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# Engineering micromotion in Floquet prethermalization via space-time symmetries

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We present a systematic framework for Floquet prethermalization under strong resonant driving, emphasizing the pivotal role of dynamical space-time symmetries. Our approach demonstrates how dynamical space-time symmetries map onto the projective static symmetry group of the prethermal Hamiltonian governing the prethermal regime. We introduce techniques for detecting dynamical symmetries through the time evolution of local observables, facilitating a detailed analysis of micromotion within each period and surpassing the limitations of conventional stroboscopic Floquet prethermal dynamics. To implement this framework, we present a prethermal protocol that preserves order-two dynamical symmetry in a spin-ladder model, confirming the predicted relationships between the expectation values of local observables at distinct temporal points in the Floquet cycle, linked by this symmetry.

## I. INTRODUCTION

Periodically driven (Floquet) systems have become a versatile platform for extending equilibrium phases to nonequilibrium settings, enabling novel phases with no equilibrium counterparts, including dynamical topological phases [1–10] and exotic spatiotemporal phases such as discrete time crystals (DTCs) [11–15] and quantum many-body scars [16–19]. Recent experimental advancements have further facilitated the realization of these phases [20–31], driving active research in Floquet engineering.

A key challenge in stabilizing these phases is managing energy absorption from periodic driving, which can lead nonintegrable many-body systems to heat up into featureless infinite-temperature states [32, 33]. To counteract this heating, methods that effectively suppress energy absorption through tunable external parameters are crucial. One approach employs strong disorder, inducing robust emergent integrability and confining the system to nonthermal states known as Floquet many-body localized (MBL) states [34–36].

Another strategy is *Floquet prethermalization*, where the system evolves into finite-temperature thermal states governed by a static effective Hamiltonian, termed the prethermal Hamiltonian, in an intermediate time regime before heating up. It has been demonstrated that Floquet many-body systems universally host long-lived prethermal quasi-equilibria in the high-frequency limit, if the local energy scale of the system remains much smaller

than the driving frequency, without requiring integrability or disorder [37–46].

However, the high-frequency limit [47, 48] does not encompass all intriguing Floquet phases. Notable examples, such as anomalous Floquet topological phases [2, 4, 49], arise under moderate to strong driving, where the local energy scale can be large. By shifting to a rotating frame where the driving appears sufficiently weakened, an approximate prethermal Hamiltonian can be constructed to describe the Floquet prethermal phase [50–52]. This approach reveals emergent symmetries in the prethermal Hamiltonian that are absent in the original Hamiltonian, paving the way for prethermal Floquet symmetry-protected topological phases [42, 43] and prethermal DTCs [14, 51].

The time-dependent nature of Floquet systems permits intertwined nonsymmorphic space-time symmetries [53–55], combining fractional temporal translations with spatial transformations, yielding a richer topological classification than static symmetries alone in noninteracting systems [56–59]. Nevertheless, the effects of these dynamical symmetries are not well explored.

In this paper, we investigate space-time symmetric quantum many-body systems, focusing on their quasi-steady prethermal states, which offer a stable framework for exploring novel phases emerging from dynamical symmetries. Specifically, we identify the symmetry group  $G_{st}^{int}$  of the prethermal Hamiltonian, derived from the dynamical space-time symmetry group  $G_{st}$  preserved by the driving protocol. Building on previous studies of static symmetry groups [42, 51], we rigorously extend this framework to encompass dynamical symmetries.

We further demonstrate that space-time symmetric driving protocols can be employed to engineer nontrivial micromotion during the prethermal stage. To concretize this idea, we analyze a quantum spin ladder subjected to

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a dual-driving protocol: one with an energy scale comparable to the driving frequency and another significantly weaker. When the driving protocol preserves an order-two dynamical symmetry, we show that the expectation values of certain observables oscillate with equal magnitude but opposite signs at integer and half-integer stroboscopic times. Crucially, this behavior is robust, independent of the initial state, and thus highly amenable to experimental realization.

## II. DYNAMICAL SPACE-TIME SYMMETRY GROUP

We consider Floquet systems with Hamiltonians  $\hat{H}(t) = \hat{H}(t+T)$  that exhibit a symmetry group  $G_{st}$ , encompassing both static and dynamical symmetries transforming both spatial and temporal coordinates [53–55]. Within  $G_{st}$ , subgroups  $M$  and  $A$  represent unitary and antiunitary symmetries, respectively. For order-2 discrete symmetries, under the assumption that  $\hat{g}_M \in M$  and  $\hat{g}_A \in A$  commute, we have

$$\begin{aligned}\hat{g}_M \hat{H}(t) \hat{g}_M^{-1} &= \hat{H}(t+T/2), \\ \hat{g}_A \hat{H}(t) \hat{g}_A^{-1} &= \hat{H}(-t+s),\end{aligned}\quad (1)$$

where  $s \in [0, T]$  is a reference point within the driving period  $T$ . Without loss of generality, we set  $s = T/2$  [57].

## III. PRETHERMALIZATION IN A DUAL-DRIVING SETUP

We consider a Hamiltonian  $\hat{H}(t) = \hat{H}_0(t) + \hat{V}(t)$ , where both  $\hat{H}_0(t)$  and  $\hat{V}(t)$  are  $T$ -periodic. The resonant drive  $\hat{H}_0(t)$  has a local energy scale comparable to the driving frequency  $\Omega = 2\pi/T$ , and we impose that its local terms commute at each moment of time. The weak drive  $\hat{V}(t)$  has a typical energy scale  $\lambda \sim 2kJ \ll \Omega$ , involving interactions up to  $k$ -local terms. Here  $J$  denotes the characteristic local interaction strength, with the cumulative interaction strength involving any site bounded by  $J$  [38, 39, 60]. The resonant drive is designed such that the one-period evolution operator generated solely by  $\hat{H}_0(t)$  is

$$\hat{U}_0(T, 0) = \mathcal{T}[e^{-i \int_0^T dt \hat{H}_0(t)}] = \hat{X}, \quad \hat{X}^N = \mathbb{I}, \quad (2)$$

which implements a spatially local  $\mathbb{Z}_N$  unitary action. In the absence of  $\hat{H}_0(t)$ , this system lies in the high-frequency regime, while turning on  $\hat{H}_0(t)$  moves it beyond this regime.

To derive the prethermal static Hamiltonian for this dual-driving setup, we turn to the interaction picture with respect to  $\hat{H}_0(t)$ , writing  $\hat{U}_0(t) \equiv \hat{U}_0(t, 0)$ . The interaction picture Hamiltonian is then

$$\hat{H}_{int}(t) = i \left[ \partial_t \hat{U}_0^\dagger(t) \right] \hat{U}_0(t) + \hat{U}_0^\dagger(t) \hat{H}(t) \hat{U}_0(t). \quad (3)$$

Since  $\hat{H}(t) = \hat{H}_0(t) + \hat{V}(t)$ , this reduces to  $\hat{H}_{int}(t) = \hat{U}_0^\dagger(t) \hat{V}(t) \hat{U}_0(t) = \hat{V}_{int}(t)$ . From  $\hat{U}_0(NT) = \mathbb{I}$  in Eq. (2) of Symmetry X, it follows that  $\hat{H}_{int}(t)$  is  $NT$ -periodic. This reformulation maps the initial strong  $T$ -periodic drive into a weak  $NT$ -periodic drive with the local energy scale  $\lambda_{int} \sim \lambda$  of  $\hat{V}_{int}(t)$  satisfying  $1 \ll \Omega/(N\lambda) = \tilde{\Omega}$ . This enables a van Vleck high-frequency expansion in terms of a unitary rotation that relates  $\hat{H}_{int}(t)$  to the Floquet Hamiltonian  $\hat{V}_{int}^{FM}$ , which is defined through  $\hat{U}_{int}(NT) = \mathcal{T}[e^{-i \int_0^{NT} dt \hat{V}_{int}(t)}] = e^{-i \hat{V}_{int}^{FM} NT}$ .

To perform this expansion, we introduce the rotation  $\hat{U}_n(t) = e^{-i \hat{K}_n(t)}$ , where the kick operator  $\hat{K}_n(t)$  is  $NT$ -periodic and expressed as  $\hat{K}_n(t) = \sum_{i=1}^n \hat{K}^{[i]}(t)$ , truncated to order  $n$ . Each term  $\hat{K}^{[i]}(t)$  scales as  $O(\tilde{\Omega}^{-i})$ , with the boundary condition  $\int_0^{NT} dt \hat{K}^{[i]}(t) = 0$  imposed in the van Vleck gauge [47, 48, 50]. This rotation yields a truncated static Hamiltonian  $\hat{D}_n$  up to order  $n \leq n_* = O(\tilde{\Omega})$  [38, 39], as given by

$$\begin{aligned}\hat{H}_{int}^R(t) &= \hat{U}_n^\dagger(t) \left[ \hat{H}_{int}(t) - i \partial_t \right] \hat{U}_n(t) \\ &\simeq \hat{D}_n + \hat{V}_{int}^{[n]}(t) + O(\tilde{\Omega}^{-n-1}),\end{aligned}\quad (4)$$

where  $\hat{D}_n = \sum_{i=0}^n \hat{V}_{vV}^{[i]}$ , with  $\hat{V}_{vV}^{[i]}$  scaling as  $O(\tilde{\Omega}^{-i}\lambda)$ . The residual time-dependent term in the dressed Hamiltonian  $\hat{H}_{int}^R(t)$ ,  $\hat{V}_{int}^{[n]}(t) = \partial_t \hat{K}^{[n+1]}(t)$ , is  $NT$ -periodic and of order  $O(\tilde{\Omega}^{-n}\lambda)$ . The evolution operator  $\hat{U}_{int}(t_2, t_1) = \hat{U}_n(t_2) \mathcal{T}[e^{-i \int_{t_1}^{t_2} dt \hat{H}_{int}^R(t)}] \hat{U}_n^\dagger(t_1)$ , governed approximately by the prethermal Hamiltonian  $\hat{D}_n$  in Eq. (4), simplifies to

$$\hat{U}_{int}(t_2, t_1) \simeq \hat{U}_n(t_2) e^{-i \hat{D}_n(t_2-t_1)} \hat{U}_n^\dagger(t_1), \quad (5)$$

defining the prethermal regime for times exponentially long in  $\tilde{\Omega}$ , provided  $1 \ll \tilde{\Omega}$ . Further details on the van Vleck expansion and explicit forms of the terms  $\hat{V}_{vV}^{[i]}$  and  $\hat{K}^{[i]}(t)$  are provided in the Supplemental Material (SM) [60] (also see Refs. [50–52]).

## IV. EXTENDED DYNAMICAL SPACE-TIME SYMMETRY GROUP IN THE INTERACTION PICTURE

Let us now derive the extended dynamical symmetry group,  $G_{st}^{int}$ , of the interaction picture Hamiltonian  $\hat{H}_{int}(t)$ , given the dynamical symmetry group  $G_{st} = M \times A = \mathbb{Z}_2 \times \mathbb{Z}_2^T$  of the original Hamiltonian  $\hat{H}(t)$ . The element  $\hat{X}$  in Eq. (2) forms a subgroup  $\mathbb{Z}_N \in G_{st}^{int}$  through the transformation

$$\hat{X}^m \hat{H}_{int}(t) \hat{X}^{-m} = \hat{H}_{int}(t - mT). \quad (6)$$

This emergent  $\mathbb{Z}_N$  symmetry of  $\hat{H}_{int}(t)$ , absent in  $\hat{H}(t)$ , arises solely from the resonant drive  $\hat{H}_0(t)$  and leads to prethermal DTCs [14, 42, 51].

