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Power System State Estimation with Line Measurements

Yu Zhang, Ramtin Madani, and Javad Lavaei

Abstract— **This paper deals with the power flow (PF) and power system state estimation (PSSE) problems, which play a central role in the analysis and operation of electric power networks. The objective is to find the complex voltage at each bus of a network based on a given set of noiseless or noisy measurements. In this paper, it is assumed that at least two groups of measurements are available: (i) nodal voltage magnitudes, and (ii) one active flow per line for a subset of lines covering a spanning tree of the network. The PF feasibility problem is first cast as an optimization problem by adding a suitable quadratic objective function. Then, the semidefinite programming (SDP) relaxation technique is used to handle the inherent non-convexity of the PF problem. It is shown that as long as voltage angle differences across the lines of the network are not too large (e.g., less than 90° for lossless networks), the SDP problem finds the correct PF solution. By capitalizing on this result, a penalized convex problem is designed to solve the PSSE problem. In addition to a linear term inherited from the SDP relaxation of the PF problem, a cost based on the weighted least absolute value is incorporated in the objective for fitting noisy measurements. The optimal solution of the penalized convex problem is shown to feature a dominant rank-one component formed by lifting the true state of the system. An upper bound on the estimation error is also derived, which depends on the noise power. It is shown that the estimation error reduces as the number of measurements increases. Numerical results for the 1354-bus European system are reported to corroborate the merits of the proposed convexification framework. The mathematical framework developed in this work can be used to study the PSSE problem with other types of measurements.**

I. Introduction

Electrical grid is an automated power system for delivering electricity from suppliers to consumers via interconnected transmission and distribution networks. Accurately determining the operating point and estimating the underlying state of the system are of paramount importance for reliable and economic operations of power networks. Power flow study and power system state estimation play a central role in monitoring the grid, whose solutions are used for major power optimization problems such as unit commitment, security-constrained optimal power flow (OPF), and network reconfiguration.

A. Power Flow Study

The power flow (PF) problem, also known as load flow problem, is a numerical analysis of the electrical power flows at steady state. PF is arguably one of the most important computations in power system analysis, and serves as a necessary prerequisite for determining the best operation of existing systems, as well as for future system planning. Specifically, having measured the voltage magnitudes and injected active/reactive powers at certain buses of a network, the classical PF problem aims to find the unknown voltage magnitude and phase angle at each bus. Using the network impedances and the obtained complex voltages, line power flows can then be determined for the entire system.

The calculation of power flows is essentially equivalent to solving a set of quadratic equations obeying the laws of physics. Finding the solution to a system of nonlinear polynomial equations is NP-hard in general. Beźout's theorem asserts that a well-behaved system can have exponentially many solutions [1]. When it comes to the feasibility of AC power flows, it is known that this problem is NP-hard for both transmission and distribution networks [2], [3].

For solving the PF problem, many iterative algorithms have been proposed and extensively studied over the last few decades [4]. The notable representative is the Newton's method, which features quadratic convergence when the initial point is sufficiently near the solution [5]. For the full Newton's method, the Jacobian matrix has to be recalculated at every iteration, which causes a heavy computational burden. Being valid in most transmission networks, a fast decoupled load flow (FDLF) method was proposed to reduce such a computational cost by means of a decoupling approximation, i.e., neglecting the off-diagonal blocks of the Jacobian [6]. Nevertheless, a fundamental drawback of various Newton-based algorithms is that there is no convergence guarantee. By leveraging advanced techniques in complex analysis and algebraic geometry, sophisticated tools have been developed for solving PF, including holomorphic embedding load flow (HELF) and numerical polynomial homotopy continuation (NPHC) [7], [8]. However, these approaches involve costly computations, and are generally not suitable for large-scale power systems. A review on recent advances in computational methods for the PF equations can be found in [9].

Facing the inherent challenge of non-convexity, convex relaxation techniques have been recently developed for finding the PF solutions [10]. In particular, a class of convex programs has been proposed to solve the PF problem whenever the solution belongs to a recovery region that contains voltage vectors with small angles. The proposed convex programs are in the form of semidefinite programming (SDP), where a suitable linear objective is designed as a surrogate of the rank-one constraint to guarantee the exactness of the SDP relaxation.

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In contrast to the classical PF problem with standard specifications at PV, PQ and slack buses, one objective of this paper is to study the effect of branch flow measurements on reducing the complexity of the PF solution. Motivated by the work [10], we contrive an SDP problem for solving the PF equations. It is shown that the proposed convex program is always exact if: (i) the specifications of nodal voltage magnitude at each bus and line active power flows over a spanning tree of the power network are available, and (ii) the line phase voltage differences are not too large (e.g., less than 90° for lossless networks). By building upon this result, we then address the power system state estimation problem.

B. Power System State Estimation

Closely related to the PF problem, the power system state estimation (PSSE) problem plays an indispensable role in grid monitoring. System measurements are acquired through the supervisory control and data acquisition (SCADA) systems, as well as increasingly pervasive phasor measurement units (PMUs). Given these measurements, the PSSE task aims at estimating complex voltages at all buses and determining the system's operating condition. As a nonlinear least-squares (LS) problem, PSSE is commonly solved by the Gauss-Newton method in practice [11]. The Gauss-Newton method is based on a linear approximation of the residuals in the LS objective, the sum of squares of which is minimized to yield a descent direction at each iteration. However, the iterations may only converge to a stationary point rather than a global optimum. Moreover, it is not easy to quantify the distance of the obtained solution from the true state as a function of noise power.

Based on the convexification technique proposed in this work for the PF problem, we develop a penalized convex problem for solving PSSE. In addition to an ℓ_1 norm penalty that is robust to outliers in the data, the objective function of the penalized convex problem features a linear regularization term whose coefficient matrix can be systematically designed according to the meter placements. We present a theoretical result regarding the quality of the optimal solution of the convex program. It is shown that the obtained optimal solution has a dominant rank-one matrix component, which is formed by lifting the true state vector of the system. The distance between the solution of the penalized convex problem and the correct rank-one component is quantified as a function of the noises. An upper bound for the tail probability of this distance is also derived, which shows the correlation between the number of measurements and the quality of the state estimation.

The focus of this paper is mainly on nodal voltage magnitude and line flow measurements. However, the mathematical framework developed here can be used to study the PSSE problem with other types of measurements.

C. Related Work

Intensive studies of the SDP relaxation technique for solving fundamental problems in power networks have been springing up due to the pioneering papers [12] and [13]. The work [13] develops an SDP relaxation for finding a global optimum of the OPF problem. A necessary and sufficient condition is derived to guarantee a zero duality gap, which is satisfied by many IEEE benchmark systems. From the perspective of physics of power systems, the follow-up papers [14] and [15] present insights into the SDP relaxation's success in handling the non-convexity of the PF equations. The work [16] shows that a global solution of the OPF problem for certain classes of mesh power networks can also be obtained by the SDP relaxation without using transformers. The works [16] and [17] develop a graph-theoretic SDP framework when the traditional SDP relaxation fails to work. The loss over problematic lines is penalized in the SDP's objective, which is instrumental in finding a near-global solution of the OPF problem.

The SDP relaxation has also been utilized for solving the PSSE problem [18], [19]. Using the Lagrangian dual method and alternating direction method of multipliers, distributed implementations are further carried out in [20], [21]. To deal with possible bad data and topology errors, a data fitting cost of weighted least absolute value (WLAV) along with a nuclear norm regularizer has been considered in [22]. Note that when the optimal solution of the SDP relaxation is not rank one, a rank-one approximation (e.g., eigenvalue decomposition) is used to recover an approximate solution for the complex voltages. The quality of the SDP's optimal solution, namely the rank-one approximation error, has not been theoretically studied in previous papers. This intriguing open problem will be studied in the present work.

