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ADVENTURES IN COULOMB GAUGE *†

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We study the phase structure of SU(2) gauge theories at zero and high temperature, with and without scalar matter fields, in terms of the symmetric/broken realization of the remnant gauge symmetry which exists after fixing to Coulomb gauge. The symmetric realization is associated with a linearly rising color Coulomb potential (which we compute numerically), and is a necessary but not sufficient condition for confinement.

There are several reasons why Coulomb gauge may be interesting and/or useful in the study of the confining force. First of all there is the speculation by Gribov [1] and Zwanziger [2] that confinement in Coulomb gauge is due to instantaneous (dressed) one-gluon exchange. Secondly, the behavior of the color Coulomb potential, defined in Coulomb gauge, is an important element in the gluon-chain model of QCD string formation [3]. Finally, as we will see below, the confining property of the color Coulomb potential is associated with the unbroken realization of a remnant gauge symmetry, and this suggests a new order parameter for studying the phase structure of lattice gauge theories.

We begin with the idea that confinement arises from one-gluon exchange in Coulomb gauge; specifically, from the instantaneous piece of the $\langle A_0 A_0 \rangle$ propagator

$$\langle A_0^a(x) A_0^b(y) \rangle = P(\vec{x}-\vec{y})\delta^{ab}\delta(x_0-y_0) + \text{non-instantaneous} \qquad (1)$$

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where

$$P(\vec{x}-\vec{y})\delta^{ab} = \left\langle \left[\frac{1}{\nabla \cdot D(A)} (-\nabla^2) \frac{1}{\nabla \cdot D(A)} \right]^{a,b}_{x,y} \right\rangle \tag{2}$$

where $D_i(A)$ is the covariant derivative. This quantity is directly related to the Coulomb interaction energy in Coulomb gauge.

Recall that the classical Hamiltonian in Coulomb gauge, $H = H_{glue} + H_{coul}$, has the form

$$\begin{aligned} H_{glue} &= \frac{1}{2} \int d^3x \; (E^{tr,a} \cdot E^{tr,a} + B^a \cdot B^a) \\ H_{coul} &= \frac{1}{2} \int d^3x d^3y \; \rho^a(x) K^{ab}(x,y;A) \rho^b(y) \\ K^{ab}(x,y;A) &= \left[\frac{1}{\nabla \cdot D(A)} (-\nabla^2) \frac{1}{\nabla \cdot D(A)} \right]^{ab}_{xy} \\ \rho^a &= \rho^a_{matter} - g f^{abc} A^b_k E^c_k \end{aligned} \tag{3}$$

Note that $\langle K \rangle$ is the instantaneous piece of the $\langle A_0 A_0 \rangle$ propagator. Gribov and Zwanziger argue that this propagator is enhanced by configurations at the Gribov horizon, defined as a boundary in function space where the operator $\nabla \cdot D(A)$ aquires a zero eigenvalue. The conjecture is that this enhancement leads to a confining Coulomb potential, and therefore confinement by one-gluon exchange.

One objection to this idea is that it is difficult to see how the string-like properties of the QCD flux tube, namely, the logarithmic growth of the flux tube cross-section (roughening), and the universal $-\pi/12R$ contribution to the static quark potential (the Lüscher term), could arise from one-gluon exchange. On the other hand, as we will see below, the color Coulomb potential is an upper bound on the confining static quark potential. This means that confinement by one-gluon exchange is a necessary condition for confinement.

Let

$$|\Psi_{qq}\rangle = \overline{q}^a(0) q^a(R) |\Psi_0\rangle \tag{4}$$

be a physical state in Coulomb gauge containing two static charges; Ψ_0 is the ground state. Then

$$\begin{aligned} \Delta E &= \langle \Psi_{qq} | H | \Psi_{qq} \rangle - \langle \Psi_0 | H | \Psi_0 \rangle \\ &= V_{coul}(R) + E_{se} \end{aligned} \tag{5}$$

is the expectation value of the excitation energy, where the R-dependent Coulomb potential $V_{coul}(R)$ can only arise from the expectation value of the non-local $\rho K \rho$ piece of the Hamiltonian. We want to address the following questions: Is $V_{coul}(R)$ confining? If so, is it asymptotically linear? If it is linear, then is the associated string tension σ_{coul} equal to string tension σ of the static quark potential? Finally, we would like to study the effect, on the Coulomb string tension, of removing center vortices.

