c\int_{\Sigma_t}\big(\gamma^{s-1}\theta^{r-1}K|\nabla\phi| + \gamma^{s-2}\theta^{r-2}K^2|\phi| + \gamma^{s-1}\theta^{r-1}K|A_t|\,|\phi|\big)|\nabla^2\phi|\,\mathrm{d}\mu_t\\
&\quad + c\int_{\Sigma_t}\big(|A_t|^2|\nabla\phi| + |A_t|\,|\nabla A_t|\,|\phi|\big)\big(\gamma^{s-1}\theta^{r-1}K|\phi| + \gamma^s\theta^r|\nabla\phi|\big)\,\mathrm{d}\mu_t\\
&\le -\int_{\Sigma_t}\gamma^s\theta^r|\nabla^2\phi|^2\,\mathrm{d}\mu_t + \frac{1}{4}\int_{\Sigma_t}\gamma^s\theta^r|\nabla^2\phi|^2\,\mathrm{d}\mu_t\\
&\quad + c\int_{\Sigma_t}\big(\gamma^s\theta^r|A_t|^2|\nabla\phi|^2 + \gamma^s\theta^r|\nabla A_t|^2|\phi|^2\big)\,\mathrm{d}\mu_t\\
&\quad + c\int_{\Sigma_t}\big(\gamma^{s-2}\theta^{r-2}K^2|\nabla\phi|^2 + \gamma^{s-4}\theta^{r-4}K^4|\phi|^2 + \gamma^{s-2}\theta^{r-2}K^2|A_t|^2|\phi|^2\big)\,\mathrm{d}\mu_t\\
&\le -\frac{3}{4}\int_{\Sigma_t}\gamma^s\theta^r|\nabla^2\phi|^2\,\mathrm{d}\mu_t\\
&\quad + c\int_{\Sigma_t}\big(\gamma^s\theta^r|A_t|^2|\nabla\phi|^2 + \gamma^s\theta^r|\nabla A_t|^2|\phi|^2\big)\,\mathrm{d}\mu_t\\
&\quad + c\int_{\Sigma_t}\big(\gamma^{s-2}\theta^{r-2}K^2|\nabla\phi|^2 + \gamma^{s-4}\theta^{r-4}K^4|\phi|^2 + \gamma^s\theta^r|A_t|^4|\phi|^2\big)\,\mathrm{d}\mu_t\,,
\end{aligned}
$$

so that the conclusion in (i) becomes

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Sigma_t}\gamma^s|\phi|^2\,\mathrm{d}\mu_t+\frac{3}{2}\int_{\Sigma_t}\gamma^s\theta^r|\nabla^2\phi|^2\,\mathrm{d}\mu_t\\
&\le c\int_{\Sigma_t}\gamma^{s-1}\partial_t\gamma|\phi|^2\,\mathrm{d}\mu_t\\
&\quad+c\int_{\Sigma_t}\gamma^s\phi*(Y+A_t*\phi*V+\theta^r|\nabla A_t|^2\phi+\theta^r|A_t|^4\phi)\,\mathrm{d}\mu_t\\
&\quad+c\int_{\Sigma_t}\big(\gamma^s\theta^r|A_t|^2|\nabla\phi|^2+\gamma^{s-2}\theta^{r-2}K^2|\nabla\phi|^2+\gamma^{s-4}\theta^{r-4}K^4|\phi|^2\big)\,\mathrm{d}\mu_t\,.
\end{aligned}
$$

(iii) Using integration by parts,

$$
\begin{aligned}
&\int_{\Sigma_t}\gamma^s\theta^r|A_t|^2|\nabla\phi|^2\,\mathrm{d}\mu_t\\
&=\int_{\Sigma_t}\langle\phi,\nabla^*\big(\gamma^s\theta^r|A_t|^2\nabla\phi\big)\rangle\,\mathrm{d}\mu_t\\
&\le c\int_{\Sigma_t}\big(\gamma^s\theta^r|A_t|^2|\phi|\,|\nabla^2\phi|+\gamma^{s-1}\theta^{r-1}K|A_t|^2|\phi|\,|\nabla\phi|\\
&\quad+\gamma^s\theta^r|A_t|\,|\nabla A_t|\,|\phi|\,|\nabla\phi|\big)\,\mathrm{d}\mu_t\\
&\le\varepsilon\int_{\Sigma_t}\gamma^s\theta^r|\nabla^2\phi|^2\,\mathrm{d}\mu_t+\varepsilon\int_{\Sigma_t}\gamma^{s-2}\theta^{r-2}K^2|\nabla\phi|^2\,\mathrm{d}\mu_t\\
&\quad+c\,\varepsilon^{-1}\int_{\Sigma_t}\gamma^s\theta^r|A_t|^4|\phi|^2\,\mathrm{d}\mu_t+\varepsilon\int_{\Sigma_t}\gamma^s\theta^r|A_t|^2|\nabla\phi|^2\,\mathrm{d}\mu_t\\
&\quad+c\,\varepsilon^{-1}\int_{\Sigma_t}\gamma^s\theta^r|\nabla A_t|^2|\phi|^2\,\mathrm{d}\mu_t
\end{aligned}
$$

for any $\varepsilon>0$, and hence by taking ε to be sufficiently small,

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Sigma_t}\gamma^s|\phi|^2\,\mathrm{d}\mu_t+\int_{\Sigma_t}\gamma^s\theta^r|\nabla^2\phi|^2\,\mathrm{d}\mu_t\\
&\le c\int_{\Sigma_t}\gamma^{s-1}\partial_t\gamma|\phi|^2\,\mathrm{d}\mu_t\\
&\quad+\int_{\Sigma_t}\gamma^s\phi*(Y+A_t*\phi*V+\theta^r|\nabla A_t|^2\phi+\theta^r|A_t|^4\phi)\,\mathrm{d}\mu_t\\
&\quad+c\int_{\Sigma_t}\big(\gamma^{s-2}\theta^{r-2}K^2|\nabla\phi|^2+\gamma^{s-4}\theta^{r-4}K^4|\phi|^2\big)\,\mathrm{d}\mu_t\,.
\end{aligned}
$$

(iv) Next, since $V = -\theta^r(\triangle H_t + A_t * A_t * A_t)$,

$$\int_{\Sigma_t} \gamma^{s-1}\partial_t\gamma|\phi|^2 \,\mathrm{d}\mu_t = \int_{\Sigma_t} \gamma^{s-1} D\widehat{\gamma}(-\theta^r \triangle H_t + \theta^r P_3^0)|\phi|^2 \,\mathrm{d}\mu_t \,.$$

On one hand, by Young's inequality,

$$\begin{aligned}
&\int_{\Sigma_t} \gamma^{s-1}\theta^r D\widehat{\gamma} P_3^0|\phi|^2 \,\mathrm{d}\mu_t \\
&\le c\int_{\Sigma_t} \gamma^{s-1}\theta^r K|A_t|^3|\phi|^2 \,\mathrm{d}\mu_t \\
&= c\int_{\Sigma_t} \big(\gamma^{3s/4}\theta^{3r/4}|A_t|^3|\phi|^{3/2}\big)\big(\gamma^{s/4-1}\theta^{r/4}K|\phi|^{1/2}\big) \,\mathrm{d}\mu_t \\
&\le c\int_{\Sigma_t} \big(\gamma^s\theta^r|A_t|^4|\phi|^2 + \gamma^{s-4}\theta^r K^4|\phi|^2\big) \,\mathrm{d}\mu_t \,.
\end{aligned}$$

On the other hand, with integration by parts,

$$\begin{aligned}
&-\int_{\Sigma_t} \gamma^{s-1}\theta^r|\phi|^2 D\widehat{\gamma}(\triangle H_t) \,\mathrm{d}\mu_t \\
&= -\int_{\Sigma_t} \gamma^{s-1}\theta^r|\phi|^2 D\widehat{\gamma}\big(D_{e_i}\nabla_{e_i}H_t - \langle A_t(e_i,e_j), \nabla_{e_j}H_t\rangle D_{e_i}f\big) \,\mathrm{d}\mu_t \\
&= \int_{\Sigma_t} \Big[\gamma^{s-1}\theta^r|\phi|^2 D^2\widehat{\gamma}(D_{e_i}f, \nabla_{e_i}H_t) + \langle D_{e_i}\big(\gamma^{s-1}\theta^r|\phi|^2\big), (D\widehat{\gamma}(\nabla_{e_i}H_t)\rangle \\
&\qquad + \gamma^{s-1}\theta^r|\phi|^2\langle A_t(e_i,e_j), \nabla_{e_j}H_t\rangle D\widehat{\gamma}(D_{e_i}f)\Big] \,\mathrm{d}\mu_t \\
&\le c\int_{\Sigma_t} \gamma^{s-2}\theta^{r-1}\big[\gamma\theta K^2|\nabla A_t|\,|\phi|^2 + (\gamma\theta|\nabla\phi| + K|\phi|)K|\nabla A_t|\,|\phi| \\
&\qquad + \gamma\theta K|A_t|\,|\nabla A_t|\,|\phi|^2\big] \,\mathrm{d}\mu_t \\
&\le c\int \gamma^{s-2}\theta^{r-1}\big[K^2|\nabla A_t|\,|\phi|^2 + \gamma\theta K|\nabla A_t|\,|\phi|\,|\nabla\phi| \\
&\qquad + \gamma\theta K|A_t|\,|\nabla A_t|\,|\phi|^2\big] \,\mathrm{d}\mu_t \\
&\le c\int_{\Sigma_t} \big[\gamma^s\theta^r|\nabla A_t|^2|\phi|^2 + \gamma^{s-4}\theta^{r-4}K^4|\phi|^2 + \gamma^{s-2}\theta^r K^2|\nabla\phi|^2 \\
&\qquad + \gamma^{s-2}\theta^r K^2|A_t|^2|\phi|^2\big] \,\mathrm{d}\mu_t \\
&\le c\int_{\Sigma_t} \big[\gamma^s\theta^r|\nabla A_t|^2|\phi|^2 + \gamma^s\theta^r|A_t|^4|\phi|^2 + \gamma^{s-4}\theta^{r-4}K^4|\phi|^2 + \gamma^{s-2}\theta^r K^2|\nabla\phi|^2\big] \,\mathrm{d}\mu_t \,.
\end{aligned}$$

As a result,

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Sigma_t}\gamma^s|\phi|^2\,\mathrm{d}\mu_t+\int_{\Sigma_t}\gamma^s\theta^r|\nabla^2\phi|^2\,\mathrm{d}\mu_t\\
&\le c\int_{\Sigma_t}\gamma^s\phi*(Y+A_t*\phi*V+\theta^r|\nabla A_t|^2\phi+\theta^r|A_t|^4\phi)\,\mathrm{d}\mu_t\\
&\qquad+c\int_{\Sigma_t}\big(\gamma^{s-2}\theta^{r-2}K^2|\nabla\phi|^2+\gamma^{s-4}\theta^{r-4}K^4|\phi|^2\big)\,\mathrm{d}\mu_t\,.
\end{aligned}
$$

(v) Finally, for arbitrary $\varepsilon>0$,

$$
\begin{aligned}
&\int_{\Sigma_t}\gamma^{s-2}\theta^{r-2}K^2|\nabla\phi|^2\,\mathrm{d}\mu_t\\
&\le c\int_{\Sigma_t}\big(\gamma^{s-2}\theta^{r-2}K^2|\nabla^2\phi|+\gamma^{s-3}\theta^{r-3}K^3|\nabla\phi|\big)|\phi|\,\mathrm{d}\mu_t\\
&\le\varepsilon\int_{\Sigma_t}\big(\gamma^s\theta^r|\nabla^2\phi|^2+\gamma^{s-2}\theta^{r-2}K^2|\nabla\phi|^2\big)\,\mathrm{d}\mu_t+c\,\varepsilon^{-1}\int_{\Sigma_t}\gamma^{s-4}\theta^{r-4}K^4|\phi|^2\,\mathrm{d}\mu_t\,,
\end{aligned}
$$

so that by taking ε to be sufficiently small,

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Sigma_t}\gamma^s|\phi|^2\,\mathrm{d}\mu_t+\frac{7}{8}\int_{\Sigma_t}\gamma^s\theta^r|\nabla^2\phi|^2\,\mathrm{d}\mu_t\\
&=c\int_{\Sigma_t}\gamma^s\phi*(Y+A_t*\phi*V+\theta^r|\nabla A_t|^2\phi+\theta^r|A_t|^4\phi)\,\mathrm{d}\mu_t\\
&\qquad+c\int_{\Sigma_t}\gamma^{s-4}\theta^{r-4}K^4|\phi|^2\,\mathrm{d}\mu_t\,,
\end{aligned}
$$

which is what we need to prove.

□

Proposition 2.3.3 (Cf. [10, Proposition 3.3]). *Let $0\le k\le m$. If $s,r\ge 2k+4$, we have*

$$
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Sigma_t}\gamma^s|\nabla^m A_t|^2\,\mathrm{d}\mu_t+\frac{3}{4}\int_{\Sigma_t}\gamma^s\theta^r|\nabla^{m+2}A_t|^2\,\mathrm{d}\mu_t
$$

$$\leq c\int_{\Sigma_t}\gamma^s\nabla^m A_t * \big[\nabla^m\big(\theta^r(P_3^2+P_5^0)\big)+\nabla^{m+1}\big(\theta^{r-1}\nabla\theta * (P_1^2+P_3^0)\big)\big]\,\mathrm{d}\mu_t$$
$$+\,c\,K^{4+2k}\int_{\Sigma_t}\gamma^{s-4-2k}\theta^{r-4-2k}|\nabla^{m-k}A_t|^2\,\mathrm{d}\mu_t\,,$$

where $c = c(s, r, m)$.

Proof. By Proposition 2.2.7, we have

$$Y = (\partial_t^\perp + \theta^r\triangle^2)\phi = \nabla^m\big(\theta^r(P_3^2+P_5^0)\big)+\nabla^{m+1}\big(\theta^{r-1}\nabla\theta * (P_1^2+P_3^0)\big).$$

In addition, $V = \theta^r(P_1^2 + P_3^0)$ implies

$$A_t * \nabla^m A_t * V + \theta^r|\nabla A_t|^2\nabla^m A_t + \theta^r|A_t|^4\nabla^m A_t = \theta^r(P_3^{m+2}+P_5^m).$$

Therefore, by taking $\phi = \nabla^m A_t$ in Lemma 2.3.2,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Sigma_t}\gamma^s|\nabla^m A_t|^2 + \frac{7}{8}\int_{\Sigma_t}\gamma^s\theta^r|\nabla^{m+2}A_t|^2\,\mathrm{d}\mu_t$$
$$\leq c\int_{\Sigma_t}\gamma^s\nabla^m A_t * \big(Y + A_t * \nabla^m A_t * V + \theta^r|\nabla A_t|^2\nabla^m A_t + \theta^r|A_t|^4\nabla^m A_t\big)\,\mathrm{d}\mu_t$$
$$+\,c\,K^4\int_{\Sigma_t}\gamma^{s-4}\theta^{r-4}|\nabla^m A_t|^2\,\mathrm{d}\mu_t$$
$$\leq c\int_{\Sigma_t}\gamma^s\nabla^m A_t * \big[\nabla^m\big(\theta^r(P_3^2+P_5^0)\big)+\nabla^{m+1}\big(\theta^{r-1}\nabla\theta * (P_1^2+P_3^0)\big)\big]\,\mathrm{d}\mu_t$$
$$+\,c\,K^4\int_{\Sigma_t}\gamma^{s-4}\theta^{r-4}|\nabla^m A_t|^2\,\mathrm{d}\mu_t\,.$$

If $k > 0$, by Proposition A.1.2,

$$K^4\int_{\Sigma_t}\gamma^{s-4}\theta^{r-4}|\nabla^m A_t|^2\,\mathrm{d}\mu_t$$
$$\leq \varepsilon\int_{\Sigma_t}\gamma^s\theta^r|\nabla^{m+2}A_t|^2\,\mathrm{d}\mu_t + c(s,r,m,\varepsilon)\,K^{4+2k}\int_{\Sigma_t}\gamma^{s-4-2k}\theta^{r-4-2k}|\nabla^{m-k}A_t|^2\,\mathrm{d}\mu_t\,,$$

and hence by taking ε to be sufficiently small,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Sigma_t}|\nabla^m A_t|^2\gamma^s\,\mathrm{d}\mu_t+\frac{3}{4}\int_{\Sigma_t}\gamma^s\theta^r|\nabla^{m+2}A_t|^2\,\mathrm{d}\mu_t\\
&\le c\int_{\Sigma_t}\gamma^s\nabla^m A_t*\big[\nabla^m\big(\theta^r(P_3^2+P_5^0)\big)+\nabla^{m+1}\big(\theta^{r-1}\nabla\theta*(P_1^2+P_3^0)\big)\big]\,\mathrm{d}\mu_t\\
&\qquad+c\,K^{4+2k}\int_{[\gamma\theta>0]}\gamma^{s-4-2k}\theta^{r-4-2k}|\nabla^{m-k}A_t|^2\,\mathrm{d}\mu_t\,.
\end{aligned}$$

□

Lemma 2.3.4. *Let $f:\Sigma\times[0,T)\to\mathbb{R}^n$ be a solution to the modified equation (1.2). If $s,r\ge 4$, then we can choose ε_0 so that assuming (2.1) for all $0\le t\le t_0$ for some $0<t_0<T$, namely*

$$\sup_{0\le t\le t_0}\int_{[\gamma\theta>0]}|A_t|^2\,\mathrm{d}\mu_t\le\varepsilon_0,\tag{2.2}$$

we have

$$\int_{[\gamma=1]}|A_t|^2\,\mathrm{d}\mu_t+\frac{1}{2}\int_0^t\int_{[\gamma=1]}\theta^r\big(|\nabla^2A_{t'}|^2+|A_{t'}|^6\big)\,\mathrm{d}\mu_{t'}\,\mathrm{d}t'\le\int_{[\gamma>0]}|A_0|^2\,\mathrm{d}\mu_0+c\,K^4et$$

for all $t\in[0,t_0)$, where $c=c(n,s,r)$ and

$$e:=\sup_{0\le t\le t_0}\int_{[\gamma\theta>0]}|A_t|^2\,\mathrm{d}\mu_t\,.$$

Proof. By Proposition 2.3.3 (with $m=0$ and $k=0$),

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Sigma_t}\gamma^s|A_t|^2\,\mathrm{d}\mu_t+\frac{3}{4}\int_{\Sigma_t}\gamma^s\theta^r|\nabla^2A_t|^2\,\mathrm{d}\mu_t+\frac{3}{4}\int_{\Sigma_t}\gamma^s\theta^r|A_t|^6\,\mathrm{d}\mu_t\\
&\le c\int_{\Sigma_t}\gamma^sA_t*\big[\theta^r(P_3^2+P_5^0)+\nabla\big(\theta^{r-1}\nabla\theta*(P_1^2+P_3^0)\big)\big]\,\mathrm{d}\mu_t\\
&\qquad+c\,K^4\int_{\Sigma_t}\gamma^{s-4}\theta^{r-4}|A_t|^2\,\mathrm{d}\mu_t+\frac{3}{4}\int_{\Sigma_t}\gamma^s\theta^r|A_t|^6\,\mathrm{d}\mu_t
\end{aligned}$$

$$
\begin{aligned}
&\le c\int_{\Sigma_t}\gamma^s\theta^r A_t * (P_3^2+P_5^0) + \nabla(\gamma^s A_t) * (\theta^{r-1}\nabla\theta) * (P_1^2+P_3^0)\,\mathrm{d}\mu_t \\
&\quad + c\,K^4\int_{\Sigma_t}\gamma^{s-4}\theta^{r-4}|A_t|^2\,\mathrm{d}\mu_t + \frac{3}{4}\int_{\Sigma_t}\gamma^s\theta^r|A_t|^6\,\mathrm{d}\mu_t \\
&\le c\int_{\Sigma_t}\gamma^s\theta^r\big(|\nabla^2 A_t|\,|A_t|^3 + |\nabla A_t|^2|A_t|^2 + |A_t|^6\big)\,\mathrm{d}\mu_t \\
&\quad + c\int_{\Sigma_t}\big(\gamma^s\theta^{r-1}K|\nabla A_t| + \gamma^{s-1}\theta^{r-1}K^2|A_t|\big)\big(|\nabla^2 A_t| + |A_t|^3\big)\,\mathrm{d}\mu_t + c\,K^4 e \\
&\le \frac{1}{12}\int_{\Sigma_t}\gamma^s\theta^r|\nabla^2 A_t|^2\,\mathrm{d}\mu_t + c\int_{\Sigma_t}\gamma^s\theta^r\big(|\nabla A_t|^2|A_t|^2 + |A_t|^6\big)\,\mathrm{d}\mu_t \\
&\quad + c\,K^2\int_{\Sigma_t}\gamma^s\theta^{r-2}|\nabla A_t|^2\,\mathrm{d}\mu_t + c\,K^4 e.
\end{aligned}
$$

Therefore by Proposition A.1.2 and Lemma 2.1.1,

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Sigma_t}\gamma^s|A_t|^2\,\mathrm{d}\mu_t + \frac{3}{4}\int_{\Sigma_t}\gamma^s\theta^r(|\nabla^2 A_t|^2 + |A_t|^6)\,\mathrm{d}\mu_t \\
&\le \frac{1}{6}\int_{\Sigma_t}\gamma^s\theta^r|\nabla^2 A_t|^2\,\mathrm{d}\mu_t + c\int_{[\gamma\theta>0]}|A_t|^2\,\mathrm{d}\mu_t\int_{\Sigma_t}\gamma^s\theta^r\big(|\nabla^2 A_t|^2 + |A_t|^6\big)\,\mathrm{d}\mu_t \\
&\quad + c\,K^4 e + c\,K^4\varepsilon_0 e \\
&\le \frac{1}{4}\int_{\Sigma_t}\gamma^s\theta^r\big(|\nabla^2 A_t|^2 + |A_t|^6\big)\,\mathrm{d}\mu_t + c\,K^4 e,
\end{aligned}
$$

and hence

$$
\begin{aligned}
&\int_{\Sigma_t}\gamma^s|A_t|^2\,\mathrm{d}\mu_t + \frac{1}{2}\int_0^t\int_{\Sigma_{t'}}\gamma^s\theta^r\big(|\nabla^2 A_{t'}|^2 + |A_{t'}|^6\big)\,\mathrm{d}\mu_{t'}\,\mathrm{d}t' \\
&\le \int_{\Sigma_0}\gamma^s\theta^r|A_0|^2\,\mathrm{d}\mu_0 + c\,K^4 e t.
\end{aligned}
$$

□

Proposition 2.3.5. *Let $m \ge 1$. If $s \ge 6$ and $r \ge 20$, then we can choose ε_0 so that* *assuming (2.1), we have that for some* $c = c(n, s, r, m, K)$*,*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Sigma_t}\gamma^s|\nabla^m A_t|^2\,\mathrm{d}\mu_t + \frac{1}{2}\int_{\Sigma_t}\gamma^s\theta^r|\nabla^{m+2}A_t|^2\,\mathrm{d}\mu_t$$
$$\leq c\,\|\theta^{r/4}A_t\|^4_{\infty,[\gamma>0]}\int_{[\theta>0]}\gamma^s|\nabla^m A_t|^2\,\mathrm{d}\mu_t + \beta_{m,\gamma},$$

where

$$\beta_{1,\gamma} = c\,\|A_t\|^2_{2,[\gamma\theta>0]},$$
$$\beta_{2,\gamma} = c\,\big(1 + \|\theta^{r/4}A_t\|^4_{\infty,[\gamma>0]}\big)\|A_t\|^2_{2,[\gamma\theta>0]},$$

and when $m \geq 3$*,* $\beta_{m,\gamma}$ *only depends on* n*,* s*,* r*,* m*,* K*, and*

$$\|\nabla^j A_t\|_{p,[\gamma\theta>0]},\ \textit{where either}\ \begin{cases} j = 0,\dots,m-2, \\ p = 2,\dots,2m+4, \end{cases} \quad \textit{or} \quad \begin{cases} j = 0,1, \\ p = \infty. \end{cases}$$

Proof. With $c = c(n, s, r, m, \varepsilon)$, we have

(i) For $m = 1$,

$$\int_{\Sigma_t}(\gamma^s\nabla A_t) * \big[\nabla\big(\theta^r(P^2_3 + P^0_5)\big) + \nabla^2\big(\theta^{r-1}\nabla\theta * (P^2_1 + P^0_3)\big)\big]\,\mathrm{d}\mu_t$$
$$\leq \int_{\Sigma_t}(\gamma^s\nabla A_t) * \nabla\big(\theta^r(P^2_3 + P^0_5)\big) + \nabla^2(\gamma^s\nabla A_t) * \big(\theta^{r-1}\nabla\theta * (P^2_1 + P^0_3)\big)\,\mathrm{d}\mu_t$$
$$\leq \int_{\Sigma_t}\gamma^s|\nabla A_t|\cdot\big[\theta^r(|\nabla^3 A_t|\,|A_t|^2 + |\nabla^2 A_t|\,|\nabla A_t|\,|A_t| + |\nabla A_t|^3 + |\nabla A_t|\,|A_t|^4)$$
$$+ \theta^{r-1}K(|\nabla^2 A_t|\,|A_t|^2 + |\nabla A_t|^2|A_t| + |A_t|^5)\big]$$
$$+ \big[\gamma^s|\nabla^3 A_t| + \gamma^{s-1}K|\nabla^2 A_t|$$
$$+ (\gamma^{s-2}K^2 + \gamma^{s-1}K|A_t|)|\nabla A_t|\big]\cdot\theta^{r-1}K(|\nabla^2 A_t| + |A_t|^3)\,\mathrm{d}\mu_t$$

$$
\begin{aligned}
&\le \varepsilon \int_{\Sigma_t} \gamma^s\theta^r|\nabla^3 A_t|^2 \,\mathrm{d}\mu_t + c\int_{\Sigma_t} (\gamma^{s-2}\theta^{r-2}K^2 + \gamma^s\theta^r|A_t|^2)|\nabla^2 A_t|^2 \,\mathrm{d}\mu_t \\
&\quad + c\int_{\Sigma_t} (\gamma^{s-4}\theta^{r-4}K^4 + \gamma^s\theta^r|A_t|^4)|\nabla A_t|^2 \,\mathrm{d}\mu_t \\
&\quad + c\int_{\Sigma_t} \gamma^s\theta^{r-2}K^2|A_t|^6 \,\mathrm{d}\mu_t + c\int_{\Sigma_t} \gamma^s\theta^r|\nabla A_t|^4 \,\mathrm{d}\mu_t \\
&\le \varepsilon \int_{\Sigma_t} \gamma^s\theta^r|\nabla^3 A_t|^2 \,\mathrm{d}\mu_t + c\int_{\Sigma_t} (\gamma^{s-2}\theta^{r-2}K^2 + \gamma^s\theta^r|A_t|^2)|\nabla^2 A_t|^2 \,\mathrm{d}\mu_t \\
&\quad + c\int_{\Sigma_t} (\gamma^{s-4}\theta^{r-4}K^4 + \gamma^s\theta^r|A_t|^4)|\nabla A_t|^2 \,\mathrm{d}\mu_t + c\int_{\Sigma_t} \gamma^s\theta^r|\nabla A_t|^4 \,\mathrm{d}\mu_t \\
&\quad + c\,K^6\|A_t\|^2_{2,[\gamma\theta>0]} \qquad \text{(Lemma 2.1.1)} \\
&\le \varepsilon \int_{\Sigma_t} \gamma^s\theta^r|\nabla^3 A_t|^2 \,\mathrm{d}\mu_t \\
&\quad + c\int_{\Sigma_t} \big(\gamma^{s-2}\theta^{r-2}K^2 + \gamma^s\theta^{r/2}\|\theta^{r/4}A_t\|^2_{\infty,[\gamma>0]}\big)|\nabla^2 A_t|^2 \,\mathrm{d}\mu_t \\
&\quad + c\int_{[\theta>0]} \big(\gamma^{s-4}\theta^{r-4}K^4 + \gamma^s\big\|\theta^{r/4}A_t\big\|^4_{\infty,[\gamma>0]}\big)|\nabla A_t|^2 \,\mathrm{d}\mu_t + c\,K^6\|A_t\|^2_{2,[\gamma\theta>0]} \\
&\qquad \text{(Proposition 2.1.3)} \\
&\le 2\varepsilon \int_{\Sigma_t} \gamma^s\theta^r|\nabla^3 A_t|^2 \,\mathrm{d}\mu_t + c\int_{\Sigma_t} \gamma^{s-2}\theta^{r-2}K^2|\nabla^2 A_t|^2 \,\mathrm{d}\mu_t \\
&\quad + c\int_{[\theta>0]} \big(\gamma^{s-4}\theta^{r-4}K^4 + \gamma^s\|\theta^{r/4}A_t\|^4_{\infty,[\gamma>0]}\big)|\nabla A_t|^2 \,\mathrm{d}\mu_t + c\,K^6\|A_t\|^2_{2,[\gamma\theta>0]} \\
&\qquad \text{(Proposition A.1.3)} \\
&\le 3\varepsilon \int_{\Sigma_t} \gamma^s\theta^r|\nabla^3 A_t|^2 \,\mathrm{d}\mu_t + c\,\|\theta^{r/4}A_t\|^4_{\infty,[\gamma>0]} \int_{[\theta>0]} \gamma^s|\nabla A_t|^2 \,\mathrm{d}\mu_t \\
&\quad + c\,K^6\|A_t\|^2_{2,[\gamma\theta>0]}. \qquad \text{(Proposition A.1.2)}
\end{aligned}
$$

