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Santa Barbara

Mixed-State Quantum Matter Under Measurement and
Decoherence

A dissertation submitted in partial satisfaction of the
requirements for the degree Doctor of Philosophy
in Physics

by

Kaixiang Su

Committee in charge:

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June 2026

The dissertation of Kaixiang Su is approved.

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May 2026

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group meetings and gave me projects to work on. Through these interactions, I learned not only how to solve particular problems, but also how a physicist develops direction and taste, and how much care is needed to turn an initial result into a mature piece of physics. His advice helped me think more clearly about the kind of physicist I wanted to become.

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Publications and Preprints

1. Jacob Hauser, Kaixiang Su, Hyunsoo Ha, Jerome Lloyd, Thomas G. Kiely, Romain Vasseur, Sarang Gopalakrishnan, Cenke Xu, and Matthew P. A. Fisher, “Strong-to-Weak Symmetry Breaking in Open Quantum Systems: From Discrete Particles to Continuum Hydrodynamics,” arXiv preprint arXiv:2602.16045 (2026).
2. Kaixiang Su, A. Sarma, M. Bintz, T. Kiely, Y. Bao, M. P. A. Fisher, and C. Xu, “Spin liquid and superconductivity emerging from steady states and measurements,” *Physical Review Letters* **135**, 050403 (2025).
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Fields of Study

- Mixed-state quantum matter, open quantum systems, and generalized symmetry. Studies with Professor Cenke Xu.
- Quantum information diagnostics and strong-to-weak symmetry breaking. Studies with Professors Cenke Xu and Matthew P. A. Fisher.
- Condensed matter theory, topology, and many-body physics. Studies with Professors Cenke Xu, Matthew P. A. Fisher, and Leon Balents.

Abstract

Mixed-State Quantum Matter Under Measurement and Decoherence

by

Kaixiang Su

Modern quantum platforms naturally realize many-body states shaped by measurement, decoherence, and dissipation. The appropriate object is often not a wave function or ground state, but a density matrix. This thesis asks how phases and phase transitions can be defined for such mixed states, especially when universal structure appears only through nonlinear, replicated, or doubled diagnostics.

The thesis develops this perspective using nonlinear diagnostics, doubled-Hilbert-space representations, statistical-mechanics mappings, and generalized symmetry. It shows that weak measurement can suppress higher-form symmetry breaking and drive mixed-state transitions; maps fidelity diagnostics of decohered Chern insulators to effective conformal-field-theory partition functions; and maps Rényi diagnostics of noisy stabilizer and CSS codes to dual statistical-mechanics models connected to decodability. It also shows that measurement and Lindbladian evolution can prepare density matrices whose doubled-space representations realize spin-liquid-like states and superconducting order. Finally, it identifies strong-to-weak symmetry breaking as a route to universal mixed-state behavior connected to hydrodynamic emergence.

Together, these results show that openness is not only a perturbation that degrades

pure-state physics. Measurement, decoherence, and dissipation can hide, preserve, transform, or generate structure in density matrices. The resulting phases and transitions are tied to information processing: what can be decoded, distinguished, recovered, or lost to local probes.

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Chapter 1

Introduction

1.1 Open quantum systems and mixed-state many-body physics

Modern quantum platforms force many-body physics to confront a basic change of object. The experimentally realized state is often not a wave function or a ground state, but a density matrix shaped by decoherence, measurement, and dissipation. Quantum simulators, noisy intermediate-scale quantum devices, and quantum computers inevitably couple to their surrounding environments and are therefore subject to noise [238]. At the same time, these platforms increasingly allow controlled measurements, including local and mid-circuit measurements. Measurement can produce ensembles of post-measurement states, or mixed states when measurement outcomes are not selected. The quantum states realized on these platforms are therefore naturally described from an open system perspective.

This change of object requires a corresponding change in the notion of many-body order. In a pure-state setting, phases of matter are usually organized by ground-state correlations, symmetry breaking, topology, and entanglement. For density matrices generated by noise, measurement, or Lindblad evolution, this framework is not sufficient

by itself. A density matrix can contain universal phase structure that is invisible to ordinary single-copy expectation values. The central problem is to identify this structure and to determine which diagnostics, representations, and symmetry principles make it visible.

This thesis develops a framework for identifying universal phase structure in density matrices, especially in regimes where ordinary single-copy observables fail to detect the relevant order. The framework combines nonlinear diagnostics, doubled-Hilbert-space representations, statistical-mechanics mappings, and generalized symmetry. Across several settings motivated by modern quantum platforms, the results demonstrate that measurement, decoherence, and open-system dynamics can suppress, preserve, or generate order in density matrices. The resulting mixed-state phases host different classes of universality whose operational meanings are often connected to information processing, decoding, and recoverability.

1.2 Mixed-state phases and nonlinear diagnostics

A central distinction between mixed-state phases and pure-state phases is that the relevant structure is not always visible in observables linear in the density matrix. Ordinary single-copy expectation values reveal averaged local quantities. They do not necessarily reveal how information is stored in the full density matrix, or whether this information can be distinguished, decoded, or locally recovered. It is therefore natural that nonlinear quantities, such as $\text{Tr } \rho^n$, Rényi entropies, replicated correlation functions, and conditional mutual information, appear repeatedly as diagnostics [40, 244]. These quantities can reveal sharp changes in mixed states even when conventional correlation functions remain smooth.

The operational meaning of these diagnostics depends on the setting. In decohered quantum error-correcting codes, Rényi entropies are tied to decodability and error

thresholds. In mixed-state topology, fidelities and strange-correlator-like overlaps probe topological information that survives decoherence. In open-system dynamics, conditional mutual information and Markov lengths diagnose whether local regions can be reconstructed from their surroundings. The common point is that nonlinear diagnostics extract internal structure of ρ that is hidden from single-copy observables.

Another recurring diagnostic is to compare a nontrivial state or density matrix with a simpler reference state. In pure-state symmetry-protected topological systems, the strange correlator demonstrates that a bulk overlap with a trivial state can reveal boundary-like structure [309]. A similar logic is useful in mixed states. Fidelities and overlap-type quantities can extract information that is not directly present in ordinary local observables, and can diagnose whether topological information survives decoherence [312]. This perspective is especially natural for decohered Chern insulators, where the overlap with a trivial insulator can be understood as the partition function of an effective conformal field theory.

1.3 Doubled-Hilbert-space representations

Many tools of quantum many-body theory were originally formulated for wave functions. It is therefore useful to represent a density matrix as an operator in a doubled Hilbert space. By vectorizing ρ , or equivalently using a Choi-type operator-state representation, one can regard the density matrix as a vector in $\mathcal{H} \otimes \mathcal{H}^*$ [55, 143]. This does not turn the mixed state into a pure state of the original system. Rather, it gives a precise representation of the operator structure of ρ .

The doubled-Hilbert-space viewpoint makes it possible to discuss symmetry actions, correlations, and ordering tendencies of a density matrix using a language close to ordinary many-body theory. This representation is especially important when order is not visible in single-copy observables but appears in nonlinear functions of ρ . In later

chapters, it provides the natural language for decohered topological states, measurement-generated spin-liquid-like states, and strong-to-weak symmetry breaking.

1.4 Duality, statistical mechanics, and universality

Duality has proven useful in obtaining structural results in phases and phase transitions, even when the system is otherwise not exactly solvable. In mixed-state settings this becomes especially valuable, since exact results are generally rarer. Nonlinear functions of density matrices often introduce replicas, Euclidean defects, or sums over measurement outcomes and noise histories. These constructions can convert mixed-state questions into partition functions of effective statistical-mechanics models. The resulting statistical weights retain information about the underlying quantum operator structure while also incorporating classical probabilities associated with noise or measurement.

Once such a mapping is obtained, duality and self-duality can constrain phase diagrams, critical points, and relations between different noise channels [69, 299]. This is not merely a formal analogy. It gives calculable access to mixed-state transitions that are difficult to diagnose directly in the density matrix. This logic underlies both the measurement-driven transitions of higher-form symmetry and the decoherence-induced transitions of quantum error-correcting codes.

1.5 Generalized symmetry in open systems

Symmetry remains an organizing principle for open quantum systems, but the relevant notion of symmetry must be enlarged in two directions. The first generalization is higher-form symmetry. Ordinary global symmetries are 0-form symmetries, whose charged objects are point-like. Many exotic quantum states instead have extended charged objects, such as lines, loops, or membranes. These states are naturally organized by higher-form symmetries [87, 202]. This language is essential for understanding why measurement can

affect topological and gauge-theoretic order differently from ordinary symmetry breaking.

The second generalization concerns how symmetry acts in an open system. For density matrices and quantum channels, it is important to distinguish strong symmetry from weak symmetry. Roughly, a strong symmetry acts separately on the two sides of the density matrix or constrains individual dynamical processes, while a weak symmetry constrains only the averaged state or channel in a diagonal way [64]. This distinction leads to phases and transitions that are absent in ordinary pure-state settings. Together, higher-form symmetry and strong-versus-weak symmetry organize the ways measurement and decoherence can suppress, preserve, or generate mixed-state order.

1.6 Main contributions

The results of this thesis apply the above ideas in several settings where density matrices arise naturally from measurement, decoherence, or open-system dynamics. The main contributions are as follows.

- The thesis demonstrates that weak local measurement can suppress spontaneous breaking of a higher-form symmetry and drive mixed-state transitions with dual descriptions.
- It maps fidelity-type diagnostics of decohered Chern insulators to effective conformal field theories, making universal data of mixed-state topology accessible through overlap-type quantities.
- It constructs statistical-mechanics models for Rényi diagnostics of decohered stabilizer and Calderbank-Shor-Steane codes, and organizes these models through duality and self-duality.

- It shows that Lindbladian evolution and strong measurement can generate density matrices whose doubled-space representations realize Gutzwiller-projected spin-liquid-like states and superconducting order.
- It identifies strong-to-weak symmetry breaking as a dynamical route to universal mixed-state behavior, and connects this transition to hydrodynamic emergence and the loss of microscopic information.

1.7 Overview of the thesis

The chapters are organized as a progression from measurement-driven mixed-state transitions, to topology and quantum error correction, and finally to open-system dynamics as a mechanism for preparing and hydrodynamically organizing mixed-state order.

Chapter 2 studies higher-form symmetries under weak local measurement. It demonstrates that weak measurement can suppress spontaneous breaking of a 1-form symmetry and drive transitions with natural dual descriptions. This chapter establishes measurement as a dynamical process that can change the phase structure of a density matrix, rather than merely read out a preexisting state.

Chapter 3 turns to decohered Chern insulators. It shows that the overlap between a decohered topological band state and a trivial reference state can be mapped to the partition function of an effective two-dimensional conformal field theory. This chapter uses fidelity-type diagnostics to make mixed-state topology visible through universal field-theoretic data.

Chapter 4 studies decohered quantum error-correcting codes. It maps Rényi diagnostics of stabilizer and Calderbank-Shor-Steane codes under Pauli decoherence to statistical-mechanics models organized by duality and self-duality. This chapter gives the operationally sharpest setting of the thesis, where mixed-state phase transitions are tied to error thresholds and decodability.

Chapter 5 studies open-system routes to preparing order. It shows that Lindbladian evolution and strong local measurements can produce density matrices whose doubled-space representations realize Gutzwiller-projected spin-liquid-like states, and in some cases superconducting order through strong-to-weak $U(1)$ symmetry breaking. This chapter demonstrates that measurement and dissipation can generate exotic mixed-state order natural to quantum platforms.

Chapter 6 studies strong-to-weak spontaneous symmetry breaking from microscopic models to continuum hydrodynamics. It identifies dimension-dependent mixed-state ordering behavior and connects the transition to the emergence of hydrodynamic descriptions and the loss of microscopic information about particle worldlines. This final technical chapter synthesizes nonlinear diagnostics, information processing, generalized symmetry, and universal open-system dynamics.

Taken together, these chapters treat openness not merely as a perturbation that degrades pure-state order, but as a setting in which density matrices acquire their own diagnostics, phase structure, and universal critical behavior.

Chapter 2

Higher-Form Symmetries Under Weak Measurement

This chapter studies the fate of higher-form symmetry breaking under weak local measurement. Starting from a state with spontaneously broken higher-form symmetry, weak measurement can drive an intrinsic transition that suppresses 1-form symmetry breaking and weakens the fluctuation of the corresponding higher-form charge. The transition has a natural Euclidean defect description: measurement produces terms localized on temporal boundaries, and after tracing the density matrix these terms become a defect in space-time. In three spatial dimensions, the resulting defect theory is a mixed-dimensional nonlocal QED and can exhibit a line of fixed points related by self-duality.

This chapter contributes to the thesis by making measurement a mechanism for changing mixed-state phase structure, not only a probe of a preexisting state. The higher-form setting makes sharp why mixed-state transitions can be invisible or trivial in ordinary local observables while remaining universal in quantities that probe the full density matrix.

2.1 Introduction

Measurement is a natural nonunitary operation in modern quantum platforms. It is not only a readout process. It can change entanglement, prepare long-range entangled states, and generate density matrices whose phase structure differs from that of the original state. In the last few years, measurement-induced entanglement transitions [20, 56, 83, 104, 105, 126, 135, 144, 156, 163, 164, 185, 222, 251, 268] and measurement-prepared long-range entangled states [136, 137, 167, 278, 279, 287, 323] have attracted substantial attention. Measurement-induced entanglement transitions have also been observed on noisy intermediate-scale quantum platforms [126, 156, 222, 237].

Symmetry is another organizing principle for phases of matter. The classic Landau paradigm is based on symmetry and spontaneous symmetry breaking (SSB). For decades, non-Landau phases such as topological order and spin liquids have motivated a broader notion of order. Generalized symmetries, including higher-form symmetries [86, 88, 128, 147, 201, 223, 259], make this broader structure more precise. In this language, topological order can often be interpreted as an SSB phase of a discrete higher-form symmetry, and the emergent photon phase [120, 133, 182] of systems such as quantum spin ice [29, 44] can be interpreted as the SSB phase of an emergent $U(1)$ 1-form symmetry.

A higher-form symmetry can have phenomenology that is sharply different from an ordinary 0-form symmetry. For example, the ground-state degeneracy associated with spontaneous breaking of a discrete 1-form symmetry, which is often interpreted as topological degeneracy, is robust against weak explicit breaking of the 1-form symmetry. The photon excitations associated with spontaneous breaking of a $U(1)$ 1-form symmetry are also robust against weak explicit breaking [124, 138, 162, 230]. The corresponding Goldstone modes can be gapped only after sufficiently strong breaking of the 1-form symmetry, through a Higgs transition. This differs from ordinary 0-form symmetry

breaking, where explicit breaking immediately gives the Goldstone mode a mass and produces a pseudo-Goldstone mode.

This chapter shows that higher-form and 0-form symmetry breaking also behave differently under weak quantum measurement. If the initial state breaks an ordinary 0-form symmetry, finite-strength local weak measurement cannot drive an intrinsic phase transition. By contrast, a state with higher-form symmetry breaking can undergo a measurement-driven transition. These transitions have a natural description as temporal defects in Euclidean space-time. For a continuous $U(1)$ 1-form symmetry, the resulting transition can have a line of fixed points connected by self-duality. For discrete 1-form symmetries, related measurement problems lead to classical statistical-mechanics models with sharp phase transitions [275].

2.2 General formula

Without loss of generality, a pure density matrix $\hat{\rho}_0$, after going through a series of local weak measurements, is transformed according to $\hat{\rho}_0 \rightarrow \hat{\rho} = \mathcal{E}[\hat{\rho}_0]$, where $\mathcal{E}[\hat{\rho}_0]$ is a composition of local measurements $\mathcal{E}[\hat{\rho}_0] = \otimes_x \mathcal{E}_x[\hat{\rho}_0]$ where

$$\mathcal{E}_x[\hat{\rho}_0] = (1 - p)\hat{\rho}_0 + p \left(\sum_m p_m \hat{P}_{m,x} \hat{\rho}_0 \hat{P}_{m,x} \right). \quad (2.1)$$

Here p measures the strength of the coupling between the system and the ancilla qubits; explicit microscopic protocols are described in the appendices to this chapter. The operator $\hat{P}_{m,x}$ projects the system to measurement outcome m , and the outcomes are summed with weight p_m . When there is no post-selection of the measurement outcomes, $p_m = 1$ and the mapping $\mathcal{E}$ is a quantum channel. With post-selection, p_m can be more general. This chapter considers weak measurements with and without post-selection.¹

¹When p_m differs from 1, $\mathcal{E}$ generically yields un-normalized density matrices that must be normalized. The resulting transformation on $\hat{\rho}_0$ becomes $\hat{\rho}_0 \rightarrow \mathcal{E}[\hat{\rho}_0] / \text{tr}\{\mathcal{E}[\hat{\rho}_0]\}$, which is a non-linear map of the density matrix.

For a generic complicated quantum many-body system, deriving the exact microscopic density matrix is difficult and often unnecessary. The Euclidean space-time path integral formalism allows us to use various analytical techniques, such as coarse-graining and the renormalization group (RG), to capture the physics in the infrared limit. In this formalism, a density matrix $\hat{\rho}_0$ of a pure state $|\Psi\rangle$ is evaluated as $[\hat{\rho}_0]_{\phi_1(\mathbf{x}), \phi_2(\mathbf{x})} \sim \lim_{\beta \rightarrow \infty} \int D\phi(\mathbf{x}, \tau) \exp(-\mathcal{S})$, with the boundary condition $\phi(\mathbf{x}, 0) = \phi_1(\mathbf{x})$ and $\phi(\mathbf{x}, \beta) = \phi_2(\mathbf{x})$. Here $\mathcal{S}$ is the action of the system whose ground state is the desired pure state $|\Psi\rangle$. After the weak measurements (with or without post-selection), the density matrix becomes [173]

$$[\hat{\rho}]_{\phi_1(\mathbf{x}), \phi_2(\mathbf{x})} \sim \lim_{\beta \rightarrow \infty} \int D\phi(\mathbf{x}, \tau) \exp(-\mathcal{S} - \mathcal{S}^{\text{int}});$$

$$\mathcal{S}^{\text{int}} = \int d\mathbf{x} \mathcal{L}^{\text{int}}(\phi(\mathbf{x}, 0), \phi(\mathbf{x}, \beta)). \quad (2.2)$$

In this path integral, the aforementioned boundary condition $\phi(\mathbf{x}, 0) = \phi_1(\mathbf{x})$ and $\phi(\mathbf{x}, \beta) = \phi_2(\mathbf{x})$ still holds. Hence the effect of weak measurements is represented by extra terms on the two temporal boundaries $\tau = 0$ and $\tau = \beta$, including the interaction between the two temporal boundaries. The form of $\mathcal{L}^{\text{int}}$ can be determined by the symmetry condition of the density-matrix transformation $\mathcal{E}$ induced by measurements, either a “doubled” (or strong) symmetry condition or a “diagonal” (or weak) symmetry condition [65, 169]. Suppose we always look at quantities linear with the density matrix such as $\text{tr}\{\hat{\rho}\hat{O}\}$, the temporal boundaries $\tau = 0$ and $\tau = \beta$ are “glued” together. Our system is mapped to a problem in Euclidean space-time with a defect at the temporal slab $\tau = 0, \beta$ (Fig.2.1a). If the system has a Lorentz invariance in the infrared, after the space-time rotation, the temporal defect will become the physical spatial defect in space-time [90, 173].

This chapter considers systems with an n -form symmetry $G^{(n)}$. We use d to la-

bel the spatial dimension and D to label the space-time dimension. A first question is what happens when the initial d -dimensional state $|\Psi\rangle$ is gapped, disordered, and symmetric under $G^{(n)}$. In the space-time picture, this is equivalent to asking how a $(d-1)$ -dimensional boundary or defect is affected by the measurement-induced defect Hamiltonian. Since the bulk is disordered and gapped, one can often integrate out the bulk and obtain an effective local problem on the defect. Examples of this sort were discussed in Ref. [23, 77, 173]. For a $2d$ Ising disordered phase, weak measurement can drive a transition in the $(2+0)d$ Ising universality class, equivalently the $(1+1)D$ Ising universality class, rather than the $(2+1)D$ Ising universality class of an equilibrium quantum phase transition. This transition is also dual to an information transition of the toric code topological order under dephasing [71].

The main focus here is a different class of questions. We consider a system with symmetry $G^{(n)}$, but start from a state $|\Psi\rangle$ with spontaneous symmetry breaking of $G^{(n)}$. The boundary-defect mapping remains valid. The difference is that the bulk can now contain gapless Goldstone modes associated with the broken symmetry, and these modes can interfere with the defect physics in a more intrinsic way.

If our system has only a 0-form symmetry, the answer to the question above is straightforward: A spontaneous breaking of a 0-form symmetry with true long-range order *cannot* be qualitatively altered by any local finite-strength weak measurement with or without post-selection; the SSB can only be destroyed with “strong” projective measurements, i.e. $p = 1$. The physical picture of this conclusion is that there is a strong “proximity effect” from the bulk, which breaks the 0-form symmetry on any defect with one lower dimension. More formally, for a true long-range order that spontaneously breaks a 0-form symmetry, the state can be exemplified as a simple product state. For example, for an Ising system whose global $\mathbb{Z}_2^{(0)}$ charge is $\prod_j X_j$, an Ising-ordered state has the following wave function at its fixed point: $|\Psi\rangle = \prod_j |Z_j = 1\rangle$. Now on every site, we perform a weak measurement on operator X_j with probability p , namely, there

is a probability p that the qubit on the site j is being measured for each j . The density matrix takes the following form when the measurement outcome $X_j = 1$ is selected

$$\rho = \otimes_j \rho_j, \quad \rho_j = (1 - p) \frac{I + Z_j}{2} + p \frac{I + X_j}{2}. \quad (2.3)$$

Then as long as $p < 1$, the expectation value of Z_j is always nonzero. And there is no intrinsic phase transition driven by the weak measurements as the system essentially reduces to a 0-dimensional problem. This conclusion holds *with or without* post-selecting the measurement outcomes.

Higher-form symmetries therefore lead to a more intrinsic measurement problem. The SSB phase of a higher-form symmetry is *not* a simple product state in space, and an intrinsic phase transition can be driven by weak measurements.

2.3 Higher-form symmetries

This section starts with SSB phases of $U(1)$ higher-form symmetries. Let us first list our main conclusions:

1. SSB of a $U(1)$ 1-form symmetry ($U(1)^{(1)}$ -SSB) in spatial dimension $d \geq 3$ (space-time dimension $D \geq 4$) can be qualitatively suppressed by finite strength weak measurements through a phase transition. In particular, at $d = 3$ the phase transition driven by weak measurements at the critical strength has a line of fixed points with self-duality. Here, the suppression of SSB is signified by the qualitatively weakened fluctuation/correlation of the 1-form symmetry charges.

2. $U(1)^{(2)}$ -SSB can be suppressed in spatial dimension $d \geq 4$ through a phase transition driven by weak measurements. A similar pattern continues to higher-form symmetries at higher dimensions.

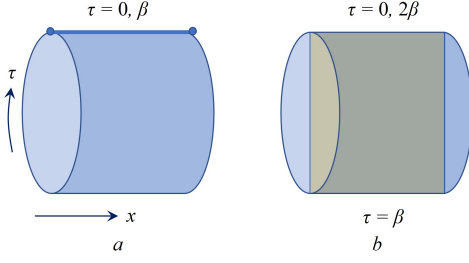


Figure 2.1: Euclidean space-time picture of quantities $\text{tr}\{\hat{\rho} \hat{O}\}$ (a), and the 2nd Rényi entropy (b).

For lower dimensional systems, the fate of $U(1)^{(n)}$ -SSB phases under weak measurements can be perceived through duality. A $U(1)^{(1)}$ -SSB phase in $2d$, and a $U(1)^{(2)}$ -SSB phase in $3d$ are both dual to a superfluid phase, i.e. they are dual to phases with SSB of a 0-form symmetry. As argued in the previous section, a generic SSB phase of a 0-form symmetry cannot be qualitatively altered by finite-strength weak measurements.

2.3.1 $U(1)^{(1)}$ 1-form symmetry in $3d$

In this section, we consider a $3d$ system with a $U(1)_m^{(1)}$ symmetry, and this $U(1)_m^{(1)}$ is spontaneously broken in state $|\Psi\rangle$. In the traditional language, the state $|\Psi\rangle$ has gapless photon excitations described by a gauge field a_μ . A $U(1)_m^{(1)}$ symmetry implies that there is no dynamical Dirac monopole in the system. We also assume that the electric charges that couple to a_μ are bosonic and gapped. Hence, below the gap of the electric charges, there is also an emergent $U(1)_e^{(1)}$ symmetry in the infrared. Both the exact $U(1)_m^{(1)}$ symmetry and the IR- $U(1)_e^{(1)}$ symmetry are spontaneously broken in the state $|\Psi\rangle$.

For a quantum system at equilibrium, the quantum phase transition from a phase with $U(1)_m^{(1)}$ and IR- $U(1)_e^{(1)}$ SSB to the $U(1)_m^{(1)}$ symmetric phase is a Higgs transition driven by the condensation of the electric gauge charge in the $(3+1)D$ space-time, by gradually increasing the kinetic energy of the electric charges. This transition is likely a weak first-order phase transition [59]. The condensation of the electric charge in the $(3+1)D$ system restores the $U(1)_m^{(1)}$ symmetry and renders the IR- $U(1)_e^{(1)}$ symmetry

absent.

The appendices to this chapter give explicit measurement protocols that can drive charge condensation at the temporal slab $\tau = 0, \beta$, both for a discrete gauge theory and for a $U(1)$ gauge theory. The discussion below focuses on the infrared universal physics of the electric-charge condensation transition driven by weak measurement. We first evaluate quantities linear in $\hat{\rho}$, with the form $\text{tr}\{\hat{\rho} \hat{O}\}$. This Higgs transition on the $(3+0)d$ defect is described by the following nonlocal bosonic QED:

$$\mathcal{S} = \int d^3x |(\nabla - i\mathbf{a})\phi|^2 + r|\phi|^2 + g|\phi|^4 + \frac{1}{2e^2} \mathbf{f} \cdot \frac{1}{|\partial|} \mathbf{f}, \quad (2.4)$$

where $\mathbf{f} = d\mathbf{a}$. Here $\mathbf{a}$ is the three spatial components of a_μ along the $(3+0)d$ defect; ϕ describes the electric charge that couples to gauge field $\mathbf{a}$. The nonlocal action of $\mathbf{a}$ in the last term of Eq. 2.4 arises from the fact that a_μ lives in the entire $(3+1)D$ bulk with an ordinary Maxwell action $\mathcal{S}_{\text{bulk}} \sim \int d^4x f_{\mu\nu} f_{\mu\nu} / (4e^2)$. After integrating the momentum orthogonal to the defect plane, we obtain the nonlocal action of a_μ in Eq. 2.4 [127].

The reason Eq. 2.4 is interesting is that the coupling constant e of the nonlocal QED is exactly marginal, and Eq. 2.4 enjoys a *self-duality* [93, 127, 212]. We can perform the standard particle-vortex duality for the ϕ field in $(3+0)d$ through the following duality relation [61, 82, 234]: $*\mathbf{J}^\phi \sim d\mathbf{b}$, where $\mathbf{J}^\phi$ is the charge current on the $(3+0)d$ defect; and $\mathbf{b}$ is the dual gauge field that couples to the vortex field ϕ_v of ϕ . After integrating out $\mathbf{a}$, Eq. 2.4 is transformed into its dual form

$$\mathcal{S}_d = \int d^3x |(\nabla - i\mathbf{b})\phi_v|^2 - r|\phi_v|^2 + \tilde{g}|\phi_v|^4 + \frac{1}{2\tilde{e}^2} \tilde{\mathbf{f}} \cdot \frac{1}{|\partial|} \tilde{\mathbf{f}}, \quad (2.5)$$

where $\tilde{\mathbf{f}} = d\mathbf{b}$. Notice that Eq. 2.5 takes the same form as Eq. 2.4, and the dual coupling constant $\tilde{e} \sim 4\pi/e$.

A $U(1)_m^{(1)}$ -SSB phase is a condensate of the magnetic flux (the 1-form charge of the $U(1)_m^{(1)}$ symmetry). The condensation of the electric charge ($r < 0$ in Eq. 2.4) would drive

the magnetic flux into an uncondensed phase on the $(2+1)D$ defect. In the dual theory, the vortex ϕ_v carries a magnetic flux orthogonal to the defect plane. Hence when the vortex condenses ($r > 0$), it drives the magnetic flux back into the condensate. Hence the vortex condensate with $r > 0$ in Eq. 2.5 corresponds to the original $U(1)_m^{(1)}$ -SSB phase.

The condensation of ϕ at the defect leads to boundary conditions for the bulk gauge field a_μ . At the $(3+0)d$ temporal defect, $\mathbf{a}$ acquires a Higgs mass in the condensate of ϕ , which becomes relevant and enforces a Dirichlet boundary condition for $\mathbf{a}$. This is analogous to the ordinary boundary condition in boundary criticality. This boundary condition implies that $\mathbf{a}$ should be replaced by $\partial_\perp \mathbf{a}$ near the defect, and hence acquires a larger scaling dimension than in the bulk. This also gives the Dirichlet boundary condition for the magnetic field $\mathbf{B} = \nabla \times \mathbf{a}$. If one measures the correlation of $\mathbf{B}$ after the phase transition driven by weak measurements, it scales as

$$\text{tr}\{\hat{\rho} \mathbf{B}(0) \cdot \mathbf{B}(\mathbf{x})\} \sim \frac{1}{|\mathbf{x}|^6}, \quad (2.6)$$

rather than $1/|\mathbf{x}|^4$ as in the bulk of the system with SSB of $U(1)_m^{(1)}$ and IR- $U(1)_e^{(1)}$. If $\mathbf{B}$ and $\mathbf{a}$ are periodically defined, we can compute the following quantity

$$\log \left(\text{tr}\{\hat{\rho} e^{iB(0)_a} e^{-iB(\mathbf{x})_b}\} \right) \sim \frac{1}{|\mathbf{x}|^6}. \quad (2.7)$$

Hence the condensation of charges induced by weak measurement still weakens the correlation of the magnetic field, which is the charge of $U(1)_m^{(1)}$; but unlike the condensation of charges in equilibrium, the correlation of magnetic field is not rendered fully short-ranged.

After a Wick rotation, the condensate of ϕ in the $(3+0)d$ defect becomes a thin-film superconductor in the $2d$ space (the XY plane), or $(2+1)D$ space-time. The in-plane components of the electric field (E_x, E_y) of the gauge field a_μ must vanish

due to screening from the condensed charges, and the out-of-plane magnetic field B_z is zero inside the superconductor due to the Meissner effect. These are equivalent to the Dirichlet boundary conditions for E_x, E_y , and B_z . After a space-time rotation in the (z, τ) plane, (E_x, E_y) becomes (B_y, B_x) . Hence the boundary condition imposed by a physical thin film superconductor precisely becomes the Dirichlet boundary condition mentioned in the previous paragraph.

Since the condensate of electric charges at the $(3+0)d$ defect does not impose the Dirichlet boundary condition for the electric field, the electric field $\mathbf{E}$ correlation function is expected to be qualitatively similar to the original pure state of the $U(1)_m^{(1)}$ -SSB phase. The correlation of $\mathbf{E}$ and $\mathbf{B}$ under weak measurement can also be derived using the explicit wave functional of the $U(1)_m^{(1)}$ -SSB state, as shown in the final appendix to this chapter.

Another quantity of interest is the 't Hooft loop operator $\exp(i \oint_{\mathcal{C}} d\mathbf{l} \cdot \mathbf{b})$ with $\mathbf{E} = \nabla \times \mathbf{b}$, whose expectation value decays as a perimeter law in the $U(1)_m^{(1)}$ -SSB pure state. Weak measurement does not qualitatively alter the perimeter law of the 't Hooft loop expectation value, even if the charges condense on the $(3+0)d$ defect in Euclidean space-time. This sharply contrasts with the Higgs phase in the entire $(3+1)D$ bulk. The physical picture is that the 't Hooft loop operator is also a membrane operator with $\oint_{\mathcal{A}} d\boldsymbol{\sigma} \cdot \mathbf{E}$, where $\mathcal{C} = \partial\mathcal{A}$. When we evaluate the 't Hooft loop $\mathcal{C}$ on the $(3+0)d$ defect, the most favorable membrane $\mathcal{A}$ is not strictly localized on the defect. It can leak into the bulk, which still yields a perimeter-law decay rather than an area law. The final appendix shows that the condensate of charges on the $(3+0)d$ defect driven by weak measurement leads to a larger coefficient of the perimeter law, which also means that the SSB of $U(1)_m^{(1)}$ is suppressed.

Quantities nonlinear in the density matrix, such as the 2nd Rényi entropy $S^{(2)} = -\log(\text{tr}\{\hat{\rho}^2\})$, have direct information-theoretic meaning and can also undergo a phase transition under weak measurement, *with or without* post-selection. In particular, when

there is a phase transition induced by weak measurement, one expects a singularity in the second Rényi entropy. The second Rényi entropy amounts to performing a path integral of fields in duplicated space-time, with $\tau \in [0, 2\beta]$, and weak measurement is mapped to the nonlocal interaction between the $3d$ temporal slabs $\tau = 0$ and $\tau = \beta$ (Fig. 2.1b).

Ref. [23, 77, 173] showed that when one considers the 2nd Rényi entropy, quantum decoherence, or weak measurement without post-selection, can drive the condensation of paired objects. These are nonlocal bound states of objects on the $\tau = 0$ and $\tau = \beta$ slabs in Fig. 2.1b. The appendices discuss the analogous transition in the 2nd Rényi entropy for systems with $U(1)$ 1-form symmetries.

2.3.2 Higher dimensions and higher-form symmetries

In general dimension $d > 3$, if there is an exact $U(1)_m^{(1)}$ symmetry in the $D = d + 1$ dimensional bulk, the electric charge field ϕ is now a $(d - 3)$ -dimensional membrane in space, which couples to a $(d - 2)$ -form $U(1)_e^{(d-2)}$ electric gauge field a . If we still start with a $U(1)_m^{(1)}$ -SSB phase, through measurement on the kinetic energy of ϕ membrane, a transition of ϕ condensation can be driven to Higgs the $(d - 2)$ -form $U(1)_e^{(d-2)}$ gauge field a . Condensation of a membrane-like object can be tedious to describe, but fortunately this transition can still be described conveniently in the dual picture, through a similar dual relation: $*J^\phi \sim db$ on the $D = d$ dimensional defect. Here J^ϕ is a $(d - 2)$ -form current of the ϕ -membrane on the defect, and b is *always* a 1-form gauge field. The 1-form gauge field b is still coupled to a scalar matter field ϕ_v . The transition we are after is still described by the same action as Eq. 2.5, with dimension $D = d$. ϕ_v always describes a segment of 1-form charge of the original $U(1)_m^{(1)}$ symmetry, which is a vector (analogous to the magnetic field $\mathbf{B}$) that orthogonally pierces through the defect. When ϕ_v condenses ($r > 0$), the ϕ membrane is suppressed; and when ϕ_v is gapped ($r < 0$), the ϕ membrane condenses.

But we can see that when $d > 3$, the coupling constant $\tilde{e}$ is *irrelevant* at the critical point $r = 0$, more precisely the scaling dimension of $\tilde{e}^2$ is $\Delta_{\tilde{e}^2} = 3 - d$. Hence the transition described by Eq. 2.5 would be at the Gaussian fixed point for general spatial dimension $d > 3$.

In spatial dimension $d > 3$, if there is an exact $U(1)_m^{(d-2)}$ -form symmetry in the bulk in its SSB phase, weak measurements can drive a transition to suppress the SSB. This transition is also described by Eq. 2.4 on the defect with $D = d$ dimensional space-time. Here a is the dual 1-form gauge field of the $(d - 2)$ -form $U(1)_m^{(d-2)}$ symmetry that is spontaneously broken in the bulk, and the ϕ field is the charge of the 1-form gauge field a_μ , or the “Dirac monopole” of the $U(1)_m^{(d-2)}$ gauge field in spatial dimension d . Now the coupling constant e is again irrelevant at the Gaussian fixed point of ϕ . Hence, the transition driven by weak measurements is also “mean-field” like.

2.4 Summary and Discussions

This chapter discusses the fate of higher-form symmetry under weak quantum measurements. A pure state with higher-form symmetry breaking can undergo a phase transition driven by weak measurements. This contrasts with the SSB state of an ordinary 0-form symmetry, whose fate under weak measurement reduces to a $(0 + 1)d$ problem and therefore does not exhibit an intrinsic phase transition. Some of the transitions driven by weak measurement can be mapped to a mixed-dimensional QED, with a line of fixed points mapped to each other under self-duality.

There is another ingredient of the mixed dimensional QED that plays a nontrivial role in the duality, i.e., the topological Θ -term in the bulk. In our current set-up, the effect of the Θ -term likely cancels out at the temporal defects due to the trace of the density matrix. But the Θ -term can make nontrivial contributions to quantities such as the strange correlator [308], which involves temporal interfaces between two

different states. It is worth exploring weak-measurement-driven transitions of strange correlators of systems with higher-form symmetries, which we expect to have an even richer structure of duality due to the nontrivial role of the Θ -term.

This chapter focused on SSB states of *continuous* higher-form symmetries. For SSB states with *discrete* higher-form symmetry, weak measurement can engineer a family of classical statistical-mechanics models under different conditions [275].

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Appendices to Chapter 2

2.5 Appendix: Weak-measurement on Z_2 gauge theory

Before discussing the measurement protocol for systems with $U(1)$ gauge fields, it is useful to first discuss a simpler weak-measurement-driven phase transition in Z_2 gauge theory. More examples of measurement-driven transitions in systems with discrete higher-form symmetries are discussed in Ref. [275]. We take the $3d$ toric code Hamiltonian as a starting point, which is defined on the $3d$ cubic lattice

$$H_{\text{TC}} = - \sum_v A_v - \sum_p B_p \quad (2.8)$$

where v denotes vertices on the cubic lattice and p denotes plaquettes on the cubic lattice. The local Hilbert space is made of qubits that reside on the links. A_v is then the product of σ^x operators connected to the vertex v ; B_p is the product of σ^z operators along the edge of plaquette p . σ_{ij}^x is the analogue of the electric field of the U(1) gauge field, and σ_{ij}^z is the analogue of $e^{ia_{ij}}$ of the lattice U(1) gauge field.

The initial state is prepared as the tensor product between the ground state of H_{TC} and $|0\rangle_a$ of the ancilla qubits. We then evolve the system qubit and ancilla qubit via the following unitary operator

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} \cos t + \sigma^z \sin t, & -\cos t + \sigma^z \sin t \\ \cos t - \sigma^z \sin t, & \cos t + \sigma^z \sin t \end{bmatrix} \quad (2.9)$$

Here, the 2×2 matrix structure of U is associated with the 2-fold Hilbert space of the ancillary qubit. When $t = 0$, U takes the form

$$U_0 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad (2.10)$$

The general $U(t)$ can be implemented via the following procedure

$$U = e^{-itH} \cdot U_0 \quad (2.11)$$

with

$$H = i\sigma^z \sigma_a^+ - i\sigma^z \sigma_a^- \quad (2.12)$$

where σ_a denotes the Pauli operator acting on the ancilla qubit. We then measure the ancilla in its computational basis which gives us an ensemble of states labeled by

measurement outcome s , a string that records the measurement outcomes on each link

$$|\psi(s)\rangle \propto \prod_{\text{link}} \frac{1}{\sqrt{2}} (\cos t + s_{ij} \sin t \sigma_{ij}^z) |\psi\rangle \quad (2.13)$$

To calculate expectation values of quantities of this ensemble, we need to evaluate

$$\overline{\langle O \rangle} = \sum_s P(s) \frac{\langle \psi(s) | O | \psi(s) \rangle}{\langle \psi(s) | \psi(s) \rangle} \quad (2.14)$$

By Born's rule (no bias), this is equivalent to

$$\begin{aligned} \text{tr} \left[\prod_{\text{link}} \frac{1}{\sqrt{2}} (\cos t + \sin t \sigma_{ij}^z) \hat{\rho}_0 \frac{1}{\sqrt{2}} (\cos t + \sin t \sigma_{ij}^z) O \right. \\ \left. + \prod_{\text{link}} \frac{1}{\sqrt{2}} (\cos t - \sin t \sigma_{ij}^z) \hat{\rho}_0 \frac{1}{\sqrt{2}} (\cos t - \sin t \sigma_{ij}^z) O \right] \end{aligned}$$

Assuming O commute with σ_{ij}^z , the equation above is equivalent to

$$\prod_{\text{link}} \text{tr}(\hat{\rho}_0 O) \quad (2.15)$$

hence the measurements without post-selection does not change the expectation value of O .

We now consider a biased selection, meaning we collect all measurement outcomes but we give a higher weight to the outcome with $s_{ij} = +1$. Now the expectation of O

becomes (again we consider quantity O that commutes with σ_{ij}^z)

$$\begin{aligned}
& \text{tr} \left[\prod_{\text{link}} \left(\frac{1}{2} + x \right) (\cos t + \sin t \sigma_{ij}^z) (\cos t + \sin t \sigma_{ij}^z) \hat{\rho}_0 O \right. \\
& \quad \left. + \prod_{\text{link}} \left(\frac{1}{2} - x \right) (\cos t - \sin t \sigma_{ij}^z) (\cos t - \sin t \sigma_{ij}^z) \hat{\rho}_0 O \right] \\
&= \text{tr} \left(\prod_{\text{link}} (1 + (4x \cos t \sin t) \sigma_{ij}^z) \hat{\rho}_0 O \right) \\
&= \text{tr} \left(\prod_{\text{link}} (1 + (\tanh \tilde{\beta}) \sigma_{ij}^z) \hat{\rho}_0 O \right)
\end{aligned}$$

Here, we make the identification $4x \cos t \sin t = \tanh \tilde{\beta}$. We can reorganize the mixed density matrix into the following form:

$$\begin{aligned}
\hat{\rho} &\sim e^{-\frac{1}{2}I(\{\sigma_{ij}^z\})} \hat{\rho}_0 e^{-\frac{1}{2}I(\{\sigma_{ij}^z\})}, \\
\text{tr}(\hat{\rho} O) &\sim \text{tr} \left(\hat{\rho}_0 e^{-I(\{\sigma_{ij}^z\})} O \right).
\end{aligned} \tag{2.16}$$

The density matrix can also be expressed in a form compatible with the general formula above:

$$\mathcal{E}_{\mathbf{x}}[\hat{\rho}_0] = (1 - p) \hat{\rho}_0 + p \left(\sum_m p_m \hat{P}_{m,\mathbf{x}} \hat{\rho}_0 \hat{P}_{m,\mathbf{x}} \right). \tag{2.17}$$

This follows from the identification

$$\begin{aligned}
p &= 2 \cos t \sin t, \quad p_+ = 1 + 2x, \quad p_- = 1 - 2x \\
\hat{P}_{m,\mathbf{x}} &= \frac{I + m \sigma_{ij}^z}{2}.
\end{aligned}$$

Here $m = \pm 1$, and $\mathbf{x}$ corresponds to the link $\langle i, j \rangle$ on the lattice.

We can view $\text{tr}(\hat{\rho})$ as a partition function of a $3d$ classical Ising model from which one can extract $\text{tr}(\hat{\rho} O)$. To see this, we note that the expectation of a string of Pauli- σ_{ij}^z operators with respect to $\hat{\rho}$ is 1 when the string is closed and 0 when the string is open. This is because the wave function of the $3d$ toric code is a superposition of

loop configurations generated by the product of B_p , which only involves closed loops. There are additional subtleties caused by logical operators, but there are only a finite number of them and they do not affect the partition function in the thermodynamic limit. Therefore, the partition function becomes

$$\sum_{\mathcal{C}} (\tanh \tilde{\beta})^{|\mathcal{C}|} \quad (2.18)$$

where $\mathcal{C}$ denotes closed loops formed by links of $3d$ cubic lattice, and this partition function is exactly the high-temperature expansion of a $3d$ classical Ising model.

Physically, the operator σ_{ij}^z enables the hopping of the charges of the Z_2 gauge theory. At larger $\tilde{\beta} > \tilde{\beta}_c$, which can be tuned by choosing x and t , the partition function of the $3d$ classical Ising model enters the low-temperature ordered phase, i.e. the condensate of the electric charge of the Z_2 gauge theory. This transition belongs to the $3d$ Ising universality class. This is consistent with the picture described above, where weak measurement drives the electric charges to condense on the temporal slab $\tau = 0$ in the path-integral formalism (Fig. 2.2).

Ref. [275] also shows that measurement without post-selection can drive a phase transition in the 2nd Rényi entropy of the Z_2 gauge theory. This is consistent with the physical picture that weak measurement can drive condensation of the bound state of gauge charges on the two temporal slabs in the path-integral formalism (Fig. 2.2).

2.6 Appendix: Lattice protocol and field theory

In this appendix we explicitly formulate the weak-measurement-driven phase transition discussed above on a lattice model. We take the $3d$ cubic lattice as an example. The local operators consist of rotors θ_i defined on the sites, and a_{ij} defined on the links of

the lattice, along with their canonical conjugates n_i and E_{ij}

$$[E_{ij}, e^{ia_{i'j'}}] = \delta_{ij,i'j'} e^{ia_{ij}}, \quad [n_i, e^{i\theta_{i'}}] = \delta_{i,i'} e^{i\theta_i} \quad (2.19)$$

Note that the rotors θ_i and a_{ij} are periodically defined, i.e. $\theta_i = \theta_i + 2\pi$, $a_{ij} = a_{ij} + 2\pi$, while their conjugates n_i and E_{ij} take discrete integer eigenvalues.

Before weak measurement or decoherence, the pure quantum state $|\Psi\rangle$ we start with is the ground state of the following lattice Hamiltonian

$$H = \frac{U}{2} \sum_i n_i^2 - J \sum_{i,\mu} \cos(\theta_i - \theta_{i+\mu} + a_{i,i+\mu}) + \frac{\tilde{U}}{2} \sum_{i,\mu} E_{i,i+\mu}^2 - K \sum_{i,\mu} \cos(\nabla \times \mathbf{a}). \quad (2.20)$$

The system is also subjected to a local "Gauss law" constraint

$$n_i - \nabla \cdot \mathbf{E} = 0. \quad (2.21)$$

One can verify that the local operator $n_i - \nabla \cdot \mathbf{E}$ commutes with the Hamiltonian. The phase diagram of the lattice gauge theory is well understood. We choose U/J and $K/\tilde{U}$ to be greater than a critical value, so that the system is in its Coulomb or photon phase. Physically this means that the rotor charge is gapped, and the gauge field $\mathbf{a}$ is in its deconfined phase with gapless photon excitations. The structure of the spatial lattice is shown in Fig. 2.2.

2.6.1 Biased Selection

We now describe a quantum channel that couples the system operator $\exp(i(\theta_i - a_{ij} - \theta_j))$ to an auxiliary qubit. One can consider more general ancilla degrees of freedom, such as a qudit or rotor, but ancilla qubits suffice for this discussion. The initial state is again prepared to be a tensor product between the ground state of H and $|0\rangle_a$ of all the ancilla qubits. First, a collective unitary acts on the auxiliary

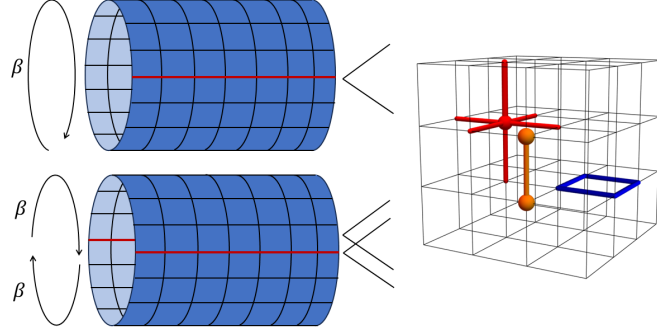


Figure 2.2: The upper-left panel shows the quantum operation with a biased selection process. This can be formulated as a defect insertion at a particular temporal slice in the Euclidean space-time path integral (red line). The lower-left panel shows the protocol for the 2nd Rényi entropy under decoherence, which is represented by defect insertions at two temporal slices separated by β with $\beta \rightarrow \infty$. The right panel shows the spatial lattice structure. The red vertex stands for the Gauss law constraint operator, the blue plaquette stands for the plaquette term controlling pure gauge fluctuations, and the orange link is the coupling of the gauge field to the rotor matter.

qubit and on the variables θ_i , a_{ij} , and θ_j . In what follows we denote $\Theta_{ij} = \theta_i - a_{ij} - \theta_j$.

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} \cos t + e^{i\Theta_{ij}} \sin t, & -\cos t + e^{i\Theta_{ij}} \sin t \\ \cos t - e^{-i\Theta_{ij}} \sin t, & \cos t + e^{-i\Theta_{ij}} \sin t \end{bmatrix} \quad (2.22)$$

One can verify this operator is unitary. When $t = 0$, U takes the simple form

$$U_0 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad (2.23)$$

We can also rewrite U in the following form

$$U = e^{-itH} \cdot U_0 \quad (2.24)$$

where

$$H = ie^{i\Theta_{ij}} \sigma^+ - ie^{-i\Theta_{ij}} \sigma^-. \quad (2.25)$$

The density matrix we start with is

$$\hat{\rho}_{\text{total}} = \hat{\rho}_0 \otimes |0\rangle \langle 0|. \quad (2.26)$$

Evolution with U will generate an entangled state between the system and ancilla

$$\frac{1}{\sqrt{2}}(\cos t + \sin t e^{i\Theta_{ij}}) |\psi\rangle |0\rangle + \frac{1}{\sqrt{2}}(\cos t - \sin t e^{-i\Theta_{ij}}) |\psi\rangle |1\rangle \quad (2.27)$$

We then measure the ancilla in its computational basis which gives us an ensemble of states labeled by measurement outcome s , a string that records the measurement outcomes on each link

$$|\psi(s)\rangle \propto \prod_{\text{link}} \frac{1}{\sqrt{2}}(\cos t + s_{ij} \sin t e^{is_{ij}\Theta_{ij}}) |\psi\rangle \quad (2.28)$$

To evaluate an average quantity of this ensemble, we need to investigate

$$\overline{\langle O \rangle} = \sum_s P(s) \frac{\langle \psi(s) | O | \psi(s) \rangle}{\langle \psi(s) | \psi(s) \rangle} \quad (2.29)$$

By Born's rule (no bias), this is equivalent to

$$\begin{aligned} \text{tr} \left[\prod_{\text{link}} \frac{1}{\sqrt{2}}(\cos t + \sin t e^{i\Theta_{ij}}) \hat{\rho}_0 \frac{1}{\sqrt{2}}(\cos t + \sin t e^{-i\Theta_{ij}}) O \right. \\ \left. + \prod_{\text{link}} \frac{1}{\sqrt{2}}(\cos t - \sin t e^{-i\Theta_{ij}}) \hat{\rho}_0 \frac{1}{\sqrt{2}}(\cos t - \sin t e^{i\Theta_{ij}}) O \right] \end{aligned}$$

Assuming O commute with $e^{i\Theta_{ij}}$, the equation above is equivalent to

$$\prod_{\text{link}} \text{tr}(\hat{\rho}_0 O) \quad (2.30)$$

hence the measurements without post-selection does not change the expectation value of O .

We now consider a biased selection, meaning we collect all measurement outcomes but we give a higher weight to the outcome with $s_{ij} = +1$. This is equivalent to

$$\begin{aligned} & \text{tr} \left[\prod_{\text{link}} \left(\frac{1}{2} + x \right) (\cos t + \sin t e^{-i\Theta_{ij}})(\cos t + \sin t e^{i\Theta_{ij}}) \hat{\rho}_0 O \right. \\ & \quad \left. + \prod_{\text{link}} \left(\frac{1}{2} - x \right) (\cos t - \sin t e^{i\Theta_{ij}})(\cos t - \sin t e^{-i\Theta_{ij}}) \hat{\rho}_0 O \right] \\ & = \text{tr} \left(\prod_{\text{link}} (1 + 4x \cos t \sin t \cos \Theta_{ij}) \hat{\rho}_0 O \right) \end{aligned}$$

We can again reorganize the mixed density matrix into the following form:

$$\begin{aligned} \hat{\rho} & \sim e^{-\frac{1}{2}I(\{\Theta_{ij}\})} \hat{\rho}_0 e^{-\frac{1}{2}I(\{\Theta_{ij}\})}, \\ \text{tr}(\hat{\rho} O) & \sim \text{tr}(\hat{\rho}_0 e^{-I(\{\Theta_{ij}\})} O). \end{aligned} \tag{2.31}$$

In the path-integral language this is equivalent to increasing the weight for configurations with $\Theta_{ij} = 0$ on a $3d$ slice at $\tau = 0$ of Euclidean space-time. By choosing x and t , the weight of configurations with $\Theta_{ij} = 0$ can become arbitrarily larger than the weight of configurations with $\Theta_{ij} = \pi$. The measurement followed by selection can therefore strongly favor charge condensation at the temporal slab $\tau = 0$, similar to the previous appendix on Z_2 gauge theory.

2.6.2 Field theory description

For a classical XY model defined on the lattice, the partition function reads

$$Z = \int_0^{2\pi} D\theta_i \exp(-I(\{\Theta_{ij}\})), \tag{2.32}$$

where $\Theta_{ij} = \theta_i - \theta_j$, and $I(\Theta_{ij})$ is a periodic function of Θ_{ij} . The simplest choice of $I(\{\Theta_{ij}\})$ is a nearest-neighbor XY model:

$$I(\{\Theta_{ij}\}) = \sum_{\text{n.n.}} -(J/T) \cos(\theta_i - \theta_j). \quad (2.33)$$

On a $3d$ lattice, by tuning J/T there is a phase transition between a condensate of $e^{i\theta_i}$ for $J > J_c$ and a disordered phase for $J < J_c$. The $3d$ XY model is not exactly solvable, but its infrared physics can be studied reliably using field theory. In the field theory formalism, we use the coarse-grained complex field $\phi(\mathbf{x})$ rather than the lattice operator $\exp(i\theta_i)$. The partition function is then mapped to

$$Z \sim \int D\phi(\mathbf{x}) \exp \left(- \int d^d \mathbf{x} \left[|\nabla \phi|^2 + r|\phi|^2 + g|\phi|^4 \right] \right). \quad (2.34)$$

This field theory corresponds to a large class of lattice models with U(1) variables. For example, one can add further-neighbor interactions in the partition function. The microscopic change modifies the UV values of the field-theory parameters, such as r and g , but the infrared physics is controlled by fixed-point values under the renormalization group. The fixed-point description is therefore generic for a large class of models with U(1) variables.

If the original lattice model also involves gauge fields defined on the links, i.e. $\Theta_{ij} = \theta_i - \theta_j - a_{ij}$, with a lattice version of the Maxwell term of a_{ij} , the theory becomes an "Abelian Higgs" model and has attracted a lot of interests in the past (see for example Ref. [41, 85]). The field theory of this model is

$$Z \sim \int D\phi(\mathbf{x}) D\mathbf{a}(\mathbf{x}) e^{-\mathcal{S}[\mathbf{a}, \phi]},$$

$$\mathcal{S}[\mathbf{a}, \phi] = \int d^d x \left[|(\nabla - i\mathbf{a})\phi|^2 + r|\phi|^2 + g|\phi|^4 \right] + \mathcal{L}_{\text{Maxwell}}. \quad (2.35)$$

The pure state density matrix $\hat{\rho}_0$ of a quantum system before weak-measurement

or decoherence can be represented in a standard path-integral form. If we are only interested in the physics in the infrared limit, the density matrix can be represented in a field theory form in the continuum, without loss of generality. In our case, the density matrix element between field configurations $|\mathbf{a}(\mathbf{x})_1, \phi(\mathbf{x})_1\rangle$ and $|\mathbf{a}(\mathbf{x})_2, \phi(\mathbf{x})_2\rangle$ reads

$$\langle \mathbf{a}(\mathbf{x})_2, \phi(\mathbf{x})_2 | \hat{\rho}_0 | \mathbf{a}(\mathbf{x})_1, \phi(\mathbf{x})_1 \rangle = \int Da_\mu(\mathbf{x}, \tau) D\phi(\mathbf{x}, \tau) e^{-S_{d+1}[a_\mu, \phi]},$$

$$S_{d+1}[a_\mu, \phi] = \int d\tau d^d x \left[|(\partial_\mu - ia_\mu)\phi|^2 + r'|\phi|^2 + g'|\phi|^4 + \frac{1}{4e^2}(f_{\mu\nu})^2 \right], \quad (2.36)$$

where $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$, and we may introduce a_0 as a Lagrange multiplier to enforce the Gauss law constraint Eq. 2.21. In the path-integral above, we also need the following boundary conditions $\mathbf{a}(\mathbf{x}, 0) = \mathbf{a}(\mathbf{x})_1$, $\mathbf{a}(\mathbf{x}, \beta) = \mathbf{a}(\mathbf{x})_2$, $\phi(\mathbf{x}, 0) = \phi(\mathbf{x})_1$, $\phi(\mathbf{x}, \beta) = \phi(\mathbf{x})_2$. In the bulk we need the system to be in the photon phase, hence we need $r' > 0$, i.e. the charged matter fields are gapped. For the physics in the infrared we can just focus on the last Maxwell term $\frac{1}{4e^2}(f_{\mu\nu})^2$ which leads to gapless photons in the bulk.

After weak-measurement described in the previous section, the evaluation of an operator reduces to the following path integral problem:

$$\text{tr}(\hat{\rho}O) \sim \int D\phi(\mathbf{x}, \tau) e^{-S[a_\mu, \phi] + \mathcal{S}[\mathbf{a}, \phi]_{\tau=0}} O_{\tau=0}. \quad (2.37)$$

Here in the path integral we need to identify $\tau = 0$ and $\tau = \beta$, due to the trace. The extra term in the action $\mathcal{S}[\mathbf{a}, \phi]_{\tau=0}$ is a coarse-grained version of $e^{-I(\{\Theta_{ij}\})}$ in Eq. 2.31, where $\Theta_{ij} = \theta_i - \theta_j - a_{ij}$ on the lattice, and the form of $\mathcal{S}[\mathbf{a}, \phi]_{\tau=0}$ is given by Eq. 2.35.

In the $d + 1$ dimensional space-time bulk, the charges are gapped out, and the low energy physics is controlled by the Maxwell term only. Since the Maxwell term is a free theory, its effects on the temporal boundary can be understood reliably. This was discussed in, for example, Ref. [127]. The bulk turns $\mathcal{S}[\mathbf{a}, \phi]_{\tau=0}$ in Eq. 2.35 into the nonlocal QED in Eq. 2.4.

As discussed above, weak measurement with or without post-selection can also drive a transition in the 2nd Rényi entropy. This transition is mapped to a path integral with interactions between the two temporal slabs $\tau = 0, \beta$ (Fig. 2.2). Ref. [23, 77, 173, 275] showed that weak measurement can drive bound states of gauge charges on the two temporal slabs to condense. In the present problem, if the initial state has an exact $U(1)_m^{(1)}$ symmetry, weak measurements may also drive condensation of a bound state of electric charges ϕ on the $\tau = 0$ and $\tau = \beta$ slabs. This bound-state condensation breaks the doubled IR- $U(1)_e^{(1)}$ symmetry down to a diagonal IR- $U(1)_e^{(1)}$ symmetry, using the language developed in Ref. [169, 173]. This transition is described by the following effective action:

$$\begin{aligned} \mathcal{S} = \int d^3x & |(\nabla - i\mathbf{a}_1 - i\mathbf{a}_2)\phi|^2 + r|\phi|^2 + g|\phi|^4 \\ & + \frac{1}{2e^2} \mathbf{f}_1 \cdot \frac{1}{|\partial|} \mathbf{f}_1 + \frac{1}{2e^2} \mathbf{f}_2 \cdot \frac{1}{|\partial|} \mathbf{f}_2. \end{aligned} \quad (2.38)$$

$\mathbf{a}_1$ and $\mathbf{a}_2$ are the gauge fields on the $\tau = 0$ and $\tau = \beta$ interfaces respectively. ϕ is effectively coupled to one gauge field $\mathbf{a}_+ = \mathbf{a}_1 + \mathbf{a}_2$ with a nonlocal action, and the transition belongs to the same universality class as Eq. 2.4. The 2nd Rényi entropy is the free energy for this transition, which should have a singularity at the transition.

