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Kinetics and Seepage Forces in Steady-State Ground-Water Flow

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ABSTRACT

Formulas for (i) the components of acceleration, (ii) the volumetric gravitational, pressure-dependent, and friction forces, and (iii) stability conditions for aquifer solids in groundwater flowing under steady-state conditions are derived in this work. The tangential acceleration governs the magnitude of the average pore velocity along the flow path of ground water. The normal acceleration component arises from changes in the direction of the average pore velocity. The frictional force exerted by the aquifer matrix on ground water is derived via Newton's second law of motion. It is shown that the friction force is of approximately the same magnitude (but opposite in direction) to the drag or seepage force exerted by moving ground water on the aquifer matrix. Loss of aquifer solids by ground-water motion may occur whenever the contact force that holds aquifer particles static is not sufficient to counterbalance the resultant of the drag force plus the buoyant unit weight of aquifer. The hydraulic shear force exerted by stream flow contributes to the detachment of channel-bottom particles, although this force is perpendicular to the ground-water induced drag force under uniform stream flow regime. Key variables governing the detachment of aquifer solids by moving ground water are the hydraulic gradient, flow-path geometry, and the unit weights of aquifer and ground water.

INTRODUCTION

A volume of ground water moving through an aquifer is subject to several forces that determine its acceleration, if any. These forces are associated with the actions of gravitation, pore pressure, the Earth's rotation, and friction exerted on the moving fluid by the aquifer's matrix. Loáiciga (2006) showed that the Earth's rotational forces acting on ground water are much smaller than the gravitational force. Concerning the friction force exerted by the aquifer matrix on ground water moving through it, little is known about the magnitude of this force under field conditions. The use of Newton's second law of motion relating the forces acting on ground water (among which the friction force) to its change in momentum is infrequent in the ground-water hydrology literature (see, however, Hubbert, 1940; Howard and McLane, 1988). Instead, the average pore velocity of ground water ($\vec{v}$, a three-dimensional vector) is customarily expressed using Darcy's law (in which K is the hydraulic conductivity, h_p is the pressure head, z is the elevation head, and ϕ denotes porosity):

$$\vec{v} = -\frac{K}{\phi}\nabla(h_p + z) \tag{1}$$

Equation (1) is combined with the continuity equation to yield the well-known equation of ground water flow expressed in terms of the hydraulic head $h = h_p + z$. The equation of ground water flow is solved (using numerical methods, usually) and, afterwards, equation (1) is used to determine groundwater velocities within an aquifer. The acceleration of groundwater ($\vec{a}$) can be determined by taking the time derivate of velocity:

$$\vec{a} = \frac{d\vec{v}}{dt} \qquad (2)$$

In the absence of an analytical expression for the velocity, the derivative in equation (2) is estimated in finite-difference form using the calculated velocity distribution throughout an aquifer.

The components of ground-water acceleration are derived analytically in this paper. Subsequently, the acceleration is related to the (volumetric) pressure-dependent, gravitational, and friction forces driving ground-water flow. Steady-state flow conditions are assumed. In addition, an analysis of forces is conducted for a generic volume of aquifer. From this analysis emerge the conditions under which losses of aquifer matrix by ground-water transport may occur. Such losses create subsurface conditions of consequence in a wide range of geologic, geomorphologic, and geotechnical processes (Terzaghi, 1931; Cedergren, 1989; Freeze and Cherry, 1979; Emery and Kuhn, 1982; Hagerty, 1991; Leipnik and Loáiciga, 2006). When ground water discharges to lake and streams the ground water drag force may suffice to detach particles on the aquifer-free water boundary, causing seepage erosion (Hagerty, 1991). Uniform flow in a gaining stream produces a hydraulic shear force (or tractive force) tangent to its wetted perimeter and in the direction of flow (Rosgen, 2006). This shear force is perpendicular to the direction of discharging ground water, and, thus, to the drag force exerted by ground water on aquifer solids, as shown in Figure 1. The conditions governing seepage erosion and hydraulic shear erosion are investigated in this paper.

THE ACCELERATION OF GROUNDWATER

The acceleration of groundwater arises from changes in the magnitude and direction of the average pore velocity along its flow path. Figure 2 shows a flow net of ground-water flow under a dam. The flow net encompasses a stream tube with vertices labeled 1-2-3-4 in which the coordinate *s* for a particular groundwater flow path is depicted. The stream tube contains a volume of aquifer a-b-c-d over which several forces act upon, as described in the next section Let (i) $\vec{e}_T$ and $\vec{e}_N$ represent unit vectors tangential and normal, respectively, to the flow path along the curvilinear coordinate *s*, (ii) *R(s)* be the principal radius of curvature at *s*, (iii) *v* denote the magnitude of the velocity vector at *s*, $v = |\vec{v}|$, and (iv) α be the angle formed between the tangential velocity *v* at *s* and a (horizontal) reference datum that passes through the point *s*, and where α is related to *R(s)* by $R(s) = \partial s / \partial \alpha$. Figure 3 shows an enlarged aquifer volume a-b-c-d with key geometric characteristics drawn on it. Kinematic analysis leads to the following expression for the groundwater acceleration (see, for example, White, 1979):

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d\left(v\,\vec{e}_T\right)}{dt} = \frac{dv}{dt}\,\vec{e}_T + v\,\frac{d\,\vec{e}_T}{dt} = \left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial s}\right)\vec{e}_T - \left(v\frac{\partial \alpha}{\partial t} + \frac{v^2}{R}\right)\vec{e}_N \quad (3)$$

The local component of the tangential acceleration, $\partial v / \partial t$, is zero in steady-state flow. It arises from changes in velocity over time at a fixed location. The local angular velocity $\partial \alpha / \partial t$ represents the temporal rate of change of the direction of the tangential velocity at a given location *s*. This term vanishes in steady-state flow. Notice however that the tangential and normal components $v\partial v / \partial s$ and v^2 / R, respectively, of acceleration may still be nonzero in steady-state groundwater flow.