Finally, apply this estimate with sufficiently small ε and $k = 1$ in Proposition 2.3.3 so that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Sigma_t} \gamma^s|\nabla A_t|^2 \,\mathrm{d}\mu_t + \frac{1}{2}\int_{\Sigma_t} \gamma^s\theta^r|\nabla^3 A_t|^2 \,\mathrm{d}\mu_t \\
&\le c\,\|\theta^{r/4}A_t\|^4_{\infty,[\gamma>0]} \int_{[\theta>0]} \gamma^s|\nabla A_t|^2 \,\mathrm{d}\mu_t + c\,K^6\|A_t\|^2_{2,[\gamma\theta>0]}.
\end{aligned}
$$

(ii) For $m = 2$,

$$
\begin{aligned}
&\int_{\Sigma_t} \gamma^s \nabla^2 A_t * \big[\nabla^2\big(\theta^r (P_3^2 + P_5^0)\big) + \nabla^3\big(\theta^{r-1}\nabla\theta * (P_1^2 + P_3^0)\big)\big]\,\mathrm{d}\mu_t \\
&\leq \int_{\Sigma_t} \Big(\nabla^2(\gamma^s \nabla^2 A_t) * \big[\theta^r P_3^2 + \nabla\big(\theta^{r-1}\nabla\theta * (P_1^2 + P_3^0)\big)\big] \\
&\qquad + \gamma^s \nabla^2 A_t * \nabla^2\big(\theta^r P_5^0\big)\Big)\,\mathrm{d}\mu_t \\
&\leq c \int_{\Sigma_t} \Big(\big[\gamma^s |\nabla^4 A_t| + \gamma^{s-1} K (|\nabla^3 A_t| + |\nabla^2 A_t|\,|A_t|) + \gamma^{s-2} K^2 |\nabla^2 A_t|\big] \\
&\qquad \cdot \big[\theta^r (|\nabla^2 A_t|\,|A_t|^2 + |\nabla A_t|^2\,|A_t|) \\
&\qquad + \theta^{r-1} K (|\nabla^3 A_t| + |\nabla^2 A_t|\,|A_t| + |\nabla A_t|\,|A_t|^2 + |A_t|^4) \\
&\qquad + \theta^{r-2} K^2 (|\nabla^2 A_t| + |A_t|^3)\big] \\
&\qquad + \gamma^s |\nabla^2 A_t| \big[\theta^r (|\nabla^2 A_t|\,|A_t|^4 + |\nabla A_t|^2\,|A_t|^3) \\
&\qquad + \theta^{r-1} K (|\nabla A_t|\,|A_t|^4 + |A_t|^6) + \theta^{r-2} K^2 |A_t|^5\big]\Big)\,\mathrm{d}\mu_t \\
&\leq \varepsilon \int_{\Sigma_t} \gamma^s \theta^r |\nabla^4 A_t|^2\,\mathrm{d}\mu_t + c \int_{\Sigma_t} (\gamma^{s-2}\theta^{r-2} K^2 + \gamma^s \theta^r |A_t|^2) |\nabla^3 A_t|^2\,\mathrm{d}\mu_t \\
&\quad + c \int_{\Sigma_t} (\gamma^{s-4}\theta^{r-4} K^4 + \gamma^{s-2}\theta^{r-2} K^2 |A_t|^2 + \gamma^s \theta^r |A_t|^4) |\nabla^2 A_t|^2\,\mathrm{d}\mu_t \\
&\quad + c \int_{\Sigma_t} \gamma^s \theta^{r-2} K^2 |\nabla A_t|^2 |A_t|^4\,\mathrm{d}\mu_t + c \int_{\Sigma_t} \gamma^s \theta^{r-4} K^4 |A_t|^6\,\mathrm{d}\mu_t \\
&\quad + c \int_{\Sigma_t} \gamma^s \theta^{r-2} K^2 |A_t|^8\,\mathrm{d}\mu_t + c \int_{\Sigma_t} \gamma^s \theta^r |\nabla A_t|^4 |A_t|^2\,\mathrm{d}\mu_t \\
&\leq \varepsilon \int_{\Sigma_t} \gamma^s \theta^r |\nabla^4 A_t|^2\,\mathrm{d}\mu_t + c \int_{\Sigma_t} (\gamma^{s-2}\theta^{r-2} K^2 + \gamma^s \theta^r |A_t|^2) |\nabla^3 A_t|^2\,\mathrm{d}\mu_t \\
&\quad + c \int_{\Sigma_t} (\gamma^{s-4}\theta^{r-4} K^4 + \gamma^s \theta^r |A_t|^4) |\nabla^2 A_t|^2\,\mathrm{d}\mu_t \\
&\quad + c \int_{\Sigma_t} \gamma^s \theta^r |\nabla A_t|^4 |A_t|^2\,\mathrm{d}\mu_t + c \int_{\Sigma_t} \gamma^s \theta^{r-8} K^8 |A_t|^2\,\mathrm{d}\mu_t \\
&\quad + c \int_{\Sigma_t} \gamma^s \theta^{r-2} K^2 |A_t|^8\,\mathrm{d}\mu_t
\end{aligned}
$$

$$
\begin{aligned}
&\le \varepsilon \int_{\Sigma_t} \gamma^s \theta^r |\nabla^4 A_t|^2 \,\mathrm{d}\mu_t \\
&\qquad + c \int_{\Sigma_t} \big(\gamma^{s-2}\theta^{r-2}K^2 + \gamma^s \theta^{r/2} \|\theta^{r/4} A_t\|^2_{\infty,[\gamma>0]}\big) |\nabla^3 A_t|^2 \,\mathrm{d}\mu_t \\
&\qquad + c \int_{[\theta>0]} \big(\gamma^{s-4}\theta^{r-4}K^4 + \gamma^s \|\theta^{r/4} A_t\|^4_{\infty,[\gamma>0]}\big) |\nabla^2 A_t|^2 \,\mathrm{d}\mu_t \\
&\qquad + c \int_{\Sigma_t} \gamma^s \theta^r |\nabla A_t|^4 |A_t|^2 \,\mathrm{d}\mu_t + c\, K^8 \|A_t\|^2_{2,[\gamma\theta>0]} && \text{(Lemma 2.1.2)} \\
&\le \varepsilon \int_{\Sigma_t} \gamma^s \theta^r |\nabla^4 A_t|^2 \,\mathrm{d}\mu_t \\
&\qquad + c \int_{\Sigma_t} \big(\gamma^{s-2}\theta^{r-2}K^2 + \gamma^s \theta^{r/2} \|\theta^{r/4} A_t\|^2_{\infty,[\gamma>0]}\big) |\nabla^3 A_t|^2 \,\mathrm{d}\mu_t \\
&\qquad + c \int_{[\theta>0]} \big(\gamma^{s-4}\theta^{r-4}K^4 + \gamma^s \|\theta^{r/4} A_t\|^4_{\infty,[\gamma>0]}\big) |\nabla^2 A_t|^2 \,\mathrm{d}\mu_t \\
&\qquad + c \big(K^8 + K^5 \|\theta^{r/4} A_t\|^3_{\infty,[\gamma>0]}\big) \|A_t\|^2_{2,[\gamma\theta>0]} && \text{(Proposition 2.1.4)} \\
&\le 2\varepsilon \int_{\Sigma_t} \gamma^s \theta^r |\nabla^4 A_t|^2 \,\mathrm{d}\mu_t + c\, \|\theta^{r/4} A_t\|^4_{\infty,[\gamma>0]} \int_{[\theta>0]} \gamma^s |\nabla^2 A_t|^2 \,\mathrm{d}\mu_t \\
&\qquad + c \big(K^8 + K^4 \|\theta^{r/4} A_t\|^4_{\infty,[\gamma>0]}\big) \|A_t\|^2_{2,[\gamma\theta>0]}. && \text{(Proposition A.1.4)}
\end{aligned}
$$

Finally, apply this estimate with sufficiently small ε and $k = 2$ in Proposition 2.3.3 so that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Sigma_t} \gamma^s |\nabla^2 A_t|^2 \,\mathrm{d}\mu_t + \frac{1}{2} \int_{\Sigma_t} \gamma^s \theta^r |\nabla^4 A_t|^2 \,\mathrm{d}\mu_t \\
&\le c\, \|\theta^{r/4} A_t\|^4_{\infty,[\gamma>0]} \int_{[\theta>0]} \gamma^s |\nabla^2 A_t|^2 \,\mathrm{d}\mu_t + c \big(K^8 + K^4 \|\theta^{r/4} A_t\|^4_{\infty,[\gamma>0]}\big) \|A_t\|^2_{2,[\gamma\theta>0]}.
\end{aligned}
$$

(iii) For $m \ge 3$,

$$
\begin{aligned}
&\int_{\Sigma_t} \gamma^s \nabla^m A_t * \big[\nabla^m\big(\theta^r (P_3^2 + P_5^0)\big) + \nabla^{m+1}\big(\theta^{r-1}\nabla\theta * (P_1^2 + P_3^0)\big)\big] \,\mathrm{d}\mu_t \\
&\le \int_{\Sigma_t} \nabla^2(\gamma^2 \nabla^m A_t) * \big[\nabla^{m-2}\big(\theta^r (P_3^2 + P_5^0)\big) + \nabla^{m-1}\big(\theta^{r-1}\nabla\theta * (P_1^2 + P_3^0)\big)\big] \,\mathrm{d}\mu_t
\end{aligned}
$$

$$
\begin{aligned}
&\leq c \int_{\Sigma_t} \big[\gamma^s |\nabla^{m+2} A_t| + \gamma^{s-1} K |\nabla^{m+1} A_t| + (\gamma^{s-2} K^2 + \gamma^{s-1} K |A_t|) |\nabla^m A_t|\big] \\
&\quad \cdot \big[\theta^r \big(|\nabla^m A_t|\, |A_t|^2 + |\nabla^{m-1} A_t|\, |\nabla A_t|\, |A_t|\big) \\
&\qquad + \theta^{r-1} K \big(|\nabla^{m-1} A_t|\, |A_t|^2 + |\nabla^{m+1} A_t| + |\nabla^m A_t|\, |A_t| + |\nabla^{m-1} A_t|\, |\nabla A_t|\big) \\
&\qquad + \theta^{r-2} K^2 \big(|\nabla^m A_t| + |\nabla^{m-1} A_t|\, |A_t|\big) + \theta^{r-3} K^3 |\nabla^{m-1} A_t| + |T|\big] \, \mathrm{d}\mu_t \,,
\end{aligned}
$$

where T is a tensor that is supported on $[\gamma\theta > 0]$ and can be described as a polynomial defined by operators $+$ and $*$, with variables being $A_t, \ldots, \nabla^{m-2} A_t$, with coefficients bounded by some $c(n, s, r, m, \varepsilon, K)$, at most of degree $(m+2)$, and without constant terms. In particular, using Hölder's inequality, $\|T\|_2^2$ is bounded above by a quantity in the same form as how $\beta_{m,\gamma}$ is described. Next, using Cauchy-Schwartz inequality and Proposition A.1.4, we have

$$
\begin{aligned}
&\int_{\Sigma_t} \gamma^s \nabla^m A_t * \big[\nabla^m \big(\theta^r (P_3^2 + P_5^0)\big) + \nabla^{m+1} \big(\theta^{r-1} \nabla\theta * (P_1^2 + P_3^0)\big)\big] \, \mathrm{d}\mu_t \\
&\leq \varepsilon \int_{\Sigma_t} \gamma^s \theta^r |\nabla^{m+2} A_t|^2 \, \mathrm{d}\mu_t + c \int_{\Sigma_t} \gamma^{s-2} \theta^{r-2} K^2 |\nabla^{m+1} A_t|^2 \, \mathrm{d}\mu_t \\
&\quad + c \int_{\Sigma_t} \big(\gamma^{s-4} \theta^{r-4} K^4 + \gamma^s \theta^r \|A_t\|_{\infty,[\gamma\theta>0]}^4\big) |\nabla^m A_t|^2 \, \mathrm{d}\mu_t \\
&\quad + c \int_{\Sigma_t} \big(\gamma^{s-6} \theta^{r-6} K^6 + \gamma^{s-2} \theta^{r-2} K^2 \|A_t\|_{\infty,[\gamma\theta>0]}^4 + \gamma^{s-2} \theta^{r-2} K^2 \|\nabla A_t\|_{\infty,[\gamma\theta>0]}^2 \\
&\qquad + \gamma^s \theta^r \|\nabla A_t\|_{\infty,[\gamma\theta>0]}^2 \|A_t\|_{\infty,[\gamma\theta>0]}^2\big) \cdot |\nabla^{m-1} A_t|^2 \, \mathrm{d}\mu_t + c\, \|T\|_2^2 \\
&\leq 2\varepsilon \int_{\Sigma_t} \gamma^s \theta^r |\nabla^{m+2} A_t|^2 \, \mathrm{d}\mu_t \\
&\quad + c \left(K^8 + \|A_t\|_{\infty,[\gamma\theta>0]}^8 + \|\nabla A_t\|_{\infty,[\gamma\theta>0]}^4\right) \int_{[\gamma\theta>0]} |\nabla^{m-2} A_t|^2 \, \mathrm{d}\mu_t + c\, \|T\|_2^2.
\end{aligned}
$$

Finally, apply this estimate with sufficiently small ε and $k = 2$ in Proposition 2.3.3 so that

$$
\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Sigma_t} \gamma^s |\nabla^m A_t|^2 \, \mathrm{d}\mu_t + \frac{1}{2} \int_{\Sigma_t} \gamma^s \theta^r |\nabla^{m+2} A_t|^2 \, \mathrm{d}\mu_t \leq \beta_{m,\gamma}
$$

for some appropriate choice for $\beta_{m,\gamma}$.

$\square$

Having $s = 6$ and $r = 20$ fixed, we will from now on omit them when describing dependence.

2.4 Short-time existence

Using Sobolev inequalities in section 3 and Gronwall's lemma, we derive a-priori estimates for L^2 norms and L^∞ norms. As a result, we can derive a short-time existence result for Willmore flow.

Convention 2.4.1. *For $j = 1, 2, 3$, let $\widehat{\gamma}_j = \sigma_j \circ \widehat{\gamma}$ and $\gamma_j = \sigma_j \circ \gamma = \widehat{\gamma}_j\big|_{\Sigma_t}$, where each σ_j is a function on $\mathbb{R}$ such that*

$$\begin{cases} \sigma_j \text{ is increasing and smooth,} \\ \sigma_j(x) = 0 \text{ for all } x \leq \frac{3-j}{3}, \\ 0 < \sigma_j(x) < 1 \text{ for all } \frac{3-j}{3} < x < \frac{4-j}{3}, \\ \sigma_j(x) = 1 \text{ for all } x \geq \frac{4-j}{3}, \text{ and} \\ |D\sigma_j(x)| \leq c \text{ and } |D^2\sigma_j(x)| \leq c \text{ for some universal constant } c. \end{cases}$$

In particular, by section 1.3, $|D\widetilde{\gamma}_j| \leq c\,K$ and $|D^2\widetilde{\gamma}_j| \leq c\,K^2$ with some universal constant c.

Lemma 2.4.2. *Let $m \geq 3$ and β_{m,γ_3} be as described in Proposition 2.3.5 but with γ replaced by γ_3. Assuming (2.1), we have*

$$\beta_{m,\gamma_3} \leq c,$$

where $c = c(n, m, K, \alpha)$ *and*

$$\alpha := \sum_{k=0}^{m-1} \|\nabla^k A_t\|_{2,[\gamma>0]}.$$

Proof. Throughout this proof, we let $c = c(n, m, K)$. First, by Lemma 2.1.5, we have

$$\|A_t\|_{\infty,[\gamma_3>0]} \leq \|A_t\|_{\infty,[\gamma_2>0]} \leq c\,\|A_t\|_{2,[\gamma>0]}^{1/2}\big(\|\nabla^2 A_t\|_{2,[\gamma>0]}^{1/2} + \|A_t\|_{2,[\gamma>0]}^{1/2}\big) \leq c\,\alpha.$$

Next, also by Lemma 2.1.5,

$$\begin{aligned}
&\|\nabla A_t\|_{\infty,[\gamma_3>0]} \\
&\leq c\,\|\nabla A_t\|_{2,[\gamma>0]}^{1/2}\big(\|\nabla^3 A_t\|_{2,[\gamma>0]}^{1/2} + \|\nabla A_t\|_{2,[\gamma>0]}^{1/2} + \|\,|A_t|^4|\nabla A_t|^2\|_{1,[\gamma_2>0]}^{1/4}\big) \\
&\leq c\,\|\nabla A_t\|_{2,[\gamma>0]}^{1/2}\big(\|\nabla^3 A_t\|_{2,[\gamma>0]}^{1/2} + \|\nabla A_t\|_{2,[\gamma>0]}^{1/2} + \|A_t\|_{\infty,[\gamma_2>0]}\|\nabla A_t\|_{2,[\gamma_2>0]}^{1/2}\big) \\
&\leq c\,(\alpha + \alpha^2).
\end{aligned}$$

Next, consider $\|\nabla^j A_t\|_{p,[\gamma_3>0]}$, where $0 \leq j \leq (m-2)$ and $3 \leq p \leq (2m+4)$ are integers. By Lemma A.2.5, we have

$$\begin{aligned}
&\|\nabla^j A_t\|_{p,[\gamma_3>0]} \\
&\leq c\,\big(\|\nabla^j A_t\|_{2,[\gamma_2>0]} + \|\nabla^{j+1} A_t\|_{2,[\gamma_2>0]} + \|\nabla^j A_t\|_{2,[\gamma_2>0]} + \|A_t\|_{\infty,[\gamma_2>0]}\|\nabla^j A_t\|_{2,[\gamma_2>0]}\big) \\
&\leq c\,(\alpha + \alpha^2).
\end{aligned}$$

By the definition of β_{m,γ_3}, we have the desired result. $\square$

Proposition 2.4.3. *For all* $k \geq 0$, *define*

$$\alpha_0(k) = \sum_{j=0}^{k} \|\nabla^j A_0\|_{2,[\gamma>0]}.$$

Assuming (2.2) for some $0 < t_0 < T$, *we have*

$$\sup_{0\le t\le t_0} \|\nabla^m A_t\|_{2,[\gamma=1]} \le c\big(n, K, m, t_0, \alpha_0(m)\big).$$

Proof. The case $m = 0$ is proved by hypothesis. We set $m > 0$ and assume that we have for each $k = 0, \dots, (m-1)$,

$$\|\nabla^k A_t\|_{2,[\gamma=1]} \le c\big(n, m, K, t_0, \alpha_0(k)\big).$$

First, we have

$$\begin{aligned}
&\int_0^t \|\theta^{r/4} A_{t'}\|^4_{\infty,[\gamma_3>0]} \,\mathrm{d}t' \\
&\le c(n,K) \int_0^t \|A_{t'}\|^2_{2,[\gamma_2>0]} \big(\|\theta^{r/2}\nabla^2 A_{t'}\|^2_{2,[\gamma_2>0]} + \|A_{t'}\|^2_{2,[\gamma_2>0]}\big)\,\mathrm{d}t' && \text{(Corollary 2.1.6)}\\
&\le c(n,K)\,\varepsilon_0 \int_0^t \int_{[\gamma_1=0]} \theta^r |\nabla^2 A_{t'}|^2 \,\mathrm{d}\mu_{t'}\,\mathrm{d}t' + \varepsilon_0\, c(n,K,t_0)\\
&\le c(n,K,t_0). && \text{(Lemma 2.3.4)}
\end{aligned}$$

In particular,

$$\beta_{2,\gamma_3} \le c(n, K, t_0).$$

By the hypothesis and Lemma 2.4.2, we also have

$$\beta_{1,\gamma_3} \le c(n, K),$$

and for all $m \ge 3$,

$$\beta_{m,\gamma_3} \le c\big(n, m, K, t_0, \alpha_0(m-1)\big).$$

In particular, for all $m \geq 1$,

$$\int_0^t \beta_{m,\gamma_3}\,\mathrm{d}t' \leq c\big(n, m, K, t_0, \alpha_0(m-1)\big).$$

Next, by replacing γ with γ_3 in Proposition 2.3.5, we have

$$\begin{aligned}
&\int_{\Sigma_t} \gamma_3^s |\nabla^m A_t|^2 \,\mathrm{d}\mu_t \\
&\leq \int_{\Sigma_t} \gamma_3^s |\nabla^m A_t|^2 \,\mathrm{d}\mu_t + \frac{1}{2}\int_0^t \int_{\Sigma_t} \gamma_3^s \theta^r |\nabla^{m+2} A_{t'}|^2 \,\mathrm{d}\mu_{t'}\,\mathrm{d}t' \\
&\leq \int_{\Sigma_0} \gamma_3^s |\nabla^m A_0|^2 \,\mathrm{d}\mu_0 + c(n,m,K) \int_0^t \left(\|\theta^{r/4} A_{t'}\|^4_{\infty,[\gamma_3>0]} \int_{\Sigma_{t'}} \gamma_3^s |\nabla^m A_{t'}|^2 \,\mathrm{d}\mu_{t'} \right) \mathrm{d}t' \\
&\quad + c(n,m,K) \sup_{0\leq t<T} \|\nabla^{m-1} A_t\|^2_{2,[\gamma>0]} \int_0^t \big(1 + \|\theta^{r/4} A_{t'}\|^4_{\infty,[\gamma_3>0]}\big)\,\mathrm{d}t' + \int_0^t \beta_{m,\gamma_3}\,\mathrm{d}t' \\
&\leq c(n,m,K) \int_0^t \left(\|\theta^{r/4} A_{t'}\|^4_{\infty,[\gamma_3>0]} \int_{\Sigma_t} \gamma_3^s |\nabla^m A_{t'}|^2 \,\mathrm{d}\mu_{t'} \right) \mathrm{d}t' + c\big(n, m, K, t_0, \alpha_0(m)\big).
\end{aligned}$$

Therefore, by Gronwall's lemma, we have

$$\begin{aligned}
&\|\nabla^m A_t\|^2_{2,[\gamma=1]} \\
&\leq \int_{\Sigma_t} \gamma_3^s |\nabla^m A_t|^2 \,\mathrm{d}\mu_t \\
&\leq c\big(n, m, K, t_0, \alpha_0(m)\big) \exp\left(c(n,m,K) \int_0^t \|\theta^r A_t\|^4_{\infty,[\gamma>0]}\,\mathrm{d}t' \right) \\
&\leq c\big(n, m, K, t_0, \alpha_0(m)\big).
\end{aligned}$$

□

Corollary 2.4.4. *Under the settings in Proposition 2.4.3, for all $m \geq 0$,*

$$\sup_{0\leq t\leq t_0} \|\nabla^m A_t\|_{\infty,[\gamma=1]} \leq c\big(n, m, K, t_0, \alpha_0(m+2)\big).$$

Proof. By Lemma 2.1.5,

$$\begin{aligned}
&\|\nabla^m A_t\|_{\infty,[\gamma=1]} \\
&\leq c(n,m,K)\, \|\nabla^m A_t\|^{1/2}_{2,[\gamma_3>0]} \big(\|\nabla^{m+2} A_t\|^{1/2}_{2,[\gamma_3>0]} + (1+\|A_t\|_{\infty,[\gamma_3>0]})\|\nabla^m A_t\|^{1/2}_{2,[\gamma_3>0]}\big) \\
&\leq c\big(n,m,K,t_0,\alpha_0(m+2)\big).
\end{aligned}$$

□

Proposition 2.4.5 (Cf. [10, Proof of Theorem 1.2]). *Let Σ be closed, and let $0 \leq \widehat{\theta} \leq 1$ be a smooth function on $\mathbb{R}^n$ such that*

$$K_2 := \sup |D\widehat{\theta}| < \infty \qquad \textit{and} \qquad \sup |D^k \widehat{\theta}| \leq c(k) K_2^k, \qquad \forall k \geq 1.$$

Then there exist $a_n > 0$ and $c_0 > 0$, both depending only on n, such that whenever $f_0 : \Sigma \to \mathbb{R}^n$ satisfies

$$\varkappa(\varrho, 0) = \sup_{x\in\mathbb{R}^n} \int_{\Sigma_0 \cap B_\varrho(x)} |A_0|^2 \,\mathrm{d}\mu_0 \leq e_0 \leq \frac{\varepsilon_0}{2a_n}$$

for some $\varrho > 0$, we can find a solution $f : \Sigma \times [0,T) \to \mathbb{R}^n$ to equation (1.2) such that

$$T \geq c_0^{-1} K^{-4},$$

where $K = \max\{2/\varrho, K_2\}$ and T is the maximum existence time. Moreover, f satisfies the following estimate for the growth of energy concentration:

$$\varkappa(\varrho, t) \leq a_n e_0 (1 + c_0 K^4 t), \qquad \forall 0 \leq t \leq c_0^{-1} K^{-4}$$

Proof. Let a_n be the number of balls of radius 1 in $\mathbb{R}^n$ required to cover a ball of radius 2. Note that without loss of generality, we can assign these balls of radius 1 to have their centers in the bigger ball.