In the condensate of ϕ , $\mathbf{a}_+ = \mathbf{a}_1 + \mathbf{a}_2$ acquires a mass and suppresses the correlation of field $\mathbf{B}_+ = \mathbf{B}_1 + \mathbf{B}_2$, i.e., a linear combination of fields on the two different temporal slabs. Hence we expect the following scaling:

$$\log \left(\text{tr} \{ \hat{\rho} e^{iB_a(\mathbf{0})} e^{-iB_b(\mathbf{x})} \hat{\rho} e^{iB_a(\mathbf{0})} e^{-iB_b(\mathbf{x})} \} \right) \sim \frac{1}{|\mathbf{x}|^6}, \quad (2.39)$$

as this is the quantity that encodes the correlation of field $\mathbf{B}_+$.

2.7 Appendix: Calculation with explicit form of density matrix

In this section we compute quantities under weak measurement using the explicit form of density matrix. Let us start with the Hamiltonian of the quantized electromagnetic field in the $3d$ space:

$$H = \int d^3x \frac{e^2}{2} \mathbf{E}(\mathbf{x})^2 + \frac{1}{2e^2} \mathbf{B}(\mathbf{x})^2. \quad (2.40)$$

The ground state wave functional of the EM field is

$$|\Psi\rangle \sim \int D\mathbf{a}(\mathbf{q}) \exp(-\mathcal{S}[\mathbf{a}(\mathbf{q})]) |\mathbf{a}(\mathbf{q})\rangle \text{ with } \mathcal{S}[\mathbf{a}] = \frac{1}{2e^2} \int \frac{d^3q}{(2\pi)^3} \frac{|\mathbf{q} \times \mathbf{a}(\mathbf{q})|^2}{|\mathbf{q}|}. \quad (2.41)$$

The correlation function between $\mathbf{B}$ fields in the ground state reads

$$\begin{aligned} \text{tr}\{\hat{\rho}_0 \mathbf{B}(\mathbf{0}) \cdot \mathbf{B}(\mathbf{x})\} &\sim \int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q} \cdot \mathbf{x}} \epsilon^{aij} \epsilon^{amn} q_i q_m \\ &\times \langle a_j^T(\mathbf{q}) a_n^T(-\mathbf{q}) \rangle. \end{aligned} \quad (2.42)$$

where $\langle a_j^T(\mathbf{q}) a_n^T(-\mathbf{q}) \rangle$ is the gauge-field propagator for the transverse modes of $\mathbf{a}$ computed according to the ‘‘action’’ $2\mathcal{S}[\mathbf{a}]$. It is straightforward to check that

$$\langle a_j^T(\mathbf{q}) a_n^T(-\mathbf{q}) \rangle = \frac{e^2}{2} \frac{\delta^{jn} - q^j q^n / |\mathbf{q}|^2}{|\mathbf{q}|}. \quad (2.43)$$

Therefore,

$$\text{tr}\{\hat{\rho}_0 \mathbf{B}(\mathbf{0}) \cdot \mathbf{B}(\mathbf{x})\} \sim e^2 \int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q} \cdot \mathbf{x}} |\mathbf{q}| \sim \frac{e^2}{|\mathbf{x}|^4}. \quad (2.44)$$

Suppose weak measurement drives electric charge to condense on the $(3+0)d$ temporal defect which effectively amounts to turning on a mass term for the gauge-field in the evaluation above, the correlation between magnetic fields reads as Eq. 2.42 but

with the propagator of the transverse components of the gauge-field computed according to the modified “action” $2\mathcal{S}[\mathbf{a}] + \frac{1}{2}m^2 \int \frac{d^3q}{(2\pi)^3} |\mathbf{a}(\mathbf{q})|^2$. This new propagator – in the presence of a gauge-field mass – takes the form for the transverse components $\langle a_j^T(\mathbf{q}) a_n^T(-\mathbf{q}) \rangle = \frac{\delta^{jn} - q^j q^n / |\mathbf{q}|^2}{m^2 + 2|\mathbf{q}|/e^2}$. Hence, at momentum much less than the scale set by the gauge-field mass

$$\begin{aligned} \text{tr}\{\hat{\rho} \mathbf{B}(\mathbf{0}) \cdot \mathbf{B}(\mathbf{x})\} &\sim \int \frac{d^3q}{(2\pi)^3} e^{iq \cdot \mathbf{x}} \frac{|\mathbf{q}|^2}{m^2 + \frac{2}{e^2}|\mathbf{q}|} \\ &\sim \int \frac{d^3q}{(2\pi)^3} e^{iq \cdot \mathbf{x}} |\mathbf{q}|^2 \left(\frac{1}{m^2} - \frac{2|\mathbf{q}|}{e^2 m^4} + O(|\mathbf{q}|^2) \right). \end{aligned} \quad (2.45)$$

Since the Fourier transformation of the quadratic piece $\mathbf{q}^2$ only leads to local correlations, the leading term that contributes at long distance is $O(|\mathbf{q}|^3)$ and yields

$$\text{tr}\{\hat{\rho} \mathbf{B}(\mathbf{0}) \cdot \mathbf{B}(\mathbf{x})\} \sim \frac{1}{e^2 m^4} \frac{1}{|\mathbf{x}|^6}. \quad (2.46)$$

These evaluations are consistent with what we expected from the boundary criticality analysis. The evaluation of the electric field is similar, but we need to treat $\mathbf{E}$ as $\delta/\delta\mathbf{a}(\mathbf{x})$ in the evaluation with the wave functional. Then we shall see that the extra mass term only causes some local correction to the correlation between electric fields.

We now turn to the evaluation of the ’t Hooft loop. We begin with the calculation in the pure state with $U(1)_m^{(1)}$ SSB. We may define the eigenstates of the $\mathbf{E}(\mathbf{x})$ field, which should obey the following relation

$$\langle \mathbf{a} | \mathbf{E} \rangle = e^{-\frac{i}{2\pi} \int d^3x \mathbf{a} \cdot \mathbf{E}} \quad (2.47)$$

The ’t Hooft loop we would like to evaluate is

$$\left\langle e^{i \int_C d\mathbf{l} \cdot \mathbf{b}} \right\rangle = \left\langle e^{i \int_{\mathcal{A}} d\boldsymbol{\sigma} \cdot \mathbf{E}} \right\rangle = \int D\mathbf{E}(\mathbf{x}) \frac{\langle \Psi | \mathbf{E} \rangle e^{i \int_{\mathcal{A}} d\boldsymbol{\sigma} \cdot \mathbf{E}} \langle \mathbf{E} | \Psi \rangle}{\langle \Psi | \Psi \rangle}, \quad (2.48)$$

where $\boldsymbol{\sigma}$ is a surface element on membrane $\mathcal{A}$. For simplicity, we choose $\mathcal{A}$ to lie on the $z = 0$ XY -plane. We can substitute in the ground-state wavefunction Eq. 2.41 to get

$$\langle e^{i \int d\boldsymbol{\sigma} \cdot \mathbf{E}} \rangle = \frac{\int D\mathbf{a} D\mathbf{a}' D\mathbf{E} e^{-\mathcal{S}[\mathbf{a}] - \mathcal{S}[\mathbf{a}'] - \frac{i}{2\pi} \int d^3x (\mathbf{a} - \mathbf{a}') \cdot \mathbf{E} + i \int d^3x \delta(z) \theta(\sqrt{x^2 + y^2} < R) E_z}}{\int D\mathbf{a} D\mathbf{a}' D\mathbf{E} e^{-\mathcal{S}[\mathbf{a}] - \mathcal{S}[\mathbf{a}'] - \frac{i}{2\pi} \int d^3x (\mathbf{a} - \mathbf{a}') \cdot \mathbf{E}}} \quad (2.49)$$

Integrating out field $\mathbf{E}$ in the numerator gives

$$\mathbf{a}(\mathbf{x}) - \mathbf{a}'(\mathbf{x}) = 2\pi\delta(z)\theta(\sqrt{x^2 + y^2} < R)\hat{e}_z = 2\pi T(\mathbf{x})\hat{e}_z = 2\pi \mathbf{T}(\mathbf{x}) \quad (2.50)$$

while the integration of $\mathbf{E}$ in the denominator simply imposes constraint $\mathbf{a}(\mathbf{x}) = \mathbf{a}'(\mathbf{x})$.

So the expectation value of the 't Hooft loop of Eq. 2.48 reduces to

$$\frac{\int D\mathbf{a} e^{-\mathcal{S}[\mathbf{a}] - \mathcal{S}[\mathbf{a} + 2\pi \mathbf{T}]}}{\int D\mathbf{a} e^{-2\mathcal{S}[\mathbf{a}]}} \quad (2.51)$$

with

$$\begin{aligned} \mathcal{S}[\mathbf{a}] + \mathcal{S}[\mathbf{a} + 2\pi \mathbf{T}] &= 2\mathcal{S}[\mathbf{a}] + \frac{1}{2e^2} \int \frac{d^3q}{(2\pi)^3} \frac{(\mathbf{q} \times 2\pi \mathbf{T}(\mathbf{q})) \cdot (\mathbf{q} \times \mathbf{a}(-\mathbf{q})) + \text{c.c.}}{|\mathbf{q}|} \\ &+ \frac{2}{e^2} \int \frac{d^3q}{(2\pi)^3} \frac{\pi^2 |\mathbf{q} \times \mathbf{T}(\mathbf{q})|^2}{|\mathbf{q}|}. \end{aligned} \quad (2.52)$$

After shifting the gauge-field in the path integral by $\mathbf{a}(\mathbf{q}) \rightarrow \mathbf{a}(\mathbf{q}) - \pi \mathbf{T}(\mathbf{q})$, we see that $\mathbf{a}(\mathbf{q})$ and $\mathbf{T}(\mathbf{q})$ decouple such that the expectation value is now merely

$$\langle e^{i \int d\boldsymbol{\sigma} \cdot \mathbf{E}} \rangle = e^{-\frac{1}{e^2} \int \frac{d^3q}{(2\pi)^3} \frac{\pi^2 (q_y^2 + q_x^2) T(\mathbf{q})^2}{|\mathbf{q}|}} \quad (2.53)$$

The Fourier transform of T is

$$T(q) = \int_0^R dr d\theta r e^{iq_2 r \cos \theta} = \frac{2\pi R J_1(q_2 R)}{q_2} \quad (2.54)$$

where $q_2 = \sqrt{q_x^2 + q_y^2}$ is the magnitude of the in-plane component of the momentum, i.e. the Fourier distribution of T is constant in q_z . $J_1(x)$ is the Bessel function of the first kind. We impose a cutoff Λ for $|\mathbf{q}|$ so that the integral 2.53 is convergent. The integral in the exponential can be simplified to

$$\int d^3q \frac{q_2^2 T(\mathbf{q})^2}{|\mathbf{q}|} = \int d^3q (2\pi R)^2 \frac{J_1^2(q_2 R)}{|\mathbf{q}|} \quad (2.55)$$

A closed form expression is tedious, so we focus on the asymptotic behaviour of $J_1(q_2 R)$ which is

$$J_1(q_2 R) \sim \sqrt{\frac{2}{\pi q_2 R}} \cos\left(q_2 R - \frac{3\pi}{4}\right) + \dots \quad (2.56)$$

The large momentum integration would then roughly give

$$\int dq_z q_2 dq_2 (2\pi R)^2 \frac{2}{\pi q_2 R} \sim 8\pi R \int dq_z dq_2 \frac{1}{q} \sim 16\pi^2 R \Lambda$$

One therefore gets a perimeter law for the 't Hooft loop as expected:

$$\left\langle e^{i \int d\sigma \cdot \mathbf{E}} \right\rangle = e^{-\frac{2\pi\Lambda}{e^2} \mathcal{P}}. \quad (2.57)$$

When the electric charges condense on the $(3+0)d$ temporal defect due to weak measurement, the evaluation of the 't Hooft loop follows the same path integral on the $(3+0)d$ as Eq. 2.51, but with a modified effective action

$$\mathcal{S} = \frac{1}{2e^2} \int \frac{d^3q}{(2\pi)^3} \frac{|\mathbf{q} \times \mathbf{a}(\mathbf{q})|^2}{|\mathbf{q}|} - m^2 \int d^3x \sum_i \cos(\mathbf{a}_i) \quad (2.58)$$

In the limit of strong m^2 (deep inside the Higgs phase) it suffices to take $\mathbf{a}(q) \sim \delta(q)$ (analogous to the derivation of Ref. [90]), which would make the second term in eq. 2.52 vanish. This leaves us with the third term which differs from the massless case (eq. 2.53)

by a factor of 2. In other words the 't Hooft loop is now

$$\left\langle e^{i \int d\sigma \cdot E} \right\rangle = e^{-\frac{4\pi\Lambda}{e^2} \mathcal{P}}. \quad (2.59)$$

Hence the 't Hooft loop still decays as a perimeter law, as expected from the argument above, but with a larger coefficient. The electric charge condensation on the $(3+0)d$ temporal defect induced by weak measurement therefore still suppresses the SSB of the $U(1)_m^{(1)}$ symmetry.

Chapter 3

Conformal Field Theories From Decohered Chern Insulators

This chapter studies a mixed-state Chern insulator through a nonlinear overlap with a pure trivial reference insulator. The central object is the fidelity-like quantity $\mathcal{Z} = \text{tr}\{\hat{\rho}_c^D \hat{\rho}_\Omega\}$, whose negative logarithm is a Rényi relative entropy. After representing the density matrix by a Euclidean path integral and performing a space-time rotation, this quantity becomes the partition function of an effective nonchiral conformal field theory. Weak decoherence or weak measurement appears as an interaction between the two interfaces, and the symmetry of the quantum operation determines which interactions are allowed.

This chapter contributes to the thesis by connecting nonlinear diagnostics of mixed states, strange-correlator logic, and universal field-theory data. It gives a concrete example where decoherence does not simply destroy topology. Instead, it changes which anomaly or CFT data are visible to a chosen nonlinear probe.

3.1 Introduction

The $2d$ Chern insulator [109] is the archetypal example of a topological insulator (TI). The Chern insulator is defined as the ground state (hence a pure state) of a two dimensional tight binding Hamiltonian of electrons. In reality, any quantum system is exposed to the environment, and experience certain level of decoherence through forming entanglement with the ancilla degrees of freedom in the environment. In the last few years TIs or more generally symmetry protected topological (SPT) states in open systems have attracted great interests, and these studies belong to a larger paradigm of classifying and characterizing topological features of density matrices [63, 101, 132, 169, 196, 205, 245, 246, 292, 316]. The strongest kind of decoherence would be thermalization, when the system reaches a thermal equilibrium with the environment after a long time evolution (compared with the microscopic time scales of the system) by interacting and entangling with the ancilla degrees of freedom in the environment. Under this type of long time and massive scale of decoherence, many of the topological features of the Chern insulator (such as its nontrivial edge states) would be lost, and the topological insulators are strictly well-defined at zero temperature.

But one can also consider much weaker decoherence caused by exposing the system to the environment for a short amount of time, and the system will form weak entanglements with the ancilla qubits. In this case, after tracing out the ancilla qubits, the pure state topological insulator will still be rendered a mixed density matrix, but in stark contrast with thermalization, it is possible that some of the topological features survive this procedure. This type of weak decoherence is equivalent to the system being “weakly measured” by the environment. In the past few years phases and phase transitions that involve quantum measurements have been actively pursued, both theoretically and experimentally [20, 24, 56, 83, 89, 104, 105, 126, 135–137, 144, 156, 163, 164, 167, 185, 222, 251, 268, 278, 279, 287, 323, 326]. The fate of the mixed density matrix after deco-

herence or measurement strongly depends on the symmetry of the decoherence channel or the measurement, as was pointed out in recent works. It was shown recently that the topological information of a mixed state density matrix can be extracted through “strange correlators”, which were originally devised to probe the pure state SPT states and TIs [169, 308, 316]. In particular, a notion of type-II strange correlator and a formalism in terms of a “doubled Hilbert space” was developed in order to extract the full anomaly of the mixed density matrix [169].

The notion of the strange correlator is based on the overlap between the SPT state and a trivial state. This overlap is then mapped, under a space-time rotation, to a correlation function on the boundary. Power-law decay of the strange correlator signifies an SPT or TI phase with gapless boundary modes, even while bulk correlation functions remain short-range and insensitive to the topological order of the bulk. Motivated by the recent progresses on understanding the topological features of mixed state density matrices, in this work, we investigate the fidelity between a pure state trivial insulator and a mixed state density matrix generated from a Chern insulator through quantum operations that correspond to decoherence and weak-measurement, and we will refer to the latter density matrix as a “mixed state Chern insulator”. We demonstrate that the fidelity can be mapped to the partition functions of a series of conformal field theories (CFT), whose central charges depend on the decoherence channel or measurement. We stress that the decoherence or measurements considered in this work are supposed to be “weak”; a strong projective measurement may drive the system into a product state. This includes weak-measurement with post-selection of measurement outcomes. We also propose a method to extract the central charge of the effective CFT, which is analogous to the finite size scaling of the actual free energy of ordinary CFTs [4, 32]. This analysis enables us to extract the information of mixed state Chern insulator density matrix without having to know exactly which order parameter to use for the strange correlator.

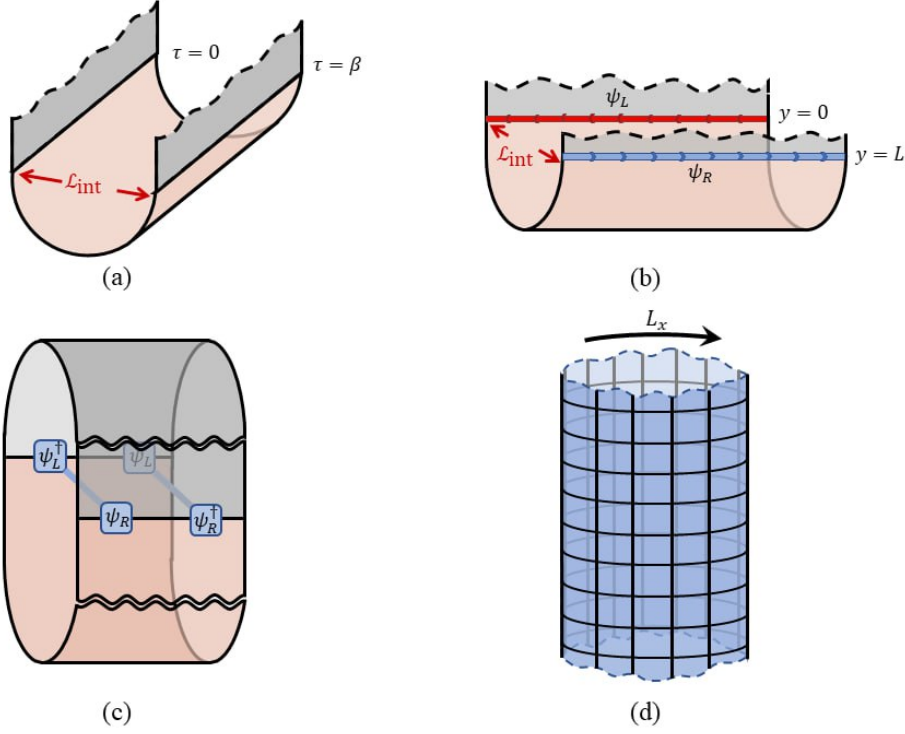


Figure 3.1: (a) The fidelity $\mathcal{Z}$ corresponds to path integral in Euclidean space-time, with interfaces at $\tau = 0$ and $\tau = \beta$; the decoherence is mapped to interaction between the two interfaces. (b) Under space-time rotation, the temporal interfaces at $\tau = 0, \beta$ become spatial interfaces $y = 0, L$, and there are chiral edge modes at each interface. (c) Example of type-II strange correlator. (d) The central charge of the effective CFT can be extracted from the finite size scaling of the Rényi relative entropy $\mathcal{F}$ on a cylindrical geometry.

3.2 Chern insulator with $\nu = 1$

We are most interested in the following quantities:

$$\mathcal{Z} = \text{tr}\{\hat{\rho}_c^D \hat{\rho}_\Omega\}, \quad \mathcal{F} = -\log \mathcal{Z}. \quad (3.1)$$

The density matrix of a pure state Chern insulator is $\hat{\rho}_c = |\Psi_c\rangle\langle\Psi_c|$, and $\hat{\rho}_\Omega$ is the density matrix of a trivial insulator. The quantity $\mathcal{Z}$ is the fidelity between a pure state trivial insulator density matrix and the Chern insulator density matrix, which is either a pure

state Chern insulator or a mixed state density matrix generated from decoherence or measurement. $\mathcal{F}$ is the Rényi relative entropy. The definition of the fidelity between two general mixed density matrices is more complicated, but it reduces to the simple form of Eq. 3.1 when one of the density matrices is pure. In our work, it suffices to always keep $\hat{\rho}_\Omega$ the density matrix of the pure state trivial insulator. Later we shall see that $\mathcal{Z}$ is mapped to the partition function of an effective $(2+0)d$ or $(1+1)D$ *nonchiral* conformal field theory (CFT), and $\mathcal{F}$ is mapped to the free energy of the CFT. If we compute the overlap between the wave functions (rather than density matrices) of a pure state Chern insulator and trivial insulator, the overlap would become the partition function of a chiral CFT [262]. Throughout this chapter we will always use d to label the spatial dimension, and D to label the space-time dimension.

The quantities $\mathcal{Z}$ and $\mathcal{F}$ encode the “distance” between the two density matrices. We start with a tight-binding Hamiltonian that realizes the Chern insulator with Chern number $\nu = 1$, but we would like to pursue generic physics that is independent of the microscopic details, hence we tune the system close to the topological transition to a trivial insulator. In the case where the system is tuned close to the topological transition, the difference between the Chern insulator and the trivial insulator can be well captured by a single Dirac fermion with a mass (Fermion doubling demands there be another Dirac fermion in the momentum space, with presumably a much larger mass), where the sign of the mass determines whether the system is a Chern insulator or a trivial insulator. We also assume that there is sufficient crystal symmetry such as spatial inversion and discrete rotation, which render terms that deviate from a simple Dirac fermion irrelevant in the infrared when the Dirac mass is small. One example of such model is the spinless version of the so-called BHZ model on the square lattice [31], which has multiple discrete symmetries such as charge-conjugation, spatial inversion, discrete rotation, etc. These symmetries ensure that close to the topological-trivial transition with band inversion, the system is well captured by a single Dirac fermion at the Gamma point of the Brillouin

zone with emergent Lorentz symmetry. The Hamiltonian and the Euclidean space-time Lagrangian of the Dirac fermion read:

$$\begin{aligned}
H &= \int d^2x \mathcal{H} = \int d^2x \psi^\dagger (\mathrm{i}\sigma^z \partial_x + \mathrm{i}\sigma^x \partial_y + m\sigma^y) \psi; \\
\mathcal{L} &= \psi^\dagger (\partial_\tau + \mathrm{i}\sigma^z \partial_x + \mathrm{i}\sigma^x \partial_y + m\sigma^y) \psi \\
&= \bar{\psi} (\gamma_0 \partial_\tau + \gamma_1 \partial_x + \gamma_2 \partial_y) \psi + m \bar{\psi} \psi \\
(\gamma_0, \gamma_1, \gamma_2) &= (\sigma^y, -\sigma^x, \sigma^z)
\end{aligned} \tag{3.2}$$

with $\bar{\psi} = \psi^\dagger \gamma_0$.

We would like to consider a mixed state density matrix generated from a Chern insulator after a quantum operation such as weak measurement, with the local quantities measured being invariant under the U(1) charge symmetry. For example, the local quantities being measured could be the current density of the system, and the measurement outcomes are summed over. After this operation, the density matrix of the mixed state Chern insulator can take the following form

$$\begin{aligned}
\hat{\rho}_c^D &= \otimes_x \mathcal{E}_x[\hat{\rho}_c], \\
\mathcal{E}_x[\hat{\rho}_c] &\sim \hat{\rho}_c + \sum_\mu p_\mu \left(\hat{J}_x^\mu \hat{\rho}_c \hat{J}_x^\mu \right).
\end{aligned} \tag{3.3}$$

where $\hat{J}^\mu$ is the three component *Hermitian* current vector:

$$\begin{aligned}
\hat{J}^\mu &= (\bar{\psi} \gamma^0 \psi, \mathrm{i} \bar{\psi} \gamma^1 \psi, \mathrm{i} \bar{\psi} \gamma^2 \psi) \\
&= (\psi^\dagger \psi, -\psi^\dagger \sigma^z \psi, -\psi^\dagger \sigma^x \psi).
\end{aligned} \tag{3.4}$$

The local charge density or current operators are part of the set of “Kraus operators”. We would like to clarify that we do not require the Kraus operators to satisfy the probability-conserving constraint, i.e. we allow certain level of post-selection in our measurement. Since the Kraus operators are invariant under the $U(1)$ symmetry, the mixed density matrix generated will be in the *canonical ensemble* with a fixed number of fermions.

The exact density matrix of a system is often tedious to work with. For a quantum many-body system, we often only care about the long wavelength behaviors of the system, and this physics can be captured through coarse-graining or renormalization group techniques. There is a rather convenient formalism that allows us to use these techniques when we study the effects of quantum operations including decoherence or weak measurement. This formalism is based on the well-known fact that the density matrix of the ground state of a system can be generated through path integral in the Euclidean space-time $[\hat{\rho}_0]_{\phi_1(\mathbf{x}), \phi_2(\mathbf{x})} \sim \lim_{\beta \rightarrow \infty} \int D\phi(\mathbf{x}, \tau) \exp(-\mathcal{S}_0)$, with the temporal boundary condition $\phi(\mathbf{x}, 0) = \phi_1(\mathbf{x})$ and $\phi(\mathbf{x}, \beta) = \phi_2(\mathbf{x})$. Here $\mathcal{S}_0$ is the action of the system whose ground state is the desired pure state $|\Psi_0\rangle$. Then under decoherence or weak measurement, the mixed density matrix becomes [89, 173] (Fig. 3.1(a))

$$[\hat{\rho}^D]_{\phi_1(\mathbf{x}), \phi_2(\mathbf{x})} \sim \lim_{\beta \rightarrow \infty} \int D\phi(\mathbf{x}, \tau) \exp(-\mathcal{S}_0 - \mathcal{S}^{\text{int}});$$

$$\mathcal{S}^{\text{int}} = \int d\mathbf{x} \mathcal{L}^{\text{int}}(\phi(\mathbf{x}, 0), \phi(\mathbf{x}, \beta)), \quad (3.5)$$

the boundary condition $\phi(\mathbf{x}, 0) = \phi_1(\mathbf{x})$, $\phi(\mathbf{x}, \beta) = \phi_2(\mathbf{x})$ must hold in this path integral. The effect of decoherence is mapped to the interaction $\mathcal{L}^{\text{int}}$ between fields at $\tau = 0$ and $\tau = \beta$.

The symmetry is the most important condition one needs to consider in order to determine the form of $\mathcal{L}^{\text{int}}$. A density matrix can have a *doubled* (or strong) symmetry condition when it is invariant under separate actions from left and right multiplication

of the symmetry operation [63, 169, 196]. When the measurement only involves quantities that are invariant under the charge-U(1) symmetry, or more technically the Kraus operators that generate the mixed state density matrix are invariant under the U(1) symmetry, $\mathcal{L}^{\text{int}}$ needs to preserve separate U(1) transformations acting from the left and right side of the density matrix, which translate to the separate U(1) symmetries of the fields at $\tau = 0$ and $\tau = \beta$. In this case $\mathcal{L}^{\text{int}}$ should include the following term:

$$\mathcal{L}^{\text{int}} \sim \sum_{\mu} -g_{\mu} J_{\mu}(\mathbf{x}, 0) J_{\mu}(\mathbf{x}, \beta) \quad (3.6)$$

where $g_{\mu} \sim p_{\mu}$ for small p_{μ} . The charge current operator $\mathbf{J} = (J_x, J_y)$ is odd under spatial inversion, and all components of J_{μ} are odd under charge conjugation. In addition to the doubled U(1) symmetry, we demand the density matrix preserve the *diagonal* (or weak) charge conjugation and spatial inversion symmetry. This condition precludes terms linear with current operators $J_{\mu}(\mathbf{x}, 0)$, $J_{\mu}(\mathbf{x}, \beta)$ in $\mathcal{L}^{\text{int}}$.

Hence the quantity $\mathcal{Z}$ can be computed through a path integral of a Chern insulator action sharing temporal interfaces with a trivial insulator at $\tau = 0, \beta$, and this computation can be approximated by a path integral with the following action on the $(2+0)d$ temporal interfaces $\tau = 0, \beta$

$$\mathcal{S} = \int d^2x \bar{\psi}(\gamma_1 \partial_1 + \gamma_2 \partial_2) \psi + \mathcal{L}^{\text{int}}, \quad (3.7)$$

as there are low energy modes localized at the temporal interfaces. The low energy modes at the temporal interface are eigensates of σ^y . It is much more physically intuitive for us to perform a space-time rotation in the (τ, y) plane, and position the interface at $y = 0, L$, and the modes localized at the spatial interface would be eigenstates of σ^z . The action that describes the low energy states at both boundaries is

$$\mathcal{S}_{1d} = \int d\tau dx \mathcal{L}_{1d} = \int d\tau dx \psi^{\dagger}(\partial_{\tau} + i\sigma^z \partial_x) \psi$$

$$= \int d\tau dx \bar{\psi}(\gamma_0 \partial_\tau + \gamma_1 \partial_x) \psi. \quad (3.8)$$

This action corresponds to a $1d$ Hamiltonian of Dirac fermion, whose bosonized form is

$$H_0 = \int dx \mathcal{H}_0(x), \quad \mathcal{H}_0(x) = \frac{K}{2\pi} (\nabla_x \theta)^2 + \frac{1}{2\pi K} (\nabla_x \phi)^2. \quad (3.9)$$

For free fermions without any interaction, $K = 1$. Here we stress that Eq. 3.8 and its bosonized form already include physics at the two opposite spatial interfaces, which correspond to the temporal interfaces $\tau = 0$ and β combined before the space-time rotation.

After the space-time rotation, $\mathcal{L}^{\text{int}}$ would take the form of interaction between currents on the spatial interfaces $y = 0$ and $y = L$:

$$\mathcal{L}^{\text{int}} \sim \sum_{\mu} -g_{\mu} J'_{\mu}(x, 0) J'_{\mu}(x, L), \quad (3.10)$$

where

$$\begin{aligned} J'_{\mu} &= (\bar{\psi} \gamma_2 \psi, i\bar{\psi} \gamma_1 \psi, i\bar{\psi} \gamma_0 \psi) \\ &= (-i\psi^{\dagger} \sigma^x \psi, -\psi^{\dagger} \sigma^z \psi, i\psi^{\dagger} \psi). \end{aligned} \quad (3.11)$$

The g_0 term does not project to an obvious nontrivial operator at the spatial boundary after space-time rotation; g_1 and g_2 have very similar effect, as they both project to the following term at the spatial boundary (Fig. 3.1(b)):

$$\mathcal{L}^{\text{int}} \sim \mathcal{H}^{\text{int}} = g \rho_L(x) \rho_R(x), \quad g \sim g_1 + g_2, \quad (3.12)$$

hence the effect of the quantum operation is now mapped to a “*repulsive*” interaction

between the left and right moving charge densities in the effective $(1+1)D$ system corresponding to the Hamiltonian density $\mathcal{H}_0 + \mathcal{H}^{\text{int}}$.

This interaction would renormalize the Luttinger parameter K , i.e. $K = \sqrt{(2\pi - g)/(2\pi + g)}$, but it will not gap out the boundary states. This is because the “doubled symmetry” condition demands that the effective action after space-time rotation be invariant under independent $U(1)_L$ and $U(1)_R$ symmetries which act on the left and right moving modes of $\mathcal{H}_0$, and it is well-known that a system with these symmetries have perturbative ’t Hooft anomaly and cannot be gapped.

3.3 Chern insulator with $\nu = 2$

We now consider two copies of Chern insulators described by the BHZ model both with Chern number $\nu = 1$, topologically this is also equivalent to a Chern insulator with Chern number $\nu = 2$. After the quantum operation which preserves the charge number, the density matrix of the Chern insulator becomes

$$\begin{aligned}
\hat{\rho}_c^D &= \otimes_{\mathbf{x}} \mathcal{E}_{\mathbf{x}}[\hat{\rho}_c], \\
\mathcal{E}_{\mathbf{x}}[\hat{\rho}_c] &\sim \hat{\rho}_c + \sum_{i=1,2} \sum_{\mu} p_{i,\mu} \left(\hat{J}_{i,\mathbf{x}}^{\mu} \hat{\rho}_c \hat{J}_{i,\mathbf{x}}^{\mu} \right) \\
&\quad + p_e \left(\hat{O}_{\mathbf{x}} \hat{\rho}_c \hat{O}_{\mathbf{x}}^{\dagger} + \hat{O}_{\mathbf{x}}^{\dagger} \hat{\rho}_c \hat{O}_{\mathbf{x}} \right), \\
\hat{O}_{\mathbf{x}} &= c_{1,\mathbf{x}}^{\dagger} \sigma^z c_{2,\mathbf{x}}, \tag{3.13}
\end{aligned}$$

with small $p_{i,\mu}$ and p_e . The subscript $i = 1, 2$ labels the two copies of the Chern insulators. In addition to the current operators of the system, the quantum operation now involves the exciton operator $\hat{O}$, which is also invariant under the total charge $U(1)$ symmetry. Our goal is to explore the possible effective CFTs that can be realized

through $\text{tr}\{\hat{\rho}_c^D \hat{\rho}_\Omega\}$, hence a specific form of exciton operators was chosen in Eq. 3.13, which ensures that the realized CFT is unitary.

Following the procedure in the previous section, after a space-time rotation in the (y, τ) plane, the quantity $\mathcal{Z} = \text{tr}\{\hat{\rho}_c^D \hat{\rho}_\Omega\}$ can be mapped to the following path integral of $(1+1)D$ interacting non-chiral fermions:

$$\begin{aligned} \mathcal{Z} &\sim \int D[\psi_i] D[\psi_i^\dagger] \exp\left(-\int d\tau dx \mathcal{L}_{1d} + \mathcal{L}^{\text{int}}\right) \\ \mathcal{L}^{\text{int}} &= \sum_{i=1,2} g_i \rho_{i,L} \rho_{i,R} + g_e (\psi_{1,L}^\dagger \psi_{2,L} \psi_{2,R}^\dagger \psi_{1,R} + h.c.), \end{aligned} \quad (3.14)$$

where $\mathcal{L}_{1d}$ is the Lagrangian of two flavors of free $(1+1)D$ Dirac fermion. We can turn the path integral problem Eq. 3.14 into a Hamiltonian formalism in $1d$:

$$\begin{aligned} H &= \int dx \sum_{i=1,2} \frac{K_i}{2\pi} (\nabla_x \theta_i)^2 + \frac{1}{2\pi K_i} (\nabla_x \phi_i)^2 \\ &+ u \cos(2\phi_1 - 2\phi_2). \end{aligned} \quad (3.15)$$

Here u is proportional to $g_e \sim p_e$, and $K_i = \sqrt{(2\pi - g_i)/(2\pi + g_i)}$ with $g_i \sim p_{i,1} + p_{i,2} > 0$. Here $\exp(i2\phi_i) \sim \psi_{i,L}^\dagger \psi_{i,R}$ corresponds to the back scattering terms, and $\exp(i2\theta_i) \sim \psi_{i,L} \psi_{i,R}$ corresponds to the Cooper pairs.

For the most natural choice of parameters where $K_i < 1$, the vertex operator $u \cos(2\phi_1 - 2\phi_2)$ is *relevant*. A relevant u will gap out the channel $\theta_- \sim \theta_1 - \theta_2$ and $\phi_- \sim \phi_1 - \phi_2$, while leaving the channel $\theta_+ \sim \theta_1 + \theta_2$ and $\phi_+ \sim \phi_1 + \phi_2$ gapless and algebraic. All the fermions will acquire a gap, for example

$$\psi_{1,L} \sim e^{i\theta_1 + i\phi_1} \sim e^{\frac{i}{2}(\theta_+ + \theta_-) + \frac{i}{2}(\phi_+ + \phi_-)}, \quad (3.16)$$

as the short range correlation of the ϕ_- channel will render all the fermion operators

short-ranged. But some composite operators will acquire quasi-long range or power-law correlation, such as the following four body operators

$$\psi_{1,L}\psi_{1,R}\psi_{2,L}\psi_{2,R} \sim e^{i2\theta_+}. \quad (3.17)$$

The correlation function $\langle e^{i2\theta_+(0,0)} e^{-i2\theta_+(0,x)} \rangle$ of the $(1+1)D$ CFT is actually a “type-II strange correlator” that was proposed in Ref. [169] in the $(2+0)d$ space before the space-time rotation:

$$\begin{aligned} & \langle e^{i2\theta_+(0,0)} e^{-i2\theta_+(\tau,x)} \rangle \\ & \sim \text{tr}\{\hat{\rho}_c^D \hat{\Delta}(0) \hat{\Delta}^\dagger(\mathbf{x}) \hat{\rho}_\Omega \hat{\Delta}(0) \hat{\Delta}^\dagger(\mathbf{x})\} \end{aligned} \quad (3.18)$$

where $\hat{\Delta}(x)$ is a Cooper pair operator $\hat{\Delta}(\mathbf{x}) \sim c_{1,\alpha}(\mathbf{x})c_{2,\alpha}(\mathbf{x})$ with the same Dirac index $\alpha = 1, 2$, and $\mathbf{x} = (x, y)$ labels the $2d$ spatial coordinate before the space-time rotation

Hence even under weak measurement, the Chern insulator experiences certain “*instability*” in the sense that the CFT whose partition function corresponds to $\text{tr}\{\hat{\rho}_c^D \hat{\rho}_\Omega\}$ can have its central charge reduced from $c = 2$ to $c = 1$ through weak measurement. The change of the central charge can be extracted through the finite size scaling of $\mathcal{F} = -\log \mathcal{Z}$, which we will discuss later in this chapter.

3.4 Chern insulator with $\nu = 2N$

Now we consider $\nu = 2N$ copies of the Chern insulator. The system can have a large $U(2N)$ flavor symmetry. There could be many possible choices of the quantum operations that generate a mixed state. As an example, we choose the following quantum operation

that acts on the Chern insulator density matrix in the following way:

$$\begin{aligned}\mathcal{E}_x[\hat{\rho}_c] &\sim \hat{\rho}_c + \sum_a \sum_\mu p_{a,\mu} \left(\hat{J}_x^{a,\mu} \hat{\rho}_c \hat{J}_x^{a,\mu} \right), \\ \hat{J}_x^{a,\mu} &= (\bar{\psi} \gamma^0 \tau^a \psi, i \bar{\psi} \gamma^1 \tau^a \psi, i \bar{\psi} \gamma^2 \tau^a \psi),\end{aligned}\tag{3.19}$$

$\hat{J}_x^{a,\mu}$ with $a = 1, 2, 3$ is the current operator for the SU(2) subgroup of the U(2N) flavor symmetry. This quantum operation corresponds to the physics that the currents of the SU(2) flavor symmetry are weakly measured. We would like to keep a *diagonal* SU(2) flavor symmetry, hence $p_{a,\mu}$ does not depend on the SU(2) index a .

Following the same procedure as before, after space-time rotation we arrive at the following effective Hamiltonian at the $(1+1)D$ space-time:

$$\mathcal{H}(x) = \mathcal{H}_0(x) + \sum_a g J_L^a(x) \cdot J_R^a(x).\tag{3.20}$$

Since the quantum process breaks the doubled SU(2) symmetry down to the diagonal SU(2), the $\sum_a g J_L^a(x) \cdot J_R^a(x)$ term is allowed, and most naturally $g > 0$. This interaction is *marginally relevant*, and its effect is to gap out the SU(2) sector of the CFT. The original U(2N)₁ CFT has the following decomposition [3]:

$$\begin{aligned}\text{U}(2N)_1 &\simeq \text{SU}(N)_2 \oplus \text{U}(1)_{2N} \oplus \text{SU}(2)_N; \text{ or} \\ \text{U}(2N) &\simeq \text{O}(4N)_1 \simeq \text{Sp}(N)_1 \oplus \text{SU}(2)_N.\end{aligned}\tag{3.21}$$

Since the SU(2)_N sector of the CFT is gapped out, eventually the central charge is reduced from $c = 2N$ to

$$c = \frac{N(2N+1)}{N+2}.\tag{3.22}$$

Other CFTs with a coset construction can also be engineered through different types of quantum operations.

3.5 Central charge of the effective CFT

In previous sections we have shown that under weak measurement the effective CFT with partition function $\mathcal{Z} = \text{tr}\{\hat{\rho}_c \hat{\rho}_\Omega\}$ can become unstable and flow to CFTs with smaller central charges. In this section we propose that, if we design our tight binding Hamiltonian on a cylinder with finite circumference L along the x direction, while infinite length along y (Fig. 3.1(d)), the “free energy” $\mathcal{F} = -\log \mathcal{Z}$ per unit length along the y direction will have the following finite size scaling

$$\frac{\mathcal{F}}{L_y} = f_0 L - \frac{\pi c}{6L} + \mathcal{O}\left(\frac{1}{L^2}\right), \quad (3.23)$$

where c is the central charge. This equation is directly inspired by the well-known formula derived in Refs. [32] and [4] for the actual $2d$ CFTs.

In the following we compute $\mathcal{F}$ for the pure state Chern insulator, and demonstrate that its finite size scaling does encode the correct central charge. In Eq. 3.2, for the Chern insulator we take its mass to be $m > 0$ while for the trivial insulator we take its mass to be $-m$. The Hamiltonian densities are

$$\mathcal{H}_c \sim \psi^\dagger(k)(-k_x \sigma^z - k_y \sigma^x + m \sigma^y) \psi(k), \quad (3.24)$$

$$\mathcal{H}_\Omega \sim \psi^\dagger(k)(-k_x \sigma^z - k_y \sigma^x - m \sigma^y) \psi(k). \quad (3.25)$$

The Bloch Hamiltonians can be diagonalized in the momentum basis labeled by

$$k_y \in \mathbb{R}; \quad k_x = \frac{2\pi(n - 1/2)}{L}, n \in \mathbb{Z}. \quad (3.26)$$

Here we have chosen an anti-periodic boundary condition for the fermions along the x direction, as one can show that the ground state wave function for the Chern insulator with an anti-periodic boundary condition has a lower energy compared with that with a periodic boundary condition.

The fidelity $\mathcal{Z}$ becomes a product of the overlap between Bloch wave functions at each momentum:

$$\begin{aligned}\mathcal{Z} &= |\langle \Psi_\Omega | \Psi_c \rangle|^2 = \prod_k b(k), \\ b(k) &= \frac{k_x^2 + k_y^2}{k_x^2 + k_y^2 + m^2}.\end{aligned}\tag{3.27}$$

The “free energy” $\mathcal{F}$ per unit length in y direction is

$$\frac{\mathcal{F}}{L_y} = -\frac{\log \mathcal{Z}}{L_y} = -\frac{1}{2\pi} \sum_{k_x} \int dk_y \log(b(k_x, k_y))\tag{3.28}$$

The integral in k_y can be performed explicitly, and yields

$$\frac{\mathcal{F}}{L_y} = \frac{1}{2\pi} \sum_{k_x} (-2\pi|k_x| + 2\pi\sqrt{k_x^2 + m^2}).\tag{3.29}$$

We examine the term $\sum_{k_x} -|k_x|$ first

$$\sum_{k_x} -|k_x| = \sum_{n=-\infty}^{\infty} -\frac{2\pi}{L} |n - 1/2| = \sum_{n=1}^{\infty} -\frac{2\pi}{L} (2n - 1)\tag{3.30}$$

In order to perform the sum of all odd integers, we need to use the Zeta function regularization:

$$\sum_{n=1}^{\infty} (2n - 1)^{-s} = (1 - 2^{-s})\zeta(s).\tag{3.31}$$

Plugging in $s = -1$ would give us $\sum_n (2n - 1) = 1/12$. Hence eventually we obtain the

following result:

$$\sum_{k_x} -|k_x| = -\frac{\pi c}{6L} \quad (3.32)$$

where $c = 1$. This result is consistent with the well-known formula of finite size scaling of $2d$ CFTs given in Ref. [4, 32].

However, there is also correction from finite mass m which is attained by analyzing the second term of Eq. 3.29. This term $\sum_{k_x} \sqrt{k_x^2 + m^2}$ would physically correspond to the Casimir effect of a massive particle in $1d$. Intuitively a massive particle would not lead to any Casimir effect with large mass, but we will test this intuition as follows. This sum can be regularized using a special form of the Abel-Plana formula for half-integer sums [36]:

$$\sum_{n=0}^{\infty} f_{n+\frac{1}{2}} = \int_0^{\infty} dt f(t) - i \int_0^{\infty} dt \frac{[f(it) - f(-it)]}{1 + e^{2\pi t}}. \quad (3.33)$$

Only the second integral contains a finite piece independent of the regularization scheme, and for a large $|m|$ the integral can be approximated as $-\sqrt{\frac{2|m|}{\pi L}} e^{-|m|L}$. Therefore, for large $|m|$, the quantity $\mathcal{F}/L_y$ with anti-periodic boundary condition reads

$$\frac{\mathcal{F}_{\text{APBC}}}{L_y} \sim -\frac{\pi c}{6L} + \sqrt{\frac{2|m|}{\pi L}} e^{-|m|L}, \quad (3.34)$$

with $c = 1$. As expected, the correction to $\mathcal{F}$ arising from the second term of Eq. 3.29 decays rapidly in the large mass limit and one recovers the exact CFT scaling.

For completeness, one can also compute the sum Eq. 3.29 for the more energetic case of periodic boundary conditions where the momentum in the x -direction now takes values of $k_x \in \frac{2\pi}{L}\mathbb{Z}$, which would yield the result

$$\frac{\mathcal{F}_{\text{PBC}}}{L_y} \sim \frac{\pi}{3L} - \sqrt{\frac{2|m|}{\pi L}} e^{-|m|L}. \quad (3.35)$$

One can also use the general formula Eq. 3.33 to evaluate the first sum of Eq. 3.29. The first integral in Eq. 3.33 diverges in the UV, and it is proportional to L ; the second integral leads to the desired result of $-\pi/(6L)$.

3.6 Chern insulator with $\nu = 4$

If we start with the Chern insulator with $\nu = 4$, the physics should be qualitatively similar to the case with other Chern numbers, as long as the quantum operation only performs measurements on quantities that preserve the $U(1)$ charge symmetry. The $U(1)$ charge symmetry will always become a doubled symmetry in the density matrix formalism, and map to the $U(1)_L$ and $U(1)_R$ symmetry after the space-time rotation. Then the perturbative 't Hooft anomaly will exclude the possibility of a gapped "partition function" $\mathcal{Z} \sim \text{tr}\{\hat{\rho}_c^D \hat{\rho}_\Omega\}$.

Let us now break the doubled $U(1)$ symmetry down to a diagonal $U(1)$ symmetry, meaning we allow the quantum operation to measure all bosonic operators, such as all fermion bilinear operators including Cooper pairs. Effectively the $1d$ system after space-time rotation has four left-moving and four right-moving complex fermions, and there is only one global (diagonal) $U(1)$ charge symmetry, when there is no post-selection that breaks the diagonal $U(1)$ symmetry. There is also a separate fermion parity of left and right chiral fermions: $Z_2^L \times Z_2^R$. The effect of the weak measurement is mapped to interacting terms of the $1d$ system. According to the classification of interacting TIs, four copies of $(1+1)D$ nonchiral complex fermions can be gapped out by four fermion interactions without breaking any symmetry or ground state degeneracy [80, 81, 103, 239]. This mechanism also generalizes to other models and higher dimensions, and is in general referred to as symmetric mass generation (SMG) [16, 17, 45, 47, 112, 193, 242, 269, 283, 294, 295, 310, 311, 313]. But the interaction terms in our case have a more restrictive form, and need to be generated by weak measurement. We have *not* found the

corresponding Kraus operators that can generate the exact form of the interactions that have been definitely proven to cause SMG for the system. We leave the possibility of SMG caused solely by decoherence or weak measurement as an open question for future study.

By considering proper Kraus operators and their Hermitian conjugates that respect fermion parity, one can indeed gap out the CFT described by $\mathcal{Z}$ and $\mathcal{F}$. However we note here that the $(1+1)D$ system so obtained from space-time rotation is not a completely trivial state, instead it would spontaneously break the $Z_2^L \times Z_2^R$ symmetry down to a diagonal fermion parity. Let us consider the following quantum operation as an example:

$$\begin{aligned}\mathcal{E}_x[\hat{\rho}_c] &\sim \hat{\rho}_c + \sum_{i=1}^4 p_i \left(\hat{O}_{i,x} \hat{\rho}_c \hat{O}_{i,x}^\dagger + \hat{O}_{i,x}^\dagger \hat{\rho}_c \hat{O}_{i,x} \right) \\ \hat{O}_{1,x} &= c_{1,x}^\dagger \sigma^z (c_{2,x}^\dagger)^T \text{ and } \hat{O}_{i \neq 1,x} = c_{1,x}^\dagger \sigma^z c_{i,x}.\end{aligned}\tag{3.36}$$

Following our previous procedure, the fidelity is mapped to a path integral problem of four flavors of $(1+1)D$ Dirac fermions with the following interactions

$$\begin{aligned}\mathcal{L}^{\text{int}} &= \sum_{i \neq 1} g_i (\psi_{1,L}^\dagger \psi_{i,L} \psi_{i,R}^\dagger \psi_{1,R} + \text{h.c.}) \\ &\quad - g_1 (\psi_{1,L}^\dagger \psi_{2,L}^\dagger \psi_{2,R} \psi_{1,R} + \text{h.c.})\end{aligned}\tag{3.37}$$

These terms after Abelian bosonization become:

$$\begin{aligned}\mathcal{L}^{\text{int}} &= \sum_{i \neq 1} u_i \cos(2\phi_1 - 2\phi_i) - u_1 \cos(2\phi_1 + 2\phi_2) \\ &= \sum_{i \neq 1} u_i \cos(\Lambda_i^T \mathcal{K} \Phi) - u_1 \cos(\Lambda_1^T \mathcal{K} \Phi).\end{aligned}\tag{3.38}$$

The vector Φ is $\Phi = (\phi_{L,1}, \dots, \phi_{L,4}; \phi_{R,1}, \dots, \phi_{R,4})^T$ where $\phi_{L,i} = \theta_i - \phi_i$ and $\phi_{R,i} = \theta_i + \phi_i$, and the K -matrix is $\mathcal{K} = \text{diag}(-1, \dots, -1; 1, \dots, 1)$. In this basis, one can verify that

the four Λ -vectors take the following form

$$\begin{aligned}
\Lambda_1 &= (1, -1, 0, 0; 1, -1, 0, 0)^T \\
\Lambda_2 &= (1, 0, -1, 0; 1, 0, -1, 0)^T \\
\Lambda_3 &= (1, 0, 0, -1; 1, 0, 0, -1)^T \\
\Lambda_4 &= (1, 1, 0, 0; 1, 1, 0, 0)^T.
\end{aligned} \tag{3.39}$$

We will outline the conditions that these vectors satisfy and show that they drive the theory into a Z_2 SSB state. A more complete discussion of these conditions and the Λ -vector formalism can be found in Refs [103, 181].

In order for these vertex operators to completely gap the system, the Λ vectors must be four linearly independent vectors that satisfy the so-called Haldane null-vector condition: $\Lambda_i^T \mathcal{K} \Lambda_j = 0$ for all i, j . This condition ensures that the fields can be rotated to a new basis where each vertex operator contains only one single field instead of a linear combination, and this condition is indeed met by the four vectors in Eq. 3.39. However, if these terms can indeed trivially gap out the $(1+1)D$ system without any degeneracy (i.e. SMG), it is necessary that the determinants of all possible 4×4 minors of the matrix with rows of Λ_i^T do not share a common factor larger than one. One can verify that this condition is *not* met by the four interaction terms generated by Kraus operators chosen above, and our choice of Kraus operators must drive the boundary into a state that spontaneously breaks the fermion parity $Z_2^L \times Z_2^R$ down to its diagonal. As a consequence, the following fermion bilinears in the effective $(1+1)D$ theory acquire a

nonzero expectation value

$$\langle \psi_{L,i}^\dagger \psi_{R,i} \rangle \neq 0. \quad (3.40)$$

The correlation of these fermion bilinears are the following type-II strange correlator first introduced in Ref. [169] (Fig. 3.1(c)):

$$\begin{aligned} & \langle \psi_{L,i}^\dagger \psi_{R,i}(0,0) \psi_{R,i}^\dagger \psi_{L,i}(\tau, x) \rangle \\ & \sim C^{\text{II}}(\mathbf{x}) = \frac{\text{tr} \left(c_{i,\alpha}^\dagger(0) c_{i,\alpha}(\mathbf{x}) \hat{\rho}_c^D c_{i,\alpha}^\dagger(\mathbf{x}) c_{i,\alpha}(0) \hat{\rho}_\Omega \right)}{\text{tr}(\hat{\rho}_c \hat{\rho}_\Omega)} \end{aligned} \quad (3.41)$$

where $i = 1, 2, 3, 4$ and α labels the Dirac index. $\mathbf{x} = (x, y)$ labels the $2d$ spatial coordinate before the space-time rotation.

3.7 Summary and Discussion

In this chapter, we investigate the fidelity between the mixed state Chern insulator, i.e. the mixed state density matrix generated from a Chern insulator under quantum operations such as decoherence or measurement, and a pure state trivial insulator. We demonstrate that the fidelity is mapped to the partition function of a $(2+0)d$ or $(1+1)D$ CFT, and the Rényi relative entropy is mapped to the free energy of the CFT. This quantum operation can lead to a variety of CFTs with different central charges. We also devised a procedure to extract the central charges, without having to know the details of the order parameters that would lead to nontrivial strange correlators.

The hermiticity of the density matrix demands that, after space-time rotation, the $(1+1)D$ theory is invariant under anti-unitary transformation $\psi_L \rightarrow \psi_R^\dagger$, $\psi_R \rightarrow \psi_L^\dagger$, and $i \rightarrow -i$. This alone does not guarantee that the effective $(1+1)D$ CFT be unitary. Indeed, the strange correlator has been utilized as a tool to engineer nonunitary

CFTs [192, 257, 316]. In our current work, the symmetries of the system and the process ensure that at least there is no *relevant* nonhermitian terms in the effective $(1+1)D$ theory.

As we have seen in this chapter, the effect of quantum operation is mapped to interaction terms in the fidelity, which corresponds to the partition function of the $(1+1)D$ CFT. The same physical picture also applies to all other free fermion topological insulators and topological superconductors (TSC). But as we have mentioned, the quantum operation only realizes certain restricted forms of interactions. We leave a complete discussion of TIs and TSCs under quantum operations for a future study.

Chapter 4

Tapestry of Dualities in Decohered Quantum Error-Correcting Codes

This chapter studies decohered quantum error-correcting codes as mixed-state quantum matter. The main diagnostic is the family of Rényi entropies of stabilizer and CSS-code density matrices under Pauli noise. These quantities map to classical statistical models with random couplings, so that decoherence-induced singularities become ordinary phase transitions in dual statistical-mechanics descriptions. For CSS codes, bit-flip and phase-flip errors generate paired models related by high-low-temperature dualities, and additional bit-phase-decoherence dualities appear at selected Rényi indices. When the code has an electric-magnetic symmetry, these relations become self-dualities with super-universal self-dual error rates.

This chapter contributes to the thesis by making the operational meaning of non-linear diagnostics more concrete. In the error-correcting-code setting, Rényi entropies organize a family of transitions connected to decodability, doubled-Hilbert-space quantum criticality, and statistical-mechanics duality. This gives one of the sharpest examples in the thesis of how information processing and mixed-state universality constrain each other.

4.1 Introduction

Quantum error correction (QEC) is a fundamental scheme in quantum computation to protect quantum information from decoherence caused by errors and noise [97, 282]. A general QEC code achieves this protection by encoding the quantum information into the “logical qubits” formed by many-body states of physical qubits with entanglement patterns resilient against errors. From the quantum matter perspective, a large class of QEC codes, epitomized by the toric code [152], can be viewed as systems with topological orders, where the long-range-entangled and topologically-protected ground states serve as the logical qubits. Recently, there has been tremendous progress in the experimental realizations of the toric code and related QEC codes on noisy intermediate-scale quantum platforms [2, 11, 255, 260]. The preparation of a QEC code on a noisy quantum platform is, in general, an out-of-equilibrium process inevitably subject to decoherence. Instead of a Gibbs ensemble, the prepared state is an “error-corrupted” mixed state, i.e., a classical mixture of different error patterns on top of the code’s logical states. Understanding such error-corrupted mixed states brings insights into the robustness of QEC codes against errors and the effect of decoherence on topological orders.

An important metric for the performance of a QEC code is the error threshold, which pertains to the “decodability” of the logical qubits from the error-corrupted mixed state. The seminal work Ref. [70] pointed out that the decodability of the 2D toric code with errors is captured by the 2D classical random-bond Ising model, and the error threshold is identified as a continuous phase transition. This method that maps the QEC code’s decodability to classical statistical models has been widely generalized to other stabilizer codes (see Ref. [33, 57, 148, 271, 293] for examples).

Recent works Ref. [22, 76, 170] provide a different perspective on the error threshold of the toric code by viewing it as a singularity intrinsic to the error-corrupted mixed state ρ . This singularity manifests a decoherence-induced phase transition (DIPT) of the

system's total von Neumann entropy $S_1 = -\text{Tr } \rho \log \rho$ as a function of the decoherence strength, i.e. the error rate. Moreover, the error threshold naturally belongs to the rich family of DIPTs of the Rényi entropies $S_R = \frac{1}{1-R} \log(\text{Tr } \rho^R)$ (for $R = 2, 3, \dots$ and $R \rightarrow 1$) in the decohered toric code [22, 76, 170]. For a general QEC code under decoherence, the DIPTs of the Rényi entropies probe the singular changes in the entanglement pattern in the error-corrupted mixed state. In light of the close relation between QEC codes and topological orders in systems free of decoherence, studying the behavior of the error-corrupted mixed state in decohered QECs and the DIPTs therein provides an interesting path toward understanding topological orders and other decoherence-induced exotic phases in beyond-Gibbs-ensemble mixed states.

In this chapter, we focus on developing a general framework to study the error-corrupted mixed states in stabilizer codes decohered by Pauli noise, including bit-flip and phase-flip errors in particular. We systematically construct the statistical models arising from the Rényi entropies S_R for a general R and identify a rich set of duality relations that weave these statistical models into a "tapestry of dualities". Our discussion contains three major parts, focusing on three classes of translation-invariant stabilizer codes: (1) general Calderbank-Shor-Steane (CSS) codes, (2) CSS codes with an "*em* symmetry" between the X -type and the Z -type stabilizers, and (3) general stabilizer codes.

