Tropical Geometry of Curves

by

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Abstract

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Algebraic geometry is a classical subject which studies shapes arising as zero sets of polynomial equations. Such objects, called varieties, may be quite complicated but many aspects of their geometry are governed by discrete data. In turn, combinatorial structures arising from particularly meaningful varieties, such as moduli spaces, are interesting in their own right. In recent years tropical geometry has emerged as a robust tool for studying varieties. As a result, many rich connections between algebraic geometry and combinatorics have developed. Tropical geometry associates polyhedral complexes, like *tropical varieties* and *skeletons*, to algebraic varieties. These encode information about the variety or the equations they came from, providing insight in to the underlying combinatorial structure.

In this thesis, I develop tropical geometry of curves from the perspectives of divisors, moduli spaces, computation of skeletons, and enumeration. Already, the realm of curves is rich to explore using the tools of tropical geometry. This thesis is divided into five chapters, each focusing on different aspects of the tropical geometry of curves.

I begin by introducing algebraic curves, tropical curves, and non-Archimedean curves. The ways in which these objects interact is a common theme of this thesis. Tropical curves are projections of non-Archimedean curves. Berkovich analytic spaces are heavenly abstract objects which can be viewed by earthly beings through their tropical shadows.

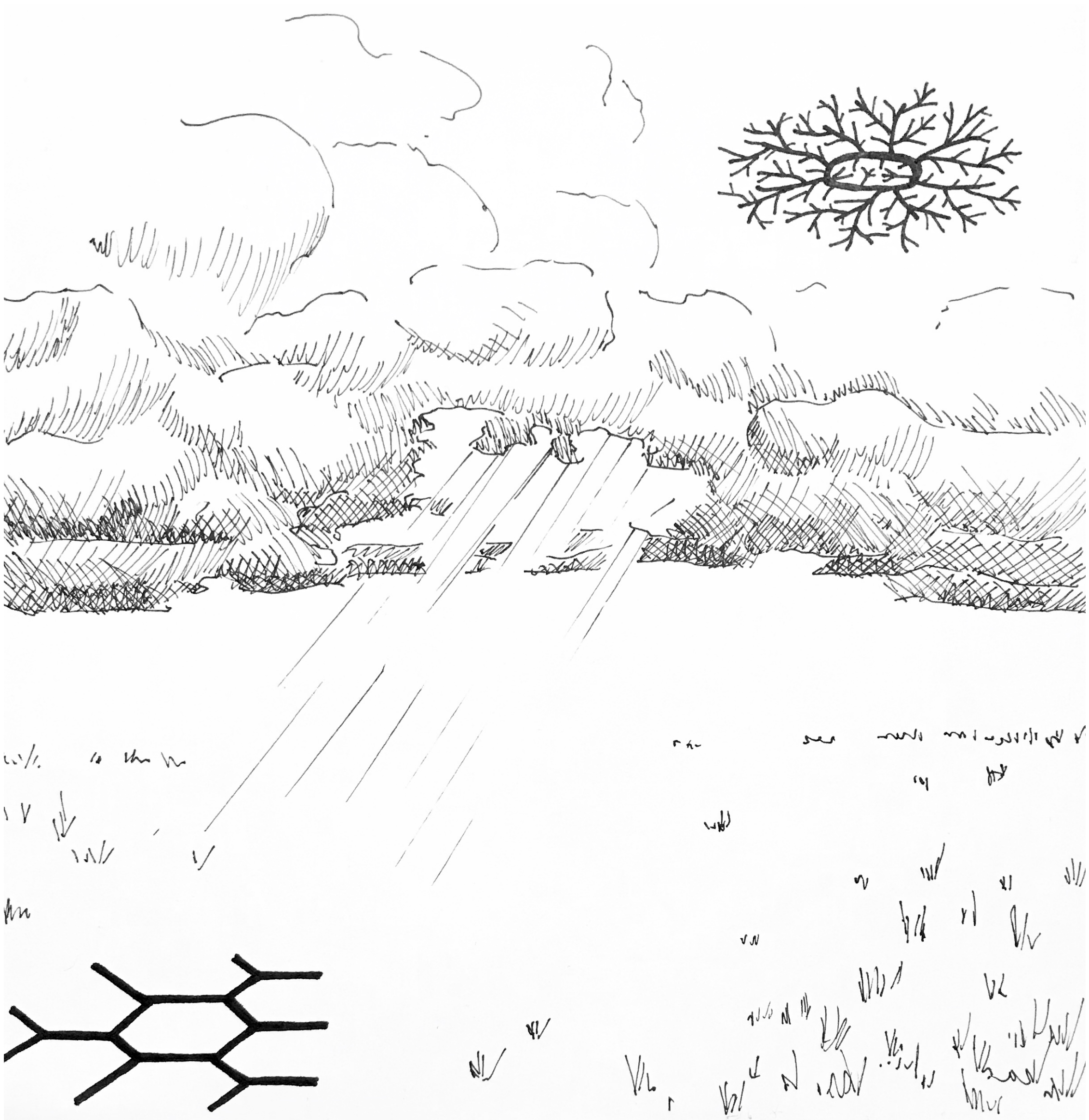
In the second chapter, I develop divisors on tropical curves and tropicalize algebraic divisors. Many constructions for classical curves related to divisors carry over to the tropical world. This will include a tropical Jacobian, a tropical version of the Riemann-Roch theorem, and a tropical Abel-Jacobi map. I first define and compute these objects. Then, I focus on the symmetric power of a curve, because this functions as a moduli space for effective divisors on the curve. I prove that the non-Archimedean skeleton of the symmetric power of a curve is equal to the symmetric power of the non-Archimedean skeleton of the curve. Using this, I prove a realizable version of the tropical Riemann-Roch Theorem.

In the third chapter I focus on moduli spaces. A recurring phenomenon in tropical geometry is that the non-Archimedean skeleton of an algebraic moduli space gives a tropical

one. I will develop detailed examples of this, following [1, 38, 99, 119]. Then, I define a divisorial motivic zeta function for marked stable curves, and prove that it is rational.

In the fourth chapter I compute abstract tropicalizations or non-Archimedean skeletons of a curve. In genus one [46, 77] and two [51, 70, 111] there are known methods for computing these tropicalizations. I develop an algorithm for computing the abstract tropicalizations of hyperelliptic and superelliptic curves. In higher genus, these are the only known results for computing abstract tropicalizations of curves.

In the final chapter I study enumerative problems, following [90, 91, 96]. Tropical geometry has proven to be a very useful tool in counting curves in the plane. I turn my attention to surfaces in space, and develop tropical counting techniques in this domain. This leads to a preliminary count of binodal cubic surfaces.



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1

Introduction

In this section, we develop the three main objects of study. Algebraic curves over valued fields can be studied through the combinatorics of their models. Tropical curves, both embedded and abstract, encode information about the algebraic curves they come from. Non-Archimedean curves carry all of the structure we are interested in, but can be difficult to compute. In Section 1.4, we summarize the main novel results in this dissertation.

1.1 Algebraic Curves

One approach to studying the geometry of a curve involves degenerating the curve. This is done by placing the curve in a one-parameter family of smooth curves which degenerate nicely into a union of irreducible components with simpler structure. For example, we may allow a curve with positive genus to degenerate into a union of projective lines. Studying degenerations in this manner can be useful because the degeneration transforms the curve of interest into a simpler one, whereby some of the initial structure of the original curve is preserved. The degenerate curve can have a rich combinatorial structure, enabling us to study the curve from a new point of view.

Valued fields provide a mechanism for studying degenerations of curves. As an example, let $f(x, y, z)$ be a homogeneous polynomial of degree four over $\mathbb{C}$ defining a smooth projective curve. Now consider the curve X_t defined over the power series ring $\mathbb{C}[[t]]$ by the equation

$$tf(x, y, z) + xy(y - 2x - 2z)(x - 2y - 2z).$$

Then, X_t gives a family of curves $X_t \to \mathbb{C}$ in the following way: the fiber over a point $a \in \mathbb{C}$ is the curve X_a defined by setting $t = a$ in the above equation. This gives a family of smooth curves degenerating to the singular curve given by the union of four lines when $t = 0$. To obtain from X_t a curve with coefficients over a field, there are two reasonable things to do. One would be to look at the curve defined over the field of fractions $\mathbb{C}((t))$, and the other would be to look at the curve defined over the residue field obtained by quotienting out by the unique maximal ideal $\langle t \rangle$ of $\mathbb{C}[[t]]$ to obtain a curve over $\mathbb{C}$, which is X_0. The field $\mathbb{C}((t))$

is an example of a valued non-Archimedean field (see Examples 1.1.5 and 1.3.5). Curves over non-Archimedean fields require special consideration and new techniques because non-Archimedean fields come with an absolute value that fails the familiar *Archimedean property* of the usual metric on the complex numbers.

To study the curve defined over $\mathbb{C}((t))$, we may do one of two things.

1. Study the combinatorics of the intersections of the components in the special fiber, or
2. Use the structure given by the absolute value on $\mathbb{C}((t))$ to analyze the geometry of the degeneration.

These two approaches will give us *tropical curves* and *non-Archimedean curves*, respectively. As we will see, these are closely related.

The above example can be generalized. We will see that curves over a valued field naturally encode degenerations to nodal curves which can be studied combinatorially.

Valued Fields

Let us begin with some background on valued fields, which are the ground fields of interest for tropical and non-Archimedean geometry. A good reference is [88, Section 2.1].

Definition 1.1.1. A *valuation* on a field K is a function $v : K \to \mathbb{R} \cup \{\infty\}$ satisfying

1. $v(a) = \infty$ if and only if $a = 0$,
2. $v(ab) = v(a) + v(b)$,
3. $v(a + b) \geq \min\{v(a), v(b)\}$.