D. Notations

Boldface lower (upper) case letters represent column vectors (matrices); calligraphic letters stand for sets. The symbols $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers, respectively. $\mathbb{R}^n$ and $\mathbb{C}^n$ denote the spaces of n-dimensional real and complex vectors, respectively. $\mathbb{S}^n$ and $\mathbb{H}^n$ stand for the spaces of $n \times n$ complex symmetric and Hermitian matrices, respectively. The symbols $(\cdot)^\top$ and $(\cdot)^*$ denote the transpose and conjugate transpose of a vector/matrix. $\mathrm{Re}(\cdot)$, $\mathrm{Im}(\cdot)$, $\mathrm{rank}(\cdot)$, $\mathrm{Tr}(\cdot)$, and $\mathrm{null}(\cdot)$ denote the real part, imaginary part, rank, trace, and null space of a given scalar or matrix. $\|\mathbf{a}\|_2$ and $\|\mathbf{A}\|_F$ denote the Euclidean norm of the vector $\mathbf{a}$ and the Frobenius norm of the matrix $\mathbf{A}$, respectively. The relation $\mathbf{X} \succeq \mathbf{0}$ means that the matrix $\mathbf{X}$ is Hermitian positive semidefinite. The (i, j) entry of $\mathbf{X}$ is shown as X_{ij}. $\mathbf{I}_n$ denotes the $n \times n$ identity matrix. The symbol $\mathrm{diag}(\mathbf{x})$ denotes a diagonal matrix whose diagonal entries are given by the vector $\mathbf{x}$, while $\mathrm{diag}(\mathbf{X})$ forms a column vector by extracting the diagonal entries of the matrix $\mathbf{X}$. The expectation operator and imaginary unit are denoted by $\mathbb{E}(\cdot)$ and j, respectively. $\mathbb{P}(\cdot)$ denotes the probability function. The notations $\measuredangle x$ and $|x|$ denote the angle and magnitude of a complex number x.

II. Preliminaries

A. System Modeling

Consider an electric power network represented by a graph $\mathcal{G} = (\mathcal{N}, \mathcal{L})$, where $\mathcal{N} := \{1, \ldots, n\}$ and $\mathcal{L} := \{1, \ldots, L\}$

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denote the sets of buses and branches, respectively. Let $v_k \in \mathbb{C}$ denote the nodal complex voltage at bus $k \in \mathcal{N}$, whose magnitude and phase are shown as $|v_k|$ and $\measuredangle v_k$. The net injected complex power at bus k is denoted as $s_k = p_k + q_k\mathrm{j}$. Define $s_{lf} = p_{lf} + q_{lf}\mathrm{j}$ and $s_{lt} = p_{lt} + q_{lt}\mathrm{j}$ as the complex power injections at the *from* and *to* ends of branch $l \in \mathcal{L}$. Moreover, define the vectors $\mathbf{v} := [v_1, \ldots, v_n]^\top \in \mathbb{C}^n$, $\mathbf{p} := [p_1, \ldots, p_n]^\top \in \mathbb{R}^n$ and $\mathbf{q} := [q_1, \ldots, q_n]^\top \in \mathbb{R}^n$, which collet nodal voltages, net injected active and reactive powers, respectively. Vector $\mathbf{i} \in \mathbb{C}^n$ collects the complex nodal current injections, whereas $\mathbf{i}_f \in \mathbb{C}^L$ and $\mathbf{i}_t \in \mathbb{C}^L$ denote the complex currents at the *from* and *to* ends of all branches. Denote the admittance of each branch (s,t) of the network as y_{st}, which is assumed to have a positive real part and negative imaginary part due to the passivity of real-world transmission lines. The Ohm's law dictates that

$$\mathbf{i} = \mathbf{Y}\mathbf{v}, \quad \mathbf{i}_f = \mathbf{Y}_f\mathbf{v}, \quad \text{and} \quad \mathbf{i}_t = \mathbf{Y}_t\mathbf{v}, \tag{1}$$

where $\mathbf{Y} = \mathbf{G} + \mathbf{B}\mathrm{j} \in \mathbb{S}^n$ is the nodal admittance matrix with the conductance $\mathbf{G}$ and susceptance $\mathbf{B}$ as its real and imaginary parts. Furthermore, $\mathbf{Y}_f \in \mathbb{C}^{L\times n}$ and $\mathbf{Y}_t \in \mathbb{C}^{L\times n}$ represent the *from* and *to* branch admittance matrices. The injected complex power can thus be expressed as $\mathbf{p} + \mathbf{q}\mathrm{j} = \mathrm{diag}(\mathbf{v}\mathbf{v}^*\mathbf{Y}^*)$. Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ denote the canonical vectors in $\mathbb{R}^n$. Define

$$\begin{aligned} &\mathbf{E}_k := \mathbf{e}_k\mathbf{e}_k^\top, \quad \mathbf{Y}_{k,p} := \frac{1}{2}(\mathbf{Y}^*\mathbf{E}_k + \mathbf{E}_k\mathbf{Y}), \\ &\mathbf{Y}_{k,q} := \frac{\mathrm{j}}{2}(\mathbf{E}_k\mathbf{Y} - \mathbf{Y}^*\mathbf{E}_k). \end{aligned} \tag{2}$$

For every $k \in \mathcal{N}$, the quantities $|v_k|^2$, p_k and q_k can be written as

$$|v_k|^2 = \mathrm{Tr}(\mathbf{E}_k\mathbf{v}\mathbf{v}^*),\ p_k = \mathrm{Tr}(\mathbf{Y}_{k,p}\mathbf{v}\mathbf{v}^*),\ q_k = \mathrm{Tr}(\mathbf{Y}_{k,q}\mathbf{v}\mathbf{v}^*). \tag{3}$$

Similarly, the branch active and reactive powers for each line $l \in \mathcal{L}$ can be expressed as

$$\begin{aligned} p_{l,f} = \mathrm{Tr}(\mathbf{Y}_{l,p_f}\mathbf{v}\mathbf{v}^*), \quad p_{l,t} = \mathrm{Tr}(\mathbf{Y}_{l,p_t}\mathbf{v}\mathbf{v}^*) \\ q_{l,f} = \mathrm{Tr}(\mathbf{Y}_{l,q_f}\mathbf{v}\mathbf{v}^*), \quad q_{l,t} = \mathrm{Tr}(\mathbf{Y}_{l,q_t}\mathbf{v}\mathbf{v}^*). \end{aligned} \tag{4}$$

The coefficient matrices $\mathbf{Y}_{l,p_f}, \mathbf{Y}_{l,p_t}, \mathbf{Y}_{l,q_f}, \mathbf{Y}_{l,q_t} \in \mathbb{H}^n$ are defined over the l-th branch from node i to node j as follows:

$$\mathbf{Y}_{l,p_f} := \frac{1}{2}(\mathbf{Y}_f^*\mathbf{d}_l\mathbf{e}_i^* + \mathbf{e}_i\mathbf{d}_l^*\mathbf{Y}_f) \tag{5a}$$

$$\mathbf{Y}_{l,p_t} := \frac{1}{2}(\mathbf{Y}_t^*\mathbf{d}_l\mathbf{e}_j^* + \mathbf{e}_j\mathbf{d}_l^*\mathbf{Y}_t) \tag{5b}$$

$$\mathbf{Y}_{l,q_f} := \frac{\mathrm{j}}{2}(\mathbf{e}_i\mathbf{d}_l^*\mathbf{Y}_f - \mathbf{Y}_f^*\mathbf{d}_l\mathbf{e}_i^*) \tag{5c}$$

$$\mathbf{Y}_{l,q_t} := \frac{\mathrm{j}}{2}(\mathbf{e}_j\mathbf{d}_l^*\mathbf{Y}_t - \mathbf{Y}_t^*\mathbf{d}_l\mathbf{e}_j^*), \tag{5d}$$

where $\{\mathbf{d}_1, \ldots, \mathbf{d}_L\}$ are the canonical vectors in $\mathbb{R}^L$.