Netx, by hypothesis, $\varkappa(\varrho, 0) \leq e_0 < \varepsilon_0$ and $\varkappa(\varrho, t)$ is continuous in t. Whenever f exists with $T > 0$, we define

$$t_0 = \max\{0 \leq t \leq T \,:\, \forall 0 \leq \tau < t,\, \varkappa(\varrho, \tau) \leq 2a_n e_0\},$$

which is always a positive number. Moreover, we either have $t_0 = T$ or $\varkappa(\varrho, t_0) = 2a_n e_0$.

For each $x \in \mathbb{R}^n$, we can find $\widehat{\gamma}$ such that

$$\chi_{B_{\varrho/2}(x)} \leq \widehat{\gamma} \leq \chi_{B_\varrho(x)}$$

as in Lemma 1.3.1 with $K_1 = 2/\varrho$, so that $K = \max\{K_1, K_2\}$. By Corollary 2.4.4,

$$\sup_{\substack{0 \leq t \leq t_0 \\ 0 \leq t < T}} \|\nabla^m A_t\|_\infty = \sup_{x \in \mathbb{R}^n} \sup_{\substack{0 \leq t \leq t_0 \\ 0 \leq t < T}} \|\nabla^m A_t\|_{\infty, \Sigma_t \cap B_{\varrho/2}(x)} \leq c(n, m, K, t_0, f_0).$$

In addition, as shown in the proof of [10, Theorem 1.2], we can show that for all $0 \leq t < t_0$,

$$|\partial_x^m f(x,t)|, |\partial_x^m \partial_t f(x,t)| \leq c\big(n, m, K, t_0, f_0\big). \tag{2.3}$$

Consider the following cases:

(i) Assume $\widehat{\theta} > 0$ and $t_0 = T$.

Since $\widehat{\theta} > 0$, f exists with $T > 0$. By the estimate above, $f(x,t)$ converges to a smooth function $f(x,T) = f_T(x)$ as $t \to T$. Therefore, by short time existence theorems (e.g. [19]), we can extend the solution to (1.2) for a longer time, a contradiction.

(ii) Assume $\widehat{\theta} > 0$ and $\varkappa(\varrho, t_0) = 2a_n e_0$.

Since $\widehat{\theta} > 0$, f exists with $T > 0$. For all $0 \le t < t_0$, by Lemma 2.3.4, we have

$$\int_{\Sigma_t \cap B_{\varrho/2}(x)} |A_t|^2 \, \mathrm{d}\mu_t \le e_0 + c\, K^4 (2a_n e_0) t = e_0(1 + c_0 K^4 t),$$

where $c = c(n)$ and we define

$$c_0 = 2a_n c,$$

which also depends only on n. Observe that we have

$$2a_n e_0 = \varkappa(\varrho, t_0) \le a_n \varkappa(\varrho/2, t_0) \le a_n e_0 (1 + c_0 K^4 t_0),$$

which implies that

$$t_0 \ge c_0^{-1} K^{-4}.$$

(iii) General case.

Let $0 < \eta < 1$ and replace $\widehat{\theta}$ with $\big(\eta + (1-\eta)\widehat{\theta}\big)$. Since (i) cannot hold, by applying case (ii), we can find

$$\widehat{f}_\eta : \Sigma \times [0, c_0^{-1} K^{-4}] \to \mathbb{R}^n$$

such that

$$\begin{cases} \partial_t \widehat{f}_\eta = -\Big(\eta + (1-\eta)\widehat{\theta} \circ \widehat{f}_\eta\Big)^r \mathbf{W}\big(\widehat{f}_\eta(\cdot, t)\big), \\ \widehat{f}_\eta\big|_{t=0} = f_0. \end{cases}$$

Moreover, we have (2.3) for all η and t without dependence on η on the right hand side. Therefore, as $\eta \to 0$, there exists a subsequential limit $\widehat{f}_\eta \to f$ such that f is defined for all

$$0 \leq t \leq c_0^{-1} K^{-4}$$

and solves (1.2). Moreover, as shown in case (ii), the concentration growth estimate holds for all $\widehat{f}_\eta$, and hence holds for their limit, f.

□

Theorem 2.4.6 (Short time existence and minimal existence time)**.** *Let $f_0 : \Sigma \to \mathbb{R}^n$ be a smooth, complete, properly immersed surface in $\mathbb{R}^n$. Then there exist $\varepsilon_1 > 0$ and $c_1 > 0$, both depending only on n, such that whenever the initial energy concentration condition*

$$\varkappa(\varrho, 0) \leq e_0 \leq \varepsilon_1$$

holds for some $\varrho > 0$ and $e_0 > 0$, there exists a solution $f : \Sigma \times [0, T) \to \mathbb{R}^n$ to the Willmore flow equation (1.1) such that $T \geq c_1^{-1} \varrho^4$. Moreover, f satisfies the following estimate for the growth of energy concentration:

$$\varkappa(\varrho, t) \leq a_n e_0 (1 + c_1 \varrho^{-4} t), \qquad \forall 0 \leq t \leq c_1^{-1} \varrho^4.$$

Proof. Define

$$\varepsilon_1 = \frac{\varepsilon_0}{2a_n} \quad \text{and} \quad c_1 = 16c_0, \text{ so that } c_0^{-1} \left(\frac{2}{\varrho}\right)^{-4} = c_1^{-1} \varrho^4.$$

Fix $K = K_2 = 2/\varrho$ and let

$$\chi_{B_{R-\varrho/2}(0)} \leq \widehat{\theta} \leq \chi_{B_R(0)}$$

as in Lemma 1.3.1, where $R > \varrho/2$.

We claim that for all R, there exists a solution

$$f_R : \Sigma \times [0, c_1^{-4}\varrho^4] \to \mathbb{R}^n$$

that solves (1.2). First, in general, we either let

$$f_R : \Sigma \times [0, T_R) \to \mathbb{R}^n$$

be a solution to (1.2) with maximum existence time $T_R > 0$, in which case we denote

$$t_R := \max\left\{0 \le t \le T_R \,:\, \sup_{0\le\tau<t} \varkappa(\varrho,\tau) \le 2a_n e_0\right\};$$

or in case such f_R doesn't exist for any $T_R > 0$, we denote $t_R = T_R = 0$ for convenience. Note that since the energy concentration doesn't change outside of $B_R(0)$, we obtain by continuity that either $t_R = T_R$ or $\varkappa(\varrho, t_R) = 2a_n e_0$.

Next, whether t_R is 0 or positive, we extend f_R to $\Sigma \times [0, t_R]$, which is already done except when $t_R = T_R > 0$. Recall that for all $0 \le t < t_R$, $\varkappa(\varrho, t) \le 2a_n e_0 \le \varepsilon_0$. As in case (i) of the proof of Proposition 2.4.5, we can derive an estimate, similarly with (2.3), and see that $f_R(\cdot, t)$ converges smoothly to some $f_R(\cdot, t_R)$ as $t \to t_R$, and that f_R can be extended as claimed.

Next, assume that $\varkappa(\varrho, t_R) < 2a_n e_0$, which implies $t_R = T_R$. We can extend the subset

$$\{f_R(x, T_R) \,:\, x \in \Sigma, f_0(x) \in B_R(0)\}$$

to a closed surface S. By Proposition 2.4.5, we can find a solution $\widehat{f}_S$ to (1.2) with initial surface S. Since $\theta > 0$ only when $f_R(x, T_R)$ agrees with S, we can extend f_R to

$$\widehat{f}_R(x,t) = \begin{cases} f_R(x,t) & \text{if } 0 \le t \le T_R, \\ \widehat{f}_S\big(f_R(x,t_R),\, t - T_R\big) & \text{if } f_0(x) \in B_R(0) \text{ and } T_R \le t < T_R + \delta, \text{ and} \\ f_0(x) & \text{if } f_0(x) \notin B_R(0) \text{ and } 0 \le t < T_R + \delta. \end{cases}$$

Despite that δ depends on S and that S depends on both R and f_R, it turns out that $\widehat{f}_R$ is another solution to (1.2) with a longer existence time than f_R, a contradiction. That is, we must have $\varkappa(\varrho, t_R) = 2a_n e_0$.

Next, by Lemma 2.3.4, we have that at $t = t_R$,

$$2a_n e_0 = \varkappa(\varrho, t_R) \le a_n \varkappa(\varrho/2, t_R) \le a_n e_0(1 + c_1 \varrho^{-4} t_R),$$

and hence

$$T_R \ge t_R \ge c_1^{-1}\varrho^4.$$

Recall that we have constructed f_R on the time interval $t \in [0, t_R]$. We will restrict it to $t \in [0, c_1^{-1}\varrho^4]$.

Finally, (2.3) holds for all R, $x \in \Sigma$, and $t \in [0, c_1^{-1}\varrho^4]$, with t_0 replaced by $c_1^{-1}\varrho^4$. Note that the right hand side doesn't depend on R. Therefore, as $R \to \infty$, there exists a subsequential limit $f_R \to f$ such that each derivative converges locally uniformly, so that f solves (1.1) and is defined on $t \in [0, c_1^{-1}\varrho^4]$. Note that Lemma 2.3.4 applies for f on $t \in [0, c_1^{-1}\varrho^4]$, and hence the concentration growth estimate follows. □

Corollary 2.4.7 (Energy inequality)**.** *If $\mathcal{W}(f_0) < \infty$ and f is the Willmore flow con-*

structed in the theorem, then we have

$$\int_{\Sigma_t} |A_t|^2 \, \mathrm{d}\mu_t + \int_0^t \int_{\Sigma_{t'}} |\mathbf{W}(f(\cdot,t'))|^2 \, \mathrm{d}\mu_{t'} \, \mathrm{d}t' \le \int_{\Sigma_0} |A_0|^2 \, \mathrm{d}\mu_0 \, .$$

Proof. Along f_R, by the definition of variational derivative, we have

$$\int_{f_R(\Sigma,t)} |A_t|^2 \, \mathrm{d}\mu_t + \int_0^t \int_{f_R(\Sigma,t')} \theta^r |\mathbf{W}(f_R(\cdot,t'))|^2 \, \mathrm{d}\mu_{t'} \, \mathrm{d}t' = \int_{\Sigma_0} |A_0|^2 \, \mathrm{d}\mu_0$$

since θ has compact support. As $R \to \infty$, both integrands on the left hand side converge pointwise to the corresponding integrands for f. Thus by Fatou's lemma,

$$\int_{\Sigma_t} |A_t|^2 \, \mathrm{d}\mu_t + \int_0^t \int_{\Sigma_{t'}} |\mathbf{W}(f(\cdot,t'))|^2 \, \mathrm{d}\mu_{t'} \, \mathrm{d}t' \le \int_{\Sigma_0} |A_0|^2 \, \mathrm{d}\mu_0 \, .$$

□

Corollary 2.4.8. *If f_0 satisfies $\mathcal{W}(f_0) < \infty$, then there exists f with $T > 0$. Moreover, if $\mathcal{W}(f_0) \le a_n \varepsilon_1 = \frac{1}{2}\varepsilon_0$, then there exists f with $T = \infty$.*

Proof. (i) For the former case, take R sufficiently large so that

$$\int_{\Sigma_0 \setminus B_R(0)} |A_0|^2 \, \mathrm{d}\mu_0 < \varepsilon_1.$$

Since f_0 is proper, we can find a finite open cover $\{B_{r_k}(x_k)\}_{k=1}^N$ of $\overline{B}_{R+1}(0)$ so that for all k,

$$\int_{\Sigma_0 \cap B_{2r_k}(x_k)} |A_0|^2 \, \mathrm{d}\mu_0 < \varepsilon_1.$$

Let $\varrho = \min\{1, r_1, \ldots, r_N\}$. As a result, for all $x \in \mathbb{R}^n$, either $x \in \overline{B}_{R+1}(0)$ so that for some $k = k(x)$ we have

$$\int_{\Sigma_0 \cap B_\varrho(x)} |A_0|^2 \, \mathrm{d}\mu_0 \le \int_{\Sigma_0 \cap B_{\varrho + r_k}(x_k)} |A_0|^2 \, \mathrm{d}\mu_0 < \varepsilon_1,$$

or $x \notin \overline{B}_{R+1}(0)$ so that

$$\int_{\Sigma_0 \cap B_\varrho(x)} |A_0|^2 \, \mathrm{d}\mu_0 \le \int_{\Sigma_0 \setminus B_R(0)} |A_0|^2 \, \mathrm{d}\mu_0 < \varepsilon_1.$$

We hence have $T \ge c_1^{-1} \varrho^4 > 0$ by Theorem 2.4.6.

(ii) For the latter case, we observe that along f_R,

$$\begin{aligned} &\int_{f_R(\Sigma,t) \cap B_R(0)} |A_t|^2 \, \mathrm{d}\mu_t \\ &= \int_{\Sigma_0 \cap B_R(0)} |A_0|^2 \, \mathrm{d}\mu_0 - \int_0^t \int_{f_R(\Sigma,t')} \theta^r |\mathbf{W}(f_R(\cdot,t'))|^2 \, \mathrm{d}\mu_{t'} \, \mathrm{d}t' \le \varepsilon_0 \end{aligned}$$

for all $0 \le t < T_R$. Corollary 2.4.4 hence applies and we have (2.3) for all $0 \le t \le t_0 = T_R$, provided $T_R < \infty$. However, $f_R(x,t)$ converges as $t \to T_R$, a contradiction against $T_R < \infty$. As a result of $T_R = \infty$, we can take a subsequential limit $f_R \to f$ with the functions being defined on $\Sigma \times [0, \infty)$.

□

2.5 Uniqueness

In this section, we consider a Willmore flow $f : \Sigma \times [0,T) \to \mathbb{R}^n$ (note that we assume continuity as $t \to 0^+$), where $T > 0$ is not necessarily the maximal existence time and

$$f(\Phi(x;t),t) = f_0(x) + \eta(x,t), \tag{2.4}$$

where η is perpendicular to $T_x\Sigma_0$, i.e., η is a 1-parameter family of sections of $N\Sigma_0$, the normal bundle of Σ_0, and for each t, Φ is an automorphism of Σ. We also let $f_0(x) = f(x, 0)$ denote the initial surface and assume $\eta\big|_{t=0} = 0$ and $\Phi(x; 0) = x$ as the initial condition. In particular, we should solve

$$\partial_t f\big|_{\Phi(x)} = -Df\big|_{\Phi(x)}(\partial_t\Phi\big|_x) + \partial_t\eta\big|_x.$$

Note that the right hand side is uniquely determined by the other side as long as $T_{\Phi(x)}\Sigma \oplus N_x\Sigma_0 = \mathbb{R}^n$.

Lemma 2.5.1. *Let $f : \Sigma \times [0, T) \to \mathbb{R}^n$ be a family of surfaces. If*

$$M = \sup_{x,t} \max_{\substack{v \in T_x\Sigma \\ |v|_g = 1}} |\partial_t D_v f| < \infty,$$

then there exists $t_1 > 0$, only depending on M, such that every term in expression (2.4) can be determined for all $0 \le t < \min(t_1, T)$.

Proof. Consider any unit tangent vector $u_0 \in T_x\Sigma_0$. Let

$$u = f_*(u_0) = v + w, \qquad \text{where } v \in T_x\Sigma_0 \text{ and } w \in N_x\Sigma_0.$$

In particular, $|v| = 1$ and $|w| = 0$ when $t = 0$. By hypothesis, we have

$$\sqrt{|\partial_t v|^2 + |\partial_t w|^2} = |\partial_t u| \le M|u|.$$

In particular,

$$|v| \ge 2 - e^{Mt} > 0 \qquad \text{when } t < \frac{1}{M}\log 2.$$

Therefore, $T_x\Sigma \oplus N_x\Sigma_0 = \mathbb{R}^n$, so that we can define π, the projection map from $\mathbb{R}^n$ onto $T_x\Sigma$

along $N_x\Sigma_0$. As a result, we can determine

$$\begin{cases} \partial_t \Phi = -\pi(\partial_t f), \\ \partial_t \eta = (I - \pi)(\partial_t f). \end{cases}$$

Moreover, for all u_0,

$$\begin{cases} \quad |D_{u_0}\Phi| = |u_0| = 1, \\ \quad |\partial_t D_u \Phi| \leq \|\pi\|_* |\partial_t D_{\Phi_* u} f| \leq M e^{Mt} \, \|D\Phi\|_*, \end{cases}$$

so that in particular,

$$2 - e^{e^{Mt}-1} \leq |D_u\Phi| \leq e^{e^{Mt}-1}.$$

In summary, η and Φ in (2.4) are well-determined within the interval $0 \leq t < \min(t_1, T)$, where

$$t_1 = \frac{1}{M}\log(1 + \log 2) < \frac{1}{M}\log 2.$$

□

If we assume further that $f(\Phi(x), t)$ solves the Willmore flow equation (1.1), then η solves the equation

$$\begin{cases} \partial_t \eta = -\mathbf{W}_N(\eta), \\ \eta\big|_{t=0} = 0; \end{cases} \tag{2.5}$$

where

$$\mathbf{W}_N(\eta)\big|_x - \mathbf{W}(f)\big|_x = Df\big(\partial_t\Phi\big|_{\Phi^{-1}(x)}\big) \in T_x\Sigma.$$

We will also need the following volume estimate.

Lemma 2.5.2. *There exists $\varepsilon_2 > 0$ such that $f = f_0 + \eta$ defines an immersed surface whenever*

$$\big\|\, |\eta|_{g_0}|A_0|_{g_0} + |\nabla\eta|^2_{g_0}\big\|_\infty \leq \varepsilon_2, \tag{2.6}$$

where g_0 and A_0 denote the metric and the second fundamental form of Σ_0, respectively. In fact, we have

$$|g - g_0|_{g_0} \leq b,$$

where $0 < b < \frac{1}{2}$ only depends on ε_2, and furthermore,

$$\frac{\det(g)}{\det(g_0)} \geq 1 - 2b > 0.$$

Proof. We can obtain

$$\partial_i f = \partial_i f_0 - (g_0)^{k\ell}\langle\eta, (A_0)_{ik}\rangle\partial_\ell f_0 + \nabla_i\eta$$

and

$$g_{ij} = (g_0)_{ij} - 2\langle\eta, (A_0)_{ij}\rangle + (g_0)^{k\ell}\langle\eta, (A_0)_{ik}\rangle\langle\eta, (A_0)_{j\ell}\rangle + \langle\nabla_i\eta, \nabla_j\eta\rangle,$$

so that in particular,

$$|g - g_0| \leq 2|\eta|\,|A_0| + |\eta|^2|A_0|^2 + |\nabla\eta|^2.$$

As a result, we can choose any

$$0 < \varepsilon_2 < \frac{\sqrt{2}-1}{2},$$

so that

$$|g - g_0| \leq b = 2\big(\varepsilon_2 + \varepsilon_2^2\big) < \frac{1}{2}$$

whenever inequality (2.6) holds. Moreover, we have

$$\frac{\det(g)}{\det(g_0)} \geq (1-b)^2 - b^2 = 1 - 2b > 0,$$

so that $f(\Sigma)$ is an immersed surface. □

Convention 2.5.3. *As tensors on Σ_0:*

- $\delta T = T - \big(T\big|_{t=0}\big)$ *as a tensor on Σ_0, where all vectors are considered as $\mathbb{R}^n$-valued so that we can pull-back from Σ to Σ_0 via the map $f_0(x) + \eta(x,t)$. When expressing δT in terms of other tensors on Σ_0, the subscript 0 that denotes $t = 0$ may be dropped.*

- *Given tensors $T_1, \ldots, T_k$ on Σ_0 and non-negative integers $\alpha_1, \ldots, \alpha_k$, the notation*

$$\widehat{P}_0\big(\nabla^{\alpha_1}T_1, \ldots, \nabla^{\alpha_k}T_k\big)$$

denotes a polynomial defined by operators $+$ and $*$, with coefficients bounded by some $c(n)$, and with variables being $\nabla^i T_j$, running through $1 \leq j \leq k$ and $0 \leq i \leq \alpha_j$.

- *When T_k is not A_0,*

$$\widehat{P}_d\big(\nabla^{\alpha_1}T_1,\dots,\nabla^{\alpha_k}T_k\big)$$

denotes a polynomial that is of the form $\widehat{P}_0\big(\nabla^{\alpha_1}T_1,\dots,\nabla^{\alpha_k}T_k\big)$ while also every term has at least degree d.

- *When T_k is A_0,*

$$\widehat{P}_d\big(\nabla^{\alpha_1}T_1,\dots,\nabla^{\alpha_{k-1}}T_{k-1},\nabla^{\alpha_k}A_0\big)$$

denotes a polynomial that is of the form $\widehat{P}_0\big(\nabla^{\alpha_1}T_1,\dots,\nabla^{\alpha_k}A_0\big)$ while also every term has at least degree d in $\nabla^i T_j$ $(1 \le j \le k-1)$ together.

We will compute in normal coordinates for Σ_0. As mentioned in the proof of Lemma 2.5.2, we already know

$$\delta(g_{\mathbb{R}^n}) = 0,$$
$$\delta(\partial f) = \eta * A_0 + \nabla\eta = \widehat{P}_1(\nabla\eta, A_0), \text{ and}$$
$$\delta(g) = \eta * A_0 + \eta * \eta * A_0 * A_0 + \nabla\eta * \nabla\eta = \widehat{P}_1(\nabla\eta, A_0).$$

We should also compute

$$\begin{aligned}\nabla_k\delta(g_{ij}) &= \nabla_k\big(-2\langle\eta,(A_0)_{ij}\rangle + g^{k\ell}\langle\eta,(A_0)_{ik}\rangle\langle\eta,(A_0)_{j\ell}\rangle + \langle\nabla_i\eta,\nabla_j\eta\rangle\big)\\ &= \widehat{P}_1(\nabla^2\eta, \nabla A_0).\end{aligned}$$

As a result,

$$\delta(\partial g) = \partial\delta(g) = \nabla\delta(g) = \widehat{P}_1(\nabla^2\eta, \nabla A_0),$$

so that

$$\delta(\Gamma) = \delta(\partial g) + \delta(g^{-1}) * \delta(\partial g) = \widehat{P}_1\big(\delta(g^{-1}), \nabla^2\eta, \nabla A_0\big),$$

and also,

$$\nabla_k \delta(g^{ij}) = \partial_k \delta(g^{ij}) = \delta(\partial_k g^{ij}) = \delta(-g^{ip}g^{jq}\partial_k g_{pq}) = \widehat{P}_1\big(\delta(g^{-1}), \nabla^2\eta, \nabla A_0\big).$$

Proposition 2.5.4. *Let $h = (h^{ij})$ be given by*

$$\delta^i_k = \big((g_0)^{ij} + h^{ij}\big)\langle \partial_j f, \partial_k f_0\rangle = \big((g_0)^{ij} + h^{ij}\big)\big((g_0)_{jk} - \langle \eta, A_{jk}\rangle\big) \qquad \forall i, k.$$

Then for η that satisfies condition (2.6),

$$\begin{aligned}\mathbf{W}_N = \mathbf{W}(f_0) &+ \triangle^4\eta + \nabla^3\eta * A_0 + \nabla^2\eta * \widehat{P}_0(\nabla A_0) + \nabla\eta * \widehat{P}_0(\nabla^2 A_0) + \eta * \widehat{P}_0(\nabla^3 A_0)\\ &+ \nabla^4\eta * \widehat{P}_1\big(\delta(g^{-1}), h, \nabla\eta, A_0\big) + \widehat{P}_2\big(\delta(g^{-1}), h, \nabla^3\eta, \nabla^3 A_0\big).\end{aligned}$$

Proof. First assume normal coordinate on Σ_0 so that $(A_0)_{ij} = \partial_i\partial_j f_0$, and hence

$$\begin{aligned}\partial_i\partial_j f &= \partial_i\big(\partial_j f_0 + \nabla_j\eta - (g_0)^{k\ell}\langle \eta, (A_0)_{jk}\rangle \partial_\ell f_0\big)\\ &= \partial_i\partial_j f_0 + \nabla^2_{ij}\eta - (g_0)^{k\ell}\langle \nabla_j\eta, (A_0)_{ik}\rangle\partial_\ell f_0 - (g_0)^{k\ell}\langle \nabla_i\eta, (A_0)_{jk}\rangle\partial_\ell f_0\\ &\quad - (g_0)^{k\ell}\langle \eta, \nabla_i(A_0)_{jk}\rangle\partial_\ell f_0 - (g_0)^{k\ell}\langle \eta, (A_0)_{jk}\rangle (A_0)_{i\ell}\\ &= \big[(A_0)_{ij} + \nabla^2_{ij}\eta + \eta * A_0 * A_0\big] + \big[\nabla\eta * A_0 + \eta * \nabla A_0\big]\end{aligned}$$

where the first bracket is normal to Σ_0, and the second bracket is tangent to Σ_0. Next, we have

$$\delta(A_{ij}) = \delta\big(\partial_i\partial_j f - g^{pq}\langle\partial_i\partial_j f, \partial_p f\rangle\partial_q f\big)$$
$$= \delta(\partial_i\partial_j f) - \big(g_0^{pq} + \delta(g^{pq})\big)\big(\langle(A_0)_{ij} + \nabla^2_{ij}\eta - \eta * A_0 * A_0, \nabla_p\eta\rangle$$
$$+ \langle\nabla\eta * A_0 + \eta * \nabla A_0, \partial_p f_0 + \eta * A_0\rangle\big)\big(\partial_q f_0 + \eta * A_0 + \nabla\eta\big)$$

$$= \nabla^2_{ij}\eta + \eta * A_0 * A_0 + \nabla\eta * A_0 + \eta * \nabla A_0$$
$$+ \nabla^2\eta * \widehat{P}_1\big(\delta(g^{-1}), \nabla\eta, A_0\big) + \widehat{P}_2(\delta(g^{-1}), \nabla\eta, \nabla A_0\big).$$

Taking contractions and derivatives, we have

$$\delta(\triangle H) = \triangle^2\eta + \nabla^3\eta * A_0 + \nabla^2\eta * (\nabla A_0 + A_0 * A_0) + \nabla\eta * (\nabla^2 A_0 + \nabla A_0 * A_0)$$
$$+ \eta * (\nabla^3 A_0 + \nabla^2 A_0 * A_0 + \nabla A_0 * \nabla A_0)$$
$$+ \nabla^4\eta * \widehat{P}_1\big(\delta(g^{-1}), \nabla\eta, A_0\big) + \widehat{P}_2\big(\delta(g^{-1}), \nabla^3\eta, \nabla^3 A_0\big),$$

and also

$$\delta\big(Q(A^0)H\big) = \nabla^2\eta * A_0 * A_0 + \nabla\eta * A_0 * A_0 * A_0 + \eta * (\nabla A_0 * A_0 * A_0$$
$$+ A_0 * A_0 * A_0 * A_0) + \widehat{P}_2\big(\delta(g^{-1}), \nabla^2\eta, \nabla A_0\big).$$

Therefore,

$$\delta(\mathbf{W}) = \triangle^4\eta + \nabla^3\eta * A_0 + \nabla^2\eta * (\nabla A_0 + A_0 * A_0) + \nabla\eta * (\nabla^2 A_0 + \nabla A_0 * A_0$$
$$+ A_0 * A_0 * A_0) + \eta * (\nabla^3 A_0 + \nabla^2 A_0 * A_0 + \nabla A_0 * \nabla A_0 + \nabla A_0 * A_0 * A_0$$
$$+ A_0 * A_0 * A_0 * A_0)$$
$$+ \nabla^4\eta * \widehat{P}_1\big(\delta(g^{-1}), \nabla\eta, A_0\big) + \widehat{P}_2\big(\delta(g^{-1}), \nabla^3\eta, \nabla^3 A_0\big).$$

Finally,

$$\begin{aligned}\delta(\mathbf{W}_N) &= \delta(\mathbf{W}) - \langle \mathbf{W}, \partial_i f_0\rangle\big((g_0)^{ij} + h^{ij}\big)\partial_j f \\ &= \triangle^4\eta + \nabla^3\eta * A_0 + \nabla^2\eta * (\nabla A_0 + A_0 * A_0) + \nabla\eta * (\nabla^2 A_0 + \nabla A_0 * A_0 \\ &\qquad + A_0 * A_0 * A_0) + \eta * (\nabla^3 A_0 + \nabla^2 A_0 * A_0 + \nabla A_0 * \nabla A_0 + \nabla A_0 * A_0 * A_0 \\ &\qquad + A_0 * A_0 * A_0 * A_0) \\ &\quad + \nabla^4\eta * \widehat{P}_1\big(\delta(g^{-1}), h, \nabla\eta, A_0\big) + \widehat{P}_2\big(\delta(g^{-1}), h, \nabla^3\eta, \nabla^3 A_0\big).\end{aligned}$$

□

Definition 2.5.5. *If η_1, η_2 are normal vector fields that satisfy condition (2.6), we denote*

$$G^{ij}(\eta_1,\eta_2) = g^{ij}\big|_{\eta=\eta_2} - g^{ij}\big|_{\eta=\eta_1} = \delta(g^{-1})\big|_{\eta=\eta_2} - \delta(g^{-1})\big|_{\eta=\eta_1}.$$

Next, by distributive law, we can derive the following.