Notation 1.1.2. The image of v is denoted by Γ_v, and is called the *value group*. The *valuation ring* R is the set of all elements with non-negative valuation. The valuation ring is a local ring with maximal ideal m given by all elements with positive valuation. The quotient R/m is denoted by k and it is called the *residue field*.

Example 1.1.3 (trivial valuation)**.** Every field has a *trivial valuation* v_0 sending $v_0(K^*)$ to 0 and $v(0) = \infty$.

Example 1.1.4 (p-adic valuation)**.** Let p be a prime. The p-adic valuation on $\mathbb{Q}$ is defined by taking $v(a/b \cdot p^l) = l$, where $\gcd(a, b) = 1$ and p does not divide a or b. The local ring R is the localization of $\mathbb{Z}$ at $\langle p \rangle$, and the residue field is $\mathbb{F}_p$.

Example 1.1.5 (Laurent series)**.** The *Laurent series* $\mathbb{C}((t))$ are the formal power series with coefficients in $\mathbb{C}$:

$$c(t) = c_1 t^{a_1} + c_2 t^{a_2} + \cdots$$

for a_i an increasing sequence of integers. The valuation is given by taking $v(c) = a_1$.

Example 1.1.6 (Puiseux series)**.** The *Puiseux series* $\mathbb{C}\{\{t\}\}$ are the formal power series with rational exponents and coefficients in $\mathbb{C}$:

$$c(t) = c_1 t^{a_1} + c_2 t^{a_2} + \cdots$$

for a_i an increasing sequence of rational numbers which have a common denominator. These are the algebraic closure of the Laurent series. The valuation is given by taking $v(c) = a_1$. This is the field which will appear most frequently in examples throughout this dissertation.

A *splitting* of a valuation is a homomorphism $\phi : \Gamma_{\text{val}} \to K^*$ such that $v(\phi(w)) = w$. The element $\phi(w)$ is denoted t^w.

Models of Curves

In this section we follow [43] for background on models of curves. We begin with a concrete example to illustrate the ideas motivating the definitions that follow.

Example 1.1.7. Let $K = \mathbb{C}((t))$ be the Laurent series. Then the valuation ring R is $\mathbb{C}[[t]]$.
Consider curve X in $\mathbb{P}^2$ over K defined by the equation

$$xy = tz^2.$$

Then X is a smooth conic over K. However, we may view X as defining a family of smooth curves with base parameter t close to but not equal to 0.

The equation $xy = tz^2$ also defines a scheme $\mathcal{X}$ over R. The topological space $\operatorname{Spec}(R)$ consists of two points. One is closed and corresponds to the maximal ideal m. The other is open and corresponds to the zero ideal, and its closure is all of $\operatorname{Spec}(R)$. So, there are two fibers to consider. The *special fiber* $\mathcal{X}_k = \mathcal{X} \times_R k$ has the equation $xy = 0$ in $\mathbb{P}^2_{\mathbb{C}}$. So, the special fiber is the union of two rational curves meeting at a node.

We now generalize the ideas present in this example. Let k be an algebraically closed field. A *curve* over k is a reduced, proper, connected scheme X of dimension 1 over k. A *node* of X is a point $p \in X(k)$ such that the completion of the local ring at p is isomorphic to $k[[x, y]]/\langle xy \rangle$. If the curve is planar, this is equivalent to the condition that the partial derivatives of its defining equation vanish but the Hessian matrix is nonsingular. A *nodal curve* is a curve whose singularities are all nodes.

Definition 1.1.8. An *n-marked curve* over k is a tuple $(X, p_1, \ldots, p_n)$ where X is a curve over k, and $p_i \in X(k)$ are distinct nonsingular points.

Definition 1.1.9. A nodal marked curve $(X, p_1, \ldots, p_n)$ is *stable* if $\operatorname{Aut}(X, p_1, \ldots, p_n)$ is finite. This means that there are only finitely many automorphisms of the curve X that fix each $p_1, \ldots, p_n$ pointwise. Put more concretely, this means that each rational component of X needs to have at least three "special points" on it, where a special point is either a node, a marked point, or a point of self-intersection (which is counted with multiplicity 2), and each genus 1 component needs at least one special point. Stated more precisely, for each irreducible component C of X, let $\phi : C^\nu \to C$ denote the normalization of C. Then $(X, p_1, \ldots, p_n)$ is stable if and only if

1. for every component C of geometric genus 0, we have
$$|C \cap \{p_1, \ldots, p_n\}| + |\{q \in C^\nu \mid \phi(q) \in X^{\text{sing}}\}| \geq 3;$$
2. for every component C of geometric genus 1, we have
$$|C \cap \{p_1, \ldots, p_n\}| + |\{q \in C^\nu \mid \phi(q) \in X^{\text{sing}}\}| \geq 1;$$

A stable n-marked curve of genus g only exists if $2g - 2 + n > 0$.

Sometimes, we will be less strict. We say that X is *semistable* if every rational component has at least two points which are marked points or points x in the normalization C^ν such that $\phi(x)$ is a singularity in X.

Let K be an algebraically closed field with a valuation. The topological space $\operatorname{Spec}(R)$ has two points η and s, corresponding to the ideals $\langle 0 \rangle$ and m respectively.

Definition 1.1.10. If $\mathcal{X}$ is a scheme over $\operatorname{Spec}(R)$ then the *generic fiber* of $\mathcal{X}$ is the fiber over the point η corresponding to the ideal $\langle 0 \rangle$ in R, and the *special fiber* is the fiber over the point s corresponding to the ideal m in R.

Definition 1.1.11. If X is any finite type scheme over K, a *model* for X is a flat and finite type scheme $\mathcal{X}$ over R whose generic fiber is isomorphic to X. We call this model *stable* if the special fiber $\mathcal{X}_k = X \times_R k$ is a stable curve over k.

The existence of stable models will allow us to degenerate curves over K into curves that can be represented in a combinatorial way.

Semistable Reduction

A curve of genus at least two always admits a (semi)stable model by the *semistable reduction theorem*. This result guarantees the existence of a model with a combinatorially tractable special fiber and gives uniqueness of this fiber up to stabilization, making it crucial for the construction of tropical curves. For more on stable and semistable reduction, see [59].

Theorem 1.1.12 (Semistable reduction theorem)**.** *Let K be a field with valuation and R its valuation ring. Let X be a smooth projective curve over K of genus at least 2. Then there exists a finite separable valued field extension K' of K such that there is a semistable model $\mathcal{X} \to R'$. Any two such extensions are dominated by a third, and so they have special fibers whose stable models are isomorphic.*

The proofs of this theorem contain somewhat algorithmic approaches, see [8] and [53]. However, computing a semistable reduction can be quite difficult. We now describe a procedure for finding a semistable model for the curve X when K has characteristic 0 by way of example. This example appears in [22]. A good reference with more examples is [68].

Example 1.1.13. Consider the curve X in $\mathbb{P}^2$ over $\mathbb{C}\{\{t\}\}$ defined by

$$xyz^2 + x^2y^2 + 29t(xz^3 + yz^3) + 17t^2(x^3y + xy^3) = 0.$$

For this curve, the special fiber is a conic with two tangent lines, depicted in Figure 1.1(a). We denote the conic by C and the two lines by l_1 and l_2.

The first step is to blow up the total space $\mathcal{X}$, removing any singularities in the special fiber, to arrive at a family whose special fiber is a nodal curve. We begin by blowing up the total space at the point p_1. The result, depicted in Figure 1.1(b), is that l_1 and C are no longer tangent, but they do intersect in the exceptional divisor, which we call e_1. The exceptional divisor e_1 has multiplicity 2, coming from the multiplicity of the point p_1. In the figures, we denote the multiplicities of the components with grey integers, and a component with no integer is assumed to have multiplicity 1. Next, we blow up the total space at the

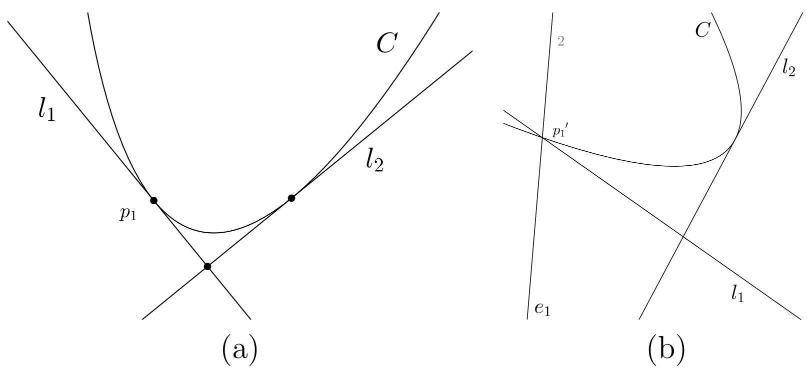


Figure 1.1: Semistable reduction in Example 1.1.13.

point labelled p_1' to get Figure 1.2(c). We denote the new exceptional divisor by e_1' with multiplicity 4, and the curves l_1 and C no longer intersect. All points except $l_2 \cap C$ are either smooth or have nodal singularities, so we repeat these two blowups here, obtaining the configuration in Figure 1.2(d).

At this point, we have a family whose special fiber only has nodes as singularities, but it is not reduced. To fix this, we make successive base changes of prime order p. Explicitly, we take the p-th cover of the family branched along the special fiber. Then, if D is a component of multiplicity q in the special fiber, either p does not divide q, in which case D is in the branch locus, or else we obtain p copies of D branched along the points where D meets the branch locus, and the multiplicity is reduced by $1/p$.