The magnitude of the average pore velocity along the flow path is expressed using Darcy's law (K and ϕ denote hydraulic conductivity and aquifer porosity, respectively):

$$v = -\frac{K}{\phi}\frac{\partial h}{\partial s} = -\frac{K}{\phi}\,i(s) \quad (4)$$

in which $i(s) = \partial h / \partial s$ denotes the hydraulic gradient along the coordinate *s*. The hydraulic gradient varies with *s*. The hydraulic head decreases in the direction of increasing *s*, so that $i < 0$, and, thus, $v > 0$.

Equation (3) coupled with equation (4) establish that the acceleration along a flow pathway has tangential (a_T) and normal (a_N) acceleration components whose magnitudes are given by (steady-state flow is assumed):

$$a_T = \frac{dv}{dt} = \frac{ds}{dt}\frac{dv}{ds} = v\frac{dv}{ds} = \frac{K^2}{\phi^2}\,i\,\frac{d\,i}{ds} \quad (5)$$

$$a_N = -\frac{v^2}{R} = -\frac{K^2}{\phi^2 R}\,i^2 \quad (6)$$

Equation (5) shows that the tangential component of the acceleration (a_T) along a flow pathway is nonzero so long as the magnitude of the velocity changes with the coordinate *s*. Equation (6) indicates that the normal acceleration component a_N is proportional to the square of the average pore velocity. The negative sign of a_N signifies that it is directed towards the center of curvature defined by the coordinate *s*. The normal acceleration arises from changes in the direction of the average pore velocity. It vanishes only when a flow path is straight or when ground water is stagnant. Equations (5) and (6) show that groundwater accelerations do exist in steady-state groundwater flow. Moreover, the accelerations (5)-(6) apply in 2- and 3-dimensional flow regimes. Further insight into the magnitude of the components of ground-water acceleration can be gained from flow nets, as shown next.

GROUNDWATER ACCELERATION FROM A FLOW NET

The acceleration of groundwater can be related to the geometry of a flow net. For the sake of argument, the flow net shown on Figure 2 is used to illustrate derivations presented in this section. The pseudo-squares of the flow net in Figure 2 are drawn so that their side along the flow path (Δs) are approximately equal to their average width (ω) perpendicular to the flow path and on the plane of the flow net (see enlarged pseudo-square a-b-c-d on Figure 3). Any other consistently constructed flow net could be used without invalidating the results below. It shall prove useful later in this paper to realize that the streamlines arriving at the discharge surface of the flow net (see points 3, 4 in Figure 2) do so perpendicular to the aquifer-free water boundary. This would also be the case if ground water discharges to an overlying stream with uniform flow regime. It is known from flow-net theory that the (constant) ground-water flow traversing the stream tube 1-2-3-4 depicted on Figure 2 is equal to:

$$\Delta Q = K\, b \frac{H}{f} \tag{7}$$

in which b is the thickness of the aquifer perpendicular to the plane of Figure 2, f is the number of head drops through the flow net (f = 8 in Figure 2), and H denotes the total head drop across the flow net (H = 7.10 in Figure 2). The total flow through the flow net of Figure 2 is $Q = p\,\Delta Q$, in which p denotes the number of stream tubes in the flow net ($p = 3$ in Figure 2).

The flow ΔQ equals the product of the area of aquifer perpendicular to the flow path (A = b w**)** times the magnitude of the specific discharge ($q = v \cdot \phi$), so that $\Delta Q = A \cdot v\,\phi$. While ΔQ is constant, the area A and specific discharge q vary with the flow-path coordinate s. Therefore:

$$\frac{d\Delta Q}{ds} = 0 = v\phi\ \frac{dA}{ds} + A\phi\ \frac{dv}{ds} \tag{8}$$

from which the change of the average pore velocity along the flow path is related to the change in the flow area by the following equation:

$$\frac{dv}{ds} = -\frac{v}{A}\frac{dA}{ds} \tag{9}$$

From equations (5) and (9) it follows that the magnitude of the tangential component of the ground water acceleration is given by the following equation:

$$a_T = v\frac{dv}{ds} = -\frac{v^2}{A}\frac{dA}{ds} \tag{10}$$

The tangential acceleration is positive (or negative) if the average pore velocity increases (or decreases) with increasing traveled distance s. The velocity v increases with s in the portion of the flow net of Figure 2 upstream from the narrowest constriction under the dam's pile. It decreases downstream from this constriction. Therefore, ground water exhibits positive (negative) tangential acceleration upstream (downstream) from the pile shown on Figure 2.

Using the fact that the flow area $A = \Delta Q/(v\,\phi)$ and the flow-net equation (7) in Equation (10) yields the following expression for the tangential acceleration along the coordinate s (recalling that A = b ω, also):

$$a_T = -\frac{v^3\ \ \phi}{b\,K\,(H/f)}\frac{dA}{ds} = -v^3\,\frac{\phi}{K\left(H/f\right)}\frac{dw}{ds} \tag{11}$$

To elucidate the order of magnitude that can be expected for a_T in aquifers, variables from the Figure 2 flow net are exemplary. With K = 1.2 x 10^{-4} m/s, ϕ = 0.3, and $i \cong 0.28$, the average pore velocity crossing the segment a-d of the stream tube 1-2-3-4 on Figure 2 is $v \cong$ 1.1 x 10^{-4} m/s. The contraction ratio at the same location is $d\omega/ds \cong$ - 1/3. These values, together with the average head drop H/f =

7.10/8 = 0.89, yield $a_T \cong$ 1.2 x 10^{-9} m/s^2. Given that the acceleration of gravity $g \simeq 9.8$ m/s^2, the a_T /g ratio is on the order of 10^{-10} for the data used.