For a unitary element  $\hat{g}_M \in M \subset G_{st}$ , defined in Eq. (1), the corresponding interaction picture element is identified as  $\hat{g}_M^{int} = \hat{U}_0^{-1}(T/2)\hat{g}_M$ , where  $\hat{g}_M^{int}$  represents the projective realization of  $\hat{g}_M$ . This is demonstrated by the relation  $[\hat{g}_M^{int}]^2 = \hat{X}^{-1}\hat{g}_M^2$ . Similarly, for an antiunitary element  $\hat{g}_A \in A \subset G_{st}$ , defined by setting  $s = T/2$  in Eq. (1), the interaction picture counterpart is  $\hat{g}_A^{int} = \hat{U}_0^{-1}(T/2)\hat{g}_A$  with the corresponding relation  $[\hat{g}_A^{int}]^2 = \hat{g}_A^2$ . These elements act on  $\hat{H}_{int}(t)$  according to symmetry relation

$$\hat{g}_{M/A}^{int} \hat{H}_{int}(t) [\hat{g}_{M/A}^{int}]^{-1} = \hat{H}_{int}(T/2 \pm t). \quad (7)$$

The algebraic structure of  $G_{st}^{int}$  is further defined by

$$\hat{g}_M^{int} \hat{X} [\hat{g}_M^{int}]^{-1} = \hat{X}, \quad \hat{g}_A^{int} \hat{X} [\hat{g}_A^{int}]^{-1} = \hat{X}^{-1}. \quad (8)$$

These relations establish the extended dynamical symmetry group in the interaction picture as  $G_{st}^{int} = \mathbb{Z}_2 \times (\mathbb{Z}_N \rtimes \mathbb{Z}_2^T)$ , encapsulating the interplay between the emergent  $\mathbb{Z}_N$  symmetry and the original symmetry group  $G_{st}$ . Detailed derivations and further discussions on these symmetry elements are provided in the SM [60].

## V. SYMMETRY GROUP OF THE PRETHERMAL HAMILTONIAN

Now we have established the dynamical symmetry group in the interaction picture, we turn to symmetry group of the truncated prethermal Hamiltonian  $\hat{D}_n$ . This inherits the symmetries  $\hat{X}$ ,  $\hat{g}_M^{int}$ , and  $\hat{g}_A^{int}$  from  $G_{st}^{int}$  of  $\hat{H}_{int}(t)$ , as specified in Eqs. (6) and (7). We define  $\hat{V}_m$  as

$$\hat{V}_m = \frac{1}{NT} \int_0^{NT} dt \hat{H}_{int}(t) e^{im(\Omega/N)t}. \quad (9)$$

The  $i$ -th order van Vleck expansion term,  $\hat{V}_V^{[i]}$ , in  $\hat{D}_n$  contains terms with numerators of the form  $\hat{V}_{m_1} \hat{V}_{m_2} \cdots \hat{V}_{m_i}$ . Symmetry operations act on these terms as follows

$$\begin{aligned} \hat{X} \hat{V}_{m_1} \cdots \hat{V}_{m_i} \hat{X}^{-1} &= e^{i\frac{2\pi}{N}(\sum_{k=1}^i m_k)} \hat{V}_{m_1} \cdots \hat{V}_{m_i}, \\ \hat{g}_M^{int} \hat{V}_{m_1} \cdots \hat{V}_{m_i} [\hat{g}_M^{int}]^{-1} &= e^{-i\frac{2\pi}{2N}(\sum_{k=1}^i m_k)} \hat{V}_{m_1} \cdots \hat{V}_{m_i}, \\ \hat{g}_A^{int} \hat{V}_{m_1} \cdots \hat{V}_{m_i} [\hat{g}_A^{int}]^{-1} &= e^{-i\frac{2\pi}{2N}(\sum_{k=1}^i m_k)} \hat{V}_{m_1} \cdots \hat{V}_{m_i}. \end{aligned} \quad (10)$$

TABLE I. Mapping dynamical space-time symmetries to static symmetries of the prethermal Hamiltonian.

|                                | $\hat{H}(t)$                         | $\hat{D}_n^a$                                               |
|--------------------------------|--------------------------------------|-------------------------------------------------------------|
| Symmetry Group                 | $G_{st}$                             | $G_{st}^{int}$                                              |
| Symmetry Elements <sup>b</sup> | $\hat{g}_M, \hat{g}_A$               | $\hat{g}_M^{int}, \hat{g}_A^{int}, \hat{X}$                 |
| Group Algebra                  | $\mathbb{Z}_2 \times \mathbb{Z}_2^T$ | $\mathbb{Z}_2 \times (\mathbb{Z}_N \rtimes \mathbb{Z}_2^T)$ |

<sup>a</sup> Prethermal Hamiltonian truncated to  $n \leq n_* = O(\tilde{\Omega})$  in Eq. (4)

<sup>b</sup>  $\hat{g}_{M,A} \in G_{st}$  in Eq. (1),  $\hat{g}_{M/A}^{int} = \hat{U}_0^{-1}(T/2)\hat{g}_{M/A}$ ,  $\hat{X}$  in Eq. (2)

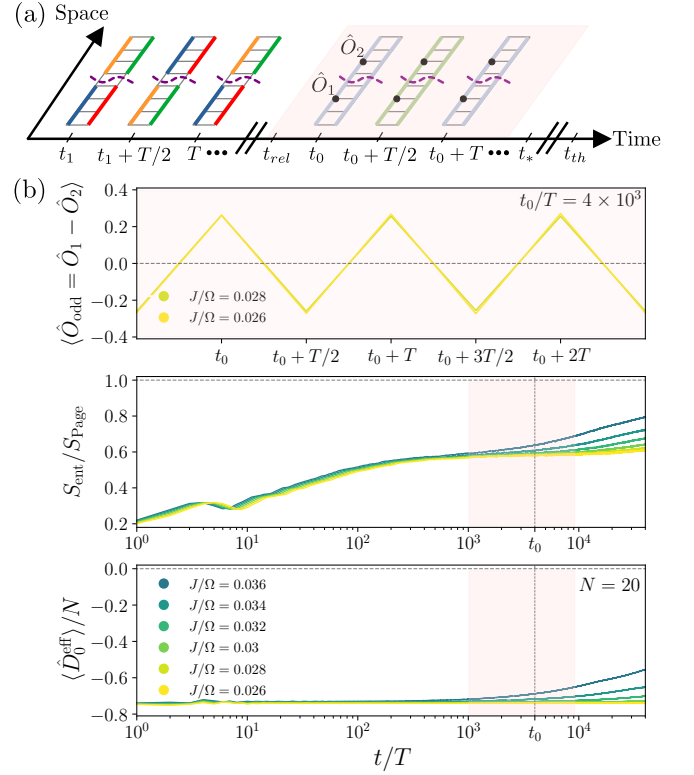


FIG. 1. (a) Schematic illustrating the Floquet thermalization of a two-leg spin-1/2 ladder model with a Hamiltonian preserving the order-2 dynamical symmetry  $\hat{g}_M$  in Eq. (1). The upper left and right legs are colored in orange and green, and the lower left and right legs are colored in blue and red. After relaxation at  $t_{rel}$ , the system enters a prethermal regime governed approximately by the static prethermal Hamiltonian in Eq. (4) until  $t_*$ , with  $t_0$  marking a reference time within this regime. Beyond  $t_*$ , heating drives the system to an infinite temperature state at  $t_{th}$ . (b) Time evolution of  $\langle \hat{O}_{odd}(t) \rangle$  with  $e^{i\alpha_M} = -1$  at  $t = mT$  and  $t = mT + T/2$  in the prethermal regime verifies Eq. (16), detecting  $\hat{g}_M$ . Shaded regions denote the prethermal regime, with bipartite entanglement and energy density of the prethermal Hamiltonian consistent across a range of frequencies. Model parameters in Eq. (18):  $\tau = 0.25$ ,  $J_1 = \Omega$ ,  $J' = J/(0.5 - \tau) = 4$ ,  $\lambda_a^{LR} = \lambda_b^{LR} = 0.5$ ,  $g_x = g_y = 0.45225$ ,  $g_z = 0.7$ , and  $g_{zz} = 1.3$  (see the SM [60] for details).

The van Vleck gauge condition imposes a constraint on the indices,  $\sum_{k=1}^i m_k = 0$  [48, 50, 51], leading to commutation relations

$$[\hat{X}, \hat{D}_n] = [\hat{g}_M^{int}, \hat{D}_n] = [\hat{g}_A^{int}, \hat{D}_n] = 0, \quad (11)$$

valid for any integer  $n \leq n_* = O(\tilde{\Omega})$ . Thus, the prethermal Hamiltonian  $\hat{D}_n$  inherits the full symmetry group  $G_{st}^{int}$  of  $\hat{H}_{int}(t)$ , as specified in Table I.

Finally, we emphasize that the rotation operator  $\hat{U}_n(t)$  in Eq. (4) also transforms under the symmetry group. In stroboscopic dynamics over  $NT$  periods, it is often treated as static by setting  $\hat{U}_n(0) = \hat{U}_n(NT)$ , assuming

commutation with the symmetry elements of Hamiltonian that has a static symmetry group [42, 51]. However, when the symmetry group includes dynamical elements, the transformations of  $\hat{U}_n(t)$  must be explicitly accounted for, as given by

$$\begin{aligned} \hat{X}^m \hat{U}_n(t) \hat{X}^{-m} &= \hat{U}_n(t - mT), \\ \hat{g}_{M/A}^{int} \hat{U}_n(t) [\hat{g}_{M/A}^{int}]^{-1} &= \hat{U}_n(T/2 \pm t). \end{aligned} \quad (12)$$

Using Eqs. (5) and (12), the time evolution over  $lT$  periods, with  $l \bmod N$ , is expressed as  $\hat{U}(lT) = \hat{X}^l \hat{U}_{int} \simeq \hat{X}^l \hat{U}_n(lT) e^{-i\hat{D}_n lT} \hat{U}_n^\dagger(0) = \hat{U}_n(0) \hat{X}^l e^{-i\hat{D}_n lT} \hat{U}_n^\dagger(0)$ , which matches the expression in Ref. [42]. Furthermore, we derive symmetry relations for  $\hat{U}_{int}(t_2, t_1)$ , given by  $\hat{g}_{M/A}^{int} \hat{U}_{int}(t_2, t_1) [\hat{g}_{M/A}^{int}]^{-1} = \hat{U}_{int}(T/2 \pm t_2, T/2 \pm t_1)$  [60]. These relations are equivalent to Eq. (7).