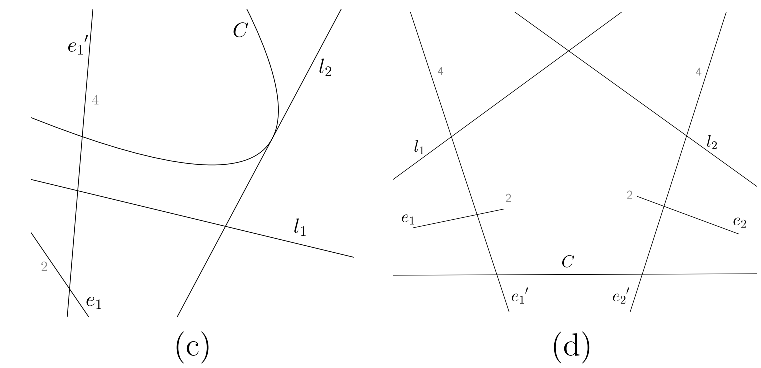


Figure 1.2: Blowing up in Example 1.1.13.

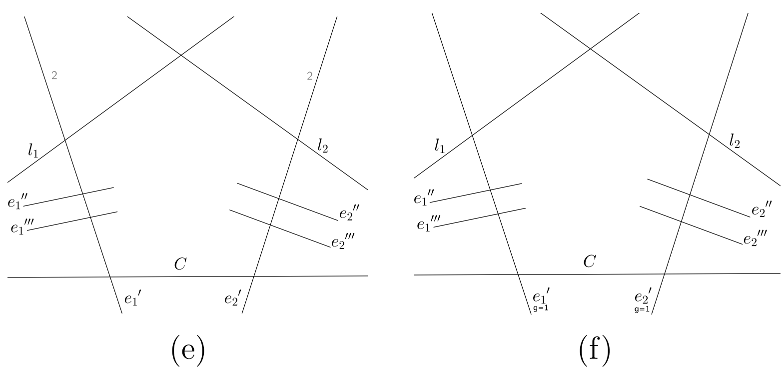


Figure 1.3: Base changes in Example 1.1.13.

In our example, we must make two base changes of order 2. Starting with Figure 1.2(d) above, we see that l_1, l_2, and C are in the branch locus. The curves e_1' and e_2' are replaced by the double cover of each of them branched at 2 points, which is again a rational curve. We continue to call these e_1' and e_2', and they each have multiplicity 2. Then, e_1 and e_2 are disjoint from the branch locus, so each one is replaced by two disjoint rational curves. The result is depicted in Figure 1.3(e).

In the second base change of order 2, all components except e_1' and e_2' are in the branch locus. The curves e_1' and e_2' each meet the branch locus in 4 points, which, by the Riemann-Hurwitz theorem, means they will be replaced by genus 1 curves, see Figure 1.3(f).

The last step is to blow down all rational curves which meet the rest of the fiber exactly once. The result is depicted in Figure 1.4. This process gives us a semistable model for X with the special fiber in Figure 1.4.

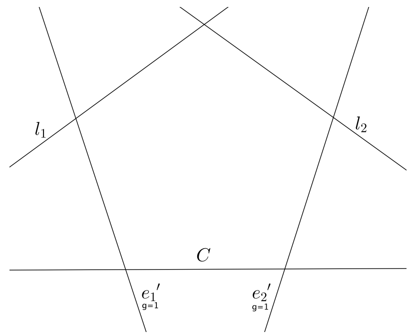


Figure 1.4: The special fiber of the semistable model obtained in Example 1.1.13

1.2 Tropical Curves

In this chapter, we develop two different notions of tropical curves. One is *embedded tropicalization*, in which a tropical curve is a one-dimensional balanced polyhedral complex. The other is *abstract tropicalization*, in which a tropical curve is a weighted metric graph. We begin with polyhedral complexes and embedded tropicalization, and then discuss metric graphs and abstract tropicalization.

Polyhedral Complexes

Definition 1.2.1. A set $X \subset \mathbb{R}^n$ is *convex* if, for any two points in the set, the line segment between them is also contained in the set. The *convex hull* $\operatorname{conv}(U)$ of a subset $U \subset \mathbb{R}^n$ is the smallest convex set containing U.

We now describe some types of convex sets which will appear often in tropical geometry. A *polytope* is a convex set which is expressible as the convex hull of finitely many points. A *polyhedral cone* C in $\mathbb{R}^n$ is the positive hull of a finite subset of $\mathbb{R}^n$:

$$C = \operatorname{pos}(v_1, \ldots, v_r) := \left\{ \sum_{i=1}^{r} \lambda_i v_i \;\middle|\; \lambda_i \geq 0 \right\}.$$

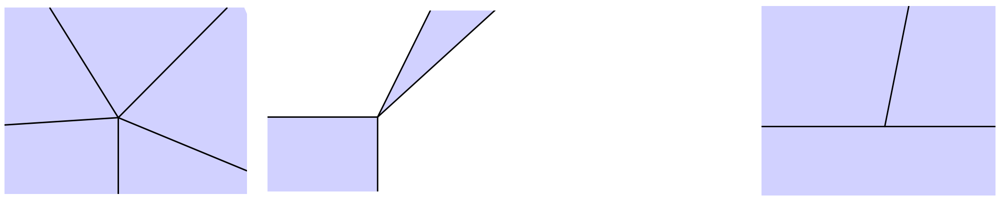
A *polyhedron* is the intersection of finitely many half spaces. An equivalent definition of a polytope is that it is a bounded polyhedron.

A *face* of a polyhedron C is determined by a linear functional $w \in \mathbb{R}^n$, by selecting the points in C where the linear functional is minimized:

$$\operatorname{face}_w(C) = \{x \in C \mid w \cdot x \leq w \cdot y \text{ for all } y \in C\}.$$

A face which is not contained in any larger proper face is called a *facet*.

Definition 1.2.2. A *polyhedral fan* is a collection $\mathcal{F}$ of polyhedral cones such that every face of a cone is in the fan, and the intersection of any two cones in the fan is a face of each. For some examples and nonexamples, see Figure 1.5.



(a) Examples of fans.

(b) Non-example of a fan.

Figure 1.5: Examples and non-examples of polyhedral fans.

Definition 1.2.3. A *polyhedral complex* is a collection Σ of polyhedra such that if $P \in \Sigma$ then every face of P is also in Σ, and if P and Q are polyhedra in Σ then their intersection is either empty or also a face of both P and Q. See Figure 1.6 for an example. The *support* $|\Sigma|$ of Σ is the union of all of the faces of Σ.

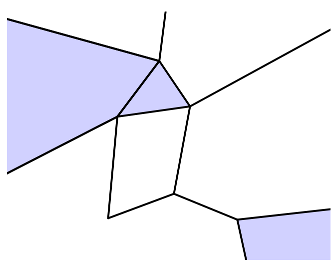


Figure 1.6: An example of a polyhedral complex.

Definition 1.2.4. Let $v_1, \dots, v_r$ be an ordered list of points in $\mathbb{R}^n$. We fix a *weight vector* $w = (w_1, \dots, w_r)$ in $\mathbb{R}^r$ assigning a weight to each point. Consider the polytope in $\mathbb{R}^{n+1}$ defined by $P = \text{conv}((v_1, w_1), \dots, (v_n, w_n))$. The *regular subdivision* of $v_1, \dots, v_r$ is the polyhedral complex on the points $v_1, \dots, v_r$ whose faces are the faces of P which are "visible from beneath the polytope". More precisely, the faces σ are the sets for which there exists $c \in \mathbb{R}^n$ with $c \cdot v_i = w_i$ for $i \in \sigma$ and $c \cdot v_i < w_i$ for $i \notin \sigma$.

Example 1.2.5. Let $\{(0,0), (1,0), (0,1), (1,1)\} \in \mathbb{R}^2$ and consider the weight vector $w = (0,1,1,0)$. Then the subdivision induced by w is given by the lower faces of the tetrahedron $\text{conv}((0,0,0), (1,0,1), (0,1,1), (1,1,0))$ which is pictured in Figure 1.7. The subdivision is pictured in Figure 1.8.

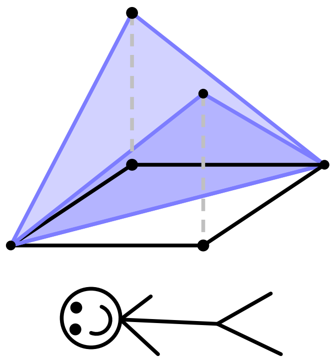


Figure 1.7: The square lifted by the weight vector w, viewed from below by a tropical geometer.

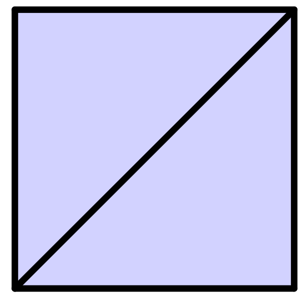


Figure 1.8: The square with subdivision induced by w.

Embedded Tropicalization

We will now see how to associate to an embedded variety its embedded tropicalization. This will be a very structured polyhedral complex. We may view embedded tropical varieties as geometry over the tropical semiring. In this section we follow [88, Chapter 3].

Definition 1.2.6. The *tropical semiring* $\mathbb{T}$ is the semiring $(\mathbb{R}, \oplus, \odot)$, where for all a and b in $\mathbb{R}$, we have the operations

$$a \oplus b := \min(a, b),$$
$$a \odot b := a + b.$$

Let K be a field with a valuation val_K which may be the trivial valuation. Consider the ring $K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ of Laurent polynomials over K.

Definition 1.2.7. Given a Laurent polynomial

$$f = \sum_{u \in \mathbb{Z}^n} c_u x^u,$$

we define its *tropicalization* $\mathrm{trop}(f) : \mathbb{T}^n \to \mathbb{T}$ to be the function that is obtained from f by replacing each c_u by its valuation and performing all additions and multiplications in the semiring $\mathbb{T}$. That is,

$$\mathrm{trop}(f)(w) = \min_{u \in \mathbb{Z}^n} \left(\mathrm{val}(c_u) + u \cdot w\right).$$

Classically, the variety of the Laurent polynomial f is a hypersurface in the algebraic torus $(K^*)^n$. In the same way, there will be a tropical hypersurface associated to $\mathrm{trop}(f)$.