Concerning the normal acceleration a_N, recall from equation (6) that its magnitude equals the square of the average pore velocity divided by the radius of curvature (R). $R \simeq 10$ m at the center of the pseudo-square a-b-c-d of the stream tube 1-2-3-4. It follows that the magnitude of the ratio a_N /g for v = 1.1 x 10^{-4} m/s is on the order of 10^{-10}. These calculations suggest that the tangential and normal components of the acceleration of ground water are very small –in fact, negligible- compared to the acceleration of gravity.

GRAVITATIONAL, PRESSURE, AND FRICTIONAL FORCES

Newton's second law of motion. The ground water contained in a pseudo-square (such as a-b-c-d in Figures 2 and 3) of a stream tube is subject to gravitational, pressure-dependent, and frictional forces, which, per unit volume of ground water, are herein denoted by F_W, F_P, and F_F, respectively. Newton's second law of motion written along the pathway of ground water flow implies that:

$$\rho_W \, a_T = F_W + F_P - F_F \qquad (12)$$

in which a_T is given by equation (11) and ρ_W is the density of ground water (approximately 1000 kg/m^3). In laminar ground-water flow the friction force (F_F) arises from viscous stresses caused by ground-water velocity gradients normal to the direction of flow within aquifer pores, akin to friction in flow through circular conduits associated with the well-known Hagen-Poiseuille flow equation (Streeter, 1966; Batchelor, 1970; Tritton, 1988). These viscous stresses are opposite to the direction of ground-water flow. The change of ground-water velocity within a highly irregular geometric array of aquifer pores induces thrust forces opposing ground-water flow, also. Given the relatively small magnitude of the changes in velocity involved, however, shear stresses are the dominant source of forces exerted by an aquifer matrix on ground water. The component of the gravitational force F_W is in the direction of flow when the average pore velocity has a downward vertical component. It opposes flow if the average pore velocity has an upward vertical component. F_W is nil when the average pore velocity is horizontal. The pressure force F_P is in the direction of decreasing pressure head.

The ground water gravitational force. From the geometry of Figure 3 – which shows an enlarged drawing of the pseudo-square a-b-c-d of Figure 2- it is deduced that the component of the gravitational force along the pathway of ground-water flow (F_W) is given by (letting γ_W denote the unit weight of water, $\gamma_W = \rho_W \, g$):

$$F_W = \gamma_W \cos\theta \qquad (13)$$

in which θ is the angle measured from a line oriented vertically downwards and counterclockwise to a line tangent to the pathway defined by the coordinate s, as shown on Figure 3. The angle θ equals 0 (or π) when ground water moves vertically downward (or upward). The gravitational force is in the direction of ground water flow whenever $0 \le \theta < \pi / 2$. It is opposite to the direction of ground water flow if $\pi / 2 < \theta \le \pi$.

The pressure force on ground water. Recall that the pressure head h_p equals P/γ_w, in which P is the pore pressure. The pressure force acting on an area $A\phi$ of pores perpendicular to the direction of ground water flow equals $P\ A\ \phi$ = $h_p\ \gamma_w\ A\phi$. The volume of ground water in the pseudo-square a-b-c-d drawn on Figure 2 and Figure 3 is approximately $A(s + \Delta s / 2)\ \phi\ \Delta s$. The net pressure force per unit volume of ground water, F_P, driving ground water flow through the pseudo-square a-b-c-d is, in the limit when $\Delta s \to 0$:

$$F_P = \frac{\gamma_w\ \phi}{\phi} \lim_{\Delta s \to 0} \left(\frac{h_p(s)A(s) - h_p(s+\Delta s)A(s+\Delta s)}{A(s + \Delta s / 2)\,\Delta s} \right) \cong \gamma_w \lim_{\Delta s \to 0} \frac{h_p(s) - h_p(s + (\Delta s))}{\Delta s} =$$

$$= -\gamma_w \frac{dh_p}{ds} \qquad (14)$$

The minus sign in equation (14) indicates that the pressure force F_P is in the direction of decreasing pressure head. Recalling that the pressure head is related to the hydraulic head and elevation head by the expression $h_p = h - z$, it follows that:

$$\frac{dh_p}{ds} = \frac{dh}{ds} - \frac{dz}{ds} = i + \cos\theta \qquad i < 0 \qquad 0 \le \theta \le \pi \qquad (15)$$

The hydraulic-head gradient i is negative because the hydraulic head decreases in the direction of increasing coordinate s. Substituting equation (15) in equation (14) yields the following equation for the pressure force (per unit volume of ground water):

$$F_p = -\gamma_w \cdot (i + \cos\theta) \qquad i < 0 \qquad 0 \le \theta \le \pi \qquad (16)$$

in which i and θ vary with the coordinate s.

The friction force on ground water. The friction force F_F can be solved from Newton's second law of motion (equation (12)), in which all forces are known except F_F. Substituting equations (13) and (16) into equation (12), and solving for the magnitude of F_F yields the following formula:

$$F_F = F_W + F_P - \rho_w a_T = \gamma_w \cos\theta - \gamma_w \cdot (i + \cos\theta) - \gamma_w \frac{a_T}{g} = -\gamma_w \cdot \left(i + \frac{a_T}{g} \right) \qquad (17)$$

The magnitude of the tangential acceleration of groundwater, a_T, is much less than the acceleration of gravity. This being the case, the frictional force per unit volume of groundwater is approximately:

$$F_f \cong -\gamma_w\, i \qquad i < 0 \qquad (18)$$

opposite to the direction of ground water flow. Result (18) is analogous to frictional forces that have been postulated for gradually-varied, steady-state, flow in channels (Chow, 1959; Cunge et al., 1980).