Corollary 2.5.6. *If η_1, η_2 satisfy (2.6) and are two solutions for equation (2.5), then we can consider $\widetilde{\eta} = \eta_2 - \eta_1$, which is a normal vector field on Σ_0 that satisfies*

$$\begin{cases}\partial_t\widetilde{\eta} + \triangle^2\widetilde{\eta} = \sum_{k=0}^4 \nabla^k\widetilde{\eta} * Q_k + G(\eta_1,\eta_2) * S, \\ \widetilde{\eta}\big|_{t=0} = 0,\end{cases}$$

where Q_4 is of the form

$$\widehat{P}_1\big(\delta(g^{-1})|_{\eta=\eta_1}, \delta(g^{-1})|_{\eta=\eta_2}, \nabla\eta_1, \nabla\eta_2\big),$$

Q_3 is of the form

$$\widehat{P}_0\big(\delta(g^{-1})|_{\eta=\eta_1}, \delta(g^{-1})|_{\eta=\eta_2}, \nabla\eta_1, \nabla\eta_2, \nabla A_0\big),$$

Q_2 is of the form

$$\widehat{P}_1\big(\delta(g^{-1})|_{\eta=\eta_1}, \delta(g^{-1})|_{\eta=\eta_2}, \nabla^2\eta_1, \nabla^2\eta_2, \nabla^3 A_0\big),$$

and Q_1, Q_0, and S are of the form

$$\begin{aligned}&\nabla\widehat{P}_1\big(\delta(g^{-1})|_{\eta=\eta_1}, \delta(g^{-1})|_{\eta=\eta_2}, \nabla^3\eta_1, \nabla^3\eta_2, \nabla A_0\big)\\ &+\widehat{P}_0\big(\delta(g^{-1})|_{\eta=\eta_1}, \delta(g^{-1})|_{\eta=\eta_2}, \nabla^3\eta_1, \nabla^3\eta_2, \nabla^3 A_0\big).\end{aligned}$$

Lemma 2.5.7. *For all η_1, η_2 satisfying (2.6),*

$$|G(\eta_1, \eta_2)| \leq c\left(|\widetilde{\eta}|\,|A_0| + |\nabla\widetilde{\eta}|\right),$$

where $\widetilde{\eta} = \eta_2 - \eta_1$ and $c = c(\varepsilon_2)$. Also,

$$|\nabla G(\eta_1, \eta_2)| \leq c\left(|\widetilde{\eta}| + |\nabla\widetilde{\eta}| + |\nabla^2\widetilde{\eta}|\right),$$

where

$$c = c\big(\varepsilon_2, \|A_0\|_{C^1}, \|\eta_1\|_{C^2}, \|\eta_2\|_{C^2}\big).$$

Proof. Letting $\eta = t\eta_2 + (1-t)\eta_1$, we have

$$\frac{\partial}{\partial t} g_{ij} = \widetilde{\eta} * A_0 + \widetilde{\eta} * \eta * A_0 * A_0 + \nabla\widetilde{\eta} * \nabla\eta,$$

and hence

$$\frac{\partial}{\partial t}g^{ij}$$
$$= \frac{(-1)^{i+j}}{\big(g_{11}g_{22}-g_{12}^2\big)^2}\left(\big(g_{11}g_{22}-g_{12}^2\big)\frac{\partial}{\partial t}g_{\underline{i}\underline{j}} - g_{\underline{i}\underline{j}}g_{22}\frac{\partial}{\partial t}g_{11} - g_{\underline{i}\underline{j}}g_{11}\frac{\partial}{\partial t}g_{22} + 2g_{\underline{i}\underline{j}}g_{12}\frac{\partial}{\partial t}g_{12}\right),$$

where $\underline{i} = 3-i$, etc. In particular,

$$\begin{aligned}\left|\frac{\partial}{\partial t}g^{ij}\right| &\le c\left((1-2b)^{-1}+(1-2b)^{-2}\right)\big(1+b^2\big)\\ &\qquad \cdot\big(|\widetilde{\eta}|\,|A_0| + |\widetilde{\eta}|\,|\eta|\,|A_0|^2 + |\nabla\widetilde{\eta}|\,|\nabla\eta|\big)\\ &\le \frac{c\,(1+\varepsilon_2)(1+b^2)}{1-2b}\big(|\widetilde{\eta}|\,|A_0| + |\nabla\widetilde{\eta}|\big).\end{aligned}$$

As a result, we have

$$\big|G^{ij}(\eta_1,\eta_2)\big| \le \sup_{0\le t\le 1}\left|\frac{\partial}{\partial t}g^{ij}\right| \le \frac{c\,(1+\varepsilon_2)(1+b^2)}{1-2b}\big(|\widetilde{\eta}|\,|A_0| + |\nabla\widetilde{\eta}|\big).$$

Next, as mentioned earlier, we have

$$\nabla\delta(g^{-1}) = \widehat{P}_1\big(\delta(g^{-1}), \nabla^2\eta, \nabla A_0\big).$$

Therefore,

$$\begin{aligned}\nabla G(\eta_1,\eta_2) = G(\eta_1,\eta_2) &* \widehat{P}_0\big(\delta(g^{-1})|_{\eta=\eta_1}, \delta(g^{-1})|_{\eta=\eta_2}, \nabla^2\eta_1, \nabla^2\eta_2, \nabla A_0\big)\\ &+ \widetilde{\eta} * \widehat{P}_0\big(\delta(g^{-1})|_{\eta=\eta_1}, \delta(g^{-1})|_{\eta=\eta_2}, \nabla^2\eta_1, \nabla^2\eta_2, \nabla A_0\big)\\ &+ \nabla\widetilde{\eta} * \widehat{P}_0\big(\delta(g^{-1})|_{\eta=\eta_1}, \delta(g^{-1})|_{\eta=\eta_2}, \nabla^2\eta_1, \nabla^2\eta_2, \nabla A_0\big)\\ &+ \nabla^2\widetilde{\eta} * \widehat{P}_0\big(\delta(g^{-1})|_{\eta=\eta_1}, \delta(g^{-1})|_{\eta=\eta_2}, \nabla^2\eta_1, \nabla^2\eta_2, \nabla A_0\big),\end{aligned}$$

and hence

$$|\nabla G(\eta_1, \eta_2)| \le c\left(|\widetilde{\eta}| + |\nabla\widetilde{\eta}| + |\nabla^2\widetilde{\eta}|\right),$$

where

$$c = c\big(\varepsilon_2, \|A_0\|_{C^1}, \|\eta_1\|_{C^2}, \|\eta_2\|_{C^2}\big).$$

□

PROPOSITION 2.5.8. *Let $f_0 : \Sigma \looparrowright \mathbb{R}^n$ be a complete, proper, immersed surface with $\|A_0\|_{C^3} < \infty$ and*

$$\liminf_{R\to\infty} R^{-4}\mu_0\big(B_R(0)\big) = 0.$$

If $\eta_i : \Sigma \times [0,T) \to \mathbb{R}^n$ solves the Willmore flow equation (2.5) while satisfying (2.6) and

$$\sup_{0\le t<T} \|\eta_i\|_{C^3} < \infty, \qquad \forall i = 1, 2,$$

then $\eta_1 = \eta_2$.

Proof. For any $R > 0$, we can find

$$\chi_{B_R(0)} \le \widehat{\gamma} \le \chi_{(A_0)_{2R}(0)}$$

as in Lemma 1.3.1 with $K = R^{-1}$ and γ to be the restriction of $\widehat{\gamma}$ on Σ. As in Lemma 2.3.2

but with $\theta = 1$ and $V = 0$, we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Sigma \gamma^s|\widetilde{\eta}|^2\,\mathrm{d}\mu + \frac{7}{8}\int_\Sigma \gamma^s|\nabla^2\widetilde{\eta}|^2\,\mathrm{d}\mu \\
&\leq c\int_\Sigma \gamma^s\widetilde{\eta} * \big(Y + |\nabla A_0|^2\widetilde{\eta} + |A_0|^4\widetilde{\eta}\big)\,\mathrm{d}\mu + c\,R^{-4}\int_\Sigma \gamma^{s-4}|\widetilde{\eta}|^2\,\mathrm{d}\mu\,, \qquad (2.7)
\end{aligned}$$

where c is universal and

$$Y = \sum_{k=0}^{4}\nabla^k\widetilde{\eta} * Q_k + G(\eta_1,\eta_2) * S$$

by Corollary 2.5.6. Next, we have

$$\begin{aligned}
\int_\Sigma \gamma^s\widetilde{\eta} * Y\,\mathrm{d}\mu \leq &\int_\Sigma \nabla^2\big(\gamma^s\widetilde{\eta} * Q_4\big) * \nabla^2\widetilde{\eta}\,\mathrm{d}\mu + \int_\Sigma \nabla\big(\gamma^s\widetilde{\eta} * Q_3\big) * \nabla^2\widetilde{\eta}\,\mathrm{d}\mu \\
&+ \int_\Sigma \gamma^s\widetilde{\eta} * \nabla^2\widetilde{\eta} * Q_2\,\mathrm{d}\mu + \int_\Sigma \big(\gamma^s\widetilde{\eta} * \nabla\widetilde{\eta}\big) * Q_1\,\mathrm{d}\mu \\
&+ \int_\Sigma \big(\gamma^s\widetilde{\eta} * \widetilde{\eta}\big) * Q_0\,\mathrm{d}\mu + \int_\Sigma \big(\gamma^s\widetilde{\eta} * G(\eta_1,\eta_2)\big) * S\,\mathrm{d}\mu\,.
\end{aligned}$$

Thus, using integration by parts, Cauchy-Schwarz inequality, Lemma 2.5.7, and Proposition A.1.3,

$$\int_\Sigma \gamma^s\widetilde{\eta} * Y\,\mathrm{d}\mu \leq \left(\frac{1}{16} + \|Q_4\|_\infty\right)\int_\Sigma \gamma^s|\nabla^2\widetilde{\eta}|^2\,\mathrm{d}\mu + c\int_\Sigma(\gamma^s + \gamma^{s-4}R^{-4})|\widetilde{\eta}|^2\,\mathrm{d}\mu\,,$$

where

$$c = c\big(\varepsilon_2, \|A_0\|_{C^3}, \|\eta_1\|_{C^3}, \|\eta_2\|_{C^3}\big).$$

Moreover, we can choose ε_2 to be sufficiently small so that $\|Q_4\|_\infty \leq \frac{1}{16}$, and hence in inequality (2.7), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Sigma \gamma^s|\widetilde{\eta}|^2\,\mathrm{d}\mu + \frac{3}{4}\int_\Sigma \gamma^s|\nabla^2\widetilde{\eta}|^2\,\mathrm{d}\mu \leq c\int_\Sigma \gamma^s|\widetilde{\eta}|^2\,\mathrm{d}\mu + c\,R^{-4}\int_\Sigma \gamma^{s-4}|\widetilde{\eta}|^2\,\mathrm{d}\mu\,.$$

In particular,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Sigma \gamma^s|\widetilde{\eta}|^2\,\mathrm{d}\mu \le c\int_\Sigma \gamma^s|\widetilde{\eta}|^2\,\mathrm{d}\mu + c\,R^{-4}\mu_0\big(B_{2R}(0)\big).$$

Since $\widetilde{\eta}=0$ at $t=0$, by Gronwall's lemma,

$$\int_\Sigma \gamma^s|\widetilde{\eta}|^2\,\mathrm{d}\mu \le c\big(e^{ct}-1\big)\,R^{-4}\mu_0\big(B_{2R}(0)\big).$$

Fixing t, we take $R\to\infty$ to conclude

$$\int_\Sigma |\widetilde{\eta}|^2\,\mathrm{d}\mu = \lim_{R\to\infty}\int_\Sigma \gamma^s|\widetilde{\eta}|^2\,\mathrm{d}\mu = 0,$$

and hence $\widetilde{\eta}=0$ for all time, i.e., $\eta_1=\eta_2$. □

Theorem 2.5.9. *Assume that $f_0:\Sigma\to\mathbb{R}^n$ is a smooth, complete, properly immersed surface in $\mathbb{R}^n$ such that*

$$\liminf_{R\to\infty} R^{-4}\mu_0\big(B_R(0)\big)=0, \text{ and}$$

for some $\varrho>0$ and $M>0$,

$$\begin{cases} \varkappa(\varrho,0)\le\varepsilon_1 \\ \displaystyle\int_{\Sigma_0\cap B_\varrho(x)} |\nabla^k A_0|^2\,\mathrm{d}\mu_0 \le M, \quad \forall x\in\mathbb{R}^n \text{ and } k=1,\ldots,5, \end{cases}$$

where ε_1 is as given in Theorem 2.4.6. Let $f=f_i:\Sigma\times[0,T)\to\mathbb{R}^n$, where $i=1,2$, be two solutions to the Willmore flow equation (1.1), then there exists $t_3>0$, only depending on n, ϱ, and M, such that $f_1=f_2$ for all $0\le t<\widehat{T}=\min(t_3,T)$.

Proof. Either let f denote f_1 or f_2. As shown in the proof of Theorem 2.4.6, we have

$$\varkappa(\varrho/2, t) \leq \varepsilon_0, \qquad \forall 0 \leq t \leq \min(t_0, T),$$

where t_0 only depends on n and ϱ. Therefore, we can apply Corollary 2.4.4 and obtain that

$$\sup_{0 \leq t < \min(t_0, T)} \|\nabla^k A\|_\infty \leq c(t_0, \varrho, M), \qquad \forall k = 0, \ldots, 3.$$

In particular, for all $0 \leq t \leq \min(t_0, T)$,

$$\sup_{x \in \Sigma} \max_{\substack{v \in T_x\Sigma \\ |v|=1}} |\partial_t D_v f| = \|\nabla \mathbf{W}(f)\|_\infty \leq \|\nabla^3 A\|_\infty + c\,\|\nabla A\|_\infty \|A\|_\infty^2 \leq c(n, \varrho, M).$$

Thus by Lemma 2.5.1, there exists t_1 such that $f = f_0 + \eta$ can be determined for all $0 \leq t < \min(t_1, T)$, where $0 < t_1 \leq t_0$ only depends on n, ϱ, and M.

Next, we claim that

$$\begin{cases} \sup_{0 \leq t < t_1} \|\partial_t \nabla^k \eta\|_\infty < \infty & \forall k = 0, 1, \text{ and} \\ \sup_{0 \leq t < t_1} \|\nabla^k \eta\|_\infty < \infty, & \forall k = 2, 3. \end{cases}$$

If the claim holds, since $\eta = 0$ at $t = 0$, there exists t_2 such that (2.6) holds for all $0 \leq t < \min(t_2, T)$, where $0 < t_2 \leq t_1$. Moreover, we can apply Proposition 2.5.8 and conclude that $f_1 = f_2$.

To prove the first part of the claim, we have

$$\|\eta\|_\infty \leq \|I - \pi\|_* \|\mathbf{W}(f)\|_\infty \qquad \text{and} \qquad \|\nabla \eta\|_\infty \leq \|I - \pi\|_* \|\nabla \mathbf{W}(f)\|_\infty,$$

so that the required upper bound can be found, where π is as defined in Lemma 2.5.1. To prove the second part of the claim, recall that

$$\begin{aligned}\delta(A_{ij}) &= \nabla^2_{ij}\eta + \nabla^2\eta * \widehat{P}_1\big(\delta(g), \nabla\eta\big) + \widehat{P}_1\big(\delta(g), \nabla\eta, \nabla A_0\big)\\ &= \nabla^2_{k\ell}\eta\big[\delta^k_i\delta^\ell_j + \widehat{P}_1\big(\delta(g), \nabla\eta\big)\big] + \widehat{P}_1\big(\delta(g), \nabla\eta, \nabla A_0\big).\end{aligned}$$

Therefore, if $0 < t_3 \leq t_2$ is chosen to be sufficiently small, then for all $0 \leq t < \min(t_3, T)$,

$$\delta^k_i\delta^\ell_j + \widehat{P}_1\big(\delta(g), \nabla\eta\big),$$

as an 4×4 matrix with rows indexed by (i, j) and columns indexed by (k, ℓ), is invertible with determinant at least c^{-1}, where $c = c(n, \varrho, M) > 0$. As a result,

$$\|\nabla^2\eta\|_\infty \leq c.$$

A similar discussion regarding $\delta(\nabla A)$ shows

$$\|\nabla^3\eta\|_\infty \leq c.$$

As mentioned, these conditions together prove the theorem. □

2.6 Type-I singularity

In this section, we give an analogous definition of type-I singularity for Willmore flows, and show that it does not exist for all sufficiently small thresholds.

Let $f : \Sigma \times [0, T) \to \mathbb{R}^n$ be a Willmore flow with maximal existence time $T \in (0, \infty]$.

Convention 2.6.1 (cf. [13, Section 4]). *Given $e > 0$, we denote the time when energy in* *a r-ball exceeds e by the formula:*

$$t(r,e) = \inf\{t \in [0,T) : \varkappa(r,t) > e\}.$$

Note that by short time existence, for all $0 < e \le \varepsilon_1$, $T - t(r,e) \ge c_1^{-1} r^4$. In particular,

$$\liminf_{r\to 0^+} r^{-1}\big(T - t(r,e)\big)^{1/4} \ge c_1^{-1/4} > 0.$$

Definition 2.6.2. *Given $0 < e \le \varepsilon_1$, we say f has a type-I singularity with respect to energy threshold e if*

$$\begin{cases} t(r,e) < T \text{ for all } r > 0, \text{ and} \\ \limsup\limits_{r\to 0^+} \left[r^{-1}\big(T - t(r,e)\big)^{1/4}\right] < \infty, \end{cases}$$

which in particular implies $T < \infty$.

We will also consider the following definition for convergence of surfaces, as described in the compactness theorem [13, Theorem 4.2], which is a generalization for [15, Compactness theorem].

Definition 2.6.3. *Let $f_i : \Sigma_i \to \mathbb{R}^n$ and $\widehat{f} : \widehat{\Sigma} \to \mathbb{R}^n$ be properly immersed surfaces without boundary. We say that Σ_i converges to $\widehat{\Sigma}$ (locally smoothly up to diffeomorphisms) if we can find numbers R_i and functions φ_i, u_i that satisfy the following.*

- *R_i is an increasing sequence of positive numbers such that $\lim\limits_{i\to\infty} R_i = \infty$;*
- *For all i, $\varphi_i : \widehat{f}(\widehat{\Sigma}) \cap B_{R_i}(0) \xrightarrow{\sim} U_i \subset \Sigma_i$ is a diffeomorphism, and u_i is a smooth normal vector field over $\widehat{f}(\widehat{\Sigma}) \cap B_{R_i}(0)$ such that $f_i \circ \varphi_i = \widehat{f} + u_i$;*
- *For all $R > 0$, there exists $i_0 = i_0(R)$ such that for all $i \ge i_0$, $f_i(\Sigma_i) \cap B_R(0) \subset f_i(U_i)$;*

and

- *For each* $k \geq 0$, $\lim_{i\to\infty} \|\nabla^k u_i\|_{\infty,\widehat{f}(\widehat{\Sigma})\cap B_{R_i}(0)} = 0$.

Lemma 2.6.4. *Let* Σ *be a closed surface. For all* $0 < e_1 < e_2 \leq \varepsilon_1$, f *cannot have a type-I singularity with respect to both energy thresholds* e_1 *and* e_2.

Proof. Pick $r_j \searrow 0$ such that

$$\lambda_i := \lim_{j\to\infty} r_j^{-1}(T - t_j^{(i)})^{1/4}$$

converges for both $i = 1, 2$, where

$$t_j^{(i)} := t(r_j, e_i).$$

By definition, we have $t_j^{(1)} < t_j^{(2)} < T$ for all j. In particular, $\lambda_1 \leq \lambda_2$. Since each Σ_t is compact, we can find $x_j \in \mathbb{R}^n$ such that

$$\int_{\Sigma_t\cap B_{r_j}(x_j)} |A_{t_j^{(2)}}|^2 \, \mathrm{d}\mu_{t_j^{(2)}} = \varkappa(r_j, t_j^{(2)}) = e_2.$$

We consider a sequence rescaled Willmore flows

$$f_j : \Sigma \times [-r_j^{-4} t_j^{(2)}, r_j^{-4}(T - t_j^{(2)}))$$

by assigning

$$f_j(p, t) = r_j^{-1}\big(f(p, t_j^{(2)} + r_j^4 t) - x_j\big).$$

While $t = t_j^2$ on the original flow corresponds to $t = 0$ on the rescaled flow, $t = t_j^1$ corresponds to

$$t = r_j^{-4}(t_j^{(1)} - t_j^{(2)}),$$

which converges to $-(\lambda_2^4 - \lambda_1^4)$.

As shown in [13, pp. 432–433], by passing to a subsequence, f_j converges to some $\widehat{f} : \widehat{\Sigma} \to \mathbb{R}^n$ for all $-(\lambda_2^4 - \lambda_1^4) \leq t \leq c_1^{-1}\varrho^4$ in the sense as in Definition 2.6.3.

Observe that on $\widehat{\Sigma}$,

$$\int_{\widehat{\Sigma}\cap B_1(0)} |A_{\widehat{\Sigma}}|^2 \, \mathrm{d}\mu_* = \lim_{j\to\infty} \int_{\Sigma_j\cap B_1(0)} |A_{j,0}|^2 \, \mathrm{d}\mu_{j,0} = \lim_{j\to\infty} \int_{\Sigma\cap B_{r_j}(x_j)} |A_{t_j^{(2)}}|^2 \, \mathrm{d}\mu_{t_j^{(2)}} = e_2.$$

Fix arbitrary $\tau < -(\lambda_2^4 - \lambda_1^4)$ so that

$$0 < t_j^{(2)} + r_j^4\tau < t_j^{(1)}$$

for all sufficiently large j. By definition, at time τ,

$$\int_{\Sigma_j\cap B_1(0)} |A_{j,\tau}|^2 \, \mathrm{d}\mu_{j,\tau} = \int_{\Sigma\cap B_{r_j}(x_j)} |A_{t_j^{(2)}+r_j^4\tau}|^2 \, \mathrm{d}\mu_{t_j^{(2)}+r_j^4\tau} \leq e_1$$

whenever $t_j^{(2)} + r_j^4\tau < t_j^{(1)}$. Therefore,

$$\int_{\widehat{\Sigma}\cap B_1(0)} |A_{\widehat{\Sigma}}|^2 \, \mathrm{d}\mu_* = \lim_{j\to\infty} \int_{\Sigma_j\cap B_1(0)} |A_{j,\tau}|^2 \, \mathrm{d}\mu_{j,\tau} \leq e_1 < e_2,$$

a contradiction. □

Theorem 2.6.5. *For all $e < \varepsilon_1$, a Willmore flow f of closed surfaces cannot have a type-I singularity with respect to energy threshold e.*

Proof. If f has a type-I singularity with respect to some energy threshold $e < \varepsilon_1$, then as

$$t(r, \varepsilon_1) \geq t(r, e),$$

we have

$$\limsup_{r \to 0^+} r^{-1} \big(T - t(r, \varepsilon_1)\big)^{1/4} \leq \limsup_{r \to 0^+} r^{-1} \big(T - t(r, e)\big)^{1/4} < \infty,$$

i.e., f also has a type-I singularity with respect to energy threshold ε_1. By the previous lemma, f cannot have a type-I singularity with respect to e, a contradiction. □

Remark 2.6.6. *The condition $e < \varepsilon_1$ is not sharp, as the choice of ε_1 in Proposition 2.4.5, Theorem 2.4.6 is not sharp.*

Chapter 3

Łojasiewicz inequality

In this chapter, we adopt from [17] the concept of weighted Sobolev spaces on complete manifolds with certain asymptotic translation invariance. We then conjecture Łojasiewicz inequality for Willmore flows near such Willmore surfaces while showing some partial results.

3.1 Weighted Sobolev spaces

Definition 3.1.1 ([17, Section 1]). *An m-dimensional differentiable manifold Σ is said to have finitely many ends if for some compact subset Σ_0 with smooth boundary, there exists a diffeomorphism*

$$\Sigma \backslash \Sigma_0 \simeq \partial\Sigma_0 \times \mathbb{R}_+.$$

For convenience, we also denote the number of ends as L, and denote

$$\Sigma_R = \Sigma_0 \cup \big(\partial\Sigma_0 \times (0, R)\big).$$

Throughout this chapter, we will always assume that Σ is a manifold with finitely many ends.