So far, all measurements of interest have been expressed as quadratic functions of the complex voltage $\mathbf{v}$. The problem formulations for the PF and PSSE will be presented next.

B. Convex Relaxation of Power Flows

The task of the PSSE problem is to estimate the complex voltages $\mathbf{v}$ based on m real measurements:

$$z_j = \mathbf{v}^*\mathbf{M}_j\mathbf{v} + \eta_j, \quad \forall j \in \mathcal{M} := \{1, 2, \ldots, m\}, \tag{6}$$

where $\{\eta_j\}_{j\in\mathcal{M}}$ are the measurement noises with possibly known statistical information. The measurement matrices $\{\mathbf{M}_j\}_{j\in\mathcal{M}}$ are arbitrary and could be any subset of the Hermitian matrices defined in (2) and (5). The PF problem is a noiseless version of the PSSE problem. More specifically, given a total of m noiseless specifications z_j for $j = 1, 2, \ldots, m$, the goal of PF is to find the nodal complex voltage vector $\mathbf{v}$ satisfying all quadratic measurement equations. That is

$$\text{find} \quad \mathbf{v} \in \mathbb{C}^n \tag{7a}$$

$$\text{subject to} \quad \mathbf{v}^*\mathbf{M}_j\mathbf{v} = z_j, \quad \forall j \in \mathcal{M}. \tag{7b}$$

After setting the phase of the voltage at the slack bus to zero, there are m power flow equations with $2n-1$ unknown parameters. The classical PF problem corresponds to the case $m = 2n-1$. An SDP relaxation of (7) can be obtained as

$$\underset{\mathbf{X}\succeq\mathbf{0}}{\text{minimize}} \quad \mathrm{Tr}(\mathbf{M}_0\mathbf{X}) \tag{8a}$$

$$\text{subject to} \quad \mathrm{Tr}(\mathbf{M}_j\mathbf{X}) = z_j, \quad \forall j \in \mathcal{M}. \tag{8b}$$

This relaxation correctly solves (7) if and only if it has a unique rank-1 solution $\mathbf{X}^{\text{opt}}$, in which case $\mathbf{v}$ can be recovered via the decomposition $\mathbf{X}^{\text{opt}} = \mathbf{v}\mathbf{v}^*$. The above problem is referred to as the SDP relaxation problem with the input vector $\mathbf{z} := [z_1, \ldots, z_m]^\top$ collecting all the measurements. The dual of (8) can be derived as

$$\underset{\boldsymbol{\mu}\in\mathbb{R}^m}{\text{maximize}} \quad -\mathbf{z}^\top\boldsymbol{\mu} \tag{9a}$$

$$\text{subject to} \quad \mathbf{H}(\boldsymbol{\mu}) \succeq \mathbf{0}, \tag{9b}$$

where $\boldsymbol{\mu} = [\mu_1, \ldots, \mu_m]^\top\mathbb{R}^m$ is the Lagrangian multiplier vector associated with the linear equality constraints (8b). The dual matrix function $\mathbf{H} : \mathbb{R}^m \to \mathbb{H}^n$ is equal to

$$\mathbf{H}(\boldsymbol{\mu}) := \mathbf{M}_0 + \sum_{j=1}^m \mu_j\mathbf{M}_j. \tag{10}$$

If strong duality holds and the primal and dual problems both attain their solutions, then every pair of optimal primal-dual solutions $(\mathbf{X}^{\text{opt}}, \boldsymbol{\mu}^{\text{opt}})$ satisfies the relation $\mathbf{H}(\boldsymbol{\mu}^{\text{opt}})\mathbf{X}^{\text{opt}} = \mathbf{0}$, due to the complementary slackness. Hence, the inequality $\mathrm{rank}(\mathbf{X}^{\text{opt}}) \leq 1$ is guaranteed to hold if $\mathrm{rank}(\mathbf{H}(\boldsymbol{\mu}^{\text{opt}})) = n-1$. In this case, the SDP relaxation can recover the solution of the PF problem.

Definition 1. *It is said that the SDP relaxation problem* (8) *recovers the voltage vector* $\mathbf{v} \in \mathbb{C}^n$ *if* $\mathbf{X} = \mathbf{v}\mathbf{v}^*$ *is its unique solution for some input* $\mathbf{z} \in \mathbb{R}^m$.

Definition 2. *A vector* $\boldsymbol{\mu} \in \mathbb{R}^m$ *is regarded as a dual SDP certificate for the voltage vector* $\mathbf{v} \in \mathbb{C}^n$ *if it satisfies the following three properties:*

$$\mathbf{H}(\boldsymbol{\mu}) \succeq 0, \quad \mathbf{H}(\boldsymbol{\mu})\mathbf{v} = 0, \quad \mathrm{rank}(\mathbf{H}(\boldsymbol{\mu})) = n-1. \tag{11}$$

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Denote the set of all dual SDP certificates for the voltage vector $\mathbf{v}$ *as* $\mathcal{D}(\mathbf{v})$.

III. Exact Recovery of Power Flow Solution

The objective of this section is to show that with appropriate nodal and branch noiseless measurements, the SDP relaxation (8) is exact and the correct complex voltage vector $\mathbf{v}$ can then be recovered. Consider a graph $\mathcal{T}$ with the vertex set $\mathcal{N}$ and the edge set $\mathcal{L}_{\mathcal{T}}$ such that $\mathcal{T}$ is a connected subgraph of $\mathcal{G}$. Throughout the rest of this paper, we assume that the available measurements consist of: (i) voltage magnitudes at all buses, (ii) active power flow at the "from" end of each branch of $\mathcal{T}$, and (iii) arbitrary additional measurements. In this case, the SDP relaxation of PF can be expressed as

$$\underset{\mathbf{X}\succeq\mathbf{0}}{\text{minimize}} \quad \text{Tr}(\mathbf{M}_0\mathbf{X}) \tag{12a}$$

$$\text{subject to} \quad X_{kk} = |v_k|^2, \qquad k \in \mathcal{N} \tag{12b}$$

$$\text{Tr}(\mathbf{Y}_{l,p_f}\mathbf{X}) = p_{l,f}, \qquad l \in \mathcal{L}_{\mathcal{T}} \tag{12c}$$

$$\text{Tr}(\mathbf{M}_j\mathbf{X}) = z_j, \qquad j \in \mathcal{M}', \tag{12d}$$

where $\mathcal{M}'$ is the index set for additional measurements.