Definition 1.2.8. The *tropical hypersurface* $\mathrm{trop}(V(f))$ is the set

$$\{w \in \mathbb{R}^n \mid \text{the minimum in } \mathrm{trop}(f)(w) \text{ is achieved at least twice}\}.$$

Example 1.2.9. We compute the *tropical line.* Let $f = x + y + 1 \in \mathbb{C}\{\{t\}\}[x^{\pm 1}, y^{\pm 1}]$. Then,

$$\begin{aligned} \operatorname{trop}(f)(x,y) &= \min\left(0 + (1,0)\cdot(x,y), 0 + (0,1)\cdot(x,y), 0\right), \\ &= \min(x,y,0). \end{aligned}$$

We must find for which x and y is this minimum achieved twice. There are 3 cases.

1. x and 0 are the winners: This happens when $x = 0$ and $y \geq 0$. This is the ray $\operatorname{pos}(e_2)$.
2. y and 0 are the winners: This happens when $y = 0$ and $x \geq 0$. This is the ray $\operatorname{pos}(e_1)$.
3. x and y are the winners: This gives the ray $\operatorname{pos}(-1,-1)$.

So, the tropical variety $\operatorname{trop}(V(f))$ is as pictured in Figure 1.9.

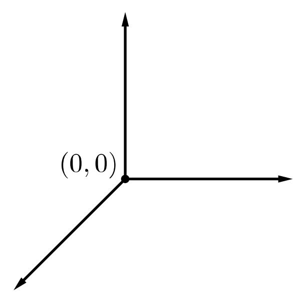


Figure 1.9: The tropical line from Example 1.2.9.

Definition 1.2.10. Let I be an ideal in $K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, and let $X = V(I)$ be its variety in the algebraic torus T^n. Then we define the *tropicalization* of X, denoted $\operatorname{trop}(X)$, to be:

$$\bigcap_{f \in I} \operatorname{trop}(V(f)).$$

A *tropical variety* is any subset of $\mathbb{R}^n$ of the form $\operatorname{trop}(X)$, where X is a subvariety of T^n.

In general, we cannot reduce the intersection to be over just any generating set of an ideal I. Any finite intersection of tropical hypersurfaces is known as a *tropical prevariety.* A *tropical basis* is a generating set for an ideal which works well with respect to tropicalization.

Definition 1.2.11. If T is a finite generating set of an ideal I in $K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, then it is a *tropical basis* if

$$\operatorname{trop}(V(I)) = \bigcap_{f \in T} \operatorname{trop}(V(f)).$$

So, the tropical prevariety defined by elements of T equals the tropical variety defined by I.

Theorem 1.2.12 ([88, Theorem 2.6.6])**.** *Let K be a valued field. Every ideal in $K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ has a finite tropical basis.*

The Fundamental Theorem of Tropical Geometry (See [88, Theorem 3.2.3] for the complete version) shows that tropical varieties can be obtained by tropicalizing classical varieties.

Theorem 1.2.13 (Fundamental Theorem of Tropical Geometry, see [88, Theorem 3.2.3]). *Let K be an algebraically closed field with a nontrivial valuation. Fix an ideal I in $K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. The following sets are the same:*

1. *the tropical variety* $\mathrm{trop}(V(I))$ *in* $\mathbb{R}^n$*,*
2. *the closure in* $\mathbb{R}^n$ *of* $\{(\mathrm{val}_K(y_1), \dots, \mathrm{val}_K(y_n)) \mid (y_1, \dots, y_n) \in V(I)\}$.

Furthermore, if $V(I)$ is irreducible and w is any point in $\Gamma_{\mathrm{val}}^n \cap \mathrm{trop}(V(I))$, then the set $\{y \in V(I) \mid \mathrm{val}(y) = w\}$ is Zariski dense in $V(I)$.

Definition 1.2.14. Let Σ be a pure d-dimensional polyhedral complex in $\mathbb{R}^n$. Then Σ is *connected through codimension 1* if, for any two d dimensional cells, there is a chain of d dimensional cells which pairwise share facets.

We now present the Structure Theorem, which gives some necessary properties for a polyhedral complex to be a tropical variety. We will not discuss the balancing condition at length here, but refer the reader to [88, Definition 3.3.1] for more details.

Theorem 1.2.15. *[Structure Theorem for Tropical Varieties] Let X be an irreducible d-dimensional subvariety of T^n. Then* $\mathrm{trop}(X)$ *is the support of a balanced, weighted,* Γ_{val}*-rational polyhedral complex pure of dimension d. Moreover, that polyhedral complex is connected through codimension 1.*

We now turn our attention to the special case of tropical hypersurfaces. In this setting, there is a nice combinatorial method for finding tropicalizations.

Definition 1.2.16. The *Newton polytope* $\mathrm{Newt}(f)$ of a Laurent polynomial $f = \sum_u c_u x^u$ in $K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ is

$$\mathrm{conv}(u \in \mathbb{Z}^n \mid c_u \neq 0).$$

Now, we are prepared to describe a way to relate a tropical hypersurface $\mathrm{trop}(V(f))$ to a regular subdivision of the Newton polytope of f.

Proposition 1.2.17 ([88, Proposition 3.1.6]). *Let $f \in K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ be a Laurent polynomial. The tropical hypersurface* $\mathrm{trop}(V(f))$ *is the support of a pure* $\Gamma_{\mathrm{val}}-$*rational polyhedral complex of dimension $n-1$ in $\mathbb{R}^n$. It is the $(n-1)$-skeleton of the polyhedral complex dual to the regular subdivision of the Newton polytope of f induced by the weights* $\mathrm{val}(c_u)$ *on the lattice points in* $\mathrm{Newt}(f)$*.*

Proposition 1.2.18 ([88, Proposition 3.3.2]). *For a tropical polynomial F in n variables, the hypersurface $V(F)$ is balanced for the weights coming from the lattice lengths of the edges in the corresponding regular subdivision of* $\mathrm{Newt}(F)$*.*

Example 1.2.19. We now compute trop($V(f)$) where $f = 7xy + 5x + 14y + 49$ with the 7-adic valuation. The corresponding subdivision of the Newton polytope and tropical curve are pictured in Figures 1.10 and 1.11.

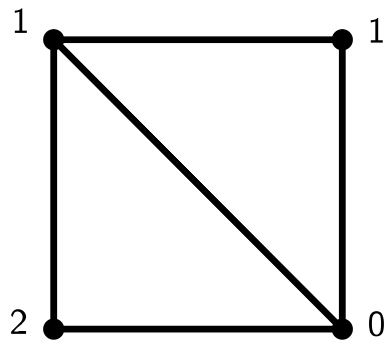


Figure 1.10: The subdivision in Example 1.2.19

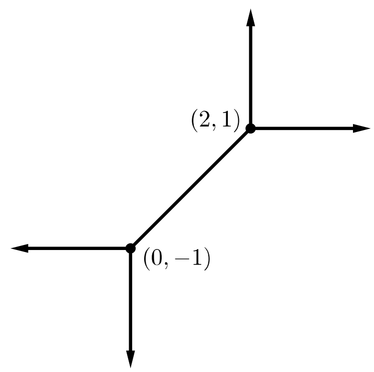


Figure 1.11: The tropical curve in Example 1.2.19

Metric Graphs

Now we develop a different type of tropicalization which can be associated to a curve. This tropicalization is intrinsic to the curve, and does not depend on the embedding. We focus on abstract tropicalizations of curves as opposed to higher dimensional varieties. In Section 1.3 we will see that this relates to something called the *Berkovich skeleton*, which can be used to define the abstract tropicalization more generally. In this section, we will see how strata in the special fiber of a semistable model correspond to cells in a polyhedral complex. For curves, this polyhedral complex will be a metric graph, and the strata consist of components and nodes.

Definition 1.2.20. A *metric graph* is a metric space Γ, together with a graph G and a length function $l : E(G) \to \mathbb{R}_{>0} \cup \{\infty\}$ such that Γ is obtained by gluing intervals e of length $l(e)$, or by gluing rays to their endpoints, according to how they are connected in G. In this case, the pair (G, l) is called a *model* for Γ. An abstract tropical curve is a metric graph Γ together with a weight function on its points $w : \Gamma \to \mathbb{Z}_{\geq 0}$, such that $\sum_{v \in \Gamma} w(v)$ is finite.

Edges of infinite length (meeting the graph in one endpoint) are called *infinite leaves*. A *bridge* is an edge whose deletion increases the number of connected components.

The *genus* of a tropical curve (Γ, w) with model G is

$$\sum_{v \in \Gamma} w(v) + |E(G)| - |V(G)| + 1. \tag{1.1}$$

We say that two tropical curves of genus greater than or equal to two are *isomorphic* if one can be obtained from the other via graph automorphisms, or by removing infinite leaves or leaf vertices v with $w(v) = 0$, together with the edge connected to it. In this way, every tropical curve has a *minimal skeleton*.

A model is *loopless* if there is no vertex with a loop edge. The *canonical loopless model* of Γ, with genus of $\Gamma \geq 2$, is the graph G with vertices

$$V(G) := \{x \in \Gamma \mid \operatorname{val}(x) \neq 2 \text{ or } w(x) > 0 \text{ or } x \text{ is the midpoint of a loop}\}. \tag{1.2}$$

If (G, l) and (G', l') are loopless models for metric graphs Γ and Γ', then a *morphism of loopless models* $\phi : (G, l) \to (G', l')$ is a map of sets $V(G) \cup E(G) \to V(G') \cup E(G')$ with:

- All vertices of G map to vertices of G'.
- If $e \in E(G)$ maps to $v \in V(G')$, then the endpoints of e must also map to v.
- If $e \in E(G)$ maps to $e' \in E(G')$, then the endpoints of e must map to vertices of e'.
- Infinite leaves in G map to infinite leaves in G'.
- If $\phi(e) = e'$, then $l'(e')/l(e)$ is an integer. These integers must be specified if the edges are infinite leaves.