Definition 3.1.2 ([17, Section 2]). *1. A tensor on $\partial\Sigma_0 \times \mathbb{R}_+$ (or the restriction of one onto $\partial\Sigma_0 \times \mathbb{R}_+$) is said to be translation-invariant if it is invariant under the $\mathbb{R}_+$-action $(\omega, z) \mapsto (\omega, z + z_0)$, $\forall z_0 > 0$.*

2. A Riemannian metric g is said to be admissible if $g = e^{2\rho}h$, where:

- *h is an asymptotically translation-invariant metric on $\partial\Sigma_0 \times \mathbb{R}_+$, i.e., for some translation-invariant metric h_∞ and all $t \in \mathbb{Z}_{\geq 0}$,*

$$\lim_{z\to\infty} \sup_{\omega\in\partial\Sigma_0} |D^t_\infty h - D^t_\infty h_\infty|_{h_\infty} = 0,$$

  where D_∞ denotes the covariant derivative induced by h_∞; and

- *$\rho \in C^\infty(\Sigma)$, and for some translation-invariant 1-form θ and all $t \in \mathbb{Z}_{\geq 0}$,*

$$\lim_{z\to\infty} \sup_{\omega\in\partial\Sigma_0} |D^{t+1}_{(h)} \rho - D^t_{(h)}\theta|_h = 0,$$

  where $D_{(h)}$ denotes the covariant derivative induced by h.

We denote the covariant derivative induced by g as $D_{(g)}$.

It is worth mentioning that for fixed g, the background data h, ρ, etc. are not unique. In fact, we have

Theorem 3.1.3 (Lockhart, [17, Theorem 2.9]). *If $g = e^{2\rho}h$ is an admissible metric, then $g = e^{2\overline{\rho}}\overline{h}$ for some asymptotically translation-invariant metric $\overline{h}$ and $\rho \in C^\infty(\Sigma)$ such that on each connected component of $\partial\Sigma_0 \times \mathbb{R}_+$, $\overline{\rho}$ only depends on z.*

Convention 3.1.4. • *Given Σ and $q, r \in \mathbb{Z} \geq 0$, denote the bundle of (r, q)-tensors* *as*

$$T^q_r\Sigma := (T^*\Sigma)^{\otimes q} \otimes (T\Sigma)^{\otimes r},$$

and, as usual, denote the bundle of differential q*-forms as* $\Lambda^q\Sigma$.

- *Given a vector bundle* E *on* Σ, *let* $\Gamma(E)$ *denote the space of measurable global sections on* E, *and let* $C_0^\infty(E)$ *denote the space of smooth global sections on* E *with compact support, correspondingly.*

Definition 3.1.5 ([17, Section 1]). *For* $s \in \mathbb{Z}_{\geq 0}$, $1 \leq p < \infty$, *and a tensor bundle* E, *we define*

$$L^{s,p}_{\text{loc}}(E) = \left\{ \sigma \in \Gamma(E) \,:\, \forall \varphi \in C_0^\infty(\Sigma), \left(\sum_{t=0}^{s} \int_\Sigma |D^t_{(g)}(\varphi\sigma)|^p_g \,\mathrm{d}\mu_g \right)^{1/p} < \infty \right\}$$

Note that $L^{s,p}_{\text{loc}}(E)$ *does not depend on the choice of* g.

Convention 3.1.6 ([17, Section 3]). *Let* $\mathbb{R}^L$ *be identified with the set of locally constant functions on* $\partial\Sigma_0 \times \mathbb{R}_+$. $\mathbb{R}^L$ *is equipped with the natural partial order:* $\delta \geq \delta'$ *if* $\delta_j \geq \delta'_j$ *for all* $j = 1, \ldots, L$, *while* $\delta > \delta'$ *if* $\delta_j > \delta'_j$ *for all* j. *In addition, given* $\delta \in \mathbb{R}^L$, δz *extends to an unspecified smooth function on* Σ, *and is hence identified with the extension.*

Definition 3.1.7 ([17, Definition 4.1]). *Given an admissible metric* g, $1 < p < \infty$, $\delta \in \mathbb{R}^L$, *and* $a \in \mathbb{R}$, *we define the weighted Sobolev space for* $s \in \mathbb{Z}_{\geq 0}$ *that*

$$W^{s,p}_{\delta,a}(E, g) := \{\sigma \in L^{s,p}_{\text{loc}}(E) \,:\, \|\sigma\|_{W^{s,p}_{\delta,a}} < \infty\},$$

where

$$\|\sigma\|_{W^{s,p}_{\delta,a}} := \left(\sum_{t=0}^{s} \int_\Sigma |e^{\delta z + (t+a)\rho} D^t_{(g)}\sigma|^p_g \,\mathrm{d}\mu_g \right)^{1/p},$$

omitting E when referring to this norm; and for $s \in \mathbb{Z}_{<0}$ that $W^{s,p}_{\delta,a}(E,g)$ is the dual space of $W^{-s,p'}_{-\delta,-a}(E,g)$, where $1/p + 1/p' = 1$.

Moreover, we identify $u \in C^\infty_0(E,g)$ with $\ell_u \in W^{s,p}_{\delta,a}(E,g)$, defined by

$$\begin{array}{rccc} \ell_u : & W^{-s,p'}_{-\delta,-a}(E,g) & \to & \mathbb{R} \\ & v & \mapsto & \displaystyle\int_\Sigma \langle u,v\rangle_g \, \mathrm{d}\mu_g \,. \end{array}$$

For convenience, we also define seminorms

$$\|\sigma\|_{W^{s,p}_{\delta,a}(S)} := \left(\sum_{t=0}^{s} \int_S |e^{\delta z + (t+a)\rho} D^t_{(g)} \sigma|^p_g \, \mathrm{d}\mu_g \right)^{1/p}$$

for all measurable set $S \subset \Sigma$.

Weighted Sobolev spaces are defined this way to make sure the following differential operators are continuous maps. See Proposition 4.6 and Corollary 4.7 of [17].

Lemma 3.1.8. *Given $s \in \mathbb{Z}_{\geq 0}$, $q \geq 1$, and V, a global smooth vector field on Σ, if*

$$\sup_\Sigma e^{(\overline{a}-a+t)\rho} |D^t_{(g)} V|_g < \infty, \; \forall t = 0, \ldots, s$$

for some $\overline{a} \in \mathbb{R}$, then contraction with V defines a continuous map

$$\iota_V : W^{s,p}_{\delta,a}(T^q_r \Sigma, g) \to W^{s,p}_{\delta,\overline{a}}(T^{q-1}_r \Sigma, g).$$

Proof. Observe that for all $t = 0, \ldots, s$,

$$\begin{aligned} &|e^{\delta z + (\overline{a}+t)\rho} D^t_{(g)} (\iota_V \sigma)|_{(g)} \\ &\leq c \sum_{b=0}^{t} e^{\delta z + (\overline{a}+t)\rho} |D^{t-b}_{(g)} V|_{(g)} |D^b_{(g)} \sigma|_{(g)} \end{aligned}$$

$$\leq c \sum_{b=0}^{t} \left(e^{(\overline{a}-a+t-b)\rho}|D^{t-b}_{(g)}V|_{(g)}\right)\left(e^{\delta z+(a+b)\rho}|D^{b}_{(g)}\sigma|_{(g)}\right)$$
$$\leq c \sum_{b=0}^{t} |e^{\delta z+(a+b)\rho}D^{b}_{(g)}\sigma|_{(g)}.$$

The rest of the proof is trivial. □

Convention 3.1.9. *For the rest of this chapter, we will consider the scenario when Σ is a m-dimensional immersed submanifold in $\mathbb{R}^n$, say $f : \Sigma \looparrowright \mathbb{R}^n$.*

Let g be the induced metric on Σ, which we will assume to be admissible. Let $\mathcal{E} = f^*(T\mathbb{R}^n)$, which is the trivial vector bundle of rank n, and let $N\Sigma$ denote the normal vector bundle, which is a sub-bundle of $\mathcal{E}$ of rank $(n-m)$. Let P denote the orthogonal projection $\mathcal{E} \to N\Sigma$.

Let $W^{s,p}_{\delta,a}(T^q_r\Sigma \otimes \mathcal{E}, g)$ be as given by Definition 3.1.7, i.e., the weighted Sobolev space of $\mathbb{R}^n$-valued q-forms on Σ. Let $W^{s,p}_{\delta,a}(T^q_r\Sigma \otimes N\Sigma, g)$ denote the closure of $C^\infty_0(T^q_r\Sigma \otimes N\Sigma)$ as a subspace of $W^{s,p}_{\delta,a}(T^q_r\Sigma \otimes \mathcal{E}, g)$.

Remark 3.1.10. *Note that the conventions and many of the following results also apply for $\Lambda^q \otimes N\Sigma$ by setting $r = 0$. However, we will focus on $(0, q)$-tensors.*

Proposition 3.1.11. *For all q, the projection map*

$$P : W^{0,p}_{\delta,a}(T^q_r\Sigma \otimes \mathcal{E}, g) \to W^{0,p}_{\delta,a}(T^q_r\Sigma \otimes \mathcal{E}, g)$$

is continuous. In addition, if for some $\beta \in \mathbb{R}$ and $s_0 \in \mathbb{Z}_{>0}$,

$$\begin{cases} \inf_{\Sigma}(\beta\rho) > -\infty, \\ \sup_{\Sigma}(e^{(t+1)(1-\beta)\rho}|\nabla^t_{(g)}A|_g) < \infty,\ \forall t = 0, \ldots, s_0, \end{cases}$$

then for all $s \in \mathbb{Z}$ such that $|s| \le s_0$,

$$P : W^{s,p}_{\delta,a}(T^q_r\Sigma \otimes \mathcal{E}, g) \to W^{s,p}_{\delta,a-\beta|s|}(T^q_r\Sigma \otimes \mathcal{E}, g)$$

is continuous, where when $s < 0$,

$$P : W^{s,p}_{\delta,a}(T^q_r\Sigma \otimes \mathcal{E}, g) \to W^{s,p}_{\delta,a+\beta s}(T^q_r\Sigma \otimes \mathcal{E}, g)$$

is given by

$$(P\sigma)(\varphi) = \sigma(P\varphi),\ \forall \varphi \in W^{-s,p'}_{-\delta,-a-\beta s}(T^q_r\Sigma \otimes \mathcal{E}, g).$$

Remark 3.1.12. *We will often assume a slightly stronger condition:*

$$\begin{cases} \beta \ge 0 \\ \rho_0 := \inf_{\Sigma} \rho > -\infty, \text{ and} \\ C_{t,\beta} := \sup_{\Sigma}(e^{(t+1)(1-\beta)\rho}|\nabla^t_{(g)}A|_g) < \infty,\ \forall t = 0, \dots, s_0. \end{cases} \tag{3.1}$$

Proof. (i) First, we claim that when $s \ge 0$, for all $\sigma \in C^\infty_0(T^q_r\Sigma \otimes N\Sigma)$,

$$\|P\sigma\|_{W^{s,p}_{\delta,a-\beta s}} \le c\,\|\sigma\|_{W^{s,p}_{\delta,a}}.$$

The case for $s = 0$ is trivial because derivative is not involved. In fact, $c = 1$.

For $s > 0$, we see that for all $0 \le t \le s$ (cf. Lemma 1.3.2),

$$|D^t_{(g)}(P\sigma)|_g \le c(t) \sum_{\substack{i_0,\dots,i_k \ge 0 \\ i_0+\dots+i_k+k=t}} \left(|D^{i_0}_{(g)}\sigma|_g \prod_{j=1}^k |\nabla^{i_j}_{(g)}A|_g \right).$$

In particular,

$$\begin{aligned}
&e^{\delta z+(a-\beta s+t)\rho}|D^t_{(g)}(P\sigma)|_g \\
&\le c(t)e^{\delta z+a\rho}\sum_{\substack{i_0,\dots,i_k\ge 0\\ i_0+\cdots+i_k+k=t}}\left(e^{\beta(t-s-i_0)\rho}(e^{i_0\rho}|D^{i_0}_{(g)}\sigma|_g)\prod_{j=1}^{k}(e^{(i_j+1)(1-\beta)\rho}|\nabla^{i_j}_{(g)}A|_g)\right)\\
&\le c(t)e^{-s\inf(\beta\rho)}\sum_{i_0=0}^{t}(e^{\delta z+(a+i_0)\rho}|D^{i_0}_{(g)}\sigma|_g),
\end{aligned}$$

so that

$$\|P\sigma\|_{W^{s,p}_{\delta,a-\beta s}}\le c(s)e^{-s\inf(\beta\rho)}\|\sigma\|_{W^{s,p}_{\delta,a}}.$$

(ii) For all $\sigma\in W^{s,p}_{\delta,a}(T^q_r\Sigma\otimes\mathcal{E},g)$, where $s\ge 0$, by Corollary 4.5 of [17], we can find a sequence $\{\sigma_j\}$ in $C^\infty_0(T^q_r\Sigma\otimes\mathcal{E})$ such that

$$\lim_{j\to\infty}\|\sigma_j-\sigma\|_{W^{s,p}_{\delta,a}}=0.$$

Thus

$$\lim_{j,k\to\infty}\|P\sigma_j-P\sigma_k\|_{W^{s,p}_{\delta,a-\beta s}}\le c\lim_{j,\to\infty}\|\sigma_j-\sigma_k\|_{W^{s,p}_{\delta,a}}=0,$$

so that $\{P\sigma_j\}$ converges to some $\tau\in W^{s,p}_{\delta,a-\beta s}(T^q_r\Sigma\otimes\mathcal{E},g)$.

It's not hard to see that for all $\psi\in C^\infty_0(\Sigma)$, $\psi P\sigma=\psi\tau$. That is, $P\sigma=\tau$.

(iii) For $s<0$, we have

$$\|P\sigma\|_{W^{s,p}_{\delta,a+\beta s}}\le\|\sigma\|_{W^{s,p}_{\delta,a}}\sup_{\varphi\ne 0}\frac{\|P\varphi\|_{W^{-s,p'}_{-\delta,-a}}}{\|\varphi\|_{W^{-s,p'}_{-\delta,-a-\beta s}}},$$

and hence P is continuous.

□

Corollary 3.1.13. *Under the same condition,*

$$\nabla_{(g)} = P \circ D_{(g)} : W^{s+1,p}_{\delta,a}(T^q_r\Sigma \otimes N\Sigma, g) \to W^{s,p}_{\delta,a+1-\beta|s|}(T^{q+1}_r\Sigma \otimes N\Sigma, g)$$

is continuous.

In view of Theorem 3.1.3, δ and a are interchangeable. Nevertheless, we can rewrite the Sobolev embedding and compactness theorems [17] for a instead of δ without using Theorem 3.1.3.

Proposition 3.1.14 (Weighted Sobolev embedding)**.** *Given $s, \overline{s} \in \mathbb{Z}$, $1 < p, \overline{p} < \infty$, $\delta, \overline{\delta} \in \mathbb{R}^L$, and $a, \overline{a} \in \mathbb{R}$, if*

(i) $\inf_{\Sigma} \rho > -\infty$,

(ii) $s - \overline{s} \geq m/p - m/\overline{p}$,

(iii) $s \geq \overline{s} \geq 0$,

(iv) $p \leq \overline{p}$ *with* $\delta \geq \overline{\delta}$ *or* $p > \overline{p}$ *with* $\delta > \overline{\delta}$, *and*

(v) $a + m/p \geq \overline{a} + m/\overline{p}$,

then the identity map

$$W^{s,p}_{\delta,a}(E, g) \to W^{\overline{s},\overline{p}}_{\overline{\delta},\overline{a}}(E, g)$$

is continuous.

Remark 3.1.15. *By Theorem 4.8 of [17],*

$$W^{s,p}_{\delta,a}(E, g) \to W^{\overline{s},\overline{p}}_{\overline{\delta},a+m(1/p-1/\overline{p})}(E, g)$$

is continuous. Thus it suffices to show continuity of

$$W^{\bar{s},\bar{p}}_{\bar{\delta},a+m(1/p-1/\bar{p})}(E,g) \to W^{\bar{s},\bar{p}}_{\bar{\delta},\bar{a}}(E,g),$$

or equivalently, the special case of the proposition above when $s=\bar{s}$, $p=\bar{p}$, and $\delta=\bar{\delta}$.

Proof of Proposition 3.1.14. As in the remark, we assume $a \geq \bar{a}$ and will show that the identity map

$$W^{s,p}_{\delta,a}(E,g) \to W^{s,p}_{\delta,\bar{a}}(E,g)$$

is continuous. Indeed, the aforementioned map is bounded because by definition,

$$\|\sigma\|_{W^{s,p}_{\delta,\bar{a}}} \leq e^{-(a-\bar{a})\rho_0}\|\sigma\|_{W^{s,p}_{\delta,a}},$$

where $\rho_0 = \inf \rho$. □

Corollary 3.1.16. *Under the same condition,*

(i) $W^{s,p}_{\delta,a}(E,g)$ *is embedded into* $W^{\bar{s},\bar{p}}_{\bar{\delta},\bar{a}}(E,g)$ *as a dense subspace, and*

(ii) $W^{s,p}_{\delta,a}(T^q_r\Sigma \otimes N\Sigma, g)$ *is embedded into* $W^{\bar{s},\bar{p}}_{\bar{\delta},\bar{a}}(T^q_r\Sigma \otimes N\Sigma, g)$ *continuously as a dense subspace.*

In particular, the embedding maps are Fredholm operators with Fredholm index 0.

Proof. (i) By the proposition, we have

$$C^\infty_0(E) \subset W^{s,p}_{\delta,a}(E,g) \subset W^{\bar{s},\bar{p}}_{\bar{\delta},\bar{a}}(E,g).$$

By Corollary 4.5 of [17], $C_0^\infty(E)$ is dense in $W^{\overline{s},\overline{p}}_{\overline{\delta},\overline{a}}(E,g)$, and hence $W^{s,p}_{\delta,a}(E,g)$ is also dense in $W^{\overline{s},\overline{p}}_{\overline{\delta},\overline{a}}(E,g)$.

(ii) For all $\sigma \in W^{s,p}_{\delta,a}(T^q_r\Sigma \otimes N\Sigma, g)$, let $\{\sigma_j\} \subset C_0^\infty(T^q_r\Sigma \otimes N\Sigma)$ such that

$$\lim_{j\to\infty} \|\sigma_j - \sigma\|_{W^{s,p}_{\delta,a}} = 0.$$

By the proposition, there exists $c > 0$ such that

$$\|\tau\|_{W^{\overline{s},\overline{p}}_{\overline{\delta},\overline{a}}} \leq c\,\|\tau\|_{W^{s,p}_{\delta,a}},\ \forall \tau \in W^{s,p}_{\delta,a}(T^q_r\Sigma \otimes N\Sigma, g).$$

This implies

$$\lim_{j\to\infty} \|\sigma_j - \sigma\|_{W^{\overline{s},\overline{p}}_{\overline{\delta},\overline{a}}} \leq c \lim_{j\to\infty} \|\sigma_j - \sigma\|_{W^{s,p}_{\delta,a}} = 0,$$

and hence $\sigma \in W^{\overline{s},\overline{p}}_{\overline{\delta},\overline{a}}(T^q_r\Sigma \otimes N\Sigma, g)$. In addition,

$$\|\sigma\|_{W^{\overline{s},\overline{p}}_{\overline{\delta},\overline{a}}} \leq c\,\|\sigma\|_{W^{s,p}_{\delta,a}},$$

so the embedding is continuous. Finally, observe that

$$C_0^\infty(T^q_r\Sigma \otimes N\Sigma) \subset W^{s,p}_{\delta}(T^q_r\Sigma \otimes N\Sigma, g) \subset W^{\overline{s},\overline{p}}_{\overline{\delta},\overline{a}}(T^q_r\Sigma \otimes N\Sigma, g),$$

while $C_0^\infty(T^q_r\Sigma \otimes N\Sigma)$ is dense in $W^{\overline{s},\overline{p}}_{\overline{\delta},\overline{a}}(T^q_r\Sigma \otimes N\Sigma, g)$. Therefore, $W^{s,p}_{\delta}(T^q_r\Sigma \otimes N\Sigma, g)$ is dense in $W^{\overline{s},\overline{p}}_{\overline{\delta},\overline{a}}(T^q_r\Sigma \otimes N\Sigma, g)$.

□

Theorem 3.1.17 (Weighted compact embedding). *Given $s, \overline{s} \in \mathbb{Z}$, $1 < p, \overline{p} < \infty$, $\delta, \overline{\delta} \in \mathbb{R}^L$, and $a, \overline{a} \in \mathbb{R}$, if* *(i)* $\lim_{z\to\infty} \inf_{\omega\in\partial\Sigma_0} \rho(\omega, z) = \infty$,

(ii) $s - \overline{s} > m/p - m/\overline{p}$,

(iii) $s > \overline{s} \geq 0$,

(iv) $p \leq \overline{p}$,

(v) $\delta \geq \overline{\delta}$, *and*

(vi) $a + m/p > \overline{a} + m/\overline{p}$,

then the embedding

$$W^{s,p}_{\delta,a}(T^q_r\Sigma \otimes N\Sigma, g) \to W^{\overline{s},\overline{p}}_{\overline{\delta},\overline{a}}(T^q_r\Sigma \otimes N\Sigma, g)$$

is compact.

Proof. Denote $\tau = a - \overline{a} + m/p - m/\overline{p}$, which is a positive number by hypothesis. First, condition (i) implies $\inf_{\Sigma} \rho > -\infty$. Thus by Proposition 3.1.14,

$$W^{s,p}_{\delta,a}(T^q_r\Sigma \otimes N\Sigma, g) \to W^{\overline{s},\overline{p}}_{\overline{\delta},\overline{a}}(T^q_r\Sigma \otimes N\Sigma, g)$$

is continuous.

Next, consider the Banach space defined in Definition 3.4 of [17]:

$$W^{s,p}_{\delta}(T^q_r\Sigma \otimes N\Sigma) := \{\sigma \in L^{s,p}_{\text{loc}}(T^q_r\Sigma \otimes N\Sigma) \,:\, \|\sigma\|_{W^{s,p}_{\delta}} < \infty\},$$

where the norm is given by

$$\|\sigma\|_{W^{s,p}_{\delta}} := \left(\sum_{t=0}^{s} \int_{\Sigma} |e^{\delta z} D_{(g)}\sigma|^p_g \, \mathrm{d}\mu_g \right)^{1/p}.$$

By Proposition/Definition 4.4 of [17],

$$\begin{aligned} K_{a,p}: \quad W^{s,p}_{\delta,a}(T^q_r\Sigma\otimes N\Sigma, g) &\to W^{s,p}_{\delta}(T^q_r\Sigma\otimes N\Sigma) \\ \sigma &\mapsto e^{(a+r-q+m/p)\rho}\sigma \end{aligned}$$

is an isomorphism. Therefore, we have the following commuting diagram:

$$\begin{array}{ccc} W^{s,p}_{\delta,a}(T^q_r\Sigma\otimes N\Sigma, g) & \longrightarrow & W^{\bar{s},\bar{p}}_{\bar{\delta},\bar{a}}(T^q_r\Sigma\otimes N\Sigma, g) \\ {\scriptstyle K_{a,p}}\downarrow\wr & & {\scriptstyle K_{\bar{a},\bar{p}}}\downarrow\wr \\ W^{s,p}_{\delta}(T^q_r\Sigma\otimes N\Sigma) & \xrightarrow{\Psi} & W^{\bar{s},\bar{p}}_{\bar{\delta}}(T^q_r\Sigma\otimes N\Sigma) \end{array}$$

where $\Psi\sigma = e^{-\tau\rho}\sigma$. Since the vertical maps are isomorphisms, it suffices to prove that Ψ is compact.

Let $\{\sigma_j\}$ be a bounded sequence in $W^{s,p}_{\delta}(T^q_r\Sigma\otimes N\Sigma)$. WLOG, let

$$\sup_j \|\sigma_j\|_{W^{s,p}_{\delta}} \leq 1.$$

Since Ψ is continuous, $\{\Psi\sigma_j\}$ is bounded in $W^{\bar{s},\bar{p}}_{\bar{\delta}}(T^q_r\Sigma\otimes N\Sigma)$. For all $R > 0$, by Rellich theorem, there exists a subsequence, which we still denote by $\{\Psi\sigma_j\}$ by abusing notation, that converges on Σ_R in the sense that

$$\lim_{j,k\to\infty} \|\Psi(\sigma_j - \sigma_k)\|_{W^{\bar{s},\bar{p}}_{\bar{\delta}}(\Sigma_R)} = 0.$$

In fact, using diagonal argument, we can find a subsequence such that

$$\lim_{j,k\to\infty} \|\Psi(\sigma_j - \sigma_k)\|_{W^{\bar{s},\bar{p}}_{\bar{\delta}}(\Sigma_R)} = 0, \forall R > 0.$$

By Lemma 3.11 of [17], we have

$$\|\Psi\sigma_j\|_{W^{\bar{s},\bar{p}}_{\bar{\delta}}(\Sigma\setminus\Sigma_{2R})} \leq c\,\|\Psi\sigma_j\|_{W^{s,p}_{\delta}(\Sigma\setminus\Sigma_R)}.$$

Moreover, the right hand side can be estimated by

$$\begin{aligned}
\|\Psi\sigma_j\|_{W^{s,p}_{\delta}(\Sigma\setminus\Sigma_R)} &= \left(\sum_{t=0}^{s}\int_{\partial\Sigma_0\times(R,\infty)} |e^{\delta z}D^t_{(h)}(e^{-\tau\rho}\sigma_j)|^p_{(h)}\,\mathrm{d}\mu_h\right)^{1/p}\\
&\leq c_R\left(\sum_{t=0}^{s}\int_{\partial\Sigma_0\times(R,\infty)} |e^{\delta z}D^t_{(h)}\sigma_j|^p_{(h)}\,\mathrm{d}\mu_h\right)^{1/p}\\
&= c_R\|\sigma_j\|_{W^{s,p}_{\delta}(\Sigma\setminus\Sigma_R)} \leq c_R,
\end{aligned}$$

where

$$c_R = c(s)\sup_{\partial\Sigma_0\times(R,\infty)}\sum_{t=0}^{s}|D^t_{(h)}e^{-\tau\rho}|_{(h)} \leq c(s)\exp(-\tau\inf_{\partial\Sigma_0\times(R,\infty)}\rho)\sum_{t=0}^{s}|D^t_{(h)}\rho|_{(h)}.$$

Since $|D^t_{(h)}\rho|_{(h)}$ is bounded by Definition 3.1.2 and

$$\lim_{R\to\infty}\inf_{\Sigma\setminus\Sigma_R}\rho = \infty,$$

which equivalent to condition (i), we have

$$\lim_{R\to\infty} c_R = 0.$$

In particular, for all $\varepsilon > 0$, we can choose sufficiently large R such that $c_R < \varepsilon/3$. Also, we can choose sufficiently large j_0 such that

$$\|\Psi(\sigma_j - \sigma_k)\|_{W^{\bar{s},\bar{p}}_{\bar{\delta}}(\Sigma_{2R})} < \frac{\varepsilon}{3},\ \forall j,k \geq j_0.$$

Therefore, by Minkowski's inequality,

$$\|\Psi(\sigma_j - \sigma_k)\|_{W^{\bar{s},\bar{p}}_{\delta}} \leq \|\Psi(\sigma_j - \sigma_k)\|_{W^{\bar{s},\bar{p}}_{\delta}(\Sigma_{2R})} + \|\Psi\sigma_j\|_{W^{\bar{s},\bar{p}}_{\delta}(\Sigma\setminus\Sigma_{2R})} + \|\Psi\sigma_k\|_{W^{\bar{s},\bar{p}}_{\delta}(\Sigma\setminus\Sigma_{2R})} < \varepsilon$$

whenever $j, k \geq j_0$. That is, the subsequence converges. □

Remark 3.1.18. *Disregarding conditions (iv) and (vi), if $\delta > \overline{\delta}$ and $a = \overline{a}$, the theorem reduces to Theorem 4.9 of [17] for arbitrary $p, \overline{p}$.*

Remark 3.1.19. *If $p = \overline{p} = 2$ and $\delta = \overline{\delta}$, then condition (ii) is implied by condition (iii), conditions (iv) and (v) are satisfied, and condition (vi) reduces to "$a > \overline{a}$".*

3.2 Łojasiewicz inequality for Willmore flows

As pointed out in Theorem 5.2 of [17], given Σ, g, p, q, and a, the Laplace operator for scalar-valued functions

$$\triangle_g : W^{s+2,p}_{\delta,a}(\Sigma, g) \to W^{s,p}_{\delta,a}(\Sigma, g)$$

is Fredholm for a.e. $\delta \in \mathbb{R}^L$, but not necessarily for all δ. Therefore, for specific choices of δ, we need to prove otherwise.