We begin by defining the correlator, for SU(N) gauge theory in Coulomb gauge, of two timelike Wilson lines

$$\begin{aligned} G(R,T) &= \langle \frac{1}{N} \mathrm{Tr}[L^\dagger(0,T)L(R,T)] \rangle \\ &= \langle \Psi_{qq} | e^{-(H-E_0)T} | \Psi_{qq} \rangle \end{aligned} \tag{6}$$

where

$$L(\vec{x},T) = P \exp \left[i \int_0^T dt A(\vec{x},t) \right] \tag{7}$$

Note that L is a timelike Wilson line (*not* a Polyakov line) of time extent T. The existence of a transfer matrix implies

$$G(R,T) = \sum_n |\langle \Psi_n | \Psi_{qq} \rangle|^2 e^{-\Delta E_n T} \tag{8}$$

where the sum is over energy eigenstates, and ΔE_n is the energy above the ground state. Denote

$$V(R,T) = -\frac{d}{dT} \log[G(R,T)] \tag{9}$$

Then its not hard to see that

$$\begin{aligned} \Delta E &= V_{coul}(R) + E_{se} \\ &= V(R,0) \end{aligned} \tag{10}$$

while

$$\begin{aligned} \Delta E_{min} &= V(R) + E'_{se} \\ &= \lim_{T \to \infty} V(R,T) \end{aligned} \tag{11}$$

where ΔE_{min} is the minimum energy of the $q\overline{q}$ system, and $V(R)$ is the static quark potential. The use of Wilson line correlators, in Coulomb gauge, to compute the static potential was first suggested by Marinari et al. [4].

With a lattice regularization, the self-energies E_{se} and E'_{se} become negligible, at large R, compared to the confining static potential $V(R)$. Then, since $\Delta E_{min} \leq \Delta E$, it follows that asymptotically

$$V(R) \leq V_{coul}(R) \tag{12}$$

This inequality was originally derived by Zwanziger [5]. It implies that if confinement exists, it exists already at the level of dressed one-gluon exchange.

With a lattice regularization, we have

$$\begin{aligned} L(x,T) &= U_0(x,a)U_0(x,2a)\cdots U_0(x,T) \\ V(R,T) &= \frac{1}{a}\log\left[\frac{G(R,T)}{G(R,T+a)}\right] \end{aligned} \tag{13}$$

so that

$$\lim_{\beta\to\infty} V(R,0) = V_{coul}(R) \ + \ \text{const.}$$

$$\lim_{T\to\infty} V(R,T) = V(R) \ + \ \text{const.} \tag{14}$$

where, in lattice units $a = 1$,

$$V(R,0) = -\log[G(R,1)] \tag{15}$$

Via lattice Monte Carlo we can then arrive at an estimate, exact in the continuum limit, of $V_{coul}(R)$ from $V(R,0)$, and compare this to the static quark potential $V(R)$.

This procedure was carried out in ref. [6] for SU(2) lattice gauge theory. The result for $V(R,0)$ at $\beta = 2.5$ is shown in Fig. 1 (upper set of data points). The two lines shown are best fits to the data by a linear, and by a linear + Lüscher, potential. The data immediately answers three of the questions posed above: The Coulomb potential is confining, and it is linear (see also [7]). However, it turns out that the associated string tension σ_{coul} is substantially greater than σ, by almost a factor of three. This means that the QCD flux tube is not simply the static charges and their associated Coulomb (longitudinal color electric) field. The minimum energy string state is more complicated than (4); it must also contain some constituent gluons, as in the gluon chain picture of string formation advocated in [3].