VOLUMETRIC AQUIFER FORCES AND MATRIX STABILITY

This section considers gravitational, pressure-induced, and contact forces (per unit volume of aquifer) acting on a volume of aquifer. Howard and McLane (1988) used an approach based on force equilibrium applied to surface grains of non-cohesive sediments to study the erosion by ground water seepage. These section's results apply to cohesive and non-cohesive sediments, which, if found on an aquifer's discharge boundary, may also be subject to a hydraulic shear or tractive exerted by overlying stream flow.

The gravitational force. The component of the gravitational force (F_{WA}) along the path of ground water flow acting on the volume of aquifer such as that delimited by the pseudo-square a-b-c-d shown on Figure 3 is:

$$F_{WA} = \gamma \cos\theta \tag{19}$$

in which $\gamma = \rho g$ is the unit weight of aquifer, ρ is the aquifer density (mass of solids plus water per unit volume of saturated aquifer), and g is the acceleration of gravity. Equation (19) differs from equation (13) in that the latter features the unit weight of ground water (γ_W) within the pseudo-square a-b-c-d. The gravitational force is positive, that is, in the direction of ground water flow whenever $0 \le \theta < \pi / 2$. It is negative, meaning opposite to the direction of ground water flow if $\pi / 2 < \theta \le \pi$. The component of the gravitational force along the path of ground-water motion is zero when $\theta = 0$.

The pressure force. Using arguments similar to those used to derive equation (16), the pressure-induced force on the aquifer volume a-b-c-d can be shown to be:

$$F_{PA} = -\gamma_W \cdot (i + \cos\theta) = -\gamma_W\, i - \gamma_W \cos\theta \qquad i < 0 \tag{20}$$

It is made explicit in equation (20) that the pressure-induced force has two components, one which is due to the hydraulic gradient ($-\gamma_W\, i$) and another which is of gravitational origin ($-\gamma_W \cos\theta$). The first component ($-\gamma_W\, i$) is a drag force exerted by moving ground water on the aquifer matrix, always acting in the direction of ground water motion. The drag force is sometimes referred to in the literature as the seepage force (Cedergren, 1989; Freeze and Cherry, 1979). This force is responsible for the erosion of subsurface particles by moving groundwater that may lead to the problem of piping in earthen embankments (Cedergen, 1980), or to sapped drainage morphology (Laity and Malin, 1985). geomorphological evolution driven by It is of approximately the same magnitude, but opposite in direction, to the friction force exerted on groundwater (see equation (18)). The drag force is perpendicular to the aquifer-free water boundary when ground water discharges into a water body with hydrostatic or nearly hydrostatic pressure distribution or a stream with prevailing uniform flow regime (shown in Figure 1).

The contact force. The contact force exerted by the solids surrounding the volume a-b-c-d on this volume arises from (i) the friction stresses (σ^*) acting on the surface of solids enveloping the volume a-b-c-d, and (ii) cohesive forces among solids on the same surface. The friction stress is caused by grain-to-grain contact forces per unit area of contact in a volume of aquifer (Terzaghi et al., 1996). The friction stress at a point within an aquifer depends on (i) the effective stress at the point (caused by natural stress, pore pressure, induced stress caused at the point by loads such as the dam shown on Figure 2, and, in some cases, stresses induced by tectonic forces (Keller and Loáiciga, 1993)), and (ii) the effective friction angle at the point. Cohesion arises from bonding among solid particles. Under stable conditions the aquifer matrix is static. In this case the (developed) contact force $F_C(s)$ along the coordinate *s* equals in magnitude and is opposite in direction to the resultant of the gravitational and pressure-dependent forces:

$$F_C(s) = \gamma \cos\theta - \gamma_w \cdot (i + \cos\theta) = -\gamma_w\, i + (\gamma - \gamma_w) \cdot \cos\theta \qquad i < 0 \qquad (21)$$

Equation (21) shows that the contact force equals the sum of the drag force (= $-\gamma_w\, i$) plus the component along the coordinate *s* of the buoyant unit weight (per unit volume) of aquifer (= $(\gamma - \gamma_w) \cdot \cos\theta$). The drag force acts in the direction of ground-water flow. The component of the buoyant unit weight acts in the direction of ground water flow whenever $0 \le \theta < \pi / 2$, that is, when ground water flow has a vertically downward component. It has a sense opposite to the direction of ground water flow whenever $\pi / 2 < \theta \le \pi$, that is, when ground water flow has an upward component (referred to as upwelling ground water).

Depending on (i) the magnitude of the friction stress, and (ii) the cohesion within the aquifer matrix, the contact force may not counterbalance the resultant of the drag force and buoyant unit weight. This takes heightened relevance for non-cohesive sediments whose particles or bundles of particles rest on the aquifer-free water boundary (into a lake or stream, for example), where the contact force is negligible or feeble. In this case the drag force –which is perpendicular to the aquifer-free water boundary under hydrostatic and uniform flow conditions- may exceed the buoyant unit weight component of aquifer-boundary particles opposing it. This leads to the separation or entrainment of boundary particles from the aquifer matrix, initiating seepage erosion in non-cohesive sediments. Hagerty (1991) presented diagrams and field photographs illustrating how a layer of non-cohesive sediments confined by an overlying cohesive formation undergoes seepage erosion. As the cohesive layer looses some of its volume to seepage erosion it undermines the structural support of the overlying formation, which eventually topples. This sequence of events causes stream bank erosion and recession. Quicksands and heave are other phenomena caused by upwelling ground water in non-cohesive sediments (Cedergren, 1989).