We call an edge $e \in E(G)$ *vertical* if ϕ maps e to a vertex of G'. We say that ϕ is *harmonic* if for every $v \in V(G)$, the *local degree*

$$d_v = \sum_{\substack{e \ni v, \\ \phi(e) = e'}} l'(e')/l(e) \tag{1.3}$$

is the same for all choices of $e' \in E(G')$. If it is positive, then ϕ is *nondegenerate*. The *degree* of a harmonic morphism is defined as

$$\sum_{\substack{e \in E(G), \\ \phi(e) = e'}} l'(e')/l(e). \tag{1.4}$$

We also say that ϕ satisfies the *local Riemann-Hurwitz condition* if:

$$2 - 2w(v) = d_v(2 - 2w'(\phi(v))) - \sum_{e \ni v} \left(l'(\phi(e))/l(e) - 1\right). \tag{1.5}$$

This is a reflection of the classical Riemann-Hurwitz condition. If ϕ satisfies this condition at every vertex v in the canonical loopless model of Γ, then ϕ is an *admissible cover* [38].

Example 1.2.21. In Figure 1.12, we have a tree T and a metric graph Γ which admits a harmonic morphism ϕ to T. All edges depicted in the image have the same length as the corresponding edges in the tree, except for the bridge, which has length equal to half the length of the corresponding edge in the tree.

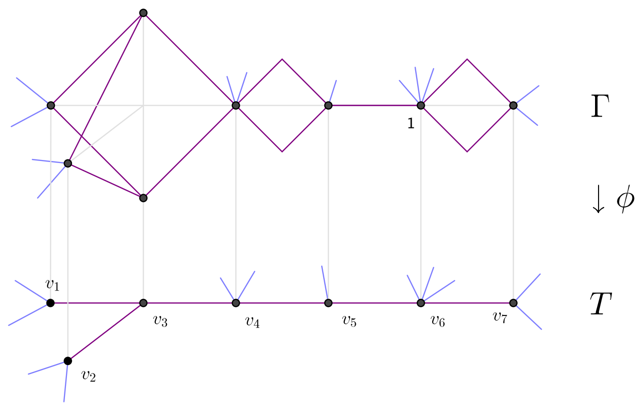


Figure 1.12: The tree T with 12 infinite leaves from Example 1.2.21 and the hyperelliptic tropical curve Γ of genus 5 which admissibly covers T by ϕ.

Abstract Tropicalization

Now, our goal is to define an object called the *abstract tropicalization* which can be associated to every curve, and does not depend on the embedding of the curve.

Let K be an algebraically closed valued field with valuation v and valuation ring R. Let X be a reduced, nodal curve over K, and let $\mathcal{X}$ be a semistable model for X.

Definition 1.2.22. Let $C_1, \ldots, C_n$ be the irreducible components of $\mathcal{X}_k$, the special fiber of $\mathcal{X}$. The *dual graph* G of $\mathcal{X}_k$ is defined with vertices v_i corresponding to the components C_i. There is an edge e_{ij} between v_i and v_j if the corresponding components C_i and C_j intersect in a node q. Then, the completion of the local ring $\mathcal{O}_{\mathcal{X},q}$ is isomorphic to $R[[x,y]]/\langle xy - f\rangle$, where R is the valuation ring of K, and $f \in m$, the maximal ideal of R. Then, we define $l(e_{ij}) = v(f)$. This gives a model (G, l) for a metric graph Γ. Setting $w(v_i) = g(C_i)$, we call the tropical curve (Γ, w) the *abstract tropicalization* of X.

Figure 1.13 gives a schematic of a special fiber. Figure 1.14 gives its dual graph.

Remark 1.2.23. In genus $g \geq 2$, any two semistable reductions are dominated by a third. So, there is a unique *stable model* obtained by contracting the smooth rational components meeting the rest of the curve in fewer than three points [68]. Hence, two different semistable models will give the same tropical curve up to isomorphism of tropical curves.

Example 1.2.24. Consider the curve E over $\mathbb{C}\{\{t\}\}$ defined by

$$y^2 = x^3 + x^2 + t^4.$$

This is a smooth elliptic curve for $t \neq 0$. However, when $t = 0$, we have the curve defined by the equation $y^2 = (x+1)x^2$. This curve has self-intersection. Extending this to $\mathbb{P}^2$ gives a

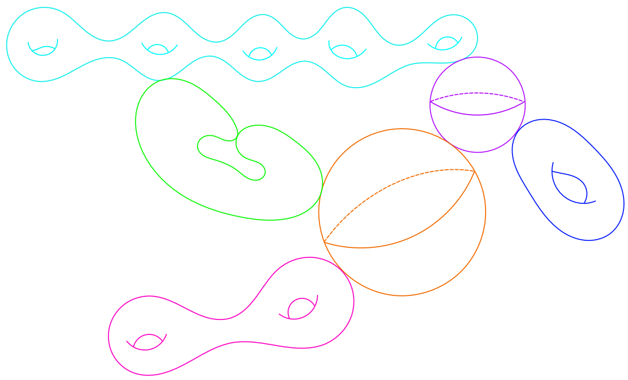


Figure 1.13: A special fiber

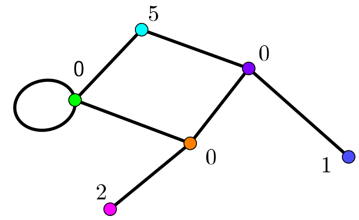


Figure 1.14: A dual graph

semistable model. The formal completion at $(0,0)$ is $R[u,v]/\langle uv - t^4 \rangle$, where $x' = x\sqrt{x+1}$, $u = y - x'$, and $v = y + x'$. Therefore, the abstract tropicalization is a cycle of length 4.

While a semistable model is guaranteed in the abstract, it is hard to compute. !n Chapter 4 we discuss the known methods for computing abstract tropicalizations of curves.

1.3 Non-Archimedean Curves

Classical complex algebraic geometry studies varieties over the complex numbers. In this setting, every algebraic set in $\mathbb{C}^n$ decomposes as a finite union of complex manifolds. Many fundamental results in complex algebraic geometry are proved by viewing varieties as manifolds and applying holomorphic functions, differential forms, Hodge theory, and Morse theory.

Now, algebraic geometers are interested in varieties over other fields, such as valued fields. As with the complex numbers, these fields come with norms. However, these fields are *non-Archimedean*, and this causes them to have a strange topology. As such, defining manifolds over these fields in a straightforward way does not yield a useful theory. This can be fixed through *Berkovich spaces*. Throughout this section, we follow the references [10, 106, 102].

Non-Archimedean Fields

First, we give some background on non-Archimedean fields. An *Archimedean field* K is one satisfying the *Archimedean axiom* that for any $x \in K^*$, there is a positive integer n such that $|nx| > 1$. While this axiom may feel natural and familiar, the real and complex numbers are essentially the only fields satisfying this axiom. To be precise, they are the only complete Archimedean fields. We now begin by introducing non-Archimedean fields, of which there are many more examples.

Definition 1.3.1. Let K be a field. A map $|\cdot| : K \to \mathbb{R}_{\geq 0}$ is a *seminorm* if

1. $|0| = 0$,
2. $|1| = 1$,
3. $|f + g| \leq |f| + |g|$.

It is called a *norm* if $|f| = 0$ if and only if $f = 0$. It is called *multiplicative* if $|f \cdot g| = |f| \cdot |g|$ for all $f, g \in K$. Lastly, it is called *non-Archimedean* if the *ultrametric inequality* holds:

$$|f + g| \leq \max\{|f|, |g|\}.$$

Remark 1.3.2. By setting $\mathrm{val}_K(\cdot) = -\log(|\cdot|)$ we obtain a valued field. Conversely, given a valued field, we can obtain a non-Archimedean norm by defining $|\cdot| = \epsilon^{\mathrm{val}_K(\cdot)}$ for $\epsilon \in (0, 1)$.

Definition 1.3.3. A field K with a norm $|\cdot|_K$ is a *non-Archimedean* field if $|\cdot|_K$ is a multiplicative non-Archimedean norm.

Example 1.3.4. Let K be any field. Then the *trivial norm*, defined as $|a|_0 = 1$ for all $a \neq 0$ and $|0|_0 = 0$ makes K a non-Archimedean field.

Example 1.3.5. Let F be any field. Then the field of formal Laurent series $K = F((t))$ is a non-Archimedean field with the norm $|f|_K = \epsilon^{\mathrm{ord}_t(f)}$ for $\epsilon \in (0, 1)$. Here, $\mathrm{ord}_t(f)$ is the smallest exponent of t with non-zero coefficient in the Laurent series expansion.

Example 1.3.6. Let p be a prime number. Consider $\mathbb{Q}_p$ which is the completion of $\mathbb{Q}$ with respect to $|\cdot|_p$. Then $|a/b|_p = \epsilon^{\mathrm{ord}_p(a) - \mathrm{ord}_p(b)}$ for $\epsilon \in (0, 1)$ makes $\mathbb{Q}_p$ a non-Archimedean field. Here, $\mathrm{ord}_p(a)$ is the largest n such that p^n divides a.