## VI. ENGINEERING PRETHERMAL MICROMOTION WITH DYNAMICAL SYMMETRIES

We propose a scheme to engineer nontrivial micromotion in the prethermal stage using dynamical symmetries in  $G_{st}$ , focusing on the order-2 unitary symmetry  $\hat{g}_M \in G_{st}$  of  $\hat{H}(t)$  in Eq. (1), which can be generalized to higher-order symmetries. For a set of Hermitian operators  $\hat{O}$  satisfying the conditions

$$\hat{g}_M \hat{O} \hat{g}_M^{-1} = e^{i\alpha_M} \hat{O}, \quad [\hat{g}_M^{int}, \hat{O}] = [\hat{D}_0, \hat{O}] = 0, \quad (13)$$

where  $e^{i\alpha_M} = \pm 1$  due to Hermiticity, we examine the expectation values  $\langle \hat{O}(mT) \rangle$  and  $\langle \hat{O}(mT + T/2) \rangle$  for integer  $m$  in the prethermal regime. Using the projective symmetry  $\hat{g}'_M = \hat{U}^{-1}(T/2) \hat{g}_M$  of the extended symmetry group for the Floquet operator  $\hat{U}(T)$  [59], with  $\hat{g}_M'^2 = \hat{U}^{-1}(T) \hat{g}_M^2$ , the expectation value at  $t = mT + T/2$  can be expressed as

$$\langle \hat{O}(mT + T/2) \rangle = e^{-i\alpha_M} \text{Tr} [\hat{g}_M'^{-1} \hat{\rho}(mT) \hat{g}_M' \hat{O}], \quad (14)$$

where  $\hat{\rho}(t)$  is the density matrix of the system at time  $t$ . By applying  $\hat{g}_M'^{-1} \hat{U}_n(0) = \hat{U}_n(0) e^{-i\hat{D}_n T/2} [\hat{g}_M^{int}]^{-1}$  (derived in SM [60]) and  $\hat{U}(mT) \simeq \hat{U}_n(0) \hat{X}^m e^{-i\hat{D}_n mT} \hat{U}_n^\dagger(0)$ , the term  $\hat{g}_M'^{-1} \hat{\rho}(mT) \hat{g}_M'$  simplifies to  $\hat{A} \hat{\rho}(0) \hat{A}^{-1}$ , where

$$\hat{A} = \hat{U}_n(0) e^{-i\hat{D}_n T/2} [\hat{g}_M^{int}]^{-1} \hat{X}^m e^{-i\hat{D}_n mT} \hat{U}_n^\dagger(0). \quad (15)$$

Approximating  $\hat{U}_n(0) \sim \mathbb{I}$ , we obtain  $\langle \hat{O}(mT + T/2) \rangle \simeq e^{-i\alpha_M} \text{Tr} [\hat{X}^m e^{-i\hat{D}_n mT} \hat{\rho}(0) e^{i\hat{D}_n mT} \hat{X}^{-m} \hat{B}]$ , where  $\hat{B} = \hat{g}_M^{int} e^{i\hat{D}_n T/2} \hat{O} e^{-i\hat{D}_n T/2} [\hat{g}_M^{int}]^{-1}$ . For comparison, the expectation value at  $t = mT$  is  $\langle \hat{O}(mT) \rangle \simeq \text{Tr} [\hat{X}^m e^{-i\hat{D}_n mT} \hat{\rho}(0) e^{i\hat{D}_n mT} \hat{X}^{-m} \hat{O}]$ . Using  $[\hat{g}_M^{int}, \hat{O}] = 0$  from Eq. (13) and the approximation  $[\hat{D}_n, \hat{O}] \simeq 0$  valid for  $1 \ll \hat{\Omega}$ , we find  $\hat{B} \simeq \mathbb{I}$ , leading to the following relation between the two expressions

$$\langle \hat{O}(mT + T/2) \rangle \simeq e^{-i\alpha_M} \langle \hat{O}(mT) \rangle. \quad (16)$$

## VII. NUMERICAL SIMULATION

As a concrete example, let us consider a two-leg, spin-1/2 ladder of length  $L$  [Fig. 1(a)] under dual driving, with an order-2 symmetry,  $\hat{g}_M \in G_{st} = \mathbb{Z}_2$ . In particular,  $\hat{g}_M$  is a mirror operation along the centerline of the ladder that maps the left and right halves of the ladder onto one other, as shown in Fig. 1(a). To enforce this symmetry in the dual-driving setup, we implement a four-step driven model,

$$\hat{H}(t) = \begin{cases} \hat{H}_{0,a} & (0 \leq t < \tau T), \\ \hat{V}_a + \hat{V}_a^{sc} & (\tau T \leq t < T/2), \\ \hat{H}_{0,b} & (T/2 \leq t < \tau T + T/2), \\ \hat{V}_b + \hat{V}_b^{sc} & (\tau T + T/2 \leq t < T), \end{cases} \quad (17)$$

satisfying the symmetry relations  $\hat{g}_M \hat{H}_{0,a} \hat{g}_M^{-1} = \hat{H}_{0,b}$  and  $\hat{g}_M \hat{V}_a \hat{g}_M^{-1} = \hat{V}_b$ . The resonant  $\hat{H}_{0,a/b}$  and the weak drive  $\hat{V}_{a/b}$ , we choose

$$\begin{aligned} \hat{H}_{0,a} &= J_1 \left[ \sum_{i=0}^{L/2-2} (S_i^z S_{i+1}^z + \sigma_i^x \sigma_{i+1}^x) + \sum_{i=L/2}^{L-2} (S_i^y S_{i+1}^y + \sigma_i^z \sigma_{i+1}^z) \right], \\ \hat{V}_a &= J' \left[ \sum_{i=0}^{L/2-2} (S_i^x S_{i+1}^x + \sigma_i^y \sigma_{i+1}^y) + \sum_{i=0}^{L/2-2} g_{zz} S_i^z \sigma_i^z \right. \\ &\quad \left. + \sum_{i=0}^{L/2-2} (g_x S_i^x + g_z S_i^z + g_y \sigma_i^y + g_z \sigma_i^z) \right], \end{aligned} \quad (18)$$

where the spins in the upper (lower) chain are labeled by  $S$  ( $\sigma$ ). Here, we choose  $J_1 = \Omega/4\tau$  to ensure that the resonant drive  $\hat{H}_0(t)$  generates the  $\mathbb{Z}_2$  symmetry operation  $\hat{X} = e^{-i\hat{H}_{0,b}\tau T} e^{-i\hat{H}_{0,a}\tau T}$ , where  $\hat{X}^2 = \mathbb{I}$ .

Finally, we introduce an additional coupling  $\hat{V}_{a/b}^{sc}$  at the center of the ladder

$$\begin{aligned} \hat{V}_{a/b}^{sc} &= J' \lambda_{a/b}^{LR} [S_*^x S_{*+1}^x + \sigma_*^y \sigma_{*+1}^y + g_{zz} S_{*+1/2}^z \sigma_{*+1/2}^z \\ &\quad + g_x S_{*+1/2}^x + g_z S_{*+1/2}^z + g_y \sigma_{*+1/2}^y + g_z \sigma_{*+1/2}^z], \end{aligned} \quad (19)$$

where  $*$  =  $L/2 - 1$ . This allows us to break the symmetry by setting  $\lambda_a^{LR} \neq \lambda_b^{LR}$ , causing  $\hat{g}_M \hat{V}_a^{sc} \hat{g}_M^{-1} \neq \hat{V}_b^{sc}$ . This Hamiltonian can be experimentally implemented using platforms such as superconducting quantum processor [61] and cold atoms [62, 63], providing a concrete avenue for its experimental realization.

To characterize Floquet thermalization dynamics [64], we monitor energy absorption by the energy density of the effective prethermal Hamiltonian,  $\langle \hat{D}_0^{\text{eff}}(t) \rangle / N$ , where  $N = 2L = 20$  is the total number of spins. Furthermore, we track the evolution of the bipartite entanglement entropy,  $S_{\text{ent}}(t) = -\text{Tr} [\hat{\rho}_{L/2} \ln \hat{\rho}_{L/2}]$ , where  $\hat{\rho}_{L/2} = \text{Tr}_{LH} [|\Psi(t)\rangle \langle \Psi(t)|]$ , normalized by the Page value,  $S_{\text{page}} = (L \ln 2 - 1)/2$  [65], signifying full thermalization to a featureless infinite temperature state. Initially,  $S_{\text{ent}}(t)$  grows during relaxation up to a time  $t_{\text{rel}}$ ,

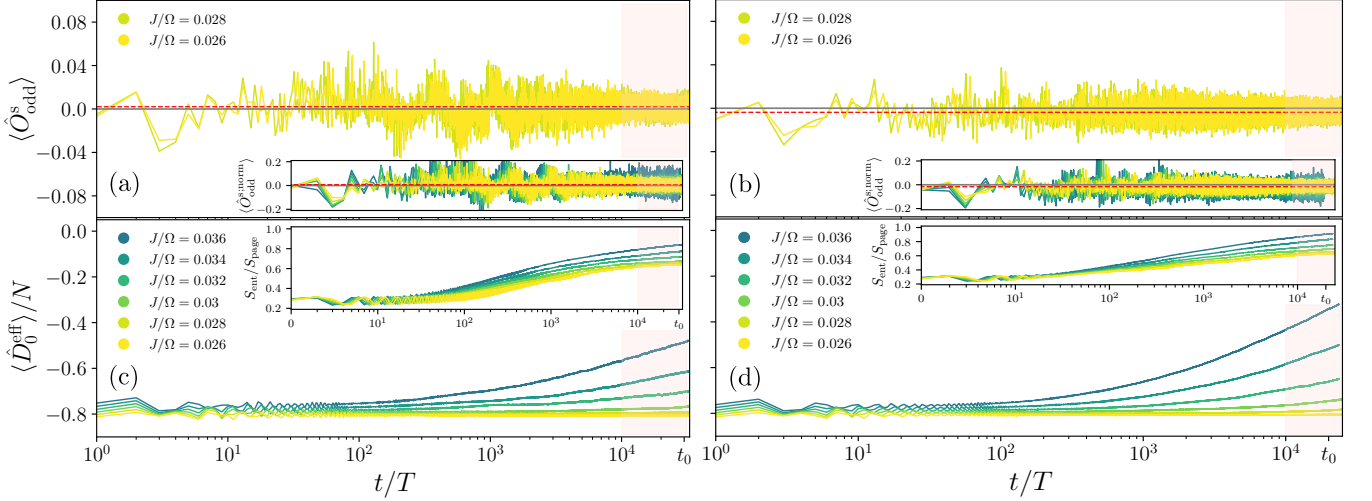


FIG. 2. (a) Time evolution of  $\hat{O}_{\text{odd}}^s(mT) = \hat{O}_{\text{odd}}(mT) + \hat{O}_{\text{odd}}(mT + T/2)$  stabilizes at zero (red line) in the prethermal regime, confirming the presence of  $\hat{g}_M$ . (b) In the symmetry-broken case,  $\hat{O}_{\text{odd}}^s(mT)$  stabilizes at a non-zero value, indicating symmetry breaking. The red line marks the center of stable oscillations in the prethermal regime. (c) and (d) Energy density and bipartite entanglement entropy (insets) highlight the prethermal regime in shaded regions. The parameters used match Fig. 1, except for the symmetry-preserving  $(\lambda_a^{LR}, \lambda_b^{LR}) = (1, 1)$  and symmetry-broken  $(\lambda_a^{LR}, \lambda_b^{LR}) = (0.8, 1.2)$  cases.

after which it reaches a prethermal plateau that persists until  $t_* \sim n_*T$ , where  $n_* = O(\bar{\Omega})$ . Beyond  $t_*$ , the system heats towards an infinite-temperature state, consistent with energy conservation during the prethermal regime and subsequent heating for a wide range of frequencies, as illustrated in Fig. 1(b).