Stability condition for cohesive sediments subject to a drag force. The resultant of the drag force and buoyant unit weight in the direction of ground water flow may exceed the contact force even in cohesive sediments. Particles on the discharge surface of ground water are separated from the aquifer matrix first, followed by internal transport of subjacent particles along preferential pathways, a phenomenon called piping. Piping is a matter of concern in the stability of earthworks subject to ground water seepage through them (Terzaghi, 1931). Let σ^* and *c* be the average friction stress and cohesion (both in N/m^2) acting on the surface of a volume of aquifer, respectively, and A^* denote the surface area enveloping a unit volume of aquifer matrix (a textural property of the aquifer matrix with dimensions L^2/L^3). The contact force F_C is expressible in terms of

$\sigma*$, *c*, and *A** as being $F_C(s) = (\sigma*(s) + c)A*$. The aquifer matrix remains stable as long as the (developed) contact force equals the resultant of the drag force and the buoyant unit weight, as expressed in equation (21), which can be re-arranged as follows:

$$\sigma*(s) + c = -\frac{\gamma_w}{A*} i + \frac{(\gamma - \gamma_w)}{A*} cos\theta \qquad i < 0 \qquad (22)$$

Equation (22) is the stability condition for cohesive sediments subject to a ground water drag force. Whenever the right-hand side of equation (22) exceeds the sum of the average friction stress and cohesion, aquifer solids may be mobilized. The loss of solids begins at the boundary of the aquifer where discharge occurs and propagates inward causing piping.

Critical hydraulic gradient for boundary particles in non-cohesive sediments. The stability condition in this case is described by setting the cohesion equal to zero in equation (22):

$$A* \cdot \sigma*(s) = -\gamma_w i + (\gamma - \gamma_w) cos\theta \qquad i < 0 \qquad (23)$$

If groundwater flow is upward vertically ($\theta = \pi$), the force $A*\sigma*$ in equation (23) reduces to the vertical effective stress (σ'). Assuming further that the upward seepage force equals the downward buoyancy force, then, the vertical effective stress is nil, and the volume of aquifer is free to move. In this instance the critical hydraulic gradient (i_c) at which the drag force becomes equal in magnitude but opposite in direction to the component of the buoyant unit weight is obtained from equation (23):

$$i_c = \frac{(\gamma - \gamma_w)}{\gamma_w} \qquad (24)$$

Entrainment of aquifer solids may occur whenever the hydraulic gradient $i < i_c$, that is, whenever *i* is more negative than i_c, meaning that its magnitude or absolute value would exceed that of i_c. Knowing the flow-net geometric and unit weights, condition (24) can be checked a priori to asses whether or not the loss of aquifer solids poses a threat. Equation (24) shows that ground water flow need not be vertically upwelling (i.e., $\theta = \pi$) for aquifer losses to take place.

Entrainment of boundary particles in non-cohesive sediments by hydraulic shear (tractive) force Sediment entrainment by stream flow has been thoroughly studied in the specialized literature of river hydraulics (see a recent summary in Rosgen, 2006). At a foundational level, the shear stress acting tangent to a stream bed in the direction of (uniform) flow where the depth of water is *d* and the slope of the stream's water surface is denoted by *S* is approximately:

$$\tau \cong \gamma_w d S \qquad (25)$$

From equation (24), the tractive force per unit volume of aquifer exerted at the bottom of a column of water of depth *d* is approximately:

$$F_{\tau} \cong \frac{\gamma_w \, d \, S}{D} \tag{26}$$

in which *D* is the effective dimension of sediment particles on the aquifer-free water boundary (or stream bed). The tractive force in equation (26) is perpendicular to the groundwater drag (or seepage) force acting of the aquifer-free water boundary (see Figure 1). The tractive force is opposed by a resistance force (per unit volume of aquifer) equal to the component of buoyant unit weight of sediment particles normal to the stream bed times a friction coefficient (μ):

$$R = (\gamma - \gamma_w)\, \mu \;\cos\beta \tag{27}$$

in which β is the angle of the stream bed's slope. Based on equilibrium of forces along the bottom of the stream in the direction of flow, entrainment of sediment particles would take place whenever the tractive force exceeds the resistance force. Using the ratio of the resistance force over the tractive force, G_{τ}, entrainment of sediment particles on the aquifer-free water boundary would occur whenever G_{τ} <1:

$$G_{\tau} = \frac{D}{d}\frac{\cos\beta}{S}\left(\frac{\gamma - \gamma_w}{\gamma_w}\right)\mu < 1 \tag{28}$$

Entrainment of non-cohesive sediment particles on the aquifer-free water boundary may be triggered by the tractive force exerted by stream flow on the stream bed according to criterion (28). Or, it may occur by the action of ground water drag or upward seepage if the hydraulic gradient is less than the critical gradient given in equation (24).These two force systems act along directions that are perpendicular to each other.

EXAMPLE CALCULATION OF VOLUMETRIC FORCES

Figure 4 shows the calculated drag force along the coordinate *s* marking the flow path of ground water through the midst of the stream tube 1-2-3-4 previously depicted on Figure 2. It is seen that the drag force has a nearly bell-shaped form with a maximum reached at the narrowest construction of the flow net, where $s \cong$ 13.6 m.

The calculated drag force, buoyant weight component along *s*, and contact force for the flow net of Figure 2 are shown on Figure 5. It is seen there that the drag and buoyant weight act in the same direction, that is, in the direction of ground-water flow up to the constriction point where $s \cong$ 13.6 m. Downstream from the constriction point the buoyant weight acts opposite to the drag force. The drag force is opposite in direction to the buoyant weight and it exceeds its magnitude between *s* = 13.6 m and *s* = 15 m, at which point they become equal in magnitude and the contact force becomes nil. This portion of the flow path is of concern from the viewpoint of matrix losses, a concern that it mitigated in the example flow net by the fact that beyond s = 15 m the buoyant weight dominates over the drag force. Evidently, potential matrix losses are exacerbated if the drag force is opposite in direction to the buoyant weight and it exceeds its magnitude near the terminus of a flow path. The lack of aquifer overburden given such imbalance of forces produces transport of solids that propagates upstream a flow path, giving rise to piping that may undermine engineering works or become an active subaerial geomorphic agent (Emery and Kuhn, 1982; Hagerty, 1991).