Definition 3. *The sparsity graph of a matrix* $\mathbf{W} \in \mathbb{H}^n$ *is a simple graph with* n *vertices whose edges are specified by the locations of the nonzero off-diagonal entries of* $\mathbf{W}$. *In other words, two arbitrary vertices* i *and* j *are connected if* W_{ij} *is nonzero.*

Assumption 1. *The edge set of the sparsity graph of* $\mathbf{M}_0$ *coincides with* $\mathcal{L}_{\mathcal{T}}$ *and in addition,*

$$-180^\circ < \measuredangle M_{0;st} - \measuredangle y_{st} < 0, \quad \forall (s,t) \in \mathcal{L}_{\mathcal{T}}, \tag{13}$$

where $M_{0;st}$ *denotes the* (s,t) *entry of* $\mathbf{M}_0$. *Moreover, the solution* $\mathbf{v}$ *being sought satisfies the relations:*

$$0 < (\measuredangle v_s - \measuredangle v_t) - \measuredangle y_{st} < 180^\circ, \quad \forall (s,t) \in \mathcal{L}_{\mathcal{T}} \tag{14a}$$

$$(\measuredangle v_s - \measuredangle v_t) - \measuredangle M_{0;st} \neq 0 \text{ or } 180^\circ, \quad \forall (s,t) \in \mathcal{L}_{\mathcal{T}}. \tag{14b}$$

For real-world transmission systems, $\measuredangle y_{st}$ is close to -90° (to reduce transmission losses), whereas $|\measuredangle v_s - \measuredangle v_t|$ is small due to thermal and stability limits. Hence, the angle condition (14a) is expected to hold. For lossless networks, (14a) requires each line voltage angle difference to be between -90° and 90°. Regarding the matrix $\mathbf{M}_0$, its entry $M_{0;st}$ can be chosen as a complex number with negative real and imaginary parts to meet the conditions (13) and (14b).

Lemma 1. *Under Assumption 1, there exists a dual SDP certificate for the voltage vector* $\mathbf{v} \in \mathbb{C}^n$.

Proof. Due to space restrictions, a sketch of the proof will be provided here only for the case where $\mathcal{T}$ is a spanning tree. Let $\mu_1, \ldots, \mu_n \in \mathbb{R}$ and $\mu_{n+1}, \ldots, \mu_{2n-1} \in \mathbb{R}$ be the Lagrange multipliers associated with the constraints (12b) and (12c), respectively.

Let $\{\tilde{\mathbf{e}}_1, \ldots, \tilde{\mathbf{e}}_m\}$ denote the canonical vectors in $\mathbb{R}^m$. To design one vector $\hat{\boldsymbol{\mu}} \in \mathcal{D}(\mathbf{v})$, for every $l = (s,t) \in \mathcal{L}_{\mathcal{T}}$ define a dual matrix

$$\mathbf{H}^{(l)} := \mathbf{M}_0^{(l)} + \mu_s^{(l)}\mathbf{E}_s + \mu_t^{(l)}\mathbf{E}_t + \mu_{n+l}^{(l)}\mathbf{Y}_{l,p_f}, \tag{15}$$

where

$$\mathbf{M}_0^{(l)} := M_{0;st}\mathbf{e}_s\mathbf{e}_t^\top + M_{0;ts}\mathbf{e}_t\mathbf{e}_s^\top \tag{16}$$

and

$$\boldsymbol{\mu}^{(l)} := -\frac{|v_t|^2\text{Im}(M_{0;st}y_{st}^*) + 2\text{Re}(y_{st})\text{Im}(v_s v_t^* M_{0;st}^*)}{\text{Im}(v_s v_t^* y_{st}^*)}\tilde{\mathbf{e}}_s - \frac{|v_s|^2\text{Im}(M_{0;st}y_{st}^*)}{\text{Im}(v_s v_t^* y_{st}^*)}\tilde{\mathbf{e}}_t + 2 \times \frac{\text{Im}(v_s v_t^* M_{0;st}^*)}{\text{Im}(v_s v_t^* y_{st}^*)}\tilde{\mathbf{e}}_{n+l}. \tag{17}$$

Note that matrix $\mathbf{Y}_{l,pf}$ has only three possible nonzero entries:

$$\mathbf{Y}_{l,pf}(s,t) = \mathbf{Y}_{l,pf}^*(t,s) = -\frac{y_{st}}{2}, \quad \mathbf{Y}_{l,pf}(s,s) = \text{Re}(y_{st}).$$

Under Assumption 1, it can be verified that

$$\mathbf{H}^{(l)}\mathbf{v} = 0 \quad \text{and} \quad \mathbf{H}^{(l)} \succeq 0, \tag{18}$$

for every $l \in \mathcal{L}_{\mathcal{T}}$. Now, set $\hat{\boldsymbol{\mu}}$ as

$$\hat{\boldsymbol{\mu}} := \boldsymbol{\mu}^{(1)} + \cdots + \boldsymbol{\mu}^{(n-1)}. \tag{19}$$

Therefore, we have $\mathbf{H}(\hat{\boldsymbol{\mu}}) = \mathbf{H}^{(1)} + \cdots + \mathbf{H}^{(n-1)}$, which satisfies the first two properties in (11). Moreover, the sparsity graph of $\mathbf{H}(\hat{\boldsymbol{\mu}})$ is the same as $\mathcal{T}$, which guarantees that $\text{rank}(\mathbf{H}(\hat{\boldsymbol{\mu}})) = n-1$. □

Theorem 1. *Under Assumption 1, the SDP relaxation problem* (8) *recovers the voltage vector* $\mathbf{v} \in \mathbb{C}^n$.

Proof. By choosing sufficiently large values for the Lagrange multipliers associated with the voltage magnitude measurements (12b), a strictly feasible point can be obtained for the dual problem (9). Therefore, strong duality holds between the primal and dual SDP problems.

According to Lemma 1, there exists a dual SDP certificate $\hat{\boldsymbol{\mu}} \in \mathcal{D}(\mathbf{v})$ that satisfies (11). Therefore,

$$\text{Tr}(\mathbf{H}(\hat{\boldsymbol{\mu}})\mathbf{v}\mathbf{v}^*) = \mathbf{0} \quad \text{and} \quad \mathbf{H}(\hat{\boldsymbol{\mu}}) \succeq \mathbf{0}. \tag{20}$$

This certifies the optimality of the point $\mathbf{X} = \mathbf{v}\mathbf{v}^*$ for the SDP relaxation problem (8). In addition, the property

$$\text{rank}(\mathbf{H}(\hat{\boldsymbol{\mu}})) = n-1 \tag{21}$$

proves the uniqueness of the primal SDP solution. □

To be able to recover a large set of voltage vectors, Theorem 1 states that there are infinitely many choices for the objective function of the SDP relaxation.

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A. Effect of Reactive Power Line Measurements

In the preceding section, the exactness of the SDP relaxation was studied in the case with the measurement of active power line flows. In what follows, it will be shown that reactive power line flows do not offer the same benefits as active power measurements. Assume that the reactive power flow at the "from" end of each branch of $\mathcal{T}$ is measured as opposed to the active power flow. Moreover, suppose that the index set $\mathcal{M}'$ for the arbitrary additional measurements is empty. In this case, Theorem 1 still holds if the conditions provided in Assumption 1 are replaced by:

$$\mathrm{Re}(M_{0;st}y_{st}^*) \neq 0, \quad \mathrm{Im}(v_s v_t^* M_{0;st}^*) \neq 0 \tag{22a}$$
$$\mathrm{Re}(v_s v_t^* y_{st}^*)\mathrm{Re}(M_{0;st}y_{st}^*) \leq 0. \tag{22b}$$

The following two different scenarios must be considered for condition (22b):

(i) If $90^\circ < (\measuredangle v_s - \measuredangle v_t) - \measuredangle y_{st} \leq 180^\circ$, then $\mathrm{Re}(v_s v_t^* y_{st}^*) < 0$ and $\mathrm{Re}(M_{0;st}y_{st}^*) > 0$, which imply that $-90^\circ \leq \measuredangle M_{0;st} - \measuredangle y_{st} \leq 90^\circ$.

(ii) If $0 \leq (\measuredangle v_s - \measuredangle v_t) - \measuredangle y_{st} < 90^\circ$, then $\mathrm{Re}(v_s v_t^* y_{st}^*) > 0$ and $\mathrm{Re}(M_{0;st}y_{st}^*) < 0$, which imply that $90^\circ \leq \measuredangle M_{0;st} - \measuredangle y_{st} \leq 270^\circ$.