To probe the dynamical symmetry, we introduce local operators  $\hat{O}_1 = S_{L/2-1}^x$  and  $\hat{O}_2 = S_{L/2}^x$ , defining the odd operator  $\hat{O}_{\text{odd}} = \hat{O}_1 - \hat{O}_2$ , which satisfies  $\hat{g}_M \hat{O}_{\text{odd}} \hat{g}_M^{-1} = -\hat{O}_{\text{odd}}$ , with  $e^{i\alpha_M} = -1$  in Eq. (13). We verify the expected relation  $\langle \hat{O}_{\text{odd}}(mT + T/2) \rangle = -\langle \hat{O}_{\text{odd}}(mT) \rangle$  in Eq. (16) through the time evolution of these operators. This confirms the emergence of nontrivial micromotion induced by the dynamical symmetry  $\hat{g}_M$ , which extends beyond the conventional stroboscopic views of prethermal Floquet phases. As shown in Fig. 1(b), the prethermal regime is indicated by a shaded region, with  $t_0$  serving as a reference temporal point within this regime.

### VIII. SENSITIVITY TO DYNAMICAL SYMMETRY BREAKING

To verify the perfect oscillation of  $\langle \hat{O}_{\text{odd}}(mT) \rangle = -\langle \hat{O}_{\text{odd}}(mT + T/2) \rangle$  is directly tied to the dynamical symmetry, we explicitly break the symmetry  $\hat{g}_M$ , by adjusting the coupling across the center bond by setting  $\lambda_a^{LR} \neq \lambda_b^{LR}$  in Eq. (19). We define  $\hat{O}_{\text{odd}}^s(mT) = \hat{O}_{\text{odd}}(mT) + \hat{O}_{\text{odd}}(mT + T/2)$  for integer  $m$ , which satisfies  $\hat{O}_{\text{odd}}^s(mT) = 0$  when the symmetry  $\hat{g}_M$  is preserved, and during the prethermal regime  $t_{\text{rel}} \leq mT \leq t_*$ . Following relaxation, as the system enters the prethermal regime,  $\hat{O}_{\text{odd}}^s(t)$  saturates to zero when the symmetry is

maintained, as shown in Fig. 2(a) for  $(\lambda_a^{LR}, \lambda_b^{LR}) = (1, 1)$ . In contrast, when the symmetry is broken by setting  $(\lambda_a^{LR}, \lambda_b^{LR}) = (0.8, 1.2)$ , the saturated value of  $\hat{O}_{\text{odd}}^s(t)$  is nonzero, as indicated by the location of the red horizontal line in Fig. 2(b).

At infinite temperature,  $\hat{O}_{\text{odd}}(mT)$  and  $\hat{O}_{\text{odd}}(mT + T/2)$  both approach zero, obscuring whether  $\hat{O}_{\text{odd}}^s(t) = 0$  genuinely reflects the symmetry preservation or simply the independent vanishing of each term. To resolve this ambiguity, we introduce the normalized quantity  $\hat{O}_{\text{odd}}^{s;\text{norm}}(mT) = \hat{O}_{\text{odd}}^s(mT) / \|\hat{O}_{\text{odd}}(mT)\|$ , shown in the insets of Fig. 2(a) and (b), which distinguishes symmetry-preserving prethermal dynamics from thermalizing states approaching infinite temperature.

### IX. CONCLUSION AND OUTLOOK

We investigate generic Floquet prethermalization under coexisting resonant and high-frequency drives, providing a framework to identify the symmetry group of the prethermal Hamiltonian and its algebra based on the dynamical space-time symmetry group of the driven system. Unlike previous studies that focused on static symmetries, our approach incorporates dynamical symmetries, with the kick operator playing a central role in the formulation. We validate this framework using a spin-ladder model with an order-two dynamical symmetry and confirm the symmetry-predicted relations between local observables at distinct temporal points within the Floquet period, revealing nontrivial micromotion that extends beyond conventional stroboscopic Floquet dynamics. Importantly, this phenomenon is independent of the initial

state, enhancing its experimental feasibility.

This work opens two promising directions for future exploration. First, the spontaneous symmetry breaking of projected static symmetries in the prethermal Hamiltonian could be extended to study discrete space-time crystalline orders, generalizing previous research on prethermal time-crystalline phases [14, 15]. Moreover, the identification of the extended symmetry group in the prethermal regime could be applied to quasiperiodically driven systems [55, 66, 67], revealing richer group algebras and allowing the discovery of novel phases characterized by enriched spatiotemporal orders.

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## SUPPLEMENTAL MATERIAL

### SI. PRETHERMALIZATION IN A DUAL-DRIVING

This section discusses the Floquet prethermalization and derives the prethermal Hamiltonian in the presence of coexisting resonant and high-frequency drives.

We consider a spin system where each spin resides in a  $d$ -dimensional Hilbert space. The spins are indexed by  $i = 1, 2, \dots, N_s$ , with the set of all spins denoted by  $\Lambda$ . The Hamiltonian is given by  $\hat{H}(t) = \hat{H}_0(t) + \hat{V}(t)$ , where both  $\hat{H}_0(t)$  and  $\hat{V}(t)$  are time-periodic with the same period  $T$ . Here,  $\hat{H}_0(t)$  is treated as the resonant drive, and  $\hat{V}(t)$  as the high-frequency drive. The Floquet operator associated with the resonant drive  $\hat{H}_0(t)$  is defined as

$$\hat{X} = \hat{U}_0(T, 0) = \mathcal{T}[e^{-i \int_0^T dt \hat{H}_0(t)}], \quad \hat{X}^N = \mathbb{I}, \quad (\text{S1})$$

which implements a spatially local  $\mathbb{Z}_N$  unitary action.

### $k$ -locality and $J$ -extensiveness constraints

We constrain the drive such that the Hamiltonian contains at most  $k$ -body interactions, where  $k$  is finite. This is expressed as

$$\hat{H}_0(t) = \sum_{|M| \leq k} \hat{h}_M(t), \quad \hat{V}(t) = \sum_{|M| \leq k} \hat{v}_M(t), \quad (\text{S2})$$

where  $\hat{h}_M(t)$  and  $\hat{v}_M(t)$  are operators acting on subsets  $M$  of the system, and  $|M|$  denotes the number of sites in the subset  $M$ . Additionally, we impose the condition that the terms in the resonant drive commute among different subsets  $M$  and  $N$  as  $[\hat{h}_M(t), \hat{h}_N(t)] = 0$ .

The local interaction strength  $J$  of the high-frequency drive is defined as

$$\sum_{M: i \in M} \|\hat{v}_M(t)\| \leq J \quad \text{for} \quad \forall i \in \Lambda = \{1, 2, \dots, N_s\}, \quad (\text{S3})$$

where the summation is over all subsets  $M$  of the system containing the spin at site  $i$ . The typical energy scale of the weak drive  $\hat{V}(t)$  is characterized by  $\lambda \sim 2kJ \ll \Omega$ , where  $\Omega = 2\pi/T$  is the driving frequency.

### Transition to the interaction picture

To examine prethermalization under this dual-driving setup, we transition to the interaction picture with respect to  $\hat{H}_0(t)$ . The interaction picture Hamiltonian is given as

$$\hat{H}_{int}(t) = i[\partial_t \hat{U}_0^\dagger(t)]\hat{U}_0(t) + \hat{U}_0^\dagger(t)\hat{H}(t)\hat{U}_0(t). \quad (\text{S4})$$

It follows that  $\hat{H}_{int}(t) = \hat{U}_0^\dagger(t)\hat{V}(t)\hat{U}_0(t) = \hat{V}_{int}(t)$ , which is  $NT$ -periodic since  $\hat{U}_0(NT) = \hat{X}^N = \mathbb{I}$  in Eq. (S1). The time evolution for one period  $NT$ , governed by  $\hat{H}_{int}(t)$ , is expressed as  $\hat{U}_{int}(NT) = \hat{U}_0^{-1}(NT)\hat{U}(NT) = \hat{U}(NT)$ . The Floquet Hamiltonian  $\hat{V}_{int}^{FM}$  in the interaction picture is defined through

$$\hat{U}_{int}(NT) = \mathcal{T}[e^{-i \int_0^{NT} dt \hat{V}_{int}(t)}] = e^{-i \hat{V}_{int}^{FM} NT}. \quad (\text{S5})$$

As  $\hat{V}(t)$  includes at most  $k$ -body interactions, as specified in Eq. (S2), and the terms in  $\hat{H}_0(t)$  commute, the interaction picture transformation via  $\hat{U}_0(t)$  ensures that  $\hat{V}_{int}(t)$  retains at most  $\tilde{k}$ -body interactions, maintaining the finiteness of  $\tilde{k} \sim k$ . Additionally, the energy at each site of  $\hat{V}_{int}(t)$  is also bounded by  $\tilde{J}$ , with  $\sum_{M: i \in M} \|\hat{v}_{int,M}(t)\| \leq \tilde{J} \sim J$ . We define

$$\hat{V}_0 := \sum_{|M| \leq \tilde{k}} \frac{1}{NT} \int_0^{NT} dt \|\hat{v}_{int,M}(t)\|, \quad (\text{S6})$$

$$O(\tilde{\Omega}^n) = \text{const} \cdot \left(\frac{\Omega}{N\lambda_{int}}\right)^n, \quad O(\tilde{T}^n) = \text{const} \cdot (NT\lambda_{int})^n,$$

where  $\lambda_{int} = 2\tilde{k}\tilde{J} \sim \lambda$  characterizes the typical properties of the weak  $NT$ -periodic Floquet system governed by  $\hat{V}_{int}(t)$ , assuming the high-frequency limit in the interaction picture ( $\tilde{T} \ll 1$  or  $1 \ll \tilde{\Omega}$ ). The driving amplitude over one period is given by  $\hat{V}_0 NT$ .

### Derivation of the prethermal Hamiltonian

Assuming  $\lambda_{int} NT < 1/4$ , the theorems in Ref. [39] can be applied. The stroboscopic dynamics can be approximated using the truncated Floquet-Magnus (FM) expansion, given by  $\hat{V}_{int}^{FM,n} = \sum_{m=0}^n \hat{\Omega}_m (NT)^m$ , where each term satisfies the bound

$$\|\hat{\Omega}_m\| \leq \frac{2\hat{V}_0 \lambda_{int}^m}{(m+1)^2} m! =: \bar{\Omega}_m, \quad (S7)$$

with  $\hat{V}_0$  defined in Eq. (S6). The error in approximating  $\hat{U}_{int}(NT) = \hat{U}(NT)$  by  $e^{-i\hat{V}_{int}^{FM,n} NT}$  is bounded by

$$\begin{aligned} \|\hat{U}(NT) - e^{-i\hat{V}_{int}^{FM,n} NT}\| &\leq 6\hat{V}_0 NT 2^{-n_0} + \bar{\Omega}_{n+1} (NT)^{n+2} \\ &\leq \frac{3N_s}{2\tilde{k}} 2^{-n_0} + \frac{2N_s}{\tilde{k}} [(n+1)\lambda_{int} NT]^{n+2} \\ &\simeq e^{-O(\tilde{\Omega})} + O(\tilde{\Omega}^{-n-2}). \end{aligned} \quad (S8)$$

Here,  $n \leq n_0 := \lfloor 1/(16\lambda_{int} NT) \rfloor = O(\tilde{\Omega})$ , where  $\lfloor \cdot \rfloor$  is the floor function, and  $N_s$  is the total number of spins. The derivations uses the following relations:  $\hat{V}_0 \leq 2\tilde{J}N_s = 2(\tilde{J}\tilde{k}) \cdot (N_s/\tilde{k})$ ,  $\lambda_{int} NT \leq 1/4$ , and  $(m+1)!/(m+2)^2 \leq (m+1)^{m+2}$ . These ensure the first two inequalities in Eq. (S8).