CONCLUSIONS

This paper has shown that the acceleration of ground water is negligible when compared to the acceleration of gravity under typical flow conditions. This fact, together with the formulation of forces acting on ground water using Newton's second law of motion, led to the determination of the friction force exerted by the aquifer matrix on moving ground water. The friction force so determined was shown to be approximately equal to the drag force exerted by ground water on the aquifer solids. The drag force arises from pressure gradients that mobilize ground water through an aquifer's pores. Force equilibrium was exploited to derive relations among pressure-dependent, buoyant, and contact forces acting on a generic aquifer volume. These relations were shown to be valuable in assessing the potential for the loss of aquifer solids and the concomitant instability of aquifers through which ground water flows subject to a substantial drag force..

REFERENCES

Batchelor, G.K. (1970). *Fluid Dynamics.* Cambridge University Press, Cambridge, UK.

Cedergren, H.E. (1989). *Seepage, Drainage, and Flow Nets.* John Wiley & Sons, New York.

Chow, V.T. (1959). *Open-Channel Hydraulics.* McGraw-Hill Book Co., New York.

Cunge, J.A., Holly, F.M. Jr., Verwey, A. (1980). *Practical Aspects of Computational River Hydraulics.* London, UK: Pitman Publishing Limited.

Freeze, R.A., Cherry, J.A. (1979). *Groundwater.* Englewood Cliffs, New Jersey: Prentice-Hall.

Keller, E.A., Loáiciga, H.A. (1993). Fluid-pressure induced seismicity at regional scales. *Geophysical Research Letters*, 20(16), 1683-1686.

Emery, K.O., Kuhn, G.G. (1982). Sea cliffs: their processes, profiles, and classification. *Geological Society of America Bulletin*, 93, 644-654.

Hagerty, D.J. (1991). Piping/sapping erosion. I: basic considerations. *Journal of Hydraulic Engineering*, 117(8), 991-1008.

Howard, A.D., McLane, C.F. (1988). Erosion of cohesionless sediments by groundwater seepage. *Water Resources Research*, 24(10), 1659-1674.

Hubbert, M. King. (1940). The theory of groundwater motion. *Journal of Geology*, XLVIII, 355-372.

Laity, J.E., Malin, M. C. (1985). Sapping processes and the development of theater-headed valley networks on the Colorado Plateau. *Geological Society of America Bulletin*, 96, 203-217.

Leipnik, R.B., Loáiciga, H.A. (2006). Radially convergent groundwater flow in sloping terrain. *Hydrologic Sciences Journal*, 51(4), 700-712

Loáiciga, H.A. (2006). The effect of the Earth's rotation on ground water motion. *Ground Water*, 45(1), 98-100.

Rosgen, D. (2006). *Watershed Assessment of River Stability and Sediment Supply*. Wildland Hydrology, Fort Collins, Colorado.

Streeter, V.L. (1966). *Fluid Mechanics*. New York, USA: McGraw-Hill.

Tritton, D.J. *Physical Fluid Dynamics.* 2nd ed., Oxford Science Publications, UK.

White, F.M. (1979). *Fluid Mechanics.* McGraw-Hill Book Co., New York.

Terzaghi, K. (1931). Underground erosion and the Corpus Christi Dam failure. *Engineering News Record,* 107, 90-92.

Terzaghi, K., Peck, R.B., Mesri, G. (1996). *Soil Mechanics in Engineering Practice.* John Wiley & Sons, New York.

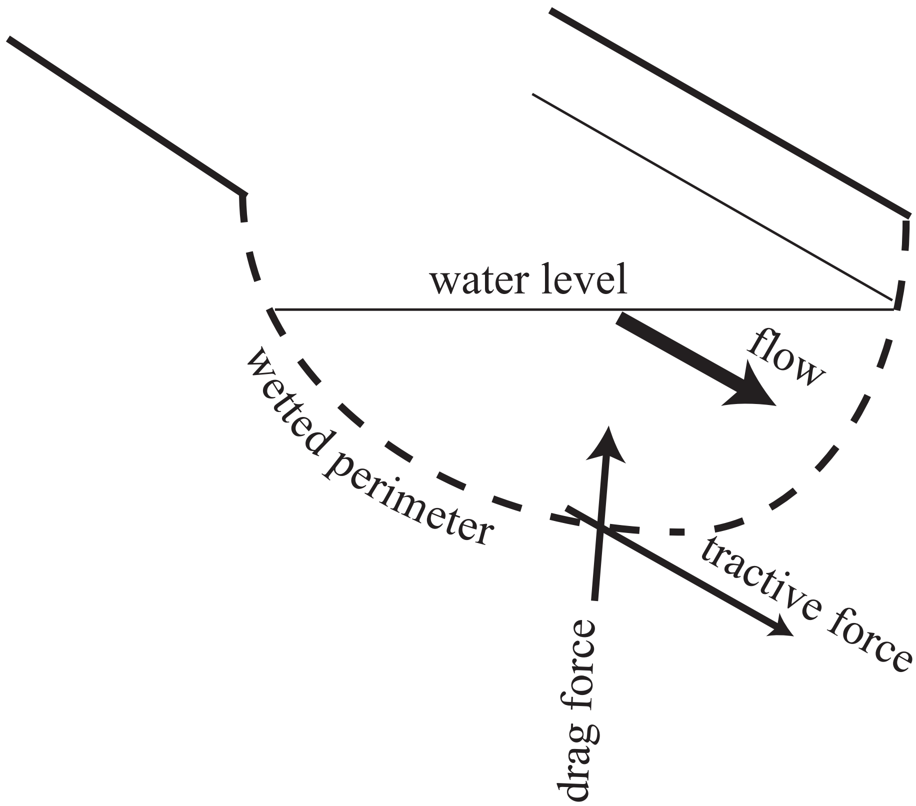

Figure 1. The drag force and shear (or tractive) force acting on the wetted perimeter of a stream with uniform flow.