We first investigate general CSS codes decohered by bit-flip and phase-flip errors. The tapestry of dualities for this class of decohered codes is shown in Fig. 4.1. The two error types cause independent decoherence in a CSS code and are, hence, studied separately. The behavior of the decohered CSS code is intimately related to a pair of dual statistical models, named SM_1 and SM_2 , which are obtained from ungauging the CSS code. Under bit-flip errors, the Rényi entropy S_R of the error-corrupted mixed state is described by a pair of dual statistical models with randomness: (1) R -replica SM_1 with real random couplings (rRC) and (2) R -replica SM_2 with imaginary random

couplings (iRC). These two statistical models are dual under a high-low-temperature (HLT) duality. Similarly, under phase-flip errors, S_R is described by another pair of random statistical models: (1) R -replica SM_1 with iRC and (2) R -replica SM_2 with rRC, which are again HLT dual to each other. The patterns of randomness in these statistical models are interchanged as we switch from bit-flip errors to phase-flip errors. Additionally, for $R = 2, 3, \infty$, we discover extra duality relations that map between (the random statistical models associated with) bit-flip and phase-flip errors. We call these dualities the bit-phase-decoherence (BPD) dualities. The random statistical models and the dualities provide powerful tools to study the phases and the DIPTs in the decohered CSS codes. We propose a conjecture on the monotonicity of the DIPTs' critical error rates as functions of the Rényi index R , and discuss the alternative interpretation of the DIPT with $R = 2$ as a quantum phase transition in the doubled Hilbert space. We also discuss two concrete examples, the decohered 3D toric code [151] and the decohered X-cube model [290], demonstrating the abovementioned general results.

When the CSS codes are endowed with additional symmetries, the duality structure under decoherence can be enriched. We analyze the case when the CSS codes have a symmetry that exchanges the X -type and Z -type stabilizers, which we dub “electric-magnetic symmetry” (em symmetry). Under the bit-flip and phase-flip errors, the tapestry of dualities of the em -symmetric CSS code is shown in Fig. 4.7, which is effectively the tapestry shown in Fig. 4.1 “folded in half” by the symmetry. The em symmetry interchanges the bit-flip and phase-flip errors. The original pair of dual statistical models SM_1 and SM_2 become the same model, call it SM , with a self-duality. Under either bit-flip or phase-flip errors, the Rényi entropies S_R are captured by a pair of statistical models with randomness: (1) R -replica SM with rRC and (2) R -replica SM with iRC. They are related by the HLT duality for any R . The BPD dualities for $R = 2, 3, \infty$ (combined with the em symmetry) become self-dualities with *super-universal* self-dual error rates shared by all em -symmetric CSS codes in different dimensions. For a given

em -symmetric CSS code (and a fixed $R = 2, 3$ or ∞), if there is a unique DIPT of S_R as the error rate varies, the critical error rate must coincide with the super-universal self-dual values. We demonstrate our general results in the concrete settings of the decohered 2D toric code [151], and the decohered Haah’s code in 3D [108].

For general stabilizer codes, bit-flip and phase-flip errors are no longer special. We therefore consider the decoherence by general Pauli noise. The general decohered stabilizer codes are tied to a single statistical model SM, which is self-dual under an HLT duality. The Rényi entropy S_R of the decohered code is captured by both (1) the R -replica SM with rRC and (2) the R -replica SM with iRC, two random statistical models related by an HLT duality. Moreover, we find extra dualities between different error rates for $R = 2, \infty$ and call them “general-Pauli-noise dualities” (GPN dualities). The GPN dualities are the generalizations of the BPD dualities of decohered CSS codes. We obtain the surface of super-universal self-dual error rates under the GPN dualities.

The rest of this chapter is organized as follows. Sec. 4.2 focuses on the general CSS codes decohered by bit-flip and phase-flip errors. Sec. 4.3 discusses the decohered CSS codes with em symmetry. Sec. 4.4 analyzes general stabilizer codes with general Pauli noise. We present our conclusion and outlook in Sec. 4.5.

4.2 CSS code under decoherence

In this section, we will first review the basics of general CSS codes and introduce the decoherence model that describes bit-flip and phase-flip errors. Next, we provide the general construction of the statistical models, the R -replica SM_1 and SM_2 with randomness, that describe the Rényi entropies S_R of the error-corrupted mixed states of the decohered CSS code. We will address how the $R \rightarrow 1$ limit recovers the disordered statistical model that captures the code’s error threshold for decodability. Then, we present the HLT and BPD dualities between these statistical models and the tapestry of

dualities they form. The DIPTs are signaled by the singular behavior of the Rényi entropies S_R and, hence, are identified with the phase transitions in the random statistical models. For $R = 2$, by generalizing Ref. [22, 168], we give an alternative interpretation of the DIPT as a quantum phase transition in the doubled Hilbert space. We also propose a conjecture on the monotonicity of the DIPTs' critical error rates as a function of the Rényi index R . Finally, we study the decohered 3D toric code and the decohered X-cube model as concrete examples.

4.2.1 Introduction to CSS codes and error models for decoherence

First, we briefly review the general description of a CSS code. Consider a set of N qubits (labeled by μ) arranged into a lattice. A general CSS code is a stabilizer code with two types of stabilizers: X -type and Z -type. Each X -type generator, denoted as $A_i[X]$, is a product of only Pauli- X operators X_μ , while each Z -type generator, denoted as $B_j[Z]$, is a product of only Pauli- Z operators Z_μ . All the stabilizers commute with each other:

$$[A_i[X], B_j[Z]] = 0, \quad \forall i, j. \quad (4.1)$$

The logical subspace of the CSS code is the ground-state Hilbert space of the CSS code Hamiltonian

$$H_{\text{css}} = - \sum_i A_i[X] - \sum_j B_j[Z]. \quad (4.2)$$

Each logical state $|\Omega\rangle$ satisfies

$$A_i[X] |\Omega\rangle = B_j[Z] |\Omega\rangle = |\Omega\rangle, \quad \forall i, j. \quad (4.3)$$

For this chapter, we work with CSS codes with the following general properties. We

assume that H_{css} is translation invariant (modulo boundary effects of the lattice if they exist) and all the terms in H_{css} are local. The number of $A_i[X]$ terms does not have to equal the number of $B_j[Z]$ terms. But we assume that there is no local logical operator and the rate of the code N_c/N vanishes in the large system limit. Here, N_c denotes the total number of logical qubits. These properties are rather general. They are broadly satisfied by commonly discussed topological CSS codes, including the toric code [151], the surface code [37, 70], color codes [35], the X -cube model [290], Haah's code [108], etc.

Next, we introduce the error models for the decoherence that we will study in this section. We consider an error model with bit-flip errors and phase-flip errors. For a single site μ , when a bit-flip (phase-flip) error occurs, the system's state is acted on by X_μ (Z_μ). The classical probabilities p_x and p_z for the appearance of these errors on a single qubit are called the error rates. The decoherence caused by these two types of errors is described by the following quantum channels respectively:

$$\begin{aligned}\mathcal{N}_{x,\mu}(\rho) &= (1 - p_x)\rho + p_x X_\mu \rho X_\mu, \\ \mathcal{N}_{z,\mu}(\rho) &= (1 - p_z)\rho + p_z Z_\mu \rho Z_\mu.\end{aligned}\tag{4.4}$$

When all the physical qubits are subject to such decoherence, a logical state $\rho_0 = |\Omega\rangle\langle\Omega|$, initially pure, becomes an error-corrupted mixed-state:

$$\rho_0 \rightarrow \mathcal{N}_x \circ \mathcal{N}_z(\rho_0),\tag{4.5}$$

a mixture of the logical state dressed by all error patterns. Here, we've defined the total decoherence channels $\mathcal{N}_{x/z} \equiv \otimes_\mu \mathcal{N}_{x/z,\mu}$. It suffices to consider the range of error rates $0 \leq p_x, p_z \leq \frac{1}{2}$ because the system experiences the maximal amount of bit-flip (phase-flip) decoherence when $p_x = 1/2$ ($p_z = 1/2$).

The bit-flip errors only affect the Z -type stabilizers, while the phase-flip errors only

influence the X -type stabilizers. Hence, the two error types cause independent decoherence effects (more details in the next subsection). Without loss of generality, it suffices to independently study the two types of mixed states ρ_b and ρ_p corrupted only by the bit-flip and phase-flip errors respectively:

$$\rho_b = \mathcal{N}_x(\rho_0), \quad \rho_p = \mathcal{N}_z(\rho_0). \quad (4.6)$$

We can write these mixed states as a summation over error chains. For example,

$$\rho_b = \sum_E P_x(E) X_E \rho_0 X_E, \quad (4.7)$$

where $\sum_E$ is a summation over all possible error chain E . An error chain E is the set of qubits where the errors occur and $X_E \equiv \prod_{\mu \in E} X_\mu$. $P_x(E)$ is the probability for the error chain E to appear:

$$P_x(E) = p_x^{|E|} (1 - p_x)^{N - |E|}. \quad (4.8)$$

Here, $|E|$ denotes the number of qubits in E . Similarly, we can write

$$\rho_p = \sum_E P_z(E) Z_E \rho_0 Z_E, \quad (4.9)$$

with $P_z(E) = p_z^{|E|} (1 - p_z)^{N - |E|}$ and $Z_E \equiv \prod_{\mu \in E} Z_\mu$. In the next subsection, we develop the statistical models that capture the behavior of the error-corrupted mixed states $\rho_{p/b}$.

As a preparation for the subsequent analysis, it is helpful to introduce an N -dimensional $\mathbb{Z}_2$ vector space $\mathcal{V}$ for the lattice of N qubits. Any subset W of qubits can be represented as an N -component vector $W \in \mathcal{V}$ whose μ th entry is 1 if the μ th site belongs to the subset W and 0 if not. Here and for the rest of the chapter, we use the same notation for a subset of qubits and its corresponding vector. For a subset W ,

its cardinality $|W|$ counts the number of nonzero entries in the vector W . With two vectors $W_1, W_2 \in \mathcal{V}$, $W_1 + W_2$ and $W_1 \cdot W_2$ represent the addition and the dot product in the $\mathbb{Z}_2$ vector space $\mathcal{V}$. In terms of subsets, $W_1 + W_2$ represents $W_1 \cup W_2 - W_1 \cap W_2$ and $W_1 \cdot W_2$ is the parity of $|W_1 \cap W_2|$.

For each stabilizer $A_i[X]$, we denote the subset of qubits it includes as $\mathbf{a}_i$. Similarly, every $B_j[Z]$ has a corresponding vector $\mathbf{b}_j$. The commutation relation Eq. (4.1) is equivalent to

$$\mathbf{a}_i \cdot \mathbf{b}_j = 0 \quad \forall i, j. \quad (4.10)$$

The collection of vectors $\{\mathbf{a}_i\}$ ($\{\mathbf{b}_j\}$) span a sub vector space denoted as $\mathcal{V}_x$ ($\mathcal{V}_z$). The relation Eq. (4.10) implies that $\mathcal{V}_x$ belongs to the orthogonal subspace of $\mathcal{V}_z$, namely $\mathcal{V}_x \subset \mathcal{V}_z^\perp$. Similarly, we have $\mathcal{V}_z \subset \mathcal{V}_x^\perp$. A logical X -operator (Z -operator) of the CSS code must be a product of X_μ (Z_μ) on a subset of qubits whose corresponding vector belongs to the quotient space $\mathcal{V}_z^\perp/\mathcal{V}_x$ ($\mathcal{V}_x^\perp/\mathcal{V}_z$). The number of logical qubits is then given by $N_c = \dim(\mathcal{V}_z^\perp/\mathcal{V}_x) = \dim(\mathcal{V}_x^\perp/\mathcal{V}_z)$. Our assumption that the code rate $N_c/N \rightarrow 0$ in the infinite system limit conceptually implies $\mathcal{V}_z^\perp$ ($\mathcal{V}_x^\perp$) is almost the same as $\mathcal{V}_x$ ($\mathcal{V}_z$), which is important for the dualities we will discuss later. The assumption that the CSS code is free of local logical operators implies that the quotient spaces $\mathcal{V}_z^\perp/\mathcal{V}_x$ and $\mathcal{V}_x^\perp/\mathcal{V}_z$ only contain vectors that represent non-local sets that involve infinitely many qubits in the infinite system limit.

In addition to characterizing the stabilizers, it proves convenient to treat any error chain E as a vector in $\mathcal{V}$ when analyzing the effects of decoherence. For example, the error chains E, E' follow the multiplication and commutation relations,

$$\begin{aligned} X_E X_{E'} &= X_{E+E'}, & Z_E Z_{E'} &= Z_{E+E'}, \\ X_{E'} Z_E &= (-1)^{E \cdot E'} Z_E X_{E'}. \end{aligned} \quad (4.11)$$

For any subset E , if $E \in \mathcal{V}_x$, X_E can be written as a product of the $A_i[X]$ stabilizers. Similarly, Z_E is product of the $B_j[Z]$ stabilizers if and only if $E \in \mathcal{V}_z$. The summation $\sum_E$ in Eq. (4.7) and (4.9) can be viewed as a summation $\sum_{E \in \mathcal{V}}$ over all $\mathbb{Z}_2$ vectors in $\mathcal{V}$.

4.2.2 Statistical models for decohered CSS codes

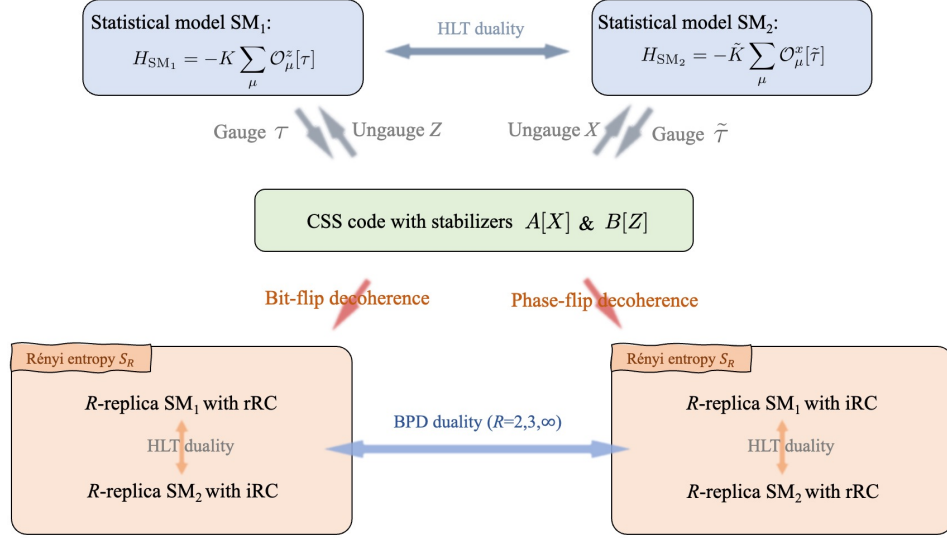


Figure 4.1: The tapestry of dualities of a general CSS code. In Sec. 4.2.2, we introduce the statistical models $SM_{1,2}$ through ungauging the CSS code and introduce the R -replica $SM_{1,2}$ with randomness (rRC and iRC) as the descriptions of the Rényi entropies of the decohered CSS code. The HLT and BPD dualities are discussed in Sec. 4.2.2 and 4.2.3.

We now develop the systematic framework to characterize the effects of decoherence in the error-corrupted mixed state $\rho = \mathcal{N}_x \circ \mathcal{N}_z(\rho_0)$ using statistical models. These decoherence effects are manifested in the behavior of $\text{Tr} \rho^R$, which are directly related to the R -th Rényi entropies $S_R = \frac{1}{1-R} \log(\text{Tr} \rho^R)$. Singularities in S_R (as functions of the error rates p_x and p_z) signal the DIPTs. In particular, the singularity in the limit $R \rightarrow 1$ is the error threshold of the decodability of the code. In the following, we show that $\text{Tr} \rho^R$ can be formulated as the partition functions of a family of statistical models of classical $\mathbb{Z}_2$ spins with random couplings. The Rényi entropies are, therefore, the respective free energies. The DIPTs can then be identified as the phase transitions in

these statistical models. This subsection is devoted to the general construction of these statistical models for the decohered CSS code in the infinite-system limit. Concrete examples will be provided in the subsequent subsections Sec. 4.2.5 and 4.2.6.

For a CSS code $\mathcal{C}$, decoherence effects of the bit-flip errors decouple from those of the phase-flip errors. The reason is that the former only affects the $B_j[Z]$ stabilizers while the latter only affects the $A_i[X]$ stabilizers. The decoupling is manifested by the following identity in the infinite-system limit

$$\text{Tr } \rho^R = \text{Tr } \rho_b^R \times \text{Tr } \rho_p^R. \quad (4.12)$$

In infinite-system limit, the absence of local logic operators implies $|\langle \Omega | X_E Z_{E'} | \Omega \rangle| = |\langle \Omega | X_E | \Omega \rangle| \times |\langle \Omega | Z_{E'} | \Omega \rangle|$ for the logical state $|\Omega\rangle$ and any pair of finite subsets of qubits E, E' . This property leads to Eq. (4.12). The decoupling of the bit-flip and phase-flip errors allows us to independently formulate the statistical models for $\text{Tr } \rho_b^R$ and $\text{Tr } \rho_p^R$ without loss of generality. Also, note that the absence of local logical operators makes all logical states locally indistinguishable. For this reason, different choices of the error-free logical state $\rho_0 = |\Omega\rangle \langle \Omega|$ will not affect the construction of the statistical models in the infinite-system limit.

As the first step towards the statistical models, we use Eqs. (4.7) and (4.9) to expand $\text{Tr } \rho_{b/p}^R$ as a sum over the vector spaces $\mathcal{V}$, $\mathcal{V}_x$, and $\mathcal{V}_z$. Take $\text{Tr } \rho_b^2$ as an example. In the infinite-system limit,

$$\begin{aligned} \text{Tr}(\rho_b^2) &= \sum_{E, E' \in \mathcal{V}} P_x(E) P_x(E') |\langle X_E X_{E'} | X_E X_{E'} \rangle_\Omega|^2 \\ &= \sum_{E \in \mathcal{V}} \sum_{C \in \mathcal{V}_x} P_x(E) P_x(E + C). \end{aligned} \quad (4.13)$$

Here, we've used the fact that $|\langle X_E X_{E'} | X_E X_{E'} \rangle_\Omega|^2 = 0$ unless $X_E X_{E'} = X_{E+E'}$ can be written as a product of the $A_i[X]$ stabilizers, i.e. $E + E' \in \mathcal{V}_x$. When $E + E' \in \mathcal{V}_x$, we

have $|\langle X_E X_{E'} | X_E X_{E'} \rangle_\Omega|^2 = 1$ and $E' = E + C$ for some $C \in \mathcal{V}_x$. For general R , we have

$$\begin{aligned}
& \text{Tr}(\rho_b^R) \\
&= \sum_{E \in \mathcal{V}} \sum_{C_2, \dots, R \in \mathcal{V}_x} P_x(E) P_x(E + C_2) \dots P_x(E + C_R) \\
&= \frac{1}{2^{\dim \mathcal{V}_x}} \sum_{E \in \mathcal{V}} \left(\sum_{C_1, 2, \dots, R \in \mathcal{V}_x} P_x(E + C_1) \dots P_x(E + C_R) \right). \tag{4.14}
\end{aligned}$$

The last line is obtained from the second by shifting $E \rightarrow E + C_1$, $C_2, \dots, R \rightarrow -C_1 + C_2, \dots, R$ with $C_1 \in \mathcal{V}_x$.

Later, we will see that $\text{Tr}(\rho_b^R)$ can be mapped to the partition function of an R -replica statistical model of classical $\mathbb{Z}_2$ spins with random couplings. $E \in \mathcal{V}$ specifies a pattern of the random couplings, and the partition function of each replica is given by $\sum_{C \in \mathcal{V}_x} P_x(E + C)$ (before averaging/summing over all randomness patterns E). The interaction between different replicas is only mediated by the random coupling E .

In the $R \rightarrow 1$ limit, the Rényi entropy S_R reduces to the von Neumann entropy S_1 :

$$\begin{aligned}
S_1(\rho_b) &= \lim_{R \rightarrow 1} \frac{1}{1 - R} \left(\text{Tr}(\rho_b^R) - 1 \right) \\
&= - \sum_{E \in \mathcal{V}} P_x(E) \log \left(\sum_{C \in \mathcal{V}_x} P_x(E + C) \right), \tag{4.15}
\end{aligned}$$

which turns out to be the quenched-disorder-average of the free energy

$$- \log \left(\sum_{C \in \mathcal{V}_x} P_x(E + C) \right) \tag{4.16}$$

in the random statistical model.

Similar to $\text{Tr}(\rho_{\text{b}}^R)$, we can expand $\text{Tr}(\rho_{\text{p}}^R)$ as

$$\begin{aligned} & \text{Tr}(\rho_{\text{p}}^R) \\ &= \frac{1}{2^{\dim \mathcal{V}_z}} \sum_{E \in \mathcal{V}} \left(\sum_{C_1, 2, \dots, R \in \mathcal{V}_z} P_z(E + C_1) \dots P_z(E + C_R) \right), \end{aligned} \quad (4.17)$$

which will also be mapped to the partition function of certain R -replica statistical models with random couplings. Consequently, the von Neumann entropy $S_1(\rho_{\text{p}}) = \lim_{R \rightarrow 1} S_R(\rho_{\text{p}})$ will be identified as the quenched-disorder-average free energy of the corresponding statistical model.

In the following, we present the systematic construction of the random statistical models of classical $\mathbb{Z}_2$ spins whose partition functions map to $\text{Tr}(\rho_{\text{b/p}}^R)$. For the sake of clarity, we introduce these statistical models in two steps. First, we construct these statistical models without randomness. These non-random statistical models naturally arise from ungauging the CSS code. In the second step, we establish the mapping between the multi-replica version of these statistical models (with randomness) and the quantities $\text{Tr}(\rho_{\text{b/p}}^R)$ (which are related to the Rényi entropies $S_R(\rho_{\text{b/p}})$).

Now, we perform the first step of our construction. For a general CSS code $\mathcal{C}$, we introduce two (non-random) classical statistical models, denoted as SM_1 and SM_2 . They can be obtained naturally by treating the CSS code as a generalized $\mathbb{Z}_2$ gauge theory and then ungauging it. As we will see, this ungauging procedure is exactly the inverse of the gauging procedure introduced in Ref. [290], which constructs CSS codes from statistical models of classical $\mathbb{Z}_2$ spins. In this step, we explain the ungauging as a procedure introduced by hand. When we consider $\text{Tr}(\rho_{\text{b/p}}^R)$ later in the second step, the ungauging is effectively implemented by the errors.

The ungauging procedure follows from the intuition based on the 2D toric code (see Fig. 4.2), which is equivalent to a conventional $\mathbb{Z}_2$ gauge theory. The general idea is as follows. For a general CSS code $\mathcal{C}$, we can treat the Z_μ 's as the gauge field variables. The

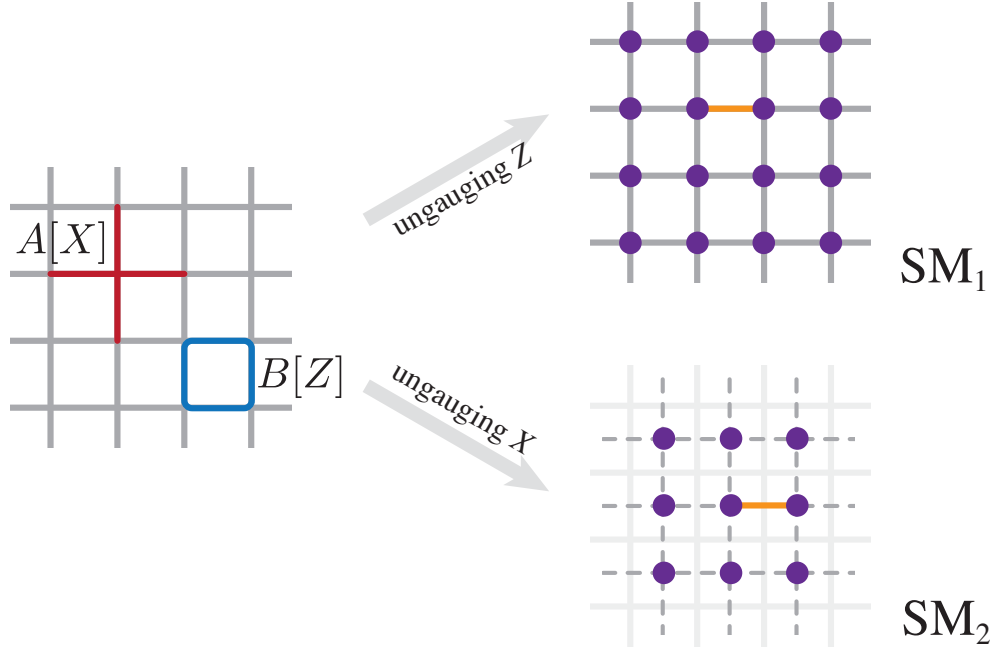


Figure 4.2: For the 2D toric code, the two types of stabilizers $A[X]$ and $B[Z]$ are pictorially represented above. Ungauging the gauge field Z and the gauge field ungauging X in the 2D toric code produce the 2D Ising model on the original lattice and that on the dual lattice, respectively. The original square lattice is depicted in gray solid lines, and the dual lattice is depicted in gray dashed lines. Each purple dot represents a classical $\mathbb{Z}_2$ spin $\tau/\tilde{\tau}$, and the orange edge presents the near-neighbor interactions in the Ising models. (see Sec. 4.3.2 for the microscopic details)

$A_i[X]$ terms in the CSS code Hamiltonian Eq. (4.2) implement the generalized Gauss law while the $B_j[Z]$ terms describe the energy costs of gauge fluxes. Ungauging the CSS code $\mathcal{C}$ from this perspective produces the classical statistical model SM_1 . If we instead treat X_μ as the gauge field and interchange the roles of $A_i[X]$ and $B_j[Z]$, ungauging the CSS code $\mathcal{C}$ results in classical statistical model SM_2 .

We now explicitly carry out the ungauging procedure to construct the non-random version of SM_1 from the general CSS code $\mathcal{C}$. We introduce a $\mathbb{Z}_2$ spin variable τ_i at the center of each $A_i[X]$ term and treat it as the matter field coupled to the generalized $\mathbb{Z}_2$ gauge field Z_μ . The operator $A_i[X]\tau_i^x$ for each i generates the gauge transformation, where τ_i^x is a Pauli- X operator. The gauge field Z_μ couples to the matter fields via the

gauge-invariant interaction $Z_\mu \mathcal{O}_\mu^z[\tau^z]$ with

$$\mathcal{O}_\mu^z[\tau^z] = \prod_i (\tau_i^z)^{(\mathbf{a}_i)_\mu}. \quad (4.18)$$

Recall that $\mathbf{a}_i \in \mathcal{V}$ is the $\mathbb{Z}_2$ vector associated with the stabilizer $A_i[X]$. $(\mathbf{a}_i)_\mu$ is the μ th component of $\mathbf{a}_i$. Due to the locality of each stabilizer $A_i[X]$, $\mathcal{O}_\mu^z[\tau^z]$ must be a local term as well for each μ . The commutation relation $[A_i[X]\tau_i^x, Z_\mu \mathcal{O}_\mu^z[\tau^z]] = 0$ for any i, μ implies the gauge-invariance of the interaction $Z_\mu \mathcal{O}_\mu^z[\tau^z]$. Hence, we can write the following Hamiltonian which describes the generalized $\mathbb{Z}_2$ gauge theory coupled to matter fields:

$$H_{\text{gt1}} = - \sum_i A_i[X] \tau_i^x - K \sum_\mu Z_\mu \mathcal{O}_\mu^z[\tau^z] - \sum_j B_j[Z], \quad (4.19)$$

where $K > 0$ is a coupling constant and the first term $\sum_i A_i[X] \tau_i^x$ leads to an emergent Gauss law (with matter fields) $A_i[X] \tau_i^x = 1$, $\forall i$ at low energies.

If we add an extra term $-g \sum_i \tau_i^x$ with a large positive g to the Hamiltonian H_{gt1} , the matter field τ_i will be gapped out and the CSS code Hamiltonian Eq. (4.2) will be recovered as the low-energy theory.

On the other hand, the model H_{gt1} allows us to ungauged by turning off all the gauge fields, namely setting $Z_\mu = 1$, and removing the Gauss-law terms $-\sum_i A_i[X] \tau_i^x$. The resulting Hamiltonian is the Hamiltonian of SM_1 (up to a constant):

$$H_{\text{SM}_1} = -K \sum_\mu \mathcal{O}_\mu^z[\tau]. \quad (4.20)$$

Here, we've suppressed the superscript z for the spin variables τ_i to emphasize that H_{SM_1} is essentially a *classical* Hamiltonian with commuting spin variables $\tau_i = \pm 1$. It is helpful to note that the lattice of the spin variable τ_i of SM_1 differs from the lattice of the qubits labeled by μ .

The construction of SM_2 is parallel to SM_1 . We simply need to interchange the roles of the $A_i[X]$ and $B_j[Z]$ stabilizers. For SM_2 , a $\mathbb{Z}_2$ spin variable $\tilde{\tau}_j$ is introduced at the center of each $B_j[Z]$ stabilizer. With X_μ treated as the generalized $\mathbb{Z}_2$ gauge field, ungauging the CSS code produces the classical Hamiltonian of SM_2 :

$$H_{\text{SM}_2} = -\tilde{K} \sum_{\mu} \mathcal{O}_{\mu}^x[\tilde{\tau}], \quad (4.21)$$

where $\tilde{K}$ is the coupling constant of SM_2 and the spin interaction $\mathcal{O}_{\mu}^x[\tilde{\tau}]$ is given by

$$\mathcal{O}_{\mu}^x[\tilde{\tau}] = \prod_j (\tilde{\tau}_j)^{(\mathbf{b}_j)_{\mu}}. \quad (4.22)$$

We emphasize that each $\mathcal{O}_{\mu}^x[\tilde{\tau}]$ term is a local spin interaction due to the locality of the $B_j[Z]$ stabilizers.

The two (non-random) statistical models SM_1 and SM_2 are dual to each other under a Kramers-Wannier-like HLT duality. We defer the details of this duality to the next subsection, where we include this duality as a component of the tapestry of dualities.

With $\text{SM}_{1,2}$ introduced, we are ready to establish the mapping between $\text{Tr } \rho_{\mathbf{b}/\mathbf{p}}^R$ for error-corrupted mixed states $\rho_{\mathbf{b}/\mathbf{p}}$ and the R -replica $\text{SM}_{1,2}$ with random couplings. There are two types of random couplings: real random couplings and imaginary random couplings, which we refer to as rRC and iRC for short.

For the error-corrupted mixed state $\rho_{\mathbf{b}}$, we obtain the following theorem.

Theorem 4.1. For the CSS code in the infinite system limit, $\text{Tr } \rho_{\mathbf{b}}^R$ of the mixed state $\rho_{\mathbf{b}}$ with bit-flip errors is proportional to the partition function of the R -replica SM_1 with real random couplings (rRC). It is also proportional to the partition

function of the R -replica SM_2 with imaginary random couplings (iRC):

$$\begin{aligned}\text{Tr}(\rho_b^R) &\propto \sum_{E \in \mathcal{V}} (Z_{\text{SM}_1}(K, E))^R \\ &\propto \sum_{E \in \mathcal{V}} \left(W_{\text{SM}_2}(\tilde{K}, E) \right)^R,\end{aligned}\tag{4.23}$$

where the coupling constants K and $\tilde{K}$ are given by the bit-flip error rate $p_x \in [0, 1/2]$ via $e^{-2K} = \tanh \tilde{K} = \frac{p_x}{1-p_x}$. The proportionality constants only depend on p_x smoothly and, hence, are unimportant for potential phases and DIPTs.

On the first line of Eq. (4.23), $Z_{\text{SM}_1}(K, E)$ is the single-replica partition function of SM_1 in the presence of the rRC configuration labeled by the error chain E :

$$Z_{\text{SM}_1}(K, E) = \sum_{\{\tau_i = \pm 1\}} \exp \left(K \sum_{\mu} (-1)^{E_{\mu}} \mathcal{O}_{\mu}^z[\tau] \right).\tag{4.24}$$

Here, E_{μ} is the μ th component of the $\mathbb{Z}_2$ vector $E \in \mathcal{V}$. E adds extra sign randomness to the real coupling constant K . Hence the name rRC. Note that $K > 0$ since $p_x \in [0, 1/2]$. One can show that

$$Z_{\text{SM}_1}(K, E) = [p_x(1-p_x)]^{-N/2} N_s \sum_{C \in \mathcal{V}_x} P_x(E + C),\tag{4.25}$$

where N_s is a factor related to internal symmetries of SM_1 . We refer to App. 4.6.1 for the derivation of this expression. The first line of Eq. (4.23) is an immediate consequence of Eqs. (4.25) and (4.14). Eq. (4.25) also shows that $Z_{\text{SM}_1}(K, E)$ is proportional to the total probability of all the error chains within the same equivalence class $[E]_x \in \mathcal{V}/\mathcal{V}_x$. Physically, the error chains within the same class $[E]_x$ differ from each other only by elements in $\mathcal{V}_x$ and, hence, cause excitations on the same set of $B_j[Z]$ stabilizers.

To provide some heuristics, we argue that the bit-flip errors suppress the coherent

quantum fluctuations of the generalized $\mathbb{Z}_2$ gauge field Z_μ . Hence, in the CSS code with bit-flip decoherence, a modified version of the Eq. (4.19) should appear. The modification includes turning off the first term (that generates coherent quantum fluctuations of the gauge field Z_μ) and treating gauge field Z_μ as a classical degree of freedom with values ± 1 . After this modification, SM_1 with rRC as defined in Eq. (4.24) emerges.

On the second line of Eq. (4.23), $W_{\text{SM}_2}(\tilde{K}, E)$ is the single-replica partition function of SM_2 in the presence of the iRC configuration labeled E :

$$\begin{aligned} W_{\text{SM}_2}(\tilde{K}, E) &= \sum_{\{\tilde{\tau}_j = \pm 1\}} \exp(-H_{\text{SM}_2}) \prod_{\mu} \left(\mathcal{O}_{\mu}^x[\tilde{\tau}] \right)^{E_{\mu}} \\ &= \sum_{\{\tilde{\tau}_j = \pm 1\}} \exp \left(-H_{\text{SM}_2} - \frac{\mathbf{i}\pi}{2} \sum_{\mu} E_{\mu} \left(\mathcal{O}_{\mu}^x[\tilde{\tau}] - 1 \right) \right), \end{aligned} \quad (4.26)$$

where H_{SM_2} is the same as in Eq. (4.21). For $p_x \in [0, 1/2]$, $\tilde{K}$ is positive. In $W_{\text{SM}_2}(\tilde{K}, E)$, a non-trivial E introduces extra imaginary coupling constants for the spin interaction $\mathcal{O}_{\mu}^x[\tilde{\tau}]$. And hence the name iRC.

Based on Theorem 4.1, the phases and the DIPTs of the decohered CSS code probed by the Rényi entropy $S_R(\rho_b)$ can be investigated via studying the phases and transitions in the R -replica SM_1 with rRC and in the R -replica SM_2 with iRC. The fact that two different statistical models describe the behavior of the same quantity $\text{Tr } \rho_b^R$ (or $S_R(\rho_b)$) indicates a duality between them. This duality is, in fact, an HLT duality, which we will discuss more about in the next subsection. For the proof of Theorem 4.1 (including the HLT duality), we refer to App. 4.6.1. The applications of Theorem 4.1 in concrete examples are provided in Sec. 4.2.5 and 4.2.6.

Using Eq. (4.25) and taking the $R \rightarrow 1$ limit of Theorem 4.1, we obtain

Corollary 4.1. The von Neumann entropy of the error-corrupted state ρ_b is given by the quenched-disorder-averaged free energy of SM_1 with rRC (up to an unim-

portant additive constant):

$$S_1(\rho_b) = - \sum_{E \in \mathcal{V}} P_x(E) \log(Z_{\text{SM}_1}(K, E)), \quad (4.27)$$

where K is given by the bit-flip error rate p_x via $e^{-2K} = \frac{p_x}{1-p_x}$.

Here, we treat the rRC configuration E as a disorder following the probability distribution $P_x(E)$. Therefore, the right-hand side of Eq. (4.27) is interpreted as a quenched-disorder-averaged free energy. In a general disordered spin model, the probability distribution of disorder and the coupling constant K can be independent parameters. The relation $e^{-2K} = \frac{p_x}{1-p_x}$ that appears in SM_1 with rRC is known as the Nishimori condition [219, 220].

We remark that the right-hand side of Eq. (4.27) is exactly the (averaged) free energy of the disordered statistical model introduced by Ref. [70] (and later generalized by Ref. [57]) to study the decodability of $\mathbb{Z}_2$ stabilizer codes with Pauli errors (bit-flip errors in this case). In particular, Ref. [70] established that the phase transition (where the quenched-disorder-averaged free energy becomes singular) in the disordered statistical model signals the decoding error threshold for the code. From the perspective of this chapter, the error threshold is viewed as the $R \rightarrow 1$ limit of the family of DIPTs labeled by R . Recall that the DIPT with index R is associated with the singularity of the Rényi entropy S_R of the decohered code. We generally expect the critical error rate $p_x^*(R)$ of the DIPT to be a function of R . The bit-flip error threshold for decoding is given by the limit $p_x^*(R \rightarrow 1)$. As a clarification, for a generic R , we use the term “critical error rate” for the error rate where the Rényi entropy S_R develops singularity and the DIPT with index R occurs. The term “(decoding) error threshold” is only associated with the DIPT in the limit $R \rightarrow 1$ and is hence given by the critical error rate in the same limit. The same terminology convention applies to all types of errors considered in this chapter.

We will discuss the physical meaning of DIPTs at different R 's and the dependence of the critical error rate $p_x^*(R)$ on R in Sec. 4.2.4.

For the mixed state ρ_p corrupted by phase-flip errors, we prove a similar theorem that maps $\text{Tr} \rho_p^R$ to the partition functions of two R -replica statistical models with randomness.

Theorem 4.2. For the CSS code in the infinite system limit, $\text{Tr} \rho_p^R$ of the mixed state ρ_p corrupted by phase-flip errors is proportional to the partition function of the R -replica SM_2 with rRC. It is also proportional to the partition function of the R -replica SM_1 with iRC:

$$\begin{aligned} \text{Tr}(\rho_p^R) &\propto \sum_{E \in \mathcal{V}} \left(Z_{\text{SM}_2}(\tilde{K}, E) \right)^R \\ &\propto \sum_{E \in \mathcal{V}} \left(W_{\text{SM}_1}(K, E) \right)^R, \end{aligned} \quad (4.28)$$

where the coupling constants K and $\tilde{K}$ are given by the phase-flip error rate $p_z \in [0, 1/2]$ via $e^{-2\tilde{K}} = \tanh K = \frac{p_z}{1-p_z}$. The proportionality constants only depend on p_z smoothly and, hence, are unimportant for potential phases and DIPTs.

Here, the partition function of the R -replica SM_2 with rRC and that of the R -replica SM_1 with iRC are defined in parallel with Eq. (4.24) and (4.26):

$$Z_{\text{SM}_2}(\tilde{K}, E) = \sum_{\{\tilde{\tau}_j = \pm 1\}} \exp \left(\tilde{K} \sum_{\mu} (-1)^{E_{\mu}} \mathcal{O}_{\mu}^x[\tilde{\tau}] \right), \quad (4.29)$$

and

$$\begin{aligned} W_{\text{SM}_1}(K, E) &= \sum_{\{\tau_i = \pm 1\}} \exp(-H_{\text{SM}_1}) \prod_{\mu} \left(\mathcal{O}_{\mu}^z[\tau] \right)^{E_{\mu}} \\ &= \sum_{\{\tau_i = \pm 1\}} \exp \left(-H_{\text{SM}_1} - \frac{i\pi}{2} \sum_{\mu} E_{\mu} \left(\mathcal{O}_{\mu}^z[\tau] - 1 \right) \right). \end{aligned} \quad (4.30)$$

Note that statistical models in Theorem 4.2 for the phase-flip errors are related to those in Theorem 4.1 for the bit-flip errors by interchanging the roles of rRC and iRC.

Similar to Eq. (4.25), we can show that

$$Z_{\text{SM}_2}(\tilde{K}, E) = [p_z(1 - p_z)]^{-N/2} \tilde{N}_s \sum_{C \in \mathcal{V}_z} P_z(E + C), \quad (4.31)$$

where $\tilde{N}_s$ is a factor associated with the internal symmetries of SM_2 . This equation implies that $Z_{\text{SM}_2}(\tilde{K}, E)$ is the total probability of phase-flip error within the same equivalence class $[E]_z \in \mathcal{V}/\mathcal{V}_z$. All error chains within the same class $[E]_z$ only differ from each other by elements in $\mathcal{V}_z$ and, hence, lead to the same pattern of excitations on the $A_i[X]$ stabilizers.

Just like the case with bit-flip errors, the two R -replica random statistical models in Theorem 4.2 are dual to each other under an HLT duality, which we will discuss more in the next subsection.

With Eq. (4.31), we can take the $R \rightarrow 1$ limit of Theorem 4.2 and obtain the following corollary.

Corollary 4.2. The von Neumann entropy of the error-corrupted state ρ_p is given by the quenched-disorder-averaged free energy of SM_2 with rRC (up to unimportant additive constants):

$$S_1(\rho_p) = - \sum_{E \in \mathcal{V}} P_z(E) \log(Z_{\text{SM}_2}(\tilde{K}, E)), \quad (4.32)$$

where $\tilde{K}$ is given by the phase-flip error rate p_z via $e^{-2\tilde{K}} = \frac{p_z}{1-p_z}$.

Again, combining Ref. [70] and Eq. (4.32), we conclude that the phase-flip error decoding threshold of the code is the critical phase-flip error rate $p_z^*(R)$ of the DIPT in the limit $R \rightarrow 1$.

4.2.3 Tapestry of dualities of a general CSS code

In this subsection, we discuss the tapestry (Fig. 4.1) woven by the dualities among the statistical models originating from a general CSS code $\mathcal{C}$.

First, there is a Kramers-Wannier-like high-low-temperature (HLT) duality between the non-random SM_1 and SM_2 obtained from the same CSS code $\mathcal{C}$ through ungauging. For bit-flip errors, Theorem 4.1 shows that two R -replica statistical models with random couplings describe the same quantity $\text{Tr } \rho_{\text{b}}^R$ associated with the R th Rényi entropy $S_R(\rho_{\text{b}})$ of the error-corrupted mixed state ρ_{b} . As shown in the bottom left corner of Fig. 4.1, these two statistical models are also dual to each other under an HLT duality, a generalization of the HLT duality that relates the non-random SM_1 and SM_2 . Similarly, for phase-flip errors, the R th Rényi entropy $S_R(\rho_{\text{p}})$ of the mixed state ρ_{p} is described by another pair of R -replica statistical models with random couplings as stated in Theorem 4.2. These two R -replica statistical models are also dual under an HLT duality, as shown in the bottom right corner of Fig. 4.1. In addition to these HLT dualities, we find that there are extra dualities, dubbed the BPD (bit-phase-decoherence) dualities, that map the random R -replica statistical models associated with the bit-flip errors to those associated with phase-flip errors. In the following, we provide a general discussion and the physical intuition for these dualities. The full technical details of the proofs of these dualities are presented in the App. 4.6.1 and 4.6.2.

Now, we discuss the HLT duality between the non-random SM_1 and SM_2 . We provide below the physical intuition of this duality by comparing the high-temperature expansion of the partition of SM_1 and the low-temperature expansion of the partition function of SM_2 . The partition function of the non-random SM_1 can be written as a high-

temperature expansion:

$$\begin{aligned}
Z_{\text{SM}_1}(K) &= \sum_{\{\tau_i = \pm 1\}} \exp \left(K \sum_{\mu} \mathcal{O}_{\mu}^z[\tau] \right) \\
&= \sum_{\{\tau_i = \pm 1\}} (\cosh K)^N \prod_{\mu} \left(1 + \mathcal{O}_{\mu}^z[\tau] \tanh K \right). \tag{4.33}
\end{aligned}$$

Recall that N is the number of qubits in the original CSS code, which equals the number of $\mathcal{O}_{\mu}^z[\tau]$ terms in SM_1 . After we expand the product over μ and sum over all the spin configurations $\{\tau_i = \pm 1\}$, the non-vanishing contribution must come from the products of $\mathcal{O}_{\mu}^z[\tau]$ terms that equal identity. Based on Eq. (4.10) and Eq. (4.18), we know that $\prod_{\mu} \left(\mathcal{O}_{\mu}^z[\tau] \right)^{(\mathbf{b}_j)_{\mu}} = 1$ for any j . In fact, we can readily see that $\prod_{\mu} \left(\mathcal{O}_{\mu}^z[\tau] \right)^{C_{\mu}} = 1$ for a local set $C \in \mathcal{V}$ implies $C \in \mathcal{V}_z$ (in the absence of any local logical operators in the code $\mathcal{C}$ which we've assumed). Hence, in the infinite system limit, we can write

$$Z_{\text{SM}_1}(K) \propto (\cosh K)^N \sum_{C \in \mathcal{V}_z} (\tanh K)^{|C|} \tag{4.34}$$

with a proportionality constant only depending on the system size. Strictly speaking, the summation here should run over the space $\mathcal{V}_x^{\perp}$ instead. Note that the space $\mathcal{V}_x^{\perp}/\mathcal{V}_z$ is associated with the logical- Z operators of the code which are all non-local. Also, the number of logical- Z operators N_c obeys $N_c/N \rightarrow 0$ as $N \rightarrow \infty$. Hence, replacing $\sum_{C \in \mathcal{V}_x^{\perp}}$ by $\sum_{C \in \mathcal{V}_z}$ does not change the free energy density of SM_1 in the infinite system limit.

For SM_2 , we perform a low-temperature expansion of the partition function. A classical ground state of the SM_2 Hamiltonian H_{SM_2} Eq. (4.21) is the state with $\tilde{\tau}_j = +1$ for all j . The ground state energy is $-N\tilde{K}$. The energy cost of a single spin flip $\tilde{\tau}_j \rightarrow -1$ at the site j is $+2\tilde{K}|\mathbf{b}_j|$. The low-temperature expansion is a summation over all possible spin flips on top of the classical ground state:

$$Z_{\text{SM}_2}(\tilde{K}) \propto e^{N\tilde{K}} \sum_{C \in \mathcal{V}_z} e^{-2\tilde{K}|C|}, \tag{4.35}$$

where the proportionality constant is given by the degeneracy of the classical Hamiltonian H_{SM_2} , which is independent of $\tilde{K}$ but related to the internal symmetry of H_{SM_2} . By comparing Eq. (4.35) and Eq. (4.34), we establish the HLT duality between the non-random SM_1 and SM_2 under the condition

$$\tanh K = e^{-2\tilde{K}}. \quad (4.36)$$

This condition relates a large (small) positive value of K to a small (large) positive value of $\tilde{K}$, which is natural for an HLT duality. Another way to obtain this HLT duality is to apply Wegner's general approach for dualities in Ising-type models [300].

Now, we discuss the dualities among the random statistical models appearing in the decohered CSS code. In the case of bit-flip errors, as shown in Theorem 4.1, the R -replica SM_1 with rRC is dual to the R -replica SM_2 with iRC under an HLT duality. The duality relation between the coupling constants K and $\tilde{K}$ is the same as Eq. (4.36) for the non-random case. The proof of the HLT duality between the two R -replica random statistical models generalizes the discussion above for the non-random case. The mathematical details of the proof are provided in App. 4.6.1. For the case of phase-flip errors, a parallel analysis can be made for the two different R -replica random statistical models that describe $\text{Tr}\rho_{\text{p}}^R$, as shown in Theorem 4.2. These two statistical models are also dual to each other under a similar HLT duality.

Next, we introduce the BPD dualities between the random statistical models associated with bit-flip errors and those associated with phase-flip errors. The statement of these BPD dualities is the following.

Theorem 4.3. For $R = 2, 3$, and ∞ , the R -replica random statistical models that describe the Rényi entropy $S_R(\rho_{\text{b}})$ caused by the bit-flip errors are dual to the statistical models that describe the Rényi entropy $S_R(\rho_{\text{p}})$ caused by the phase-flip

errors. We can summarize these BPD dualities as

$$\text{Tr}(\rho_b^R) \propto \text{Tr}(\rho_p^R), \quad \text{for } R = 2, 3, \text{ and } \infty, \quad (4.37)$$

(up to unimportant proportionality constants) when the error rates $0 < p_{x,z} < \frac{1}{2}$ satisfy the duality relations

$$\left[(1 - p_x)^R + p_x^R\right] \left[(1 - p_z)^R + p_z^R\right] = \frac{1}{2^{R-1}}. \quad (4.38)$$

In particular, in $R \rightarrow \infty$ limit, the duality condition reduces to

$$(1 - p_x)(1 - p_z) = \frac{1}{2} \quad (4.39)$$

for the range of error rates $p_{x,z} \in (0, 1/2)$ under consideration. Note that duality relation Eq. (4.38) maps the weak bit-flip decoherence (p_x close to 0) to the strong phase-flip decoherence (p_z close to $1/2$), and vice versa. Hence, the BPD duality is a “strong-weak” duality for the strength of decoherence. We provide the mathematical details of the proof of this duality in App. 4.6.2. Here, we sketch the general idea that leads to these BPD dualities. Based on Theorem 4.1 and 4.2, we can use the R -replica SM_1 with rRC to describe $S_R(\rho_b)$ and the R -replica SM_1 with iRC to describe $S_R(\rho_p)$. Integrating out the randomness, rRC or iRC, results in the interactions between the R replicas of SM_1 (see App. 4.6.3). For a general R , the inter-replica interactions mediated by rRC differ from those mediated by iRC. However, the cases with $R = 2, 3$ are exceptions. The inter-replica interactions mediated by rRC and iRC are identical when the error rates $p_{x,z}$ satisfy the relations Eq. (4.38). Therefore, there are BPD dualities between the R -replica random statistical models for $S_R(\rho_b)$ and $S_R(\rho_p)$ when $R = 2, 3$.

We can also understand the BPD duality with $R = 2$ using the HLT duality between the non-random SM_1 and SM_2 . Combining Eqs. (4.13), (4.24) and (4.25), we can

integrate out the randomness E (see App. 4.6.3) and obtain

$$\text{Tr } \rho_{\text{b}}^2 \propto Z_{\text{SM}_1}(K') \quad \text{with } K' = \text{arctanh} \left((1 - 2p_x)^2 \right), \quad (4.40)$$

where $Z_{\text{SM}_1}(K')$ is the partition function of the non-random SM_1 . Essentially, integrating out the real random-coupling “renormalizes” the coupling constant from $K = -\frac{1}{2} \log \frac{p_x}{1-p_x}$ to K' . Similarly, we can show that

$$\text{Tr } \rho_{\text{p}}^2 \propto Z_{\text{SM}_2}(\tilde{K}') \quad \text{with } \tilde{K}' = \text{arctanh} \left((1 - 2p_z)^2 \right). \quad (4.41)$$

The BPD duality with $R = 2$ shown in Theorem 4.3 is equivalent to the HLT duality between the non-random statistical models in Eq. (4.40) and (4.41).

The BPD duality at $R \rightarrow \infty$ can also be justified using the HLT duality between the non-random SM_1 and SM_2 . From Eq. (4.23), we notice that $\text{Tr } \rho_{\text{b}}^R \Big|_{R \rightarrow \infty}$ is dominated by the randomness pattern $E \in \mathcal{V}$ that maximizes the partition function $Z_{\text{SM}_1}(K, E)$. A high-temperature expansion similar to Eq. (4.34) implies that the maximum of $Z_{\text{SM}_1}(K, E)$ can be reached by the trivial randomness pattern $E = 0$. Therefore, the behavior of $\text{Tr } \rho_{\text{b}}^R \Big|_{R \rightarrow \infty}$ is dominated by (the R th power of) the partition function $Z_{\text{SM}_1}(K)^R$ of the non-random SM_1 . Similarly, $\text{Tr } \rho_{\text{p}}^R \Big|_{R \rightarrow \infty}$ is dominated by the partition function $Z_{\text{SM}_2}(\tilde{K})^R$ without randomness. Therefore, the HLT duality between the non-random SM_1 and SM_2 leads to the BPD duality between the Rényi entropies $S_R(\rho_{\text{b}})$ and $S_R(\rho_{\text{p}})$ in the limit $R \rightarrow \infty$.

We comment that a technically similar approximate duality was discovered first in the spin glass context using the “replica trick” [221] and later applied to the error threshold of stabilizer codes [33, 272] (at $R \rightarrow 1$ in our language). In contrast, the dualities at $R = 2, 3, \infty$ discussed in this chapter are exact and are directly associated with the Rényi entropies S_R , which are physical quantities by themselves instead of models appearing in the intermediate steps of the replica trick.

4.2.4 Discussion on the interpretation of DIPTs and the monotonicity of critical error rates

In this subsection, we briefly discuss the physical interpretation of DIPTs with different values of R . Also, we present a conjecture on the monotonicity of the critical error rates $p_{x,z}^*(R)$ of the DIPTs as functions of the Rényi index R .

As discussed in the previous subsections, the critical error rates $p_{x,z}^*$ of the DIPT in the limit $R \rightarrow 1$ matches the error threshold for the decodability of the logical information in the error-corrupted CSS code. For the DIPT with $R = 2$, in addition to the singularity of the 2nd Rényi entropy, we can also view it from the perspective of a quantum phase transition in the doubled Hilbert space, which we explain in the following.

The discussions of bit-flip errors and phase-flip errors are completely parallel. Let's take the former as an example. Using the Choi-Jamiołkowski isomorphism [55, 143], we can map the mixed-state density matrix ρ_b to a pure state $|\rho_b\rangle\rangle$, called the Choi representation of ρ_b , in the doubled Hilbert space. $|\rho_b\rangle\rangle$ is related to its error-free counterpart $|\rho_0\rangle\rangle$ via

$$|\rho_b\rangle\rangle \propto \left(e^{\tilde{K} \sum_{\mu} X_{\mu} \otimes X_{\mu}} \right) |\rho_0\rangle\rangle, \quad (4.42)$$

with $\tilde{K} = \operatorname{arctanh}\left(\frac{p_x}{1-p_x}\right)$. One can show that $|\rho_b\rangle\rangle$ is a ground state of the following parent Hamiltonian in the doubled Hilbert space

$$H_b^D = H_{\text{css}} \otimes \mathbb{I} + \mathbb{I} \otimes H_{\text{css}} + 2 \sum_j e^{-2\tilde{K} \sum_{\mu} (b_j)_{\mu} X_{\mu} \otimes X_{\mu}}, \quad (4.43)$$

where H_{css} is the CSS-code Hamiltonian given in Eq. (4.2). The construction of this parent Hamiltonian generalizes the construction for the decohered cluster states studied in Ref. [168]. A similar construction for the decohered 2D toric code was given in Ref.

[22]. App. 4.6.6 contains the details of the construction of Eq. (4.43) in the most general setting. An interesting property is that H_b^D is frustration-free up to additive constants. In other words, we can decompose this Hamiltonian into a sum of positive-semidefinite local terms, and each of such local terms annihilates the ground state $|\rho_b\rangle\rangle$.

Let's analyze the phase diagram of this model Eq. (4.43) in the doubled Hilbert space. When p_x is close to 0, we can treat the last term of H_b^D as a perturbation. Since H_{css} by itself is a gapped Hamiltonian, the ground state $|\rho_b\rangle\rangle$ for small p_x is smoothly connected to $|\rho_0\rangle\rangle$ which contains two decoupled copies of the same CSS code (in the doubled Hilbert space). Another regime is represented by the limit $p_x = 1/2$. The ground state at $p_x = 1/2$ is a stabilizer state with stabilizers,

$$A_i[X] \otimes \mathbb{K}, \mathbb{K} \otimes A_i[X], X_\mu \otimes X_\mu, \text{ and } B_j[Z] \otimes B_j[Z].$$

This stabilizer state at $p_x = 1/2$ is equivalent to a single copy of the original CSS code $\mathcal{C}$ (embedded inside the doubled Hilbert space). If two copies of the original CSS codes represent a quantum phase of matter different from a single copy, there must be a quantum phase transition between them. This quantum phase transition must occur at the critical error rate $p_x^*(R=2)$ of the DIPT with $R=2$. Note that this quantum phase transition lives in the same spatial dimension as the statistical model but has non-trivial temporal dynamics in the doubled Hilbert space. The spatial correlation of this quantum phase transition can be equivalently captured by the random statistical models with $R=2$ discussed in previous subsections. For example, the equal-time correlation function $\langle\langle\rho_b|O_1\otimes O_2|\rho_b\rangle\rangle \propto \text{Tr}\left(\rho_b O_1 \rho_b O_2^T\right)$ can be translated into a correlation function of the 2D random statistical models with $R=2$. Here, O_2^T is the transpose of the operator O_2 . When the statistical models for $R=2$ exhibit critical behavior, the same critical behavior appears in the spatial correlation of the quantum system in the doubled Hilbert space, indicating a quantum phase transition.

The physical implications of the DIPTs with a general R (beyond the singularities of the Rényi entropies) and the relations among the DIPTs with different R 's are both interesting future research directions. Pertaining to the latter, we would like to present the following conjecture.

Conjecture: The critical error rates $p_{x,z}^*(R)$ are both monotonically increasing functions of R .

Here, we've assumed that for every error type, bit-flip or phase-flip, and for every Rényi index R , there is a unique DIPT in the range of error rates $p_{x,z} \in (0, 1/2)$. The associated critical error rates are the subject of the conjecture above.

A piece of evidence for this conjecture is given by the relation

$$p_{x,z}^*(R = 2) < p_{x,z}^*(R \rightarrow \infty), \quad (4.44)$$

which we prove in the following. Take the case of bit-flip errors as an example. As discussed in Sec. 4.2.3, the behavior of S_R for both $R = 2$ and $R \rightarrow \infty$ are related to the non-random SM_1 . The "renormalized" coupling constant in the $R = 2$ case is $K' = \text{arctanh}((1 - 2p_x)^2)$ (see Eq. (4.40)) and the coupling constant in the $R \rightarrow \infty$ case is $K = \frac{1}{2} \log \frac{1-p_x}{p_x}$. The fact $K > K'$ for $p_x \in (0, \frac{1}{2})$ implies $p_x^*(R = 2) < p_x^*(R \rightarrow \infty)$.

As we will see in Sec. 4.3, for the class of CSS codes with an em symmetry, the critical error rates $p_{x,z}^*(R)$ for $R = 2, 3, \infty$ are exactly fixed (see Eq. (4.64)) by the dualities (assuming the uniqueness of DIPT for each R in the range $p_{x,z} \in (0, 1/2)$). These exact critical error rates also obey our conjecture.

More evidence of our conjecture will be provided when we discuss specific examples, including the 3D toric code, the X-cube model, and the 2D toric code. In particular, we will see that our conjecture is consistent with the error thresholds, i.e., $p_{x,z}^*(R \rightarrow 1)$, of these models obtained in previous literature.

If true, our conjecture establishes a general relation between DIPTs with different

Rényi indices R . It would provide an interesting method to upper bound the error threshold $p_{x,z}^*(R \rightarrow 1)$ using $p_{x,z}^*(R = 2)$. Note that studying the limit $R \rightarrow 1$ requires the averaging over quenched disorders/randomness in the corresponding statistical models, while the $R = 2$ case is essentially captured by the same statistical model (with a renormalized coupling) but without randomness. We expect the latter to be generally simpler to analyze than the former.

Additionally, we remark that, since the statistical models for $R = 2, \infty$ can be mapped to the non-random SM_1 (see Sec. 4.2.3), the DIPTs at $R = 2$ and $R = \infty$ share the same universality class as the phase transition of the non-random SM_1 (when such phase transition exists).

4.2.5 Example: 3D toric code

As a pedagogical example, let us apply our formalism to the decohered toric code model. The 2D toric code belongs to a special class of CSS codes with an *em* symmetry, which we will focus on in Sec. 4.3. Here, we consider the random statistical models that describe the 3D toric code with bit-flip and phase-flip errors.