Next, we apply a unitary rotation to shift from the FM gauge to the van Vleck (vV) gauge, as described in [50] and further discussed in [51]. When truncating both high-frequency expansions to a finite order  $n \leq n_0$ , the two gauges are related as

$$\begin{aligned} &\|e^{-i\hat{K}_n(NT)} e^{-i\hat{V}_{int}^{vV,n} NT} e^{i\hat{K}_n(0)} - e^{-i\hat{V}_{int}^{FM,n} NT}\| \quad (S9) \\ &\leq \frac{16N_s}{3\tilde{k}} [4(n+1)\lambda_{int} NT]^{n+2} \simeq O(\tilde{\Omega}^{-n-2}), \end{aligned}$$

where  $\hat{K}_n(NT) = \hat{K}_n(0)$  is the  $NT$ -periodic truncated kick operator up to the  $n$ -th order, with  $\hat{K}_n(t) = \sum_{i=1}^n \hat{K}^{[i]}(t)$ . Each term  $\hat{K}^{[i]}(t)$  scales as  $O(\tilde{\Omega}^{-i})$ , and the boundary condition  $\int_0^{NT} dt \hat{K}^{[i]}(t) = 0$  is imposed in the vV gauge. Additionally,  $\hat{V}_{int}^{vV,n} = \sum_{i=0}^n \hat{V}_{vV}^{[i]}$ , with  $\hat{V}_{vV}^{[i]}$  scaling as  $O(\tilde{\Omega}^{-i}\lambda_{int})$ , represents the  $n$ -th order truncated vV expansion. This corresponds to the  $n$ -th order effective prethermal Hamiltonian  $\hat{D}_n$  in Ref. [42]. Accordingly, the error in approximating  $\hat{U}(NT)$  by  $e^{-i\hat{K}_n(NT)} e^{-i\hat{V}_{int}^{vV,n} NT} e^{i\hat{K}_n(0)}$  is

$$\begin{aligned} &\|\hat{U}(NT) - e^{-i\hat{K}_n(NT)} e^{-i\hat{V}_{int}^{vV,n} NT} e^{i\hat{K}_n(0)}\| \quad (S10) \\ &\leq \|\text{Eq. (S8)}\| + \|\text{Eq. (S9)}\| \simeq e^{-O(\tilde{\Omega})} + O(\tilde{\Omega}^{-n-2}). \end{aligned}$$

In the limit  $1 \ll \tilde{\Omega}$ , this validates that the stroboscopic time evolution over  $NT$  periods is well approximated by the effective  $n$ -th order truncated prethermal Hamiltonian  $\hat{V}_{int}^{vV,n} = \hat{D}_n$ .

### Van Vleck high-frequency expansion

A rotated frame, known as the Floquet reference frame as described in Ref. [52], connects  $\hat{H}_{int}(t)$  to  $\hat{V}_{int}^{FM}$ . This connection is established via the transformation  $\hat{V}_{int}^{FM} = \hat{U}^\dagger(t) [\hat{H}_{int}(t) - i\partial_t] \hat{U}(t)$ , where  $\hat{U}(t) = e^{-i\hat{K}(t)}$ . In the main text, we truncate the rotation  $\hat{U}_n(t)$  to order  $n$  and introduce the dressed Hamiltonian  $\hat{H}_{int}^R(t)$ , given by

$$\begin{aligned} \hat{H}_{int}^R(t) &= e^{i\hat{K}_n(t)} \left[ \hat{H}_{int}(t) - i\partial_t \right] e^{-i\hat{K}_n(t)} \quad (S11) \\ &\simeq \hat{D}_n + \hat{V}_{int}^{[n]}(t) + O(\tilde{\Omega}^{-n-1}), \end{aligned}$$

where  $\hat{V}_{int}^{[n]}(t) = \partial_t \hat{K}^{[n+1]}(t)$  is  $NT$ -periodic and satisfies  $\int_0^{NT} dt \hat{V}_{int}(t) = 0$ . We present the  $n$ -th order truncation of the high-frequency expansions for the van Vleck (vV) effective Hamiltonian,  $\hat{D}_n = \hat{V}_{int}^{vV,n} = \sum_{i=0}^n \hat{V}_{vV}^{[i]}$ , and the kick operator,  $\hat{K}_n(t) = \sum_{i=1}^n \hat{K}^{[i]}(t)$ , for  $n \leq 2$ , as follows [50–52]

$$\begin{aligned} \hat{V}_{vV}^{[0]} &= \hat{V}_0, \quad (S12) \\ \hat{V}_{vV}^{[1]} &= \sum_{m \neq 0} \frac{[\hat{V}_{-m}, \hat{V}_m]}{2m(\Omega/N)}, \\ \hat{V}_{vV}^{[2]} &= \sum_{m \neq 0} \frac{[[\hat{V}_{-m}, \hat{V}_0], \hat{V}_m]}{2m^2(\Omega/N)^2} + \sum_{m \neq 0} \sum_{n \neq 0, m} \frac{[[\hat{V}_{-m}, \hat{V}_{m-n}], \hat{V}_n]}{3mn(\Omega/N)^2}, \\ \hat{K}^{[1]}(t) &= i \sum_{m \neq 0} \frac{\hat{V}_m}{m(\Omega/N)} e^{-im(\Omega/N)t}, \\ \hat{K}^{[2]}(t) &= -i \sum_{m \neq 0} \frac{[\hat{V}_m, \hat{V}_0]}{m^2(\Omega/N)^2} e^{-im(\Omega/N)t} \\ &\quad - i \sum_{m \neq 0} \sum_{n \neq 0, m} \frac{[\hat{V}_n, \hat{V}_{m-n}]}{2mn(\Omega/N)^2} e^{-im(\Omega/N)t}. \end{aligned}$$

Here,  $\hat{V}_m$  is defined as

$$\hat{V}_m = \frac{1}{NT} \int_0^{NT} dt \hat{H}_{int}(t) e^{im(\Omega/N)t}. \quad (S13)$$

## SII. SYMMETRY OF PRETHERMAL HAMILTONIAN

In this section, we derive the symmetry group of the prethermal Hamiltonian  $\hat{D}_n$  and identify its group algebra, starting from the dynamical space-time symmetry group  $G_{st}$  of the original Hamiltonian  $\hat{H}(t)$ . We

first establish the extended dynamical space-time symmetry group  $G_{st}^{int}$  of the interaction picture Hamiltonian  $\hat{H}_{int}(t)$ , and then identify the symmetry group of the truncated prethermal Hamiltonian  $\hat{D}_n$ .

### Dynamical symmetry group $G_{st}$

We consider Floquet systems that exhibit the symmetry group  $G_{st}$ , encompassing both static and dynamical symmetries transforming both spatial and temporal coordinates. Within  $G_{st}$ , subgroups  $M$  and  $A$  represent unitary and antiunitary symmetries, respectively. For order-2 discrete symmetries, under the assumption that  $\hat{g}_M \in M$  and  $\hat{g}_A \in A$  commute, we have

$$\begin{aligned}\hat{g}_M \hat{H}(t) \hat{g}_M^{-1} &= \hat{H}(t + T/2), \\ \hat{g}_A \hat{H}(t) \hat{g}_A^{-1} &= \hat{H}(-t + s),\end{aligned}\quad (\text{S14})$$

where  $s \in [0, T]$  is a reference point within the driving period  $T$ . Without loss of generality, we set  $s = T/2$ .

### Extended dynamical symmetry group $G_{st}^{int}$

The element  $\hat{X}$ , with  $\hat{X}^N = \mathbb{I}$  in Eq. (S1), forms a subgroup  $\mathbb{Z}_N \subset G_{st}^{int}$ , with transformations

$$\begin{aligned}\hat{X}^m \hat{H}_{int}(t) \hat{X}^{-m} &= \hat{H}_{int}(t - mT), \\ \hat{X}^m \hat{U}_{int}(t, 0) \hat{X}^{-m} &= \hat{U}_{int}(t - mT, -mT).\end{aligned}\quad (\text{S15})$$

This emergent  $\mathbb{Z}_N$  symmetry of  $\hat{H}_{int}(t)$ , absent in  $\hat{H}(t)$ , arises solely from the resonant drive  $\hat{H}_0(t)$  and underpins the realization of prethermal discrete time crystals [42, 51].

For a unitary element  $\hat{g}_M \in M \subset G_{st}$ , as defined in Eq. (S14) and  $\hat{U}_{int}(t, 0) = \mathcal{T}[e^{-i \int_0^t \hat{H}_{int}(t)}] = \hat{U}_0^{-1}(t) \hat{U}(t)$ , the action of  $\hat{g}_M$  is expressed as

$$\begin{aligned}\hat{g}_M \hat{U}_{int}(t) \hat{g}_M^{-1} &= \hat{U}_0(T/2, t + T/2) \hat{U}(t + T/2, T/2) \\ &= \hat{U}_0(T/2) \hat{U}_{int}(t + T/2) \hat{U}^{-1}(T/2).\end{aligned}\quad (\text{S16})$$

We define the unitary symmetry element  $\hat{g}_M^{int} \in G_{st}^{int}$  as  $\hat{g}_M^{int} = \hat{U}_0^{-1}(T/2) \hat{g}_M$ , which satisfies

$$\begin{aligned}\hat{g}_M^{int} \hat{U}_{int}(t, 0) [\hat{g}_M^{int}]^{-1} &= \hat{U}_{int}(t + T/2) \hat{U}^{-1}(T/2) \hat{U}_0(T/2), \\ &= \hat{U}_{int}(t + T/2, T/2),\end{aligned}\quad (\text{S17})$$

and  $[\hat{g}_M^{int}]^2 = \hat{U}_0^{-1}(T) \hat{g}_M^2 = \hat{X}^{-1} \hat{g}_M^2$ . Therefore,  $\hat{g}_M^{int}$  acts on  $\hat{H}_{int}(t)$  as

$$\hat{g}_M^{int} \hat{H}_{int}(t) [\hat{g}_M^{int}]^{-1} = \hat{H}_{int}(T/2 + t). \quad (\text{S18})$$

For  $\hat{g}_A \in A \subset G_{st}$ , defined as in Eq. (S14), the symmetry transformations satisfy  $\hat{g}_A \hat{U}(t, 0) \hat{g}_A^{-1} = \hat{U}(T/2 -$