As a result, $\measuredangle M_{0;st}$ must belong to one of the two complementary intervals $[\measuredangle y_{st} + 90^\circ, \measuredangle y_{st} + 270^\circ]$ and $[\measuredangle y_{st} - 90^\circ, \measuredangle y_{st} + 90^\circ]$, depending on the value of $\measuredangle v_s - \measuredangle v_t$. Therefore, it is impossible to design the matrix $\mathbf{M}_0$ without the knowledge of the phase difference $\measuredangle v_s - \measuredangle v_t$.

B. Three-Bus Example

Consider the 3-bus power system shown in Figure 1. Suppose that the measured signals consist of two active power line flow p_{12} and p_{23}, as well as the squared voltage magnitudes $|v_1|^2$, $|v_2|^2$ and $|v_3|^2$. Theorem 1 states that the SDP relaxation problem (8) is able to find the unknown voltage vector $\mathbf{v}$, using an appropriately designed coefficient matrix $\mathbf{M}_0$. It turns out that the vector $\mathbf{v}$ can also be found through a direct calculation. More precisely, one can write

$$\begin{aligned} p_{12} &= \mathrm{Re}(v_1(v_1 - v_2)^* y_{12}^*) = |v_1|^2 \mathrm{Re}(y_{12}) \\ &\quad - |v_1||v_2||y_{12}|\cos(\measuredangle v_1 - \measuredangle v_2 - \measuredangle y_{12}) \end{aligned} \tag{23a}$$
$$\begin{aligned} p_{23} &= \mathrm{Re}(v_2(v_2 - v_3)^* y_{23}^*) = |v_2|^2 \mathrm{Re}(y_{23}) \\ &\quad - |v_2||v_3||y_{23}|\cos(\measuredangle v_2 - \measuredangle v_3 - \measuredangle y_{23}), \end{aligned} \tag{23b}$$

which yields that

$$\measuredangle v_1 - \measuredangle v_2 = \arccos\left(\frac{p_{12} - |v_1|^2\mathrm{Re}(y_{12})}{|v_1||v_2||y_{12}|}\right) + \measuredangle y_{12} \tag{24a}$$
$$\measuredangle v_2 - \measuredangle v_3 = \arccos\left(\frac{p_{23} - |v_2|^2\mathrm{Re}(y_{23})}{|v_2||v_3||y_{23}|}\right) + \measuredangle y_{23}. \tag{24b}$$

Each phase difference $\measuredangle v_1 - \measuredangle v_2$ or $\measuredangle v_2 - \measuredangle v_3$ can have two possible solutions, but only one of them satisfies the angle condition (14a). Hence, all voltage phases $\{\measuredangle v_i\}_{i \in \{1,2,3\}}$ can be easily recovered. This argument applies to all power systems. In other words, the PF problem considered in this paper can be solved by a direct calculation of the phase angles, without having to solve the SDP relaxation (8). Nevertheless, as soon as the measurements are noisy, the equations (24) cannot be used (because quantities $p_{12}, p_{23}, |v_1|^2, |v_2|^2, |v_3|^2$ are no longer available as they are corrupted by noise). In contrast, the proposed SDP relaxation works in both noiseless and noisy cases. This will be elaborated in the next section.

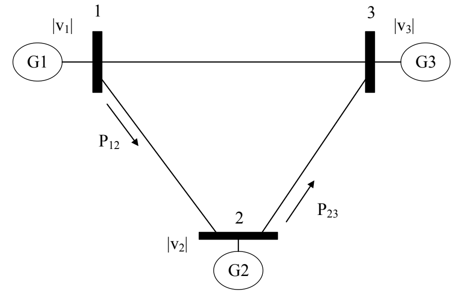

Fig. 1: A 3-bus power system with two active power line measurements p_{12} and p_{23}, as well as three nodal voltage magnitude measurements $|v_1|$, $|v_2|$, and $|v_3|$.

IV. Convexification of State Estimation

Consider the PSSE problem with noisy measurements. It is desirable to find a solution that optimally fits the measurements under a given criterion. For instance, the optimality criterion can be maximum likelihood (ML), weighted least-squares (WLS) or maximum a posteriori (MAP). This amounts to an optimization problem of the form

$$\underset{\mathbf{v}\in\mathbb{C}^n,\, \boldsymbol{\nu}\in\mathbb{R}^m}{\text{minimize}} \quad f(\boldsymbol{\nu}) \tag{25a}$$
$$\text{subject to} \quad z_j - \mathbf{v}^*\mathbf{M}_j\mathbf{v} = \nu_j, \quad \forall j \in \mathcal{M}, \tag{25b}$$

where $\boldsymbol{\nu} := [\nu_1, \ldots, \nu_m]^\top$ and the function $f(\cdot)$ quantifies the estimation criterion. Common choices of $f(\cdot)$ are the weighted ℓ_1 and ℓ_2 norm functions:

$$f_{\mathrm{WLAV}}(\boldsymbol{\nu}) = \|\boldsymbol{\nu}\|_{1,\boldsymbol{\Sigma}^{-\frac{1}{2}}} = \sum_{j=1}^{m} |\nu_j|/\sigma_j, \tag{26}$$
$$f_{\mathrm{WLS}}(\boldsymbol{\nu}) = \|\boldsymbol{\nu}\|_{2,\boldsymbol{\Sigma}^{-1}} = \sum_{j=1}^{m} \nu_j^2/\sigma_j^2, \tag{27}$$

where $\boldsymbol{\Sigma}$ is a constant diagonal matrix with the diagonal entries $\sigma_1^2, ..., \sigma_m^2$. The above functions correspond to the weighted least absolute value (WLAV) and WLS estimators, respectively. Note that the latter estimator is equivalent to the ML estimator if the noise $\boldsymbol{\eta} := [\eta_1, \ldots, \eta_m]^\top$ is normal distributed with zero mean and the covariance matrix $\boldsymbol{\Sigma}$.

Due to the inherent quadratic relationship between the voltage $\mathbf{v}$ and the measured quantities $\{|v_i|^2, \mathbf{p}, \mathbf{q}, \mathbf{p}_l, \mathbf{q}_l\}$, the quadratic equality constraints (25b) make the problem (25) nonconvex and NP-hard in general. To remedy this drawback, consider the penalized convex problem

$$\underset{\mathbf{X}\succeq\mathbf{0},\, \boldsymbol{\nu}\in\mathbb{R}^m}{\text{minimize}} \quad \rho f(\boldsymbol{\nu}) + \mathrm{Tr}(\mathbf{M}_0\mathbf{X}) \tag{28a}$$
$$\text{subject to} \quad \mathrm{Tr}(\mathbf{M}_j\mathbf{X}) + \nu_j = z_j, \quad \forall j \in \mathcal{M}, \tag{28b}$$

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where $\rho > 0$ is a pre-selected coefficient that balances the data fitting cost $f(\boldsymbol{\nu})$ against the regularization term $\mathrm{Tr}(\mathbf{M}_0\mathbf{X})$ that is inherited from the SDP relaxation of the PF problem.

In this paper, we assume that $f(\mathbf{x}) = \|\mathbf{x}\|_{1,\boldsymbol{\Sigma}^{-\frac{1}{2}}}$, and develop strong theoretical results on the estimation error. Note that the penalized convex problem (28) can be expressed as

$$\min_{\mathbf{X}\succeq\mathbf{0}} \mathrm{Tr}(\mathbf{M}_0\mathbf{X})+\rho\sum_{j=1}^{m}\sigma_j^{-1}\left|\mathrm{Tr}\left(\mathbf{M}_j(\mathbf{X}-\mathbf{v}\mathbf{v}^*)\right)-\eta_j\right|. \quad (29)$$

A. Bounded Estimation Error

In this section, we aim to show that the solution of the penalized convex problem estimates the true solution of PSSE, where the estimation error is a function of the noise power.