We will fix $p = 2$ and assume condition (3.1) for $\beta = 0$ and $s_0 = 1$. In particular, by Corollary 3.1.13 and Proposition 3.1.14,

$$\triangle : W^{2,2}_{\delta,a-2}(N\Sigma, g) \to W^{0,2}_{\delta,a}(N\Sigma, g)$$

is continuous. In addition, we fix arbitrary $\delta \in \mathbb{R}^n$ and $a \in \mathbb{R}$.

We define a bilinear form on $W^{1,2}_{\delta,a-2}(N\Sigma, g)$:

$$B(u,v) := \langle \nabla u, \nabla v \rangle_{W^{0,2}_{\delta,a-1}}.$$

Also, for all $u \in W^{1,2}_{\delta,a-2}(N\Sigma, g)$, we define $\hat{B}(u) = v$ if $v \in W^{0,2}_{\delta,a}(N\Sigma, g)$ and

$$B(u,\psi) = -\langle v, e^{-2\rho}\psi \rangle_{W^{0,2}_{\delta,a}} \qquad \text{for all } \psi \in C^\infty_0(N\Sigma).$$

We can immediately see the following:

Lemma 3.2.1. *For all $u \in W^{2,2}_{\delta,a-2}(N\Sigma, g)$, we have $u \in \mathrm{Dom}(\hat{B})$ and*

$$\hat{B}u = \triangle u - 2\nabla_{(\delta\partial_z + (a-1)\mathrm{grad}\rho)} u \in W^{0,2}_{\delta,a}(N\Sigma, g)$$

In particular,

$$W^{2,2}_{\delta,a-2}(N\Sigma, g) \subset \mathrm{Dom}(\hat{B}).$$

Proof. Observe that for all $\psi \in C^\infty_0(N\Sigma)$,

$$\begin{aligned}
&- \langle \triangle u, e^{-2\rho}\psi \rangle_{W^{0,2}_{\delta,a}} \\
&= \langle \nabla^*\nabla u, e^{-2\rho}\psi \rangle_{W^{0,2}_{\delta,a}} \\
&= \int_\Sigma e^{2\delta z + 2a\rho} \langle \nabla^*\nabla u, e^{-2\rho}\psi \rangle \, \mathrm{d}\mu_g \\
&= \int_\Sigma \langle \nabla u, \nabla (e^{2\delta z + 2(a-1)\rho}\psi) \rangle_g \, \mathrm{d}\mu_g \\
&= B(u,\psi) + \int_\Sigma e^{2\delta z + 2(a-1)\rho} \langle \nabla u, (2\delta \, \mathrm{d}z + 2(a-1)\,\mathrm{d}\rho) \otimes \psi \rangle_g \, \mathrm{d}\mu_g \\
&= B(u,\psi) - \langle \nabla_{(2\delta\partial_z + 2(a-1)\mathrm{grad}\rho)} u, e^{-2\rho}\psi \rangle_{W^{0,2}_{\delta,a}}
\end{aligned}$$

and hence the aforementioned formula for $\hat{B}$ follows. ∎

Lemma 3.2.2 (Coercivity)**.** *There exist sufficiently large $\zeta > 0$ and sufficiently small $\eta > 0$ such that for all $u \in W^{1,2}_{\delta,a-2}(N\Sigma, g)$,*

$$B(u,u) + \zeta \|u\|^2_{W^{0,2}_{\delta,a-2}} \geq \eta \|u\|^2_{W^{1,2}_{\delta,a-2}}.$$

Proof. For all $u \in W^{1,2}_{\delta,a-2}(N\Sigma, g)$, we have

$$\begin{aligned}
&B(u,u) + \zeta \|u\|^2_{W^{0,2}_{\delta,a-2}} \\
&= \Big(\int_\Sigma e^{2\delta z + 2(a-1)\rho} |\nabla_{(g)} u|^2_g \, \mathrm{d}\mu_g \Big) + \zeta \|u\|^2_{W^{0,2}_{\delta,a}} \\
&= \int_\Sigma e^{2\delta z + 2(a-1)\rho} \big(|D_{(g)} u|^2_g - \langle P D_{(g)} u, D_{(g)} u \rangle_g \big) \, \mathrm{d}\mu_g + \zeta \|u\|^2_{W^{0,2}_{\delta,a-2}} \\
&= \int_\Sigma e^{2\delta z + 2(a-1)\rho} \big(|D_{(g)} u|^2_g - g^{ij} g^{k\ell} \langle D_{(g)i} u, D_{(g)k} f \rangle \langle D_{(g)\ell} f, D_{(g)j} u \rangle \big) \, \mathrm{d}\mu_g + \zeta \|u\|^2_{W^{0,2}_{\delta,a-2}} \\
&= \int_\Sigma e^{2\delta z + 2(a-1)\rho} \big(|D_{(g)} u|^2_g + g^{ij} g^{k\ell} \langle D_{(g)i} u, D_{(g)k} f \rangle \langle D_{(g)j} D_{(g)\ell} f, u \rangle \big) \, \mathrm{d}\mu_g + \zeta \|u\|^2_{W^{0,2}_{\delta,a-2}} \\
&\geq \int_\Sigma e^{2\delta z + 2(a-1)\rho} \big(|D_{(g)} u|^2_g - |A|_g |D_{(g)} u|_g |u| \big) \, \mathrm{d}\mu_g + \zeta \|u\|^2_{W^{0,2}_{\delta,a-2}} \\
&\geq \int_\Sigma e^{2\delta z + 2(a-1)\rho} \big(\frac{3}{4} |D_{(g)} u|^2_g - |A|^2_g |u|^2 \big) \, \mathrm{d}\mu_g + \zeta \|u\|^2_{W^{0,2}_{\delta,a-2}} \\
&= \int_\Sigma \big[\frac{3}{4} e^{2\delta z + 2(a-1)\rho} |D_{(g)} u|^2_g + e^{2\delta z + 2(a-2)\rho} \big(-e^{2\rho} |A|^2_g + \zeta \big) |u|^2 \big] \, \mathrm{d}\mu_g \\
&\geq \int_\Sigma \big[\frac{3}{4} e^{2\delta z + 2(a-1)\rho} |D_{(g)} u|^2_g + e^{2\delta z + 2(a-2)\rho} \big(\zeta - C^2_{0,0} \big) |u|^2 \big] \, \mathrm{d}\mu_g \\
&\geq \eta \|u\|^2_{W^{1,2}_{\delta,a-2}}
\end{aligned}$$

for some sufficiently large $\zeta > 0$ and sufficiently small $\eta > 0$. □

Lemma 3.2.3 (Regularity)**.** *Let $s \geq 0$. For all $u \in \mathrm{Dom}(\hat{B})$, if $\hat{B} u \in W^{s,2}_{\delta,a}(N\Sigma, g)$, then $u \in W^{s+2,2}_{\delta,a-2}(N\Sigma, g)$. Moreover, for all $u \in \mathrm{Dom}(\hat{B})$,*

$$\|u\|_{W^{s+2,2}_{\delta,a-2}} \leq c \big(\|u\|_{W^{1,2}_{\delta,a-2}} + \|\hat{B} u\|_{W^{s,2}_{\delta,a}} \big).$$

Moreover,

$$\mathrm{Dom}(\hat{B}) \subset W^{2,2}_{\delta,a-2}(N\Sigma, g).$$

Proof. Let $\mathcal{U} = \{(U_\alpha, \varphi_\alpha)\}$ be an atlas, i.e., an open cover $\{U_\alpha\}$ for $\partial\Sigma_0$ and coordinate maps

$$\varphi_\alpha : U_\alpha \xrightarrow{\sim} B_2(0) \subset \mathbb{R}^{m-1}.$$

We denote $V_\alpha = \varphi_\alpha^{-1}\big(B_1(0)\big)$ and

$$\begin{array}{rccc} \Phi_\alpha : & U_\alpha \times \mathbb{R}_+ & \xrightarrow{\sim} & B_2(0) \times \mathbb{R}_+ \\ & (\omega, z) & \mapsto & (\varphi_\alpha(\omega), z). \end{array}$$

Since $\partial\Sigma_0$ is compact, we can choose $\mathcal{U}$ such that $\{U_\alpha\}$ is a finite open cover of $\partial\Sigma_0$, that $\{V_\alpha\}$ also covers $\partial\Sigma_0$, and that

$$c^{-1}\mathrm{Id} \le (\Phi_\alpha^{-1})^* h \le c\,\mathrm{Id}$$

for some $0 < c < \infty$. We will abuse the notation and identify h with $\Phi_\alpha^* h$, etc.

For all $u \in \mathrm{Dom}(\hat{B})$, consider $e^{-2\delta z-(2a-4+n)\rho}\psi$ as a test function, where $\psi \in C_0^\infty(N\Sigma)$ is supported on $U_\alpha \times \mathbb{R}_+$. We hence obtain

$$\begin{aligned} &\int_{U_\alpha \times \mathbb{R}_+} e^{2\delta z+(2a-2)\rho} \big\langle \nabla_{(g)} u, \nabla_{(g)}\big(e^{-2\delta z-(2a-4+n)\rho}\psi\big)\big\rangle_g \, \mathrm{d}\mu_g \\ &= -\int_{U_\alpha \times \mathbb{R}_+} e^{2\delta z+2a\rho} \langle \hat{B} u, e^{-2\delta z-(2a-2+n)\rho}\psi\rangle \, \mathrm{d}\mu_g\,. \end{aligned} \tag{3.2}$$

For the left hand side of equation (3.2), we see that

$$\big\langle \nabla_{(g)} u, \nabla_{(g)}\big(e^{-2\delta z-(2a-4+n)\rho}\psi\big)\big\rangle_g$$

$$
\begin{aligned}
&= g^{ij}\langle \partial_i u, \nabla_{(g)_j}(e^{-2\delta z-(2a-4+n)\rho}\psi)\rangle \\
&= e^{-2\delta z-(2a-4+n)\rho} g^{ij}\big\langle \partial_i u, (\partial_j\psi - \partial_j[2\delta z+(2a-4+n)\rho]\psi + g^{k\ell}\langle A_{jk},\psi\rangle\partial_\ell f)\big\rangle \\
&= e^{-2\delta z-(2a-4+n)\rho} g^{ij}\big(\langle \partial_i u, \partial_j\psi\rangle - \partial_j[2\delta z+(2a-4+n)\rho]\langle \partial_i u,\psi\rangle \\
&\quad - g^{k\ell}\langle A_{i\ell},u\rangle\langle A_{jk},\psi\rangle\big).
\end{aligned}
$$

Therefore, equation (3.2) can be rewritten on the coordinates as the following:

$$
\begin{aligned}
&\int_{B_2(0)\times\mathbb{R}_+} e^{(2-n)\rho}\sqrt{\det(g)}g^{ij}\langle \partial_i u, \partial_j\psi\rangle \,\mathrm{d}x \\
&\quad - \int_{B_2(0)\times\mathbb{R}_+} e^{(2-n)\rho}\sqrt{\det(g)}\big\langle g^{ij}\partial_j[2\delta z+(2a-4+n)\rho]\partial_i u, \psi\big\rangle \,\mathrm{d}x \\
&\quad + \int_{B_2(0)\times\mathbb{R}_+} e^{(2-n)\rho}\sqrt{\det(g)}\big\langle g^{ij}g^{k\ell}\langle A_{i\ell},u\rangle A_{jk}, \psi\big\rangle \,\mathrm{d}x \\
&= -\int_{B_2(0)\times\mathbb{R}_+} e^{-n\rho}\sqrt{\det(g)}\langle e^{2\rho}\hat{B}u, \psi\rangle \,\mathrm{d}x
\end{aligned}
$$

By construction, we have

$$
\begin{aligned}
&c^{-1}\mathrm{Id} \le e^{(2-n)\rho}\sqrt{\det(g)}g^{ij} \le c\,\mathrm{Id}, \\
&e^{(2-n)\rho}\sqrt{\det(g)}g^{ij}\partial_j[2\delta z+(2a-4+n)\rho] \le c \text{ for all } i, \\
&e^{(2-n)\rho}\sqrt{\det(g)}g^{ij}g^{k\ell}|A_{i\ell}|\,|A_{jk}| \le c\,e^{2\rho}|A|_g^2 \le c, \text{ and} \\
&e^{-n\rho}\sqrt{\det(g)} \le c,
\end{aligned}
$$

where $c = c(C_{0,0}, \mathcal{U}, h)$. As a result, by Theorem 8.10 in [8], we have

$$
\|u\|_{W^{s+2,2}(\Omega')} \le c\left(\|u\|_{W^{1,2}(\Omega)} + \|e^{2\rho}\hat{B}u\|_{W^{s,2}(\Omega)}\right),
$$

where $\Omega = B_2(0)\times(K, K+3)$ and $\Omega' = B_1(0)\times(K+1, K+2)$ for arbitrary $K \ge 0$. By definition of the weighted Sobolev norms, we have

$$
\begin{aligned}
&\|u\|_{W^{s+2,2}_{\delta,a-2}(V_\alpha\times(K+1,K+2))}\\
&= \left(\sum_{t=0}^{s+2}\int_{V_\alpha\times(K+1,K+2))} |e^{\delta z+(a-2+t)\rho} D^t_{(g)} u|_g^2 \,\mathrm{d}\mu_g\right)^{1/2}\\
&\le c\left(\sum_{t=0}^{s+2}\int_{\Omega'} e^{2\delta z+(2a-4+n)\rho}|\partial^t u|^2\,\mathrm{d}x\right)^{1/2}\\
&\le c\left(\sup_{z\in(K+1,K+2)} e^{2\delta z+(2a-4+n)\rho}\right)\|u\|_{W^{s+2,2}(\Omega')}\\
&\le c\left(\sup_{z\in(K+1,K+2)} e^{2\delta z+(2a-4+n)\rho}\right)\big(\|u\|_{W^{1,2}(\Omega)}+\|e^{2\rho}\hat{B}u\|_{W^{s,2}(\Omega)}\big)\\
&= c\left(\sup_{z\in(K+1,K+2)} e^{2\delta z+(2a-4+n)\rho}\right)\\
&\qquad\cdot\left[\left(\sum_{t=0}^{1}\int_{\Omega}|\partial^t u|^2\,\mathrm{d}x\right)^{1/2}+\left(\sum_{t=0}^{s}\int_{\Omega}|\partial^t(e^{2\rho}\hat{B}u)|^2\,\mathrm{d}x\right)^{1/2}\right]\\
&\le c\left(\sup_{z\in(K+1,K+2)} e^{2\delta z+(2a-4+n)\rho}\right)\\
&\qquad\cdot\Bigg[\left(\sum_{t=0}^{1}\int_{U_\alpha\times(K,K+3)} e^{-2\delta z-(2a-2+n)\rho}|e^{\delta z+(a-4+t)\rho}D^t_{(g)}u|_g^2\,\mathrm{d}\mu_g\right)^{1/2}\\
&\qquad\quad+\left(\sum_{t=0}^{s}\int_{U_\alpha\times(K,K+3)} e^{-2\delta z-(2a-4+n)\rho}|e^{\delta z+(a+t)\rho}D^t_{(g)}(\hat{B}u)|_g^2\,\mathrm{d}\mu_g\right)^{1/2}\Bigg]\\
&\le c\left(\sup_{z\in(K+1,K+2)} e^{2\delta z+(2a-4+n)\rho}\right)\left(\sup_{z\in(K,K+3)} e^{-2\delta z-(2a-4+n)\rho}\right)\\
&\qquad\cdot\big(\|u\|_{W^{1,2}_{\delta,a-2}(U_\alpha\times(K,K+3))}+\|\hat{B}u\|_{W^{s,2}_{\delta,a}(U_\alpha\times(K,K+3))}\big)
\end{aligned}
$$

Note that ρ only depends on z. Observe that

$$
\begin{aligned}
&\log\left[\left(\sup_{z\in(K+1,K+2)} e^{2\delta z+(2a-4+n)\rho}\right)\left(\sup_{z\in(K,K+3)} e^{-2\delta z-(2a-4+n)\rho}\right)\right]\\
&=\sup_{z\in(K+1,K+2)}(2\delta z+(2a-4+n)\rho)-\inf_{z\in(K,K+3)}(2\delta z+(2a-4+n)\rho)
\end{aligned}
$$

$$\leq 2 \sup_{z\in(K,K+3)} |\partial_z(2\delta z + (2a+n)\rho)| \leq c,$$

and hence

$$\|u\|_{W^{s+2,2}_{\delta,a-2}(V_\alpha\times(K+1,K+2))} \leq c\left(\|u\|_{W^{1,2}_{\delta,a-2}(U_\alpha\times(K,K+3))} + \|\hat{B}u\|_{W^{s,2}_{\delta,a}(U_\alpha\times(K,K+3))}\right).$$

Similarly,

$$\|u\|_{W^{s+2,2}_{\delta,a-2}(\Sigma_1)} \leq c\left(\|u\|_{W^{1,2}_{\delta,a-2}(\Sigma_2)} + \|\hat{B}u\|_{W^{s,2}_{\delta,a}(\Sigma_2)}\right).$$

As a result, we have

$$\begin{aligned}
\|u\|_{W^{s+2,2}_{\delta,a-2}} &= \left(\|u\|^2_{W^{s+2,2}_{\delta,a-}(\Sigma_1)} + \sum_\alpha \sum_{K=0}^{\infty} \|u\|^2_{W^{s+2,2}_{\delta,a-2}(V_\alpha\times(K+1,K+2))}\right)^{1/2} \\
&\leq c\Bigg(\|u\|^2_{W^{1,2}_{\delta,a-2}(\Sigma_2)} + \sum_\alpha \sum_{K=0}^{\infty} \|u\|^2_{W^{1,2}_{\delta,a-2}(U_\alpha\times(K,K+3))} \\
&\qquad + \|\hat{B}u\|^2_{W^{s,2}_{\delta,a}(\Sigma_2)} + \sum_\alpha \sum_{K=0}^{\infty} \|\hat{B}u\|^2_{W^{s,2}_{\delta,a}(U_\alpha\times(K,K+3))}\Bigg)^{1/2} \\
&\leq c\left(\|u\|^2_{W^{1,2}_{\delta,a-2}} + \|\hat{B}u\|^2_{W^{s,2}_{\delta,a}}\right)^{1/2} \\
&\leq c\left(\|u\|_{W^{1,2}_{\delta,a-2}} + \|\hat{B}u\|_{W^{s,2}_{\delta,a}}\right).
\end{aligned}$$

In particular, $u \in W^{s+2,2}_{\delta,a-2}(N\Sigma, g)$.

Finally, we can take $s = 0$ and obtain $\mathrm{Dom}(\hat{B}) \subset W^{2,2}_{\delta,a-2}(N\Sigma, g)$. □

By comparing Lemmas 3.2.1 and 3.2.3, we conclude that

$$\mathrm{Dom}(\hat{B}) = W^{2,2}_{\delta,a-2}(N\Sigma, g).$$

Proposition 3.2.4. *Assume* $\lim_{z\to\infty} \inf_{\omega\in\partial\Sigma_0} \rho = \infty$ *and condition (3.1) with* $\beta = 0$ *and* $s_0 = 1$.

Then given $k \in \mathbb{Z}$ and $\varepsilon > 0$,

$$\triangle : W^{2,2}_{\delta,a-2}(N\Sigma, g) \to W^{0,2}_{\delta,a-\varepsilon}(N\Sigma, g)$$

is Fredholm with Fredholm index 0.

Proof. By Lemma 3.2.2 and Lax–Milgram theorem,

$$\begin{array}{rcl} W^{2,2}_{\delta,a-2}(N\Sigma, g) & \to & W^{0,2}_{\delta,a}(N\Sigma, g) \\ u & \mapsto & \widehat{B}u + \zeta e^{-2\rho}u \end{array}$$

is an isomorphism, where $\zeta > 0$ is as given in Lemma 3.2.2. As a result, for all $\varepsilon > 0$,

$$\begin{array}{rcl} W^{2,2}_{\delta,a-2}(N\Sigma, g) & \to & W^{0,2}_{\delta,a-\varepsilon}(N\Sigma, g) \\ u & \mapsto & \widehat{B}u + \zeta e^{-2\rho}u \end{array}$$

is Fredholm with Fredholm index 0.

In addition,

$$\begin{array}{rcl} W^{2,2}_{\delta,a-2}(N\Sigma, g) & \to & W^{1,2}_{\delta,a}(N\Sigma, g) \\ u & \mapsto & -2\nabla_{(\delta\partial_z+(a-1)\mathrm{grad}\rho)}u + \zeta e^{-2\rho}u \end{array}$$

is continuous. Thus by Theorem 3.1.17,

$$\begin{array}{rcl} W^{2,2}_{\delta,a-2}(N\Sigma, g) & \to & W^{0,2}_{\delta,a-\varepsilon}(N\Sigma, g) \\ u & \mapsto & -2\nabla_{(\delta\partial_z+(a-1)\mathrm{grad}\rho)}u - \zeta e^{-2\rho}u \end{array}$$

is compact.

Therefore, by Lemma 3.2.1, $\triangle : W^{2,2}_{\delta,a-2}(N\Sigma, g) \to W^{0,2}_{\delta,a-\varepsilon}(N\Sigma, g)$ is Fredholm with Fredholm index 0. □

Corollary 3.2.5. *For all $\varepsilon > 0$,*

$$\triangle^2_{(g)} : W^{4,2}_{\delta,a-4}(N\Sigma, g) \to W^{0,2}_{\delta,a-\varepsilon}(N\Sigma, g)$$

is also Fredholm with Fredholm index 0.

Proof. First, observe that by Proposition 3.2.4,

$$\triangle_{(g)} : W^{2,2}_{\delta,-a+2\varepsilon}(N\Sigma, g) \to W^{0,2}_{-\delta,-a+\varepsilon}(N\Sigma, g)$$

is Fredholm with Fredholm index 0, and hence its dual map,

$$\triangle_{(g)} : W^{0,2}_{\delta,a-\varepsilon}(N\Sigma, g) \to W^{-2,2}_{\delta,a-2\varepsilon}(N\Sigma, g),$$

is also Fredholm with Fredholm index 0.

Finally, since both operators are Fredholm with Fredholm index 0, the composition is also Fredholm with Fredholm index 0. □

Convention 3.2.6. *Let $f_W : \Sigma \to \mathbb{R}^n$ be a complete, properly immersed Willmore surface. Let F denote the Willmore energy in any of the following forms:*

- $F(\eta) = \frac{1}{2}\int_\Sigma |A|^2 \,\mathrm{d}\mu,$
- $F(\eta) = \frac{1}{2}\int_\Sigma |H|^2 \,\mathrm{d}\mu,$
- $F(\eta) = \frac{1}{2}\int_\Sigma |A^0|^2 \,\mathrm{d}\mu,$ *or*
- $F(\eta) = \frac{1}{2}\int_\Sigma (|A|^2 \,\mathrm{d}\mu - |A_W|^2 \,\mathrm{d}\mu_W),$

where A_W and A denote the second fundamental forms of $f_W(\Sigma)$ and $(f_W + \eta)(\Sigma)$, respectively, and μ_W and μ denote the volume forms of $f_W(\Sigma)$ and $(f_W + \eta)(\Sigma)$, respectively. As *usual,* $H = \text{tr}A$ *and* $A^0 = A - \frac{1}{2}Hg$.

Proposition 3.2.7. *If* A*,* ∇A*,* $\nabla^2 A$*, and* $\nabla^3 A$ *are pointwise bounded:*

(i) There exists an open neighborhood $0 \in U \subset W^{4,2}_{0,\varepsilon}(N\Sigma, g_W)$ *where* $g = (f_W + \eta)^* g_{\mathbb{R}^n}$ *is uniformly equivalent to* g_W*,*

(ii) F *is well-defined and analytic on* U*, and*

(iii) The second derivative

$$D^2F(0) : W^{4,2}_{0,\varepsilon}(N\Sigma, g_W) \to W^{-4,2}_{0,-\varepsilon}(N\Sigma, g_W)$$

is Fredholm with Fredholm index 0.

Proof. (i) First, let $\widehat{\gamma}$ satisfy

$$\begin{cases} \chi_{B_R(0)} \leq \widehat{\gamma} \leq \chi_{B_{R+1}(0} \text{ for some } R > 0 \text{ and} \\ |D\widehat{\gamma}| \leq 2, \end{cases}$$

and let $\gamma = \widehat{\gamma}\big|_\Sigma$. Then by Lemma 2.1.5, we have that for all $\eta \in W^{2,2}_{0,0}(N\Sigma, g_W)$,

$$\begin{aligned} \|\eta\|_\infty &= \lim_{R\to\infty} \|\eta\|_{\infty,[\gamma=1]} \\ &\leq c \liminf_{R\to\infty} \|\eta\|^{1/2}_{2,[\gamma>0]} \big(\|\nabla^2\eta\|^2_{2,[\gamma>0]} + \|\eta\|^2_{2,[\gamma>0]} + \|\,|A_W|^4|\eta|\|_{1,[\gamma>0]}\big)^{1/4} \\ &\leq c\,(1 + \|A_W\|_\infty)\|\eta\|_{W^{2,2}_{0,0}}. \end{aligned}$$

Similarly,

$$\|\nabla\eta\|_\infty \leq c\,(1 + \|A_W\|_\infty)\|\eta\|_{W^{3,2}_{0,0}}.$$

Thus by Lemma 2.5.2, we can find an open neighborhood $U \subset W^{4,2}_{0,\varepsilon}(N\Sigma, g_W)$ of 0 where mapping $\eta \in U$ to $g^{-1} \in L^\infty(T^2_0\Sigma, g_W)$ is well-defined and continuous.