$t, T/2)$  and  $\hat{g}_A \hat{U}_0(t, 0) \hat{g}_A^{-1} = \hat{U}_0(T/2 - t, T/2)$ , which yield

$$\begin{aligned}\hat{g}_A \hat{U}_{int}(t) \hat{g}_A^{-1} &= \hat{U}_0(T/2, T/2 - t) \hat{U}(T/2 - t, T/2), \\ &= \hat{U}_0(T/2) \hat{U}_{int}(T/2 - t) \hat{U}^{-1}(T/2).\end{aligned}\quad (\text{S19})$$

Define the antiunitary symmetry element  $\hat{g}_A^{int} \in G_{st}^{int}$  as  $\hat{g}_A^{int} = \hat{U}_0^{-1}(T/2) \hat{g}_A$ , which transforms  $\hat{U}_{int}(t)$  as

$$\begin{aligned}\hat{g}_A^{int} \hat{U}_{int}(t) [\hat{g}_A^{int}]^{-1} &= \hat{U}_{int}(T/2 - t) \hat{U}^{-1}(T/2) \hat{U}_0(T/2), \\ &= \hat{U}_{int}(T/2 - t, T/2),\end{aligned}\quad (\text{S20})$$

and satisfies  $[\hat{g}_A^{int}]^2 = \hat{g}_A^2$ . The action of  $\hat{g}_A^{int}$  on  $\hat{H}_{int}(t)$  follows

$$\hat{g}_A^{int} \hat{H}_{int}(t) [\hat{g}_A^{int}]^{-1} = \hat{H}_{int}(T/2 - t). \quad (\text{S21})$$

The algebraic structure of  $G_{st}^{int}$  is further defined by

$$\hat{g}_M^{int} \hat{X} [\hat{g}_M^{int}]^{-1} = \hat{X}, \quad \hat{g}_A^{int} \hat{X} [\hat{g}_A^{int}]^{-1} = \hat{X}^{-1}, \quad (\text{S22})$$

from

$$\begin{aligned}\hat{g}_M^{int} \hat{X} [\hat{g}_M^{int}]^{-1} &= \hat{U}_0^{-1}(T/2) \hat{g}_M \hat{U}_0(T) [\hat{g}_M]^{-1} \hat{U}_0(T/2), \\ &= \hat{U}_0^{-1}(T/2) \hat{U}_0(T + T/2, T/2) \hat{U}_0(T/2), \\ &= \hat{X}\end{aligned}\quad (\text{S23})$$

and

$$\begin{aligned}\hat{g}_A^{int} \hat{X} [\hat{g}_A^{int}]^{-1} &= \hat{U}_0^{-1}(T/2) \hat{g}_A \hat{X} \hat{g}_A^{-1} \hat{U}_0(T/2), \\ &= \hat{U}_0^{-1}(T/2) \hat{U}_0(-T/2, T/2) \hat{U}_0(T/2), \\ &= \hat{U}_0(0, T) = \hat{X}^{-1}.\end{aligned}\quad (\text{S24})$$

These relations establish the extended dynamical symmetry group in the interaction picture as  $G_{st}^{int} = \mathbb{Z}_2 \times (\mathbb{Z}_N \rtimes \mathbb{Z}_2^T)$ , encapsulating the interplay between the emergent  $\mathbb{Z}_N$  symmetry and the original symmetry group  $G_{st} = \mathbb{Z}_2 \times \mathbb{Z}_2^T$ .

### Symmetry group of the prethermal Hamiltonian

The symmetry group of the truncated prethermal Hamiltonian  $\hat{D}_n$  inherits symmetry elements  $\hat{X}$ ,  $\hat{g}_M^{int}$ , and  $\hat{g}_A^{int}$  from  $G_{st}^{int}$  of  $\hat{H}_{int}(t)$ , as specified in Eqs. (S15), (S18), and (S21). The actions of these elements on the Fourier component  $\hat{V}_m$  in Eq. (S13) are expressed as

$$\begin{aligned}\hat{X} \hat{V}_m \hat{X}^{-1} &= e^{i2\pi m/N} \hat{V}_m, \\ \hat{g}_{M/A}^{int} \hat{V}_m [\hat{g}_{M/A}^{int}]^{-1} &= e^{-i2\pi m/2N} \hat{V}_m.\end{aligned}\quad (\text{S25})$$

The  $i$ -th order van Vleck expansion term,  $\hat{V}_v^{[i]}$ , in  $\hat{D}_n = \hat{V}_{int}^{vV,n}$  includes terms with numerators of the form  $\hat{V}_{m_1} \hat{V}_{m_2} \cdots \hat{V}_{m_i}$ , as shown in Eq. (S12). Symmetry operations act on these terms as follows

$$\begin{aligned}\hat{X} \hat{V}_{m_1} \cdots \hat{V}_{m_i} \hat{X}^{-1} &= e^{i\frac{2\pi}{N}(\sum_{k=1}^i m_k)} \hat{V}_{m_1} \cdots \hat{V}_{m_i}, \\ \hat{g}_{M/A}^{int} \hat{V}_{m_1} \cdots \hat{V}_{m_i} [\hat{g}_{M/A}^{int}]^{-1} &= e^{-i\frac{2\pi}{2N}(\sum_{k=1}^i m_k)} \hat{V}_{m_1} \cdots \hat{V}_{m_i}.\end{aligned}\quad (\text{S26})$$

The van Vleck gauge condition imposes a constraint on the indices,  $\sum_{k=1}^i m_k = 0$  [50, 51], leading to the commutation relations

$$[\hat{X}, \hat{D}_n] = [\hat{g}_M^{int}, \hat{D}_n] = [\hat{g}_A^{int}, \hat{D}_n] = 0, \quad (\text{S27})$$

valid for any integer  $n \leq n_0 = O(\tilde{\Omega})$ . Thus, the prethermal Hamiltonian  $\hat{D}_n$  inherits the full symmetry group  $G_{st}^{int}$  of  $\hat{H}_{int}(t)$ .

From Eqs. (S12) and (S25), the transformations of  $\hat{K}^{[i]}(t)$  under symmetry operations are given by

$$\begin{aligned} \hat{X} \hat{K}^{[i]}(t) \hat{X}^{-1} &= \hat{K}^{[i]}(t - mT), \\ \hat{g}_{M/A}^{int} \hat{K}^{[i]}(t) [\hat{g}_{M/A}^{int}]^{-1} &= \pm \hat{K}^{[i]}(T/2 \pm t). \end{aligned} \quad (\text{S28})$$

These relations yield the following transformations presented in the main text,  $\hat{X}^m \hat{U}_n(t) \hat{X}^{-m} = \hat{U}_n(t - mT)$  and  $\hat{g}_{M/A}^{int} \hat{U}_n(t) [\hat{g}_{M/A}^{int}]^{-1} = \hat{U}_n(T/2 \pm t)$ . The time evolution over  $lT$  periods, with  $l \bmod N$ , is expressed as  $\hat{U}(lT) = \hat{X}^l \hat{U}_{int} \simeq \hat{X}^l \hat{U}_n(lT) e^{-i\hat{D}_n lT} \hat{U}_n^\dagger(0) = \hat{U}_n(0) \hat{X}^l e^{-i\hat{D}_n lT} \hat{U}_n^\dagger(0)$ , which matches the expression in Ref. [42]. In addition, the symmetry transformations for  $\hat{U}_{int}(t_2, t_1)$  follow as

$$\begin{aligned} \hat{X} \hat{U}_{int}(t_2, t_1) \hat{X}^{-1} &= \hat{X} [\hat{U}_n(t_2) e^{-i\hat{D}_n(t_2-t_1)} \hat{U}_n^\dagger(t_1)] \hat{X}^{-1} \\ &= \hat{U}_n(t_2 - T) \hat{X} e^{-i\hat{D}_n(t_2-t_1)} \hat{X}^{-1} \hat{U}_n^\dagger(t_1 - T) \\ &= \hat{U}_{int}(t_2 - T, t_1 - T), \end{aligned} \quad (\text{S29})$$

and

$$\begin{aligned} \hat{g}_{M/A}^{int} \hat{U}_{int}(t_2, t_1) [\hat{g}_{M/A}^{int}]^{-1} \\ &= \hat{U}_n(T/2 \pm t_2) \hat{g}_{M/A}^{int} e^{-i\hat{D}_n(t_2-t_1)} [\hat{g}_{M/A}^{int}]^{-1} \hat{U}_n^\dagger(T/2 \pm t_1) \\ &= \hat{U}_{int}(T/2 \pm t_2, T/2 \pm t_1), \end{aligned} \quad (\text{S30})$$

using  $[\hat{X}, \hat{D}_n] = [\hat{g}_{M/A}^{int}, \hat{D}_n] = 0$ . These results are equivalent to Eqs. (S15), (S17) and (S20).

### III. DETECTION OF DYNAMICAL SYMMETRY

This section introduces a scheme to engineer nontrivial micromotion during the prethermal stage, using dynamical symmetries in  $G_{st}$ . Specifically, we focus on the order-2 unitary symmetry  $\hat{g}_M \in G_{st}$  of  $\hat{H}(t)$  in Eq. (S14), which can be generalized to higher-order symmetries.

#### Derivation of key symmetry relations

We derive key relations for  $\hat{g}_M' \hat{U}_n(0) [\hat{g}_M^{int}]^{-1}$  and  $\hat{g}_M'^{-1} \hat{U}_n(0) \hat{g}_M^{int}$ , which are essential for developing schemes to detect dynamical symmetries. Here,  $\hat{g}_M' = \hat{U}^{-1}(T/2) \hat{g}_M$  is a projective symmetry element of the extended symmetry group for the Floquet operator  $\hat{U}(T)$ ,

satisfying  $\hat{g}_M'^2 = \hat{U}^{-1}(T) \hat{g}_M^2$ , where  $\hat{g}_M^2 = e^{i\beta} \mathbb{I}$ , as introduced in Ref. [59]. The expression for  $\hat{g}_M' \hat{U}_n(0) [\hat{g}_M^{int}]^{-1}$  is obtained as follows

$$\begin{aligned} \hat{g}_M' \hat{U}_n(0) [\hat{g}_M^{int}]^{-1} &= \hat{g}_M [\hat{g}_M^{int}]^{-1} \hat{U}_n(T/2) \\ &= \hat{U}^{-1}(T/2) \hat{U}_0(T/2) \hat{U}_n(T/2) \\ &= \hat{U}_{int}(0, T/2) \hat{U}_n(T/2) = \hat{U}_n(0) e^{i\hat{D}_n T/2}. \end{aligned} \quad (\text{S31})$$