Theorem 2. *Assume that the solution* $\mathbf{v}$ *of the PSSE problem and the matrix* $\mathbf{M}_0$ *satisfy Assumption 1. Consider an arbitrary dual SDP certificate* $\hat{\boldsymbol{\mu}} \in \mathcal{D}(\mathbf{v})$. *Let* $(\mathbf{X}^{\mathrm{opt}}, \boldsymbol{\nu}^{\mathrm{opt}})$ *denote an optimal solution of the penalized convex program* (28) *for a coefficient* ρ *satisfying the inequality*

$$\rho \geq \max_{j\in\mathcal{M}} |\sigma_j\hat{\mu}_j|. \quad (30)$$

There exists a scalar $\beta > 0$ *such that*

$$\|\mathbf{X}^{\mathrm{opt}}-\beta\mathbf{v}\mathbf{v}^*\|_F \leq 2\sqrt{\frac{\rho\times f_{\mathrm{WLAV}}(\boldsymbol{\eta})\times\mathrm{Tr}(\mathbf{X}^{\mathrm{opt}})}{\lambda}}, \quad (31)$$

where λ *is the second smallest eigenvalue of the matrix* $\mathbf{H}(\hat{\boldsymbol{\mu}})$.

Proof. Observe that

$$\begin{aligned}
&\mathrm{Tr}(\mathbf{M}_0\mathbf{X}^{\mathrm{opt}})+\rho\sum_{j=1}^{m}\sigma_j^{-1}\big|\mathrm{Tr}\big(\mathbf{M}_j(\mathbf{X}^{\mathrm{opt}}-\mathbf{v}\mathbf{v}^*)\big)\big|-\rho f_{\mathrm{WLAV}}(\boldsymbol{\eta})\\
&\overset{(a)}{\leq} \mathrm{Tr}(\mathbf{M}_0\mathbf{X}^{\mathrm{opt}})+\rho\sum_{j=1}^{m}\sigma_j^{-1}\left|\mathrm{Tr}\left(\mathbf{M}_j(\mathbf{X}^{\mathrm{opt}}-\mathbf{v}\mathbf{v}^*)\right)-\eta_j\right|\\
&\overset{(b)}{\leq} \mathrm{Tr}(\mathbf{M}_0\mathbf{v}\mathbf{v}^*)+\rho f_{\mathrm{WLAV}}(\boldsymbol{\eta}), \quad (32)
\end{aligned}$$

where the relation (a) follows from a triangle inequality and the inequality (b) is obtained by evaluating the objective of (29) at the feasible point $\mathbf{v}\mathbf{v}^*$. Therefore, we have

$$\begin{aligned}
&\mathrm{Tr}\left(\mathbf{M}_0(\mathbf{X}^{\mathrm{opt}}-\mathbf{v}\mathbf{v}^*)\right)+\rho\sum_{j=1}^{m}\sigma_j^{-1}\left|\mathrm{Tr}\left(\mathbf{M}_j(\mathbf{X}^{\mathrm{opt}}-\mathbf{v}\mathbf{v}^*)\right)\right|\\
&\leq 2\rho f_{\mathrm{WLAV}}(\boldsymbol{\eta}). \quad (33)
\end{aligned}$$

Recall that $\mathbf{M}_0 = \mathbf{H}(\hat{\boldsymbol{\mu}}) - \sum_{j=1}^{m}\hat{\mu}_j\mathbf{M}_j$ and $\mathbf{H}(\hat{\boldsymbol{\mu}})\mathbf{v} = \mathbf{0}$ (cf. (10) and (20)). Upon defining $\vartheta_j := \mathrm{Tr}\left(\mathbf{M}_j(\mathbf{X}^{\mathrm{opt}}-\mathbf{v}\mathbf{v}^*)\right)$, one can write

$$\sum_{j=1}^{m}(\rho\sigma_j^{-1}|\vartheta_j|-\hat{\mu}_j\vartheta_j)+\mathrm{Tr}(\mathbf{H}(\hat{\boldsymbol{\mu}})\mathbf{X}^{\mathrm{opt}})\leq 2\rho f_{\mathrm{WLAV}}(\boldsymbol{\eta}). \quad (34)$$

Hence, it follows from (30) that

$$\mathrm{Tr}(\mathbf{H}(\hat{\boldsymbol{\mu}})\mathbf{X}^{\mathrm{opt}}) \leq 2\rho f_{\mathrm{WLAV}}(\boldsymbol{\eta}). \quad (35)$$

Now, consider the eigenvalue decomposition of $\mathbf{H}(\hat{\boldsymbol{\mu}}) = \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^*$, where $\boldsymbol{\Lambda} = \mathrm{diag}(\lambda_1,\ldots,\lambda_n)$ collects the eigenvalues of $\mathbf{H}(\hat{\boldsymbol{\mu}})$, which are sorted in descending order. The matrix $\mathbf{U}$ is a unitary matrix whose columns are the corresponding eigenvectors. Define

$$\check{\mathbf{X}} := \begin{bmatrix}\tilde{\mathbf{X}} & \tilde{\mathbf{x}}\\ \tilde{\mathbf{x}}^* & \alpha\end{bmatrix} = \mathbf{U}^*\mathbf{X}^{\mathrm{opt}}\mathbf{U}, \quad (36)$$

where $\tilde{\mathbf{X}} \in \mathbb{H}_+^{n-1}$ is the $(n-1)$-th order leading principal submatrix of $\check{\mathbf{X}}$. It can be concluded from (35) that

$$2\rho f_{\mathrm{WLAV}}(\boldsymbol{\eta}) \geq \mathrm{Tr}(\mathbf{H}(\hat{\boldsymbol{\mu}})\mathbf{X}^{\mathrm{opt}}) = \mathrm{Tr}(\boldsymbol{\Lambda}\check{\mathbf{X}}) \geq \lambda\mathrm{Tr}(\tilde{\mathbf{X}}),$$

where the last inequality follows from the equation $\mathrm{rank}(\mathbf{H}(\hat{\boldsymbol{\mu}})) = n-1$. Therefore, an upper bound for the trace and Frobenius norm of the matrix $\tilde{\mathbf{X}}$ can be obtained as

$$\|\tilde{\mathbf{X}}\|_F \leq \mathrm{Tr}(\tilde{\mathbf{X}}) \leq \frac{2\rho}{\lambda}f_{\mathrm{WLAV}}(\boldsymbol{\eta}).$$

By defining $\tilde{\mathbf{v}} = \mathbf{v}/\|\mathbf{v}\|_2$, the matrix $\mathbf{X}^{\mathrm{opt}}$ can be decomposed as follows:

$$\begin{aligned}
\mathbf{X}^{\mathrm{opt}} &= \mathbf{U}\check{\mathbf{X}}\mathbf{U}^* = \begin{bmatrix}\tilde{\mathbf{U}} & \tilde{\mathbf{v}}\end{bmatrix}\begin{bmatrix}\tilde{\mathbf{X}} & \tilde{\mathbf{x}}\\ \tilde{\mathbf{x}}^* & \alpha\end{bmatrix}\begin{bmatrix}\tilde{\mathbf{U}}^*\\ \tilde{\mathbf{v}}^*\end{bmatrix}\\
&= \tilde{\mathbf{U}}\tilde{\mathbf{X}}\tilde{\mathbf{U}}^* + \tilde{\mathbf{v}}\tilde{\mathbf{x}}^*\tilde{\mathbf{U}}^* + \tilde{\mathbf{U}}\tilde{\mathbf{x}}\tilde{\mathbf{v}}^* + \alpha\tilde{\mathbf{v}}\tilde{\mathbf{v}}^*. \quad (37)
\end{aligned}$$