The 3D toric code is a CSS code defined on a cubic lattice where the qubits, labeled by μ , are located on the edges of the lattice. As shown in Fig. 4.3, the 3D toric code has an X -type stabilizer $A_i[X]$ for every site i and a Z -type stabilizer $B_p[Z]$ for every plaquette p . Each $A_i[X]$ is a product of the Pauli- X operators on the six edges connected to the site i , while each $B_p[X]$ is a product of the Pauli- Z operators on the four edges that belong to the plaquette p :

$$A_i = \prod_{\mu \in \star_i} X_\mu, \quad B_p = \prod_{\mu \in \square_p} Z_\mu, \quad (4.45)$$

Now we construct SM_1 following the recipe in Sec. 4.2.2. We introduce a classical spin τ_i on each site i (where the $A_i[X]$ stabilizer is located). Eq. (4.18) tells us that the

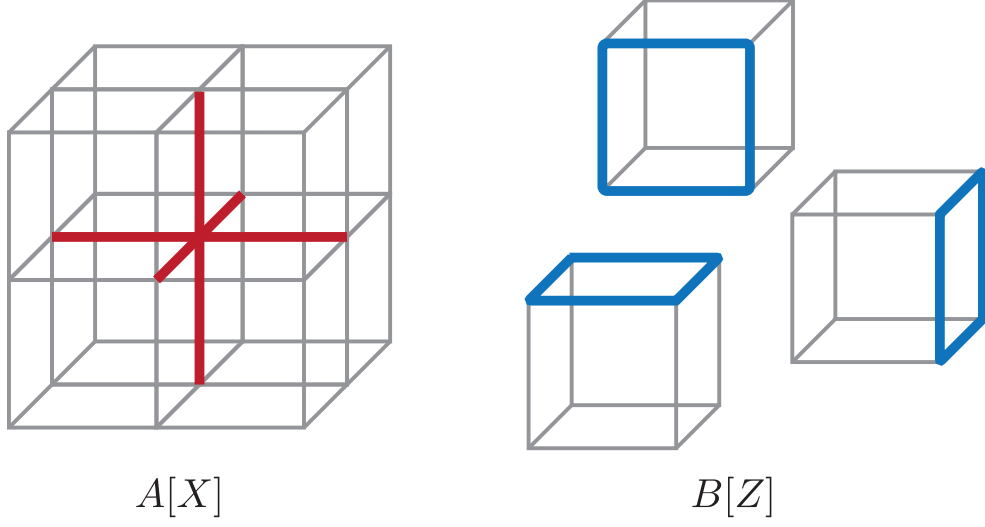


Figure 4.3: Stabilizers of the 3D toric code. Qubits live on the edges of the lattice. An X -type stabilizer (red) is a product of 6 Pauli- X operators on the edges connected to the same site. A Z -type stabilizer (blue) is a product of 4 Pauli- Z operators on the edges of a plaquette.

spin interaction $\mathcal{O}_\mu^z$ is a product of τ_i 's whose associated stabilizers A_i overlap with the edge μ . Hence, we have

$$\mathcal{O}_{\mu=\langle i, i' \rangle}^z = \tau_i \tau_{i'}, \quad (4.46)$$

where $\mu = \langle i, i' \rangle$ means μ is the edge between the nearest-neighbor pair of sites i and i' (see Fig. 4.4 (a)). The classical Hamiltonian of the non-random SM_1 is then given by

$$H_{\text{SM}_1} = -K \sum_{\langle i, i' | i, i' \rangle} \tau_i \tau_{i'}, \quad (4.47)$$

which is exactly the Hamiltonian of the 3D classical Ising model (with the nearest-neighbor interaction).

For SM_2 , we introduce a classical spin $\tilde{\tau}_p$ for each plaquette p . Each spin interaction $\mathcal{O}_\mu^x$ is the product of the four spins $\tilde{\tau}_p$ on the four plaquettes bordering the edge μ .

Hence, the Hamiltonian of SM_2 is given by

$$H_{\text{SM}_2} = -\tilde{K} \sum_{\mu} \prod_{p \text{ s.t. } \mu \in \partial p} \tilde{\tau}_p, \quad (4.48)$$

where “ $\prod_{p \text{ s.t. } \mu \in \partial p}$ ” represents the product over the plaquette p such that the edge μ belongs to the boundary ∂p of the plaquette. As shown in Fig. 4.4 (b), each spin interaction term in H_{SM_2} , when depicted on the dual cubic lattice, involves four classical spins on the four dual edges that form a dual plaquette. Hence, H_{SM_2} describes a 3D classical $\mathbb{Z}_2$ gauge theory on the dual lattice. It is well-known that the non-random 3D Ising model and the 3D classical $\mathbb{Z}_2$ gauge theory are dual to each other under an HLT duality (or Kramers-Wannier duality)[300].

In the following, we discuss the DIPTs in the decohered 3D toric code using the multi-replica SM_1 and SM_2 with random couplings. We will focus on the cases with $R \rightarrow 1$ and $R = 2, 3, \infty$. Note that the rRC in the Ising model is also known as the random-bond disorder, while the rRC in the 3D $\mathbb{Z}_2$ gauge theory is often referred to as the random-plaquette disorder.

For $R \rightarrow 1$, it follows from Corollary 4.1 and 4.2 that, the von Neumann entropy $S_1(\rho_b)$ of the error-corrupted mixed state ρ_b is the quenched-disorder-averaged free energy of the 3D random-bond Ising model, while $S_1(\rho_p)$ is quenched-disorder-averaged free energy of the 3D random-plaquette $\mathbb{Z}_2$ gauge theory. The respective disorder distribution satisfies the Nishimori condition. As mentioned earlier, these quenched-disordered statistical models are exactly the ones previously used to study the bit-flip and phase-flip error thresholds $p_{x/z}^*(R \rightarrow 1)$ for the 3D toric code [70, 226, 293]. The phase diagram of the 3D random-bond Ising model (in the $R \rightarrow 1$) was studied numerically in Refs. [140, 229]. The 3D random-plaquette $\mathbb{Z}_2$ gauge theory was numerically investigated in Refs. [226, 293]. These numerical studies obtained the following error thresholds for the

3D toric code:

$$p_x^*(R \rightarrow 1) \approx 0.233, \quad p_z^*(R \rightarrow 1) \approx 0.033. \quad (4.49)$$

For $R = 2$, as described around Eqs. (4.40) and (4.41), the 2-replica random statistical models that describes the 2nd Rényi entropy $S_2(\rho_{b/p})$ can be reduced to the non-random $SM_{1,2}$ with renormalized couplings $K' = \text{arctanh}((1 - 2p_x)^2)$ and $\tilde{K}' = \text{arctanh}((1 - 2p_z)^2)$. The critical point of SM_1 , i.e. the 3D classical Ising model in this case, has been numerically studied (see Ref. [79] for example), which allows us to extract the critical bit-flip error rate $p_x^*(2)$ for the DIPT with $R = 2$. The critical phase-flip error rate $p_z^*(2)$ is related to the $p_x^*(2)$ via the BPD duality Eq. (4.38). Therefore, we conclude

$$p_x^*(2) \approx 0.266, \quad p_z^*(2) \approx 0.099, \quad (4.50)$$

for the DIPTs with $R = 2$. These DIPTs belong to the 3D Ising universality class and, hence, share the same critical exponents. These transitions at $R = 2$ can also be understood from the perspective of the spontaneous breaking of $\mathbb{Z}_2$ higher-form symmetries. A counterpart with continuous higher-form symmetries was studied in Ref. [274].

For $R = 3$, the 3-replica disordered statistical models that describe the behaviors of $S_3(\rho_{b/p})$ can be reduced to Askin-Teller-type statistical models, namely two coupled copies of $SM_{1,2}$, after the rRC or iRC are integrated out (see general discussions in App. 4.6.3). For bit-flip errors, $S_3(\rho_b)$ is captured by the 3D Askin-Teller model on the cubic lattice:

$$H_{\text{AT,b}} = -K'' \sum_{\langle ii' \rangle} \left(\tau_i^{(1)} \tau_{i'}^{(1)} + \tau_i^{(2)} \tau_{i'}^{(2)} + \tau_i^{(1)} \tau_{i'}^{(1)} \tau_i^{(2)} \tau_{i'}^{(2)} \right). \quad (4.51)$$

Here, the coupling constant K'' is given by the relation $\tanh K'' + (\tanh K'')^{-1} = (1 - 2p_x)^{-2} + 1$. $\tau_i^{(1)}$ and $\tau_i^{(2)}$ are the classical $\mathbb{Z}_2$ variables on the two copies of cubic lattices

respectively. Previous numerical studies showed that this model exhibits a first-order transition between the paramagnetic and the ferromagnetic phase at the critical coupling $K_c'' \approx 0.157$ [15]. This result implies that, with $R = 3$, the DIPT driven by bit-flip errors is a first-order transition occurring at the critical bit-flip error rate

$$p_x^*(3) \approx 0.288. \quad (4.52)$$

For phase-flip errors, $S_3(\rho_p)$ is described by a 3D Askin-Teller-type $\mathbb{Z}_2$ gauge theory with the Hamiltonian

$$H_{\text{AT,p}} = -\tilde{K}'' \sum_{\mu} \left(\prod_{\mu \in \partial p} \tilde{\tau}_p^{(1)} + \prod_{\mu \in \partial p} \tilde{\tau}_p^{(2)} + \prod_{\mu \in \partial p} \tilde{\tau}_p^{(1)} \tilde{\tau}_p^{(2)} \right). \quad (4.53)$$

To the best of our knowledge, this model has not been studied before. The BPD duality implies that this model is dual to the model in Eq. (4.51). Therefore, the DIPT with $R = 3$ driven by phase-flip errors is also first-order. It occurs at the critical phase-flip error rate

$$p_z^*(3) \approx 0.135. \quad (4.54)$$

As shown in Sec. 4.2.3, in the limit $R \rightarrow \infty$, the behavior of the Rényi entropies $S_{\infty}(\rho_{b/p})$ are effectively described by the non-random $\text{SM}_{1,2}$ with the couplings $K = \frac{-1}{2} \log \frac{p_x}{1-p_x}$ and $\tilde{K} = \frac{-1}{2} \log \frac{p_z}{1-p_z}$. Hence, we can obtain the critical error rates:

$$p_x^*(\infty) \approx 0.391, \quad p_z^*(\infty) \approx 0.179. \quad (4.55)$$

The corresponding DIPTs belong to the 3D Ising universality class.

We've discussed the critical error rates of the DIPTs with several different R 's in the decohered 3D toric code. The results are summarized in Table 4.1. As we can see,

R	$p_x^*(R)$	$p_z^*(R)$
1	0.233	0.033
2	0.266	0.099
3	0.288	0.135
∞	0.391	0.179

Table 4.1: Critical error rates of the DIPTs in the decohered 3D toric code

these critical error rates agree with our conjecture that $p_{x,z}^*(R)$ monotonically increase as functions of R .

4.2.6 Example: X-cube model

Our second example is the X-cube model, which is a CSS code that exhibits fracton topological order [290]. The X-cube model is defined on a 3D cubic lattice with a qubit located on each edge of the lattice. Each X -stabilizer involves four nearby edges forming a cross, while each Z -stabilizer involves twelve edges of a unit cube (see Fig. 4.5):

$$A_{i,a} = \prod_{\mu \in +_{i,a}} X_{\mu}, \quad B_c = \prod_{\mu \in \square_c} Z_{\mu}, \quad (4.56)$$

where $+_{i,a}$ denotes a cross formed by the four edges centered at site i , and $a = x, y, z$ labels the perpendicular direction to the cross. $\square_c$ denotes the cube labeled by c .

In SM_1 , there should be one classical spin per $A_{i,a}$ stabilizer. Hence, we introduce three colors of classical spins $\tau_{i,a=x,y,z}$ for each site i on the cubic lattice and represent them as the purple arrows in Fig. 4.6 (a). The ungauging procedure in Sec. 4.2.2 produces the following SM_1 Hamiltonian, which describes a tricolor Ising model with the four-spin interactions

$$H_{\text{SM}_1} = -K \left(\sum_{\langle ii'|ii'\rangle \parallel \hat{z}} \tau_{i,x} \tau_{i,y} \tau_{i',x} \tau_{i',y} + \sum_{\langle ii'|ii'\rangle \parallel \hat{x}} \tau_{i,z} \tau_{i,y} \tau_{i',z} \tau_{i',y} + \sum_{\langle ii'|ii'\rangle \parallel \hat{y}} \tau_{i,x} \tau_{i,z} \tau_{i',x} \tau_{i',z} \right). \quad (4.57)$$

For each nearest-neighbor pair of sites $\langle i, i' \rangle$, only the spins with their colors different from the direction of the edge $\langle i, i' \rangle$ are involved in the nearest-neighbor four-spin interaction (see Fig. 4.6 (a)).

This tricolor Ising model has a classical gauge symmetry that changes $\tau_{i,a} \rightarrow -\tau_{i,a}$ independently for each site i . A change of variables effectively imposes a gauge fixing:

$$\eta_{i,x} \equiv \tau_{i,y}\tau_{i,z}, \quad \eta_{i,y} \equiv \tau_{i,x}\tau_{i,z}, \quad (4.58)$$

where $\eta_{i,x/y} = \pm 1$ are gauge invariant classical $\mathbb{Z}_2$ variables. Using these new variables, we can rewrite the Hamiltonian as an anisotropic 3D Ashkin-Teller model [146, 272],

$$H_{\text{SM}_1} = -K \left(\sum_{\langle ii' | ii' \rangle \parallel \hat{z}} \eta_{i,x} \eta_{i,y} \eta_{i',x} \eta_{i',y} + \sum_{\langle ii' | ii' \rangle \parallel \hat{x}} \eta_{i,x} \eta_{i',x} + \sum_{\langle ii' | ii' \rangle \parallel \hat{y}} \eta_{i,y} \eta_{i',y} \right). \quad (4.59)$$

Note we can also directly obtain this statistical model if we remove the $A_{i,z}$ stabilizers from the X-cube model. Removing the $A_{i,z}$ stabilizers does not change the nature of this CSS code because $A_{i,z}$ can be expressed using the remaining stabilizers, i.e., $A_{i,z} = A_{i,x}A_{i,y}$.

For SM_2 , we introduce a classical spin $\tilde{\tau}_c$ to each cube c . The cube label c can also be viewed as the site index for the dual cubic lattice. The Hamiltonian of SM_2 is given by

$$H_{\text{SM}_2} = -\tilde{K} \sum_{\tilde{\square}} \prod_{c \in \tilde{\square}} \tilde{\tau}_c, \quad (4.60)$$

where $\tilde{\square}$ sums over the dual plaquettes $\tilde{\square}$ on the dual lattice and " $c \in \tilde{\square}$ " indicates that c is a dual site on one of the corners of $\tilde{\square}$. H_{SM_2} represents a 3D plaquette Ising model with a four-spin interaction for each dual plaquette (see Fig. 4.6 (b)).

We are now ready to examine the DIPTs of the X-cube model. The non-random

R	$p_x^\star$	$p_z^\star$
1	0.075	0.152
2	0.120	0.241
∞	0.212	0.365

Table 4.2: Critical error rates for the X-cube model. The results for $R = 2, \infty$ are derived from the study of the (non-random) anisotropic Ashkin-Teller model and the plaquette Ising model in Ref. [213]. The error thresholds in the limit $R \rightarrow 1$ were obtained in Ref. [272]

anisotropic Ashkin-Teller model (Eq. (4.59)) and the 3D plaquette Ising model (Eq.(4.60)) were studied numerically in Ref. [213]. First-order phase transitions were found at the critical couplings $K_c \approx 0.657$ and $\tilde{K}_c \approx 0.276$. We can convert these results into the critical error rates of the DIPTs with $R = 2$ and $R = \infty$. The $R \rightarrow 1$ limit was numerically studied by Ref. [272] as the error thresholds of the X-cube model. We summarize these results in Table 4.2. These numerically obtained critical error rates agree with our conjectured monotonicity of $p_{x,z}^\star(R)$.

The $R = 3$ case requires the consideration of two coupled copies of plaquette Ising models on the dual cubic lattice:

$$H_{\text{AT,p}} = -\tilde{K}'' \sum_{\tilde{\square}} \left(\prod_{c \in \tilde{\square}} \tilde{\tau}_c^{(1)} + \prod_{c \in \tilde{\square}} \tilde{\tau}_c^{(2)} + \prod_{c \in \tilde{\square}} \tilde{\tau}_c^{(1)} \tilde{\tau}_c^{(2)} \right) \quad (4.61)$$

and its BPD dual. To our knowledge, this model has not been studied before.

4.3 Decohered CSS code with electric-magnetic symmetry

In this section, we study the consequences of the dualities for the CSS codes with an electric-magnetic (em) symmetry, a symmetry relating the X -type and Z -type stabilizers. We will see that the em symmetry effectively folds the tapestry of dualities in Fig. 4.1 in half, resulting in a new tapestry shown in Fig. 4.7. The new tapestry allows us to

pin down the *super-universal* self-dual error rates for the Rényi entropies $S_R(\rho_{\text{b/p}})$ with $R = 2, 3, \infty$. These super-universal self-dual error rates must coincide with the critical error rates $p_{x/z}^*$ of the DIPTs if there is a unique DIPT for each R . We will also discuss concrete examples of *em*-symmetric CSS codes, including the 2D toric code and Haah's code in 3D.

4.3.1 Tapestry of dualities and self-dual error rates in *em*-symmetric CSS codes

Let's first define the *em* symmetry. A CSS code is *em*-symmetric if there is a unitary transformation $\mathcal{U}_{em}$ such that

$$\begin{aligned}\mathcal{U}_{em}A_i[X]\mathcal{U}_{em}^\dagger &= B_{j=f(i)}[Z], \\ \mathcal{U}_{em}B_j[X]\mathcal{U}_{em}^\dagger &= A_{i=g(j)}[Z].\end{aligned}\tag{4.62}$$

Here, $\mathcal{U}_{em}$ is a product of the Hadamard gate $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ on each qubit, which exchanges the Pauli- X and the Pauli- Z operators, and a spatial action that permutes the locations of the qubits. The functions f and g are bijections between the locations of X - and Z -type stabilizers induced by the spatial action in $\mathcal{U}_{em}$.

Since the *em* symmetry maps between the X -type and Z -type stabilizers, the two statistical models $\text{SM}_{1,2}$ produced by the ungauging procedure in Sec. 4.2.2 become identical (up to some spatial rearrangement of the classical spins) in an *em*-symmetric CSS code. Hence, we will not distinguish $\text{SM}_{1,2}$ in this section and call them both SM. The original HLT duality between the non-random SM_1 and SM_2 becomes an HLT self-duality of the non-random SM.

The *em* symmetry maps the bit-flip errors and the phase-flip errors into each other. Therefore, the decoherence effects caused by the two error types are effectively identical. We remark that the *em* symmetry is an intrinsic property of the CSS code under the

current discussion. We do *not* require the error model to respect this symmetry. As followed from Theorem 4.1 and 4.2, the R th Rényi entropies $S_R(\rho_{\text{b/p}})$ are described by both the R -replica SM with rRC and R -replica SM with iRC. The two types of the R -replica random statistical models are dual to each other under the HLT dualities.

The BPD duality leads to profound consequences for *em*-symmetric CSS codes. Combining the *em* symmetry action and the BPD duality, both of which map between bit-flip errors and phase-flip errors, we conclude that the random statistical models for $S_R(\rho_{\text{b}})$ with $R = 2, 3, \infty$ are self-dual, and so are those for $S_R(\rho_{\text{p}})$.

Corollary 4.3. For an *em*-symmetric CSS code with decoherence, the Rényi entropies $S_R(\rho_{\text{b/p}})$ with $R = 2, 3, \infty$ are described by R -replica random statistical models with self-dualities. The dualities map between the two error rates $0 < p, \tilde{p} < \frac{1}{2}$ related by:

$$\left[(1-p)^R + p^R\right] \left[(1-\tilde{p})^R + \tilde{p}^R\right] = \frac{1}{2^{R-1}}. \quad (4.63)$$

Here, both of the error rates p and $\tilde{p}$ can be either the bit-flip error rates or the phase-flip error rates, depending on the error type considered.

Similar to Theorem 4.3, the duality relations Eq. (4.63) are dualities between weak and strong decoherence strength. By solving the self-dual condition $p = \tilde{p}$ together with Eq. (4.63), we obtain the super-universal self-dual error rates (listed below) shared by *all* *em*-symmetric CSS codes. If we further assume that there is a unique DIPT for each R (within the range $p \in (0, 1/2)$), the self-dual error rates must coincide with the critical error rates of the corresponding DIPTs:

$$p^*(R) = \begin{cases} \frac{1}{2} \left(1 - \sqrt{\sqrt{2} - 1} \right) \approx 0.178 & \text{for } R = 2, \\ \frac{1}{6} (3 - \sqrt{3}) \approx 0.211 & \text{for } R = 3, \\ \frac{1}{2} (2 - \sqrt{2}) \approx 0.293 & \text{for } R \rightarrow \infty. \end{cases} \quad (4.64)$$

It is worth noting that these super-universal critical error rates are consistent with the conjectured monotonicity of $p_{x/z}^*(R)$. Also, we called these values super-universal because they encompass DIPTs of different universality classes and in different dimensions, as demonstrated by the examples discussed below.

If there is more than one DIPT for $p \in [0, 1/2]$, the values given Eq. (4.64) should be viewed as the self-dual error rates. The critical error rates that differ from the self-dual error rates must form pairs. For every pair, there must be a critical error rate smaller than the self-dual value. Therefore, in the case of an *em*-symmetric CSS code with multiple DIPTs for $p \in [0, 1/2]$ and at a given R , the self-dual error rate in Eq. (4.64) upper bounds the critical error rate of the first DIPT encountered as the error rate increases from 0.

The tapestry summarizing all the dualities of an *em*-symmetric CSS code is shown in Fig. 4.7. Conceptually, this tapestry is Fig. 4.1 folded in half in the middle by the *em* symmetry.

In subsequent subsections, as demonstrations, we discuss two concrete models where our general results of *em*-symmetric CSS codes apply.

4.3.2 Example: 2D toric code

The 2D toric code is a familiar example of an *em*-symmetric CSS code. It is defined on a 2D square lattice with a qubit on each edge. As shown in Fig. 4.8, each *X*-type stabilizer is a product of four Pauli-*X* operators on the four edges forming a cross while

each Z -type stabilizer is a product of four Pauli- Z operators on the four edges forming a plaquette:

$$A_i = \prod_{\mu \in +_i} X_\mu \quad B_p = \prod_{\mu \in \square_p} Z_\mu, \quad (4.65)$$

where $+_i$ labels the cross centered at the site i and $\square_p$ is the plaquette labeled by p .

It is straightforward to visualize the em symmetry by comparing the pictorial representations of the stabilizers on the square lattice and their representations on the dual square lattice (see Fig. 4.8). This em symmetry is generated by the unitary operator $\mathcal{U}_{em}$ that interchanges the X and Z operators on each qubit and translates the system both in the horizontal and vertical directions by half of the lattice spacing.

For the 2D toric code, SM is the 2D classical Ising model with the nearest-neighbor coupling:

$$H_{\text{SM}} = -K \sum_{\langle i, i' | i, i' \rangle} \tau_i \tau_{i'}, \quad (4.66)$$

where the classical spins τ_i are located on the site of the square lattice. The HLT self-duality of this model (without randomness) is the well-known Kramers-Wannier duality.

Now, we discuss the Rényi entropies $S_R(\rho_{\text{b/p}})$ in the decohered 2D toric code. The $R \rightarrow 1$ limit is described by SM with quenched random-coupling disorder, which exactly recovers the random-bond Ising model introduced in Ref. [70] for the study of the error thresholds $p^*(R \rightarrow 1)$. The cases with $R = 2$ and $R = \infty$ can be effectively captured by the non-random SM (see App. 4.6.3). Hence, there is a unique DIPT for $R = 2, \infty$, whose universality class is equivalent to the Ising critical point in 2D. The corresponding critical error rates, fixed by the self-dualities, are given by Eq. (4.64).

For $R = 3$, integrating out the randomness in the statistical model yields the 2D

square-lattice Ashkin-Teller model, whose Hamiltonian is given by

$$H_{\text{AT}} = -K'' \sum_{\langle ii' \rangle} \left(\tau_i^{(1)} \tau_{i'}^{(1)} + \tau_i^{(2)} \tau_{i'}^{(2)} + \tau_i^{(1)} \tau_{i'}^{(1)} \tau_i^{(2)} \tau_{i'}^{(2)} \right). \quad (4.67)$$

This Ashkin-Teller model tuned by the coupling constant K'' has a unique second-order transition [157], corresponding to $p^*(3)$.

For $R = 4, 5, 6$, Ref. [76] numerically simulated the corresponding multi-replica statistical models and found a unique phase transition in each case. The values of $p^*(R)$ with $R \rightarrow 1$ and $R = 2, 3, 4, 5, 6, \infty$ are consistent with the conjectured monotonicity of $p^*(R)$ as a function of R .

4.3.3 Example: Haah's code

Haah's code [108] provides an example of 3D *em*-symmetric CSS code. It is defined on a 3D cubic lattice with two qubits per site. There is one X -type and one Z -type stabilizer per cube. They are pictorially represented in Fig. 4.9 (a). Each $A[X]$ stabilizer is a product of eight Pauli- X operators on the qubits in red, while each $B[Z]$ stabilizer is a product of eight Pauli- Z operators on the qubits in blue. The *em* symmetry action $\mathcal{U}_{em}$ of Haah's code is the product of (1) a Hadamard gate on each qubit, (2) a spatial inversion $\vec{r} \rightarrow -\vec{r}$, and (3) a swap of the two qubits on each site.

Apply the ungauging procedure in Sec. 4.2.2, we obtain a 3D SM with a classical $\mathbb{Z}_2$ spin per site on the dual cubic lattice. The Hamiltonian of this SM contains two types of four-spin interaction per unit cell, each associated with a tetrahedron inside a cube. They are pictorially represented in Fig. 4.9. This SM is also known as the "fractal Ising model". It was introduced in Ref. [290] as the classical model that produces Haah's code after gauging. The (non-random) fractal Ising model was recently shown through numerical studies to exhibit a unique first-order transition [42].

Following the recipe in Sec. 4.2.2, the Rényi entropies $S_R(\rho_{b/p})$ are mapped to the

R -replica random fractal Ising models. As discussed earlier, the cases with $R = 2$ and $R \rightarrow \infty$ can be effectively reduced to the non-random fractal Ising model, which has a unique first-order transition. Therefore, the DIPTs with $R = 2$ and $R = \infty$ are first-order transitions occurring at the self-dual error rates listed in Eq. (4.64). For $R = 3$, the 3-replica random fractal Ising model can be reduced to an Ashkin-Teller-type model after integrating out the randomness (see App. 4.6.3). The phase diagram of this model and the nature of the self-dual error rate $p^*(3)$ are both interesting problems that we leave for future investigation.

4.4 General stabilizer codes under decoherence

The mapping from the Rényi entropies to statistical models can be generalized to general stabilizer codes subject to a general Pauli noise channel (to be explained below). As we will see, we can associate a statistical model SM with a general stabilizer code without the CSS structure. The non-random version of SM has an HLT self-duality. The R -th Rényi entropy of the decohered stabilizer code is mapped to both the R -replica SM with rRC and the R -replica SM with iRC, which are related by an HLT duality. For $R = 2$ and $R \rightarrow \infty$, there is a general-Pauli-noise (GPN) duality that maps between two different Pauli noise channels with different sets of error rates. We obtain the self-dual conditions of the Pauli noise channels and discuss their relations to DIPTs. We also include the Chamon model [46] as a concrete example to demonstrate our general construction.

4.4.1 Stabilizer codes and general Pauli noise

Let's start our discussion with the basics of stabilizer codes and the model for general Pauli noise. We still focus on the stabilizer code defined on a system of qubits (with each carrying a 2-fold local Hilbert space).

For a general stabilizer code, each stabilizer $A_J[X, Z]$ can be written as a product of

the Pauli- X and Z operators. In this convention, any Pauli- Y operator in $A_J[X, Z]$ is decomposed into $\mathbf{i}XZ$. The subscript J of $A_J[X, Z]$ labels the stabilizer's center location (and the species of stabilizers if applicable). Each stabilizer $A_J[X, Z]$ can be represented by a pair of $\mathbb{Z}_2$ vectors $(\mathbf{a}_J, \mathbf{b}_J)$ with $\mathbf{a}_J, \mathbf{b}_J \in \mathcal{V}$:

$$A_J[X, Z] = (\mathbf{i})^{\mathbf{a}_J \cdot \mathbf{b}_J} \prod_{\mu} (X_{\mu})^{(\mathbf{a}_J)_{\mu}} \prod_{\mu} (Z_{\mu})^{(\mathbf{b}_J)_{\mu}}. \quad (4.68)$$

Recall $\mathcal{V} = \mathbb{Z}_2^N$, the $\mathbb{Z}_2$ vector space associated with a system of N qubits. The prefactor $(\mathbf{i})^{\mathbf{a}_J \cdot \mathbf{b}_J}$ ensures that $A_J[X, Z]^2 = \mathbb{I}$. The fact that all the stabilizers commute with each other is equivalent to the condition

$$\mathbf{a}_J \cdot \mathbf{b}_{J'} + \mathbf{a}_{J'} \cdot \mathbf{b}_J = 0, \quad \forall J, J'. \quad (4.69)$$

Note that the " $\cdot$ " above is the dot product in the $\mathbb{Z}_2$ vector space $\mathcal{V}$. This $\mathbb{Z}_2$ -vector-based representation of the stabilizers is called "binary symplectic representation" in the literature [9]. Eq. (4.69) defines the binary symplectic inner product between $(\mathbf{a}_J, \mathbf{b}_J)$ and $(\mathbf{a}_{J'}, \mathbf{b}_{J'})$. The binary symplectic representation is applicable to any Pauli strings. In other words, any product of Pauli operators can be represented by a pair of $\mathbb{Z}_2$ vectors $(\mathbf{e}, \mathbf{f}) \in \mathcal{V} \oplus \mathcal{V}$.

In this section, we consider the general Pauli noise that induces the Pauli noise channel $\mathcal{E} = \otimes_{\mu} \mathcal{E}_{\mu}$, a product of the local Pauli noise channel $\mathcal{E}_{\mu}$ on each qubit μ :

$$\begin{aligned} \mathcal{E}_{\mu}[\rho_0] = & (1 - p_x - p_y - p_z)\rho_0 + p_x X_{\mu} \rho_0 X_{\mu} \\ & + p_y Y_{\mu} \rho_0 Y_{\mu} + p_z Z_{\mu} \rho_0 Z_{\mu}. \end{aligned} \quad (4.70)$$

where $p_{x,y,z} > 0$ are the probability/error rates of the X -, Y -, Z -type errors on each qubit. The general Pauli noise channel $\mathcal{E}$ includes the two quantum channels $\mathcal{N}_{x,z}$ induced by the bit-flip and the phase-flip errors and studied in Sec. 4.2 and 4.3. $\mathcal{E}$ reduces to the

depolarizing channel when we set $p_x = p_y = p_z$. For the following analysis, we make a technical assumption that $(1 - p_x - p_y - p_z) > p_{x,y,z}$ without the loss of generality. For a general Pauli noise channel, one can always combine it with an appropriate choice of unitary global action $\prod_\mu X_\mu$, $\prod_\mu Y_\mu$, or $\prod_\mu Z_\mu$ to construct a new quantum channel that satisfies the condition $(1 - p_x - p_y - p_z) > p_{x,y,z}$. Since this global action does not change the system's entropy, studying the decoherence effect of the new channel is equivalent to studying the original channel.

In what follows, we investigate the Rényi entropies $S_R(\rho_m)$ of the error-corrupted mixed state $\rho_m = \mathcal{E}(\rho_0)$ where ρ_0 is a pure logical state of the stabilizer code. Like the case of CSS codes, we will construct the statistical models associated with $S_R(\rho_m)$ and study their dualities.

4.4.2 Statistical models for general stabilizer codes under general Pauli decoherence

To motivate the statistical model associated with a general decohered stabilizer code $\mathcal{C}$, it is useful first to consider the setting of CSS code with the general Pauli noise channel $\mathcal{E}$ defined above. In the case where the decoherence channel is generated by independent bit-flip and phase-flip errors, namely when $\mathcal{E} = \mathcal{N}_x \circ \mathcal{N}_z$, we have

$$p_x = (1 - q_z)q_x, \quad p_y = q_x q_z, \quad p_z = (1 - q_x)q_z, \quad (4.71)$$

where $q_{x/z}$ are the respective rates of the bit-flip and phase-flip errors in $\mathcal{N}_{x/z}$. The Y -type error, now occurring with probability $p_y = q_x q_z$, is the consequence of a bit-flip and a phase-flip error occurring on the same qubit. As discussed in Sec. 4.2, the effects of independent bit-flip and phase-flip errors decouple on a CSS code. Therefore, the entropy of the entire system can be captured by two decoupled (random and multi-replica) statistical models SM_1 and SM_2 . The former lives on the lattice formed by

the centers of the X -type stabilizers, while the latter lives on the lattice formed by the Z -type stabilizers.

It turns out that, when p_y deviates from $q_x q_z$, namely when the Y -type errors start to become independent from the bit-flip and phase-flip errors, SM_1 couples to SM_2 forming a single statistical model, which will be called SM in the following, living on the lattice formed by the centers of *all* stabilizers. The duality between SM_1 and SM_2 becomes the self-duality of SM. In fact, as we show below, even for a general stabilizer code $\mathcal{C}$ with Pauli decoherence, there is a corresponding self-dual SM whose R -replica random versions describe the Rényi entropies of the error-corrupted mixed state.

Like Sec. 4.2, we first introduce the non-random version of the statistical model SM associated with the general stabilizer code $\mathcal{C}$. We assign one classical $\mathbb{Z}_2$ spin τ_J to each stabilizer $A_J[X, Z]$. The Hamiltonian of SM is given by

$$H_{\text{SM}} = - \sum_{\mu} K_x \mathcal{O}_{\mu}^x[\tau] + K_y \mathcal{O}_{\mu}^y[\tau] + K_z \mathcal{O}_{\mu}^z[\tau], \quad (4.72)$$

which contains three types of spin interactions

$$\begin{aligned} \mathcal{O}_{\mu}^z[\tau] &= \prod_J (\tau_J)^{(\mathbf{a}_J)_{\mu}}, \quad \mathcal{O}_{\mu}^x[\tau] = \prod_J (\tau_J)^{(\mathbf{b}_J)_{\mu}}, \\ \mathcal{O}_{\mu}^y[\tau] &= \prod_J (\tau_J)^{(\mathbf{a}_J + \mathbf{b}_J)_{\mu}}, \end{aligned} \quad (4.73)$$

with coupling constants $K_{x,y,z}$. These spin interactions are the generalizations of Eqs. (4.18) and (4.22). They satisfy the relation

$$\mathcal{O}_{\mu}^x[\tau] \mathcal{O}_{\mu}^y[\tau] \mathcal{O}_{\mu}^z[\tau] = 1. \quad (4.74)$$

For the non-random SM, the comparison of the high-temperature and low-temperature expansion of the partition function $Z_{\text{SM}} = \sum_{\{\tau_J = \pm 1\}} e^{-H_{\text{SM}}}$ yields a self-

duality of SM that maps the coupling constants $\mathbf{K} = (K_x, K_y, K_z)$ to $\tilde{\mathbf{K}} = (\tilde{K}_x, \tilde{K}_y, \tilde{K}_z)$,

$$\begin{aligned} e^{-2(\tilde{K}_x + \tilde{K}_y)} &= \frac{\tanh K_z + \tanh K_x \tanh K_y}{1 + \tanh K_x \tanh K_y \tanh K_z}, \\ e^{-2(\tilde{K}_z + \tilde{K}_x)} &= \frac{\tanh K_y + \tanh K_z \tanh K_x}{1 + \tanh K_x \tanh K_y \tanh K_z}, \\ e^{-2(\tilde{K}_y + \tilde{K}_z)} &= \frac{\tanh K_x + \tanh K_y \tanh K_z}{1 + \tanh K_x \tanh K_y \tanh K_z}. \end{aligned} \quad (4.75)$$

The detailed derivation of this duality is presented in App. 4.6.4.

Under the Pauli noise channel, the error-corrupted mixed state of the stabilizer code is denoted as ρ_m . We prove the following theorem pertaining to the R -th Rényi entropy $S_R(\rho_m) = \frac{1}{1-R} \log \text{Tr}(\rho_m^R)$ of the decohered stabilizer code.

Theorem 4.4. For the general stabilizer code in the infinite system limit, $\text{Tr} \rho_m^R$ of the error-corrupted mixed state ρ_m generated by the Pauli noise channel is proportional to the partition function of the R -replica SM with the rRC. $\text{Tr} \rho_m^R$ is also proportional to the partition function of the R -replica SM with iRC:

$$\text{Tr}(\rho_m^R) \propto \sum_{E \in \mathcal{V} \oplus \mathcal{V}} \left(Z_{\text{SM}}(\mathbf{K}, E) \right)^R \quad (4.76)$$

$$\propto \sum_{E \in \mathcal{V} \oplus \mathcal{V}} \left(W_{\text{SM}}(\tilde{\mathbf{K}}, E) \right)^R, \quad (4.77)$$

with the coupling constants $\mathbf{K}$ and $\tilde{\mathbf{K}}$ given by the error rates $p_{x,y,z}$ via

$$\begin{aligned} e^{-2(K_x + K_y)} &= \frac{\tanh \tilde{K}_z + \tanh \tilde{K}_x \tanh \tilde{K}_y}{1 + \tanh \tilde{K}_x \tanh \tilde{K}_y \tanh \tilde{K}_z} = \frac{p_z}{1 - p_x - p_y - p_z}, \\ e^{-2(K_y + K_z)} &= \frac{\tanh \tilde{K}_x + \tanh \tilde{K}_y \tanh \tilde{K}_z}{1 + \tanh \tilde{K}_x \tanh \tilde{K}_y \tanh \tilde{K}_z} = \frac{p_x}{1 - p_x - p_y - p_z}, \\ e^{-2(K_x + K_z)} &= \frac{\tanh \tilde{K}_y + \tanh \tilde{K}_x \tanh \tilde{K}_z}{1 + \tanh \tilde{K}_x \tanh \tilde{K}_y \tanh \tilde{K}_z} = \frac{p_y}{1 - p_x - p_y - p_z}. \end{aligned} \quad (4.78)$$

Here, $E \equiv (\mathbf{e}, \mathbf{f}) \in \mathcal{V} \oplus \mathcal{V}$ is the pair of $\mathbb{Z}_2$ vectors that represents the random couplings

in the statistical models. We have defined the partition function of SM with the rRC pattern E (and the coupling constants $\mathbf{K}$) as

$$Z_{\text{SM}}(\mathbf{K}, E) = \sum_{\{\tau_J = \pm 1\}} \exp \left[\sum_{\mu} \left(K_x (-1)^{f_{\mu}} \mathcal{O}_{\mu}^x[\tau] + K_y (-1)^{(e+f)_{\mu}} \mathcal{O}_{\mu}^y[\tau] + K_z (-1)^{e_{\mu}} \mathcal{O}_{\mu}^z[\tau] \right) \right], \quad (4.79)$$

and the SM partition function with the iRC pattern E (and the coupling constants $\tilde{\mathbf{K}}$) as

$$\begin{aligned} W_{\text{SM}}(\tilde{\mathbf{K}}, E) &= \sum_{\{\tau_J = \pm 1\}} \exp(-H_{\text{SM}}(\tilde{\mathbf{K}})) \prod_{\mu} \left(\mathcal{O}_{\mu}^x[\tau] \right)^{e_{\mu}} \left(\mathcal{O}_{\mu}^z[\tau] \right)^{f_{\mu}}. \\ &= \sum_{\{\tau_J = \pm 1\}} \exp \left(-H_{\text{SM}}(\tilde{\mathbf{K}}) - \frac{i\pi}{2} \sum_{\mu} e_{\mu} \left(\mathcal{O}_{\mu}^x[\tau] - 1 \right) - \frac{i\pi}{2} \sum_{\mu} f_{\mu} \left(\mathcal{O}_{\mu}^z[\tau] - 1 \right) \right). \end{aligned} \quad (4.80)$$

The two types of R -replica random SM are dual to each other under the HLT duality. The proof of Theorem 4.4 (and the HLT duality) is provided in App. 4.6.4.

The phase diagram of the decohered stabilizer code $\mathcal{C}$ has three tuning parameters $p_{x,y,z}$, which can potentially lead to even richer physics than the case of CSS codes with only bit-flip and phase-flip errors. The DIPTs of the decohered stabilizer code $\mathcal{C}$, indicated by the singularities in the Rényi entropy $S_R(\rho_m)$, corresponds to the phase transitions in these R -replica random statistical models.

Taking the limit of $R \rightarrow 1$ of Theorem 4.4, we obtain the relation between the von Neumann entropy $S_1(\rho_m)$ and the quenched-disorder-average partition function of SM with random couplings (derivation summarized in App. 4.6.4):

Corollary 4.4. The von Neumann entropy of the error-corrupted state ρ_m is given by the quenched-disorder-averaged free energy of SM with rRC (up to an unimpor-

tant additive constant):

$$S_1(\rho_m) = - \sum_{E \in \mathcal{V} \oplus \mathcal{V}} P(E) \log(Z_{\text{SM}}(\mathbf{K}, E)), \quad (4.81)$$

with the generalized Nishimori condition Eq. (4.78) relating the probability distribution of the randomness and the coupling constant $\mathbf{K}$.

Here, the error chain $E \in \mathcal{V} \oplus \mathcal{V}$ represents a Pauli string in a similar fashion as Eq. (4.68). The probability $P(E)$ of the error chain is given by

$$P(E) = p_0^{N-N_x(E)-N_y(E)-N_z(E)} p_x^{N_x(E)} p_y^{N_y(E)} p_z^{N_z(E)}, \quad (4.82)$$

where $N_{x/y/z}(E)$ are the number of Pauli $X/Y/Z$ -operators contained in the Pauli string E , $p_0 \equiv (1 - p_x - p_y - p_z)$, and N is the total number of qubits. Similar to the cases of CSS code, the $R \rightarrow 1$ limit recovers the statistical models introduced in the previous literature to specifically study the decodability and the error thresholds in stabilizer codes (see Ref. [34] for example).

At this point, we have finished the general construction of the random statistical models for $S_R(\rho_m)$. These statistical models offer tools to study emergent quantum matters in decohered stabilizer codes and the DIPTs between them. The physical implication of the DIPTs for a general R will be left for future studies.

For $R = 2$, based on the same reasoning as in Sec. 4.2.4, we can use the Choi-Jamiołkowski isomorphism to map the error-corrupted matrix state ρ_m to its Choi representation $|\rho_m\rangle\rangle$ in the doubled Hilbert space. We can show that

$$|\rho_m\rangle\rangle \propto \left(\prod_{\mu} e^{\tilde{K}_x X_{\mu} \otimes X_{\mu} - \tilde{K}_y Y_{\mu} \otimes Y_{\mu} + \tilde{K}_z Z_{\mu} \otimes Z_{\mu}} \right) |\rho_0\rangle\rangle, \quad (4.83)$$

where $\tilde{K}_{x,y,z}$ are given by $p_{x,y,z}$ through Eq. (4.78). The negative sign in front of the $Y_{\mu} \otimes$

Y_μ terms is the consequence of $Y_\mu^\top = -Y_\mu$ under the Choi-Jamiołkowski isomorphism. The spatial correlation function on $|\rho_m\rangle\rangle$ is identical to the correlation functions in the 2-replica random SM. One can construct a frustration-free parent Hamiltonian similar to Eq. (4.43) for which $|\rho_m\rangle\rangle$ is an exact ground state (see App. 4.6.6 for details). Therefore, the $R = 2$ DIPT for a decohered stabilizer code is related to a quantum phase transition in the doubled Hilbert space.

As a simple illustration of the general construction of the statistical models, we briefly discuss its application to the Chamon model [46] decohered by the Pauli noise channel. The Chamon model is defined on a cubic lattice with a single qubit per site. There is one stabilizer for each cube on the lattice. Each stabilizer is a product of 6 Pauli operators (see Fig. 4.10 (a)). The SM associated with the decohered Chamon model is defined on the dual cubic lattice with one classical $\mathbb{Z}_2$ spin per site. The interactions $\mathcal{O}^{x,y,z}$ in the classical Hamiltonian of SM are four-spin interactions illustrated in Fig. 4.10 (b). To the best of our knowledge, the phase diagram of SM (with or without randomness) has not been studied before. It is interesting to study the possible phases of the decohered Chamon model using SM in the future.

4.4.3 Dualities for general decohered stabilizer codes

Now, we discuss the GPN (general Pauli noise) duality, a generalization of the BPD dualities (associated with the decohered CSS codes) to more general stabilizer codes. The statement of the GPN duality is the following.

Theorem 4.5. For $R = 2$, and ∞ ,
the R -replica random statistical models
that map to the Rényi entropy $S_R(\rho_m)$
at the error rates (p_x, p_y, p_z) are dual
to those statistical models at the error

rates $(\tilde{p}_x, \tilde{p}_y, \tilde{p}_z)$. The two sets of dual error rates obey

$$\tilde{\mathcal{N}} \equiv \tilde{p}_0^2 + \tilde{p}_x^2 + \tilde{p}_y^2 + \tilde{p}_z^2. \quad (4.84)$$

For $R = 2$,

$$\begin{aligned} (1 - 2p_y - 2p_z)^2 &= \frac{2\tilde{p}_0\tilde{p}_x + 2\tilde{p}_y\tilde{p}_z}{\tilde{\mathcal{N}}}, \\ (1 - 2p_z - 2p_x)^2 &= \frac{2\tilde{p}_0\tilde{p}_y + 2\tilde{p}_z\tilde{p}_x}{\tilde{\mathcal{N}}}, \end{aligned} \quad (4.85)$$

$$(1 - 2p_x - 2p_y)^2 = \frac{2\tilde{p}_0\tilde{p}_z + 2\tilde{p}_x\tilde{p}_y}{\tilde{\mathcal{N}}}.$$

and

$$\begin{cases} 1 - 2p_y - 2p_z = \tilde{p}_x/\tilde{p}_0, \\ 1 - 2p_z - 2p_x = \tilde{p}_y/\tilde{p}_0, \\ 1 - 2p_x - 2p_y = \tilde{p}_z/\tilde{p}_0. \end{cases} \quad \text{for } R = \infty, \quad (4.86)$$

Here, $p_0 \equiv 1 - p_x - p_y - p_z$ and $\tilde{p}_0 \equiv 1 - \tilde{p}_x - \tilde{p}_y - \tilde{p}_z$.

A proof of this theorem is given in App. 4.6.5. Conceptually, the GPN dualities with $R = 2$ and ∞ can be effectively viewed as the descendants of the self-duality of the non-random SM. For $R = 2$, after we integrate out the random couplings E , the two-replica random SM reduces to a single copy of non-random SM. For the limit $R = \infty$, the partition function of the R -replica random SM is dominated by the trivial random coupling configuration, i.e. $E = 0$. Therefore, the self-duality of the non-random SM

implies the GPN dualities for $R = 2, \infty$. We caution that Theorem 4.4 is applicable only when the error rates $p_{x,y,z}$ and $\tilde{p}_{x,y,z}$ are both physical, namely they are between 0 and 1. In App. 4.6.5, we also discuss how the GPN duality recovers the BPD duality for $R = 2, \infty$ for CSS codes.

For both $R = 2$ and $R = \infty$, there is a surface of self-dual error rates. The error rates (p_x, p_y, p_z) that satisfy the following equations map back to themselves under the GPN duality

$$\begin{cases} (1 - p_x - p_y - p_z)^2 + p_x^2 + p_y^2 + p_z^2 = \frac{1}{2}, & \text{for } R = 2, \\ p_x + p_y + p_z = \frac{1}{2}, & \text{for } R = \infty. \end{cases} \quad (4.87)$$

Note that these self-dual conditions are derived within the regime with $p_{x,y,z} < p_0 = 1 - p_x - p_y - p_z$. As commented earlier, the roles of p_0 , p_x , p_y , and p_z can be permuted once the Pauli noise channel is followed by a global unitary action $\prod_\mu X_\mu$, $\prod_\mu Y_\mu$, or $\prod_\mu Z_\mu$. With these permutations taken into account, the surfaces of self-dual error rates are depicted in Fig. 4.11.

The surfaces of self-dual error rates are expected to constrain the phase diagram of the decohered stabilizer codes. One can argue that the renormalization group flow cannot cross this self-dual surface. A possible scenario is that this self-dual surface matches the critical surface of the DIPTs. The specific role of this self-dual surface in the phase diagram of the decohered stabilizer code should depend on the details of the model.

4.5 Conclusions and Outlook

In this chapter, we systematically develop the formalism that studies the stabilizer codes decohered by Pauli noise/errors using statistical models. We focus on the Rényi entropies S_R of the decohered codes as a probe of the systems entanglement structure. We find

a general mapping between the S_R 's and classical statistical models that can be systematically constructed from the code's defining data. The phase transitions in these statistical models indicate non-trivial DIPTs in the decohered code. We discover intricate tapestries of dualities emerging among these statistical models. These dualities cast strong constraints on the phase diagram of the decohered quantum matter hosted in the stabilizer codes with noise. More specifically, this chapter focuses on three general cases: (1) CSS codes decohered by bit-flip and phase-flip errors (2) *em*-symmetric CSS codes under bit-flip and phase-flip decoherences. (3) General stabilizer codes under generic Pauli-noise decoherence.

First, for generic CSS codes, we construct a pair of statistical models SM_1 and SM_2 , related to each other by an HLT duality (Eq. (4.36)), through an ungauging procedure. We show that the R -th Rényi entropy $S_R(\rho_b)$ for the error-corrupted mixed state ρ_b under the bit-flip decoherence is described by (1) R -replica SM_1 with rRC and (2) R -replica SM_2 with iRC. The two R -replica statistical models with different types of randomness are dual to each other through an HLT duality. Similarly, $S_R(\rho_p)$ for the error-corrupted mixed state ρ_p under phase-flip decoherence is described by (1) R -replica SM_2 with rRC and (2) R -replica SM_1 with iRC. These two models are also the HLT dual of each other. Moreover, for $R = 2, 3, \infty$, we find BPD dualities that relate $S_R(\rho_b)$ and $S_R(\rho_p)$, connecting the decoherence effects from the two error types. It is a "strong-weak" duality in that it maps strong bit-flip decoherence to weak phase-flip decoherence and vice versa. The tapestry of dualities associated with a general CSS code is illustrated in Fig. 4.1.

The classical statistical models that describe S_R provide us with powerful tools to investigate the phase diagrams of decohered CSS codes. In particular, the DIPTs in a decohered CSS code are identified as the phase transitions in the corresponding statistical models. The DIPTs with $R = 2, \infty$ are particularly simple as we have shown their equivalence to the phase transitions of $SM_{1,2}$ without randomness. In general, DIPTs

for different R 's happen at different critical error rates $p_{x/z}^*(R)$. The universality classes with different R 's are also different. Thus, there is a family of DIPTs indexed by R for a single CSS code. $p_{x/z}^*(R \rightarrow 1)$ corresponds to the decoding error thresholds of the decohered CSS code, which has been extensively studied in many earlier works (for example Ref. [70, 226, 293]). The DIPTs for $R \geq 2$ reveal other interesting singularities in the entanglement structure of the error-corrupted mixed states. For $R = 2$, the DIPT can also be interpreted as a quantum phase transition in a doubled Hilbert space. The nature and physical implication of the DIPTs at $p^*(R)$ for $R \geq 3$ are interesting questions for the future. We also propose a conjecture on the monotonicity of the critical error rates $p_{x/z}^*(R)$ as a function of R and present several pieces of evidence for this conjecture.

Second, for CSS codes with an *em* symmetry between the X -type and Z -type stabilizers, the corresponding SM_1 and SM_2 become the same model SM . The tapestry of dualities of *em*-symmetric CSS code becomes Fig. 4.7, which is effectively the tapestry of a general CSS code (Fig. 4.1) folded in half. The Rényi entropy S_R under either bit-flip or phase-flip decoherence is described by (1) R -replica SM with rRC and (2) R -replica SM with iRC. The two R -replica statistical models with iRC and rRC are related to each other through an HLT duality. The BPD duality for $R = 2, 3, \infty$ becomes a self-duality that relates strong bit-flip (phase-flip) decoherence to weak bit-flip (phase-flip) decoherence. Remarkably, these dualities yield super-universal self-dual error rates for $R = 2, 3, \infty$ (Eq. (4.64)). If there is a unique DIPT (at a given R), the self-dual error rate must coincide with the critical error rate of the transition. In the two examples we examined, 2D toric code and Haah's code in 3D, the self-dual error rates indeed match the critical error rates of the DIPTs. The properties of the DIPTs in these two codes are very different, demonstrating that the self-dual error rates encompass different universality classes of phase transitions and are super-universal. For cases with multiple DIPTs for single R , the BPD self-duality implies that the DIPT away from the self-dual error rates must appear in pairs, and the self-dual error rates are upper bounds to the

transition with the lowest error rate.

Finally, we extend our analysis to general stabilizer codes under decoherence of general Pauli noises, i.e., independent X -, Y -, Z -errors with error rates p_x, p_y, p_z . The description of the decohered code is centered around a single statistical model SM. For a CSS code, SM is the previously constructed $SM_{1,2}$ coupled together. The entropy $S_R(\rho_m)$ caused by the general Pauli noise is mapped to (1) R -replica SM with rRC and (2) R -replica SM with iRC. The two R -replica statistical models with randomness are related to each other by an HLT duality. In addition, we find a GPN self-duality that relates different error rates for $R = 2, \infty$, analogous to the BPD duality. Super-universal self-dual surfaces of error rates are identified for $R = 2, \infty$ (Fig. 4.11). These self-dual surfaces exist for a general stabilizer code and constrain the possible universal properties of the decohered code. One potential scenario is that the self-dual surfaces of error rates are exactly the phase boundary between different phases of the decohered code. A possible next step of investigation is to understand the physical meaning of these self-dual surfaces in specific stabilizer codes.

Our systematic construction of the statistical models provides powerful tools to investigate the DIPTs of QEC codes. For future research, an important question to address concerns the quantum-information-theoretic interpretation of these DIPTs indexed by R . As explained, in the $R \rightarrow 1$ limit, the DIPT is associated with the decodability of the logical information from the decohered code. Does the DIPT with a given $R > 1$ mark the limit of decoherence for some concrete R -dependent quantum information processing protocol? Finding such protocols will deepen our understanding of the physical meaning of this family of DIPTs and will help refine the notion of topological order in error-corrupted mixed states. Moreover, such protocols will provide guidance on how to experimentally observe the DIPTs in a noisy intermediate-scale quantum platform. Additionally, it is an interesting future direction to establish more detailed relations between the critical behavior of statistical models at different R , including the criti-

cal correlation functions and scaling exponents, and the quantum-information-theoretic properties of the decohered code.

Another topic for future research pertains to our conjectured monotonic dependence of the critical error rates for DIPTs on the Rényi index R . The current conjecture concerns the bit-flip and phase-flip errors. If the conjecture is true, it will offer a tool to bound the critical error rates for a specific R , particularly the error threshold (in the limit of $R \rightarrow 1$), using results for larger R values. We have presented a few pieces of evidence of the conjecture. Developing a deeper understanding of the relationship between statistical models at different R will be important for resolving this conjecture. Moreover, it is also interesting to investigate if there is a similar relation between the critical error rates at different R for other types of errors, such as the general Pauli noise.

In this chapter, we have focused on the decoherence effects caused by Pauli noise/errors on the physical qubits of the QEC code. Investigating the effect of coherent errors, such as random-angle X - or Z -rotations, is a natural direction to generalize our framework. It was found that 2D surface code with coherent errors can be mapped to a 2D Ising model with complex couplings [286]. It would be interesting to generalize the systematic construction of the statistical models and the tapestry of dualities to general QEC codes with coherent errors. For a QEC code, another type of error that affects its performance is the measurement read-out error. For the decodability problem, taking the read-out error into account results in a statistical model living in one higher dimension than the spatial dimension of the QEC code [70]. An interesting open question is whether the measurement read-out errors can be investigated from the perspective of decohered quantum matter and if there is also a family of phase transitions similar to the family of DIPTs indexed by R .

Considering the decoherence-induced phases and transitions beyond stabilizer codes of qubit systems is another natural direction for future expeditions. For instance, it should be feasible to generalize our study to decohered stabilizer codes with qudits

(each having a local Hilbert space of dimension $d > 2$). Exploring the decoherence effect in subsystems codes [160, 161] and the recently discovered Floquet codes [115] can potentially uncover new forms of decoherence-induced quantum matter.

4.6 Appendix: Proof Roadmap and Technical Details

This appendix keeps the proof logic needed for the thesis version of the chapter. The longer paper-style derivations are retained in the source file below this section but are not printed in the filed thesis version.

4.6.1 CSS statistical-model construction

Let ρ_b be the mixed state after bit-flip decoherence with error rate p and $\lambda = p/(1-p)$. The starting point is the symmetrized Rényi expression

$$\text{Tr}(\rho_b^R) = \frac{(1-p)^{NR}}{2^{\dim \mathcal{V}_x}} \sum_{E \in \mathcal{V}} \sum_{C_1, 2, \dots, R \in \mathcal{V}_x} \lambda^{|E+C_1|} \dots \lambda^{|E+C_R|}. \quad (4.88)$$

The two ingredients are the low-temperature expansion of SM_1 , which generates $\mathcal{V}_x$, and the high-temperature expansion of SM_2 , which generates $\mathcal{V}_z^\perp$. Since $\mathcal{V}_z^\perp/\mathcal{V}_x$ consists of logical operators, its contribution changes the free energy density only subextensively under the assumptions of no local logical operators and vanishing code rate. We therefore write $\sum_{\mathcal{V}_z^\perp} \rightarrow \sum_{\mathcal{V}_x}$ with $\xrightarrow{N \rightarrow \infty}$ when comparing thermodynamic singularities.

For SM_1 , the low-temperature expansion gives

$$Z_{\text{SM}_1}(K, E) = N_s \sum_{C \in \mathcal{V}_x} \exp\left(K(N - 2|E + C|)\right) = \lambda^{-N/2} N_s \sum_{C \in \mathcal{V}_x} \lambda^{|E+C|}, \quad (4.89)$$

where N_s is the symmetry factor of the domain-wall expansion. Combining this with

Eq. (4.88) yields the first representation of $\text{Tr}(\rho_b^R)$ in Theorem 4.1. For SM_2 , the high-temperature expansion with an insertion determined by E gives

$$W_{\text{SM}_2}(\tilde{K}, E) = (1 - \lambda^2)^{-N/2} 2^{N_B} \sum_{C \in \mathcal{V}_x^\perp} \lambda^{|E+C|} \stackrel{N \rightarrow \infty}{\equiv} (1 - \lambda^2)^{-N/2} 2^{N_B} \sum_{C \in \mathcal{V}_x} \lambda^{|E+C|}. \quad (4.90)$$

Thus $Z_{\text{SM}_1}(K, E)$ and $W_{\text{SM}_2}(\tilde{K}, E)$ compute the same error-chain sum up to nonuniversal prefactors. This is the HLT duality behind Theorem 4.1. The von Neumann expression in Corollary 4.1 follows by taking the $R \rightarrow 1$ limit after the same expansion.