The same figure shows the effect, on $V(R,0)$, of center vortex removal. The center vortex theory of confinement has been studied very actively in recent years; the theory and the numerical evidence in its favor are reviewed in ref. [8]. Center vortices are identified by first fixing to an adjoint gauge,

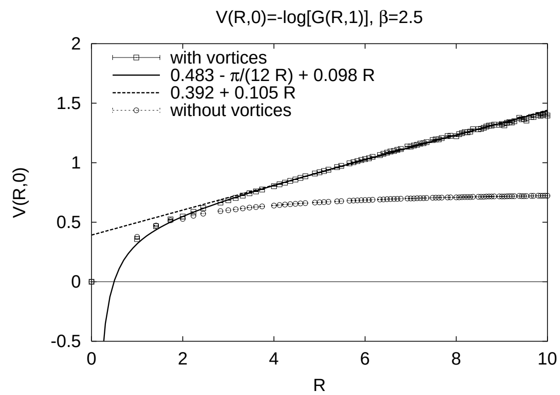

Figure 1. Lattice approximation $V(R,0)$ (upper data set, open squares) to the color Coulomb potential $V_{coul}(R)$ + constant. The lines are best fits to linear, and linear + Lüscher, potentials. The lower data set (open circles) for $V(R,0)$ is obtained after removing center vortices from the lattice configurations, as described in the text.

and then projecting link variables to the center subgroup of the gauge group. An example is the direct maximal center gauge (= Landau gauge in the adjoint representation), where the procedure is to gauge fix to a local maximum of

$$R = \sum_{x,\mu} \Big| \mathrm{Tr}[U_\mu(x)] \Big|^2 \tag{16}$$

and then to project each link to the closest center element, e.g. for SU(2)

$$U_\mu(x) \to Z_\mu(x) = \mathrm{signTr}[U_\mu(x)] \tag{17}$$

Vortices are removed from a given lattice configuration by multiplying the adjoint gauge-fixed configuration by the projected configuration, i.e.

$$U_\mu(x) \to U'_\mu(x) = Z_\mu(x) U_\mu(x) \tag{18}$$

In ref. [9] it was shown that after vortex removal the string tension vanishes, chiral symmetry breaking is eliminated, and each vortex-removed configuration has zero topological charge.

One can then ask what vortex removal does to the color Coulomb potential. In this connection, a relevant fact is that thin center vortices can be shown [10] to lie on the Gribov horizon, which is thought to play an important role in the enhancement of the Coulomb energy. In our numerical study, the modified configuration $U'_\mu(x)$ is gauge-fixed to Coulomb gauge, the timelike link correlators are calculated, and $V(R,0)$ is extracted. The result at $\beta = 2.5$ is shown in the lower data set (open circles) in Fig. 1. It is clear that vortex removal completely removes the confining property of the Coulomb potential. Further details can be found in ref. [6].

Of course, pure gauge theory at zero temperature is a special, albeit very important, case. In general gauge theories can exist in various phases, and it is useful to characterize these in terms of the distribution of the electric field emanating from a static isolated source.

- *Massless Phase:* The electric field is spherically symmetric, and falls off like $1/R^2$. This is the case for compact QED_4, and for lattice SU(N) gauge theory in $D > 4$ dimensions, at weak couplings.
- *Confined Phase:* The color electric field is collimated into a flux tube; global center symmetry is unbroken. Examples include SU(N) pure gauge theories at low temperature, and SU(N) gauge theories with matter in the adjoint representation of the gauge group.
- *Screened Phases:* There is a Yukawa-like falloff of the color electric field. This happens in SU(N) gauge theories when the Z_N center symmetry is broken spontaneously, as in high temperature gauge theory and gauge-Higgs theories with the Higgs field in the adjoint representation. Gauge theories with only a trivial center symmetry (consisting only of the identity element) are also in the screened phase; these theories include SU(N) gauge theories with matter in the fundamental representation, and G_2 gauge theory with or without matter fields.

For the purpose of studying Coulomb energy in these various phases, we find it useful to introduce a new order parameter, related to the realization of a remnant gauge symmetry in Coulomb gauge.

Let $U_\mu(x)$ be a lattice configuration fixed to Coulomb gauge. Note that the Coulomb gauge condition is preserved by the gauge transformation

$$\begin{aligned} U_k(x,t) &\rightarrow g(t)U_k(x,t)g^\dagger(t) \\ U_0(x,t) &\rightarrow g(t)U_0(x,t)g^\dagger(t+1) \end{aligned} \tag{19}$$

On any time slice, this is a global transformation, and therefore can be spontaneously broken in the following sense: At any fixed time t, in the infinite volume limit, the average of timelike link variables $U_0(x,t)$ is non-zero in any thermalized configuration. This means that

$$\begin{aligned}\lim_{R\to\infty} G(R,1) &> 0 \\ \lim_{R\to\infty} V(R,0) &= \text{finite const.} \\ \sigma_{coul} &= 0\end{aligned} \tag{20}$$

in the broken phase. Therefore Coulomb confinement or non-confinement can be understood as the symmetric or spontaneously broken realization, respectively, of the remnant gauge symmetry in Coulomb gauge.