Since $\check{\mathbf{X}}$ is positive semidefinite, the Schur complement dictates the relation $\tilde{\mathbf{X}} - \alpha^{-1}\tilde{\mathbf{x}}\tilde{\mathbf{x}}^* \succeq \mathbf{0}$. Using the fact that $\alpha = \mathrm{Tr}(\mathbf{X}^{\mathrm{opt}}) - \mathrm{Tr}(\tilde{\mathbf{X}})$, one can write

$$\|\tilde{\mathbf{x}}\|_2^2 \leq \alpha\mathrm{Tr}(\tilde{\mathbf{X}}) = \mathrm{Tr}(\mathbf{X}^{\mathrm{opt}})\mathrm{Tr}(\tilde{\mathbf{X}}) - \mathrm{Tr}^2(\tilde{\mathbf{X}}). \quad (38)$$

Therefore,

$$\begin{aligned}
\|\mathbf{X}^{\mathrm{opt}} - \alpha\tilde{\mathbf{v}}\tilde{\mathbf{v}}^*\|_F^2 &= \|\tilde{\mathbf{U}}\tilde{\mathbf{X}}\tilde{\mathbf{U}}^*\|_F^2 + 2\|\tilde{\mathbf{v}}\tilde{\mathbf{x}}^*\tilde{\mathbf{U}}^*\|_F^2 && (39a)\\
&= \|\tilde{\mathbf{X}}\|_F^2 + 2\|\tilde{\mathbf{x}}\|_2^2 && (39b)\\
&\leq \|\tilde{\mathbf{X}}\|_F^2 - 2\mathrm{Tr}^2(\tilde{\mathbf{X}}) + 2\mathrm{Tr}(\mathbf{X}^{\mathrm{opt}})\mathrm{Tr}(\tilde{\mathbf{X}}) && (39c)\\
&\leq 2\mathrm{Tr}(\mathbf{X}^{\mathrm{opt}})\mathrm{Tr}(\tilde{\mathbf{X}}) && (39d)\\
&\leq \frac{4\rho f_{\mathrm{WLAV}}(\boldsymbol{\eta})}{\lambda}\mathrm{Tr}(\mathbf{X}^{\mathrm{opt}}), && (39e)
\end{aligned}$$

where (39a) follows from the fact that $\tilde{\mathbf{U}}^*\tilde{\mathbf{v}} = \mathbf{0}$, (39b) is due to $\tilde{\mathbf{U}}^*\tilde{\mathbf{U}} = \mathbf{I}_{n-1}$, and (39d) is in light of $\|\tilde{\mathbf{X}}\|_F \leq \mathrm{Tr}(\tilde{\mathbf{X}})$. The proof is completed by choosing β as $\alpha/\|\mathbf{v}\|_2^2$. □

Theorem 2 bounds the estimation error as a function of the noise vector $\boldsymbol{\eta}$. In particular, the error is zero if $\boldsymbol{\eta} = 0$. Define

$$\zeta := \frac{\|\mathbf{X}^{\mathrm{opt}}-\beta\mathbf{v}\mathbf{v}^*\|_F}{\sqrt{n\times\mathrm{Tr}(\mathbf{X}^{\mathrm{opt}})}}. \quad (40)$$

Since $\mathrm{Tr}(\mathbf{X}^{\mathrm{opt}}) \approx n$ holds for the PF problem, the denominator of the above equation is expected to be around n. Hence, the quantity ζ acts as a root-mean-square estimation error. Observe that the estimation error ζ is a random quantity that depends on the realization of the measurements $\mathbf{z}$. Using Theorem 2, we bound the tail probability of ζ below. To this end, define κ as $\frac{m}{n}$. If m were the number of lines in the network, κ was between 1.5 and 2 for most real-world power networks [23].

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Corollary 1. *Under the assumptions of Theorem 2, the tail probability of the root-mean-square estimation error ζ is upper bounded as*

$$\mathbb{P}(\zeta > t) \leq \mathrm{e}^{-\gamma m} \tag{41}$$

for every $t > 0$, where $\gamma = \frac{t^4 \lambda^2}{32 \kappa^2 \rho^2} - \ln 2$.

Proof. Define $\tilde{\eta}_i := \eta_i / \sigma_i$ for $i = 1, \ldots, m$. Then, $\tilde{\boldsymbol{\eta}}$ is a standard normal random vector, and

$$f_{\mathrm{WLAV}}(\boldsymbol{\eta}) = \|\tilde{\boldsymbol{\eta}}\|_1 . \tag{42}$$

Applying the Chernoff's bound [24] to $\tilde{\boldsymbol{\eta}}$ yields that

$$\begin{aligned} \mathbb{P}(\|\tilde{\boldsymbol{\eta}}\|_1 > t) &\leq \mathrm{e}^{-\psi t} \mathbb{E}\, \mathrm{e}^{\psi \|\tilde{\boldsymbol{\eta}}\|_1} = \mathrm{e}^{-\psi t} (\mathbb{E}\, \mathrm{e}^{\psi |\tilde{\eta}_1|})^m \\ &= \mathrm{e}^{-\psi t} \left(\mathrm{e}^{\frac{\psi^2}{2}} \operatorname{erfc}\left(\frac{-\psi}{\sqrt{2}} \right) \right)^m \leq 2^m \mathrm{e}^{(m\psi^2 - 2\psi t)/2} , \end{aligned} \tag{43}$$

which holds for every $\psi > 0$. Note that the complementary error function $\operatorname{erfc}(a) := \frac{2}{\sqrt{\pi}} \int_a^\infty \mathrm{e}^{-x^2}\, \mathrm{d}x \leq 2$ holds for all $a \in \mathbb{R}$. The minimization of the upper bound (43) with respect to ψ gives the optimal solution $\psi^{\text{opt}} = \frac{t}{m}$. Now, it follows from Theorem 2 that

$$\begin{aligned} \mathbb{P}(\zeta > t) &\leq \mathbb{P}\left(2 \sqrt{\frac{\rho \|\tilde{\boldsymbol{\eta}}\|_1}{n \lambda}} > t \right) = \mathbb{P}\left(\|\tilde{\boldsymbol{\eta}}\|_1 > \frac{t^2 n \lambda}{4 \rho} \right) \\ &\leq \exp\left(m \ln 2 - \frac{t^4 n^2 \lambda^2}{32 m \rho^2} \right) . \end{aligned} \tag{44}$$

The proof is completed by substituting $n = \frac{m}{\kappa}$ into (44). □

Recall that the measurements used for solving the PSSE problem include one active power flow per each line of the subgraph $\mathcal{T}$. The graph $\mathcal{T}$ could be as small as a spanning tree of $\mathcal{G}$ or as large as the entire graph $\mathcal{G}$. Although the results developed in this paper work in all these cases, the number of measurements could vary a lot for different choices of $\mathcal{T}$. A question arises as to whether more measurements reduce the estimation error. To address this problem, notice that if it is known that some measurements are corrupted with high values of noise, it is obviously preferable to discard such bad measurements. Now, assume that we have two sets of measurements with similar noise levels. It is aimed to show that the set with a higher cardinality would lead to a better estimation error.