Example 1.3.7. If K is a non-Archimedean field with norm $|\cdot|$, then $|\cdot|$ extends uniquely to a norm on any algebraic extension K' of K. The field K' is complete if it is a finite extension of K and K is complete. If K' is an algebraic closure of K, then its completion $\hat{K'}$ is still algebraically closed [23, Proposition 3.4.1.3].

In the theory of manifolds over the complex numbers, we have a process of taking a smooth complex algebraic variety X and associating to it an analytic space (manifold) X^{an}. If X is proper, then no information is lost in analytification: two smooth and proper varieties X and Y are isomorphic if and only if X^{an} and Y^{an} are isomorphic as complex manifolds (Serre's GAGA principle). Additionally, we can apply the tools of complex analysis, differential geometry and algebraic topology to X^{an}.

We would like to be able to do something similar over here, namely, to make manifolds over non-Archimedean fields. If we naively proceed with what was done in the complex case, many of the nice properties such as the GAGA principle fail. In part, this is due to the bizarre nature of the metric topology on a non-Archimedean field, which we now discuss.

Let K be a non-Archimedean field with the topology induced by the metric from $|\cdot|_K$.

Definition 1.3.8. Let $r \in \mathbb{R} > 0$. An *open ball* in K is a subset of the form

$$B^-(a,r) = \{x \in K \mid |x-a| < r\}.$$

A *closed ball* in K is a subset of the form

$$B^+(a,r) = \{x \in K \mid |x-a| \leq r\}.$$

In each case, a is called the center of the ball and r is the radius.

Proposition 1.3.9. *Let $b \in B^\pm(a,r)$ for $a \in K$ and $r \in \mathbb{R}_{>0}$.*

1. *Then $B^\pm(b,r) = B^\pm(a,r)$. In other words, any point of a ball is the center of the ball.*
2. *If two balls in K intersect, then one is included in the other.*
3. *Every ball in K is both open and closed in the metric topology.*
4. *The only connected subsets of K are the singletons and the empty set.*

The result of Proposition 1.3.9 is that if we try to make K-analytic manifolds, we will run into problems because they can be broken up into arbitrarily small disjoint open pieces with no global geometric structure. For instance, let f and g be any two polynomials. Then the piecewise function

$$h(x) = \begin{cases} f(x) & \text{if } x \in B(0,1) \\ g(x) & \text{otherwise} \end{cases}$$

is continuous, and analytic in the sense that it is given by a convergent power series in a neighborhood of every point. Ultimately, creating K-analytic manifolds in this way leads to having too few isomorphism classes, and so the GAGA principle fails. The following theorem gives an example of this.

Theorem 1.3.10 (Serre)**.** *Let K be a finite extension of $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$ whose residue field has order q. If X is a compact n-dimensional "naive" non-Archimedean analytic variety then X is isomorphic to a disjoint union of S copies of $B^+(0,1)$ for some unique $S \in \{1,\ldots,q-1\}$.*

In order to overcome this, we will need to put a lot more work into defining analytic spaces over non-Archimedean fields.

Berkovich Spaces

The field of rational numbers $\mathbb{Q}$ is totally disconnected in its metric topology. In passing to $\mathbb{R}$, we are adding in new points that fill in the gaps in $\mathbb{Q}$. Around 1990 Vladimir Berkovich formulated the idea that there were too many points missing from non-Archimedean manifolds for the naive definition to work. Adding in the missing points produces a path-connected, locally compact Hausdorff space that contains all of the original points. If the field has a nontrivial norm and is algebraically closed, the original points are dense in the whole space. Throughout this subsection, we follow the references [10, 102, 106].

We begin by defining Berkovich spaces on affine varieties. Let K be a non-Archimedean field, and let $f_1, \ldots, f_r \in K[x_1, \ldots, x_n]$. Let $X = V(f_1, \ldots, f_r)$ be the variety defined by $f_1, \ldots, f_r$. If L is an extension field of K, then we write $X(L)$ to denote the points $y \in L^n$ such that $f_i(y) = 0$ for $1 \leq i \leq r$. Let $x \in X(K)$. Then the *seminorm associated to* x is a multiplicative seminorm

$$|\cdot|_x : K[X] = K[x_1, \ldots, x_m]/\langle f_1, \ldots, f_r\rangle \to \mathbb{R}_{\geq 0}$$

given by

$$|f|_x = |f(x)|_K,$$

where the norm on the right hand side is the norm $|\cdot|_K$ on K. There are other multiplicative seminorms on $K[X]$. We will only consider multiplicative seminorms with the property that the restriction of the seminorm to K is $|\cdot|_K$. Since the norm $|\cdot|_K$ on K is non-Archimedean, any seminorm on $K[X]$ extending $|\cdot|_K$ will also satisfy the ultrametric inequality.

Definition 1.3.11 (Berkovich space, affine case)**.** The *Berkovich analytification* X^{an} is the space of all multiplicative seminorms on $K[X]$ that extend the given norm on K. For a point $x \in X^{an}$, we denote the corresponding norm by $|\cdot|_x$. We endow X^{an} with the coarsest topology such that for each $f \in K[x]$, and $x \in X^{an}$, we have that the function on X^{an} given by $x \mapsto |f|_x$ is continuous.

If $K[X]$ is nonzero, then X^{an} is nonempty. We see this as follows. The system of polynomials $f_1, \ldots, f_r$ will have a common zero over the algebraic closure $\overline{K}$. Since K is complete, its norm extends uniquely to $\overline{K}$. Then, a solution x with coordinates in $\overline{K}$ determines a point of X^{an} given by $|f|_x = |f(x)|_{\overline{K}}$. This observation is useful more generally. If L is an extension field of K with a norm that extends the given one on K, then any solution to $f_1, \ldots, f_r$ with coordinates in L determines a point on X^{an}.

Remark 1.3.12. The topology on X^{an} is the subspace topology for the natural inclusion $X^{an} \subset (\mathbb{R}_{\geq 0}^{K[X]})$, where the topology on $\mathbb{R}_{\geq 0}^{K[X]}$ is the product topology of the topology on $\mathbb{R}$.

We now see our first example of a Berkovich space.

Example 1.3.13 (Berkovich affine line, trivial valuation)**.** Consider the affine line $\mathbb{A}^1_K = \operatorname{Spec}(K[y])$ in the case where the norm on K is trivial and K is algebraically closed. Then any seminorm $|\cdot|_x \in (\mathbb{A}^1_K)^{an}$ is determined by the value of $|y-a|_x$ for all $a \in K$. Let us analyze the possibilities.

1. If for $f \neq 0$ we have $|f|_x = 1$, then $|\cdot|_x = |\cdot|_0$ is trivial. This gives a point in $(\mathbb{A}^1_K)^{an}$.

2. If $|y|_x = r$ for some $r > 1$, then for all $a \in K$, we have

$$|y-a| = \max\{|y|_x, |a|_x\} = \max\{r, 1\} = r.$$

This gives ray of norms in $(\mathbb{A}^1_K)^{an}$ parameterized by the number r. The ray is connected to the point corresponding to the trivial norm at the point corresponding to $r = 0$.

3. If there exists a such that $|y-a|_x = r$ for some $r \in [0,1]$, then for all $b \neq a$, we have

$$|y-b|_x = \max\{|y-a|_x, |a-b|\} = \max\{r, 1\} = 1.$$

This gives, for each $a \in K$, an interval of norms in $(\mathbb{A}^1_K)^{an}$ parametrized by r. These intervals are connected to the point corresponding to the trivial norm at the points corresponding to $r = 0$.

So, the structure of $(\mathbb{A}^1_K)^{an}$ is a tree with an infinite stem, and infinite branches each corresponding to elements of K (See Figure 1.15).

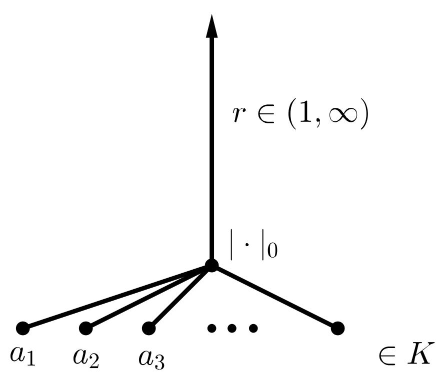


Figure 1.15: The space $(\mathbb{A}^1_K)^{an}$ when K is algebraically closed and trivially valued.

Example 1.3.14 (Berkovich affine line, [11])**.** Now suppose K is algebraically closed and that it has a non-trivial, non-Archimedean valuation. We have the types of points described in Theorem 1.3.15 in the Berkovich affine line.

Theorem 1.3.15 (Berkovich's Classification Theorem [19])**.** *Every point in $(\mathbb{A}^1_K)^{an}$ corresponds to one of the four following types.*

Type I: *For $a \in K$ and $f \in K[y]$, we have norms $|f|_a = |f(a)|_K$.*
Type II: *For $a \in K$ and $r \in |K^*|$ we have norms $|f|_{B(a,r)} = \sup_{z \in B(a,r)} |f(z)|$.*
Type III: *For $a \in K$ and $r \notin |K^*|$ we have norms $|f|_{B(a,r)} = \sup_{z \in B(a,r)} |f(z)|$.*
Type IV: *If $B(a_n, r_n)$ is a family of nested balls with empty interior, then we have norms $|f|_x = \inf_n |f|_{B(a_n,r_n)}$.*

These different types of points fit in to a tree as pictured in Figure 1.16. Let $x, x' \in (\mathbb{A}^1_K)^{an}$ be distinct type I or II points corresponding to the disks $B(a, r)$ and $B(a', r')$. There is a unique path in $(\mathbb{A}^1_K)^{an}$ between x, x' which can be described as follows. If $B(a, r) \subset$

$B(a',r')$, then this path corresponds to all points of $(\mathbb{A}^1_K)^{an}$ corresponding to disks B such that $B(a,r) \subset B \subset B(a',r')$. These are totally ordered by containment. If $a = a'$, then this is the disks $\{B(a,t) \mid r \leq t \leq r'\}$, which is homeomorphic to an interval. On the other hand, if $B(a,r)$ and $B(a',r')$ are disjoint, the unique path between x and x' consists of all points in $(\mathbb{A}^1_K)^{an}$ corresponding to disks $B(a,t)$ with $r \leq t \leq |a-a'|$ or $B(a',t')$ with $r' \leq t' \leq |a-a'|$.