4.6.2 BPD duality

The BPD duality is obtained by summing over the error chain E before interpreting the result as a statistical model. For bit-flip decoherence,

$$\text{Tr}(\rho_b^R) = (1 - p)^{RN} \sum_{C_{2,\dots,R} \in \mathcal{V}_x} \prod_{\mu=1}^N (\lambda^{n_\mu} + \lambda^{R-n_\mu}), \quad (4.91)$$

where $n_\mu = \sum_{\alpha=2}^R C_{\alpha,\mu}$. Resolving the constraint $C \in \mathcal{V}_x$ by Fourier transforming over $\mathcal{V}_x^\perp$ reorganizes the same expression as a sum over dual variables in $\mathcal{V}_z$:

$$\text{Tr}(\rho_b^R) \stackrel{N \rightarrow \infty}{\equiv} \frac{1}{2^{(R-1)\dim \mathcal{V}_x^\perp}} \sum_{\tilde{C}_{2,\dots,R} \in \mathcal{V}_z} \prod_{\mu=1}^N \left[\frac{1 + (-1)^{\tilde{n}_\mu} \lambda}{1 + \lambda} \left(\frac{1 - \lambda}{1 + \lambda} \right)^{\tilde{n}_\mu} \right]. \quad (4.92)$$

Comparing this weight with the analogous phase-flip expression gives the matching condition

$$\frac{1 + (-1)^{\tilde{n}_\mu} \lambda_x}{1 + \lambda_x} \left(\frac{1 - \lambda_x}{1 + \lambda_x} \right)^{\tilde{n}_\mu} = \frac{\lambda_z^{\tilde{n}_\mu} + \lambda_z^{R-\tilde{n}_\mu}}{1 + \lambda_z^R}. \quad (4.93)$$

For $R = 2$ this gives

$$\left[(1 - p_x)^2 + p_x^2 \right] \left[(1 - p_z)^2 + p_z^2 \right] = \frac{1}{2}, \quad (4.94)$$

and for $R = 3$ it gives the corresponding cubic relation

$$\left[(1 - p_x)^3 + p_x^3\right] \left[(1 - p_z)^3 + p_z^3\right] = \frac{1}{4}. \quad (4.95)$$

For $R \geq 4$ the matching is overdetermined in general. In the limit $R \rightarrow \infty$, the random partition functions are dominated by their clean sectors, giving the strong-weak relation $(1 - p_x)(1 - p_z) = 1/2$ for $p_x, p_z \in (0, 1/2)$.

4.6.3 Integrating out replica randomness

The same BPD structure can be seen directly in the replica statistical models. For rRC in the bit-flip problem, integrating out E gives

$$\text{Tr}(\rho_b^R) \propto \sum_{\{\tau_j^\alpha = \pm 1\}} \prod_{\mu} 2(\cosh K)^R \sum_{r=0}^{R-1} \sum_{\alpha_1 < \dots < \alpha_r} (\tanh K)^r \left(\frac{1 + (-1)^r e^{-2K}}{1 + e^{-2K}} \right) \mathcal{O}^{\alpha_1} \dots \mathcal{O}^{\alpha_r}. \quad (4.96)$$

For $R = 2$ there is only one replica, and the model reduces to a clean copy with renormalized coupling,

$$\text{Tr}(\rho_b^2) \propto \sum_{\{\tau_j = \pm 1\}} \prod_{\mu} \left(1 + (\tanh K)^2 \mathcal{O}_{\mu}^z[\tau] \right) \propto Z_{\text{SM}_1}(K'), \quad (4.97)$$

where $\tanh K' = (\tanh K)^2$. For $R = 3$, the two replicas form an Ashkin-Teller-type model,

$$\text{Tr}(\rho_b^3) \propto \sum_{\{\tau_j^\alpha = \pm 1\}} \prod_{\mu} e^{K''(\mathcal{O}^1 + \mathcal{O}^2 + \mathcal{O}^1 \mathcal{O}^2)}. \quad (4.98)$$

For iRC, the effective interaction is

$$\text{Tr}(\rho_{\text{b}}^R) \propto \sum_{\{\tilde{\tau}_j^\alpha = \pm 1\}} \prod_{\mu} \frac{e^K}{\cosh \tilde{K}} \exp \left(\tilde{K} \left(\sum_{\alpha=1}^{R-1} \tilde{\mathcal{O}}^\alpha + \prod_{\beta=1}^{R-1} \tilde{\mathcal{O}}^\beta \right) \right). \quad (4.99)$$

The rRC and iRC interactions coincide for $R = 2, 3$, which is the statistical-model reason that the BPD dualities exist at these Rényi indices.

4.6.4 General stabilizer codes and Pauli noise

For a non-CSS stabilizer code, a Pauli error chain is specified by $E = (\mathbf{e}, \mathbf{f}) \in \mathcal{V} \oplus \mathcal{V}$. Its probability under independent Pauli noise is

$$P(E) = (1 - p_x - p_y - p_z)^N \lambda_x^{N_x(E)} \lambda_y^{N_y(E)} \lambda_z^{N_z(E)} \quad (4.100)$$

$$= (1 - p_x - p_y - p_z)^N \left(\sqrt{\frac{\lambda_y \lambda_z}{\lambda_x}} \right)^{|\mathbf{f}|} \left(\sqrt{\frac{\lambda_x \lambda_y}{\lambda_z}} \right)^{|\mathbf{e}|} \left(\sqrt{\frac{\lambda_x \lambda_z}{\lambda_y}} \right)^{|\mathbf{e} + \mathbf{f}|}. \quad (4.101)$$

The Rényi quantity can be symmetrized as

$$\text{Tr}(\rho_{\text{m}}^R) = \sum_{E \in \mathcal{V} \oplus \mathcal{V}} \sum_{C_2, \dots, C_R \in \mathcal{V}_s} P(E) P(E + C_2) \cdots P(E + C_R) \quad (4.102)$$

$$= \frac{1}{2^{\dim \mathcal{V}_s}} \sum_{E \in \mathcal{V} \oplus \mathcal{V}} \left[\sum_{C \in \mathcal{V}_s} P(E + C) \right]^R. \quad (4.103)$$

The LTE of the general stabilizer statistical model with rRC produces

$$Z_{\text{SM}}(\mathbf{K}, E) = N_s [p_x p_y p_z (1 - p_x - p_y - p_z)]^{-N/4} \sum_{C \in \mathcal{V}_s} P(E + C), \quad (4.104)$$

while the HTE of the same model with iRC gives

$$W_{\text{SM}}(\tilde{\mathbf{K}}, E) \stackrel{N \rightarrow \infty}{\simeq} [(1 - 2p_x - 2p_y)(1 - 2p_x - 2p_z)(1 - 2p_y - 2p_z)]^{-N/4} \times 2^{N_A} \sum_{C \in \mathcal{V}_s} P(E + C). \quad (4.105)$$

These two expansions prove Theorem 4.4 and its $R \rightarrow 1$ corollary. They also show that the non-random model is HLT self-dual after replacing the HTE space by the stabilizer space up to nonlocal logical sectors.

4.6.5 GPN duality and self-dual surfaces

For $R = 2$, the purity under general Pauli noise may be written as a sum over stabilizer differences. After summing the unconstrained error variables locally, the Boltzmann weights are

$$f(C_\mu, \mathbf{p}) = \frac{1}{p_0^2} \begin{pmatrix} p_0^2 + p_x^2 + p_y^2 + p_z^2 \\ 2p_x p_0 + 2p_y p_z \\ 2p_y p_0 + 2p_x p_z \\ 2p_z p_0 + 2p_x p_y \end{pmatrix}, \quad (4.106)$$

where $p_0 = 1 - p_x - p_y - p_z$ and the rows correspond to $C_\mu = I, X, Y, Z$. Performing the dual resolution of the same stabilizer constraint gives

$$\tilde{f}(C_\mu, \mathbf{p}) = \frac{1}{p_0^2} \begin{pmatrix} 1 \\ (1 - 2p_y - 2p_z)^2 \\ (1 - 2p_x - 2p_z)^2 \\ (1 - 2p_x - 2p_y)^2 \end{pmatrix}. \quad (4.107)$$

The $R = 2$ GPN duality is the equality of normalized weights,

$$\frac{f(C_\mu, \mathbf{p})}{f(I, \mathbf{p})} = \frac{\tilde{f}(C_\mu, \tilde{\mathbf{p}})}{\tilde{f}(I, \tilde{\mathbf{p}})}, \quad (4.108)$$

for $C_\mu = X, Y, Z$. For $R \rightarrow \infty$, the same duality descends from the clean-model bound

$$0 \leq Z(\mathbf{K}, E) \leq Z(\mathbf{K}), \quad 0 \leq W(\mathbf{K}, E) \leq Z(\mathbf{K}). \quad (4.109)$$

In the pure bit-flip limit, the GPN relation reduces to the BPD duality through

$$\frac{2\lambda_x}{1 + \lambda_x^2} = \left(\frac{1 - \tilde{\lambda}_z}{1 + \tilde{\lambda}_z} \right)^2. \quad (4.110)$$

The self-dual surface at $R = \infty$ is $p_x + p_y + p_z = 1/2$, equivalently

$$e^{-2(K_x + K_y)} + e^{-2(K_y + K_z)} + e^{-2(K_x + K_z)} = 1. \quad (4.111)$$

For $R = 2$, the self-dual surface becomes

$$(1 - 2p_y - 2p_z)^2 + (1 - 2p_x - 2p_z)^2 + (1 - 2p_x - 2p_y)^2 = 1, \quad (4.112)$$

or, equivalently, $(1 - p_x - p_y - p_z)^2 + p_x^2 + p_y^2 + p_z^2 = 1/2$.

4.6.6 Parent Hamiltonian in the doubled Hilbert space

For a stabilizer code with

$$A_J[X, Z] = (\mathbf{i})^{\mathbf{a}_J \cdot \mathbf{b}_J} \prod_{\mu} (X_{\mu})^{(\mathbf{a}_J)_{\mu}} \prod_{\mu} (Z_{\mu})^{(\mathbf{b}_J)_{\mu}}, \quad (4.113)$$

we use the projector Hamiltonian

$$H_s = \sum_J \frac{\mathbb{I} - A_J}{2}. \quad (4.114)$$

The Choi-Jamiołkowski map sends $A\rho B$ to $A \otimes B^\top |\rho\rangle\rangle$, where the transpose is in the Pauli- Z basis. Thus the pure logical density matrix maps to a ground state of

$$H_0^D = H_s \otimes \mathbb{K} + \mathbb{K} \otimes H_s^\top. \quad (4.115)$$

For the decohered state,

$$|\rho_m\rangle\rangle \propto \exp\left(\sum_\mu \tilde{K}_x X_\mu \otimes X_\mu - \tilde{K}_y Y_\mu \otimes Y_\mu + \tilde{K}_z Z_\mu \otimes Z_\mu\right) |\rho_0\rangle\rangle \equiv \hat{\mathcal{E}} |\rho_0\rangle\rangle. \quad (4.116)$$

Writing

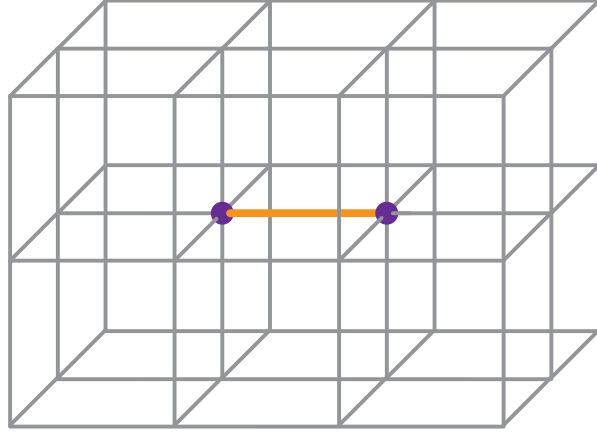
$$\hat{\mathcal{E}}_J = e^{\sum_\mu \tilde{K}_x (\mathbf{b}_J)_\mu X_\mu \otimes X_\mu - \tilde{K}_y ((\mathbf{a}_J)_\mu + (\mathbf{b}_J)_\mu) Y_\mu \otimes Y_\mu + \tilde{K}_z (\mathbf{a}_J)_\mu Z_\mu \otimes Z_\mu}, \quad (4.117)$$

one obtains a local frustration-free parent Hamiltonian

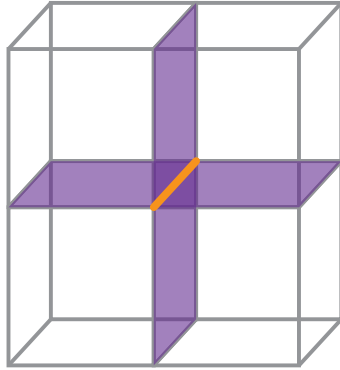
$$H^D = \sum_J \hat{\mathcal{E}}_J^{-1} \left(\frac{\mathbb{K} - A_J}{2} \otimes \mathbb{K} + \mathbb{K} \otimes \frac{\mathbb{K} - A_J^\top}{2} \right) \hat{\mathcal{E}}_J^{-1}. \quad (4.118)$$

Each term annihilates $|\rho_m\rangle\rangle$, so $|\rho_m\rangle\rangle$ is an exact ground state. Expanding for small $\tilde{K}$ shows that the doubled Hamiltonian is H_0^D perturbed by inter-copy couplings $X \otimes X$, $-Y \otimes Y$, and $Z \otimes Z$, which is the quantum doubled-space counterpart of the $R = 2$ statistical model.

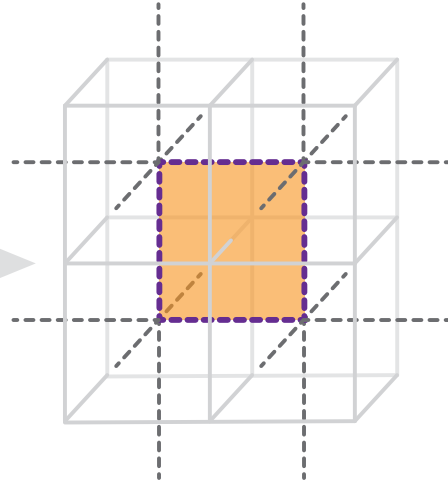
(a) SM_1 :



(b) SM_2 :



cubic lattice



dual cubic lattice

Figure 4.4: Statistical models associated with the 3D toric code: (a) The 3D Ising model has a spin on each site of the cubic lattice and nearest-neighbor interactions (depicted as the two spins (purple dots) “coupled” by the orange edge). (b) Classical 3D $\mathbb{Z}_2$ gauge theory on the dual cubic lattice has a spin per dual edge (dashed lines). Each spin interaction involves the four classical spins on the boundary of a dual plaquette (orange).

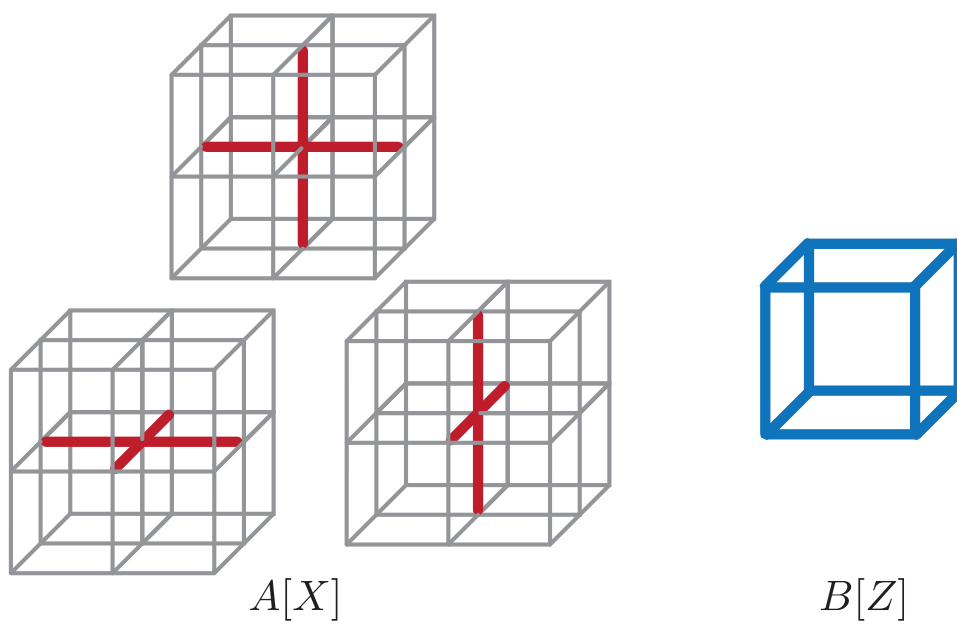
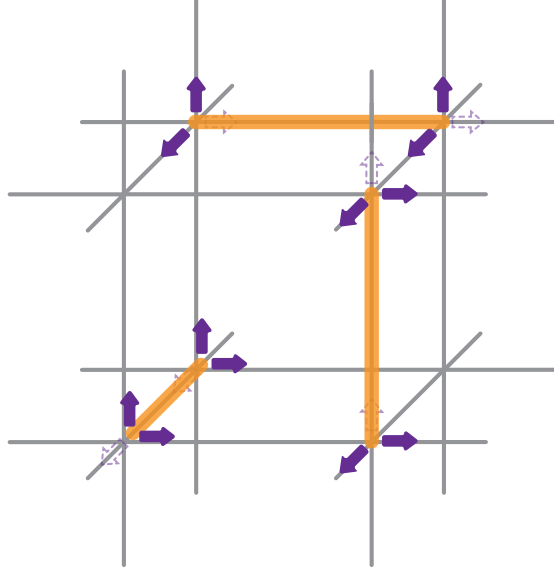


Figure 4.5: X -type stabilizers (red) and Z -type stabilizers (blue) of the X-cube model

(a) SM_1 :



(b) SM_2 :

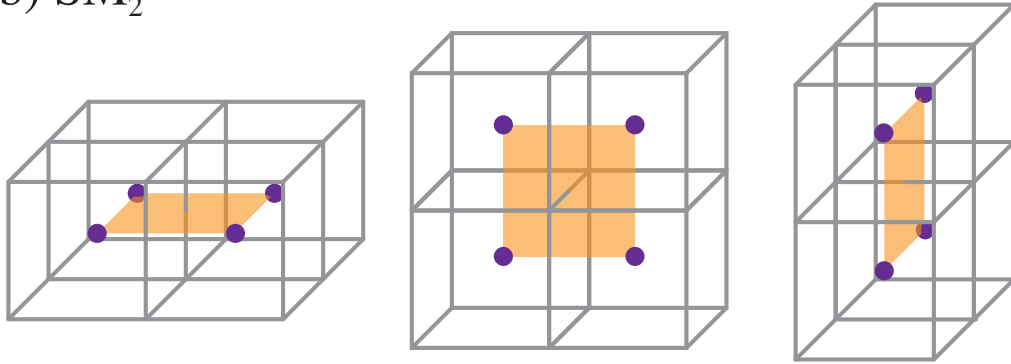


Figure 4.6: Two statistical models originate from the X-cube model. (a) SM_1 is a model with three colors of classical spins (depicted by the purple arrows along the three directions) per site on a cubic lattice. Each nearest-neighbor interaction (orange) involves the four spins on the two neighboring sites whose colors (arrow directions) are perpendicular to the edge connecting the neighboring sites. (b) SM_2 is a plaquette Ising model on the dual cubic lattice. There is one classical spin per dual site. Each spin interaction term involves the four spins on the four corners of a dual plaquette.

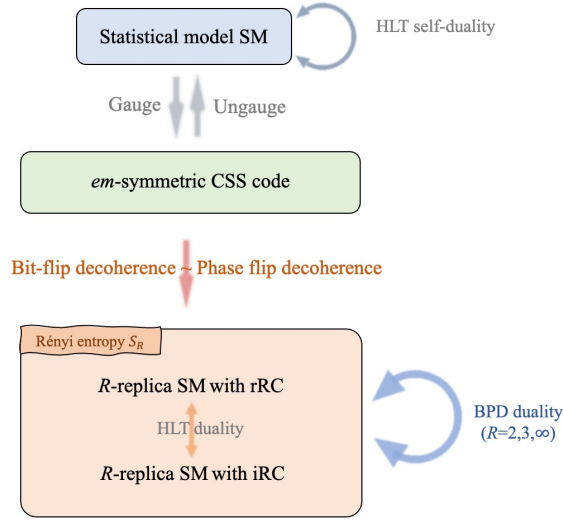


Figure 4.7: The tapestry of dualities of an *em*-symmetric CSS code. The ungauging procedure leads to SM with a self-duality. Under bit-flip or phase-flip decoherence, R -replica SM with rRC and R -replica SM with iRC emerge as the description of the system's Rényi entropy. These random statistical models are related by HLT dualities for a general R . For $R = 2, 3, \infty$, the BPD dualities (combined with the *em* symmetry) relate different values of error rates.

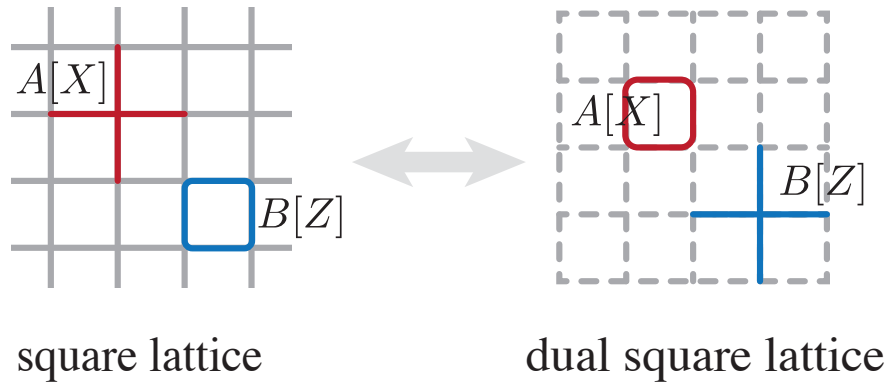
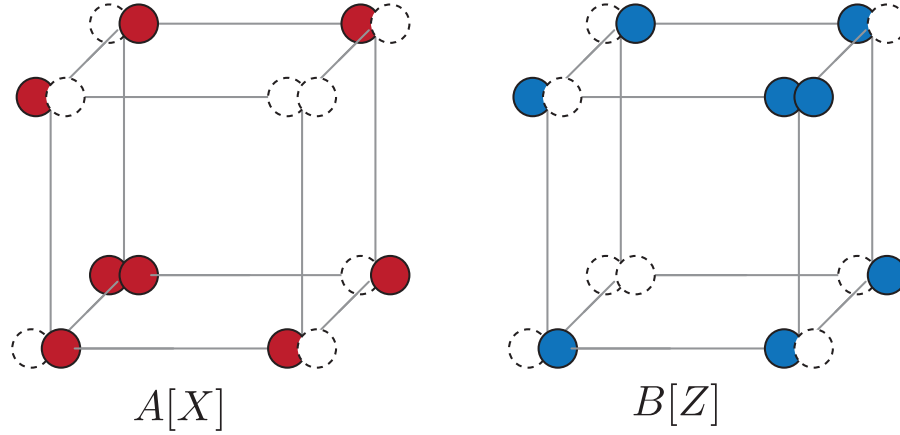


Figure 4.8: The *em* symmetry of the 2D toric code is evident from the pictorial representations of the X - and Z -type stabilizers on the square lattice and its dual. The unitary operator $\mathcal{U}_{em}$ is a product of Hadamard gates on all sites and a translation action by half of a lattice spacing in both the horizontal and vertical directions.

(a) Haah's code:



(b) Spin interactions in SM:

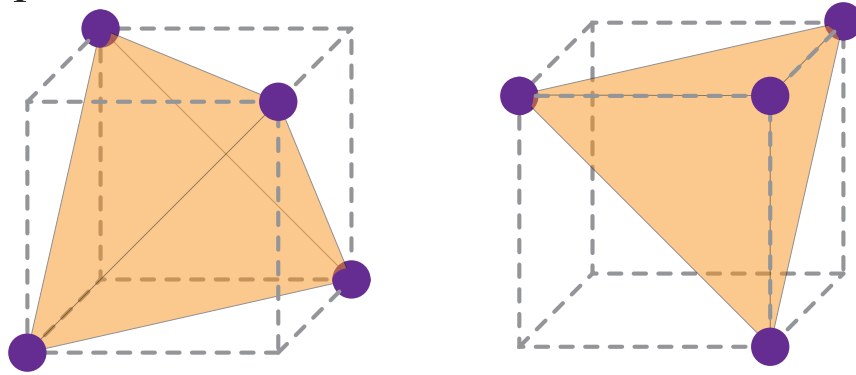
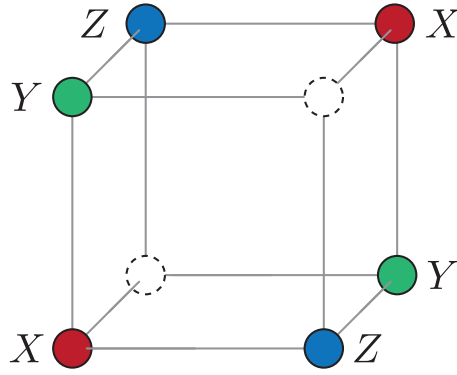


Figure 4.9: (a) Haah's code is defined on a 3D cubic lattice with two qubits (depicted as the red, blue, and dashed circles) per site. For each cube, the X -type stabilizer is the product of the Pauli- X operators on the 8 red qubits, and the Z -type stabilizer is the product of the Pauli- Z operators on the 8 blue qubits. (b) In SM, which originates from Haah's code, there are two types of four-spin interactions in each unit cell. Each type is a product of the 4 classical spins (purple dots) on the corners of the corresponding tetrahedron (orange).

(a) Stabilizers in Chamon model:



(b) Spin interactions in SM:

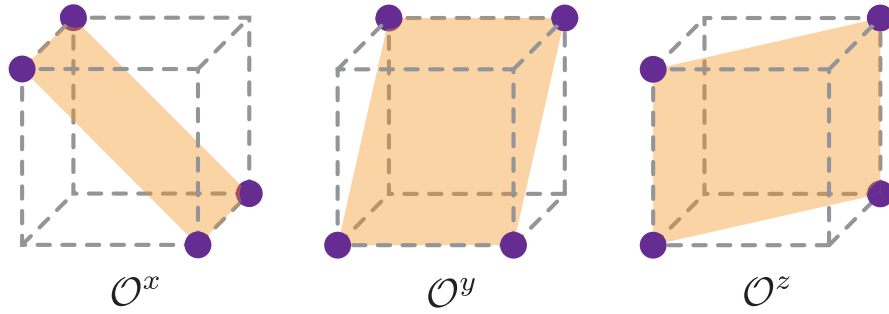


Figure 4.10: (a) The stabilizer of Chamon model (b) Three types of interaction $\mathcal{O}^{x,y,z}$ in SM: Each term (orange) is a product of four classical spins (purple dots). The dashed lines represents the edges of the dual cubic lattice.

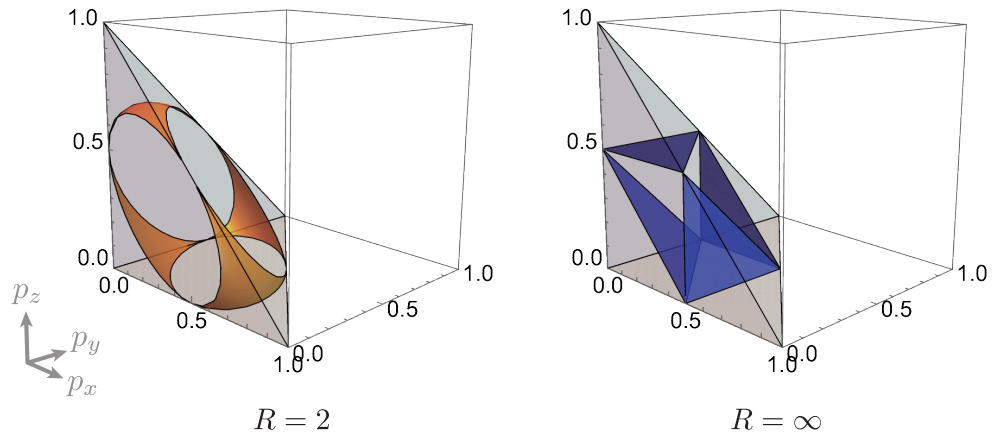


Figure 4.11: The surfaces of self-dual error rates for $R = 2$ and $R = \infty$. The shaded area is the physical parameter regime with $p_{0,x,y,z} > 0$.

Chapter 5

Spin Liquid and Superconductivity From Steady States and Measurements

This chapter studies measurement and dissipation as mechanisms for preparing mixed-state quantum matter. Starting from a simple fermion wave function, local density dephasing or strong density measurements without post-selection impose a Gutzwiller-type constraint on the Choi state of the density matrix. For gapless fermions this construction produces an algebraic Choi-spin liquid whose nonlinear Rényi-2 correlations inherit the universal behavior of projected spin-liquid wave functions. For spinless fermions with a Fermi surface, the doubled-space projection can generate superconducting order, which appears in the density matrix as strong-to-weak $U(1)$ symmetry breaking.

This chapter contributes to the thesis by adding a constructive direction to the earlier diagnostic chapters. The same doubled-Hilbert-space viewpoint becomes a preparation principle: measurements and Lindbladian evolution can impose local constraints that generate exotic order in a density matrix.

5.1 Introduction

Quantum spin liquids, a class of highly nontrivial disordered phase of quantum spins, have been the subject of an extremely active subfield of condensed matter physics since their early theoretical proposal [12, 13]. Despite great progress made in the field, many open questions and challenges remain (for a review, please refer to [19, 256, 322]). First of all, the theoretical description of spin liquids usually involves a strongly coupled gauge theory, which is a formidable analytical problem except for certain theoretical limits. Secondly, though it is certain that spin liquids do exist in some elegant theoretical models, e.g. the Kitaev model [154], numerical simulation of quantum spin liquid on more realistic models often suffers from sign problems as the models that potentially realize the spin liquid usually have geometric frustration. Hence controversies continue to persist about the existence or nature of the spin liquids in various important realistic models. Thirdly, the signal of quantum spin liquid in real correlated materials may be obscured by the inevitable disorders and impurities.

Rather than the parent Hamiltonian, one can instead focus on the spin liquid wave function. One standard construction of spin liquid wave function is the so-called Gutzwiller projected state [100, 122, 207, 209, 210, 240, 291]:

$$|\text{SL}\rangle = \prod_i \hat{\mathcal{P}}(\hat{n}_{i,\uparrow} + \hat{n}_{i,\downarrow} = 1) |f_{i,\uparrow}, f_{i,\downarrow}\rangle, \quad (5.1)$$

where $|f_{i,\uparrow}, f_{i,\downarrow}\rangle$ is a simple spin-1/2 fermion state, and it can be often taken as a free fermion state with on average one fermion per-site. The projection $\hat{\mathcal{P}}$ ensures that there is one and only one fermion per site, which matches the onsite Hilbert space of a spin-1/2 system.

Though the Gutzwiller projection is broadly used as a trial mean field wave functions of spin liquids, it is never exactly realized in condensed matter systems. This chapter demonstrates that the Gutzwiller wave function can be realized in a completely different

set-up: it can be constructed as the steady mixed state of a Lindbladian evolution, or as the ensemble created by strong measurements of local operators, starting with a simple fermion wave function. The mixed state density matrix obtained from both constructions in the doubled Hilbert space (the Choi-Jamiołkowski representation [54, 142]), becomes exactly a Gutzwiller wave function. Predictions of the constructed mixed state will be made based on our theoretical understanding of quantum spin liquids.

We also demonstrate that, sometimes the constructed mixed state density matrix is expected to become a superconductor in the doubled Hilbert space, which corresponds to the “strong-to-weak” spontaneous symmetry breaking, a subject that has attracted great interests very recently [171, 178, 179, 197, 224, 248]. These theoretical predictions and expectations can be tested using the experimental protocols summarized in Sec. 5.7.

5.2 Basic Formalism

5.2.1 Construction with Lindbladian

We consider the nonunitary Lindbladian evolution of a density matrix. Let us assume that the density matrix at $t = 0$ corresponds to a pure quantum state $\rho_0 = |\Psi_0\rangle\langle\Psi_0|$ of fermions. For simplicity we will start with a non-interacting spinless fermion wave function $|\Psi_0\rangle$, and we assume that $|\Psi_0\rangle$ is the ground state of a Hamiltonian H_0 on a d -dim lattice: $H_0 = \sum_{\langle i,j \rangle} -t_{ij}c_i^\dagger c_j + \text{h.c.}$ We will first assume that H_0 is a *gapless* free fermion tight-binding model, later we will discuss interactions in H_0 .

We consider the situation where the jump operators are the local fermion density operators $L_i = \sqrt{\gamma}\hat{n}_i = \sqrt{\gamma}c_i^\dagger c_i$. The nonunitary evolution of a density matrix under a general Lindbladian reads (we assume that the unitary part of the evolution is absent)

$$\partial_t \rho(t) = \sum_i L_i \rho L_i^\dagger - \frac{1}{2} \left(L_i^\dagger L_i \rho + \rho L_i^\dagger L_i \right). \quad (5.2)$$

Exploring physics of mixed-states using the Choi-Jamiołkowski isomorphism of the density matrix, i.e. representing the density matrix as a pure quantum state in the doubled Hilbert space [54, 142], has attracted a great deal of interests [25, 60, 94, 134, 174, 189, 261, 296, 307]. The Choi-Jamiołkowski isomorphism maps a density matrix $\rho = \sum_n p_n |\psi_n\rangle\langle\psi_n|$ to a state $|\rho\rangle\rangle \sim \sum_n p_n |\psi_{n,L}\rangle|\psi_{n,R}\rangle$. In our current case it is also convenient to take the Choi-Jamiołkowski representation of the density matrix. The evolution of the Choi state in the doubled Hilbert space is given by

$$|\rho_t\rangle\rangle \sim e^{t\mathcal{L}}|\rho_0\rangle\rangle, \quad (5.3)$$

where

$$\begin{aligned} \mathcal{L} &= \sum_i \gamma \left(\hat{n}_{i,L} \hat{n}_{i,R} - \frac{1}{2} (\hat{n}_{i,L}^2 + \hat{n}_{i,R}^2) \right) \\ &= \sum_i -\frac{\gamma}{2} (\hat{n}_{i,L} - \hat{n}_{i,R})^2. \end{aligned} \quad (5.4)$$

Here L and R label two copies of fermionic modes in the doubled Fock space of the Choi-Jamiołkowski isomorphism of the density matrix. The initial Choi state $|\rho_0\rangle\rangle$ for the pure state $\rho_0 = |\Psi_0\rangle\langle\Psi_0|$ is the free fermion state with the parent Hamiltonian

$$\begin{aligned} \mathcal{H}_0 &= H_0(c_{i,L}) + H_0^*(c_{i,R}) \\ &= \sum_{\langle i,j \rangle} -t_{ij} c_{i,L}^\dagger c_{j,L} - t_{ij}^* c_{i,R}^\dagger c_{j,R} + \text{h.c.} \end{aligned} \quad (5.5)$$

In the limit $t \rightarrow \infty$, the steady state $|\rho_\infty\rangle\rangle$ satisfies the constraint $\hat{n}_{i,L} - \hat{n}_{i,R} = 0$.

To more explicitly reveal the implication of the constraint $\hat{n}_{i,L} - \hat{n}_{i,R} = 0$, let us perform a particle-hole (PH) transformation on $c_{i,R}$, and formally relabel the fermions

in the L, R spaces as “ $\uparrow$ ” and “ $\downarrow$ ”:

$$\begin{aligned} c_{i,L} &\rightarrow f_{i,\uparrow}, \quad c_{i,R} \rightarrow \eta_i f_{i,\downarrow}^\dagger, \\ \hat{n}_{i,L} &\rightarrow \hat{n}_{i,\uparrow}, \quad \hat{n}_{i,R} \rightarrow 1 - \hat{n}_{i,\downarrow}, \end{aligned} \tag{5.6}$$

where $\eta_i = \pm 1$ can be chosen to depend on the sites i . Then in the limit $t \rightarrow \infty$, the constraint $\hat{n}_{i,L} - \hat{n}_{i,R} = 0$ becomes

$$\hat{n}_{i,\uparrow} + \hat{n}_{i,\downarrow} = 1, \tag{5.7}$$

which is precisely the *Gutzwiller projection*. In this case, the steady Choi state is mapped to a spin wave function in the standard spin liquid literature:

$$|\rho_\infty\rangle\rangle = \prod_i \hat{\mathcal{P}}(\hat{n}_{i,\uparrow} + \hat{n}_{i,\downarrow} = 1) |\rho_0\rangle\rangle, \tag{5.8}$$

where $\hat{\mathcal{P}}(\hat{n}_{i,\uparrow} + \hat{n}_{i,\downarrow} = 1)$ is the projector onto the subspace with single occupation. From this point onward, because every site contains exactly one fermion $f_{i,\alpha}$, the projected wave function $|\rho_\infty\rangle\rangle$ lives in an effective spin-1/2 Hilbert space, with spin-1/2 operators $\hat{S}_i^\mu = \frac{1}{2} f_{i,\alpha}^\dagger \sigma_{\alpha\beta}^\mu f_{i,\beta}$. Under certain conditions, for example, (1) the fermions in $|\Psi_0\rangle$ are at half-filling, i.e. there are on average 1/2 fermions per site; and (2) under particle-hole transformation H_0^* becomes H_0 with a certain choice of η_i , then the parent tight-binding Hamiltonian of $|\rho_0\rangle\rangle$ in the doubled Hilbert space enjoys a full effective SU(2) “spin” symmetry. For example, if H_0 is a tight-binding model on a bipartite lattice with nearest neighbor hopping, then choosing $\eta_i = -1$ on one sublattice will result in a SU(2) symmetric $\mathcal{H}_0$ in Eq. 5.5.

5.2.2 Construction with measurements

When the jump operators are Hermitian and all commute with each other, the steady state density matrix also describes the ensemble of wave functions generated by strong measurements on the jump operators *without post-selection*. For example, when we perform strong measurement of local densities $\hat{n}_i$ on a pure fermion quantum state $|\Psi_0\rangle$, *without* post-selecting the measurement outcomes, the density matrix of the ensemble becomes

$$\rho = \sum_{\mathbf{n}} \hat{P}_{\mathbf{n}} \rho_0 \hat{P}_{\mathbf{n}}, \quad (5.9)$$

where $\mathbf{n}$ represents a configuration of fermion numbers $\hat{n}_i$, and $\hat{P}_{\mathbf{n}}$ is a projection operator to the particular configuration. Such an ensemble can be obtained in Fermi gas microscope with high spatial resolution [53, 74, 98, 99, 110, 227, 232].

Through measurements, the system wave function collapses to the eigenstate of local density. Without postselection, the final mixed state is a classical mixture of eigenstates of local density with all eigenvalues, which becomes a quantum superposition in the doubled space, except now the fermion density in the left and right spaces are identified. This classical mixture of density eigenstates is precisely given by the Gutzwiller projection in the doubled space.

We also note that, construction of spin liquids as the doubled state of a mixed state was discussed before [94, 134, 261], but to the best of our knowledge, the general connection to the Gutzwiller wave function, which is a broadly used construction of spin liquids, has not been explored.

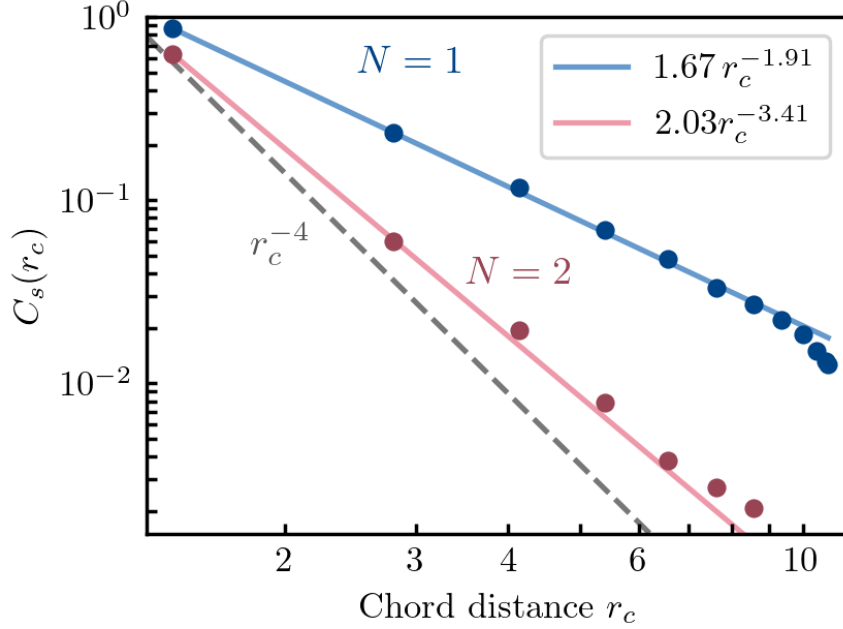


Figure 5.1: The staggered spin-spin correlation of the projected π -flux state on the square lattice (24×24 torus), which is a Dirac spin liquid constructed with $N = 1$ and $N = 2$ species of fermions. The power-law exponent is considerably enhanced compared with the unprojected wave function (dashed line), consistent with predictions of spin liquid theory. The analytical calculation based on $1/N$ -expansion is given in Eq. 5.13, note that there is another “valley number” N_v in the equation. In our case $N_v = 2$.

5.3 Dirac spin liquid

A spin liquid state typically involves an emergent dynamical gauge field [100, 122, 139, 207, 209, 210, 240, 291]. When the emergent gauge symmetry is $U(1)$, and the fermions $f_{i,\alpha}$ have Dirac nodes at the Fermi level, the low energy action describing the spin liquid is a QED_3 with Dirac fermion matter fields coupled to a $U(1)$ gauge field ¹

QED_3 Dirac spin liquids have attracted great interest [113, 118, 121, 122, 129, 139, 240, 291, 305]. The dynamical gauge field A_μ may lead to confinement and hence destabilize the spin liquid states. In our construction, to ensure a deconfined phase of the

¹The Gutzwiller wave function may not faithfully correspond to the exact full gauge theory. For example, Ref. [280] discussed a case where the Gutzwiller wave function does not capture the full fluctuations of gauge field. We will include a more detailed discussion between these two approaches in a future work.

gauge field, one can start with a state $|\Psi_0\rangle$ that is the ground state of N flavors of degenerate fermions $c_{i,I}$ at half-filling, where $I = 1 \cdots N$. For example, one can start with a state of the alkaline earth cold atoms, which enjoy a large flavor symmetry [96]. Then eventually the effective Gutzwiller projection imposed on the mixed state in the doubled space is

$$\sum_{I=1 \cdots N} \hat{n}_{i,I,\uparrow} + \hat{n}_{i,I,\downarrow} = N, \quad (5.10)$$

which ensures that $|\rho\rangle\rangle$ is a wave function of an effective $SU(2N)$ spin system, with self-conjugate representation of $SU(2N)$ on each site.

Assuming the dispersion of $|\Psi_0\rangle$ has Dirac nodes in the momentum space with N -fold flavor degeneracy, the spin liquid that $|\rho\rangle\rangle$ simulates is described by the following theory:

$$\mathcal{S} = \int d^2x d\tau \sum_{v=1}^{N_v} \sum_{a=1}^{2N} \bar{\psi}_{a,v} \gamma_\mu (\partial_\mu - iA_\mu) \psi_{a,v} + \frac{1}{4e^2} F_{\mu\nu}^2. \quad (5.11)$$

Here v labels the N_v Dirac nodes/valleys in the momentum space. For example, if $|\Psi_0\rangle$ is the ground state of a nearest neighbor tight binding model on the honeycomb lattice at half-filling, or on the square lattice with π -flux per square, then $N_v = 2$, i.e. there are two independent Dirac cones in the momentum space. It is known that for large enough N , Eq. 5.11 describes a stable algebraic liquid state which is also a $(2+1)d$ conformal field theory. We refer to this liquid state in the doubled Hilbert space as the algebraic Choi-spin liquid.

Predictions for the Choi-spin liquid described above can be made based on our understanding of the spin liquid. Let us still start with a $SU(2N)$ spin liquid on the square lattice with π -flux, whose low energy physics is described by QED₃ in Eq. 5.11. The local spin operator $\hat{S}_i^z = \sum_{I=1 \cdots N} \hat{n}_{i,I,\uparrow} - \hat{n}_{i,I,\downarrow}$ has a nonzero overlap with the fermion-bilinear composite field $\bar{\psi} S^z \mu^z \psi$, where $\mu^z = \pm 1$ denotes the two Dirac valleys. $\bar{\psi} S^z \mu^z \psi$

is an $SU(2N)$ Néel order parameter. Then at long distance the spin correlation functions is given by

$$\langle \hat{S}_0^z \hat{S}_x^z \rangle \sim (-1)^x \frac{1}{|\mathbf{x}|^{2\Delta}} + \dots . \quad (5.12)$$

Note that the sign of the correlation function above oscillates with the sublattice of $\mathbf{x}$. With sufficiently large N , the scaling dimension Δ can be computed using the standard $1/N$ expansion, and it is smaller than the scaling dimension of the free Dirac fermion [241]:

$$\Delta = 2 - \frac{32}{3NN_v\pi^2} + O\left(\frac{1}{N^2}\right). \quad (5.13)$$

We have conducted numerical study on the Gutzwiller projected π -flux state on the square lattice, and we see a power-law scaling with a considerably smaller scaling dimension compared with free Dirac fermion (Fig. 5.1), qualitatively consistent with the prediction of QED₃, and consistent with previous numerics [231]. If we take $N_v = 2$ in Eq. 5.13, the first order $1/N$ expansion leads to power-law $2\Delta_1 \sim 2.92$ and $2\Delta_2 \sim 3.46$ for $N = 1, 2$ respectively, both feature enhanced correlation compared with the free Dirac fermion (with $2\Delta = 4$). And $N = 2$ gives better agreement with the numerics (Fig. 5.1), as expected.

The $SU(2N)$ spin correlation function above can be used to make predictions for the following ‘‘Renyi-2’’ correlation in the current context:

$$\begin{aligned} \langle\langle \rho | \delta \hat{n}_{0,\uparrow} \delta \hat{n}_{x,\uparrow} | \rho \rangle\rangle &\sim \langle\langle \rho | \hat{S}_0^z \hat{S}_x^z | \rho \rangle\rangle \\ &\sim \text{Tr} \left(\rho^2 \delta \hat{n}_0 \delta \hat{n}_x \right) \sim (-1)^x \frac{1}{|\mathbf{x}|^{2\Delta}}. \end{aligned} \quad (5.14)$$

Here $\delta \hat{n}(\mathbf{x}) = \hat{n}(\mathbf{x}) - N/2$, and we have used the fact that $\delta n_{\uparrow}(\mathbf{x}) \sim \hat{S}_x^z/2$, since $\hat{n}_{\uparrow}(\mathbf{x}) +$

$\hat{n}_\downarrow(\mathbf{x})$ is a constant. Since the Choi state $|\rho\rangle\rangle$ has a $SU(2N)$ symmetry, all the $SU(2N)$ spin correlation function should have the same scaling as Eq. 5.14. For example,

$$\text{Tr} \left(\rho^2 (\hat{O}_0)_J^I (\hat{O}_x)_I^J \right) \quad (5.15)$$

is expected to decay in the same manner as Eq. 5.14 for sufficiently large N , where $\hat{O}_J^I = c_I^\dagger c_J$ with $I \neq J$ is a $SU(N)$ operator that operates on the index $I = 1 \cdots N$. Note that $\hat{O}$ does not change the total number configuration $\mathbf{n}$.

5.4 Experimental Protocol

The correlation Eq. 5.14 can be probed experimentally through the protocol summarized in Sec. 5.7. Our goal is to verify whether the prepared experimental system corresponds to a spin liquid in the doubled space. Building on insights from QED₃, the nature of the actual experimental system can be verified by comparing the “classical-classical” (CC) and the “quantum-classical” (QC) correlator proposed in Ref. [91]. Compared with other scenarios, the advantage of our proposal is that we do not require a precise simulation of the actual interacting experimental system in classical computer, which is often intractable. In our case, it suffices to simulate a noninteracting reference system. The key steps are:

- (1) Prepare state $|\Psi_0\rangle$ (fermionic atoms in optical lattice) with parent Hamiltonian H_0 . H_0 has a gapless spectrum, and can include short-range interactions.
- (2) In each experimental run, measure the fermion density on each site, obtain density configuration $\mathbf{n}_m$.
- (3) Input $\mathbf{n}_m$ in computer, find $p_{\mathbf{n}_m} = \langle \tilde{\Psi}_0 | \hat{P}_{\mathbf{n}_m} | \tilde{\Psi}_0 \rangle$, where $|\tilde{\Psi}_0\rangle$ is the *free fermion* state whose parent Hamiltonian is the noninteracting part of H_0 .
- (4) Average $p_{\mathbf{n}_m} \delta n_{0,m} \delta n_{x,m}$ over all experimental runs, and the result corresponds to the QC-correlator.

(5) Independently compute the CC-correlator using $|\tilde{\Psi}_0\rangle$. If the CC-correlator and QC-correlator exhibit the same power-law scaling, based on our understanding of QED₃, the experimentally prepared $|\rho\rangle\rangle$ should be a spin liquid state, and its scaling is captured by the QC and CC correlators.

5.5 SU(2) gauge symmetry and superconductivity

It is well-known in the field of spin liquid that (see for example Ref. [303]), a U(1) projection may actually lead to SU(2) gauge symmetry. In our context, for a class of free fermion states $|\Psi_0\rangle$ with $N = 1$ whose parent Hamiltonian H_0 only has real hopping t_{ij} , the seemingly U(1) gauge projection in the doubled space $\prod_i \hat{\mathcal{P}}(\hat{n}_{i,\uparrow} + \hat{n}_{i,\downarrow} = 1)$ would also lead to a SU(2) gauge symmetry. For example, the π -flux state with $N = 1$ in Fig. 5.1 actually has SU(2) gauge symmetry.

To expose the explicit SU(2) gauge symmetry, we no longer need the PH transformation of $H_0^*(c_{i,R})$. Instead, we just need to (trivially) relabel $L \rightarrow 1, R \rightarrow 2$, then the U(1) gauge projection $\hat{n}_{i,1} - \hat{n}_{i,2} = 0$ automatically implies a SU(2) gauge constraint:

$$c_{i,\alpha}^\dagger \tau_{\alpha,\beta}^l c_{i,\beta} = 0, \quad l = 1, 2, 3. \quad (5.16)$$

The reason is that, in the Fock space of $c_{i,1}$ and $c_{i,2}$, the only states that survive the projection are $|0, 0\rangle_i$, and $c_{i,1}^\dagger c_{i,2}^\dagger |0, 0\rangle_i$, both are singlets under a local SU(2) rotation on $[c_{i,1}; c_{i,2}]$. This SU(2) gauge structure *only* emerges when $N = 1$, i.e. when the original state $|\Psi_0\rangle$ is a spinless fermion state.

Motivated by experiments in candidate spin liquid materials, one of the spin liquid states discussed most is the “spinon Fermi surface state” [52, 111, 141, 149, 176, 177, 187, 209, 264, 320, 321]. A “spinon Fermi surface” state can be naturally produced in our context when the original state $|\Psi_0\rangle$ is a spinless fermion state with a Fermi surface. Note that in our current case, the Fermi surface state after projection would

most naturally have a $SU(2)$ gauge symmetry as long as H_0 is real, rather than a $U(1)$ gauge symmetry as was often discussed in the literature of spin liquids.

A $(2 + 1)d$ Fermi surface coupled to a dynamical bosonic field (e.g. a dynamical gauge field) is a challenging theoretical problem in general, and it has continuously attracted enormous theoretical interests and efforts [58, 123, 166, 175, 203, 204, 206, 211, 216, 217, 235, 243]. In our current case, intuitively, the $SU(2)$ gauge field would yield an attractive interaction between c_1 and c_2 , and this attractive interaction may lead to pairing instability, i.e. in the doubled Hilbert space there could be condensate of the Cooper pair operator $\hat{\Delta} = c_1 c_2$. A Cooper pair condensate leads to the following long-range correlation:

$$\begin{aligned} \langle\langle \rho_\infty | (c_1 c_2)_0 (c_1 c_2)_x^\dagger | \rho_\infty \rangle\rangle &= \text{Tr} \left(c_0^\dagger \rho_\infty c_0 c_x^\dagger \rho_\infty c_x \right) \\ &\xrightarrow{x \rightarrow \infty} \text{Const.} \end{aligned} \quad (5.17)$$

Superconductivity of matter fields coupled to a non-Abelian gauge field is referred to as color superconductivity (for reviews, see Ref. [10, 14]). The superconductivity can be obtained through an ϵ -expansion in theory [200, 325], and an extrapolation to $\epsilon = 1$. An experimental realization of the state in the quantum simulator can serve as a test for this theoretical prediction.

We have also numerically studied the projected Fermi surface state with two flavors of fermions, and indeed we observe a long range correlation of Cooper pairs (Fig. 5.2), consistent with our theoretical prediction.

In the doubled space, a Cooper pair condensate spontaneously breaks the charge $U(1)_e$ symmetry, which is also the so-called “strong” $U(1)$ symmetry of the density matrix [39, 62]. The operator $c_x^\dagger \rho c_x$ transforms nontrivially under the strong $U(1)$ symmetry, but invariant under the “weak” $U(1)$ symmetry. Hence the long-range correlation of Eq. 5.17 implies a “strong-to-weak” spontaneous breaking (SW-SSB) of the

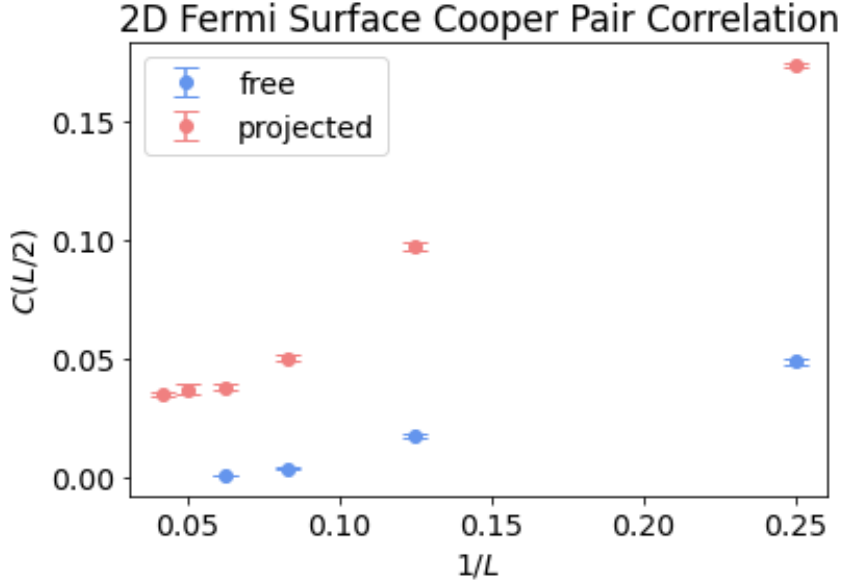


Figure 5.2: The projected free fermion wave function with two-flavors of fermions and a Fermi surface. There is a long range correlation of Cooper pair, as expected. The numerics is performed on a square lattice tight binding model with nearest neighbor and 2nd neighbor hopping, $t_2/t_1 = 0.2$. We fix the system size at $L \times L$, and measure Cooper pair correlation at $r = L/2$.

U(1) symmetry in the mixed state, a subject that has attracted enormous interest recently [171, 178, 179, 197, 224, 248]. The SW-SSB physically can be viewed as a signature of a quantum system becoming more classical, through interacting with the environment, or measurement. In the Keldysh formalism, the SW-SSB corresponds to the “locking” of the forward and backward paths through condensation of a field that couples the two paths, the condensed field precisely corresponds to the superconductor order parameter in our case.

5.6 Summary and Discussion

This chapter has demonstrated that starting with a simple fermion wave function, the steady state density matrix of the Lindbladian evolution, or the ensemble generated from strong measurement of local densities, is a Gutzwiller projected wave function in

the doubled Fock space. The Gutzwiller projection has been broadly used as a numerical trial wave function of spin liquid. We propose that this new construction of spin liquids can be realized in real experimental platforms, e.g. the Fermi gas microscope. We also predict that in certain scenarios, the constructed mixed state has a strong-to-weak spontaneous $U(1)$ symmetry breaking, which corresponds to a superconductor in the doubled Fock space. Predictions made by our understanding of spin liquids, as well as the measurement protocol for our predictions, are summarized in Sec. 5.7.

With Hermitian jump operators, the maximally mixed density matrix is always one of the steady states. For local dephasing that we considered, local density configurations are always conserved. This is why the identity matrix is not the only solution of the steady state, and the true steady state will depend on the initial state. At time t , the deviation of the particle number on a given site from the Gutzwiller projection is approximately $\exp(-\gamma t)$ (Eq. 5.4). Hence we need $t \gg 1/\gamma$ for the system to be well-approximated by the Gutzwiller wave function. If we turn on perturbations with strength g in the Lindbladian that do not commute with local density, their effect will become significant for $t > 1/g$. Hence as long as $g \ll \gamma$, there is a window at finite time where we can observe the desired physics of spin liquid.

We have focused on mixed states prepared from initial *gapless* states $|\Psi_0\rangle$. Gapped fermion states can also be very interesting – even free fermion insulators can feature nontrivial topology, e.g. the Chern insulator. The Choi-representation of the Chern insulator in the doubled space is a quantum spin Hall state, and the procedure proposed in this chapter will lead to a Gutzwiller projection of the QSH state. We leave the generalizations of this chapter to topological insulators to a future work.

5.7 Protocols and Conventions

The experimentally useful diagnostic of the algebraic Choi-spin liquid is the Rényi-2 density correlator

$$\begin{aligned} \langle\langle \rho | \delta \hat{n}_{0,\uparrow} \delta \hat{n}_{\mathbf{x},\uparrow} | \rho \rangle\rangle &\sim \langle\langle \rho | \hat{S}_0^z \hat{S}_{\mathbf{x}}^z | \rho \rangle\rangle \\ &\sim \text{Tr} \left(\rho^2 \delta \hat{n}_0 \delta \hat{n}_{\mathbf{x}} \right), \end{aligned} \quad (5.18)$$

where $\delta \hat{n}_{\mathbf{x}} = \hat{n}_{\mathbf{x}} - \bar{n}$. If the initial state is free, this quantity can be evaluated directly from the Gaussian wave function. The more useful case for experiments is when $|\Psi_0\rangle$ may include short-range interactions. Then the diagnostic should avoid assuming that the interacting state can be efficiently simulated classically. We instead compare a classical-classical (CC) correlator in a free reference state with a quantum-classical (QC) correlator obtained from experimental measurement outcomes, following the logic of Ref. [91].