We now introduce the order parameter Q, as the modulus of the spatial average of timelike links, i.e.

$$\begin{aligned}U_0^{av}(t) &= \frac{1}{L^3}\sum_{\vec{x}} U_0(\vec{x},t) \\ Q &= \left\langle \sqrt{\text{Tr}[U_0^{av}(t)U_0^{av\dagger}(t)]} \right\rangle\end{aligned} \tag{21}$$

On general grounds

$$Q = c + \frac{b}{L^{3/2}} \quad \text{with} \quad \begin{cases} c = 0 \text{ symmetric phase} \\ c > 0 \text{ broken phase} \end{cases} \tag{22}$$

Thus $Q > 0$ in the infinite volume limit implies that $V_{coul}(R)$ is non-confining. Since $V_{coul}(R)$ is an upper bound on $V(R)$, this implies that $Q = 0$ is a necessary (but not sufficient) condition for confinement.

It is useful to try out this order parameter in compact QED_4, where there is a transition from the confining to the massless phase at $\beta \approx 1$. Figure 2 shows our results for Q vs. the root inverse 3-volume $L^{-3/2}$ at $\beta = 0.7$ (confining phase) and $\beta = 1.3$ (massless phase). In this case the Q parameter seems to nicely distinguish between the two phases, extrapolating to zero only in the confined phase.

Next, we consider SU(2) gauge-Higgs theory with a "frozen" Higgs field in the adjoint representation. The lattice Lagrangian is

$$\begin{aligned}S &= \beta \sum_{plaq} \frac{1}{2}\text{Tr}[UUU^\dagger U^\dagger] \\ &\quad + \frac{\gamma}{4}\sum_{x,\mu} \phi^a(x)\phi^b(x+\widehat{\mu})\text{Tr}[\sigma^a U_\mu(x)\sigma^b U_\mu^\dagger(x)]\end{aligned} \tag{23}$$

where ϕ is a real 3-component field satisfying the constraint $\sum_a (\phi^a)^2 = 1$. This is a theory with a confining, center symmetric phase, and a non-confining phase with spontaneously broken center symmetry. Our finding

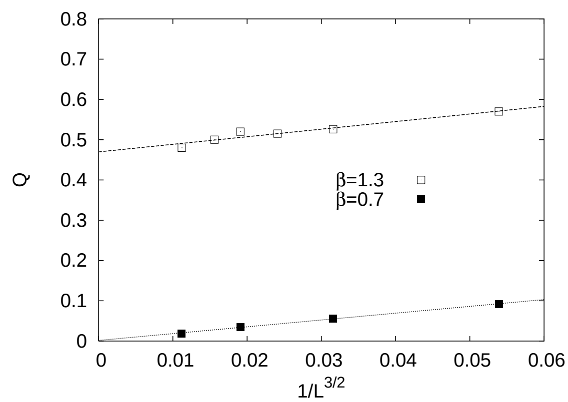

Figure 2. Extrapolation of Q to infinite volume in QED_4, for $\beta = 0.7$ (confining phase) and $\beta = 1.3$ (massless phase).

is that the transition line in the $\beta - \gamma$ phase diagram corresponding to the remnant symmetry-breaking transition is identical to the transition line for confinement-deconfinement, mapped out long ago by Brower et al. [11] from measurements of the plaquette energy. In the confined phase we find $Q = 0$ (when extrapolated to infinite volume), and $Q > 0$ in the Higgs phase, as indicated schematically in Figure 3.