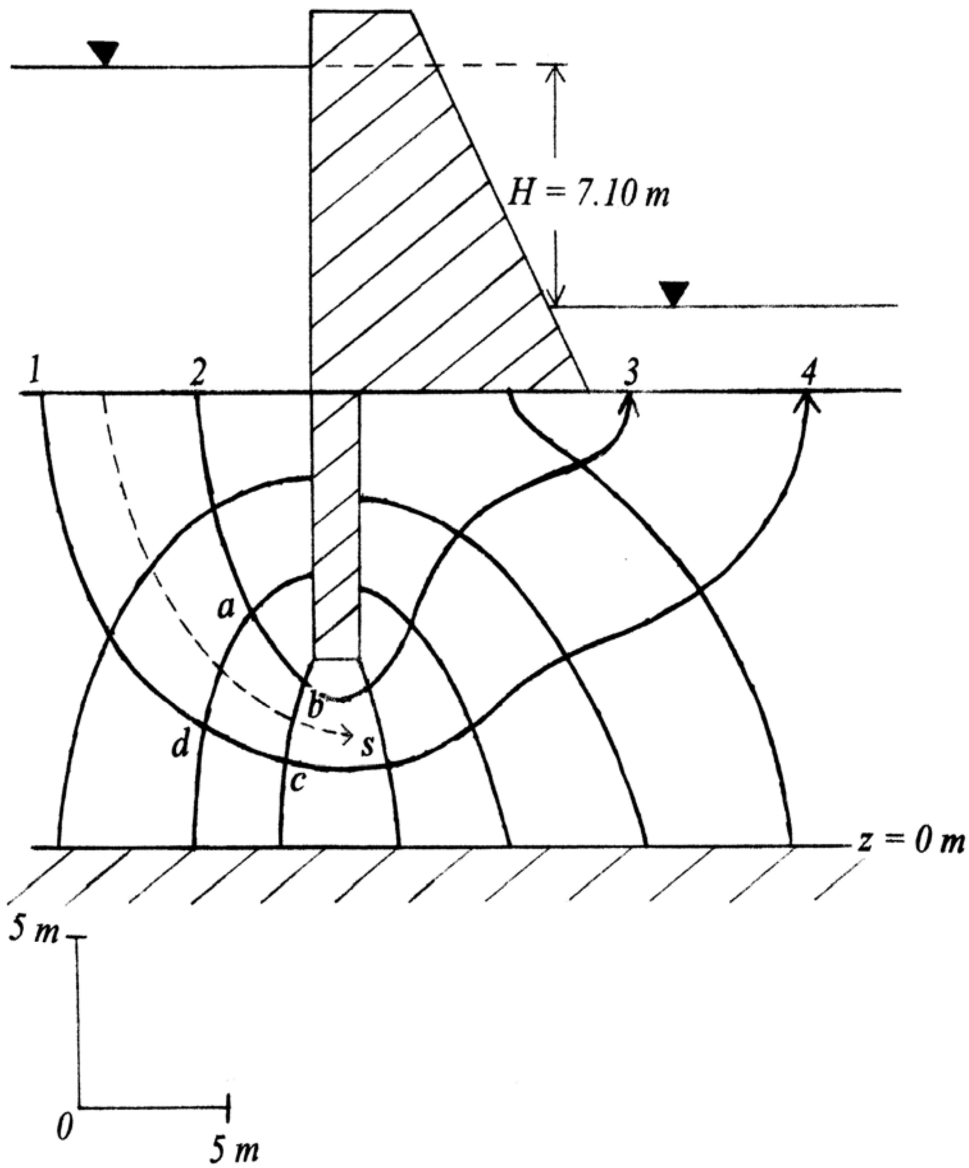

Figure 2. Flow net under a gravity dam, showing stream tube 1-2-3-4, typical aquifer volume a-b-c-d, and the curvilinear coordinate *s* depicting the pathway of ground water through the center of the stream tube.