We can visualize this path for any distinct points $a, a' \in K$ as follows. We start increasing the radius of the degenerate disk $B(a,0)$ until it contains a'. This happens at the point corresponding to $B(a,|a-a'|) = B(a',|a-a'|)$. Then, we decrease the radius to the point $B(a',0)$. So, we have added more points to fill the gaps in the totally disconnected space K.

Before defining Berkovich spaces more generally, we see how to relate X^{an} to $\mathrm{Spec}(K[X])$. Recall that $\mathrm{Spec}(K[X])$ is the set of prime ideals of $K[X]$ with the Zariski topology. Let x be a point in X^{an}. The set of functions $f \in K[X]$ such that $|f|_x = 0$ is a prime ideal p. Let κ_p be the *residue field*, meaning it is the fraction field of $K[X]/p$. The seminorm $|\cdot|_x$ factors through a norm on κ_p that restricts to the given norm on K.

For any extension field L over K, let $V_{\mathbb{R}}(L)$ be the space of all norms on L that extend the given norm on K. The map taking a point in X^{an} to the kernel of the corresponding seminorm gives a surjection $X^{an} \to \mathrm{Spec}(K[X])$. The fiber over a point p is $V_{\mathbb{R}}(\kappa_p)$. So,

$$X^{an} = \bigsqcup_{p \in X} V_{\mathbb{R}}(\kappa_p).$$

This decomposition motivates the following definition for the non-affine case.

Definition 1.3.16. Let X be an algebraic variety over a non-Archimedean field K. Then we define the *Berkovich analytification* of X to be the set of multiplicative seminorms on residue fields of points of X. That is,

$$X^{an} = \left\{ x = (\xi_x, |\cdot|_x) \;\middle|\; \begin{array}{c} \xi_x \in \mathrm{Spec}(K[X]),\ \text{with residue field } \kappa(\xi_x), \\ |\cdot|_x : \kappa(\xi_x) \to \mathbb{R}_{\geq 0} \text{ is a seminorm extending } |\cdot|_K \end{array} \right\}.$$

It has the coarsest topology such that the following two maps are continuous:

1. The map $X^{an} \to X$ sending $x \mapsto \xi_x$;
2. For all open $U \subset X$, and for all $f \in \mathcal{O}_X(U)$, the map $x \mapsto |f(\xi_x)|_x$ is continuous.

It is essential that we allow ξ_x to be a non-closed point of the scheme X. This is because if ξ is a closed point of X then $\kappa(\xi)$ is a finite extension of k and there is a unique extension of the absolute value on K to $\kappa(\xi)$. So, there is only one point x in X^{an} with $\xi_x = \xi$, and we obtain nothing new. When ξ is not closed, $\kappa(\xi)$ is a transcendental extension of K and there will be infinitely many extensions of the norm on K to $\kappa(\xi)$.

Remark 1.3.17. We may instead wish to view X^{an} in terms of points of X defined over normed extensions L of K. A normed extension L of K is a field L together with a norm

$|\cdot|: L \to \mathbb{R}_{\geq 0}$ that extends the given norm on K. Consider triples $(L, |\cdot|, x)$ where L is an extension of K, $|\cdot|$ is a norm extending the given norm on K, and x is a point of X over L. We will say two such triples $(L, |\cdot|, x)$ and $(L', |\cdot|', x')$ are equivalent if there is an embedding $L \subset L'$ such that the restriction of $|\cdot|'$ is $|\cdot|$ and x is identified with x' by the induced inclusion $X(L) \subset X(L')$. Then,

$$X^{an} = \{(L, |\cdot|, x)\}/\sim .$$

Example 1.3.18 (Berkovich projective line [11])**.** We obtain the Berkovich projective line $(\mathbb{P}^1_K)^{an}$ from $(\mathbb{A}^1_K)^{an}$ by adding a type I point at infinity, denoted ∞. The topology of $(\mathbb{P}^1_K)^{an}$ is the one-point compactification of $(\mathbb{A}^1_K)^{an}$.

We can place a partial order on $(\mathbb{A}^1_K)^{an}$ by saying that $x \leq x'$ if and only if $|f|_x \leq |f|_{x'}$ for all $f \in K[y]$. In terms of disks, if x, x' are points of type I, II, or III, then $x \leq x'$ if and only if the disk corresponding to x is contained in the disk corresponding to x'. There is a unique least upper bound $x \vee x'$ corresponding to the smallest disk containing both the disks corresponding to x and x'. We extend this partial order to $(\mathbb{P}^1_K)^{an}$ by declaring that $x < \infty$ for all $x \in (\mathbb{A}^1_K)^{an}$. Then, we write

$$[x, x'] = \{z \in (\mathbb{P}^1_K)^{an} \mid x \leq z \leq x'\} \cup \{z \in (\mathbb{P}^1_K)^{an} \mid x' \leq z \leq x\}.$$

So that the unique path between $x, y \in (\mathbb{P}^1_K)^{an}$ is given by

$$l_{x,y} = [x, x \vee y] \cup [x \vee y, y].$$

We now describe the space $(\mathbb{P}^1_K)^{an}$. Starting at a type II point corresponding to $B(a, r)$, we can navigate the space as follows. The branches emanating from this point are in one-to-one correspondence with elements of $\mathbb{P}^1_k$. There is no branching at points of type III, and these branches terminate at points of type I and IV. See Figure 1.16.

The following proposition tells us that X^{an} has reasonable topological properties and that it retains the basic geometric properties of X.

Theorem 1.3.19 ([19])**.** *Let X be a K-scheme of finite type. The topological space X^{an} is locally compact and locally path connected. Moreover, we have that X is connected if and only if X^{an} is connected, X is separated if and only if X^{an} is Hausdorff, and X is proper if and only if X^{an} is compact. The induced topology on the subset $X(K)$ of points with coordinates in K is the metric topology. If K is algebraically closed with nontrivial valuation, then this subset is dense in X^{an}.*

Analytifying Curves

We now focus on Berkovich analytifications of curves, following [15]. Let K be a complete, algebraically closed non-Archimedean field with non-trivial valuation and let X be a smooth, proper, and geometrically integral algebraic curve over K.

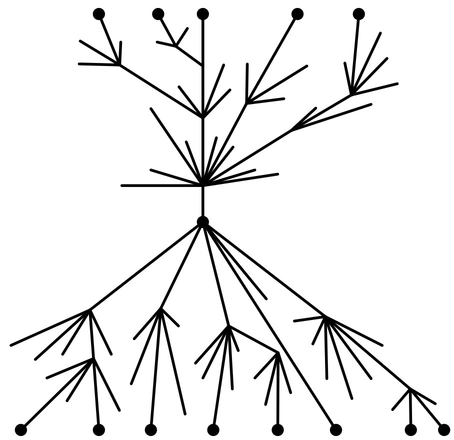


Figure 1.16: The Berkovich projective line when K is non-trivially valued.

If the genus of X is at least 1, there is a canonically defined subset $\Sigma \subset X^{an}$ called the *skeleton* of X^{an}, which is homeomorphic to a finite metric graph. The space X^{an} admits a deformation retraction onto Σ. The fiber of each point in Σ of this retraction is a tree. Then, X^{an} may have non-trivial global topology. See Figure 1.17.

First, we will describe the skeleton starting with a fixed model of the curve. Later, we will define it using semistable vertex sets of X^{an}. Let $R = \{f \in K \mid |f| \leq 1\}$ denote the valuation ring of K, and let k be its residue field. Let $\mathcal{X}$ be a semistable model for X. Let $Z = \mathcal{X}_0$ denote its special fiber, and let $\pi : X(K) \to Z(k)$ denote the reduction map. Suppose $Z = \cup_{i=1}^{t} Z_i$, where the Z_i are smooth and irreducible projective curves, each containing at least two singular points of Z. Let Z_i^* be the nonsingular affine curve obtained from Z_i by removing all singular points of Z from Z_i. Denote by $X_i^* = \pi^{-1}(Z_i^*)$. This is a *rigid analytic space*. For each ordinary double point $p \in Z(k)$ on components Z_i and Z_j, we have $X_p = \pi^{-1}(Z_i^* \cup Z_j^* \cup \{p\})$, which is also a rigid analytic space. As a rigid space, then X is obtained by gluing the X_p along the subsets X_i^*.

Then, X^{an} can be described by gluing $(X_p)^{an}$ along the subsets $(X_i^*)^{an}$. The reduction map $\pi : X(K) \to Z(k)$ then extends to a reduction map $\mathrm{red} : X^{an} \to Z$ as follows. Recall that points of X^{an} correspond to equivalence classes of points x of $X(L)$ for normed extensions L of K. This gives a map $x : \mathrm{Spec}(L) \to K$, which extends in a unique way by the valuative criterion of properness to a map $\mathrm{Spec}(\mathcal{O}_L) \to \mathcal{X}$ from the valuation ring of L. Then, the reduction map $\mathrm{red}(x)$ is defined to be the image of the closed point. Then by [19, Proposition 2.4.4], for each irreducible component Z_i of Z, there is a unique point ξ_i of X^{an} corresponding to the generic point of Z_i.