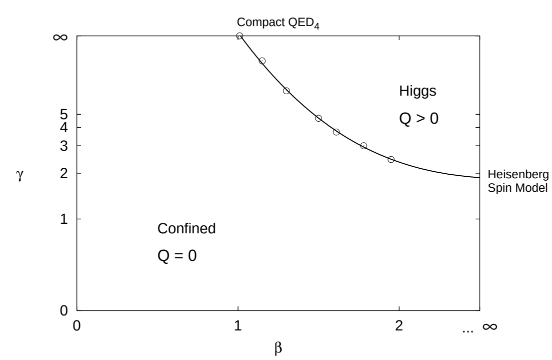

Figure 3. Phase diagram of the SU(2) adjoint Higgs model. The plaquette energy and the remnant symmetry order parameter Q locate the same transition line between the confined and Higgs phases.

One might guess that the transition from the confined to the deconfined phase is always accompanied by remnant symmetry breaking. Surprisingly,

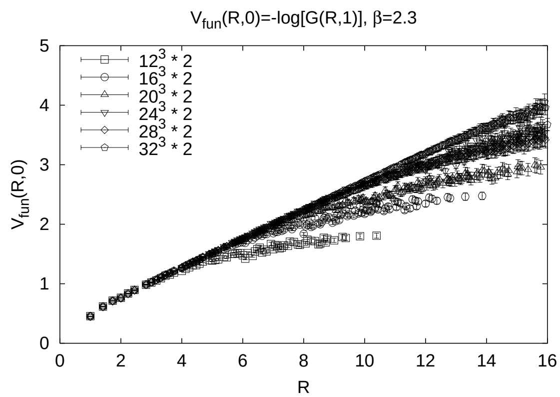

Figure 4. Coulomb potential in the deconfined phase, at $\beta = 2.3$ and $L_t = 2$ lattice spacings in the time direction, with spatial volumes from 12^3 to 32^3.

this turns out not to be true. We have also computed $V(R,0)$ and Q in the deconfined phase of pure SU(2) lattice gauge theory, with the results shown in Figs. 4 and 5. This data was taken at $\beta = 2.3$ on lattices with time extension of two lattice spacings, well within the deconfined phase. Yet the Coulomb potential is clearly linear and confining at large lattice volume, while the extrapolation of Q to infinite volume seems compatible with zero. A possible reason for this behavior is the fact that $K(x, y; A)$, whose expectation value gives the instantaneous Coulomb propagator, depends only on the spacelike components A_k at a fixed time. On the lattice, this translates to dependence only on spacelike links on a time slice. But we know that spacelike links on a time slice are a confining ensemble even in the deconfined phase, since spacelike Wilson loops are known to have an area law falloff at any temperature. If the Coulomb propagator depends only on the confining properties of spacelike links, then it is not so surprising that the Coulomb potential is confining in the deconfined regime (nor is this a paradox: the Coulomb potential is only an upper limit on the static potential). A test of this explanation is to remove the confining properties of the spacelike links by removing center vortices, via the de Forcrand/D'Elia procedure explained above. Then one expects the Coulomb potential to be non-confining, and this is, in fact, what is observed.

Finally, we study a gauge-Higgs system with the radially frozen Higgs field in the fundamental representation. For the SU(2) gauge group, the

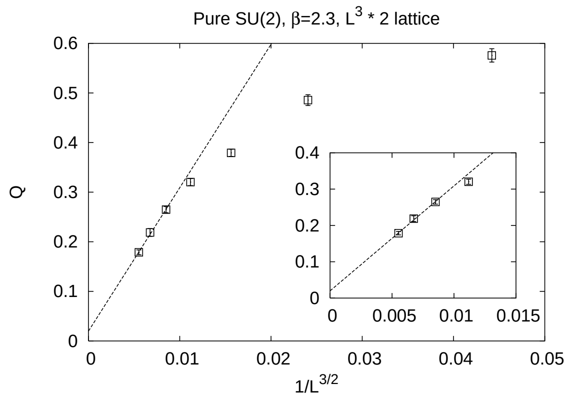

Figure 5. The Q parameter vs. root inverse 3-volume in the deconfined phase, $\beta = 2.3$ and $L_t = 2$ lattice spacings.

lattice Lagrangian can be expressed as [12]

$$S = \beta \sum_{plaq} \frac{1}{2}\mathrm{Tr}[UUU^\dagger U^\dagger] + \gamma \sum_{x,\mu} \frac{1}{2}\mathrm{Tr}[\phi^\dagger(x)U_\mu(x)\phi(x+\widehat{\mu})] \tag{24}$$

with ϕ an SU(2) group-valued field. This is a theory with only a screened phase; it can be proven that no transition to a confined phase is possible [13]. There is a first-order phase transition line in the $\beta - \gamma$ phase diagram, but this line has an endpoint, and does not divide the diagram into thermodynamically separate phases.