Let $p_{\mathbf{n}} = \langle \tilde{\Psi}_0 | \hat{P}_{\mathbf{n}} | \tilde{\Psi}_0 \rangle$ be the probability of the density configuration $\mathbf{n}$ in the free reference state $|\tilde{\Psi}_0\rangle$, and let $q_{\mathbf{n}}$ be the probability in the experimentally prepared state $|\Psi_0\rangle$. The reference CC correlator is

$$\langle\langle \delta \hat{n}_0 \delta \hat{n}_{\mathbf{x}} \rangle\rangle_{\tilde{\rho}} = \frac{\text{Tr} \tilde{\rho}^2 \delta \hat{n}_0 \delta \hat{n}_{\mathbf{x}}}{\text{Tr} \tilde{\rho}^2} = \frac{\sum_{\mathbf{n}} p_{\mathbf{n}}^2 \delta n_0 \delta n_{\mathbf{x}}}{\sum_{\mathbf{n}} p_{\mathbf{n}}^2}. \quad (5.19)$$

Equivalently, sampling M configurations from the free reference gives

$$\langle\langle \delta \hat{n}_0 \delta \hat{n}_{\mathbf{x}} \rangle\rangle_{\tilde{\rho}} = \frac{\sum_{m=1}^M p_{\mathbf{n}_m} \delta n_{0,m} \delta n_{\mathbf{x},m}}{\sum_{m=1}^M p_{\mathbf{n}_m}}. \quad (5.20)$$

The same free-fermion samples can also be used for the $\text{SU}(N)$ bilinear in Eq. 5.15,

$$\langle\langle (\hat{O}_0)_J^I (\hat{O}_{\mathbf{x}})_I^J \rangle\rangle_{\tilde{\rho}}$$

$$= \frac{\sum_{m=1}^M \langle \tilde{\Psi}_0 | \hat{P}_{\mathbf{n}_m} (\hat{O}_0)^I_J (\hat{O}_{\mathbf{x}})^J_I | \tilde{\Psi}_0 \rangle}{\sum_{m=1}^M p_{\mathbf{n}_m}}. \quad (5.21)$$

The QC correlator is the corresponding Rényi-2 quantity for

$$\sigma = \sum_{\mathbf{n}} \hat{P}_{\mathbf{n}} |\Psi_0\rangle \langle \tilde{\Psi}_0 | \hat{P}_{\mathbf{n}}, \quad (5.22)$$

or, in doubled-space notation,

$$|\sigma\rangle\rangle = \prod_i \hat{\mathcal{P}}(n_{i,L} = n_{i,R}) |\Psi_0\rangle_L |\tilde{\Psi}_0\rangle_R. \quad (5.23)$$

It is given by

$$\langle\langle \delta \hat{n}_0 \delta \hat{n}_{\mathbf{x}} \rangle\rangle_{\sigma} = \frac{\text{Tr} \sigma^{\dagger} \sigma \delta \hat{n}_0 \delta \hat{n}_{\mathbf{x}}}{\text{Tr} \sigma^{\dagger} \sigma} = \frac{\sum_{\mathbf{n}} q_{\mathbf{n}} p_{\mathbf{n}} \delta n_0 \delta n_{\mathbf{x}}}{\sum_{\mathbf{n}} q_{\mathbf{n}} p_{\mathbf{n}}}. \quad (5.24)$$

In an experiment one samples $\mathbf{n}_m$ from the interacting state, computes $p_{\mathbf{n}_m}$ classically from $|\tilde{\Psi}_0\rangle$, and estimates

$$\langle\langle \delta \hat{n}_0 \delta \hat{n}_{\mathbf{x}} \rangle\rangle_{\sigma} = \frac{\sum_{m=1}^M p_{\mathbf{n}_m} \delta n_{0,m} \delta n_{\mathbf{x},m}}{\sum_{m=1}^M p_{\mathbf{n}_m}}. \quad (5.25)$$

If the CC and QC correlators show the same power-law decay, the natural interpretation is that the short-range interactions inherited from H_0 are irrelevant at the spin-liquid fixed point. The QC state has lower microscopic symmetry than the QQ state $|\rho\rangle\rangle$, so agreement between CC and QC is a stringent check that the experimentally prepared mixed state is controlled by the same long-distance algebraic spin-liquid physics. The Cooper-pair correlator in Eq. 5.17 can be treated in the same way: for a measured configuration $\mathbf{n}_m$, one evaluates the free-reference probability of the shifted configuration $\mathbf{n}'_m$ that differs by moving one fermion from $\mathbf{x}$ to 0.

The same sampling logic also gives routes to other gapless liquids. A spinon Fermi

surface can be probed by preparing a half-filled spinless fermion state with a Fermi surface and pairing the experimental configurations with the particle-hole transformed reference configurations,

$$|\sigma\rangle\rangle \sim \prod_i \hat{\mathcal{P}}(n_{i,L} + \tilde{n}_{i,R} = 1) |\Psi_0\rangle_L |\tilde{\Psi}_0\rangle_R. \quad (5.26)$$

For a d -wave Bose liquid, one instead pairs two anisotropic fermion states, with the projection

$$|\sigma\rangle\rangle \sim \hat{\mathcal{P}}(\hat{n}_{i,x,L} = \hat{n}_{i,y,R}) |\Psi_{0,x}\rangle_L |\tilde{\Psi}_{0,y}\rangle_R. \quad (5.27)$$

These variants are not separate thesis stories; they record how the same Choi/Gutzwiller mechanism extends beyond the Dirac spin liquid discussed in the main text.

The non-Abelian gauge structure can also be generalized. If the initial state contains kN fermions $c_{i,l,J}$, with $l = 1, \dots, k$ and $J = 1, \dots, N$, one may choose Hermitian jump operators

$$L_i^\mu = \hat{T}_i^\mu = c_{i,l,J}^\dagger T_{ll'}^\mu c_{i,l',J}, \quad (5.28)$$

where T^μ are generators of $SU(k)$. In the Choi representation the Lindbladian becomes

$$\mathcal{L} = \sum_{i,\mu} -\frac{1}{2} (\hat{T}_{i,1}^\mu - (\hat{T}_{i,2}^\mu)^T)^2 = \sum_{i,\mu} -\frac{1}{2} (\hat{T}_{i,1}^\mu - (\hat{T}_{i,2}^\mu)^*)^2. \quad (5.29)$$

The long-time state is therefore projected to an onsite $SU(k)$ singlet,

$$|\rho\rangle\rangle = \prod_i \hat{\mathcal{P}}(SU(k))_i |\rho_0\rangle\rangle, \quad (5.30)$$

and realizes a QCD₃-type Choi-spin liquid with $2N$ matter flavors coupled to an emergent $SU(k)$ gauge field.

5.7.1 Fermionic doubled-space convention

For fermions, the Choi map needs a definite convention because the two copies of the fermionic operators have mutual fermionic statistics [21]. We write a density matrix in the Fock basis as

$$\rho(c_i) = \sum_{\mathbf{n}_L, \mathbf{n}_R} \rho_{\mathbf{n}_L, \mathbf{n}_R} \mathcal{O}_{\mathbf{n}_L}^\dagger |\text{vac}\rangle\langle\text{vac}| \mathcal{O}_{\mathbf{n}_R}, \quad (5.31)$$

with $\mathcal{O}_{\mathbf{n}_L}^\dagger := \prod_{i=1}^N (c_i^\dagger)^{n_{i,L}}$ and $\mathcal{O}_{\mathbf{n}_R} := \prod_{i=N}^1 (c_i)^{n_{i,R}}$. We choose the reference state

$$|\mathcal{I}\rangle\rangle = \sum_{\mathbf{n}} \prod_{i=1}^N (c_{i,L}^\dagger)^{n_i} (c_{i,R}^\dagger)^{n_i} |\text{vac}\rangle\rangle = \prod_{i=1}^N (1 + c_{i,L}^\dagger c_{i,R}^\dagger) |\text{vac}\rangle\rangle. \quad (5.32)$$

Then $|\rho\rangle\rangle := \rho(c_{i,L})|\mathcal{I}\rangle\rangle$ takes the form

$$|\rho\rangle\rangle = \sum_{\mathbf{n}_L, \mathbf{n}_R} \rho_{\mathbf{n}_L, \mathbf{n}_R} (\mathcal{O}_{\mathbf{n}_L}^\dagger)_L (\mathcal{O}_{\mathbf{n}_R}^T)_R |\text{vac}\rangle\rangle. \quad (5.33)$$

The transpose is defined by

$$c_i^T = c_i^\dagger, \quad (c_i^\dagger)^T = -c_i, \quad (5.34)$$

$$\left(\sum_i \alpha_i c_i + \beta_i c_i^\dagger \right)^T = \sum_i \alpha_i c_i^\dagger - \beta_i c_i, \quad (5.35)$$

and satisfies

$$\mathcal{O}_L |\mathcal{I}\rangle\rangle = (\mathcal{O}^T)_R |\mathcal{I}\rangle\rangle. \quad (5.36)$$

For the pure Gaussian initial state $\rho_0 = |\Psi_0\rangle\langle\Psi_0|$ with $|\Psi_0\rangle = \prod_{k=1}^N (c_k^\dagger)^{n_k} |\text{vac}\rangle$, this convention gives

$$|\rho_0\rangle\rangle = \prod_{k=1}^N (c_{k,L}^\dagger c_{k,R}^T)^{n_k} |\text{vac}\rangle\rangle, \quad (5.37)$$

which is the convention used throughout this chapter.

Chapter 6

Strong-to-Weak Symmetry Breaking in Open Quantum Systems

This chapter studies the dynamics of strong-to-weak spontaneous symmetry breaking in open quantum systems with a conserved $U(1)$ charge. The central question is how a density matrix can retain an exact global charge while losing local recoverability of that charge information. The chapter shows that this loss is not governed by ordinary diffusion. In one dimension, the strong symmetry is not spontaneously broken at any finite time, but the nonlinear diagnostics develop a correlation length that grows linearly in time. In two dimensions, both field theory and numerical evidence support a finite-time Berezinskii–Kosterlitz–Thouless-like transition. In sufficiently high dimensions, the onset of strong-to-weak symmetry breaking is tied to the emergence of a classical continuum hydrodynamic description.

This chapter contributes the last conceptual step in the thesis. Earlier chapters use nonlinear diagnostics, doubled representations, and statistical-mechanics mappings to reveal mixed-state structure. Here those tools are connected to hydrodynamic emergence: strong-to-weak symmetry breaking becomes a sharp way to ask when microscopic charge information has become inaccessible to local recovery, so that a continuum clas-

sical description contains all locally usable information.

6.1 Introduction

Systems coupled to the outside world can undergo information-theoretic transitions, which differ in many crucial ways from equilibrium phase transitions. The quantum error correction threshold [6, 68, 153, 155, 265] is the most studied example of such a transition, but recent work has revealed a much broader class of transitions occurring in systems coupled to external environments. Beyond quantum error correction, known information-theoretic transitions include measurement-induced criticality [184, 186, 267], separability transitions [49, 50], teleportation transitions [8, 26], complexity transitions [215, 285], topological phase transitions driven by decoherence [75, 106, 194, 195, 198, 258, 270, 297], and strong-to-weak spontaneous symmetry breaking (SW-SSB) [48, 102, 117, 131, 150, 172, 180, 225, 247, 249, 314, 315, 324]. The latter two transitions are also known in the literature as mixed-state phase transitions. A crucial distinction between information-theoretic (or mixed-state) transitions and conventional phase transitions is that the former can occur at finite time: for example, an error-correcting code subjected to errors at rate γ (absent error correction) will typically accumulate enough errors by time $\sim 1/\gamma$ that the logical information is irretrievable. Under spatially local dynamics, whether unitary or dissipative, standard observables such as correlation functions—which are linear in the system’s density matrix—can only spread inside a causal light cone, so the associated correlation lengths cannot diverge at finite time. Accordingly, such observables are not singular at information-theoretic transitions. Instead, information-theoretic phase transitions can be diagnosed in one of two ways: either by the performance of information-processing tasks (such as error correction [68]), or by nonlinear functions of the system’s density matrix. In the latter case, various Rényi correlation measures [172, 190, 277, 302, 317], as well as the

so-called “Markov length” which measures the decay scale of conditional mutual information [51, 191, 218, 252, 254, 318], have emerged as powerful diagnostics of mixed state phase transitions.

SW-SSB is an information-theoretic analogue of spontaneous symmetry breaking that occurs in mixed states [180]. Unlike pure states, mixed states can be invariant under symmetries in two ways—strong and weak. Here we focus on the case of $U(1)$ symmetry, associated with a conventional scalar charge. In this context, a (globally) strongly symmetric state is one with a definite value of the total charge in the system, while a weakly symmetric state can be any incoherent mixture of distinct charge sectors. Intuitively, SW-SSB occurs when a state has a sharp value of the global charge, but this information is not stored (or retrievable) locally. An instructive example of a state with SW-SSB is the maximally mixed state in a fixed charge sector. This is an ensemble of all bit-strings with a global charge constraint. Locally, this constraint is invisible and the state looks maximally mixed; however, if part of the state is lost, one needs *global* data (namely, the total charge of the rest of the system) to correctly restore it. By contrast, a highly-structured bit string like “010101...” can be restored by a local dissipation which forces each bit to take its assigned value: in states like this, we say the strong symmetry is “intact” and not spontaneously broken.

As the example above might suggest, SW-SSB has profound connections to thermalization and (in the $U(1)$ case) hydrodynamics [131, 225]. Under generic nonintegrable dynamics on the lattice, a conserved charge will spread diffusively¹. The local density matrix looks thermal on length scales that are limited by diffusion, i.e., $x \sim \sqrt{t}$, and conventional correlation functions also only spread on this scale. It is tempting to conjecture that since the system looks maximally mixed on the diffusive scale, probes of SW-SSB will also seem to develop order on this scale. We argue that, in fact, such probes

¹One exception to this is the case of traditional $U(1)$ SSB, where a superfluid with ballistic charge modes can arise. However, such transitions (where a weak symmetry is spontaneously broken down to nothing) are not the focus of this chapter.

expose interesting dynamics that are entirely distinct from the diffusion of charge. In one dimension, SW-SSB does not occur at any finite time, but the length scale associated with SW-SSB correlators grows parametrically faster than charge spreads (specifically, it grows linearly in time). In two or more dimensions, this length scale grows still faster, diverging at a finite t_c , past which the SW-SSB correlators are quasi-long-range ordered (in $2d$) or long-range ordered (in higher d). These results are closely parallel to those for conventional superfluidity in d dimensions; we will make this analogy explicit below.

We will provide evidence for these claims in three models, which provide complementary perspectives on the universal features of $U(1)$ -symmetric open quantum dynamics. Two of these models are purely dissipative: the first is a microscopic spin-1/2 Lindblad dynamics and the second is a phenomenological decohered $U(1)$ rotor model. The spin-1/2 model maps to a well-known classical model (the “simple symmetric exclusion process”, or SSEP), and may be efficiently simulated, as we do in one and two spatial dimensions. The rotor model admits an analytical replica field theory and renormalization-group (RG) treatment, allowing a direct comparison to the numerics in $1d$ and $2d$. The third model adds coherent Hamiltonian evolution on top of the $U(1)$ -symmetric decoherence. This extension lets us discuss how finite-time SW-SSB transitions in higher dimensions can be accompanied by an emergent long-wavelength classical hydrodynamics.

In one dimension, a continuous $U(1)$ symmetry forbids spontaneous symmetry breaking at any finite evolution time. Accordingly, the SW-SSB order parameters we study do not develop true long-range order. Instead, both the mixed-state correlators and the information-theoretic diagnostics decay exponentially at long distances. While their short-distance behavior can be nonuniversal, the asymptotic decay is characterized by a correlation length that grows linearly in time. This linear growth is reminiscent of one-dimensional finite-temperature bosonic systems, where β -periodic worldlines of a conserved $U(1)$ charge lead to an exponential correlation length set by the inverse tem-

perature β . For the mixed-state correlators, the relevant length scale is controlled by boundary-condition constraints on these charge worldlines, rather than by ordinary diffusive charge spreading. Finally, from a decoding perspective on charge fluctuations, we explain why the information-theoretic diagnostics are governed by the same ballistic length scale rather than by diffusion.

In higher dimensions we expect qualitatively different behavior, including a finite-time SW-SSB transition. We develop an analytical framework based on a replica rotor field theory that applies in arbitrary spatial dimension. In one dimension, we benchmark this theory against numerics and find that it captures the correct physics, including the predicted factor-of-two relation between the correlation lengths extracted from the Rényi-1 and Rényi-2 correlators. In two dimensions, the same framework predicts a Berezinskii–Kosterlitz–Thouless (BKT)-type transition [30, 158, 159]. A key feature is that the replica constraint changes the relevant topological defects from single vortices to vortex dipoles across replicas.

By including coherent Hamiltonian evolution together with decoherence in the rotor setting, we can address the higher-dimensional regime where a finite-time SW-SSB transition is expected. We consider a generic $U(1)$ -symmetric open-system dynamics built from simple Lindblad generators, including on-site dephasing and decoherence of charge-conserving hopping terms. While at early times the density matrix can remain highly nontrivial due to coherent evolution, we argue that once SW-SSB occurs at a finite time, the long-wavelength dynamics is effectively described by a classical stochastic hydrodynamic description. In our model, the resulting long-wavelength theory is the standard Hohenberg-Halperin Model F, a well-known stochastic classical hydrodynamic description of a $U(1)$ order parameter coupled to an additional conserved density [125]. We conjecture that, for generic $U(1)$ -symmetric systems with decoherence in sufficiently high spatial dimension, a finite-time SW-SSB drives a quantum-to-classical transition, with a long-wavelength hydrodynamic description emerging.

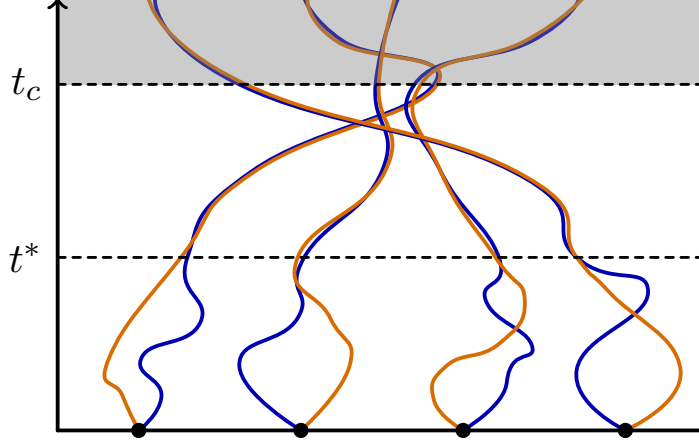


Figure 6.1: Illustration of the “two-step” dynamical process underlying SW-SSB in an open quantum system. In the density matrix representation of a quantum system, each particle has two worldlines, corresponding to the ket and bra space of the density matrix, or the forward and backward paths in the Keldysh formalism. At a crossover time scale t^* , the two worldlines merge together, meaning the density matrix becomes approximately diagonal. In two and three dimensions, at a later time, $t_c \gtrsim t^*$, there is a finite-time *phase transition* whereupon the merged worldlines intertwine, exchange and condense, losing track of their original positions, marking the onset of SW-SSB and emergent classical hydrodynamics.

We finally study the possibility of SW-SSB starting directly from classical continuum hydrodynamic equations. We verify the scalings of the Rényi correlators computed in the previous models in $d = 1$ to 3; however, we observe that there is no possibility for a finite-time transition in $d > 1$ (the SW-SSB occurs at infinitesimal times). Moreover, we find that the Markov length diverges at all times in the 1d hydrodynamic theory. These findings point to discreteness of charge as being an essential feature for the SW-SSB *transition* to occur, at finite times.

We propose that an intrinsic open quantum system undergoes a “two-step” process during decoherence. Unlike a ground-state closed quantum system, each particle in an open system must be represented by two worldlines, one in the ket and one in the bra space, or equivalently, the forward and backward paths in the Keldysh formalism. As decoherence proceeds, the system first reaches a crossover time scale t^* , at which the two worldlines effectively merge into a bound state. In this sense, the density matrix becomes

diagonal in the configuration basis. In two and three dimensions, at a later critical time t_c there is a SW-SSB phase transition, where the merged worldlines become sufficiently scrambled that information about the initial positions is irreversibly lost: the particles' initial positions become undecodable, and a classical hydrodynamic description emerges. This two-step process is depicted schematically in Fig. 6.1.

The two time scales t^* and t_c are not always well separated in a generic open quantum system. Within this chapter, two of the three quantum models we study (the decohered spin-1/2 model and the rotor model) effectively begin at times $t \gtrsim t^*$, since their density matrices are always diagonal in the occupation basis; equivalently, their dynamics can be mapped to a classical model. The third model discussed in Sec. 6.5 includes the regime $t < t^*$ in Fig. 6.1, and in that case t^* and t_c are not clearly separated.

We note that an ordinary zero-temperature superfluid (a phase that spontaneously breaks the strong symmetry to nothing) corresponds to a different situation: the ket and bra worldlines scramble and condense separately, without first merging.

We begin in Sec. 6.2 with an overview of relevant observables and the worldline picture for charge-conserving dynamics. Next, in Sec. 6.3, we discuss our decohered spin-1/2 model and provide numerical results for linear and nonlinear observables in $1d$ and $2d$. In Sec. 6.4, we discuss our decohered rotor model and provide analytical results in $1d$ and $2d$. In Sec. 6.5, we define a richer $U(1)$ -symmetric model and show how Model-F dynamics arises after a finite-time phase transition. In Sec. 6.6 we directly analyze the hydrodynamics as an effective classical description of a system with SW-SSB. Finally, in Sec. 6.7, we provide a discussion and outlook, and discuss consequences of our results for the onset of a continuum hydrodynamic description in $U(1)$ -symmetric many-body systems.

Model/Theory	Dim.	Key results	Section
spin-1/2	$1d$	Diffusion, No SW-SSB short-range CMI, efficient charge decoding	Sec. 6.3.2
	$2d$	SW-SSB, BKT-like transition	Sec. 6.3.3
rotor	$1d$	No SW-SSB, $\xi^{(1)}/\xi^{(2)} = 2$	Sec. 6.4.1
	$2d$	SW-SSB, BKT-like transition, $\Delta^{(2)}/\Delta^{(1)} = 2$	Sec. 6.4.2
including Hamiltonian	$\geq 2d$	emergent Model F	Sec. 6.5
hydrodynamics	$1d$	No SW-SSB, long-range CMI	Sec. 6.6
	$\geq 2d$	SW-SSB	

Table 6.1: Summary of models, dimensions, results, and the corresponding sections.

6.2 Diagnostics and the worldline picture

6.2.1 Observables

6.2.1.1 SW-SSB measures

Here we focus exclusively on systems with $U(1)$ symmetry, or equivalently with a single conserved charge $\hat{Q}$. A state is strongly symmetric if it satisfies $e^{i\theta\hat{Q}}\hat{\rho} = e^{i\theta Q_0}\hat{\rho}$; i.e., if $\hat{\rho}$ has a definite charge value Q_0 . A state is weakly symmetric if $e^{i\theta\hat{Q}}\hat{\rho}e^{-i\theta\hat{Q}} = \hat{\rho}$; i.e., if $\hat{\rho}$ contains no off-diagonal matrix elements between states of different charge. Dynamics preserves strong (weak) symmetry if a strongly (weakly) symmetric density matrix retains this symmetry at all times. Under strongly symmetric quantum channels, the system does not exchange charge with its environment. If one decomposes these channels in Kraus form, each Kraus operator separately commutes with $\hat{Q}$. We will take the charge $\hat{Q}$ to be a sum of local terms, $\hat{Q} = \sum_r \hat{q}_r$, and restrict ourselves to spatially local dynamics. Under spatially local dynamics with a strong $U(1)$ symmetry, charge typically spreads diffusively: $\langle \hat{q}_r(t) \hat{q}_{r'}(0) \rangle = t^{-d/2} \phi(|r - r'|/\sqrt{t})$, where $\phi(\cdot)$ is a scaling function that is asymptotically Gaussian. As we will argue, this diffusive hydrodynamics does not control the length- or timescales associated with SW-SSB.

Physically, $U(1)$ SW-SSB is a statement about the *distinguishability* of quantum states, i.e., the number of measurements that one needs to tell two states apart. The distinguishability of two general quantum states $\hat{\rho}$ and $\hat{\sigma}$ is canonically measured by the fidelity, $\mathcal{F}(\hat{\rho}, \hat{\sigma}) = \left(\text{Tr} \sqrt{\sqrt{\hat{\rho}} \hat{\sigma} \sqrt{\hat{\rho}}} \right)^2$. The “fidelity correlator” $C_{ij}^{(F)} \equiv \mathcal{F}(\hat{\rho}, \hat{O}_{ij} \hat{\rho} \hat{O}_{ij}^\dagger)$ [180] measures the fidelity between the reference state $\hat{\rho}$ and a state $\hat{O}_{ij} \hat{\rho} \hat{O}_{ij}^\dagger$ derived from it by moving a charge from site i to site j . Thus, if $\hat{S}_i^+$ and $\hat{S}_i^-$ are charge raising and lowering operators, one can choose $\hat{O}_{ij} = \hat{S}_i^+ \hat{S}_j^- + \hat{S}_i^- \hat{S}_j^+$. A state is said to exhibit SW-SSB if $\lim_{|i-j| \rightarrow \infty} C_{ij}^{(F)} \neq 0$, i.e., if moving a particle across the system does not lead to a sharply distinguishable state. Interestingly, we will find that states are indistinguishable even after moving charges across distances much larger than the diffusive length scale. This indicates that SW-SSB is not simply a measure of localization of charge information.

The fidelity correlator is of particular interest because long-range order in the fidelity correlator is stable under strongly symmetric finite-depth local channels [254]. Thus, it is a suitable diagnostic of mixed-state phases, which may be organized in terms of two-way local, short-depth channel connectivity [254] (analogous to the classification of pure-state phases in terms of short-depth unitary connectivity [114]). In practice, fidelities are hard to compute, so one works with proxies for the fidelity correlator. The most robust of these diagnostics is the Rényi-1 correlator,

$$C^{(1)}(i, j) \equiv \text{Tr} \left(\sqrt{\hat{\rho}} \hat{O}_{ij} \sqrt{\hat{\rho}} \hat{O}_{ij}^\dagger \right). \quad (6.1)$$

which is tightly related to $C^{(F)}$ but is easier to compute. Since $C^{(F)}(i, j) \leq C^{(1)}(i, j) \leq [C^{(F)}(i, j)]^{1/2}$, long-range order in $C^{(1)}(i, j)$ provides a stable diagnosis of SW-SSB, and we adopt it throughout this chapter [302]. When $C^{(1)}$ decays exponentially, we denote its correlation length by $\xi^{(1)}(t)$, where t is the evolution time.

Another commonly-used diagnostic is the Rényi-2 correlator. For our operator $\hat{O}_{ij}$

acting on sites i and j , it is defined as

$$C^{(2)}(i, j) \equiv \frac{\text{Tr}(\hat{O}_{ij} \hat{\rho} \hat{O}_{ij}^\dagger \hat{\rho})}{\text{Tr}(\hat{\rho}^2)}. \quad (6.2)$$

This quantity is convenient to calculate, but has no direct information-theoretic interpretation and does not satisfy the same stability properties as the fidelity and Rényi-1 correlators under strongly symmetric finite-depth channels. As a diagnostic of a mixed-state phase, therefore, Rényi-2 correlators are merely suggestive. The Rényi-2 correlator is also the simplest example of the replica generalization of the Rényi-1 correlator, a formalism often used in analytical studies of SW-SSB, as discussed in Sec. 6.4. When $C^{(2)}$ decays exponentially, we denote its correlation length by $\xi^{(2)}(t)$.

Much of our numerical analysis for the spin-1/2 model concerns density matrices that are diagonal in the S^z basis, $\hat{\rho} = \sum_s P(s) |s\rangle \langle s|$, where s denotes a spin configuration and $P(s)$ is a classical probability distribution. It is convenient to introduce the associated vectors

$$|\rho\rangle \equiv \sum_s P(s) |s\rangle, \quad |\sqrt{\rho}\rangle \equiv \sum_s \sqrt{P(s)} |s\rangle. \quad (6.3)$$

The vector $|\rho\rangle$ has a nonstandard normalization, since $\langle \rho | \rho \rangle = \sum_s P(s)^2$ is generally not equal to one. Instead, $\langle \mathbb{I} | \rho \rangle = 1$ (with $\langle \mathbb{I} | = \sum_s \langle s |$). In contrast, $\langle \sqrt{\rho} | \sqrt{\rho} \rangle = \sum_s P(s) = 1$.

For the operator insertion $\hat{O}_{ij} = \hat{S}_i^+ \hat{S}_j^-$, define s' as the configuration obtained from s by shifting only the two spins at i and j by $s'_i = s_i + 1$ and $s'_j = s_j - 1$ while all other sites are unchanged. Then the diagonal-sector correlators reduce to

$$C^{(2)}(i, j) = \frac{\langle \rho | \hat{S}_i^+ \hat{S}_j^- | \rho \rangle}{\langle \rho | \rho \rangle} = \frac{\sum'_s P(s) P(s')}{\sum_s P(s)^2}, \quad (6.4)$$

and

$$C^{(1)}(i, j) = \langle \sqrt{\rho} | \hat{S}_i^+ \hat{S}_j^- | \sqrt{\rho} \rangle = \sum_s' \sqrt{P(s)P(s')}. \quad (6.5)$$

Here $\sum_s'$ denotes a restricted sum over configurations s for $s_i = -\frac{1}{2}$ and $s_j = +\frac{1}{2}$ so that s' is well-defined. Note that in this diagonal (classical) sector, the fidelity correlator is identical to the Rényi-1 correlator. Furthermore, these definitions coincide precisely with the Bhattacharyya coefficient $B(P(s), P'(s))$, where we have defined $P'(s) = P(s')$, which is a standard metric for the similarity of classical probability distributions [73]. The Bhattacharyya coefficient is related to a more familiar metric of similarity, the Kullback-Leibler divergence $D_{\text{KL}}(P\|P') \equiv \text{Tr}(P(\log P - \log P'))$, by the inequality $D_{\text{KL}}(P\|Q) \geq -\log B(P, Q)$. The latter quantity is an estimate of the number of samples one would need from distribution P' to determine if it were distinct from P . For Gaussian distributions, the inequality above is saturated, so the two measures carry the same information.

6.2.1.2 Conditional mutual information

So far, we have focused on direct order parameters for SW-SSB: these incorporate our knowledge of the expected pattern of symmetry breaking via the choice of the operator $\hat{O}_{ij}$. For equilibrium phase transitions, the mutual information between two distant regions offers an observable-agnostic way of probing a diverging length-scale. The mutual information can be defined as follows. For a region $\mathcal{R}$, we define the reduced density matrix $\hat{\rho}_{\mathcal{R}} \equiv \text{Tr}_{\mathcal{R}^c}(\hat{\rho})$ and its von Neumann entropy

$$S_{\mathcal{R}} \equiv S(\hat{\rho}_{\mathcal{R}}) = -\text{Tr}(\hat{\rho}_{\mathcal{R}} \log \hat{\rho}_{\mathcal{R}}). \quad (6.6)$$

The mutual information is defined as $I(A : B) \equiv S_A + S_B - S_{AB}$. It upper-bounds all connected correlations between operators supported in regions A and B [306]. Like in-

dividual correlation functions, the mutual information is bounded by the Lieb-Robinson light cone [188], and thus cannot diverge at finite time. In recent work Ref. [252] it was argued that an analog of the mutual information for information-theoretic transitions is the conditional mutual information (CMI). The CMI for three regions A , B , and C is

$$I(A : C|B) = I(A : BC) - I(A : B) = S_{AB} + S_{BC} - S_B - S_{ABC}, \quad (6.7)$$

where A and C are separated by the intermediate region B . In many cases the CMI is expected to be short ranged, decaying exponentially in the separation between A and C :

$$I(A : C|B) \propto f(|A|, |C|) e^{-R_B/\xi^M}. \quad (6.8)$$

Here R_B is the distance separating A from C , which in $1d$ is $R_B = |B|$. The length scale ξ^M is called the Markov length [252, 254]. Provided the Markov length stays finite as a state is evolved, *no* mixed-state phase transition can occur: any two mixed states that can be connected via a finite-Markov-length (FML) path are in the same mixed-state phase.

The Markov length has an operational meaning in terms of recoverability [254]. States within the same mixed-state phase are connected by local quantum channels, so one can map between them in both directions. Across a phase transition, this two-way connectivity is lost. In particular, an initial state evolved under a non-FML path cannot be recovered using a short depth circuit comprised of only local channels. This can be viewed as an inference or decoding problem relevant to error correction [253]. Starting from a state with no errors, decoherence evolves it to a mixed state, and recoverability asks whether one can decode back. The amount of information required for successful recovery is controlled by the Markov length ξ^M .

Mixed state phase transitions associated with diverging Markov length might encompass a broader class of transitions than those associated with SW-SSB. However, the Markov length upper-bounds the length-scale associated with the decay of the fidelity correlator, by the following argument (made precise in Theorem 4 of Ref. [180]). Suppose the Markov length is ξ^M . We now consider moving a particle from site i to site j with $|i - j| \gg \xi^M$, and consider the fidelity between the initial state ρ and the state with the particle moved, ρ' . Recall that fidelity increases monotonically under quantum channels. Consider erasing site i , and trying to recover it using all information available within a distance ξ^M of site i . Since $\xi^M \ll |j - i|$, the recovery map does not have access to the charge on site j , so it must act the same way regardless of the charge on that site. Therefore it will recover ρ and ρ' to states of distinct total charge. The fidelity between these is zero, so (by monotonicity) the fidelity between the initial states leading to these is also zero (to within exponential error). Thus the SW-SSB correlators all decay on the scale of ξ^M or faster—as we will see below, in many cases they in fact decay strictly faster.

In our numerics we will study the time evolution of the Markov length as a probe of SW-SSB. The standard geometry used in discussions of mixed-state phases and recoverability takes $A \cup B \cup C$ to cover the entire system. In one dimension, this corresponds to three consecutive intervals A , B , and C that partition the full chain [Fig. 6.2(a)]. Here, instead, we choose A , B , and C to be consecutive intervals that occupy only part of the chain, with $|A|$ and $|C|$ short (typically $O(1)$, often a single site), and we vary the length R_B [Fig. 6.2(b)]. Equivalently, one may view this as a two-point correlator in which A and C are separated by B while the rest of the system is traced out. Although the two geometries lead to conceptually different quantities, once there is no true long-range order they are expected to track the same characteristic length scale.

To build intuition for these nonlinear diagnostics, we next re-express them in a world-line representation. This viewpoint also clarifies how nonlinear observables differ from



Figure 6.2: CMI geometries in $1d$. (a) In the first case, regions A , B , and C cover the entire system. This is the standard geometry for CMI in the context of $1d$ mixed state phases and recoverability. (b) In the second case, the three regions do not cover the entire system, with regions A and C typically of size $O(1)$. In both cases, we denote the separation between regions A and C as $R_B = |B|$.

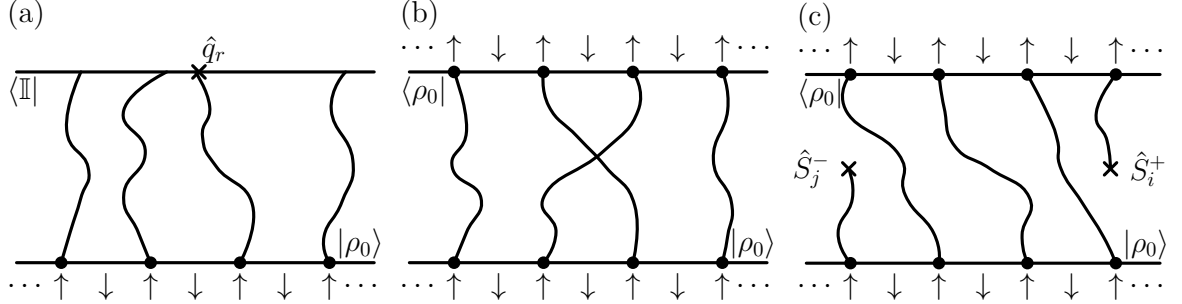


Figure 6.3: Worldline representation of the local $U(1)$ charge dynamics (with time running upward), illustrated for the spin-1/2 model. The gray arrows indicate the Néel initial state $\hat{\rho}_0$. Panel (a) shows the local charge expectation value, where the final-time boundary is free and is probed by the insertion of a charge operator $\hat{q}_r$. Panels (b) and (c) show the corresponding representation for the Rényi-2 correlator $C^{(2)}(i, j)$, where the temporal boundaries are both $\hat{\rho}_0$ at the bottom and the top. They correspond to the denominator and numerator of $C^{(2)}(i, j)$, respectively; in (c) the operator insertions $\hat{S}_i^+$ and $\hat{S}_j^-$ create and annihilate a defect worldline.

linear observables, even though both are evaluated from the same underlying density matrix.

6.2.2 The worldline picture

It is useful to distinguish observables that are *linear* in the density matrix, such as the local charge density $\langle \hat{q}_r(t) \rangle$, from the *nonlinear* observables discussed above. In $U(1)$ -symmetric open dynamics, the conserved charge typically exhibits diffusive hydrodynamics [131, 225]. This behavior can be understood from the structure of the $U(1)$ -symmetric Lindbladian, which supports a quadratic long-wavelength Goldstone mode [225], and it can sometimes also admit an explicit mapping to an effective random-walking classical

dynamics in the diagonal sector as described in the next section for our first model.

Given the charge dynamics, one might wonder whether mixed-state correlators and information-theoretic quantities such as the CMI are also controlled by the same diffusion with dynamical exponent $z = 2$. The worldline representation (Fig. 6.3) provides a simple explanation for why this need not be the case. Here the worldlines represent a bound-state of the forward and backward paths, that were illustrated in Fig. 6.1 for times $t > t^*$. The dynamics is a path-integral sum over all trajectories of the local $U(1)$ charge (for a diagonal density matrix), with weights set by the local update rules and boundary conditions fixed by the observable.

Much of our analysis focuses on the diagonal sector, where $\hat{\rho}$ is diagonal in the $\hat{q}_r$ basis and can be viewed as a classical probability distribution over charge configurations. In this setting, the charge expectation value can be written in the vectorized notation as

$$\langle \hat{q}_r(t) \rangle = \text{Tr} (\hat{q}_r \hat{\rho}(t)) = \langle \mathbb{I} | \hat{q}_r e^{\mathcal{L}^{\text{diag}} t} | \rho_0 \rangle, \quad (6.9)$$

where $\langle \mathbb{I} |$ is the vector with all components equal to one in the configuration basis which implements the trace, and $\mathcal{L}^{\text{diag}}$ is the effective generator governing the diagonal-sector evolution. In the worldline picture, this belongs to a *free* boundary condition at the final time. Worldlines are not required to match prescribed endpoints, and the resulting behavior is therefore governed by ordinary diffusion. Fig. 6.3(a) illustrates this structure for the spin-1/2 model, where we are taking an initial pure state $\hat{\rho}_0$ corresponding to a Néel arrangement diagonal in the $\hat{S}_j^z$ basis.

By contrast, mixed-state correlators impose a different boundary condition. In particular for the Rényi correlators, even though the *bulk* evolution is generated by the same diagonal-sector updates from $\mathcal{L}^{\text{diag}}$, the observable fixes both the initial and the final boundary conditions to be $\hat{\rho}_0$. As a consequence, worldlines cannot simply wander

freely. They must be paired up consistently between the two boundaries, and exchange processes become well defined. This change in boundary conditions is the basic reason why these quantities are not described by simple diffusion.

Fig. 6.3(b,c) illustrates the worldline picture for the spin-1/2 Rényi-2 correlator $C^{(2)}(i, j)$. In the denominator [Fig. 6.3(b)], the initial and final boundary conditions are both set by $\hat{\rho}_0$. The doubled construction produces a bulk of duration $2t$, and worldlines are constrained to connect the bottom boundary to the top boundary rather than terminate freely. In the numerator [Fig. 6.3(c)], there is an additional insertion at time t , inside the bulk: $\hat{S}_i^+$ creates a defect worldline at site i , while $\hat{S}_j^-$ annihilates it at site j . Because of this additional creation/annihilation, other worldlines must rearrange to accommodate the defect, and the correlator measures the cost of this rearrangement. The Rényi-1 correlator has the same boundary-conditions, but with a different probabilistic weight.

This structure is reminiscent of the worldline representation of quantum fluids. In that setting, superfluidity is tied to the appearance of large cycles, where many worldlines permute and become indistinguishable. In a normal fluid, such exchanges are typically rare and worldlines remain mostly separated. The key difference for us is the boundary condition and the ensemble. In the usual finite-temperature path integral one computes $\text{tr}(e^{-\beta\hat{H}})$, so the trace imposes β -periodic boundary conditions and automatically sums over all worldline permutations. Also, one often works in a grand-canonical ensemble, so the particle number can fluctuate. In our case, the boundaries are fixed by $\hat{\rho}_0$ at both ends. This effectively selects a canonical sector in which the number of worldlines is fixed by the boundary conditions rather than being summed over, but the boundary conditions being the same at the initial and final times automatically sums over all worldline permutations.

For periodic boundary conditions in space, the worldline ensemble splits into sectors labeled by an integer winding number W . Here W counts the net number of worldlines

that wind around the periodic direction during the evolution. In equilibrium path integrals, the second moment of winding-number fluctuations, $\langle W^2 \rangle$, is proportional to the superfluid stiffness [236], and a nonzero $\langle W^2 \rangle$ in the thermodynamics limit quantifies the world-line indistinguishability. In our setting, instead of a β -periodic temporal boundary condition, the bottom and top boundaries are fixed to the same state $\hat{\rho}_0$. Nevertheless, one can still organize worldline trajectories by their spatial winding sectors and study the statistics of W . This motivates us to consider winding-number observables for both the Rényi-2 and Rényi-1 constructions. The distinction between them is the probabilistic weight assigned to trajectories, which we spell out in the next section. These winding observables are especially important in two dimensions, where the stiffness inferred from winding-number fluctuations captures the BKT-like transition.

6.3 Decohered spin-1/2 model

The first model we consider is a minimal interacting $U(1)$ -symmetric Lindblad dynamics acting on a d -dimensional spin-1/2 system. Decoherence is generated by the nearest-neighbor jump operators $\hat{L}_{ij} = \hat{S}_i^+ \hat{S}_j^-$, with the Lindbladian taking the form

$$\mathcal{L}(\rho) = \gamma \sum_{\langle i,j \rangle} \left[\hat{L}_{ij} \hat{\rho} \hat{L}_{ij}^\dagger - \frac{1}{2} \{ \hat{L}_{ij}^\dagger \hat{L}_{ij}, \hat{\rho} \} + (i \leftrightarrow j) \right], \quad (6.10)$$

where $\langle i, j \rangle$ denotes a pair of adjacent sites. This dynamics has a strong $U(1)$ symmetry because each jump operator commutes with $\hat{U}_\theta = e^{i\theta \sum_j \hat{S}_j^z}$. The strong symmetry also implies a weak symmetry, enabling the possibility of SW-SSB during time evolution starting from a $U(1)$ -symmetric initial state (i.e., a state with fixed initial charge).

A simplifying feature of this model is that the diagonal and off-diagonal sectors (in the charge basis) of the density matrix evolve independently. A short calculation, provided in Appendix 6.8.1, shows that the diagonal density matrix $|\rho\rangle$ evolves under

the effective Lindblad operator

$$\mathcal{L}^{\text{diag}} = 2\gamma \sum_{\langle ij \rangle} \left(\hat{\mathbf{S}}_i \cdot \hat{\mathbf{S}}_j - \frac{1}{4} \right), \quad |\rho_t\rangle = e^{\mathcal{L}^{\text{diag}} t} |\rho_0\rangle, \quad (6.11)$$

where $\langle ij \rangle$ denotes nearest-neighbor sites i and j . Furthermore, as discussed in Appendix 6.8.1, all off-diagonal sectors feature a finite dissipative gap relative to the diagonal sector. Therefore the late-time dynamics is governed only by the diagonal evolution. In the remainder of this section, we restrict our attention to initial states diagonal in the charge basis.

Within the diagonal sector, the Lindblad dynamics is therefore equivalent to imaginary-time evolution under the *ferromagnetic* Heisenberg model [107]. Note that $-\mathcal{L}^{\text{diag}}$ is positive semi-definite and has a lowest eigenvalue zero, so gives a well-defined Markovian evolution on diagonal density matrices. The ferromagnetic Heisenberg model supports quadratic Goldstone modes, which is even true for higher spin S [225, 298], implying diffusive scaling for linear observables.

The diagonal dynamics can be cast in another form which is particularly instructive. Using the identity for two spins, $\hat{\mathbf{S}}_i \cdot \hat{\mathbf{S}}_j = \frac{1}{2} \text{SWAP}_{ij} - \frac{1}{4} \mathbb{I}$, where $\text{SWAP}_{i,j}$ exchanges the neighboring spins, we obtain

$$\begin{aligned} e^{\mathcal{L}^{\text{diag}} t} &= \exp \left(\gamma t \sum_{\langle i,j \rangle} (\text{SWAP}_{i,j} - 1) \right) = \lim_{\delta t \rightarrow 0} \exp \left(\gamma \delta t \sum_{\langle i,j \rangle} (\text{SWAP}_{i,j} - 1) \right)^{t/\delta t} \\ &= \lim_{\delta t \rightarrow 0} \left(\prod_{\langle i,j \rangle} \left(1 - \gamma \delta t + \gamma \delta t \text{SWAP}_{i,j} \right) \right)^{t/\delta t} \\ &= \int \rho_{[0,t]}(d\omega) \prod_{(\langle i,j \rangle, t) \in \omega} \text{SWAP}_{i,j}. \end{aligned} \quad (6.12)$$

In other words, within a short time window δt on a single bond $(i, i+1)$, the operator $\text{SWAP}_{i,i+1}$ is applied with average rate γ . The resulting history can be visualized as a ladder where swap events are rungs that are placed randomly in time and space according

to a Poisson process. We denote a realization of this stochastic update pattern by ω . Identifying an occupied site $\bullet$ with $\uparrow$ and an empty site $\circ$ with $\downarrow$, the dynamics leads to hopping of particles with a rate γ , modulo the exclusion constraint that at most one particle can occupy a given site. Since the left and right hopping rates are symmetric, this results in the classical symmetric simple exclusion process [273] (SSEP) in the diagonal charge sector. This dynamics is illustrated in Fig. 6.3. In the remainder of this section, we will study the dynamics of SW-SSB in this model in one and two dimensions. We impose the strong symmetry starting from an initial state with fixed charge; specifically, we consider dynamics of the initial state $|\rho_0\rangle = |\text{Néel}\rangle$, in both one and two dimensions. In the particle language, this is the half-filled state with particles on every other site.

Our results are summarized as follows:

- In $1d$, SW-SSB does not occur at finite time, but the dynamics of linear and nonlinear observables are parametrically different. Charge spreads diffusively, but length scales associated with nonlinear observables ($\xi^{(1)}$ and $\xi^{(2)}$ for Rényi correlations, the Markov length ξ^M for the CMI, and the region size ξ^{dec} required for accurate charge decoding) all grow ballistically. Intuitively, successful recovery of information becomes increasingly difficult at late times, but is always possible given access to a sufficiently large snapshot of the system.
- In $2d$, we find signatures of a BKT-like SW-SSB transition at finite time by evaluating Rényi susceptibilities and stiffness. We expect the associated state recovery and decoding tasks to fail in $2d$ past this transition.

6.3.1 Numerical method: sampling trick

In the following sections, we use the $U(1)$ -symmetric spin-1/2 model as a numerically accessible setting. Throughout, we take the initial state $\hat{\rho}_0$ to be a Néel state. In most of the cases we consider, the dynamics is effectively classical in the sense that the density

matrix remains diagonal in the charge basis [see Eq. (6.3)]. We therefore treat $\hat{\rho}_t$ as a classical probability distribution over charge configurations, evolved by a diagonal-sector generator, $|\rho_t\rangle = e^{\mathcal{L}^{\text{diag}}_t}|\rho_0\rangle$. We simulate this evolution using time-evolving block decimation (TEBD) with matrix product states (MPS) in one dimension, and quantum Monte Carlo (QMC) in two dimensions. In particular, for the $2d$ case we use a worm algorithm to sample and average over spacetime trajectories with the specified boundary conditions.

Rényi-2 observables are straightforward in both approaches. In $1d$ they reduce to standard MPS contractions. In $2d$ they correspond to a spacetime trajectory ensemble with the temporal boundary condition fixed as $\hat{\rho}_0$ at both the bottom and the top, over a total temporal extent $2t$. Operator insertions are implemented directly in both cases, so Rényi-2 correlators are easy to evaluate. In two dimensions, winding-number observables are likewise straightforward to measure within the same trajectory ensemble.

Rényi-1 observables are less direct because they involve $|\sqrt{\rho}\rangle$ and therefore do not reduce to a single standard contraction. There are two ways to overcome this issue. First, one can use tensor cross interpolation (TCI) to infer $|\sqrt{\rho}\rangle$ by sampling $|\rho\rangle$, after which Rényi-1 correlations may be computed in the standard way [228]. Second, one can sample Rényi-1 correlations directly from $|\rho\rangle$ by the following approach. In $1d$, this can be done by rewriting the configuration sum as an average over the spins away from i and j . Denoting by $\bar{s}_{ij}$ the configuration on all sites except i and j , one can write

$$C^{(1)}(i, j) = \sum_{\bar{s}_{ij}} P(\bar{s}_{ij}) \sqrt{P(\downarrow_i \uparrow_j | \bar{s}_{ij}) P(\uparrow_i \downarrow_j | \bar{s}_{ij})}. \quad (6.13)$$

Then, $\bar{s}_{ij}$ can be sampled from its marginal $P(\bar{s}_{ij})$ by sequential sampling of the MPS [191]. For each sampled $\bar{s}_{ij}$, one fixes all spins except those at i and j , and evaluates the two conditional probabilities, and then averages the resulting estimator over samples. In this chapter, we primarily use TCI, but also verify the results using

direct sampling of $|\rho\rangle$.

We evaluate the CMI using the same sampling method [191]. In particular, we sample a configuration s_B on region B from its marginal, and for each sample we compute the conditional distribution on A and C needed to estimate the CMI using $I(A : C|B) = \sum_{s_B} P(s_B) I(A : C|B = s_B)$.

In $2d$, we evaluate Rényi-1 and related disorder-averaged quantities using a two-step sampling procedure. We first generate an outcome configuration $\{s\}$ by evolving the diagonal-sector classical dynamics defined by $\mathcal{L}^{\text{diag}}$ up to time t , which samples $\{s\}$ with the correct weight. Conditioned on this sampled $\{s\}$, we then run a second QMC with fixed temporal boundaries given by $|\rho_0\rangle$ at the bottom and $|s\rangle$ at the top, and we measure the desired conditional quantities within this fixed-boundary ensemble. This gives the probability ratio $P(s')/P(s)$ entering the estimator for $C^{(1)}(i, j)$. We also compute a stiffness-like quantity by measuring the integrated current (total displacement) in a chosen direction within the fixed-boundary QMC run and then averaging its fluctuations over the outer samples of $\{s\}$. In the classical limit, this reduces to the winding-number definition used in Ref. [281].

6.3.2 1d results

6.3.2.1 Diffusion and Rényi correlations

In this section, we present numerical data for the evolution over time of local charge, Rényi-2 correlations, Rényi-1 correlations, and the infinite MPS Rényi-2 transfer matrix gap. As is well known, the spreading of local charge displays a diffusive length scale $\xi^D(t) \sim \sqrt{t}$. Nevertheless, the other observables all indicate the presence of ballistic length scales $\xi^{(1)}(t) \sim t$ and $\xi^{(2)}(t) \sim t$ for Rényi-1 and Rényi-2 correlations respectively. Notably, we find numerically that $\xi^{(1)}(t) = 2\xi^{(2)}(t)$ in the spin-1/2 model, matching an analytical result obtained for the 1d decohered rotor model in Sec. 6.4.1.

As expected from the Mermin–Wagner theorem, in one dimension a continuous $U(1)$ symmetry does not exhibit symmetry breaking after any finite time evolution, neither in the strong nor in the weak sense. Therefore, SW-SSB is not observed at finite times, and long-range order can emerge only in the asymptotic limit of infinite time. This is reflected in our numerical results for the nonlinear observables, whose correlation lengths diverge only after extensive time.

Although there is no SW-SSB at finite times, we still observe the expected diffusive spreading of the conserved charge. This is directly seen from domain-wall melting starting from a sharp domain-wall initial state $|\uparrow \dots \uparrow \downarrow \dots \downarrow\rangle$. The magnetization profiles $\langle S_x^z \rangle$ at different times collapse under the diffusive rescaling $x/\sqrt{t}$, indicating that $\langle S_x^z \rangle = f(x/\sqrt{t})$ for suitable times (after a constant initial onset time of diffusion, and before the boundaries become relevant). These data are given in Fig. 6.4 with $L = 100$ and $\gamma = 0.1$.

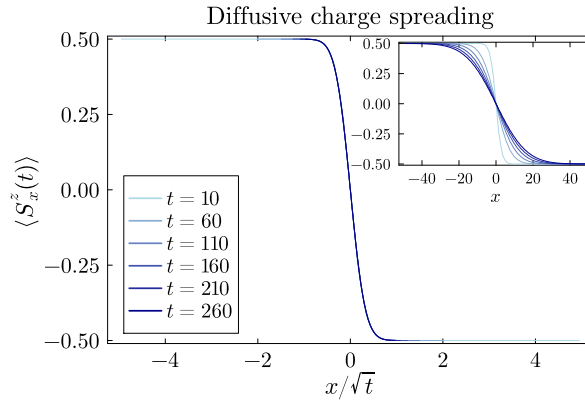


Figure 6.4: Spatial charge profiles over time from a domain wall initial state, with $L = 100$ and $\gamma = 0.1$ from $t = 10$ to $t = 260$. In the inset, the horizontal axis is not scaled and it is apparent that the initial domain wall melts over time, with charge spreading from the left to the right. When the horizontal axis is rescaled by $\sqrt{t}$, the charge profiles at different times all collapse to one curve. This indicates a scaling form $\langle S_i^z(t) \rangle = f(x/\sqrt{t})$ over this time range, indicative of diffusion. In both cases, $x = 0$ is taken to be halfway between site 50 and site 51.

The behavior of nonlinear observables is markedly different. In particular, we find that the Rényi-1 and Rényi-2 correlators decay exponentially in space, with a decay

length that grows *linearly* with the evolution time. This behavior is reminiscent of one-dimensional free bosonic systems at finite temperature. There, thermal fluctuations introduce a finite thermal length scale, so correlators such as $\text{Tr}(b_0 b_r^\dagger e^{-\beta H})$ decay exponentially at large r , with a correlation length ξ^β proportional to β . As discussed in the Sec. 6.2.2, the Rényi-2 correlation length is akin to the finite-temperature correlation length of the Heisenberg model. We remark that if the environment were non-Markovian such that symmetric feedback could be applied to the system, the system could develop standard SSB instead of SW-SSB in the steady state [116].

Numerical data for Rényi-1 and Rényi-2 correlators in our spin-1/2 model are given in Fig. 6.5(a) for $L = 128$ and $\gamma = 0.1$. We observe that $C^{(1)}(x; t)$ and $[C^{(2)}(x; t)]^{1/2}$ are linear on a semi-log plot when x is sufficiently large (but not so large that boundary effects arise), indicating that $C^{(1)}(x; t) \propto e^{-x/\xi^{(1)}(t)}$ and $C^{(2)}(x; t) \propto e^{-x/\xi^{(2)}(t)}$. More specifically, we observe that the data collapse reasonably well under the ballistic rescaling x/t . Both $C^{(1)}(x; t)$ and $[C^{(2)}(x; t)]^{1/2}$ are proportional to a scaling form $f(x/t)$, which implies that $\xi^{(1)}(t)$ and $\xi^{(2)}(t)$ both grow ballistically. Furthermore, since $C^{(1)}(x; t)$ and $[C^{(2)}(x; t)]^{1/2}$ have the same slope, it follows that $\xi^{(1)}(t) = 2\xi^{(2)}(t)$. The Rényi-1 data in Fig. 6.5(a) are obtained using TCI, and we show in Appendix 6.8.2 that the results are consistent with data obtained from direct sampling.

The long-distance decay of Rényi-2 correlators is directly related to the gap of the MPS transfer matrix, similar to the case of standard observables in quantum ground states [38]. We can work directly with infinite MPS (iMPS) so that the transfer matrix is translationally invariant. The largest eigenvalue of the transfer matrix, if appropriately normalized, will be 1; let the next-largest eigenvalue be λ_2 . Then, (Rényi-2) correlations at sufficiently large distance r will be suppressed by λ_2^r , from which we can infer a correlation length $\xi = -(\log \lambda_2)^{-1}$. The growth of this correlation length over time is given in Fig. 6.5(b), and is clearly ballistic.

In Sec. 6.4.1, we study both the Rényi-1 and Rényi-2 observables analytically in a

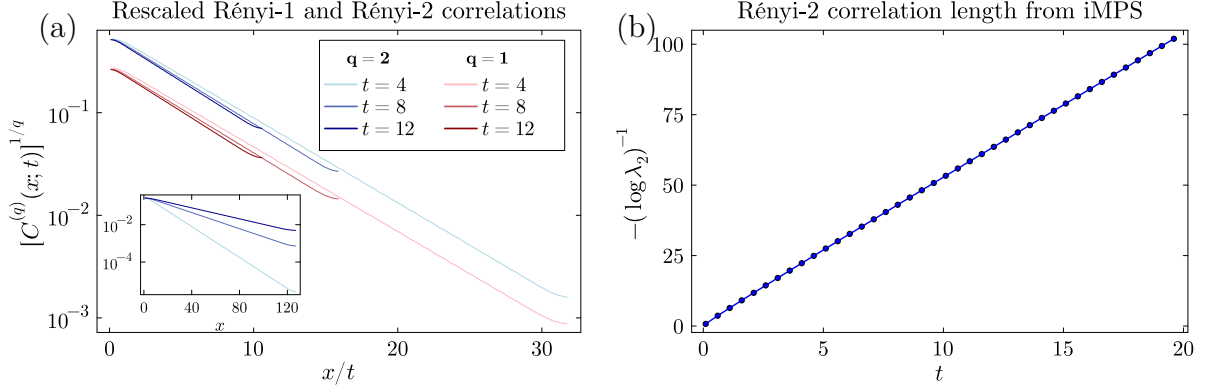


Figure 6.5: (a) Evolution of Rényi-1 and Rényi-2 correlations over time, with $L = 128$ and with $\gamma = 0.1$. In the inset, $[C^{(2)}(x; t)]^{1/2}$ is given on a semi-log plot vs. x for a range of times, demonstrating that $[C^{(2)}(x; t)]^{1/2} \propto e^{-x/(2\xi^{(2)}(t))}$. Upon rescaling the horizontal axis, we see that both $C^{(1)}(x; t)$ and $[C^{(2)}(x; t)]^{1/2}$ are functions of x/t , and specifically that $C^{(1)}(x; t) \propto [C^{(2)}(x; t)]^{1/2} \propto e^{-x/(2\xi^{(2)}(t))}$. This implies that $\xi^{(1)}(t) = 2\xi^{(2)}(t) \propto t$. (b) The infinite MPS Rényi-2 transfer matrix gap is given on a log-log scale. The data indicates saturation to t^{-1} scaling. Specifically, applying a linear fit to the data from $t = 2$ to $t = 20$ yields a slope of $0.97 \approx 1$.

phenomenological rotor model. This theory predicts that the correlation lengths extracted from the Rényi-1 and Rényi-2 correlators are related by a factor of two, as we observed in Fig. 6.5(a).

6.3.2.2 Conditional mutual information (CMI)

We now turn to a numerical study of the system's conditional correlations. The CMI is expected to decay according to the Markov length, as reviewed in Section 6.3.2.1. The SW-SSB of a zero-form symmetry (such as $U(1)$) is known to imply the divergence of the Markov length [180]. We expect the Markov length to follow a similar scaling to the Rényi correlation lengths, growing as $\xi^M \propto t$.

We use the method of Ref. [191] to numerically sample the CMI, representing the time-evolved state as an MPS. We work directly in the thermodynamic limit using the framework of uniform MPS [284]. Results below are for a maximum bond dimension $\chi = 48$, which we have checked is well-converged for the CMI data and density correlators, and a hopping rate $\gamma = 0.1$. The results for the CMI are shown with a contour plot

in Fig. 6.6(a), as a function of evolution time t and size of the buffer region $R_B = |B|$. The data are plotted on a logarithmic color scale and normalized at each timestep by $I_{0,t} = \max_{R_B} I_t(A : C|B)$. It is clear that the Markov length increases with time, consistent with a divergence at infinite time suggested by our results of the previous section. To demonstrate the Markov length scaling, we show a scaling collapse of the CMI data at different times with respect to the ballistic scaling variable $x_b = t$, in Fig. 6.6(b). The CMI data are rescaled by a factor of t^2 , to better highlight the collapse. The data shows a good collapse to an exponential decay for $R_B \gg x_b$, consistent with a Markov length $\xi^M \propto t$. Interestingly, a collapse with respect to the diffusive scaling variable $x_d = \sqrt{t}$, shown in the inset to Fig. 6.6(b), reveals an intermediate regime $x_d \lesssim R_B \ll x_b$ where the CMI decays *algebraically*, $I_t(A : C|B) \propto R_B^{-\alpha}$. Numerically we find $\alpha \approx 1.2 - 1.4$, but the exponent drifts continuously toward larger values over the accessible times. We will connect this surprising behavior to the emergence of a Gaussian hydrodynamic regime in Section 6.6, which predicts $\alpha = 2$.