Now, consider the expression  $\hat{g}_M'^{-1} \hat{U}_n(0) \hat{g}_M^{int}$ . It is derived as

$$\begin{aligned} [\hat{g}_M^{int} \hat{U}_n^\dagger(0) \hat{g}_M'^{-1}] [\hat{g}_M'^{-1} \hat{U}_n(0) \hat{g}_M^{int}] &= \hat{g}_M^{int} \hat{U}_n^\dagger(0) \hat{g}_M'^{-2} \hat{U}_n(0) \hat{g}_M^{int} \\ &= \hat{g}_M^{int} \hat{U}_n^\dagger(0) \hat{g}_M'^{-2} \hat{U}(T) \hat{U}_n(0) \hat{g}_M^{int} = e^{-i\beta} \hat{g}_M^{int} \hat{X} e^{-i\hat{D}_n T} \hat{g}_M^{int} \\ &= e^{-i\beta} \hat{X} e^{-i\hat{D}_n T} [\hat{g}_M^{int}]^2 = e^{-i\hat{D}_n T}, \end{aligned} \quad (\text{S32})$$

using  $[\hat{g}_M^{int}, \hat{X}] = [\hat{g}_M^{int}, \hat{D}_n] = [\hat{X}, \hat{D}_n] = 0$  and  $[\hat{g}_M^{int}]^2 = \hat{X}^{-1} \hat{g}_M^2 \hat{X} = e^{i\beta} \hat{X}^{-1}$ . Combining this result with Eq. (S31), we obtain the expression  $\hat{g}_M'^{-1} \hat{U}_n(0) \hat{g}_M^{int}$  as

$$\begin{aligned} \hat{g}_M'^{-1} \hat{U}_n(0) \hat{g}_M^{int} &= [\hat{g}_M^{int} \hat{U}_n^\dagger(0) \hat{g}_M'^{-1}]^{-1} e^{-i\hat{D}_n T} \\ &= [\hat{U}_n(0) e^{i\hat{D}_n T/2}] e^{-i\hat{D}_n T} = \hat{U}_n(0) e^{-i\hat{D}_n T/2}. \end{aligned} \quad (\text{S33})$$

**Detection scheme for the initial state:**  $[\hat{\rho}(0), \hat{D}_n] = 0$

For a set of Hermitian operators  $\hat{O}$  that satisfy the relation  $\hat{g}_M \hat{O} \hat{g}_M^{-1} = e^{i\alpha_M} \hat{O}$ , where  $e^{i\alpha_M} = \pm 1$  due to Hermiticity, we examine the expectations values  $\langle \hat{O}(mT) \rangle$  and  $\langle \hat{O}(mT + T/2) \rangle$  for integer  $m$  in the prethermal regime, and derive the relation between them imposed by the presence of dynamical symmetries. The expectation value at  $t = mT + T/2$  is given by

$$\begin{aligned} \langle \hat{O}(mT + T/2) \rangle &= \text{Tr} [\hat{U}(T/2) \hat{\rho}(mT) \hat{U}^{-1}(T/2) \hat{O}] \\ &= e^{-i\alpha_M} \text{Tr} [\hat{g}_M^{-1} \hat{U}(T/2) \hat{\rho}(mT) \hat{U}^{-1}(T/2) \hat{g}_M \hat{O}] \\ &= e^{-i\alpha_M} \text{Tr} [\hat{g}_M'^{-1} \hat{\rho}(mT) \hat{g}_M' \hat{O}], \end{aligned} \quad (\text{S34})$$

where  $\hat{\rho}(t)$  is the density matrix of the system at  $t$ . By applying  $\hat{g}_M'^{-1} = \hat{U}_n(0) e^{-i\hat{D}_n T/2} [\hat{g}_M^{int}]^{-1} \hat{U}_n^\dagger(0)$  derived in Eq. (S33) and  $\hat{U}(mT) \simeq \hat{U}_n(0) \hat{X}^m e^{-i\hat{D}_n mT} \hat{U}_n^\dagger(0)$ , the term  $\hat{g}_M'^{-1} \hat{\rho}(mT) \hat{g}_M'$  simplifies to  $\hat{A} \hat{\rho}(0) \hat{A}^{-1}$ , where

$$\hat{A} = \hat{U}_n(0) e^{-i\hat{D}_n T/2} [\hat{g}_M^{int}]^{-1} \hat{X}^m e^{-i\hat{D}_n mT} \hat{U}_n^\dagger(0). \quad (\text{S35})$$

Approximating  $\hat{U}_n^\dagger(0) \hat{\rho}(0) \hat{U}_n(0) \simeq \hat{\rho}(0) + i[\hat{K}^{[1]}(0), \hat{\rho}(0)] + O(\tilde{\Omega}^{-2}) \hat{\rho}(0) \simeq \hat{\rho}(0)$  and using  $[\hat{g}_M^{int}, \hat{D}_n] = 0$ , we obtain

$$\begin{aligned} \langle \hat{O}(mT + T/2) \rangle &\simeq e^{-i\alpha_M} \text{Tr} [\hat{\rho}(mT) \hat{O}] \\ &= e^{-i\alpha_M} \langle \hat{O}(mT) \rangle, \end{aligned} \quad (\text{S36})$$

if the initial state  $|\Psi(0)\rangle$  is taken as the eigenstate of the truncated prethermal Hamiltonian  $\hat{D}_n$  for  $n \leq n_0$ .

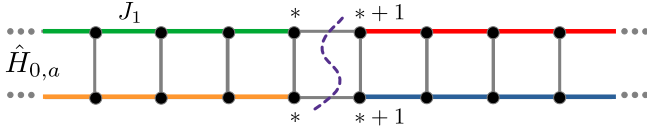


FIG. S1. Schematic of one-dimensional spin-1/2 ladder of length  $L$ .  $S$  spins reside in the upper chain and  $\sigma$  spins in the lower chain. Along each chain, the spins interact via nearest neighbor couplings with strength  $J_1 \sim \Omega$ , as described in  $\hat{H}_{0,a}$  of Eq. (S38). The symmetry operation  $\hat{g}_M$  represents a mirror reflection about the centerline of the ladder. Here, the notation  $*$  =  $L/2 - 1$  denotes the left sites of the upper and lower chains adjacent to the center bond of the ladder.

**Detection scheme for the initial state:**  $[\hat{\rho}(0), \hat{D}_n] \neq 0$

We generalize the above detection scheme to be independent of the initial state, making it highly amenable to experimental realization, by imposing additional conditions on the chosen Hermitian operator  $\hat{O}$ , which satisfies  $[\hat{g}_M^{int}, \hat{O}] = [\hat{D}_0, \hat{O}] = 0$ . Starting from Eq. (S35) and approximating  $\hat{U}_n(0) \sim \mathbb{I}$ , we obtain  $\langle \hat{O}(mT + T/2) \rangle \simeq e^{-i\alpha_M} \text{Tr}[\hat{X}^m e^{-i\hat{D}_n mT} \hat{\rho}(0) e^{i\hat{D}_n mT} \hat{X}^{-m} \hat{B}]$ , where

$$\hat{B} = \hat{g}_M^{int} e^{i\hat{D}_n T/2} \hat{O} e^{-i\hat{D}_n T/2} [\hat{g}_M^{int}]^{-1}. \quad (\text{S37})$$

For comparison, the expectation value at  $t = mT$  is  $\langle \hat{O}(mT) \rangle \simeq \text{Tr}[\hat{X}^m e^{-i\hat{D}_n mT} \hat{\rho}(0) e^{i\hat{D}_n mT} \hat{X}^{-m} \hat{O}]$ . Using  $[\hat{g}_M^{int}, \hat{O}] = 0$  and the approximation  $[\hat{D}_n, \hat{O}] \simeq 0$  valid for  $1 \ll \tilde{\Omega}$ , we find  $\hat{B} \simeq \mathbb{I}$ , leading to the following relation between the two expressions as  $\langle \hat{O}(mT + T/2) \rangle \simeq e^{-i\alpha_M} \langle \hat{O}(mT) \rangle$ , which is equivalent to Eq. (S36).

#### SIV. ADDITIONAL DETAILS ON NUMERICS

##### Model

As a concrete example, we introduce a model of a two-leg, spin-1/2 ladder of length  $L$  under dual driving, with an order-2 symmetry,  $\hat{g}_M \in G_{st} = \mathbb{Z}_2$ . Specifically,  $\hat{g}_M$  represents a mirror operation along the centerline of the ladder, as shown in Fig. S1. The spins in the upper (lower) chain are labeled by  $S$  ( $\sigma$ ). We design a four-step driven model defined by

$$\hat{H}(t) = \begin{cases} \hat{H}_{0,a} & (0 \leq t < \tau T), \\ \hat{V}_a + \hat{V}_a^{sc} & (\tau T \leq t < T/2), \\ \hat{H}_{0,b} & (T/2 \leq t < \tau T + T/2), \\ \hat{V}_b + \hat{V}_b^{sc} & (\tau T + T/2 \leq t < T), \end{cases} \quad (\text{S38})$$

satisfying the symmetry relations  $\hat{g}_M \hat{H}_{0,a} \hat{g}_M^{-1} = \hat{H}_{0,b}$  and  $\hat{g}_M \hat{V}_a \hat{g}_M^{-1} = \hat{V}_b$ . The resonant  $\hat{H}_{0,a/b}$  and the weak drive  $\hat{V}_{a/b}$  are provided in the main text. The coupling strength

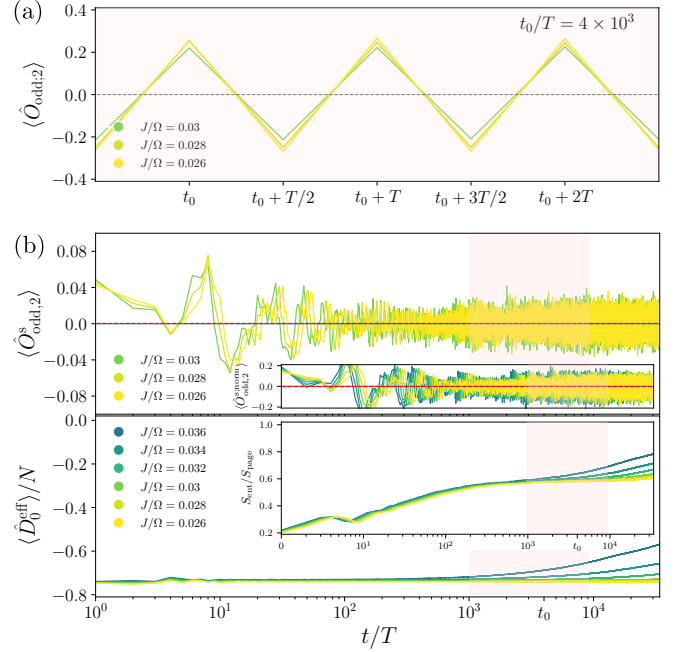


FIG. S2. After relaxation, the system enters a prethermal regime governed approximately by the effective prethermal Hamiltonian  $\hat{D}_0$  in Eq. (S42) until  $t_0 = n_0 T$  ( $n_0 = O(\tilde{\Omega})$ ), with  $t_0$  marking a reference temporal point within this regime. (a) Time evolution of  $\langle \hat{O}_{\text{odd},2}(t) \rangle$  at  $t = mT$  and  $t = mT + T/2$  within the prethermal regime verifies Eq. (S36) and detects  $\hat{g}_M$ . The shaded regions indicate the prethermal regime, with bipartite entanglement and energy density consistent across a range of frequencies ( $J/\Omega = 0.026, 0.028$ ). (b) Time evolution of  $\hat{O}_{\text{odd},2}^s(mT) = \hat{O}_{\text{odd},2}(mT) + \hat{O}_{\text{odd},2}(mT + T/2)$  stabilizes at zero (red horizontal line) in the prethermal regime, confirming the presence of  $\hat{g}_M$ . The normalized quantity  $\hat{O}_{\text{odd},2}^{s,\text{norm}}(mT) = \hat{O}_{\text{odd},2}^s(mT) / \|\hat{O}_{\text{odd},2}(mT)\|$ , shown in the inset, distinguishes symmetry-preserving prethermal dynamics from thermalizing states approaching infinite temperature. The parameters used match Fig. 1 in the main text.