The remnant symmetry transition line coincides with the (thermodynamic) line of first-order transitions found by Lang et al. [12], but it then extends beyond the thermodynamic line all the way to $\beta = 0$ and $\gamma = 2$. This line divides the phase diagram into $Q = 0$ and $Q > 0$ regions, as indicated schematically in Fig. 6. In Fig. 7 we plot Q vs. γ at $\beta = 0$. If Q were the magnetization of an Ising spin system, this would surely be a second order phase transition, with the solid line in the figure representing the infinite volume limit. Nevertheless, there is no thermodynamic transition. At $\beta = 0$ one can easily compute the free energy exactly, which is found to be

$$F(\gamma) = 4V \log\left[\frac{2I_1(\gamma)}{\gamma}\right] \tag{25}$$

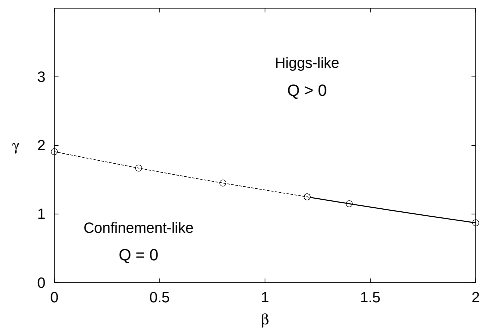

Figure 6. Phase diagram of the fundamental Higgs model. There is a thermodynamic transition and a Q transition along the solid line, but a non-thermodynamic transition (Kertész line) in Q along the dashed line.

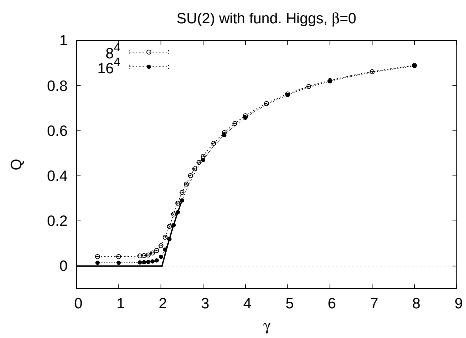

Figure 7. Q vs. γ at $\beta = 0$ in the SU(2) fundamental Higgs model, on 8^4 and 16^4 lattices. The solid line is the presumed extrapolation to infinite volume.

This expression is perfectly analytic at all $\gamma > 0$. On the other hand, a strong-coupling analysis of $G(R, 1)$ at fixed $\beta \ll 1$ [10] arrives at an exponential decay to zero as $R \to \infty$ at small γ, but a non-zero large-R limit at large γ. This implies a symmetry-breaking transition at some critical value $\gamma_{cr}(\beta)$, which motivated our numerical study of Q in this model.

The remnant symmetry breaking transition in the gauge-Higgs system, in the absence of a thermodynamic transition, is probably an example of a *Kertész line* [14] in statistical mechanics. This possibility was first suggested by Langfeld [15], who discovered remnant gauge symmetry breaking in Landau gauge in a closely related model (see also Satz [16]). Kertész lines are associated with percolation transitions, and it is natural to ask, in this case, what is percolating. Recent investigations [17] indicate that in the gauge-Higgs system, the Kertész line locates center vortex percolation transitions.

This concludes a summary of our investigations of phase structure in lattice gauge theory, as seen by Coulomb energy and remnant symmetry. Our study has uncovered Coulomb "over-confinement" ($\sigma_{coul} \approx 3\sigma$) in the low temperature confined phase, the persistence of Coulomb confinement in the deconfined phase, connections between vortex and Coulomb confinement, and (in accord with Langfeld [15]) symmetry breaking in the absence of a thermodynamic phase transition. These aspects of non-perturbative gauge theory are somewhat surprising, and merit further study.

Acknowledgments

Results reported in the second half of this talk were obtained in collaboration with Daniel Zwanziger. These results will be presented in more detail in a later publication [10].

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