Our data suggests that there is no mixed-state phase transition at finite time in the 1d model, but instead the Markov length demonstrates the same ballistic scaling $\xi^M \propto t$ as the Rényi correlation lengths studied in the previous section. The possibility remains for a finite-time transition in higher dimensions. We provide numerical evidence for this scenario by extracting the Markov length for the spin-1/2 model in quasi-1d ladders of increasing width W . We compute the Markov length using the Lyapunov spectrum of the MPS (see [191]), which provides a cleaner extrapolation than fitting the CMI data. We show in Fig. 6.6(c) the Markov length scaling for the 1d chain ($W = 1$) and several ladders of width $W = 2$ to $W = 4$. The Markov lengths display clear linear-time growth, with the larger-width ladders growing as $\xi_W^M \propto W \xi_1^M$. Extrapolating to the 2d limit, we expect that a mixed-state phase transition occurs at finite time in $d > 1$, consistent with our finding in Sec. 6.3.3.

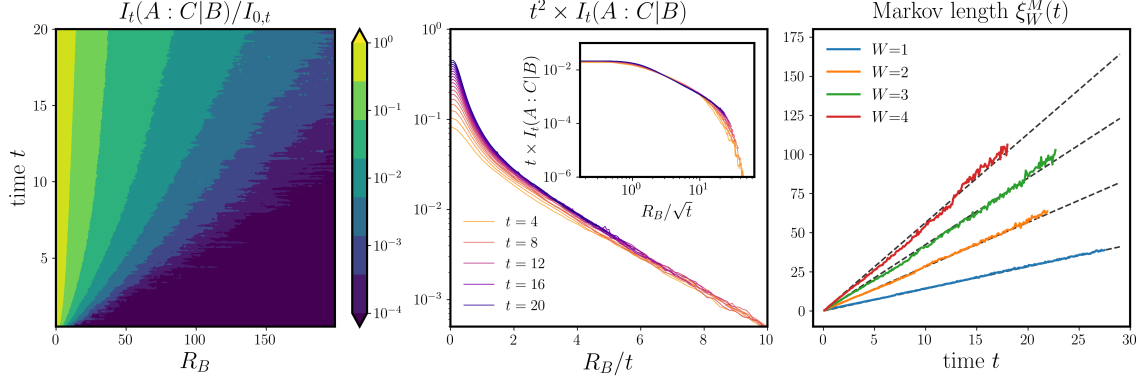


Figure 6.6: (a) CMI $I_t(A : C|B)$ for the spin-1/2 model, as a function of evolution time t and subsystem size $R_B = |B|$. The colorbar is log scaled. The lengthscale of the CMI (Markov length) diverges as a function of the evolution time, $\xi^M(t) \propto t$. (b) Scaling collapse of CMI vs ballistic lengthscale $x_b = t$, demonstrating the linearly growing Markov length. *Inset*: Scaling collapse of CMI vs diffusive lengthscale $x_d = \sqrt{t}$. The data shows a pre-Markov regime where conditional correlations decay algebraically (see Sec. 6.6). (c) Markov length $\xi_W^M(t)$ as a function of time for the chain ($W = 1$) and ladders of increasing width ($W = 2 - 4$). The Markov length follows the scaling $\xi_W^M(t) \sim Wt$ – dashed lines display the lines $W\xi_1^M(t)$.

6.3.2.3 Decoding charge

To complete our 1d results, we consider several decoding protocols which connect the SW-SSB transition to the difficulty of learning the charge of a subregion, which we will denote A . We note that the Markov length studied in the previous section sets the size of subregion B needed in order to accurately recover the *entire* state in region A , given access to the state on B [78, 252]. Here we consider the simpler task of decoding the total charge N_A of the region, ignoring information about the spatial distribution of charges. As we will see, the accuracy of charge decoding increases with the size R_B of the accessible region B , enabling us to pull out a decoding length scale, $\xi^{\text{dec}}(t)$. We find that $\xi^{\text{dec}}(t)$ diverges linearly with time, matching our results for the Rényi and Markov length scalings. Our results provide an intuitive picture for the ballistic growth of these length-scales in 1d.

We take a system of $2L$ spin-1/2 degrees of freedom initialized in the Néel state, with open boundary conditions. In the particle language, this corresponds to a total of

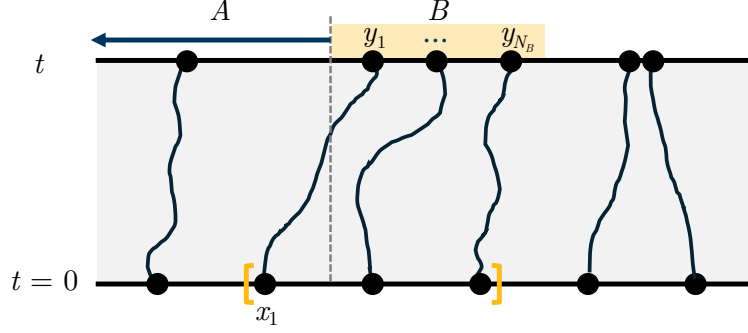


Figure 6.7: Cartoon of the decoding task. At a fixed time t we observe occupations in the window B (yellow) and aim to infer the number of particles remaining in A . The observed particle positions are $y_1, \dots, y_{N_B}$. From these, we infer the originating block of N_B consecutive odd sites with leftmost site x_1 and obtain the total particle number passed through the boundary between A and B .

L particles occupying the odd sites and the dynamics is given by the SSEP. We work in the regime $L \gg t^2$ so boundary effects are negligible. We take the region A to consist of the first L sites, and region B as the sites $[L+1, \dots, L+R_B]$, so $R_B = |B|$. The decoding task is to correctly infer the number of particles in A , given the charge configuration of region B . Note that for $R_B = L - |A|$, the answer follows immediately from number conservation, $N_A = L - N_B$.

Since the dynamics is an exclusion process, particle worldlines cannot cross. Therefore, the N_B particles observed in B at the final time t must originate from N_B consecutive particles in the initial Néel pattern. Let x_1 denote the leftmost site of this candidate block. Given an estimate of x_1 , we obtain the inferred number of particles in A by counting how many positive odd integers lie strictly to the left of x_1 . The decoder succeeds if this inferred value equals the true count: see Fig. 6.7. An optimal decoder would evaluate the likelihood for every possible x_1 of producing the observed configuration in B and then choose the maximum-likelihood estimate. Below, we analyze three transparent non-optimal decoders that may be tested with very large system sizes. Although they are suboptimal, all three capture the correct scaling of the minimal size of B needed for successful decoding, namely $\xi^{\text{dec}} \sim t$. Lastly, we implement the

optimal decoder using MPS simulations of the dynamics and analyze its performance, again finding scaling consistent with $\xi^{\text{dec}} \sim t$.

Center-of-mass (COM) decoder. The COM decoder infers the leftmost site x_1 from the center of mass of the N_B occupied sites observed in B at time t . Denote these occupied positions by $\{y_1, \dots, y_{N_B}\}$ and define $\bar{y} \equiv \frac{1}{N_B} \sum_{j=1}^{N_B} y_j$. If $N_B = 0$, the decoder trivially fails. If the inference length R_B is large, then typically N_B is large, so $\bar{y}$ is self-averaging and fluctuates weakly around its initial position.

The decoder models the occupied sites as originating from a length- N_B block of consecutive odd sites $\{x_1, x_1 + 2, \dots, x_1 + 2(N_B - 1)\}$. The center of mass of this block is $\bar{x} \equiv x_1 + (N_B - 1)$. Since x_1 is odd, $\bar{x}$ is an integer with parity $\bar{x} \equiv N_B \pmod{2}$. We therefore round the observed $\bar{y}$ to the nearest integer x^* with the same parity as N_B as an estimator of $\bar{x}$ and infer

$$x_1 = x^* - (N_B - 1). \quad (6.14)$$

The decoder fails when $\bar{y}$ crosses a decision boundary, halfway between adjacent admissible integers, which are spaced by 2. On hydrodynamic scales SSEP is diffusive, implying $\text{Var}(\bar{y}) \sim t/N_B$. Thus the success probability is controlled by t/N_B . Since $N_B \sim R_B$ at finite density, this suggests the decoding success rate is controlled by R_B/t , consistent with Fig. 6.8(a).

Minimal-weight perfect matching (MWPM) decoder. Let the occupied positions in B be $y_1 \leq \dots \leq y_{N_B}$. For each candidate odd integer x_1 , we define the ℓ_1

matching cost

$$\sum_{i=1}^{N_B} |y_i - (x_1 + 2(i-1))|. \quad (6.15)$$

The MWPM decoder chooses an x_1 that minimizes this cost. If there are multiple minimizers, one is selected uniformly at random.

Height decoder. Instead of matching individual particles, we summarize the configuration in B by an integrated *height* profile. We infer how many particles have crossed the cut by estimating the height offset, which we fix by requiring that the average height in the window stays close to its initial value $1/2$ from the Néel state. A useful intuition is that the height is a cumulative sum. A fluctuation entering the window from the left boundary shifts the entire profile inside B , whereas a fluctuation near the right edge affects the profile only locally. This asymmetry makes the height profile sensitive to the net particle transfer across the cut.

Let $n_j \in \{0, 1\}$ denote the occupation at site j at time t ($n_j = 1$ if occupied and $n_j = 0$ if empty). We define the height field by

$$h(0) = 0, \quad h(j) = h(j-1) + (2n_j - 1), \quad j = 1, \dots, 2L, \quad (6.16)$$

so the height increases (decreases) by one on occupied (empty) sites. We anchor a relative height at the cut and build it across the window,

$$h_{\text{rel}}(j) = \sum_{x=L+1}^j (2n_x - 1), \quad j \in B. \quad (6.17)$$

This determines the profile inside B up to an unknown additive constant, $h(L)$. We

estimate this constant by shifting the profile so that its window average is closest to $1/2$. Once $h(L)$ is inferred, we obtain the number of particles remaining in A .

Optimal decoder. At time t the density matrix of the system will be some distribution

$$|\rho_t\rangle = \sum_{s_A, s_B, s_C} P(s_A, s_B, s_C; t) |s_A\rangle |s_B\rangle |s_C\rangle. \quad (6.18)$$

Thus for each configuration s_B , there is a distribution of possible total charges N_A in region A :

$$P(N_A | s_B; t) = \sum_{s_A, s_C} P(s_A, s_B, s_C; t) \delta_{|s_A|, N_A} \delta_{|s_C|, N - N_A - |s_B|} \quad (6.19)$$

where $|s|$ counts the total charge in configuration s and N is the total charge of the whole system. The optimal decoder is a *maximum likelihood* decoder which guesses that the total charge in region A is

$$N_A^* = \arg \max_{N_A} P(N_A | s_B; t). \quad (6.20)$$

This decoder will succeed with probability $P(N_A^*)$, which is the quantity plotted in Fig. 6.8(d).

The sequence of Eqs. (6.18)-(6.20) essentially describes the protocol for efficiently computing N_A^* . In 1d, we may efficiently represent $|\rho_t\rangle$ as a state. Within this framework, we can erase all information in region A except for N_A by applying a sequence of gates that map $|n_1\rangle |n_2\rangle \rightarrow |n_1 + n_2\rangle$ until the sites in region A have been replaced by a single qudit with dimension $R_A + 1$. Then, when region B is projected down to configuration s_B and region C is traced out, the result is a vector of dimension $R_A + 1$ encoding

$P(N_A|s_B;t)$. Consequently, the optimal decoder may be efficiently benchmarked and efficiently applied in practice.

Fig. 6.8 shows the performance of the four decoders. The success probabilities collapse well when plotted as a function of R_B/t , indicating the scaling $\xi^{\text{dec}}(t) \sim t$. This highlights that although local charge correlations develop diffusively under the SSEP dynamics, successfully inferring the charge fluctuation at a given cut requires access to a window whose size grows proportionally to the evolution time. We note that the task of estimating N_A is closely related to the task of estimating the (“disorder”-dependent) winding number W defined in Eq. 6.25 (Section 6.3.3). In other words, $N_A(t) - N_A(0)$ counts the total flux of particle worldlines crossing the cut shown by the dashed line in Fig. 6.7; in the Figure, $W = +1$. The possibility of decoding accurately ($p \approx 1$) for $R_B \gg \xi^{\text{dec}}$ implies the variance of W vanishes for system sizes $L \gg \xi^{\text{dec}}$, consistent with there being no sharp SW-SSB transition in 1d. We will show next that the situation is very different in 2d, and the winding number fluctuations do not decay with the system size.

6.3.3 $2d$ results

According to the Mermin-Wagner theorem, in a Gibbs state at equilibrium there cannot be a sharp transition that spontaneously breaks the $U(1)$ symmetry in $1d$. In $2d$ there can be a quasi-long-range order in the low temperature phase below the BKT phase transition. From the formalism we developed we have seen the time t of the Lindbladian plays the role of the inverse temperature in the imaginary time evolution. The goal of this subsection is to pursue evidence for a BKT-like transition at finite time during the evolution, resulting in a BKT algebraic mixed state phase.

Even though the evolution Eq. (6.11) can be regarded as an imaginary time evolu-

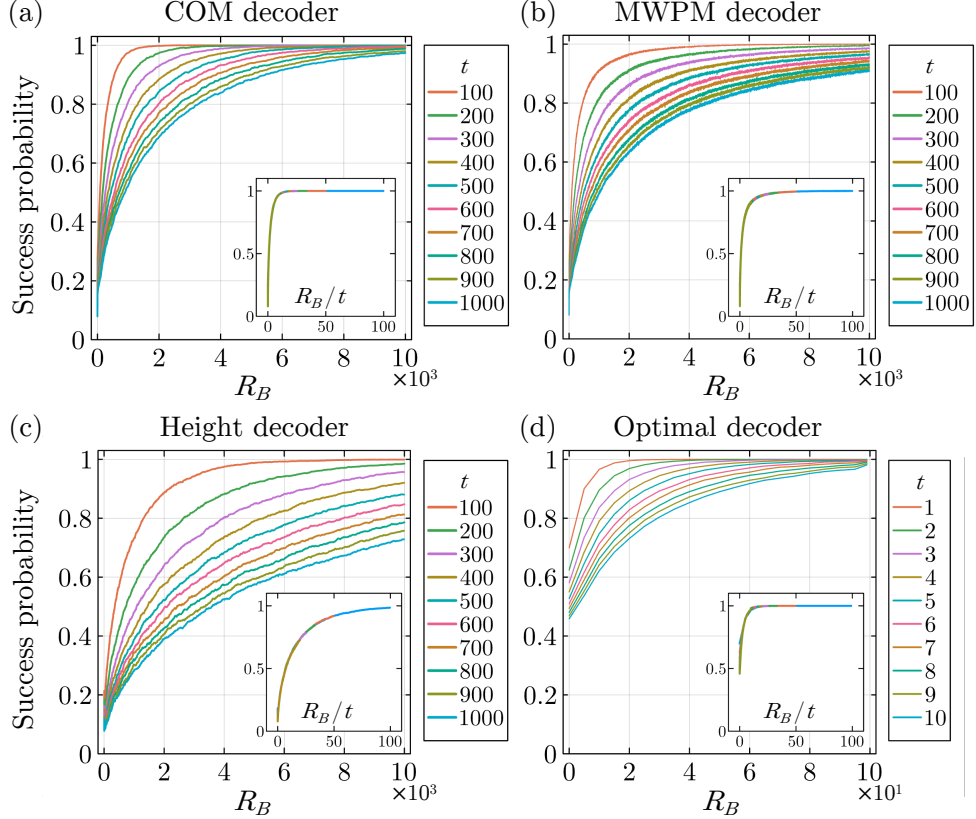


Figure 6.8: Success probability versus size of B region R_B , for several times t and for (a) the COM decoder, (b) the MWPM decoder, (c) the height decoder, and (d) the optimal decoder. Insets show the same data plotted against the rescaled width R_B/t , exhibiting an excellent collapse. Simulations for (a,b,c) use $L = 2 \times 10^4$ and each curve is averaged over $\sim 10^4$ samples. Simulations for (d) use $L = 2 \times 10^2$ and each curve is averaged over 2000 samples. We note that the large probabilities of success at $R_B = 0$ in (d) would be suppressed at larger system sizes and later times.

tion with t identified as inverse temperature, the physics can be quite different from an equilibrium problem defined by the effective Hamiltonian $-\mathcal{L}^{\text{diag}}$. Indeed various quantities we will be considering show properties different from their equilibrium counterparts. Moreover, due to the $SU(2)$ invariance, at equilibrium the equal time correlation function will never be long range ordered in $2d$ (We note at this point that the $SU(2)$ symmetry only occurs for spin-1/2 systems. If we instead consider a general decohered spin- S system, there is no $SU(2)$ symmetry even in the diagonal subspace). However our numerics suggest that $\hat{\rho}_t$ does undergo a finite time transition, and develop a quasi-long-range

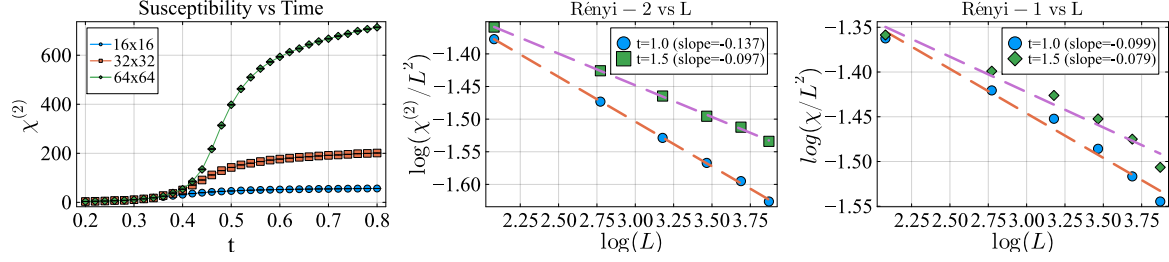


Figure 6.9: Rényi susceptibilities defined in Eq. (6.21) for our spin-1/2 model under Lindblad evolution.

order, for the nonlinear observables we consider.

We consider a $2d$ square lattice of size $L \times L$. To numerically access the various nonlinear observables related to SW-SSB, we use a quantum Monte Carlo method for results obtained in this subsection. The QMC method samples spacetime trajectories from which an estimator of the various observables can be obtained. To detect SW-SSB of $U(1)$ symmetry, we first examine the Rényi-2 and Rényi-1 correlation functions (defined in Section 6.2.1). We define the Rényi susceptibility as

$$\chi^{(Q)}(t) = \frac{1}{L^2} \sum_{i,j} C^{(Q)}(i, j; t), \quad (6.21)$$

for $Q = 1, 2$. The Rényi susceptibilities are expected to saturate to a constant in the disordered phase, while inside the BKT algebraic phase it grows with system size as a power-law. We plot the computed susceptibilities in Fig. 6.9. The expected behavior is verified for the Rényi-2 susceptibility. In particular, $\chi^{(2)}/L^2$ decays with L as a power law in the BKT algebraic SW-SSB phase. We provide further evidence with a rotor-model calculation performed in Appendix 6.8.4. The Rényi-1 susceptibility is less definitive, as $\chi^{(1)}/L^2$ does not show a clear power-law scaling with L . We comment on this below.

Another sharp diagnosis for the BKT-like transition is the jump of the dimensionless stiffness. The stiffness can be defined as the winding number fluctuation in the space-time path integral. The winding number can be accessed more concretely in Stochastic

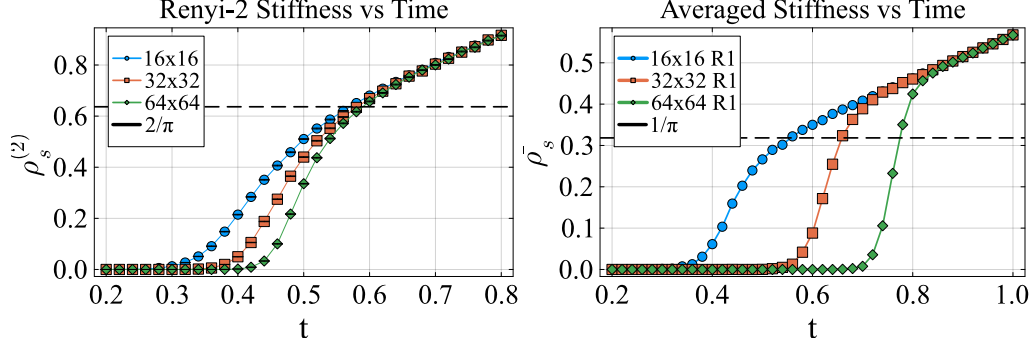


Figure 6.10: The Rényi-2 stiffness defined in Eq. (6.24) (left) and the “disorder-averaged” stiffness defined in Eq. (6.25) (right). The Rényi-2 stiffness behaves consistently with our theoretical expectation, with a jump at the transition with amplitude $2/\pi$. The disorder-averaged stiffness exhibits a stronger finite-size effect, which we attribute to the modified RG equation to be presented in the next section.

Series Expansion, which samples the operator strings

$$\frac{t^n}{n!} \langle \rho_0 | \mathcal{L}_1 \cdots \mathcal{L}_n | \rho_0 \rangle, \quad (6.22)$$

obtained from Taylor expansion of $e^{t\mathcal{L}}$. Here $\mathcal{L}_i$ labels a local term in the Lindbladian. The weight of the operator string sampled is determined both by the matrix element in Eq. (6.22) and the factor $\frac{t^n}{n!}$. This can be viewed as a way to sample configurations in spacetime. For each configuration consisting of a list of operators, the winding number in μ direction is defined as [250]

$$W_\mu = \frac{N_+ - N_-}{L}, \quad (6.23)$$

where N_+ denotes the amount of $S_{i+\mu}^+ S_i^-$ in the operator string (6.22) and N_- denotes the amount of $S_i^+ S_{i+\mu}^-$ in the operator string. By averaging over the configurations sampled, we can obtain the stiffness from the statistics of the winding number. Using other discretized representations of path integrals (e.g. the one used in continuous time worm algorithm) would give equivalent results.

We define two different measures of the stiffness depending on the final boundary

condition. The Rényi-2 stiffness is defined as

$$\rho_s^{(2)} = \langle W^2 \rangle_{R2} \quad (6.24)$$

with the initial configuration evolved for time $2t$ and the boundary conditions at time 0 and $2t$ fixed to be the Néel state — the expectation with the Rényi-2 boundary condition is denoted as $\langle \cdot \rangle_{R2}$. This is in slight contrast to the equilibrium stiffness which imposes periodic boundary conditions in imaginary time. In the equilibrium setting, this quantity is known to exhibit a universal jump at the transition, and therefore serves as a clear diagnostic for BKT-like physics.

Similarly, we define a “disorder-averaged” stiffness with free final boundary condition, evolving the state for time t and averaging over the final spin configuration s at time t :

$$\bar{\rho}_s = \sum_s P(s) \left(\langle W^2 \rangle_s - \langle W \rangle_s^2 \right). \quad (6.25)$$

The average for fixed disorder configuration s is denoted $\langle \cdot \rangle_s$. The procedure to evaluate this quantity is again to sample s , and then to use another QMC sampling to estimate the winding number fluctuation. We display results for both the Rényi-2 and disorder-averaged stiffness in Fig. 6.10. We observe a clear jump in the Rényi-2 stiffness with magnitude $2/\pi$ as t is increased. The disorder-averaged stiffness is less conclusive: there is a strong finite size effect and the apparent transition point t_c drifts to larger values as the system size increases. Our finding for the Rényi-2 observables are consistent with rotor model calculations carried out in the next section. The strong finite-size effect of the disorder-averaged stiffness is also suggested within the RG analysis in the replica limit discussed in the next section, as we find that an extra term in the RG equation would lead to even longer correlation length near the critical point, compared with the BKT transition.

6.4 Decohered rotor model

In this section we turn to our second model which consists of a lattice of $U(1)$ quantum rotors, with number operators $\hat{n}_j$ and phases $\hat{\phi}_j$ which satisfy commutation relations,

$$[\hat{n}_i, e^{i\hat{\phi}_j}] = \delta_{ij} e^{i\hat{\phi}_i}. \quad (6.26)$$

We will again explore Lindbladian dynamics with an initial state of the evolution to be $|0\rangle = \prod_i |\hat{n}_i = 0\rangle$, and the jump operator $L_{ij} = e^{i\hat{\phi}_i} e^{-i\hat{\phi}_j}$. In the absence of a coherent Hamiltonian (which we shall explore in the following Section) the master equation is $\partial_t \hat{\rho} = \frac{1}{2} \sum_{i,j} \hat{L}_{ij} \hat{\rho} \hat{L}_{ij}^\dagger - \hat{\rho}$, here the sum i, j includes both orientations. At time t the density matrix can be represented as,

$$\hat{\rho}(t) \sim \int D\phi_L D\phi_R e^{\sum_{ij} t(\cos(\phi_{L,i} - \phi_{R,i} - \phi_{L,j} + \phi_{R,j}) - 1)} |\phi_L\rangle \langle \phi_R|. \quad (6.27)$$

We first note that just like the spin-1/2 models studied in the previous sections, the system has a “weak-local- $U(1)$ ” symmetry $\phi_{L,i} \rightarrow \phi_{L,i} + \alpha_i$, $\phi_{R,i} \rightarrow \phi_{R,i} + \alpha_i$, therefore the field $\phi \sim \phi_{L,i} + \phi_{R,i}$ is a redundant “gauge mode” and can be integrated out. In the next section we will see that ϕ corresponds to the “classical field”, while $\tilde{\phi} = \phi_L - \phi_R$ is the “quantum field”. A more generic channel or Lindbladian would break the weak-local- $U(1)$ symmetry, but we later will argue that breaking this symmetry will not lead to qualitative change of the discussion, at least at the perturbative level.

Due to the weak-local- $U(1)$ symmetry, the density matrix is always diagonal in the occupational basis $|\mathbf{m}\rangle = \prod_i |\hat{n}_i = m_i\rangle$, and the probability of finding configuration $\mathbf{m}$ at time t is

$$P(\mathbf{m})_t = \langle \mathbf{m} | \hat{\rho}(t) | \mathbf{m} \rangle \sim \int D\tilde{\phi} e^{\sum_{ij} t \cos(\tilde{\phi}_i - \tilde{\phi}_j)} e^{i \sum_i m_i \tilde{\phi}_i} = \langle e^{i \sum_i m_i \tilde{\phi}_i} \rangle_t, \quad (6.28)$$

where $\tilde{\phi} = \phi_L - \phi_R$.

In this section we will often use the Gaussian theory after a spin wave expansion of the rotor model, i.e. $\cos(\tilde{\phi}_i - \tilde{\phi}_j) \approx 1 - \frac{1}{2}(\tilde{\phi}_i - \tilde{\phi}_j)^2$, as an approximation of Eq. (6.27) at large- t . The Gaussian field theory in d -dim space reads:

$$S = \int d^d x \frac{\tilde{t}}{2} (\partial_\mu \phi)^2. \quad (6.29)$$

$\tilde{t}$ is a parameter in the continuum field theory, we expect it to be largely proportional to t albeit renormalization. In this Gaussian field theory we are interested in the correlation function $C(\mathbf{x}) = \langle e^{-i\phi_0} e^{i\phi_{\mathbf{x}}} \rangle$, whose behaviors in $1d$ and $2d$ are evaluated as

$$\begin{aligned} d = 1 & : C(x) \sim e^{-|x|/\xi}, \quad \xi = 2\tilde{t}; \\ d = 2 & : C(\mathbf{x}) \sim \left(\frac{1}{|\mathbf{x}|} \right)^{1/(2\pi\tilde{t})}. \end{aligned} \quad (6.30)$$

Our results are summarized as follows:

- For the $1d$ decohered rotor model, though the system remains strongly-symmetric at any finite t , the correlation length of the Rényi-1 and Rényi-2 correlators grows linearly in time, $\xi \propto t$. Moreover, they satisfy a universal relation at large t : $\xi^{(1)}/\xi^{(2)} = 2$, which is confirmed numerically in the previous section.
- For the $2d$ decohered rotor model, there is a phase transition between the strongly symmetric phase and a quasi-long-range SW-SSB phase. We perform two calculations for these correlators, with and without using the replica formalism, and they lead to the same conclusion. We also perform the RG calculation to demonstrate that the phase transition belongs to a modified BKT-like universality class, with an unusual scaling at the critical point.

6.4.1 1d rotor model

We first discuss the $U(1)$ rotor model in $1d$. This model should not have a $U(1)$ -SW-SSB phase, not even a quasi-long-range order except for $t \rightarrow \infty$. However, we will show that with large t , this model still exhibits universal behaviors: the correlation length of the Rényi-1 and Rényi-2 correlators satisfy a simple relation $\xi^{(1)} = 2\xi^{(2)}$, as was reported in our numerical study of the $s = 1/2$ model.

6.4.1.1 Rényi-2 correlator

We first evaluate the Rényi-2 correlator of the $1d$ rotor model:

$$C^{(2)}(x)_t = \int D\phi_L D\phi_R e^{\sum_{ij} 2t(\cos(\phi_{L,i}-\phi_{R,i}-\phi_{L,j}+\phi_{R,j})-1)} e^{-i(\phi_{L,0}-\phi_{R,0})+i(\phi_{L,x}-\phi_{R,x})}. \quad (6.31)$$

As we discussed already, the system has a weak-local- $U(1)$ symmetry and the field $\phi = \phi_L + \phi_R$ is redundant. After integrating out ϕ , the Rényi-2 correlator becomes

$$C^{(2)}(x)_t \sim \int D\tilde{\phi} e^{\sum_{ij} 2t(\cos(\tilde{\phi}_i-\tilde{\phi}_j)-1)} e^{-i\tilde{\phi}_0+i\tilde{\phi}_x}, \quad (6.32)$$

In $1d$ we focus on the regime with large t , and perform a spin-wave expansion of Eq. (6.32). The Rényi-2 correlator reduces to the correlation of the Gaussian theory Eq. (6.29), with $\tilde{t}$ replaced by $2\tilde{t}$. Therefore the Rényi-2 correlator decays exponentially as

$$d = 1 \quad : \quad C^{(2)}(x) \sim e^{-|x|/\xi^{(2)}}, \quad \xi^{(2)} = 4\tilde{t}. \quad (6.33)$$

The Gaussian theory after spin-wave expansion is a very convenient way of evaluating the Rényi-2 correlator, but here the spin-wave expansion is not necessary. One can take

the standard Villain form analysis of the rotor model, and the Rényi-2 correlator becomes

$$C^{(2)}(x) \sim \sum_{E_{\bar{i}}} \delta_{\nabla E, \rho} \exp \left(\sum_{\bar{i}} -\frac{1}{4t} E_{\bar{i}}^2 \right). \quad (6.34)$$

Here $E_{\bar{i}}$ can be viewed as the discrete electric field on the links $\bar{i}$, and it is constrained by the Gauss law $\nabla E = \mathbf{d}_{0,x}$, where $\mathbf{d}_{0,x} = \{(-1)_0; (+1)_x\}$ is a configuration of dipole with ± 1 charges residing at 0 and x . Therefore $E_{\bar{i}}$ is a step function, and $C^{(2)}(x)$ reads

$$C^{(2)}(x) \sim \sum_E \exp \left(-\frac{L-x}{4t} E^2 - \frac{x}{4t} (E+1)^2 \right) \sim e^{-\frac{x}{4t} \theta_3 \left(\frac{ix}{4t}, \frac{iL}{4\pi t} \right)}, \quad (6.35)$$

where L is the length of the chain, and θ_3 is the third Jacobi theta function. In the limit of large L , the sum of E is evidently dominated by $E = 0$, and the Rényi-2 correlator reduces to the same form as the Gaussian theory.

6.4.1.2 Rényi-1 correlator

To evaluate the Rényi-1 correlator at large t , we first note that the probability of finding configuration $\mathbf{m} = (\dots m_i \dots)$ at time t is given by

$$P(\mathbf{m})_t = \langle \prod_i e^{im_i \hat{\phi}_i} \rangle_t \sim \delta_{\sum_i m_i, 0} \exp(-\mathcal{E}_1[\mathbf{m}]), \quad \mathcal{E}_1[\mathbf{m}] = -\frac{1}{2t} \sum_{j < k} m_j m_k |x_j - x_k|, \quad (6.36)$$

where we have used the spin-wave approximation. Here, $\mathcal{E}_1[\mathbf{m}]$ is the energy of 1d Coulomb gas with charge configuration $\mathbf{m}$, which interact with each other through a linear Coulomb potential energy.

The Rényi-1 correlator is defined as:

$$C^{(1)}(x)_t = \text{Tr} \left(\sqrt{\rho(t)} e^{-i\hat{\phi}_0 + i\hat{\phi}_x} \sqrt{\rho(t)} e^{i\hat{\phi}_0 - i\hat{\phi}_x} \right). \quad (6.37)$$

Since the density matrix is always diagonal in the occupational basis, the Rényi-1 cor-

relator can also be written as

$$C^{(1)}(x, t) = \sum_{\mathbf{m}} \sqrt{P(\mathbf{m})P(\mathbf{m}')}, \quad (6.38)$$

where $\mathbf{m}'$ is the charge configuration $\mathbf{m}$ after moving one particle from 0 to x : $\mathbf{m}' = \mathbf{m} + \mathbf{d}_{0,x}$, with dipole $\mathbf{d}_{0,x}$. Using Eq. (6.36), the Rényi-1 correlator can be written as:

$$\begin{aligned} C^{(1)}(x)_t &= \sum_{\mathbf{m}} \sqrt{\langle \prod_i e^{im_i \hat{\phi}_i} \rangle \langle \prod_i e^{im_i \hat{\phi}_i} e^{-i\hat{\phi}_0 + i\hat{\phi}_x} \rangle} \\ &= \sum_{\mathbf{m}} \sqrt{\exp(-\mathcal{E}_1[\mathbf{m}]) \exp(-\mathcal{E}_1[\mathbf{m} + \mathbf{d}_{0,x}])} \\ &= \sum_{\mathbf{m}} \exp(-\mathcal{E}_1[\mathbf{m} + \mathbf{d}_{0,x}]) \times \exp\left(-\mathcal{E}_1\left[\frac{1}{2}\mathbf{d}_{0,x}\right]\right) \\ &= A(x)_t \times B(x)_t. \end{aligned} \quad (6.39)$$

The two factors in the last equations above are:

$$A(x)_t = \sum_{\{m\}} \exp\left(\frac{1}{2\tilde{t}} \sum_{j < k} m_j^A m_k^A |x_j - x_k|\right), \quad B(x)_t \sim e^{-|x|/\xi^{(1)}}, \quad \xi^{(1)} = \frac{2}{\tilde{t}} \quad (6.40)$$

Here $\mathbf{m}^A$ is a charge configuration that is obtained from configuration $\mathbf{m}$ by inserting “test-charges” $\pm 1/2$ at position 0 and x :

$$\mathbf{m}^A = \mathbf{m} + \frac{1}{2}\mathbf{d}_{0,x}. \quad (6.41)$$

The factor $A(x)_t$ is the partition function of $1d$ Coulomb gas, with extra test charges. We will show in Appendix 6.8.3 that the decay length of $A(x)_t$ increases very rapidly with $\tilde{t}$, so the Rényi-1 correlator is dominated by $B(x)_t$. The physical picture is that the test charges of a $1d$ Coulomb gas are screened, therefore the partition function (and

$A(x)_t$) does not vary strongly with x . Then the dominant factor of $C^{(1)}(x)_t$ is $B(x)_t$, and it decays exponentially with correlation length $\xi^{(1)} = 8\tilde{t}$, which is twice of that of $\xi^{(2)}$, i.e. $\xi^{(1)} = 2\xi^{(2)}$, exactly what was observed numerically for our $s = 1/2$ model.

6.4.2 $2d$ rotor model

Under the Lindbladian evolution Eq. (6.27), the $2d$ rotor model will have a quasi-long-range $U(1)$ -SW-SSB at large t . We will perform a renormalization group analysis later to discuss the phase transition, but let us assume that we are already in the quasi-long-range SW-SSB phase, and evaluate the Rényi-1 and Rényi-2 correlators. We still first ignore the compactness of $\tilde{\phi}$, and take a spin-wave expansion. When evaluating the Rényi-2 correlator, the effective theory is given by Eq. (6.29), with $\tilde{t}$ replaced by $2\tilde{t}$. Therefore the Rényi-2 correlator in the SW-SSB phase reads

$$C^{(2)}(\mathbf{x})_t \sim \frac{1}{|\mathbf{x}|^{1/(4\pi\tilde{t})}}, \quad (6.42)$$

and at the critical point $\tilde{t}_c = 1/\pi$ predicted through an RG calculation (to be presented, below), the Rényi-2 correlator has a universal scaling $C^{(2)}(\mathbf{x})_t \sim 1/|\mathbf{x}|^{1/4}$. We note that the Rényi-2 physics of the decohered $2d$ rotor model was also discussed in Ref. [180].

6.4.2.1 Rényi-1 correlator: direct calculation

Now let us evaluate the Rényi-1 correlator. In the large- t SW-SSB phase, the formalism we used for the $1d$ rotor model still applies, provided we replace the probability of finding configuration $\mathbf{m}$ at time t by

$$\begin{aligned} P(\mathbf{m})_t &= \left\langle \prod_i e^{im_i \hat{\phi}_i} \right\rangle \sim \delta_{\sum_i m_i, 0} \exp(-\mathcal{E}_2[\mathbf{m}]), \\ \mathcal{E}_2[\mathbf{m}] &= -\frac{1}{2\pi\tilde{t}} \sum_{j < k} m_j m_k \ln |x_j - x_k|. \end{aligned} \quad (6.43)$$

Here, $\mathcal{E}_2[\mathbf{m}]$ is the energy of $2d$ Coulomb gas with charge configuration $\mathbf{m}$, which interact with each other through a logarithmic Coulomb potential energy. We can then write,

$$\begin{aligned}
C^{(1)}(\mathbf{x})_t &= \sum_{\mathbf{m}} \sqrt{P(\mathbf{m})P(\mathbf{m}')} \\
&= \sum_{\mathbf{m}} \exp\left(-\mathcal{E}_2[\mathbf{m} + \frac{1}{2}\mathbf{d}_{0,x}]\right) \times \exp\left(-\mathcal{E}_2[\frac{1}{2}\mathbf{d}_{0,x}]\right) \\
&= A(\mathbf{x})_t \times B(\mathbf{x})_t.
\end{aligned} \tag{6.44}$$

The factor $B(\mathbf{x})_t$ gives a power-law decay,

$$B(\mathbf{x})_t = \exp\left(-\mathcal{E}_2[\frac{1}{2}\mathbf{d}_{0,x}]\right) = \left(\frac{1}{|\mathbf{x}|}\right)^{1/(8\pi\tilde{t})}, \tag{6.45}$$

while the factor $A(\mathbf{x})_t$ can be evaluated as

$$A(\mathbf{x})_t \sim \int D\chi(\mathbf{x}) \exp\left(-\int d^2x \frac{\tilde{t}}{2}(\partial_\mu\chi)^2 - \lambda \cos(\chi)\right) e^{-\frac{i}{2}\chi_0 + \frac{i}{2}\chi_x}. \tag{6.46}$$

Here the sum over $\mathbf{m}$ will be recovered once we expand in powers of the fugacity λ . Since the sum over $\mathbf{m}$ has no bound, $\lambda \rightarrow \infty$. With infinite λ , the fluctuation of χ in Eq. (6.46) is pinned completely, and $A(\mathbf{x})_t$ does not decay with $\mathbf{x}$.

Thus in the Gaussian phase for $t > t_c$, the long distance behavior of the Rényi-1 correlator is given by $B(\mathbf{x})_t$ only, and it decays as

$$C^{(1)}(\mathbf{x})_t \sim \left(\frac{1}{|\mathbf{x}|}\right)^{1/(8\pi\tilde{t})}. \tag{6.47}$$

We can see that the relation $C^{(2)}(\mathbf{x})_t \sim \left(C^{(1)}(\mathbf{x})_t\right)^2$ still holds in 2d, or in other words, inside a Gaussian phase (the phase with $U(1)$ SW-SSB), the scaling dimension of the Rényi-1 and Rényi-2 correlator satisfy $\Delta^{(2)} = 2\Delta^{(1)}$. At the critical point $\tilde{t}_c = 1/\pi$, as

predicted through the RG calculation to be presented below, the Rényi-1 correlator has scaling $C^{(1)}(x)_t \sim 1/|x|^{1/8}$.

6.4.2.2 Rényi-1 correlator: replica formula

In this subsection we reevaluate the Rényi-1 correlator of the $2d$ rotor model with the replica formula. We first generalize the Rényi-1 correlator to the following Rényi- Q correlator:

$$C^{(Q)}(\mathbf{x})_t = \text{Tr} \left(\rho(t)^{Q/2} e^{-i\hat{\phi}_0 + i\hat{\phi}_{\mathbf{x}}} \rho(t)^{Q/2} e^{i\hat{\phi}_0 - i\hat{\phi}_{\mathbf{x}}} \right). \quad (6.48)$$

This Rényi- Q correlator quantity can be conveniently evaluated with even integer Q . We will eventually take the replica limit $Q \rightarrow 1$.

The generating theory for the Rényi- Q correlator is the following partition function:

$$\begin{aligned} Z_t^{(Q)} &= \frac{1}{(\mathcal{N}_t)^Q} \int \prod_{a=1}^Q D\tilde{\phi}_a \exp \left(- \left[\sum_{a=1}^Q \sum_{ij} -t \cos(\nabla_{ij}(\tilde{\phi}_a - \tilde{\phi}_{a+1})) \right] \right), \\ &= \frac{1}{(\mathcal{N}_t)^Q} \int \prod_{a=1}^Q D\varphi_a \delta \left(\sum_{a=1}^Q \varphi_a \right) \exp \left(- \left[\sum_{a=1}^Q \sum_{ij} -t \cos(\nabla_{ij}\varphi_a) \right] \right). \end{aligned} \quad (6.49)$$

In the second line above we have defined $\varphi_a = \tilde{\phi}_a - \tilde{\phi}_{a+1}$, and the new fields φ_a are subject to a hard constraint $\sum_{a=1}^Q \varphi_a(\mathbf{x}) = 0$.

After a spin wave expansion of Eq. (6.49), the Rényi- Q correlator is now evaluated as:

$$\begin{aligned} C^{(Q)}(\mathbf{x})_t &\sim \int \prod_{a=1}^Q D\varphi_a \delta \left(\sum_{a=1}^Q \varphi_a \right) \exp \left(i \sum_{a=1}^{Q/2} (\varphi_{a,0} - \varphi_{a,\mathbf{x}}) \right) \\ &\quad \times \exp \left(- \sum_{a=1}^Q \int d^2x \frac{\tilde{t}}{2} (\partial_\mu \varphi_a)^2 \right). \end{aligned} \quad (6.50)$$

Due to the hard constraint $\sum_{a=1}^Q \varphi_a = 0$, the correlation of the Gaussian fields φ_a is

$$\begin{aligned} \langle \varphi_a(0) \varphi_b(\mathbf{x}) \rangle &= - \left(\delta_{ab} - \frac{1}{Q} \right) \frac{1}{2\pi\tilde{t}} \ln |\mathbf{x}|, \\ \sum_{a,b=1}^{Q/2} \langle \varphi_a(0) \varphi_b(\mathbf{x}) \rangle &= - \frac{Q}{4} \frac{1}{2\pi\tilde{t}} \ln |\mathbf{x}|. \end{aligned} \quad (6.51)$$

This eventually leads to

$$C^{(Q)}(x)_t \sim \left(\frac{1}{|\mathbf{x}|} \right)^{Q/(8\pi\tilde{t})}. \quad (6.52)$$

In the replica limit $Q \rightarrow 1$, it recovers exactly the same result Eq. (6.47) obtained without using the replica formula.

6.4.2.3 Renormalization group

Now let us approach the SW-SSB phase transition from the quasi-long-range $U(1)$ -SW-SSB phase. In an ordinary $2d$ superfluid, the transition is a BKT transition driven by proliferation of vortices, with critical $\tilde{t}_c = 2/\pi$. The critical $\tilde{t}_c$ corresponds to the well-known universal jump of the dimensionless stiffness (also called the helicity modulus) at the BKT transition. The replica- Q partition function Eq. (6.49) are Q -copies of Gaussian fields with a constraint $\sum_a \varphi_a = 0$. This constraint enforces that the total vorticity of system is zero everywhere, i.e. $\sum_{a=1}^Q v_{a,\tilde{i}} = 0$. Therefore the minimal vortices are “replica-dipoles” such as $(+1, -1, 0, \dots)$. Since a replica-dipole costs twice the energy of a single vortex, the standard “energy-entropy” argument concludes that the transition associated with the vortex proliferation should be at $\tilde{t}_c = 1/\pi$, half of that of a single component superfluid. A similar structure of inter-replica dipoles was observed in field theories describing monitored $U(1)$ quantum [27] and classical systems [95, 214], where the corresponding “charge-sharpening” transition [5] can also be associated with SW-SSB [266, 289]. Our finite time transition is a $(2+0)d$ version of this monitored $(1+1)d$

problem, where conditioning occurs only at the final time step.

For a more systematic RG, let us return to Eq. (6.49). The most standard RG for a BKT transition is performed in the dual theory. We first consider the following modified theory:

$$S = \int d^2x \sum_a \frac{\tilde{t}}{2} (\partial_\mu \varphi_a)^2 + \frac{m}{2} (\partial_\mu \sum_a \varphi_a)^2 = \int d^2x \frac{K_{ab}}{2} \partial_\mu \varphi_a \partial_\mu \varphi_b, \quad (6.53)$$

where $K_{ab} = \tilde{t} \delta_{ab} + m$ is a matrix in the replica space. We eventually will need to take $m \rightarrow \infty$, to freeze the mode $\sum_a \varphi_a$, and effectively enforce the constraint $\sum_a \varphi_a = 0$. If we keep m finite, the dual theory of Eq. (6.53) is

$$S_d = \int d^2x \left(\frac{K_{ab}^d}{2} \partial_\mu \theta_a \partial_\mu \theta_b - \sum_a v \cos(2\pi \theta_a) \right), \quad (6.54)$$

here $K^d = K^{-1}$, and $e^{i2\pi\theta_a}$ creates a vortex of boson species φ_a .

How do we see that the vortices need to form a replica-dipole? Suppose we evaluate the correlation function of $\cos(2\pi\theta_a)$, i.e. the single vortex operator, we will see that its scaling dimension is

$$\Delta_a = \pi \tilde{t} + \pi m, \quad (6.55)$$

and when we take large- m , the single vortex operator is highly irrelevant. The most relevant vortex configuration turns out to be exactly a replica-dipole:

$$V_{ab} = e^{i2\pi\theta_a - i2\pi\theta_b}, \quad \Delta_{ab} = 2\pi \tilde{t}. \quad (6.56)$$

If we only include the most relevant vortices, the dual theory should be

$$S_d = \int d^2x \left(\frac{K_{ab}^d}{2} \partial_\mu \theta_a \partial_\mu \theta_b - \sum_{a \neq b} \tilde{v} \cos(2\pi \theta_a - 2\pi \theta_b) \right). \quad (6.57)$$

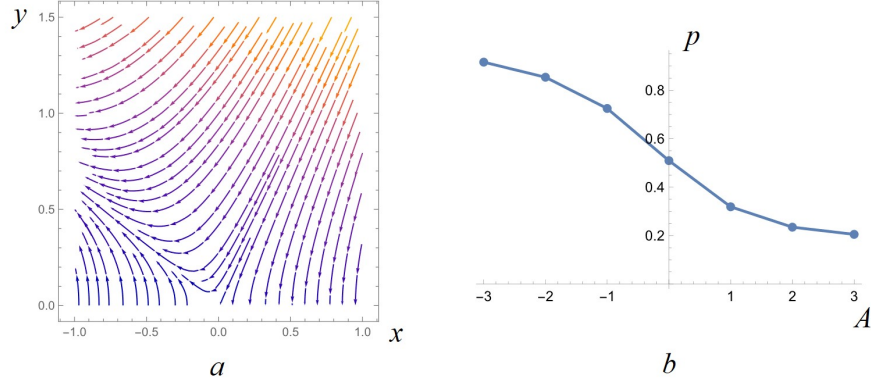


Figure 6.11: (a) The RG flow of Eq. (6.64) with $A < 0$. The separatrix and line of fixed points still exist. (b) The exponent p defined in Eq. (6.65) as a function of A in Eq. (6.64), which is extracted from numerical solution of the RG equations Eq. (6.64).

Using $\lim_{m \rightarrow \infty} K_{ab}^d = \hat{\Pi}_{ab}/\tilde{t}$ with $\hat{\Pi}_{ab} = \delta_{ab} - \frac{1}{Q}$ the projector onto inter-replica modes, we can also write the action directly enforcing the constraint

$$S = \int d^2x \left(\frac{1}{2\tilde{t}} \sum_a (\partial_\mu \hat{\theta}_a)^2 - \sum_{a \neq b} \tilde{v} \cos(2\pi \hat{\theta}_a - 2\pi \hat{\theta}_b) \right), \quad (6.58)$$

where we have introduced the fields $\hat{\theta}_a = \hat{\Pi}_{ab}\theta_b$, which satisfy $\sum_a \hat{\theta}_a = 0$. This form matches the field theory of monitored 1+1d systems with $U(1)$ symmetry discussed in Refs. [27, 95, 214].

The RG flow for $\tilde{v}$ near the critical point includes two contributions:

$$\frac{d\tilde{v}}{d \ln l} = (2 - 2\pi \tilde{t})\tilde{v} + 2\pi(Q - 2)\tilde{v}^2. \quad (6.59)$$

The first term is the standard BKT beta function, and it confirms that the critical point for the transition happens at $\tilde{t}_c = 1/\pi$, rather than $2/\pi$ as is the case for an ordinary rotor model. The second term stems from the OPE between $V_{ab} \times V_{bc} \rightarrow V_{ac}$, where V_{ab} is the vortex dipole, and the factor $Q - 2$ comes from choosing $b \neq a, c$ with $a \neq c$.

To derive the RG equation for $\tilde{t}$, we follow the standard RG steps in the dual theory,

and use the OPE:

$$\cos(2\pi\theta_a - 2\pi\theta_b)_x \cos(2\pi\theta_a - 2\pi\theta_b)_0 \sim -\frac{\pi^2}{|\mathbf{x}|^4} (x^\mu \partial_\mu (\theta_a - \theta_b))^2 + \dots \quad (6.60)$$

Under rescaling $x \rightarrow x l$ in real space we generate a new term

$$\begin{aligned} \delta S &= \int d^2x \frac{\pi^3}{2} \tilde{v}^2 \ln l \left(\sum_{a,b} (\partial_\mu \theta_a - \partial_\mu \theta_b)^2 \right) \\ &= \int d^2x \frac{\pi^3}{2} \tilde{v}^2 \ln l \left(Q \sum_a (\partial_\mu \theta_a)^2 - (\partial_\mu \sum_a \theta_a)^2 \right). \end{aligned} \quad (6.61)$$

This means that at the new scale l , K^d becomes

$$K^d(l) = K^d(0) + \delta_{ab} \pi^3 Q \tilde{v}^2 \ln l - \pi^3 \tilde{v}^2 \ln l. \quad (6.62)$$

Then at the new scale, $K(l)$ in the original formalism is $K(l) = K^d(l)^{-1}$. After expanded to the leading order of $\ln l$, $K(l)$ takes the same form as the original K , with separately renormalized $\tilde{t}(l)$ and $m(l)$:

$$\begin{aligned} K(l)_{ab} &= (K^d(l))_{ab}^{-1} = \tilde{t}(l) \delta_{ab} + m(l) \\ \rightarrow \quad \frac{d\tilde{t}}{d \ln l} &= -Q \pi^3 \tilde{v}^2 \tilde{t}^2, \quad \frac{dm}{d \ln l} = \pi^3 \tilde{v}^2 \tilde{t}^2. \end{aligned} \quad (6.63)$$

Though m will be renormalized under RG, it won't affect the physics, as we eventually take m to infinity to enforce the constraint. Eq. (6.59) and Eq. (6.63) together constitute the RG flow, and after rescaling $\tilde{v}$ and $\tilde{t}$, the RG equations near the transition point $\tilde{t} \sim 1/\pi$ reduce to

$$\frac{dy}{d \ln l} = -sy + Ay^2, \quad \frac{ds}{d \ln l} = -y^2, \quad (6.64)$$

$A = (Q-2)\sqrt{\frac{2}{Q}}$. The same form of RG equations were obtained in previous literature [43, 214].

At $Q = 2$, the coefficient A vanishes, therefore the Rényi-2 quantities should undergo a standard BKT transition. But for other Q these RG equations are expected to lead to different solutions. Like the BKT transition, there is still a line of fixed points at $\tilde{v} \sim y = 0$ and $\tilde{t} > t_c = 1/\pi$, therefore in the thermodynamic limit we still expect a stiffness jump with the size $1/\pi$. However, the modified RG equations lead to different scaling of the correlation length:

$$\xi \sim \exp\left(\frac{c}{|\tilde{t} - \tilde{t}_c|^p}\right), \quad (6.65)$$

where the exponent p is a continuous function of A (Fig. 6.11b), with $p = 1/2$ for $A = 0$.

The BKT transition is associated with a jump of the dimensionless stiffness. Our theory predicts that the jump of the Rényi-2 stiffness defined in Eq. (6.24) should be $2\tilde{t}_c = 2/\pi$, while the jump of the “disorder-averaged” stiffness in Eq. (6.25) is expected to be $\tilde{t}_c = 1/\pi$, i.e. the stiffness in the replica limit $Q \rightarrow 1$. Our simulation of the spin-1/2 model in the previous section confirms the jump of the Rényi-2 stiffness (Fig. 6.10), but exhibits more complicated behavior of the disorder-averaged stiffness. There is evidently a strong finite-size effect for the disorder averaged stiffness, qualitatively consistent with prediction from the RG equation with the extra Ay^2 term.

As we discussed previously, our Lindbladian for rotor model has a weak-local- $U(1)$ symmetry $\phi_{L,i} \rightarrow \phi_{L,i} + \alpha_i$, $\phi_{R,i} \rightarrow \phi_{R,i} + \alpha_i$. A generic channel or Lindbladian should break this local symmetry. For example, one can consider an extra jump operator $L'_{ij} = \cos(\hat{\phi}_i - \hat{\phi}_j)$, which adds another perturbation to the action Eq. (6.27):

$$\delta S \sim (\cos(\nabla_{ij}\phi_L) - \cos(\nabla_{ij}\phi_R))^2 \sim \sin\left(\frac{1}{2}\nabla_{ij}\tilde{\phi}\right)^2 \sin\left(\frac{1}{2}\nabla_{ij}\phi\right)^2, \quad (6.66)$$

where $\phi = \phi_L + \phi_R$. When δS is weak, ϕ remains massive (i.e. there is no spontaneous

breaking of the global weak symmetry), and we can safely replace $\sin\left(\frac{1}{2}\nabla_{ij}\phi\right)^2$ by its expectation value, and the role of δS is to renormalize $\tilde{t}$ in our rotor theory. All our results remain qualitatively unchanged.

6.5 Model F rotor dynamics – emergent classicality

In this section we consider our third model, consisting of Lindblad evolution for the rotor model with open system $U(1)$ dynamics that includes coherent Hamiltonian dynamics, and also targets several classical long-time steady states. As we shall see, if we start in a pure state with (strong) $U(1)$ symmetry, in $2d$ and $3d$ we can describe a finite-time SW-SSB transition where the system transitions between a regime of quantum dynamics at short times to a long-time dynamical regime which is described by classical Langevin dynamics corresponding to Model F of Halperin-Hohenberg [125].

As in Sec. 6.4, we will consider a lattice of quantum rotors, with number operators $\hat{n}_j$ and phases $\hat{\phi}_j$, but now with coherent Hamiltonian dynamics,

$$\hat{H}_0 = \frac{K}{2} \sum_j \hat{n}_j^2 - J \sum_{\langle ij \rangle} \cos(\hat{\phi}_i - \hat{\phi}_j). \quad (6.67)$$

We choose two sets of jump operators to target two different classical steady states,

$$\hat{\rho}_\phi = \int \prod_j d\phi_j e^{-\beta F_\phi(\phi)} |\phi\rangle\langle\phi|; \quad F_\phi = -J \sum_{\langle ij \rangle} \cos(\phi_i - \phi_j) \quad (6.68)$$

and

$$\hat{\rho}_n = \sum_{\mathbf{n}} e^{-\beta F_n(\mathbf{n})} |\mathbf{n}\rangle\langle\mathbf{n}|; \quad F_n = \frac{K}{2} \sum_j n_j^2. \quad (6.69)$$

Here ϕ and $\mathbf{n}$ again stand for a configuration of ϕ_i and n_i respectively.

To target $\hat{\rho}_\phi$ we introduce our first set of (on-site) jump operators,

$$\hat{L}_{\phi_j}^a = e^{-ia\hat{n}_j} e^{-\frac{\beta}{4}[F_\phi(\hat{\phi}_j+a) - F_\phi(\hat{\phi}_j)]}, \quad (6.70)$$

which shifts the phase ϕ_j by a constant “ a ”, weighted by the change in classical free energy. The corresponding term in the Lindbladian is,

$$\mathcal{L}_\phi[\hat{\rho}] = \gamma_\phi \sum_j \int_a \partial_a^2 \delta(a) [\hat{L}_{\phi_j}^a \hat{\rho} \hat{L}_{\phi_j}^{a\dagger} - \frac{1}{2} \{\hat{L}_{\phi_j}^{a\dagger} \hat{L}_{\phi_j}^a, \hat{\rho}\}], \quad (6.71)$$

and one can show that $\mathcal{L}_\phi[\hat{\rho}_\phi] = 0$. We remark that $\partial_a^2 \delta(a)$ has three peaks: positive peaks at infinitesimal positive and negative values of a , and a negative peak at $a = 0$. Since $\hat{L}_{\phi_j}^{a=0} = \mathbb{K}$, for which the integrand is identically zero, the distribution selects an infinitesimal positive rotation and an infinitesimal negative rotation from the integral.

To target $\hat{\rho}_n$ we choose jump operators acting on links of the form,

$$\hat{L}_{n_{ij}}^\pm = e^{\pm i(\hat{\phi}_i - \hat{\phi}_j)} e^{-\frac{\beta}{4}[F_n(\hat{n}_i \pm 1, \hat{n}_j \mp 1) - F_n(\hat{n}_i, \hat{n}_j)]}, \quad (6.72)$$

which hops a boson weighted by the change in the classical (charging) free energy. The corresponding contribution to the Lindbladian is,

$$\mathcal{L}_n[\hat{\rho}] = \gamma_n \sum_{\langle i,j \rangle} \sum_{\pm} [\hat{L}_{n_{ij}}^\pm \hat{\rho} \hat{L}_{n_{ij}}^{\pm\dagger} - \frac{1}{2} \{\hat{L}_{n_{ij}}^{\pm\dagger} \hat{L}_{n_{ij}}^\pm, \hat{\rho}\}], \quad (6.73)$$

which has the steady state, $\mathcal{L}_n[\hat{\rho}_n] = 0$.