$J_1 = \Omega/4\tau$  ensures that the resonant drive  $\hat{H}_0(t)$  generates the  $\mathbb{Z}_2$  symmetry operation  $\hat{X} = e^{-i\hat{H}_{0,b}\tau T} e^{-i\hat{H}_{0,a}\tau T}$ , where  $\hat{X}^2 = \mathbb{I}$ . For consistency with prior works [69, 70] on hydrodynamics, we adopt the following parameter values for the high-frequency drive  $\hat{V}_{a/b}$ :  $g_x = g_y = 0.45225$ , and  $g_z = 0.7$ . We set  $\tau = 0.25$  and  $J = 1$ , leading to  $J' = J/(0.5 - \tau) = 4$ .

We introduce an additional coupling  $\hat{V}_{a/b}^{sc}$  at the center of the ladder

$$\hat{V}_{a/b}^{sc} = J' \lambda_{a/b}^{LR} [S_*^x S_{*+1}^x + \sigma_*^y \sigma_{*+1}^y + g_{zz} S_{*+1/2}^z \sigma_{*+1/2}^z + g_x S_{*+1/2}^x + g_z S_{*+1/2}^z + g_y \sigma_{*+1/2}^y + g_z \sigma_{*+1/2}^z], \quad (\text{S39})$$

where  $*$  =  $L/2 - 1$  (see Fig. S1). This allows symmetry breaking by setting  $\lambda_a^{LR} \neq \lambda_b^{LR}$ , causing  $\hat{g}_M \hat{V}_a^{sc} \hat{g}_M^{-1} \neq \hat{V}_b^{sc}$ . For the numerical results in the main text, we set  $(\lambda_a^{LR}, \lambda_b^{LR}) = (0.5, 0.5)$  for Fig. 1 and (1, 1) and (0.8, 1.2) in Fig. 2(a) and (b). Additionally, we set an Ising inter-

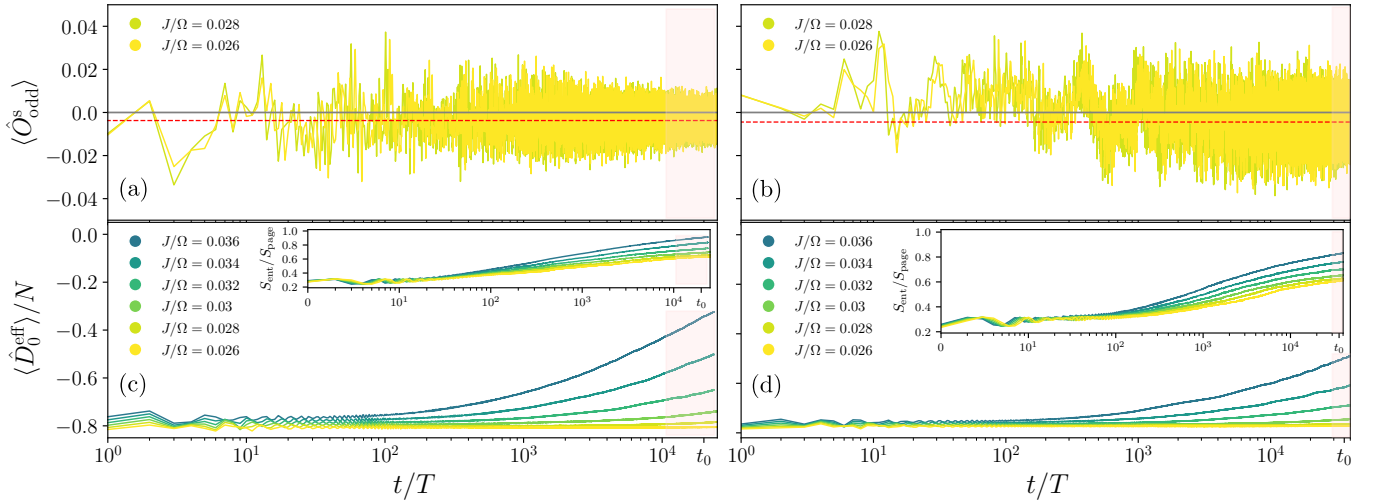


FIG. S3. In the symmetry-broken cases (a)  $(\lambda_a^{LR}, \lambda_b^{LR}) = (0.8, 1.2)$  and (b)  $(\lambda_a^{LR}, \lambda_b^{LR}) = (0.5, 1)$ ,  $\hat{O}_{\text{odd}}^s(mT)$  stabilizes at a non-zero value, signaling symmetry breaking. The red horizontal line denotes the center of stable oscillations within the prethermal regime. With increased symmetry breaking, the offset from zero becomes more pronounced. (c) and (d) Energy density and bipartite entanglement entropy (insets) indicate the prethermal regime, highlighted by shaded regions. The parameters used are consistent with Fig. 1, except for the symmetry-breaking values of  $(\lambda_a^{LR}, \lambda_b^{LR})$ .

action strength  $g_{zz} = 1.3$  to couple the spins across each rung of the ladder.

For the model described in Eq. (S38), the symmetry operator  $\hat{g}_M$  is given by

$$\hat{g}_M = \prod_{i=0}^{L/2-1} e^{-i\frac{\pi}{4}(\mathbb{I} + \vec{S}_i \cdot \vec{S}_{L-1-i})} e^{-i\frac{\pi}{4}(\mathbb{I} + \vec{\sigma}_i \cdot \vec{\sigma}_{L-1-i})}, \quad (\text{S40})$$

and the symmetry operator  $\hat{X}$  takes the form

$$\hat{X} = [S_0^x S_{*+1}^x S_{L-1}^x][\sigma_0^y \sigma_{*+1}^y \sigma_{L-1}^y], \quad \hat{X}^2 = \mathbb{I}. \quad (\text{S41})$$

The zeroth-order effective prethermal Hamiltonian,  $\hat{D}_0 = [\int_0^T dt \hat{V}_{\text{int}}(t) + \hat{X}^{-1} \int_0^T dt \hat{V}_{\text{int}}(t) \hat{X}] / 2T$ , can be decomposed as  $\hat{D}_0 = \hat{D}_0^L + \hat{D}_0^R + \hat{D}_0^c$ , where  $\hat{D}_0^L$ ,  $\hat{D}_0^R$ , and  $\hat{D}_0^c$  correspond to the left half, right half, and center of the ladder, respectively, as depicted in Fig. S1. These terms are defined as follows

$$\begin{aligned} \hat{D}_0^L &= J[-S_0^x S_1^x - \sigma_0^y \sigma_1^y + \sum_{i=1}^{L/2-3} (S_i^x S_{i+1}^x + \sigma_i^y \sigma_{i+1}^y) \\ &\quad - S_{*-1}^x S_*^x - \sigma_{*-1}^y \sigma_*^y - g_{zz} S_0^z \sigma_0^z + \sum_{i=1}^{L/2-2} g_{zz} S_i^z \sigma_i^z \\ &\quad - g_x S_0^x - g_y \sigma_0^y + \sum_{i=1}^{L/2-2} (g_x S_i^x + g_z S_i^z + g_y \sigma_i^y + g_z \sigma_i^z)], \\ \hat{D}_0^R &= J[\sum_{i=L/2}^{L-2} (S_i^x S_{i+1}^x + \sigma_i^y \sigma_{i+1}^y) + \sum_{i=L/2+1}^{L-1} g_{zz} S_i^z \sigma_i^z + \\ &\quad \sum_{i=L/2+1}^{L-2} (g_x S_i^x + g_z S_i^z + g_y \sigma_i^y + g_z \sigma_i^z) + g_x S_{L-1}^x + g_y \sigma_{L-1}^y], \end{aligned}$$

and

$$\begin{aligned} \hat{D}_0^c &= J[(\lambda_a^{LR} + \lambda_b^{LR}) + (S_*^x S_{*+1}^x + \sigma_*^y \sigma_{*+1}^y) + \\ &\quad (\lambda_b^{LR} - 1)g_{zz} S_*^z \sigma_*^z + (1 - \lambda_a^{LR})g_{zz} S_{*+1}^z \sigma_{*+1}^z + \\ &\quad (\lambda_b^{LR} - 1)(g_x S_*^x + g_y \sigma_*^y) + (1 - \lambda_a^{LR})(g_x S_{*+1}^x + g_y \sigma_{*+1}^y)]. \end{aligned} \quad (\text{S42})$$

### Dynamical symmetry-preserving prethermalization

In the main text, to probe the dynamical symmetry, we define the odd operator  $\hat{O}_{\text{odd}} = S_{L/2-1}^x - S_{L/2}^x$ , which satisfies  $\hat{g}_M \hat{O}_{\text{odd}} \hat{g}_M^{-1} = -\hat{O}_{\text{odd}}$  with  $e^{i\alpha_M} = -1$ . We also impose the conditions  $[\hat{g}_M^{\text{int}}, \hat{O}_{\text{odd}}] = [\hat{D}_0, \hat{O}_{\text{odd}}] = 0$ , ensuring that the detection scheme is independent of the initial state:  $[\hat{\rho}(0), \hat{D}_0] \neq 0$ . The expected relation  $\langle \hat{O}_{\text{odd}}(mT + T/2) \rangle = -\langle \hat{O}_{\text{odd}}(mT) \rangle$  in Eq. (S36) is verified through the time evolution of these operators, demonstrating the emergence of nontrivial micro-motion induced by the dynamical symmetry  $\hat{g}_M$  within the prethermal regime (see Fig. 1 in the main text).

Here, we confirm this result for another odd operator,  $\hat{O}_{\text{odd},2} = \sigma_{L/2-1}^y - \sigma_{L/2}^y$  [Fig. S2(a)]. We define  $\hat{O}_{\text{odd},2}^s(mT) = \hat{O}_{\text{odd},2}(mT) + \hat{O}_{\text{odd},2}(mT + T/2)$  for integer  $m$ , and observe that  $\hat{O}_{\text{odd},2}^s(t)$  saturates to zero, as indicated by the red horizontal line as the system enters the prethermal regime following relaxation [Fig. S2(b)]. The shaded region marks the prethermal regime for  $J/\Omega = 0.026, 0.028$ , while for  $J/\Omega = 0.03$ , the dynamics exit the prethermal regime. Here,  $t_0$  serves as a reference temporal point within this regime.

### Sensitivity to dynamical symmetry breaking

To verify the perfect oscillation of  $\langle \hat{O}_{\text{odd}}(mT) \rangle = -\langle \hat{O}_{\text{odd}}(mT + T/2) \rangle$  is directly tied to the dynamical symmetry, we explicitly break the symmetry  $\hat{g}_M$ , by adjusting the coupling across the center bond by setting

$\lambda_a^{LR} \neq \lambda_b^{LR}$  in Eq. (S39). When the symmetry is broken by setting  $(\lambda_a^{LR}, \lambda_b^{LR}) = (0.8, 1.2)$ , the saturated value of  $\hat{O}_{\text{odd}}^s(t)$  is nonzero, as shown by the location of the red horizontal line in Fig. S3(a). As the symmetry breaking increases to  $(\lambda_a^{LR}, \lambda_b^{LR}) = (0.5, 1)$ , the offset from zero becomes larger, as seen in Fig. S3(b).

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