Using the formula for A in the proof of Proposition 2.5.4, by abusing notation,

$$A_{ij} = (A_W)_{ij} + \nabla^2_{ij}\eta + (\nabla^2\eta + \nabla\eta + \eta) * \widehat{P}_0\big(\delta(g^{-1}), \nabla\eta, \nabla A_W\big).$$

Therefore, the integrand in $F(\eta)$ can be rewritten as

$$|A|^2\,\mathrm{d}\mu - |A_W|^2\,\mathrm{d}\mu_W = (\nabla^2\eta + \nabla\eta + \eta)^{\otimes 2} * \widehat{P}_0\big(\delta(g^{-1}, \nabla\eta, \nabla A_W\big)\,\mathrm{d}\mu_W\,.$$

Recall that $\eta \in L^2$, $\nabla\eta \in L^2$, $\nabla^2\eta \in L^2$, and everything else involved in L^∞ are continuous with respect to $\eta \in W^{4,2}_{0,\varepsilon}(N\Sigma, g_W)$, and hence F is continuous on U.

Moreover, every term involved are analytic (cf. [4, Lemma 3.2]), and hence F is also analytic.

(ii) By Proposition 2.5.4, $DF : U \to W^{-2,2}_{0,-\varepsilon}(N\Sigma, g_W)$ is given by

$$\begin{aligned}
DF(\eta) &= \mathbf{W}_N(\eta) \\
&= \triangle^2\eta + \nabla^3\eta * A_W + \nabla^2\eta * (\nabla A_W + A_W * A_W) \\
&\quad + \nabla\eta * (\nabla^2 A_W + \nabla A_W * A_W + A_W * A_W * A_W) \\
&\quad + \eta * (\nabla^3 A_W + \nabla^2 A_W * A_W + \nabla A_W * \nabla A_W \\
&\qquad + \nabla A_W * A_W * A_W + A_W * A_W * A_W * A_W) \\
&\quad + \widehat{P}_2\big(\delta(g^{-1}), h, \nabla^4\eta, \nabla^3 A_W\big).
\end{aligned}$$

Taking derivative, we obtain that $D^2\mathcal{W}(0) : W^{4,2}_{0,\varepsilon}(N\Sigma, g_W) \to W^{0,2}_{0,0}(N\Sigma, g_W)$ is given by

$$
\begin{aligned}
D^2\mathcal{W}(0)(v) = {} & \triangle^2 v + \nabla^3 v * A_W + \nabla^2 v * (\nabla A_W + A_W * A) \\
& + \nabla v * (\nabla^2 A_W + \nabla A_W * A + A_W * A_W * A_W) \\
& + v * (\nabla^3 A_W + \nabla^2 A_W * A_W + \nabla A_W * \nabla A_W + \nabla A_W * A_W * A_W \\
& \quad + A_W * A_W * A_W * A).
\end{aligned}
$$

In particular, by hypothesis and Theorem 3.1.17, $(D^2\mathcal{W}(0) - \triangle^2)$ is compact. Therefore, by Corollary 3.2.5, $D^2\mathcal{W}$ is Fredholm with Fredholm index 0.

□

In view of [3, Corollary 3.11] and [17, Theorem 5.2] (cf. [18]), the following is conjectured:

Conjecture 3.2.8. *Let $f_W : \Sigma \to \mathbb{R}^n$ be a Willmore immersion that is complete and proper, and satisfies condition (3.1) for some $\beta \geq 0$ and $s_0 = 1$. Then:*

(1) There exists $\mathcal{D}_\triangle \subset \mathbb{R}^L$, which is a union of hyperplanes and is of measure zero, such that for all $\delta \in \mathbb{R}^L \backslash \mathcal{D}_\triangle$,

$$\triangle : W^{2,2}_{\delta, a-2}(N\Sigma, g) \to W^{0,2}_{\delta, a-\beta}(N\Sigma, g)$$

is Fredholm with Fredholm index 0.

(2) Let F be as described in Convention 3.2.6. Then for all $\delta \in \mathbb{R}^L \backslash \mathcal{D}_\triangle$,

$$D^2 F : W^{2,2}_{\delta, -2+\beta}(N\Sigma, g) \to W^{-2,2}_{\delta, 2-\beta}(N\Sigma, g)$$

is Fredholm with Fredholm index 0.

(3) For all $\delta \in \mathbb{R}^L \backslash \mathcal{D}_\triangle$, the restriction

$$D^2F : W^{4,2}_{\delta,-4+6\beta}(N\Sigma, g) \to W^{0,2}_{\delta,0}(N\Sigma, g)$$

is well-defined, and the image is a direct summand for a finite-dimensional subspace.

(4) Moreover, for all $\delta \in \mathbb{R}^L \backslash \mathcal{D}_\triangle$, F satisfies the Łojasiewicz–Simon inequality: there exists $\theta \in (0, \frac{1}{2}]$ such that for all $v \in U$,

$$|F(\eta) - F(0)|^{1-\theta} \leq C\|\mathbf{W}(f_W + \eta)\|_{W^{0,2}_{\delta,0}}.$$

Remark 3.2.9. *When $f_W(\Sigma)$ is a plane, the normal bundle is trivial, and hence statements (1) and (2) are proved in [17].*

3.3 Examples

Example 3.3.1 (Plane)**.** *Let Σ be the plane $\{x_3 = 0\}$ in $\mathbb{R}^3$. $\Sigma_0 = \overline{B}^2_1(0) \times \{0\}$ and $\Sigma \backslash \Sigma_0$ is parameterized by the logarithmic polar coordinate $\Phi(\omega, z) = (e^z \cos\omega, e^z \sin\omega, 0)$ for all $\omega \in \mathbb{R}/2\pi$ and $z \in \mathbb{R}_+$. Also, $L = 1$.*

The induced metric $g = e^z(\mathrm{d}z^2 + \mathrm{d}\omega^2)$ is equal to $e^{2\rho}h$, where:

- *$h = h_\infty = \mathrm{d}z^2 + \mathrm{d}\omega^2$, which is a translation invariant metric, and*
- *$\rho = \frac{1}{2}$, making $D\rho = \theta = \frac{1}{2}\,\mathrm{d}z$ a translation invariant 1-form.*

Example 3.3.2 (Catenoid)**.** *Consider a catenoid*

$$\Sigma = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \,:\, \sqrt{x_2^2 + x_3^2} = \cosh(x_1)\}.$$

We let $\Sigma_0 = \{x = 0, y^2 + z^2 = 1\}$*, and let* $\Sigma \backslash \Sigma_0$ *be parameterized by*

$$\Phi_j(\omega, z) = ((-1)^j z\,,\, \cosh z \cos\omega\,,\, \cosh z \sin\omega)$$

for all $j = 1, 2$*,* $\omega \in \mathbb{R}/2\pi$*, and* $z \in \mathbb{R}_+$*. Also,* $L = 2$*.*

The induced metric $g = \cosh z\,(\mathrm{d}z^2 + \mathrm{d}\omega^2)$ *is equal to* $e^{2\rho}h$*, where:*

- $h = h_\infty = \mathrm{d}z^2 + \mathrm{d}\omega^2$*, which is a translation invariant metric, and*
- $\rho = \log(\cosh z/2)$*, where* $D\rho = \tanh z\,\mathrm{d}z$ *satisfies that* $\lim_{z\to\infty}\sup_\omega |D\rho - \mathrm{d}z\,|_h = 0$*.*

In addition, the second fundamental form is given by

$$A_{(g)} = (-\,\mathrm{d}z^2 + \mathrm{d}\omega^2) \otimes \hat{n},$$

where the unit normal vector is

$$\hat{n}\big|_{\Phi_j(\omega,z)} = \big((-1)^j \tanh z, -\operatorname{sech} z \cos\omega, -\operatorname{sech} z \sin\omega\big).$$

In particular,

$$|A_{(g)}|^2_{(g)} = 2\operatorname{sech} z = 2e^{-2\rho}.$$

In addition, it may be useful to know

$$|\nabla A_{(g)}|^2_{(g)} = 4\tanh^2 z \operatorname{sech}^3 z \le 4e^{-3\rho},$$
$$|\nabla^2 A_{(g)}|^2_{(g)} = 18\tanh^4 z \operatorname{sech}^4 z - 12\tanh^2 z \operatorname{sech}^6 z + 4\operatorname{sech}^8 z \le 22e^{-4\rho},$$
$$|\nabla^3 A_{(g)}|^2_{(g)} = 144\tanh^6 z - 288\tanh^4 z \operatorname{sech}^2 z + 242\tanh^2 z \operatorname{sech}^4 z \le 386.$$

Example 3.3.3 (Costa surface). *Consider the Weierstrass representation for minimal surfaces:*

$$X(\zeta) = \mathrm{Re} \begin{bmatrix} \int (1-G^2)F \, \mathrm{d}\zeta \\ \int i(1+G^2)F \, \mathrm{d}\zeta \\ \int 2FG \, \mathrm{d}\zeta \end{bmatrix},$$

where $F = F(\zeta)$ and $G = G(\zeta)$ are meromorphic functions. Writing $\zeta = u + iv$, we have

$$\|\partial_u X\| = \|\partial_v X\| = (1+|G|^2)|F|, \quad \langle \partial_u X, \partial_v X \rangle = 0.$$

Costa showed in [6] that for all $a \in \mathbb{R}\backslash\{0\}$,

$$F(\zeta) = \wp(\zeta) \quad \text{and} \quad G(\zeta) = \frac{a}{\wp'(\zeta)},$$

where $\wp(\zeta)$ denotes the Weierstrass $\wp$-function with respect to the lattice $\Lambda = \mathbb{Z} + i\mathbb{Z}$, defines a complete minimal surface of genus 1 with 3 ends and total curvature -12π. This surface is referred to as the Costa surface. One of the ends is "planar" while the other two are "catenoidal."

Denote $\Sigma = (\mathbb{C}/\Lambda)\backslash\{Q_1, Q_2, Q_3\}$, where Q_1, Q_2, Q_3 are represented by $\frac{1}{2}, 0, \frac{i}{2}$, respectively. Let $\Sigma_0 = \Sigma\backslash\big(B_{1/4}(Q_1) \cup B_{1/4}(Q_2) \cup B_{1/4}(Q_3)\big)$. We let $\Sigma\backslash\Sigma_0$ be parameterized by

$$\Phi_j(\omega, z) = Q_j + \frac{1}{4}e^{-z+i\omega}$$

for all $j = 1, 2, 3$, $\omega \in \mathbb{R}/2\pi$, and $z \in \mathbb{R}_+$. Also, $L = 3$. Consider the translation invariant metric

$$h_\infty = \mathrm{d}z^2 + \mathrm{d}\omega^2 = \frac{1}{16}e^{-2z}(\mathrm{d}u^2 + \mathrm{d}v^2).$$

As pointed out by Costa, F has a order-2 pole at Q_2, G has order-1 poles at Q_1 and Q_3, while both are holomorphic elsewhere. Consider the metric induced by $X : \Sigma \to \mathbb{R}^3$:

$$g = (1 + |G|^2)^2 |F|^2 (\mathrm{d}u^2 + \mathrm{d}v^2) = e^{2\rho} h,$$

where $\rho = 3z + b$ with b being a real constant on each component of $\Sigma \backslash \Sigma_0$. This makes g satisfy item 2 of Definition 3.1.2.

Since $X(\Sigma)$ is minimal, by [27, Lemma 9.1],

$$|A|_g = \sqrt{-2K} = \frac{4\sqrt{2}|G'|}{|F|(1 + |G|^2)^2}.$$

In particular, in a sufficiently small neighborhood of each Q_j,

$$|A|_g \leq Ce^{-2z} \leq Ce^{-\frac{2}{3}\rho}.$$

For more examples and properties of complete minimal surfaces in $\mathbb{R}^3$, see [27], [33], etc.

Chapter 4

Stability of Willmore surfaces

4.1 Gap phenomena and low energy convergence

Assuming small total energy, we derive rigidity results for a Willmore surface to be a plane and a Willmore flow to converge to a plane when given sufficient conditions to converge to some surface.

Proposition 4.1.1. *If $f : \Sigma \times [0,T) \to \mathbb{R}^n$ is a Willmore flow and $\mathcal{W}(f_0) < \frac{1}{2}\varepsilon_0$, then*

$$\int_{\Sigma_t} |A_t|^2 \, \mathrm{d}\mu_t + \frac{1}{2} \int_0^t \int_{\Sigma_{t'}} \left(|\nabla A_{t'}|^2 + |A_{t'}|^6 \right) \mathrm{d}\mu_{t'} \, \mathrm{d}t' \le \int_{\Sigma_0} |A_0|^2 \, \mathrm{d}\mu_0$$

for all t.

Proof. For any $R > 0$, we can find

$$\chi_{B_R(0)} \le \widehat{\gamma} \le \chi_{B_{2R}(0)}$$

as in Lemma 1.3.1 with $K = R^{-1}$ and γ to be the restriction of $\widehat{\gamma}$ on Σ. Consider

$$e(t,R) = \int_{\Sigma_t \cap B_R(0)} |A_t|^2 \,\mathrm{d}\mu_t\,,$$

and

$$t_0(R) = \max\{t \in [0,T] \,:\, \forall \tau \in [0,t),\, e(\tau,R) \le \varepsilon_0\}.$$

As in part (ii) of the proof of Proposition 2.4.5, a continuity argument using Lemma 2.3.4 shows that

$$t_0(R) \ge \min\left\{T,\, c^{-1}R^4\left(1 - \varepsilon_0^{-1}\int_{\Sigma_0} |A_0|^2 \,\mathrm{d}\mu_0\right)\right\}.$$

However, by definition, $t_0(R)$ is decreasing in R, so we can take $R \to \infty$ on the right hand side and obtain $t_0(R) = T$ for all R. Equivalently, for all $0 \le t < T$, $e(t,R) \le \varepsilon_0$ and hence

$$\int_{\Sigma_t} |A_t|^2 \,\mathrm{d}\mu_t = \lim_{R\to\infty} e(t,R) \le \varepsilon_0.$$

Next, we use Lemma 2.3.4 on all $0 \le t < T$ and obtain by monotone convergence theorem that:

$$\begin{aligned}
&\int_{\Sigma_t} |A_t|^2 \,\mathrm{d}\mu_t + \frac{1}{2}\int_0^t \int_{\Sigma_{t'}} \left(|\nabla^2 A_{t'} + |A_{t'}|^6\right) \mathrm{d}\mu_{t'}\,\mathrm{d}t' \\
&= \lim_{R\to\infty}\left(\int_{\Sigma_t \cap B_R(0)} |A_t|^2 \,\mathrm{d}\mu_t + \frac{1}{2}\int_0^t \int_{\Sigma_{t'} \cap B_R(0)} \left(|\nabla^2 A_{t'} + |A_{t'}|^6\right) \mathrm{d}\mu_{t'}\,\mathrm{d}t'\right) \\
&\le \liminf_{R\to\infty}\left(\int_{\Sigma_0 \cap B_{2R}(0)} |A_0|^2 \,\mathrm{d}\mu_0 + c\,R^{-4}\varepsilon_0 t\right) \\
&= \int_{\Sigma_0} |A_0|^2 \,\mathrm{d}\mu_0\,.
\end{aligned}$$

□

As mentioned in the introduction, the following theorem is a special case of both [13, Theorem 2.7] and [36, Theorem 1, (2)].

THEOREM 4.1.2 (Gap rigidity)**.** *If Σ_0 is a complete, smooth, properly immersed Willmore surface with $\mathcal{W}(f_0) < \frac{1}{2}\varepsilon_0$, then Σ_0 is a plane.*

Proof. By hypotheses, $f(x,t) = f_0(x)$ is a Willmore flow. Since $\mathcal{W}(f) = \mathcal{W}(f_0)$, by Proposition 4.1.1,

$$\int_{\Sigma_0} |A_0|^6 \, \mathrm{d}\mu_0 = 0.$$

That is, A_0 vanishes globally. Therefore, Σ_0 is a plane. □

REMARK 4.1.3. *Alternatively, one can show that $f(x,t) = f_0(x)$ is the Willmore flow in Theorem 2.4.6.*

COROLLARY 4.1.4 (Low energy convergence)**.** *Let $f : \Sigma \times [0,\infty) \to \mathbb{R}^n$ be a solution to (1.1). Assume that $\mathcal{W}(f_0) < \frac{1}{2}\varepsilon_0$ and that*

$$\sup_{t\geq 0} \mu_t\big(B_R(0)\big) < \infty \text{ for all } R > 0.$$

Then as $t \to \infty$, any subsequence has a further subsequence such that Σ_t converges to a plane $L : \mathbb{R}^2 \to \mathbb{R}^n$ in the sense as in Definition 2.6.3.

Proof. By Proposition 4.1.1,

$$\int_{\Sigma_t} |A_t|^2 \, \mathrm{d}\mu_t \leq \int_{\Sigma_0} |A_0|^2 \, \mathrm{d}\mu_0 \leq \varepsilon_0.$$

Note that the proof for [13, Theorem 3.5] doesn't require Σ to be closed, and in fact holds as long as the boundary of Σ is not involved. Therefore, by taking ϱ to be arbitrarily big in the aforementioned theorem, we have

$$\|\nabla^k A_t\|_\infty \le c(k)\varepsilon_0 t^{-\frac{k+1}{4}}, \quad \text{for all } k \ge 0.$$

Now we have upper bounds for area and derivatives of curvature in $B_R(0)$, and the bounds depends on R but not on t. As $t \to \infty$, we can use [13, Theorem 4.2] to find a properly immersed surface L to be the limit of Σ. In particular,

$$\|A_L\|_{\infty,L} \le \limsup_{t\to\infty} \|A_t\|_{\infty,\Sigma_t} = 0,$$

where A_L denotes the second fundamental form of L. Therefore, L is a plane. □

Alternatively, we also obtain the following convergence result:

Corollary 4.1.5. *Let $f : \Sigma \times [0,\infty) \to \mathbb{R}^n$ be a solution to (1.1). Assume that $\mathcal{W}(f_0) < \frac{1}{2}\varepsilon_0$ and that*

$$\liminf_{R\to\infty} R^{-2}\mu_0\big(B_R(0)\big) < \infty.$$

Then as $t \to \infty$, any subsequence has a further subsequence such that Σ converges to a plane $L : \mathbb{R}^2 \to \mathbb{R}^n$ in the sense as in Definition 2.6.3.

Proof. We denote $c = c(n)$. For any $R > 0$, we can find

$$\chi_{B_R(0)} \le \widehat{\gamma} \le \chi_{B_{2R}(0)}$$

as in Lemma 1.3.1 with $K = R^{-1}$ and γ to be the restriction of $\widehat{\gamma}$ on Σ. Along the Willmore flow, by [13, Theorem 3.5],

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Sigma_t}\gamma^4\,\mathrm{d}\mu_t\\
&=\int_{\Sigma_t}\big(\langle-\triangle H_t-Q(A^0_t)H_t,-\gamma^4H_t\rangle+s\gamma^3\langle-\triangle H_t-Q(A^0_t)H_t,D\widehat{\gamma}\rangle\big)\,\mathrm{d}\mu_t\\
&\le\int_{\Sigma_t}\gamma^4\big(-|\nabla H_t|^2+|A^0_t|^2|H_t|^2\big)\,\mathrm{d}\mu_t\\
&\quad+c\int_{\Sigma_t}\Big(|\nabla H_t|\left[\gamma^3|D\widehat{\gamma}|\,|A_t|+\gamma^2\big(|D\widehat{\gamma}|^2+|D^2\widehat{\gamma}|\big)\right]+\gamma^3|D\widehat{\gamma}|\,|A^0_t|^2|H_t|\Big)\,\mathrm{d}\mu_t\\
&\le\int_{\Sigma_t}\gamma^4\big(-|\nabla H_t|^2+|A^0_t|^2|H_t|^2\big)\,\mathrm{d}\mu_t+R^{-2}\int_{\Sigma_t}\gamma^4\,\mathrm{d}\mu_t\\
&\quad+c\,R^{-1}\big(\|A_t\|^2_{2,[\gamma>0]}+\|\nabla H_t\|^2_{2,[\gamma>0]}+\|H_t\|_{\infty,[\gamma>0]}\|A^0_t\|^2_{2,[\gamma>0]}\big)+c\,R^{-2}\|\nabla H_t\|^2_{2,[\gamma>0]}\\
&\le\int_{\Sigma_t}\gamma^4\big(-|\nabla H_t|^2+|A^0_t|^2|H_t|^2\big)\,\mathrm{d}\mu_t+R^{-2}\int_{\Sigma_t}\gamma^4\,\mathrm{d}\mu_t\\
&\quad+c\,R^{-1}\big(\varepsilon_0+\varepsilon_0t^{-\frac{1}{2}}+\varepsilon_0^{\frac{3}{2}}t^{-\frac{1}{4}}\big)+c\,R^{-2}\varepsilon_0t^{-\frac{1}{2}}.
\end{aligned}$$

Using (b) of Lemma 2.2.3, Gauss–Codazzi equations, and [13, Theorem 3.5], we have (cf. [13, equation 68]):

$$\begin{aligned}
&\int_{\Sigma_t}\gamma^4\big(-|\nabla H_t|^2+|A^0_t|^2|H_t|^2\big)\,\mathrm{d}\mu_t\\
&\le-2\int_{\Sigma_t}\gamma^4|\nabla A^0_t|^2\,\mathrm{d}\mu_t+c\,R^{-1}\int_{\Sigma_t}\gamma^3|A^0_t|\big(|\nabla H_t|+|\nabla A^0_t|\big)\,\mathrm{d}\mu_t+c\int_{\Sigma_t}\gamma^4|A^0_t|^4\,\mathrm{d}\mu_t\\
&\le c\,\|A^0_t\|^4_{\infty,[\gamma>0]}\int_{\Sigma_t}\gamma^4\,\mathrm{d}\mu_t+c\,R^{-1}\big(\varepsilon_0+\varepsilon_0t^{-\frac{1}{2}}\big).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Sigma_t}\gamma^4\,\mathrm{d}\mu_t\\
&\le\big(R^{-2}+c\,\|A^0_t\|^4_{\infty,[\gamma>0]}\big)\int_{\Sigma_t}\gamma^4\,\mathrm{d}\mu_t+c\,R^{-1}\big(\varepsilon_0+\varepsilon_0t^{-\frac{1}{2}}+\varepsilon_0^{\frac{3}{2}}t^{-\frac{1}{4}}\big)+c\,R^{-2}\varepsilon_0t^{-\frac{1}{2}}.
\end{aligned}$$

In addition, since

$$\int_{[\gamma>0]} |A_0^0|^2 \,\mathrm{d}\mu_0 \le \int_{\Sigma_0} |A_0|^2 \,\mathrm{d}\mu_0 \le \varepsilon_0,$$

by [13, Proposition 3.4], we have

$$\int_0^t \|A_\tau^0\|_{\infty,[\gamma>0]}^4 \,\mathrm{d}\tau \le c\,\varepsilon_0(1+R^{-4}t),$$

and hence by Gronwall's lemma,

$$\int_{\Sigma_t} \gamma^4 \,\mathrm{d}\mu_t \le \left(\int_{\Sigma_0} \gamma^4 \,\mathrm{d}\mu_0 + c\,R^{-1}\big(\varepsilon_0 t + \varepsilon_0 t^{\frac{1}{2}} + \varepsilon_0^{\frac{3}{2}} t^{\frac{3}{4}}\big) + c\,R^{-2}\varepsilon_0 t^{\frac{1}{2}}\right) e^{R^{-2}t + c\,\varepsilon_0(1+R^{-4}t)}.$$

In particular,

$$\mu_t\big(B_R(0)\big) \le \left(\mu_0\big(B_{2R}(0)\big) + c\,R^{-1}\big(\varepsilon_0 t + \varepsilon_0 t^{\frac{1}{2}} + \varepsilon_0^{\frac{3}{2}} t^{\frac{3}{4}}\big) + c\,R^{-2}\varepsilon_0 t^{\frac{1}{2}}\right) e^{R^{-2}t + c\,\varepsilon_0(1+R^{-4}t)}.$$

Next, by monotonicity formula, for any $0 < r < R$,

$$r^{-2}\mu_t\big(B_r(0)\big) \le c\left(R^{-2}\mu_t\big(B_R(0)\big) + \int_{\Sigma_t\cap B_R(0)} |H_t|^2 \,\mathrm{d}\mu_t\right).$$

Moreover, fixing r, t and letting $R \to \infty$,

$$\mu_t\big(B_r(0)\big) \le c\left(\liminf_{R\to\infty} \big[R^{-2}\mu_0\big(B_{2R}(0)\big)\big]\, e^{c\,\varepsilon_0} + \varepsilon_0\right) r^2,$$

which is an area bound that is independent of the time variable t. Therefore, the statement can be proved by Corollary 4.1.4. □

4.2 Global existence

Finally, in view of [4, Lemma 4.1], Conjecture 3.2.8 (Łojasiewicz inequality) may lead to another stability result:

Conjecture 4.2.1. *Let* $f_W : \Sigma \to \mathbb{R}^n$ *be a Willmore immersion that is complete and proper. Assume that the induced metric* $g = e^{2\rho}h$ *is admissible, as in Definition 3.1.2. Assume condition (3.1) for some* $\beta \geq 0$ *and* $s_0 = 1$.

If $f : \Sigma \times [0, T)$ *is a a Willmore flow, where:*

- T *is the maximal existence time,*
- $\mathcal{W}(f_t) \geq \mathcal{W}(f_W)$ *whenever* $\|K(f_t \circ \Phi - f_W)\|_{C^k(\Sigma,h)} \leq \eta$ *up to some diffeomorphism* $\Phi \in \mathrm{Aut}(\Sigma)$*, and*
- $\|f_0 - f_W\|_{W^{2,2}_{\delta,a} \cap C^1} < \varepsilon$*, where* $\varepsilon = \varepsilon(n, k, \eta)$*,*

then $T = \infty$*, and as* $t \to \infty$*,* f_t *converges locally smoothly up to diffeomorphisms to a Willmore surface* f_∞ *that satisfies* $\mathcal{W}(f_\infty) = \mathcal{W}(f_W)$.

Remark 4.2.2. *[1] may suggest a different statement that has stronger assumption while is more likely to be true.*

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Appendix A

Geometric inequalities

In this appendix, we derive several different variants of inequalities regarding L^p norms of tensors with the cutoff functions on manifolds.

A.1 Interpolation inequalities

Interpolation inequalities characterize the convexity property of certain sequences. In the context of this article, we consider the sequence of L^p norms of different derivatives of a given tensor.

Lemma A.1.1. *Let $\{a_m\}_{m=0}^{M}$ be a sequence of non-negative real numbers, and $c_1(\varepsilon)$, ..., $c_{M-1}(\varepsilon)$ be a sequence of functions taking non-negative values such that for all $\varepsilon > 0$ and $1 \leq m \leq (M-1)$,*

$$a_m \leq \varepsilon a_{m+1} + c_m(\varepsilon)\, a_{m-1}.$$

Then for all $1 \leq m_0 \leq (M-1)$ *we also have*

$$a_{m_0} \leq \varepsilon a_M + c\, a_0,$$

where $c = c(\varepsilon, M, c_1, \ldots, c_{M-1})$.