The full Lindblad equation of motion is taken as,

$$\partial_t \hat{\rho} = -i[\hat{H}_0, \hat{\rho}] + \mathcal{L}_\phi[\hat{\rho}] + \mathcal{L}_n[\hat{\rho}]. \quad (6.74)$$

This dynamics has a strong $U(1)$ symmetry since the unitary operator, $\hat{U}_\theta = e^{i\theta \sum_j \hat{n}_j}$ commutes with the Hamiltonian as well as both sets of jump operators,

$$[\hat{U}_\theta, \hat{H}_0] = 0; \quad [\hat{U}_\theta, \hat{L}_{\phi_j}^a] = 0; \quad [\hat{U}_\theta, \hat{L}_{n_{ij}}^\pm] = 0. \quad (6.75)$$

In the limit with $J = K = 0$ we have no coherent Hamiltonian dynamics and our

two sets of jump operators simplify to $\hat{L}_{n_{ij}}^\pm = e^{\pm i(\hat{\phi}_i - \hat{\phi}_j)}$, and $\hat{L}_{\phi_j}^a = e^{-ia\hat{n}_j}$. The former is the jump operator that we explored in detail in Sec. 6.4.

6.5.1 Keldysh path integral for full Lindbladian dynamics

We now consider a Keldysh type path integral for the dynamics of the density matrix. First consider the Hamiltonian dynamics, $\hat{\rho}(t) = e^{-i\hat{H}_0 t} \hat{\rho}(0) e^{i\hat{H}_0 t}$, and focus on the propagator for the matrix elements of the density matrix; $\rho(\phi, \phi'; t) \equiv \langle \phi | \hat{\rho}(t) | \phi' \rangle$,

$$\rho(\phi_f, \phi'_f; t) = \int d\phi_0 d\phi'_0 \mathcal{K}(\phi_f, \phi'_f, t; \phi_0, \phi'_0, 0) \rho(\phi_0, \phi'_0; 0), \quad (6.76)$$

with

$$\mathcal{K}_0(t, 0) = \sum_{W_R, W_L} \int \prod_j D\phi_{Rj}(t) D\phi_{Lj}(t) Dn_{Rj}(t) Dn_{Lj}(t) e^{S_0(t)}, \quad (6.77)$$

where W_R, W_L are integer winding numbers, and the functional integrals are over real fields, $\phi_{R/L}, n_{R/L}$ running from $[-\infty, \infty]$. The boundary conditions on the phase-fields at the initial and final times are,

$$\phi_{Rj}(0) = \phi_{0j}; \quad \phi_{Lj}(0) = \phi'_{0j}; \quad (6.78)$$

$$\phi_{Rj}(t) = \phi_{fj} + 2\pi W_{Rj}; \quad \phi_{Lj}(t) = \phi'_{fj} + 2\pi W_{Lj}. \quad (6.79)$$

The “action” is given by

$$S_0(t) = \int_0^t dt' [\sum_j i(n_{Rj} \dot{\phi}_{Rj} - n_{Lj} \dot{\phi}_{Lj}) + iH_0(\phi_R, \mathbf{n}_R) - iH_0(\phi_L, \mathbf{n}_L)]. \quad (6.80)$$

Next, we add in the Lindbladian dissipation terms. To simplify matters, we assume that J and K are small such that we may approximate to linear order in these parameters.

Then, the full action is $S = S_0 + S_\phi + S_n$, where

$$S_\phi = \gamma_\phi \int_t \sum_j [-\tilde{n}_j^2 - i\beta J \tilde{n}_j \sum_{i \in j} (\sin(\phi_{Ri} - \phi_{Rj}) + \sin(\phi_{Li} - \phi_{Lj}))], \quad (6.81)$$

with $i \in j$ denoting i being a nearest neighbor of j , and

$$S_n = \gamma_n \int_t \sum_{\langle ij \rangle} [2 \cos(\tilde{\phi}_i - \tilde{\phi}_j) - i\beta K(n_i - n_j) \sin(\tilde{\phi}_i - \tilde{\phi}_j)]. \quad (6.82)$$

Here, we have defined "classical" and "quantum" fields,

$$\phi = (\phi_R + \phi_L)/2; \quad \tilde{\phi} = \phi_R - \phi_L, \quad (6.83)$$

and

$$n = (n_R + n_L)/2; \quad \tilde{n} = n_R - n_L. \quad (6.84)$$

We also note that

$$i(n_{Rj} \dot{\phi}_{Rj} - n_{Lj} \dot{\phi}_{Lj}) = i\tilde{n}_j \dot{\phi}_j + i n_j \dot{\tilde{\phi}}_j, \quad (6.85)$$

so that the quantum number field, $\tilde{n}_j$, and the classical phase field, ϕ_j are canonically conjugate pairs, while likewise for the classical number field, n_j and the quantum phase field, $\tilde{\phi}_j$.

Under the strong $U(1)$ symmetry, $\phi_R \rightarrow \phi_R + \theta$, the quantum field transforms appropriately,

$$\tilde{\phi} \rightarrow \tilde{\phi} + \theta, \quad (6.86)$$

and under the weak symmetry, $\phi_{R/L} \rightarrow \phi_{R/L} + \theta$ the classical field transforms, but not the quantum field;

$$\phi \rightarrow \phi + \theta; \quad \tilde{\phi} \rightarrow \tilde{\phi}. \quad (6.87)$$

Thus $e^{i\phi}$ carries weak $U(1)$ charge, while $e^{i\tilde{\phi}}$ carries strong, but not weak charge. Long-

ranged order in the Rényi-1 correlator of the field $e^{i\tilde{\phi}}$ signals a breaking of the strong down to the weak symmetry. If $\langle e^{i\phi} \rangle \neq 0$ the weak symmetry breaks.

6.5.2 Breaking the strong to weak symmetry and emergence of Model F classicality

As in Sec. 6.4 we take our initial state to be pure with $\prod_j |n_j = 0\rangle$. In the simpler rotor model studied therein, for $d \geq 2$ we found a finite time transition, t_c , where the strong symmetry was spontaneously broken down to the weak symmetry. We anticipate that this will likewise occur in the more complicated model studied in this Section. In that case, for $t > t_c$ we can perform a spin-wave expansion for the quantum field, $\tilde{\phi}$, expanding the cosine and sine potentials for small spatial gradients of $\tilde{\phi}_j$, i.e.,

$$\cos(\tilde{\phi}_i - \tilde{\phi}_j) \approx 1 - \frac{1}{2}(\tilde{\phi}_i - \tilde{\phi}_j)^2, \quad (6.88)$$

and

$$\sin(\tilde{\phi}_i - \tilde{\phi}_j) \approx (\tilde{\phi}_i - \tilde{\phi}_j). \quad (6.89)$$

In $d \geq 2$ we are thus precluding vortices in the quantum phase field, which correspond, for example, to a vortex in ϕ_R but not ϕ_L . In $1d$ at finite-time, the presence of quantum phase slips, say in ϕ_R but not ϕ_L , invalidates the spin-wave expansion.

After making the spin-wave expansion we can introduce Hubbard-Stratanovich noise fields, $\eta(t), \xi(t)$ to decouple the terms $\delta S_\phi = -\gamma_\phi \int_t \sum_j \tilde{n}_j^2$ and $\delta S_n = -\gamma_n \int_t \sum_{\langle i,j \rangle} (\tilde{\phi}_i - \tilde{\phi}_j)^2$. The full action then becomes,

$$\begin{aligned} S = & \int_t \sum_j i\tilde{n}_j [\dot{\phi}_j - \partial F / \partial n_j + \beta \gamma_\phi \partial F / \partial \phi_j - \eta_j] \\ & + \int_t \sum_j i\tilde{\phi}_j [\dot{n}_j + \partial F / \partial \phi_j - \beta \gamma_n \nabla_j^2 \partial F / \partial n_j - \xi_j] + S_\eta + S_\xi, \end{aligned} \quad (6.90)$$

where S_η and S_ξ are quadratic in η and ξ . Upon integration over $\tilde{n}_j$ and $\tilde{\phi}_j$ we thereby arrive at two coupled classical Langevin equations,

$$\dot{\phi}_j = \partial F / \partial n_j - \beta \gamma_\phi \partial F / \partial \phi_j + \eta_j, \quad (6.91)$$

$$\dot{n}_j = -\partial F / \partial \phi_j + \beta \gamma_n \nabla_j^2 \partial F / \partial n_j + \xi_j. \quad (6.92)$$

Here, the classical free energy is,

$$F = \frac{1}{2} K \sum_j n_j^2 - J \sum_{\langle i,j \rangle} \cos(\phi_i - \phi_j), \quad (6.93)$$

which is of the identical form as the original Hamiltonian, except here both ϕ and n are classical fields. The noise terms have correlators,

$$\langle \eta_i(t) \eta_j(t') \rangle = 2\gamma_\phi \delta_{ij} \delta(t - t'), \quad (6.94)$$

$$\langle \xi_i(t) \xi_j(t') \rangle = 2\gamma_n \nabla_i^2 \delta_{ij} \delta(t - t'). \quad (6.95)$$

This is a lattice “hard-spin” (rotor) model corresponding to Halperin-Hohenberg Model F dynamics [125]. The classical dynamical steady state satisfies, $\mathcal{P}_{eq}[\phi, \mathbf{n}] \propto e^{-\beta F(\phi, \mathbf{n})}$.

It is worth emphasizing that the field n_j in the classical hydrodynamical description takes on all real values - that is, it is not quantized. This “unquantization” necessarily occurs when there is a SW-SSB transition (as there is in $2d$ and $3d$), since under the spin-wave approximation in the quantum phase field, $\tilde{\phi}$, the conjugate classical density field, n , becomes continuous.

In the superfluid phase, with $\langle e^{i\phi} \rangle \neq 0$, we can perform a spin-wave expansion in the classical phase field. After taking the spatial continuum limit, we end up with a damped

Goldstone hydrodynamic mode, with dispersion,

$$\omega_k = \pm\sqrt{JK}k + i(\gamma_\phi + \gamma_n)k^2 + \dots. \quad (6.96)$$

In the normal phase we can drop the boson hopping terms, $J \rightarrow 0$, as it will renormalize to zero under RG, whence, after taking the spatial continuum limit, we end up with a noisy diffusion equation,

$$\dot{n} = D\nabla^2 n + \xi(x, t); \quad D = \beta K \gamma_n, \quad (6.97)$$

as well as a spatially local phase temporal-diffusion equation of motion,

$$\dot{\phi} = Kn(x) + \eta(x, t). \quad (6.98)$$

In contrast to our spin-1/2 model where classical diffusion sets in for earlier times before SW-SSB, here we might anticipate non-trivial quantum dynamics for $t < t_c$, with completely classical diffusive dynamics only setting in for $t > t_c$. Moreover, if we change the Lindbladian dynamics to explicitly break the strong $U(1)$ symmetry, the density relaxes quickly (as does the order parameter) and we also will not have a propagating hydrodynamic Goldstone mode in the superfluid, while the phase dynamics will have a diffusive mode.

6.5.3 Wigner function and steady state density matrix

As noted above, in the long-time steady state the probability distribution function over the classical fields ϕ and n approaches an equilibrium distribution, $\mathcal{P}_{eq} \propto e^{-\beta F(\phi, \mathbf{n})}$. As we show in Appendix 6.8.4, this distribution function is proportional to the Wigner distribution, $W(\phi, \mathbf{n})$, associated with the long-time fully quantum density matrix $\hat{\rho}_\infty$, with the understanding that here n_j lives on the reals (due to our SW-SSB expansion/approx-

imation) while the *exact* density matrix will have n being (half-)integer. If this is the case, we can Fourier transform to obtain expressions for the (quantum) density matrix,

$$\left\langle \phi + \frac{\tilde{\phi}}{2} \left| \hat{\rho}_\infty \right| \phi - \frac{\tilde{\phi}}{2} \right\rangle = \int \prod_j dn_j e^{i \sum_j n_j \tilde{\phi}_j} W(\phi, \mathbf{n}). \quad (6.99)$$

The integration over real n_j then gives,

$$\left\langle \phi + \frac{\tilde{\phi}}{2} \left| \hat{\rho}_\infty \right| \phi - \frac{\tilde{\phi}}{2} \right\rangle \propto e^{\beta J \sum_{\langle ij \rangle} \cos(\phi_i - \phi_j)} e^{-\sum_j \tilde{\phi}_j^2 / 2K}, \quad (6.100)$$

which shows a suppression of the off-diagonal matrix elements in the ϕ -basis. Notice that the right-hand-side is not 2π -periodic in $\tilde{\phi}_j$, which it would be if we had not made the spin-wave expansion/approximation after assuming SW-SSB.

In the density basis, $\mathbf{n}$, the analogous expression is,

$$\left\langle \mathbf{n} + \frac{\tilde{\mathbf{n}}}{2} \left| \hat{\rho}_\infty \right| \mathbf{n} - \frac{\tilde{\mathbf{n}}}{2} \right\rangle = \int \prod_j d\phi_j e^{-i \sum_j \phi_j \tilde{n}_j} W(\phi, \mathbf{n}). \quad (6.101)$$

While the ϕ_j integrals cannot be performed in general, if we break the weak symmetry and go into the superfluid, replacing $\cos(\phi_i - \phi_j) \approx 1 - (\phi_i - \phi_j)^2/2$, we have

$$\left\langle \mathbf{n} + \frac{\tilde{\mathbf{n}}}{2} \left| \hat{\rho} \right| \mathbf{n} - \frac{\tilde{\mathbf{n}}}{2} \right\rangle \approx e^{-\frac{K}{2} \sum_j n_j^2} e^{-\frac{1}{2J} \int_k |\tilde{n}_k|^2 / k^2}. \quad (6.102)$$

To demonstrate (conventional) “off-diagonal long range order” ODLRO in the superfluid, we consider the correlator,

$$G(x) \equiv \text{Tr}(e^{i(\hat{\phi}_x - \hat{\phi}_0)} \hat{\rho}_\infty) \propto \int \prod_j dn_j \left\langle \mathbf{n} + \frac{\tilde{\mathbf{n}}(x)}{2} \left| \hat{\rho}_\infty \right| \mathbf{n} - \frac{\tilde{\mathbf{n}}(x)}{2} \right\rangle, \quad (6.103)$$

with $\tilde{n}_j(x) = \delta(x_j - x) - \delta(x_j)$, corresponding to “moving” a boson from site 0 to site x .

Plugging this into above gives,

$$G(x) \approx e^{-\frac{1}{2J} \int_k \frac{2(1-\cos(kx))}{k^2}}, \quad (6.104)$$

as could be obtained directly upon using $\hat{\rho}_\infty$ expressed in the phase basis. In $d = 2$ this gives power-law ODLRO $G(x) \sim |x|^{-1/2\pi J}$.

6.6 SW-SSB in classical systems

6.6.1 Generalities

In the previous sections, we discussed SW-SSB as a sharp transition of a quantum system that can occur during decoherence, beyond which classical physics emerges. A hallmark of this classical regime is Model-F dynamics, which appears only after the SW-SSB transition. Within the SW-SSB phase, the system loses its memory of initial particle positions as well as the discrete nature of individual particles, and classical hydrodynamics becomes a legitimate effective description. In this section, we will adopt classical hydrodynamics as an effective theory for the system deep inside the SW-SSB phase. We then verify that the Rényi correlators can be evaluated within this hydrodynamic framework. Our main results are as follows:

- For spatial dimensions $d = 2$ and $d = 3$, the Rényi correlators computed from the hydrodynamic equations are consistent with our earlier analyses, i.e., they exhibit long-range or quasi-long-range order associated with SW-SSB.
- For $d = 1$, the Rényi correlators obtained from hydrodynamics show exponentially decaying correlations. However, fixing the exponential decay constant requires one to pick a “quantum” of charge, thus implicitly introducing some notion of charge discreteness (while hydrodynamics is inherently a continuum theory). Moreover,

despite the exponential decay of Rényi correlators, the CMI computed within hydrodynamics is long-range, inconsistent with what we expect in a strongly symmetric phase. Together, these results suggest that in $1d$ hydrodynamics is not a faithful description of an underlying system with discrete charges, even at late times and large scales: information-theoretic quantities always remain sensitive to discreteness. (We note that at short times, it was proved for discrete classical systems with locally finite configuration spaces that the CMI remains finite, again inconsistent with hydrodynamics [319].)

We begin by reviewing the definition of SW-SSB for classical distributions; see also Sec. 6.2.1.1. We consider a classical system of particles indexed by density configuration $\mathbf{n} = n(x)$, whose probability distribution is $P(\mathbf{n})$. We assume that we only have access to information of the positions of the particles, without labeling each individual particle. Within the diagnoses of SW-SSB discussed in the previous sections, the Bhattacharyya coefficient $B(P(\mathbf{n}) : P(\mathbf{n}')) \equiv \sum_{\mathbf{n}} \sqrt{P(\mathbf{n})P(\mathbf{n}')}$ is a standard measure of the statistical similarity of two classical distributions. Here, the distribution $P(\mathbf{n}')$ is defined with $\mathbf{n}'$ given by $\mathbf{n}$ after “moving one particle” from 0 to r , in a sense that will be made precise below.

While the Bhattacharyya coefficient is associated with the indistinguishability of distributions with shifted charge profiles, it is itself not explicitly associated with any symmetry features. In order to manifest the connection to the strong and weak symmetry, one can map the classical distribution to the following fictitious quantum state in the doubled space:

$$|\Psi^{(Q)}\rangle \sim \sum_{\mathbf{n}} \psi(\mathbf{n})^{(Q)} |\mathbf{n}\rangle_1 \otimes |\mathbf{n}\rangle_2, \quad \psi(\mathbf{n})^{(Q)} = P(\mathbf{n})^{Q/2}. \quad (6.105)$$

The strong and weak symmetries are now defined as the separate and diagonal symmetry

transformations of the doubled space. We can also define the Rényi-correlators:

$$C^{(Q)}(r) \sim \langle\langle \Psi^{(Q)} | e^{-iq(\phi_1+\phi_2)_0} e^{+iq(\phi_1+\phi_2)_r} | \Psi^{(Q)} \rangle\rangle \sim \sum_{\mathbf{n}} P(\mathbf{n})^{Q/2} P(\mathbf{n}')^{Q/2}, \quad (6.106)$$

where ϕ is the conjugate variable of n , and if n is continuous (integer), then the charge q also needs to be continuous (integer). Here, $\mathbf{n}'(x) = \mathbf{n}(x) + q\delta(x) - q\delta(x-r)$. The Rényi correlators can be defined for any Q , but the most accurate diagnosis for SW-SSB is given by $Q = 1$: when $C^{(1)}(x)$ has long range (or quasi-long-range) correlation, i.e. $\lim_{r \rightarrow \infty} C^{(1)}(r)$ does not decay exponentially, the classical system exhibits SW-SSB. Note that Rényi-1 correlator $C^{(1)}$ is precisely the Bhattacharyya coefficient.

It is also illuminating to write the Rényi correlators in the "first-quantized" form, when the particles are discrete (that is, $\mathbf{n}$ is an integer, say given by $\mathbf{n} = 0, 1$):

$$C^{(Q)}(r) \sim \int d^d x_2 \cdots d^d x_n (P(0, x_2, \cdots x_n))^{Q/2} (P(r, x_2, \cdots x_n))^{Q/2}. \quad (6.107)$$

Here the probability distribution function is assumed to be symmetrical under exchange of any two particles, because we are assuming the particles are not labelled. As such, it can be viewed as a (boson) wave function, and the Rényi-correlators take the same form as the more familiar ODLRO of ordinary quantum systems.

Mapping the classical distribution to a fictitious quantum systems has the following benefits:

- The classical-quantum mapping endows a classical probability distribution with well-defined notions of symmetry and symmetry breaking, and allows us to classify classical distributions within the familiar Ginzburg-Landau (GL) paradigm. When the classical distribution undergoes a phase transition, the mapping also provides a natural framework to describe this transition in GL terms. Indeed, in the examples discussed in the previous sections, the transition of the distribution $P(\mathbf{m})$ falls into

a generalized class of conventional phase transitions.

- The mapping also highlights a connection between the “emergent indistinguishability” of classical particle world lines and the familiar indistinguishability of quantum particles. The indistinguishability of quantum bosons underlies the existence of superfluid or ODLRO in ordinary quantum systems. Correspondingly, SW-SSB is possible only when an emergent indistinguishability arises among classical particle world lines. In our setting, this emergent indistinguishability manifests itself through a transition in the winding-number fluctuations $\langle W^2 \rangle$, which we have used as a numerical signature of SW-SSB.
- As was noted in Ref. [276], the doubled state associated with a classical distribution takes the form of a “Gutzwiller-projected wave function”, a widely studied trial wave function in condensed matter theory. This mapping therefore connects very different areas of physics and provides useful insights for the behavior of distributions. For example, motivated by the connection to Gutzwiller wave functions, one naturally expects that if $P(\mathbf{n})$ is a classical density distribution generated by an insulating wave function, then the Bhattacharyya coefficient $B(P(\mathbf{n}) : P(\mathbf{n}'))$ decays exponentially with x , whereas if $P(\mathbf{n})$ is generated by a metallic wave function, then $B(P(\mathbf{n}) : P(\mathbf{n}'))$ saturates to a constant at large x .

6.6.2 Classical fluctuating hydrodynamics

We take as our canonical example classical hydrodynamics in d -dimensions; other classical Gaussian theories can be treated with similar methods. In the hydrodynamic limit, the dynamics of the continuum density field $n_t(x)$ obeys the fluctuating hydrodynamic equation

$$\partial_t n_t(x) = D \nabla^2 n_t(x) + \nabla \cdot \zeta_t(x), \quad \langle \zeta_t(x) \zeta_s(x') \rangle = \gamma_n \delta(t-s) \delta(x-x'). \quad (6.108)$$

Here D is the diffusion constant, and $\zeta_t(x)$ is a Gaussian noise field, with the variance γ_n fixed by the fluctuation-dissipation theorem. We define the structure factor $S_t(k) = \langle n_t(k)n_t(-k) \rangle$, whose solution from the hydrodynamic equation with initial condition $S_0(k)$ is given by

$$S_t(k) = e^{-2Dk^2t} S_0(k) + \frac{\gamma_n}{2D} (1 - e^{-2Dk^2t}). \quad (6.109)$$

Below we will assume that the initial charge configuration is uniform and that $n_t(x)$ is defined relative to this global offset i.e. $n_0(x) = 0$, and $S_0(x) = 0$. Then in real space, the structure factor takes the form

$$S_t(x - x') = \frac{\gamma_n}{2D} \delta(x - x') - \frac{\gamma_n}{2D} \frac{1}{(8\pi Dt)^{d/2}} e^{-(x-x')^2/8Dt}. \quad (6.110)$$

Eq. (6.110) expresses the diffusion of charge correlations in the hydrodynamic limit, as discussed in Sec. 6.3.2.1. At time t , the probability functional is Gaussian:

$$P_t(n(x)) \propto \exp \left(-\frac{1}{2} \int d^d x d^d x' n(x) V_t(x - x') n(x') \right), \quad (6.111)$$

$$V_t(x - x') = S_t^{-1}(x - x') = \int \frac{d^d k}{(2\pi)^d} S_t^{-1}(k) e^{ik \cdot (x-x')}. \quad (6.112)$$

Equivalently, all n -point correlation functions can be calculated in terms of S_t via Wick's theorem.

6.6.2.1 Rényi correlators

We now turn to the computation of Rényi correlators for the Gaussian hydrodynamic theory. For a general Gaussian distribution given by Eq. (6.111), the evaluation is quite

straightforward:

$$C_t^{(Q)}(r) \sim \exp\left(\frac{q^2 Q^2}{4} V_t(r)\right). \quad (6.113)$$

If $V(r)$ is a long range interaction, like the Coulomb, $C^{(Q)}(r)$ will decay exponentially in $1d$ (strongly symmetric), and with a power-law in $2d$ (quasi-long-range SW-SSB). This is the case for the hydrodynamic correlator Eq. (6.109) (with $S_0(k) = 0$):

$$C_t^{(Q)}(r) \sim \exp\left(\frac{q^2 Q^2 D}{2 \gamma_n} \int \frac{d^d k}{(2\pi)^d} \frac{e^{ik \cdot r}}{1 - e^{-2Dk^2 t}}\right). \quad (6.114)$$

If we are interested in the long distance behavior of the Rényi correlator, we can look at the small- k behavior of the quantity above,

$$C_t^{(Q)}(r) \sim \exp\left(\frac{q^2 Q^2}{4 \gamma_n t} \int \frac{d^d k}{(2\pi)^d} \frac{e^{ik \cdot r}}{k^2}\right). \quad (6.115)$$

Its scaling in each dimension is roughly:

$$\begin{aligned} d = 1 & \quad : \quad C_t^{(Q)}(r) \sim \exp\left(-\frac{c}{t}|r|\right), \\ d = 2 & \quad : \quad C_t^{(Q)}(r) \sim \frac{1}{|r|^{c/t}}, \\ d = 3 & \quad : \quad C_t^{(Q)}(r) \sim \text{const.} \end{aligned} \quad (6.116)$$

This scaling matches the long-time behavior of the decohered spin and rotor models discussed in previous sections: in $1d$, the Rényi correlators decay exponentially with a decay length that grows linearly with time, while in $d = 2$ (resp. $d = 3$), Rényi are (quasi)-long-range ordered.

It is interesting to explore the possibility of a finite-time SW-SSB transition, driven

by the hydrodynamics equation. For example, in $2d$ and $3d$, if one starts with a state (a density distribution) with strong $U(1)$ symmetry, and evolves the system according to hydrodynamics, is there a SW-SSB transition at finite t ? The solution Eq. (6.109) suggests that this is not going to happen, even if we start with a distribution with strong symmetry, e.g. $V_0(x) \sim -|x|$, the long wavelength behavior of structure factor $S_t(k)$ is always dominated by the second term of Eq. (6.109) in $d > 1$, therefore the SW-SSB emerges at infinitesimal t . On the other hand, for $d = 1$, if we start with an initial distribution with SW-SSB, like $V_0(x) \sim e^{-m|x|}$, the structure factor $S_t(k)$ is dominated by S_0 in Eq. (6.109), therefore the SW-SSB will persist.

6.6.2.2 Rényi-1 correlator in 1d; numerical evaluation

Below we perform an explicit numerical calculation of the Rényi-1 correlator, for the one-dimensional case within the hydrodynamic description. The density field is defined on a discrete lattice of L sites, with the structure factor given by Eq. (6.109) with the quantized momenta $k_n = 2\pi n/L$, $0 \leq n < L$. We take a lattice spacing a with $x_j = ja$. We will compute the Bhattacharyya distance, $\mathcal{B}_r(P_q^{+-}(\mathbf{n}) : P_q^{-+}(\mathbf{n})) = -\log B(P_q^{+-}(\mathbf{n}) : P_q^{-+}(\mathbf{n}))$; here B is the Bhattacharyya coefficient (Rényi-1 correlator) with charge q “moved” across the system by distance r . Concretely, we have defined conditional probability distributions;

$$P_q^{+-}(\mathbf{n}) = P(\mathbf{n}' | n(0) = q, n(r) = -q), \quad P_q^{-+}(\mathbf{n}) = P(\mathbf{n}' | n(0) = -q, n(r) = q), \quad (6.117)$$

with $\mathbf{n}'$ labelling the density on all sites except $x = 0, r$.

The Bhattacharyya distance can be efficiently computed for Gaussian distributions, and in our case gives

$$\mathcal{B}_r(P_q^{+-}(\mathbf{n}) : P_q^{-+}(\mathbf{n})) = \frac{1}{2} \sum_{ij} \mu_q(x_i) S_q^{-1}(x_i - x_j) \mu_q(x_j), \quad (6.118)$$

where μ_q is the mean density field in the conditional state $P_q^{+-}(\mathbf{n})$, and S_q^{-1} the conditional covariance matrix. The sum runs over all $i, j \neq 0, r/a$. We refer to Appendix 6.8.4 for further details. The important thing to note is that after conditioning, the density average μ_q is displaced into a dipole-like configuration, with negative μ in the vicinity (set by the correlation length $\xi \approx \sqrt{Dt}$) of $x = 0$ and positive μ in the vicinity of $x = r$. The long-distance behavior $r \gg t$ of S_q^{-1} approximates the 1d Coulombic interaction $S_q^{-1} \approx |x - x'|/t$. The Bhattacharyya distance therefore measures the energy of the μ_q -dipole configuration, which therefore goes as $\mathcal{B}_r(t) \approx r/t$ in 1d.

We confirm this physical picture numerically, computing the Bhattacharyya distance for several different times as shown in Fig. 6.12(a). We plot the coefficient rescaled by a factor of t/cL , where $L = 3000$ and $c = a^2 q^2 / \gamma_n$ follows from dimensional analysis. The finite-size scaling collapse $\mathcal{B}_r(t) = \frac{1}{4t} \frac{r}{L} (1 - \frac{r}{L})$ is immediate (orange dashed line). We have checked this result for larger L . In the limit of infinite system size, we observe the result

$$\mathcal{B}_r(t) = \frac{a^2 q^2}{\gamma_n} \frac{r}{2t} \sim \frac{r}{\xi^{(1)}}. \quad (6.119)$$

Here, $\xi^{(1)}(t)$ is the decay length for the fidelity and Rényi-1 correlator, which grows linearly with time, t , as we found in previous sections for our 1d microscopic discrete particle models. This is perhaps surprising, since 1d continuum hydrodynamics does not emerge from our microscopic models, unless we ignore the discrete phase-slip events which will be present in the dynamics. We can well ask whether information theoretic quantities like CMI for the 1d continuum hydrodynamics will be different from what we found in our microscopic discrete particle models. As we shall now show, this is indeed the case.

6.6.2.3 Conditional mutual information

Although the classical diagnostics of SW-SSB for the continuum Gaussian hydrodynamics in the previous section yield a lengthscale which grows linearly in time, the CMI

behaves in a qualitatively distinct way.

We consider a $1d$ geometry where $A \cup B \cup C$ is a proper subset of the system, and B separates A from C , as shown in Fig. 6.6. We denote the size of B by $R_B = |B|$. To compute the CMI for the hydrodynamic theory, we again make use of the Gaussian property of the correlations. We treat the structure factor as a matrix $\mathbf{S}_{ij} = S(x_i - x_j)$ and define the corresponding covariance matrix restricted to a region X as $\mathbf{S}^X$ — for example the matrix of region $A \cup B$ is $\mathbf{S}^{AB}$ (our notation is slightly different from the previous section due to the geometry we consider here). When $\mathbf{n}$ is normally distributed with covariance $\mathbf{S}$, the differential entropy of a subsystem A is given by [7] $H^A = \frac{1}{2} \log \|\mathbf{S}^A\|$, where $\|\cdot\|$ denotes the matrix determinant. Using the CMI definition $I(A : C|B) = H^{AB} + H^{BC} - H^B - H^{ABC}$, we obtain the useful formula

$$I(A : C|B) = \frac{1}{2} \log \frac{\|\mathbf{S}^{AB}\| \|\mathbf{S}^{BC}\|}{\|\mathbf{S}^{ABC}\| \|\mathbf{S}^B\|}. \quad (6.120)$$

As a consequence of the charge conservation law $\int dx S(x, t) = 0$, the spectrum of the full system covariance matrix is gapless with a quadratic zero, $S(k) \approx ak^2$ at small k . This feature leads to surprising behavior in the CMI — namely algebraic decay in space, as we will show below. From a mathematical point of view, this result can be understood from the theory of Toeplitz determinants, and in particular corrections to subsystem entropies which arise due to the “Fisher-Hartwig singularities” in certain critical systems [28, 84, 145, 233, 288]. Here we will give a simple argument in order to show the essential mechanism.

To see the effect of the gapless covariance on the CMI, we replace $S(k)$ with the approximation

$$S_0(k) = c(2 - 2 \cos k), \quad (6.121)$$

where $c = 2\gamma_n Dt$. This dispersion matches the true one for $k \ll (2Dt)^{-1/2} \ll 1$. Intuitively, we expect that correlations contributing to the CMI over a range R_B will be

long-wavelength modes with $k \lesssim R_B^{-1}$. Therefore we expect our approximation to hold for $R_B \gg (2Dt)^{1/2} \gg 1$. The covariance $S_0(k)$ has the advantage that the resulting $\mathbf{S}$ is a tri-diagonal matrix $S_{i,i} = 2c$, $S_{i,i+1} = S_{i,i-1} = -c$, and its determinant can be computed from the recurrence relation $D(R_B) = 2D(R_B - 1) - D(R_B - 2)$, giving $D(R_B) = R_B + 1$. The resulting entropy for region B of size R_B is

$$H_0^B = \frac{1}{2} R_B \log(2\pi ea) + \frac{1}{2} \log(R_B + 1). \quad (6.122)$$

Aside from a volume law term $H_0^B \propto R_B$ (which is sensitive to UV modes but does not contribute to the CMI), we see that the entropy obtains a universal logarithmic correction due to the low energy modes. This directly results in the CMI decaying algebraically in R_B : taking A and C each as a single site for simplicity, we find for $R_B \gg 1$

$$I(A : C|B) = \log(R_B + 2) - \frac{1}{2} \log(R_B + 1) - \frac{1}{2} \log(R_B + 3) = \frac{1}{2} R_B^{-2} + O(R_B^{-3}). \quad (6.123)$$

While the CMI is expected to show a non-universal correction in the region $R_B \lesssim (2Dt)^{1/2}$, the long-distance decay will be governed by the power law $I(A : C|B) \approx \frac{1}{2} R_B^{-2}$ for $R_B \gg (2Dt)^{1/2}$. In Appendix 6.8.3, we perform a calculation of the Rényi-2 CMI for the 1d decohered rotor model, finding agreement with the Gaussian power law decay.

We demonstrate the predicted scaling numerically, considering a finite ring of L sites and evaluating entropies via matrix determinants. We fix A and C each to be regions of a single site, and compute the CMI for increasing separations R_B using formula (6.120). Results are shown in Fig. 6.12(b), for several different times, and rescaled according to $I(A : C|B) = t^{-1} f(R_B/\sqrt{t})$. We find a short-distance regime $R_B \lesssim (2Dt)^{1/2}$ where the CMI is $O(t^{-1})$, and a crossover to the predicted algebraic scaling, including the prefactor $1/2$.

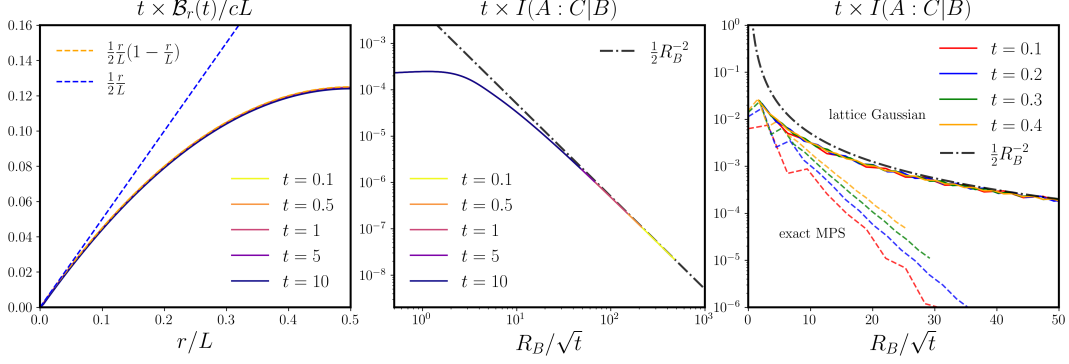


Figure 6.12: (a) Bhattacharyya distance computed according to Eq. (6.118). We show collapsed data $t \times \mathcal{B}^r(t)/cL$, with $c = a^2 q^2 / \gamma_n$, as a function of r/L for different times, and a system size $L = 3000$. The blue dashed line is the curve $y(x) = x/2$ and gives the infinite size scaling limit, from which we obtain $\mathcal{B}_r(t) \sim r/2t$. (b) CMI calculated in the Gaussian hydrodynamic theory, for several different times and a system size $L = 3000$. The CMI exhibits a universal collapse onto the power law decay $I(A : C|B) \approx \frac{1}{2} R_B^{-2}$ for $R_B \gg (2Dt)^{1/2}$. (c) The CMI computed using the Gaussian hydrodynamic formula Eq. (6.120) applied to the exact lattice covariance matrix (solid lines), and the CMI computed from MPS for the spin-1/2 (SSEP) simulations (dashed lines). The Gaussian formula gives the power-law decay (dot-dashed line), while the spin-1/2 CMI decays exponentially at large distances.

Our result should be compared to the numerically obtained CMI for the spin-1/2 model obtained in Section 6.3.2. There, we found an intermediate power-law regime for $(2Dt)^{1/2} \lesssim R_B \ll t$, followed by exponential decay on a scale set by the Markov length, $\xi^M \propto t$. The numerical exponent was found to have a maximum value ≈ 1.4 at the latest accessible times, while the Gaussian prediction gives an exponent of 2; nevertheless we expect the algebraic decay can be traced to the emergence of a continuum hydrodynamic regime, with the discrepancy attributed to finite time deviations. In the hydrodynamic Gaussian theory, the CMI decays algebraically even for $R_B \gg t$. Note, however, that when starting from the microscopic spin-1/2 model, the diffusive hydrodynamic limit is only justified for $a_{UV} \ll R_B \sim (Dt)^{1/2} \ll Dt$ with a_{UV} the UV spatial cutoff. To highlight this point, we perform exact MPS simulations for short times, starting from the Néel state, and exactly evaluate the CMI for small R_B . We also compute the CMI using the Gaussian formula Eq. (6.120) applied to the exact lattice covariance

matrix, which can be obtained either from the MPS or from an exact calculation. The results are displayed in Fig. 6.12(c): the CMI computed for the spin-1/2 model exhibits a clear exponential decay with increasing R_B , whereas the CMI within the Gaussian hydrodynamics approximation matches the expected algebraic decay.

We note that one fundamental difference between the spin-1/2 model and the hydrodynamic description is that the particles are discrete in the former, whereas the hydrodynamic density field, $n(x)$, takes on a continuum of values. In Section 6.5 it was shown that the “spin-wave” approximation for the quantum phase field, $\tilde{\phi}$, is tantamount to ignoring the discreteness of the particles. It is plausible that on spatial scales with $r \ll t$ the spin-wave approximation might be valid, but on the longest scales phase slips cannot be ignored, and the spin-wave approximation which is necessary for obtaining the continuous density field hydrodynamic description is not valid - so does not give a faithful description of the spin-1/2 model dynamics. In any event, it is striking that the exponential decay of the CMI in the spin-1/2 model is not obtained in hydrodynamics.

6.7 Summary, discussion and outlook

6.7.1 Summary

In this chapter we considered three complementary models for the Lindbladian dynamics of systems with $U(1)$ symmetry, focusing on SW-SSB and emergent classicality. The spin-1/2 model with nearest-neighbor $S_i^+ S_j^-$ jump operators is perhaps the simplest model for such physics, and we numerically computed information-theoretic diagnostics of SW-SSB, including the Rényi-1, the Rényi-2 correlators in $1d$ and $2d$, as well as the conditional mutual information and accuracy of charge decoding in $1d$.

We also studied a quantum rotor model under Lindblad evolution with the strong $U(1)$ symmetry in both $1d$ and $2d$. In both dimensions, we uncovered a universal relation between the Rényi-1 and Rényi-2 correlator. This relation implies that in $1d$,

though the system has no $U(1)$ SW-SSB, the correlation lengths of the Rényi-1 and Rényi-2 correlators satisfy $\xi^{(1)} = 2\xi^{(2)} \sim t$ at large t , consistent with our numerical simulation for the spin-1/2 model. In $2d$, there is a finite-time SW-SSB transition into a phase with quasi-long-range algebraic Rényi-1 and Rényi-2 correlators. In this phase the scaling dimension of the Rényi-1 correlator is half of that of the Rényi-2 correlator. Our RG calculation based on the rotor model shows that the $U(1)$ SW-SSB driven by the Lindblad evolution belongs to the same BKT-like universality class that controls the “charge-sharpening” transition in monitored $(1+1)d$ systems with $U(1)$ symmetry [27].

Thirdly, we studied Lindbladian dynamics for rotors that included coherent Hamiltonian evolution and carefully chosen jump operators to target two different long-time steady states. For this model, we found that in $2d$ and $3d$ for times larger than the finite-time SW-SSB transition the dynamics became classical, described by Langevin dynamics corresponding to Model F of Halperin and Hohenberg. In this instance, the finite-time SW-SSB transition in $2d$ and $3d$ is actually a transition where classicality emerges!

In all three of our microscopic models we have demonstrated that in $2d$ (and $3d$) there is a finite-time SW-SSB transition, and at times thereafter classical continuum hydrodynamics emerges, wherein the discreteness of the particles is unimportant and the particle density could be described by a continuous field. In $1d$, due to the presence of phase slips continuum hydrodynamics is not expected to be emergent at any finite time, at least for information-theoretic quantities that are nonlinear in the state/probability distribution, such as CMI.

In Sec. 6.6 we formulated SW-SSB directly within the classical continuum hydrodynamic theory, and computed SW-SSB correlators and CMI. In $2d$ and $3d$ we found that SW-SSB occurs at infinitesimal time (no finite time transition); this is not surprising since the very existence of hydrodynamics required that SW-SSB had already occurred. In $d = 1$, despite our belief that continuum hydrodynamics does not emerge, we used

the theory to compute Rényi correlators and CMI. We found that although the Rényi correlators decay exponentially, with a length scale that grows linearly in time, the CMI within continuum hydrodynamics decays subexponentially at all times: thus, the Markov length in continuum hydrodynamics is always infinite in $1d$, despite the absence of SW-SSB. These results are inconsistent with the behavior we found for $1d$ microscopic models where both Rényi correlators and CMI fall off exponentially, and indicate that $1d$ continuum hydrodynamics does not faithfully capture the long-distance and late-time physics of microscopic models with discrete charges in $1d$. This is consistent with our expectations, however, since phase slips in $1d$ microscopic models always preclude the emergence of continuum hydrodynamics.

6.7.2 Experimental protocol

To measure Rényi correlators in quantum systems, we can perform shadow tomography of the density matrix [1, 130], or we can use the “quantum-classical” estimator [92]. We also propose an alternative variational approach, described below. This variational approach gives an upper bound to the Rényi-1 correlator. It applies directly to classical distributions. For a quantum state, one first replaces the quantum state by a classical bit-string distribution in the computational basis (which is fixed by $U(1)$ symmetry), by applying a fully dephasing channel. This dephased Rényi-1 correlator gives an upper bound to the quantum Rényi-1 correlator as well, by the following argument. The Rényi-1 correlator measures the Holevo fidelity $\mathcal{F}(\rho, \mathcal{Q}_x(\rho))$ between two states ρ and $\mathcal{Q}_x(\rho)$, where $\mathcal{Q}_x(\cdot) = \sigma_x^+ \sigma_0^- \cdot \sigma_0^+ \sigma_x^-$ [302]. Importantly, the linear operation $\mathcal{Q}$ commutes with the dephasing channel $\mathcal{D}$ (since it maps diagonal entries to diagonal entries), so $\mathcal{Q}_x(\mathcal{D}(\rho)) = \mathcal{D}(\mathcal{Q}_x(\rho))$. Thus the fully dephased Rényi-1 correlator can be written as $\mathcal{F}(\mathcal{D}(\rho), \mathcal{D}(\mathcal{Q}_x(\rho))) \geq \mathcal{F}(\rho, \mathcal{Q}_x(\rho))$ because fidelity is non-decreasing under quantum channels.

Beyond being simpler to estimate, the classical Rényi correlator is easier to probe in

near-term experiments, since it only requires computational-basis snapshots. For example, in a quantum gas microscope experiment [18, 263, 301, 304], one takes a simultaneous snapshot of all the particles in the system, leading to a classical probability distribution over classical configurations. Alternatively, one could consider SW-SSB in purely classical systems such as active matter (treating the classical particles as indistinguishable in practice, see discussion below).

Our approach exploits the information-theoretic interpretation of the Rényi-1 correlator as a measure of distinguishability. Specifically, the Rényi-1 correlator is equivalent to the Bhattacharyya coefficient (BC) between the original distribution $P(\mathbf{n})$ and a displaced distribution $P'(\mathbf{n}) = P(\mathbf{n}')$, where $\mathbf{n}'$ is obtained by displacing a single particle within the configuration by certain distance. The BC quantifies the similarity between these two distributions: as $\text{BC} \rightarrow 0$, the distributions become perfectly distinguishable, allowing an ideal classifier to achieve a zero failure rate. Conversely, a finite BC implies an overlap, ensuring that any classifier will suffer from a non-zero failure rate.

Based on this observation, we outline the following protocol:

- Collect snapshots of the density configurations $\mathbf{n}$, or other relevant local degrees of freedom, from the experimental system.
- For each observed configuration $\mathbf{n}$, generate a new configuration $\mathbf{n}'_x$ by shifting a single particle from a random origin by a distance x . This procedure generates a displaced distribution $P'_x(\mathbf{n}) = P(\mathbf{n}'_x)$.
- Train a classifier to distinguish between the ensembles P and P'_x and record the resulting failure rate $R(x)$. More specifically, the classifier will decide whether a given configuration belongs to P or P'_x . The Bayes error bound relates the failure rate of the optimal classifier to the BC (the Rényi-1 correlator) via [72]:

$$\frac{1}{2}C^{(1)}(x) \geq R(x) \geq \frac{1}{2} \left(1 - \sqrt{1 - |C^{(1)}(x)|^2} \right). \quad (6.124)$$

Because any realized classifier may be sub-optimal, $R(x)$ provides an experimental upper bound on the Rényi-1 correlator (i.e., an upper bound on the ideal failure rate). A direct consequence of this relation is that if the failure rate $R(x)$ decays exponentially with x , the Rényi-1 correlator $C^{(1)}(x)$ must also be short-ranged, signaling strongly symmetric, or no SW-SSB.

The classical conditional mutual information can, analogously, be measured by a decoding experiment. One takes a snapshot of all the particles in the system, and feeds the configuration of region B into a decoder (either one of the human-constructed decoders we discussed in Sec. 6.3.2.3, or a suitable machine-learning algorithm). The decoder’s task is to use this information to predict the number of particles in A . As we showed in Sec. 6.3.2.3, its success rate is close to unity when the size of region B exceeds the Markov length. Since a general decoder will be suboptimal, this protocol yields an upper bound on the Markov length.

6.7.3 Discussion and outlook

Our results suggest a natural conjecture as to what properties a classical world might need to have for its dynamics to exhibit SW-SSB transitions. The objects in this classical world might be distinguishable particles, continuum waves, or discrete but indistinguishable objects, like the particles in a lattice gas. If the classical particles are distinguishable, then SW-SSB cannot occur, which is apparent from our world-line/trajectory discussion for SSEP. Because worldlines can be tagged by the particle “labels,” any positional uncertainty can only spread as far as an individual (labelled) particle, and the state is strongly symmetric at the longest distances at all times. In the opposite limit of a continuous classical density field evolving according to (linear) diffusive hydrodynamics, as we showed SW-SSB is present at all times or never present, as discussed in Sec. 6.6. We thus conjecture that “discrete and indistinguishable” is a necessary (but not sufficient) condition required to have “intrinsic” SW-SSB *transitions*, be it a classical or a quantum

world. Of course, even in a world of distinguishable particles, SW-SSB might *appear* to take place, for an imperfect observer without access to the charge labels, but it would not correspond to any fundamental property of the world.

In the quantum world, where particles are truly indistinguishable, our results have implications for the status of continuum hydrodynamics as a theory. Conventionally, hydrodynamics is regarded as an approximate description, which involves coarse-graining and losing information. However, the existence of SW-SSB transitions suggests that continuum hydrodynamics might be valid in a precise information-theoretic sense after the SW-SSB timescale: continuum hydrodynamics contains all the information about charge that can possibly be locally accessed past the SW-SSB timescale. (Note that at finite time after the SW-SSB time scale, this continuum hydrodynamic description will include infinitely many couplings describing nonlinearities, higher-derivatives *etc.*; conventional hydrodynamics like Eq. (6.108) involving only a few parameters emerges only at long times.) Conventionally, the validity of the continuum description does not depend on dimensionality: our results suggest that there are natural questions about state reconstruction that lie beyond hydrodynamics, but only in one dimension. Importantly, the fact that even the continuum diffusion equation does not exhibit SW-SSB in one dimension shows that one cannot fully identify the diffusion mode with the Goldstone mode of $U(1)$ SW-SSB, as has been suggested [131, 225] (outside the steady-state setting explored in that work); in one dimension, the diffusion mode is well defined even though SW-SSB is absent. It would be interesting to revisit recent proposals on the onset of hydrodynamics [67, 119, 165] in light of these results. We note that the connection between SW-SSB and hydrodynamics was discussed in the past (see for instance [66]), but our approach emphasizes a different objective: we aim to derive the hydrodynamics emerging from a completely microscopic theory on the lattice with discrete charges, rather than assuming continuum theory from the beginning.

Our focus was entirely on SW-SSB of the simplest continuous symmetry, $U(1)$. In

this case, the SW-SSB phase need not have long-range entanglement. For nonabelian symmetries like $SU(2)$, even the maximally mixed state in a given charge sector is known to be long-range entangled [183, 208]. How this long-range entanglement emerges under $SU(2)$ -symmetric dissipative dynamics, whether it exhibits singularities at the SW-SSB transition, and whether SW-SSB in $SU(2)$ symmetric systems is related in some way to monitored $SU(2)$ dynamics [199] remain open questions. In the case we explored, the failure of continuum Model-F hydrodynamics to emerge in $1d$ could be directly attributed to charge discreteness being a relevant perturbation, but it is not clear how far this feature generalizes. Understanding the *dynamics* of SW-SSB transitions for higher-form symmetries (relevant, e.g., to the error correction threshold for topological codes) remains another interesting open question. Already for $U(1)$ symmetry the SW-SSB and ordinary SSB transitions are in different universality classes, but the nature of SW-SSB phases and phase transitions with different symmetry groups could be even richer. Finally, while most work on SW-SSB so far has focused on lattice models with internal symmetries, it would be interesting to explore the dynamics of SW-SSB for continuous spatial symmetries like translation or rotation. For example, the Navier–Stokes equations—hydrodynamics based on momentum conservation—might emerge as a consequence of SW-SSB of translational symmetry. A more thorough discussion of this connection could shed light on the sharp onset of hydrodynamics in nature. We leave a detailed exploration of this possibility for future work.

6.8 Appendix: Technical Details for SW-SSB Dynamics

This appendix records the technical steps that are useful for reading this chapter as part of the thesis. The longer paper-style appendices are retained below in the source file but

are not printed in the filed thesis version.

6.8.1 Diagonal sector of the spin-1/2 Lindbladian

In the doubled representation of the density matrix, the spin-1/2 Lindbladian separates into sectors labeled by $S_{L,i}^z - S_{R,i}^z$. The diagonal sector has $S_{L,i}^z = S_{R,i}^z$ and is governed by

$$\mathcal{L}^{\text{diag}} = 2\gamma \sum_{\langle ij \rangle} \left(\mathbf{S}_i \cdot \mathbf{S}_j - \frac{1}{4} \right). \quad (6.125)$$

Thus the diagonal density matrix evolves as an imaginary-time Heisenberg problem. This is the reason that the numerical calculations can use matrix-product-state and stochastic-series-expansion methods while still probing nonlinear diagnostics of the mixed state.

6.8.2 Numerical cross-checks

The Rényi-1 correlator in one dimension was evaluated both by tensor cross interpolation of $|\sqrt{\rho}\rangle$ and by direct sampling from $|\rho\rangle$. The two methods agree within the numerical resolution of the manuscript. The same appendix material also records a spin-1 quantum Monte Carlo check in two dimensions, where the Rényi-2 stiffness shows the expected BKT-type jump while the disorder-averaged stiffness has stronger finite-size drift.

6.8.3 Rotor transfer matrix and CMI

For the decohered rotor model in one dimension, the Rényi-1 correlator can be written as a Coulomb-gas partition function with half-integer electric fields between the two operator insertions. This gives the asymptotic correlation length

$$\xi^{-1} \sim -\log \left(\frac{\theta_2(0, i/(2\pi\tilde{t}))}{\theta_3(0, i/(2\pi\tilde{t}))} \right) + \frac{1}{8\tilde{t}}. \quad (6.126)$$

The same transfer-matrix formalism gives the Rényi- Q conditional mutual information. In the Rényi-2 case, the long-distance decay is exponential when $R_B/t \gg 1$, while finite embedded intervals in a larger chain can show the algebraic behavior discussed in the main text.

6.8.4 Stiffness, Model F, and hydrodynamic distinguishability

The printed chapter uses three additional technical inputs. First, the Rényi-2 stiffness of the two-dimensional rotor model is obtained by imposing a twist in the replicated rotor partition function; in the Gaussian phase it equals $2\tilde{t}$, while the disorder-averaged stiffness equals $\tilde{t}$. Second, after SW-SSB in the rotor Lindbladian with coherent Hamiltonian evolution, the Keldysh action reduces to a classical Model-F stochastic dynamics. Third, the Bhattacharyya distance for Gaussian hydrodynamic distributions is computed from conditional covariance matrices and becomes the energy of a charge dipole. These facts underlie the chapter’s interpretation of SW-SSB as an information-theoretic onset of continuum hydrodynamics.

Chapter 7

Conclusions and Outlook

7.1 Summary

This thesis has studied open quantum systems from the point of view of density matrices as many-body objects. In this setting, measurement, decoherence, and dissipation are not only imperfections that obscure pure-state physics. They define physical processes through which quantum information is lost, redistributed, or made inaccessible to local recovery. The central claim of the thesis is that this process can have universal structure. A density matrix can therefore support phases and phase transitions that are not visible in ordinary expectation values, but become sharp once one probes the nonlinear, replicated, or doubled structure of the state.

The chapters develop this claim in several connected directions. Weak measurement can drive intrinsic transitions in higher-form symmetry order. Decoherence can leave topological information visible to fidelity-type diagnostics even when conventional observables are no longer sufficient. Quantum error-correcting codes under noise provide a setting where mixed-state singularities are directly tied to decodability and statistical-mechanics duality. Measurement and Lindbladian evolution can also generate exotic order in the doubled representation of a density matrix. Finally, strong-to-weak symmetry

breaking gives a dynamical example in which the loss of locally recoverable microscopic charge information is connected to the emergence of hydrodynamic descriptions.

Taken together, these examples demonstrate that mixed-state phases are not a small modification of pure-state phases. They require their own diagnostics and organizing principles. The same density matrix can appear featureless to single-copy observables while retaining sharp structure in fidelities, Rényi quantities, conditional mutual information, or doubled-space order parameters. This distinction is essential in modern quantum platforms, where measurement and noise are intrinsic parts of the physical evolution rather than external nuisances.

Several themes run through these results. Nonlinear diagnostics are necessary for seeing the internal structure of density matrices. Quantities such as $\text{Tr } \rho^n$, Rényi entropies, strange-correlator-like fidelities, and conditional mutual information compare different copies, different boundary conditions, or different local reconstructions of the same mixed state. They therefore ask questions that ordinary linear observables cannot ask. Is topological information still distinguishable after decoherence? Can a local region be recovered from its surroundings? Does a density matrix retain a memory of a symmetry sector even when averaged observables are smooth? These are nonlinear questions in the density matrix, and the phase structure found in this thesis is tied to precisely these questions.

Doubled-Hilbert-space representations provide a natural language for mixed-state order. The Choi representation does not make a mixed state pure in the original physical system. Rather, it exposes the operator structure of the density matrix as a many-body object in a doubled space. This makes it possible to adapt familiar concepts such as correlations, symmetry breaking, and projected wave functions to problems where the physical object is ρ rather than a wave function. The same viewpoint appears in decohered topology, measurement-generated spin-liquid-like states, and strong-to-weak symmetry breaking. It gives a common language for results that otherwise look like

separate phenomena.

Statistical mechanics also reappears in a robust way in open quantum systems. Non-linear functions of density matrices introduce replicas, sums over measurement outcomes, disorder averages, and histories of decoherence. These structures often become partition functions of effective statistical-mechanics models. This is not merely a calculational accident. It reflects the fact that a density matrix contains both quantum operator structure and classical probabilistic structure. Duality, self-duality, and defect constructions then become ways of constraining mixed-state phase diagrams. In the examples studied here, this perspective gives access to universal data that would be difficult to extract directly from the density matrix.

Generalized symmetry remains essential, but its meaning changes in open systems. Higher-form symmetry organizes the response of extended operators to measurement and explains why certain mixed-state transitions are natural in gauge-theoretic settings. Strong and weak symmetries distinguish whether symmetry information is preserved at the level of individual dynamical processes or only after averaging. This distinction has no direct analog in ordinary closed-system ground-state physics, yet it controls the emergence of strong-to-weak symmetry breaking and its connection to recoverability. Generalized symmetry is therefore not only a classification tool. It also identifies what kind of information is protected, hidden, or lost under open-system evolution.

This perspective connects the thesis to information processing. Many of the transitions studied here have an operational meaning. They diagnose whether errors are decodable, whether topological information remains visible to a fidelity probe, whether a local density matrix can be reconstructed from nearby data, or whether microscopic charge histories remain distinguishable after coarse graining. This gives mixed-state universality a concrete interpretation. The singularity is not only a change in a correlation function. It is also a change in what can be learned, recovered, or distinguished from the state.

7.2 Outlook

One immediate direction is experimental. Quantum platforms now allow controlled decoherence, local measurement, postselection, and mid-circuit operations. This makes it realistic to probe nonlinear mixed-state diagnostics, although the protocols are more demanding than measuring ordinary observables. Fidelity-type quantities, Rényi correlations, and decoding-based observables are natural candidates for near-term implementations. A practical challenge is to identify diagnostics that retain the universal content of the theory while requiring the fewest copies, samples, or reconstruction steps. The results of this thesis suggest that the most promising observables are those with a direct information-processing interpretation.

A second direction is numerical. The mixed-state diagnostics emphasized here are often expensive because they involve replicas, square roots of density matrices, or conditional recovery problems. Tensor-network methods, Monte Carlo sampling, tensor cross interpolation, and decoding algorithms each access part of this structure. A systematic numerical toolkit for mixed-state phases would combine these approaches and treat nonlinear diagnostics as primary observables rather than derived quantities. In particular, Markov lengths and conditional mutual information provide a useful bridge between many-body physics and recoverability, but their behavior in higher dimensions and in less idealized dynamics remains poorly understood.

A third direction is classification. Pure-state phases are organized by symmetry, topology, and entanglement, but mixed-state phases require additional distinctions. Strong versus weak symmetry, finite versus divergent Markov length, nonlinear versus linear detectability, and recoverable versus unrecoverable information all appear as organizing principles. A complete classification should clarify which of these distinctions are stable under local quantum channels and which depend on the chosen diagnostic. The results in this thesis suggest that mixed-state phases are best organized not only

by what correlations exist, but by what information can be accessed under physically allowed operations.

A fourth direction is the relation between mixed-state order and hydrodynamics. The strong-to-weak symmetry breaking results suggest that hydrodynamic emergence can be understood as an information-theoretic transition: after a characteristic time, the continuum description may contain all locally recoverable information about the conserved charge. This idea should be tested beyond the $U(1)$ setting. Nonabelian symmetries, higher-form symmetries, spatial symmetries, and systems with momentum conservation may display sharper or qualitatively different versions of this phenomenon. Such examples could clarify when hydrodynamics is only an approximation, and when it is the effective theory of all locally accessible information.

The broader conclusion is that openness changes the object of many-body physics. Once the state is a density matrix, universal structure can live in places that ordinary observables do not see. Measurement and decoherence can erase information, but they can also impose constraints, generate order, and define new phase transitions. The appropriate language is therefore not simply the language of damaged pure states. It is a theory of density matrices, nonlinear diagnostics, generalized symmetry, and recoverability. This thesis develops several pieces of that theory and shows that they are already visible in concrete models motivated by present quantum platforms.

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