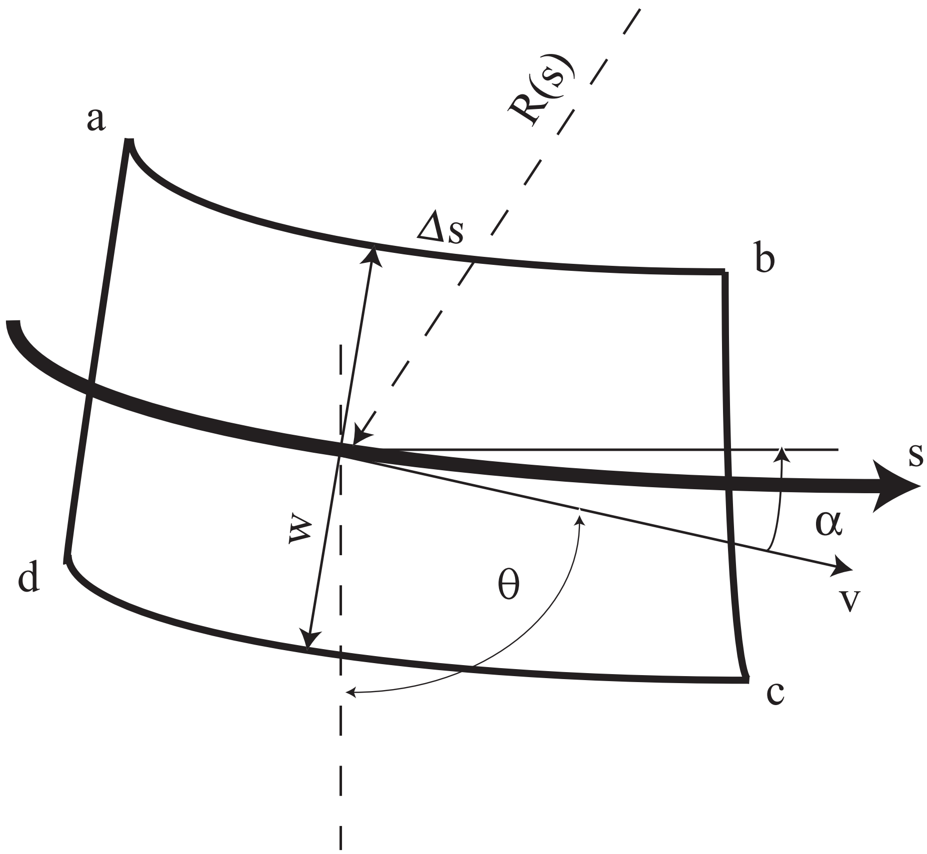

Figure 3. Enlarged aquifer volume a-b-c-d showing its dimensions, the average pore velocity *v* tangent to the path flow of ground water (*s*), and the angle θ .

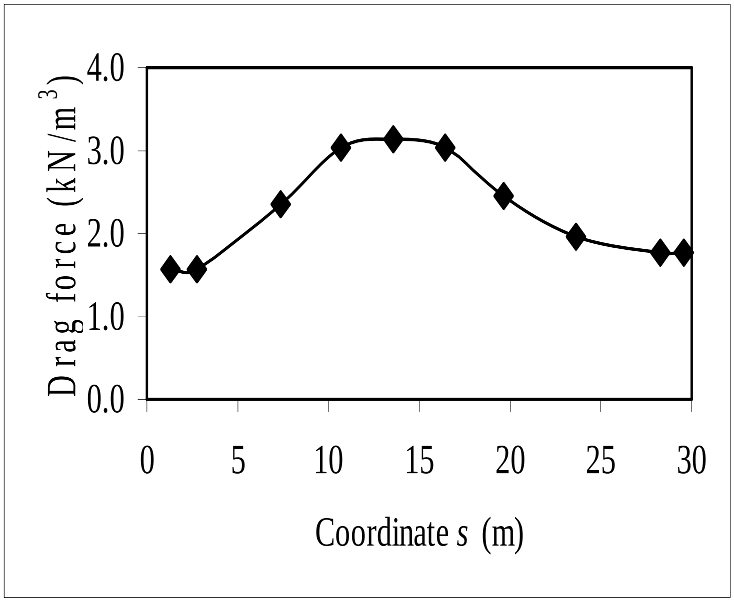

Figure 4. The drag force ($-\gamma_w\ i$) along the coordinate *s* shown on Figure 2.

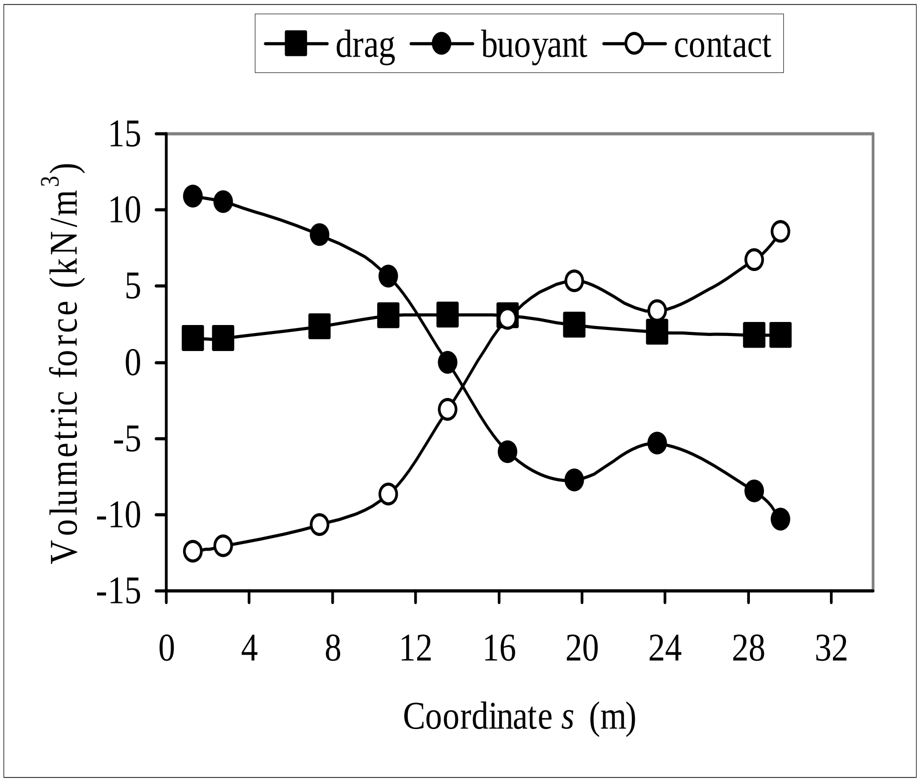

Figure 5. The drag force ($-\gamma_w\, i$), buoyant unit weight along s ($(\gamma - \gamma_w) \cdot \cos\theta$), and contact force along the coordinate s shown on Figure 2. The contact force is opposite in direction to the resultant of the drag and buoyant unit weight.