Definition 4. *Define $\omega(\mathcal{T})$ as the minimum of $2\sqrt{\frac{\rho}{n\lambda}}$ over all dual SDP certificates $\hat{\boldsymbol{\mu}} \in \mathcal{D}(\mathbf{v})$, where $\rho = \max_{j \in \mathcal{M}} |\sigma_j \hat{\mu}_j|$ and λ is the second smallest eigenvalue of $\mathbf{H}(\hat{\boldsymbol{\mu}})$.*

In light of Theorem 2, the root-mean-square estimation error ζ satisfies the inequality $\zeta \leq \omega(\mathcal{T}) \sqrt{f_{\mathrm{WLAV}}(\boldsymbol{\eta})}$ if an optimal coefficient ρ is used. The term $\sqrt{f_{\mathrm{WLAV}}(\boldsymbol{\eta})}$ is related to the noise power. If this term is kept constant, then the estimation error is a function of $\omega(\mathcal{T})$.

Theorem 3. *Consider two choices of the graph $\mathcal{T}$, denoted as $\mathcal{T}_1$ and $\mathcal{T}_2$, such that $\mathcal{T}_1$ is a subgraph of $\mathcal{T}_2$. Then, the relation $\omega(\mathcal{T}_2) \leq \omega(\mathcal{T}_1)$ holds.*

Proof. The proof follows from the fact that the feasible set for the dual certificate $\hat{\boldsymbol{\mu}}$ is bigger for $\mathcal{T} = \mathcal{T}_2$ than $\mathcal{T} = \mathcal{T}_1$. □

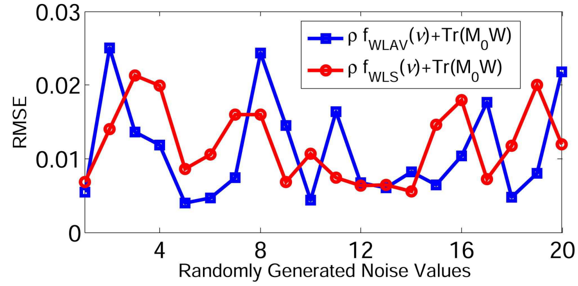

Fig. 2: The root-mean-square error (RMSE) of the estimated voltages obtained by solving problem (28) with different objective functions. The level of noise is set to $c = 0.01$.

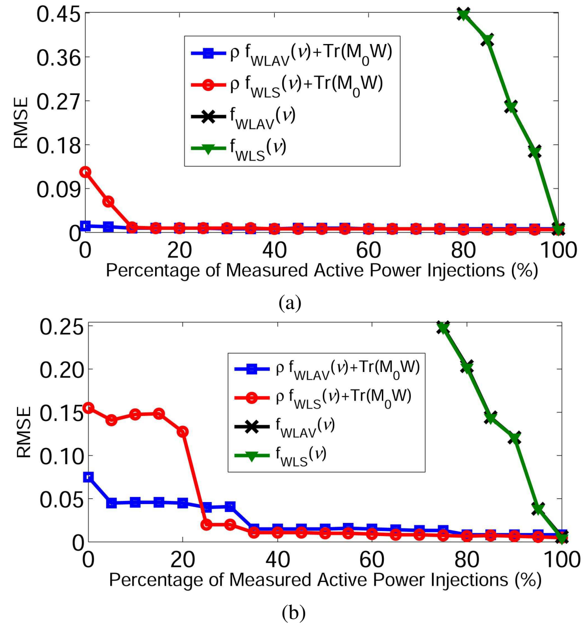

Fig. 3: The RMSE of the estimated voltages obtained by solving problem (28) with different objective functions. The level of noise is set to (a): $c = 0.01$ and (b): $c = 0.02$.

V. Simulations

Recently, it has been shown that certain SDP relaxation based approaches outperform the Newton's method for solving the PSSE [18]–[22]. Those algorithms are special cases of our proposed convex problem (28), where $\rho = +\infty$. In this section, we conduct large-scale simulations on the PEGASE 1354-bus system [25] to show the merits of the penalized convex problem with a finite coefficient ρ. More precisely, we compare the performance of four objective functions:

$$g_1(\mathbf{X}, \boldsymbol{\nu}) := \rho\, f_{\mathrm{WLS}}(\boldsymbol{\nu}) + \operatorname{Tr}(\mathbf{M}_0 \mathbf{X}) \tag{45a}$$

$$g_2(\mathbf{X}, \boldsymbol{\nu}) := \rho\, f_{\mathrm{WLAV}}(\boldsymbol{\nu}) + \operatorname{Tr}(\mathbf{M}_0 \mathbf{X}) \tag{45b}$$

$$g_3(\mathbf{X}, \boldsymbol{\nu}) := f_{\mathrm{WLS}}(\boldsymbol{\nu}) \tag{45c}$$

$$g_4(\mathbf{X}, \boldsymbol{\nu}) := f_{\mathrm{WLAV}}(\boldsymbol{\nu}), \tag{45d}$$

where $\mathbf{M}_0$ is a randomly generated real symmetric matrix with negative values at entries corresponding to the line flow measurements and zero values elsewhere.

In all simulations, we assume at least two groups of measurements are available: (i) nodal voltage magnitudes,

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and (ii) one active flow per line for a subset of lines of the network covering a spanning tree. All measurements are subject to zero-mean Gaussian noises. For squared voltage magnitudes $\{|v_k|^2\}_{k\in\mathcal{N}}$, the standard deviations of the noises are chosen c times the noiseless values of $\{|v_k|^2\}_{k\in\mathcal{N}}$, where $c > 0$ is a pre-selected constant. Likewise, the standard deviations for nodal active/reactive power and line flow measurements are $1.5c$ and $2c$ times the corresponding noiseless values, respectively. When the optimal solution of the penalized SDP is not rank one, the rank-one approximation algorithm in [17] is adopted to recover an estimate solution $\hat{\mathbf{v}}$. Furthermore, we exploit the sparsity structure of the problem to reduce the computational complexity of the SDP. Specifically, through a graph-theoretic algorithm [17], we choose a small subset of the entries of the matrix variable $\mathbf{X}$, and formulate an equivalent reduced-order SDP with respect to those entries.

In Figure 2, the root-mean-square error $\|\hat{\mathbf{v}} - \mathbf{v}\|/\sqrt{n}$ is plotted for 20 realizations of randomly generated noises, where we show the performance of two objective functions g_1 and g_2. The errors corresponding to the functions g_3 and g_4 are so high that they are not plotted in this figure. The parameter ρ is set to 0.5 and the measurements are corrupted by Gaussian noises with $c = 0.01$.

In Figures 3, the effect of the number of measurements on the quality of the estimation is shown for two instances of randomly generated noise values corresponding to (a) $c = 0.01$ and (b) $c = 0.02$, respectively. All four objective functions in (45) are considered. In this experiment, different numbers of additional active power injections are included. The estimation accuracy for each objective function is depicted as a curve with respect to the percentage of nodes with measured active power injections. When the number of measurements is close to the number of unknown parameters, the objective functions g_3 and g_4 produce very high errors that are out of the ranges of the above plots.

VI. Conclusions

In this paper, a convex optimization framework is developed for the power flow (PF) and power system state estimation (PSSE) problems with nodal and line measurements. The quadratic power flow equations are lifted into a higher-dimensional space, which enables their formulation as linear functions of a rank-one positive semidefinite matrix variable. By adding a meticulously-designed linear objective, the PF feasibility problem is relaxed into a convex minimization program. It is shown that as long as voltage angle differences across the lines of the network are not too large (e.g., less than $90°$ for lossless networks), the designed convex problem finds the correct solution of the PF problem. This result along with the proposed framework is then extended to the PSSE problem. Aside from the well-designed linear term, a weighted least absolute value loss is added as the data fitting cost for the noisy and possible bad measurements. This leads to a penalized convex problem. The distance between the optimal solution of the SDP and the true state of the system is quantified in terms of the noise power, and shown to decay as the number of measurements increases. Simulation results on a benchmark test system corroborate the merits of the proposed approach.

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