Proof. We prove inductively and start with an observation that the statement is trivial for $M = 2$. Let $M_0 \geq 3$ and assume that the statement is true for all $2 \leq M \leq (M_0 - 1)$. For the case $M = M_0$, if $2 \leq m \leq (M_0 - 1)$, we have

$$\begin{cases} a_m \leq \dfrac{\varepsilon}{2} a_M + b_1 a_1 \\ a_1 \leq \dfrac{1}{2b_1} a_m + b_2 a_0 \end{cases}$$

so that $a_m \leq \varepsilon a_M + b_1 b_2 a_0$; and if $m = 1$, we have

$$\begin{cases} a_{M-1} \leq \varepsilon a_M + b_1 a_m \\ a_m \leq \dfrac{1}{b_1 + 1} a_{M-1} + b_2 a_0 \end{cases}$$

so that $a_m \leq \varepsilon a_M + (b_1 + 1) b_2 a_0$, where in both cases, b_1 and b_2 only depend on $\varepsilon, M, c_1, \ldots,$ and c_{M-1}. This proves that the statement holds for $M = M_0$. Hence by induction, it holds for all $M \geq 2$. □

Proposition A.1.2. *Let* $0 < m_0 < M$ *be integers,* $2 \leq j < \infty$*, and* $p, q \geq j$*. If* $s \geq Mp$ *and* $r \geq Mq$*, then for all* $\varepsilon > 0$*,*

$$\begin{aligned} &K^{M-m_0} \left(\int_\Sigma \gamma^{s-(M-m_0)p} \theta^{r-(M-m_0)q} |\nabla^{m_0} \phi|^j \, \mathrm{d}\mu \right)^{1/j} \\ &\leq \varepsilon \left(\int_\Sigma \gamma^s \theta^r |\nabla^M \phi|^j \, \mathrm{d}\mu \right)^{1/j} + c\, K^M \left(\int_\Sigma \gamma^{s-Mp} \theta^{r-Mq} |\phi|^j \, \mathrm{d}\mu \right)^{1/j} \end{aligned}$$

where $c = c(s, r, \varepsilon, r_\phi, M, j)$.

Proof. Throughout this proof, we let $c = c(s, r, r_\phi, M, j)$. Define

$$a_m = K^{M-m}\left(\int_\Sigma \gamma^{s-(M-m)p}\theta^{r-(M-m)q}|\nabla^m\phi|^j\,\mathrm{d}\mu\right)^{1/j}.$$

Then for each $m_0 + 1 \le m \le M - 1$, using integration by parts and Hölder's inequality,

$$\begin{aligned}
a_m^j &\le c\,K^{(M-m)j}\int_\Sigma \Big(\gamma^{s-(M-m)p}\theta^{r-(M-m)q}|\nabla^{m+1}\phi| \\
&\qquad\qquad + \left|\nabla\left(\gamma^{s-(M-m)p}\theta^{r-(M-m)q}\right)\right|\,|\nabla^m\phi|\Big)|\nabla^m\phi|^{j-2}\,|\nabla^{m-1}\phi|\,\mathrm{d}\mu \\
&\le c\,K^{(M-m)j}\int_\Sigma \gamma^{s-(M-m)p}\theta^{r-(M-m)q}|\nabla^{m+1}\phi|\,|\nabla^m\phi|^{j-2}\,|\nabla^{m-1}\phi|\,\mathrm{d}\mu \\
&\quad + c\,K^{(M-m)j+1}\int_\Sigma \gamma^{s-(M-m)p-1}\theta^{r-(M-m)q-1}|\nabla^m\phi|^{j-1}\,|\nabla^{m-1}\phi|\,\mathrm{d}\mu \\
&\le c\,(a_{m+1} + a_m)a_m^{j-2}a_{m-1}.
\end{aligned}$$

Thus for arbitrary $\varepsilon > 0$,

$$a_m \le c\sqrt{(a_{m+1} + a_m)a_{m-1}} \le \frac{\varepsilon}{2}a_{m+1} + \frac{1}{2}a_m + c\,(1 + \varepsilon^{-1})a_{m-1},$$

which implies

$$a_m \le \varepsilon a_{m+1} + c\,(1 + \varepsilon^{-1})a_{m-1}.$$

By Lemma A.1.1, we can conclude the result. □

Proposition A.1.3. *Let $M \ge 2$ be an integer, $\alpha \ge 0$, $2 \le j < \infty$, and $p, q \ge 0$. If*

$s \geq \max(2p, Mj)$ *and* $r \geq \max(2q, Mj)$, *then for all* $\varepsilon > 0$,

$$
\begin{aligned}
&\alpha \left(\int_\Sigma \gamma^{s-p} \theta^{r-q} |\nabla^{M-1}\phi|^j \, \mathrm{d}\mu \right)^{1/j} \\
&\leq \varepsilon \left[\left(\int_\Sigma \gamma^{s} \theta^{r} |\nabla^{M}\phi|^j \, \mathrm{d}\mu \right)^{1/j} + c\, K^M \left(\int_\Sigma \gamma^{s-Mj} \theta^{r-Mj} |\phi|^j \, \mathrm{d}\mu \right)^{1/j} \right] \\
&\qquad + c\, \alpha^2 \varepsilon^{-1} \left(\int_\Sigma \gamma^{s-2p} \theta^{r-2q} |\nabla^{M-2}\phi|^j \, \mathrm{d}\mu \right)^{1/j},
\end{aligned}
$$

where $c = c(s, r, r_\phi, M)$.

Proof. Using integration by parts,

$$
\begin{aligned}
&\alpha^j \int_\Sigma \gamma^{s-p} \theta^{r-q} |\nabla^{M-1}\phi|^j \, \mathrm{d}\mu \\
&\leq c\, \alpha^j \int_\Sigma \left(\gamma^{s-p} \theta^{r-q} |\nabla^{M}\phi| + \gamma^{s-p-1} \theta^{r-q-1} K |\nabla^{M-1}\phi| \right) |\nabla^{M-1}\phi|^{j-2} |\nabla^{M-2}\phi| \, \mathrm{d}\mu \\
&\leq c\, \alpha^j \left(\int_\Sigma \gamma^{s} \theta^{r} |\nabla^{M}\phi|^j \, \mathrm{d}\mu + \int_\Sigma \gamma^{s-j} \theta^{r-j} K^j |\nabla^{M-1}\phi|^j \, \mathrm{d}\mu \right)^{1/j} \\
&\qquad \cdot \left(\int_\Sigma \gamma^{s-p} \theta^{r-q} |\nabla^{M-1}\phi|^j \, \mathrm{d}\mu \right)^{(j-2)/j} \left(\int_\Sigma \gamma^{s-2p} \theta^{s-2q} |\nabla^{M-2}\phi|^j \, \mathrm{d}\mu \right)^{1/j} \\
&\leq \frac{\varepsilon^j}{4} \int_\Sigma \gamma^{s} \theta^{r} |\nabla^{M}\phi|^j \, \mathrm{d}\mu + \frac{\varepsilon^j}{2} \int_\Sigma \gamma^{s-j} \theta^{r-j} K^j |\nabla^{M-1}\phi|^j \, \mathrm{d}\mu \\
&\qquad + \frac{\alpha^j}{2} \int_\Sigma \gamma^{s-p} \theta^{r-q} |\nabla^{M-1}\phi|^j \, \mathrm{d}\mu + c\, \alpha^{2j} \varepsilon^{-j} \int_\Sigma \gamma^{s-2p} \theta^{s-2q} |\nabla^{M-2}\phi|^j \, \mathrm{d}\mu \,,
\end{aligned}
$$

and hence

$$
\begin{aligned}
&\alpha^j \int_\Sigma \gamma^{s-p} \theta^{r-q} |\nabla^{M-1}\phi|^j \, \mathrm{d}\mu \\
&\leq \frac{\varepsilon^j}{2} \int_\Sigma \gamma^{s} \theta^{r} |\nabla^{M}\phi|^j \, \mathrm{d}\mu + \varepsilon^j \int_\Sigma \gamma^{s-j} \theta^{r-j} K^j |\nabla^{M-1}\phi|^j \, \mathrm{d}\mu \\
&\qquad + c\, \alpha^{2j} \varepsilon^{-j} \int_\Sigma \gamma^{s-2p} \theta^{s-2q} |\nabla^{M-2}\phi|^j \, \mathrm{d}\mu \,.
\end{aligned}
$$

Next, by Proposition A.1.2,

$$\begin{aligned}
&\alpha^j \int_\Sigma \gamma^{s-p}\theta^{r-q}|\nabla^{M-1}\phi|^j \,\mathrm{d}\mu \\
&\le \varepsilon^j \int_\Sigma \gamma^s\theta^r|\nabla^M\phi|^j \,\mathrm{d}\mu + c\,\alpha^{2j}\varepsilon^{-j}\int_\Sigma \gamma^{s-2p}\theta^{s-2q}|\nabla^{M-2}\phi|^j \,\mathrm{d}\mu \\
&\qquad + c\,\varepsilon^j K^{Mj}\int_\Sigma \gamma^{s-Mj}\theta^{r-Mj}|\phi|^j \,\mathrm{d}\mu\,,
\end{aligned}$$

which is equivalent to the inequality to be proved. □

Proposition A.1.4. *Let $0 \le m_1 < m_0 < M$ be integers, $\alpha \ge 0$, $2 \le j < \infty$, and $p, q \ge 0$. If $s \ge \max\big(Mp - m_1(p-j), Mj\big)$ and $r \ge \max\big(Mq - m_1(q-j), Mj\big)$, then for all $\varepsilon > 0$,*

$$\begin{aligned}
&\alpha^{M-m_0}\left(\int_\Sigma \gamma^{s-(M-m_0)p}\theta^{r-(M-m_0)q}|\nabla^{m_0}\phi|^j \,\mathrm{d}\mu\right)^{1/j} \\
&\le \varepsilon\left(\int_\Sigma \gamma^s\theta^r|\nabla^M\phi|^j \,\mathrm{d}\mu\right)^{1/j} + c\,\alpha^{M-m_1}\left(\int_\Sigma \gamma^{s-(M-m_1)p}\theta^{r-(M-m_1)q}|\nabla^{m_1}\phi|^j \,\mathrm{d}\mu\right)^{1/j} \\
&\qquad + c\left(K^M + \alpha^{M-m_1}K^{m_1}\right)\left(\int_{[\gamma\theta>0]} |\phi|^j \,\mathrm{d}\mu\right)^{1/j},
\end{aligned}$$

where $c = c(s, r, \varepsilon, r_\phi, M)$.

Proof. By Proposition A.1.3, for all $m = (m_1+1), \ldots, (M-1)$, there exists $b_m \ge 0$ such that

$$\begin{aligned}
&\alpha\left(\int_\Sigma \gamma^{s-(M-m)p}\theta^{r-(M-m)q}|\nabla^m\phi|^j \,\mathrm{d}\mu\right)^{1/j} \\
&\le \varepsilon\Bigg[\left(\int_\Sigma \gamma^{s-(M-m-1)p}\theta^{r-(M-m-1)q}|\nabla^{m+1}\phi|^j \,\mathrm{d}\mu\right)^{1/j} \\
&\qquad\qquad + b_{m+1}\,K^{m+1}\left(\int_\Sigma \gamma^{s-Mp+(m+1)(p-j)}\theta^{r-Mq+(m+1)(q-j)}|\phi|^j \,\mathrm{d}\mu\right)^{1/j}\Bigg] \\
&\qquad + \alpha^2\varepsilon^{-1}b_{m-1}\left(\int_\Sigma \gamma^{s-(M-m+1)p}\theta^{s-(M-m+1)q}|\nabla^{m-1}\phi|^j \,\mathrm{d}\mu\right)^{1/j}.
\end{aligned}$$

We can construct a sequence $\{\widehat{b}_m\}$ inductively by

$$\widehat{b}_m = \begin{cases} \max\{b_m, 1\} & \text{if } m = m_1 \text{ or } m_1 + 1\text{, and} \\ b_m + \dfrac{\widehat{b}_{m-1}^2}{4\widehat{b}_{m-2}} & \text{if } m_1 + 2 \le m \le M, \end{cases}$$

so that

$$\begin{cases} \widehat{b}_m \ge b_m & \text{for all } m = m_1, \ldots, M\text{, and} \\ \widehat{b}_m \le 2\sqrt{\widehat{b}_{m-1}(\widehat{b}_{m+1} - b_{m+1})} & \text{for all } m = m_1 + 1, \ldots, M - 1, \end{cases}$$

and hence

$$\begin{aligned} &\alpha\,\widehat{b}_m K^m \left(\int_\Sigma \gamma^{s-Mp+m(p-j)} \theta^{r-Mq+m(q-j)} |\phi|^j \,\mathrm{d}\mu \right)^{1/j} \\ &\le \alpha\,\widehat{b}_m K^m \left(\int_\Sigma \gamma^{s-Mp+(m+1)(p-j)} \theta^{r-Mq+(m+1)(q-j)} |\phi|^j \,\mathrm{d}\mu \right)^{1/(2j)} \\ &\qquad\qquad \cdot \left(\int_\Sigma \gamma^{s-Mp+(m-1)(p-j)} \theta^{r-Mq+(m-1)(q-j)} |\phi|^j \,\mathrm{d}\mu \right)^{1/(2j)} \\ &\le \varepsilon(\widehat{b}_{m+1} - b_{m+1}) K^{m+1} \left(\int_\Sigma \gamma^{s-Mp+(m+1)(p-j)} \theta^{r-Mq+(m+1)(q-j)} |\phi|^j \,\mathrm{d}\mu \right)^{1/j} \\ &\quad + \alpha^2 \varepsilon^{-1} \widehat{b}_{m-1} K^{m-1} \left(\int_\Sigma \gamma^{s-Mp+(m-1)(p-j)} \theta^{r-Mq+(m-1)(q-j)} |\phi|^j \,\mathrm{d}\mu \right)^{1/j}. \end{aligned}$$

Let

$$\begin{aligned} a_m = \alpha^{M-m} \Bigg[&\left(\int_\Sigma \gamma^{s-(M-m)p} \theta^{r-(M-m)q} |\nabla^m \phi|^j \,\mathrm{d}\mu \right)^{1/j} \\ &+ \widehat{b}_m K^m \left(\int_\Sigma \gamma^{s-Mp+m(p-j)} \theta^{r-Mq+m(q-j)} |\phi|^j \,\mathrm{d}\mu \right)^{1/j} \Bigg] \end{aligned}$$

so that by the two inequalities above, we get

$$a_m \le \varepsilon a_{m+1} + \varepsilon^{-1} a_{m-1}.$$

Therefore, by Lemma A.1.1, there exists some $c = c(s, r, \varepsilon, r_\phi, M)$ such that

$$
\begin{aligned}
&\alpha^{M-m_0}\left(\int_\Sigma \gamma^{s-(M-m_0)p}\theta^{r-(M-m_0)q}|\nabla^{m_0}\phi|^j\,\mathrm{d}\mu\right)^{1/j}\\
&\leq a_{m_0}\\
&\leq \varepsilon a_M + c\,a_{m_1}\\
&\leq \varepsilon\left(\int_\Sigma \gamma^s\theta^r|\nabla^M\phi|^j\,\mathrm{d}\mu\right)^{1/j}\\
&\quad + c\,\alpha^{M-m_1}\left(\int_\Sigma \gamma^{s-(M-m_1)p}\theta^{r-(M-m_1)q}|\nabla^{m_1}\phi|^j\,\mathrm{d}\mu\right)^{1/j}\\
&\quad + c\left(K^M + \alpha^{M-m_1}K^{m_1}\right)\\
&\quad \cdot\left(\int_\Sigma \gamma^{s-\max\left(Mp-m_1(p-j),Mj\right)}\theta^{r-\max\left(Mq-m_1(q-j),Mj\right)}|\phi|^j\,\mathrm{d}\mu\right)^{1/j}.
\end{aligned}
$$

□

A.2 Multiplicative Sobolev inequalities

Sobolev inequalities provide an upper bound of the target norm of a function in terms of the given norm of the function, and hence characterize embeddings from a Sobolev space to another, say, from $W^{2,2}$ to $L^p = W^{0,p}$ for some $p > 2$. In general, constants that are involved in Sobolev inequalities would depend on the domain. In our case, the constants only depend on the mean curvature of the surface and the dimension of the ambient Euclidean space.

Theorem A.2.1 (Michael–Simon Sobolev inequality [26]). *Let $f : \Sigma^2 \to \mathbb{R}^n$ be a smooth immersion. Then for any $u \in C^1_c(\Sigma)$ we have*

$$
\left(\int_\Sigma u^2\,\mathrm{d}\mu\right)^{1/2} \leq c_n\left(\int_\Sigma |\nabla u|\,\mathrm{d}\mu + \int_\Sigma |H|\,|u|\,\mathrm{d}\mu\right),
$$

where c_n is a constant only depending on n.

Convention A.2.2. *Let $h \in C^1_c(\Sigma)$ satisfy $\|\nabla h\|_\infty \le c\,K$, where $c = c(n, s, r)$. For example, $h = \gamma^s \theta^r$.*

We can rewrite [10, Lemma 5.1] as the following.

Lemma A.2.3. *Let $1 \le p, q, w \le \infty$ satisfy $\frac{1}{p} + \frac{1}{q} = \frac{1}{w}$ and $\alpha, \beta \in \mathbb{R}$ satisfy $\alpha + \beta = 1$. For any $b \ge \max(\alpha q, \beta p)$ and $-\frac{1}{p} \le t \le \frac{1}{q}$, we have*

$$\|h^{b/(2w)}\nabla\phi\|_{2w}^2 \le c\,\|h^{b(p^{-1}+t)}\nabla^2\phi\|_p\,\|h^{b(q^{-1}-t)}\phi\|_q + c\,bK\|h^{b/p-\beta}\nabla\phi\|_p\,\|h^{b/q-\alpha}\phi\|_q,$$

where $c = c(n, w, r_\phi)$.

Lemma A.2.4 (Cf. [10, Theorem 5.6])**.** *For all $u \in C^1(\Sigma)$ and $h \in C^1(\Sigma)$ such that their product has compact support, $0 < m < \infty$, and $2 < p < \infty$,*

$$\|h^\alpha u\|_\infty \le c\,\|u\|_m^{1-\alpha}\big(\|h\nabla u\|_p + K\|u\|_p + \|huH\|_p\big)^\alpha,$$

where

$$\alpha = \frac{2p}{(p-2)m + 2p}$$

and $c = c(n, m, p)$.

Proof. Let

$$q = \frac{1}{1 - \frac{1}{p}} \in (1, 2),$$

$$\tau_0 = m/q \in (0, \infty),$$

$$\beta_0 = 0,$$

$$\tau_\nu = \left(\frac{2}{q}\right)^\nu \cdot \tau_0 + \frac{\left(\frac{2}{q}\right)^{\nu+1} - \frac{2}{q}}{\frac{2}{q} - 1} \in (0, \infty) \qquad \text{for all } \nu = 1, 2, \ldots, \text{ and}$$

$$\beta_\nu = \frac{\left(\frac{2}{q}\right)^{\nu+1} - \frac{2}{q}}{\frac{2}{q} - 1} \in (0, \infty) \qquad \text{for all } \nu = 1, 2, \ldots;$$

so that the numbers solve the inductive formulas (where $\nu = 0, 1, \ldots$)

$$\begin{cases} \tau_{\nu+1} q = 2(1 + \tau_\nu), \\ \dfrac{\beta_{\nu+1}}{\tau_{\nu+1}} = \dfrac{1 + \beta_\nu}{1 + \tau_\nu}, \end{cases}$$

namely,

$$\frac{\beta_{\nu+1}}{1 + \beta_\nu} = \frac{\tau_{\nu+1}}{1 + \tau_\nu} = \frac{2}{q}.$$

As a result, we see that

$$\begin{aligned}
&\left\| h^{\beta_{\nu+1}/\tau_{\nu+1}} u \right\|_{\tau_{\nu+1} q}^{1+\tau_\nu} \\
&= \left\| h^{(1+\beta_\nu)/(1+\tau_\nu)} u \right\|_{2(1+\tau_\nu)}^{1+\tau_\nu} \\
&= \left\| h^{1+\beta_\nu} u^{1+\tau_\nu} \right\|_2 \\
&\le c_n \left(\left\| \nabla \left(h^{1+\beta_\nu} u^{1+\tau_\nu} \right) \right\|_1 + \left\| h^{1+\beta_\nu} u^{1+\tau_\nu} H \right\|_1 \right) \qquad \text{(Theorem A.2.1)} \\
&\le c_n \left((1 + \tau_\nu) \left\| h^{1+\beta_\nu} u^{\tau_\nu} \nabla u \right\|_1 + (1 + \beta_\nu) \left\| h^{\beta_\nu} u^{1+\tau_\nu} \nabla h \right\|_1 + \left\| h^{1+\beta_\nu} u^{1+\tau_\nu} H \right\|_1 \right) \\
&\le c_n \left\| h^{\beta_\nu} u^{\tau_\nu} \right\|_q \Big((1 + \tau_\nu) \left\| h \nabla u \right\|_p + (1 + \beta_\nu) \left\| u \nabla h \right\|_p + \left\| h u H \right\|_p \Big) \\
&\le A B_\nu \left\| h^{\beta_\nu/\tau_\nu} u \right\|_{\tau_\nu q}^{\tau_\nu},
\end{aligned}$$

where

$$A = c_n \left(\left\| h \nabla u \right\|_p + \left\| u \nabla h \right\|_p + \left\| h u H \right\|_p \right) \quad \text{and} \quad B_\nu = 1 + \tau_\nu \ge 1 + \beta_\nu.$$

Observe that

$$\left(\frac{q}{2}\right)^{\nu} B_\nu \leq c_* := \tau_0 + \frac{2}{2-q}.$$

Therefore,

$$\left\|h^{\beta_{\nu+1}/\tau_{\nu+1}} u\right\|_{\tau_{\nu+1}q} \leq \left(Ac_*\left(\frac{2}{q}\right)^{\nu}\right)^{1/(1+\tau_\nu)} \left\|h^{\beta_\nu/\tau_\nu} u\right\|_{\tau_\nu q}^{\tau_\nu/(1+\tau_\nu)}.$$

Define $\varepsilon_\nu = \tau_\nu/(1+\tau_\nu)$ so that we get from the previous inequality that

$$\left\|h^{\beta_\nu/\tau_\nu} u\right\|_{\tau_\nu q} \leq \|u\|_{\tau_0 q}^{\varepsilon_0\times\cdots\times\varepsilon_{\nu-1}} \prod_{j=0}^{\nu-1} \left(Ac_*\left(\frac{2}{q}\right)^{j}\right)^{\varepsilon_j\times\cdots\times\varepsilon_{\nu-1}/\tau_j}$$

On the left hand side, we have

$$\lim_{\nu\to\infty} \tau_\nu q = \infty \text{ and } \lim_{\nu\to\infty} \frac{\beta_\nu}{\tau_\nu} = \frac{2}{(2-q)\tau_0+2} = \alpha,$$

and hence

$$\lim_{\nu\to\infty} \left\|h^{\beta_\nu/\tau_\nu} u\right\|_{\tau_\nu q} = \left\|h^{\alpha} u\right\|_{\infty}.$$

On the right hand side, observe that

$$\varepsilon_j \times \cdots \times \varepsilon_{\nu-1} = \frac{\tau_j}{1+\tau_{\nu-1}} \left(\frac{2}{q}\right)^{\nu-j-1}$$

so that we have

$$\lim_{\nu\to\infty} \left(\varepsilon_0 \times \cdots \times \varepsilon_{\nu-1}\right) = \frac{(2-q)\tau_0}{(2-q)\tau_0+2} = 1-\alpha,$$

which implies

$$\lim_{\nu\to\infty} \left\|h^{\beta_0/\tau_0}u\right\|_{\tau_0 q}^{\varepsilon_0\times\cdots\varepsilon_{\nu-1}} \leq \left\|u\right\|_m^{1-\alpha},$$

$$\lim_{\nu\to\infty} \sum_{j=0}^{\nu-1} \left(\varepsilon_j \times \cdots \times \varepsilon_{\nu-1}/\tau_j\right) = \frac{2}{(2-q)\tau_0+2} = \alpha,$$

which implies

$$\lim_{\nu\to\infty} \prod_{j=0}^{\nu-1} (Ac_*)^{\varepsilon_j\times\cdots\times\varepsilon_{\nu-1}/\tau_j} = (Ac_*)^{\alpha},$$

and

$$\lim_{\nu\to\infty} \sum_{j=0}^{\nu-1} \left(j\times\varepsilon_j \times \cdots \times \varepsilon_{\nu-1}/\tau_j\right) = \frac{4}{(2-q)^2\tau_0+2(2-q)}.$$

In summary,

$$\begin{aligned}\left\|h^{\alpha}u\right\|_{\infty} &\leq c\left\|u\right\|_m\Big(\left\|h\nabla u\right\|_p + \left\|u\nabla h\right\|_p + \left\|huH\right\|_p\Big)\\ &\leq c\left\|u\right\|_m\Big(\left\|h\nabla u\right\|_p + K\left\|u\right\|_p + \left\|huH\right\|_p\Big),\end{aligned}$$

where

$$c = (c_n c_*)^{\alpha}\cdot\left(\frac{2}{q}\right)^{4/[(2-q)^2\tau_0+2(2-q)]}$$

only depends on n, m, and p. □

Lemma A.2.5. *For all $u\in C_c^1(\Sigma)$ and $2<p<\infty$,*

$$\|u\|_p \leq c\,\|u\|_2^{2/p}\big(\|\nabla u\|_2 + \|Hu\|_2\big)^{1-2/p},$$

where $c = c(n,p)$.

Proof. For all positive integers τ, by Theorem A.2.1, we have

$$\begin{aligned}\|u^{1+\tau}\|_2 &\le c_n\,(1+\tau)\left(\int_\Sigma |u|^\tau|\nabla u|\,\mathrm{d}\mu + \int_\Sigma |H|\,|u|^{1+\tau}\,\mathrm{d}\mu\right)\\ &\le c_n\,(1+\tau)\|u^\tau\|_2\big(\|\nabla u\|_2 + \|Hu\|_2\big).\end{aligned}$$

As a result, by induction,

$$\|u^\tau\|_2 \le c_n^{\tau-1}\,(\tau!)\|u\|_2\big(\|\nabla u\|_2 + \|Hu\|_2\big)^{\tau-1},$$

or equivalently,

$$\|u\|_{2\tau} \le c_n\left[\tau!\,\|u\|_2\big(\|\nabla u\|_2 + \|Hu\|_2\big)^{\tau-1}\right]^{1/\tau}.$$

Finally, take $\tau = \left\lceil \frac{p}{2} \right\rceil$ so that

$$\begin{aligned}\|u\|_p &\le \|u\|_2^{\frac{2\tau-p}{(\tau-1)p}}\|u\|_{2\tau}^{\frac{\tau(p-2)}{(\tau-1)p}}\\ &\le c_n(\tau!)^{1/\tau}\|u\|_2^{\frac{2\tau-p}{(\tau-1)p}}\|u\|_2^{\frac{p-2}{(\tau-1)p}}\big(\|\nabla u\|_2 + \|Hu\|_2\big)^{\frac{p-2}{p}}\\ &= c_n(\tau!)^{1/\tau}\|u\|_2^{\frac{2}{p}}\big(\|\nabla u\|_2 + \|Hu\|_2\big)^{1-\frac{2}{p}}.\end{aligned}$$

□