We now describe the topological space X^{an} in terms of the dual graph G of Z (Definition

1.2.22). Then the gluing data for X^{an} corresponds to the graph G. The spaces $(X_i^*)^{an}$ are contractible, and there is a deformation retraction $r_i : (X_i^*)^{an} \twoheadrightarrow \{\xi_i\}$. For each singular point p of Z corresponding to the intersection of Z_i and Z_j, the space $(X_p)^{an}$ deformation retracts onto a line segment e_p with endpoints ξ_i and ξ_j.

Then, the entire Berkovich space X^{an} admits a deformation retraction onto a metric graph Γ with model G. Let $r : X^{an} \to \Gamma$ be this retraction map. For each $x \in \Gamma$, each connected component of $r^{-1}(x)\backslash\{x\}$ is a topological tree isomorphic to the open disk

$$\mathcal{B}(0,1)^- = \{x \in K^{an} \mid |y|_x < 1\}.$$

So far, we have seen how the skeleton is a subset of X^{an} associated to a semistable model of X. We now define the skeleton in terms of *semistable vertex sets* of X, only making reference to the space X^{an} itself, instead of starting with a model of X. This will provide better understanding of the geometry of Berkovich curves, since semistable vertex sets will allow us to break X^{an} in to simpler pieces. Each of these pieces is the preimage of an interval or ray under the following tropicalization map.

Let $\mathbb{A}^1_K = \text{Spec}(K[y])$. We define the *tropicalization map*

$$\text{trop} : (\mathbb{A}^1_K)^{an} \to \mathbb{R} \cup \{\infty\}$$

by $\text{trop}(x) = -\log(|y|_x)$. Now, we describe several subsets of $(\mathbb{A}^1_K)^{an}$, which we will use to describe Berkovich curves.

For $a \in K^*$, the *standard closed ball of radius* $|a|$ is $B(a) = \text{trop}^{-1}([\text{val}(a), \infty])$. The *standard open ball of radius* $|a|$ is $B(a)_+ = \text{trop}^{-1}((\text{val}(a), \infty])$. For $a, b \in K^*$ with $|a| \leq |b|$, the *standard closed annulus of inner radius* $|a|$ *and outer radius* $|b|$ is defined to be $S(a,b) = \text{trop}^{-1}([\text{val}(b), \text{val}(a)])$. The *standard open annulus of inner radius* $|a|$ *and outer radius* $|b|$ is $S(a,b)_+ = \text{trop}^{-1}((\text{val}(b), \text{val}(a)))$.

We now define a section $\sigma : \mathbb{R} \to (\mathbb{A}^1_K)^{an}$ of the tropicalization map by

$$\sigma(r) = |\cdot|_r \quad \text{where} \quad \left|\sum_{n=-\infty}^{\infty} a_n t^n\right|_r = \max\{|a_n| \exp(-rn) \mid n \in \mathbb{Z}\}.$$

The map σ is the only continuous section of trop.

Definition 1.3.20. Let A be a standard open or closed disk or annulus. Then the *skeleton* of A is the closed subset $\Sigma(A) = \sigma(\mathbb{R}) \cap A = \sigma(\text{trop}(A))$.

Let X denote a smooth connected complete algebraic curve over K. We will define a skeleton in X^{an} relative to a *semistable vertex set.*

Definition 1.3.21. A *semistable vertex set of* X is a finite set V of type II points of X^{an} such that $X^{an}\backslash V$ is a disjoint union of open balls and finitely many open annuli. A decomposition of X^{an} into a semistable vertex set and a disjoint union of open balls and finitely many open annuli is called a *semistable decomposition of* X.

Definition 1.3.22. Let V be a semistable vertex set of X. The *skeleton* of X with respect to V is

$$\Sigma(X, V) = V \cup \bigcup_A \sigma(A),$$

where A runs over all connected components of $X^{an} \backslash V$ that are generalized open annuli.

The skeleton $\Sigma(X, V)$ is compact. The connected components of $X^{an} \backslash \Sigma(X, V)$ are open balls B. The limit boundary of any connected component B is a single point $x \in \Sigma(X, V)$ [15, Lemma 3.4].

Definition 1.3.23. Let V be a semistable vertex set of X. We define a retraction $\tau_V : X^{an} \to \Sigma(X, V)$ as follows. Let $x \in X^{an} \backslash \Sigma(X, V)$ and let B_x be the connected component of x in $X^{an} \backslash \Sigma(X, V)$. Then the single point $y \in \Sigma(X, V)$ in the boundary limit of B_x is the image $\tau_V(x) = y$.

This retraction is continuous. If A is a generalized open annulus in the semistable decomposition of X then τ_V restricts to the retraction $\tau_A : A \to \Sigma(A)$ as defined in Definition 1.3.20 [15, Lemma 3.8]. Then, the skeleton $\Sigma(X, V)$ has the structure of a metric graph, with model given by (V, E), where the edges $e \in E$ correspond to the closures of the skeletons of the generalized open annuli in the semistable decomposition of X. In particular, $\Sigma(X, V)$ is a metric space. The length of an edge e coming from a standard open annulus of inner radius $|a|$ and outer radius $|b|$ is equal to $\mathrm{val}_K(a) - \mathrm{val}_K(b)$.

We now discuss the relationship between semistable vertex sets of X and semistable models of X. Let $\mathcal{X}$ be a semistable model for X. Let $V(\mathcal{X})$ denote the inverse image of the set of generic points of the special fiber $\mathcal{X}_k$ under the reduction map $\mathrm{red} : X^{an} \to X_k$.

Theorem 1.3.24 ([15])**.** *Let $\mathcal{X}$ be a semistable model of X. Then $V(\mathcal{X})$ is a semistable vertex set of X. We have that $\Sigma(X, V(\mathcal{X}))$ is the incidence graph of $\mathcal{X}_k$. The association $\mathcal{X} \mapsto V(\mathcal{X})$ gives a bijection between the set of semistable models of X and the set of semistable vertex sets of X.*

Let $\mathcal{X}$ be a semistable model of X. The points of X^{an} corresponding to generic points of irreducible components of $\mathcal{X}_k$ having arithmetic genus at least 1 is denoted $S(X^{an})$, and it is independent of the model $\mathcal{X}$. This is called the set of *marked points* of X^{an}. This set is empty if and only if X is a *Mumford curve*.

Let $x \in S(X^{an})$ be a marked point of X^{an}. Let C be the irreducible component of $\mathcal{X}_k$ with generic point $\xi = \mathrm{red}(x)$. We define the *genus* of x, denoted $g(x)$, to be the genus of the normalization of C.

Any semistable vertex set of X must contain all type II points with positive genus.

Proposition 1.3.25 (Genus formula [15])**.** *Let $\mathcal{X}$ be a semistable model for X. Then*

$$g(X) = g(\Sigma(X, V)) + \sum_{x \in V(\mathcal{X})} g(x),$$

where $g(X)$ is the genus of X, and $g(\Sigma(X, V))$ is the genus of $\Sigma(X, V)$ as a graph.

Definition 1.3.26. If the genus of X is at least one, there is a maximal subgraph $\Sigma \subset \Sigma(X, V)$ containing the marked points $S(X^{an})$ and having no unmarked vertices of degree 1. The graph Σ is called the *minimal skeleton* of X^{an}. The skeleton does not depend on the choice of model $\mathcal{X}$ of X. The graph Γ admits a deformation retraction onto Σ.

Example 1.3.27. Let X be an elliptic curve with multiplicative reduction, so that $\mathrm{val}_K(j) < 0$, where $j \in K$ is the *j-invariant.* Then the space X^{an} is as pictured in Figure 1.17. The minimal skeleton is a circle of circumference $-\mathrm{val}_K(j)$. Any type II point is a (minimal) semistable vertex set of X.

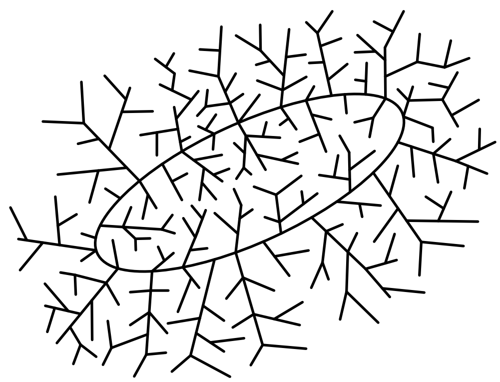


Figure 1.17: The Berkovich analytification of an elliptic curve.

Faithful Tropicalization

As before, let K be a complete non-Archimedean field, and let X be a smooth proper algebraic curve over K. Let $i : X \hookrightarrow \mathbb{A}^n$ be an embedding of X, which gives generators $f_1, \ldots, f_m$ of the coordinate ring $K[X]$. The *tropicalization corresponding to i* is the map $\mathrm{trop}_i : X \to \mathbb{R}^n$ is given by

$$\mathrm{trop}_i(x) = (-\log|f_1(x)|, \ldots, -\log|f_n(x)|).$$

This map can be extended to the analytification X^{an} to get a map $\pi_i : X^{an} \to \mathbb{R}^n$ given by

$$\pi_i(x) = (-\log|f_1|_x, \ldots, -\log|f_n|_x).$$

The image of this map is equal to $\mathrm{trop}_i(X)$. This explains the picture in the frontmatter of this dissertation: shadows of the Berkovich skeleton can be viewed through various embedded tropicalizations. We can relate the Berkovich analytification X^{an} to embedded tropicalizations $\mathrm{trop}_i(X)$ in the following way.

Theorem 1.3.28. *[105, Theorem 1.1] Let X be an affine variety over K. Then there is a homeomorphism (given by the inverse limit of maps $\pi_i : X^{an} \to \mathbb{A}^n$)*

$$X^{an} \to \varprojlim \operatorname{trop}_i(X). \tag{1.6}$$

Given just one embedded tropicalization, how can we detect information about the Berkovich skeleton? This is the problem of *certifying faithfulness*, as studied in [14, Section 5.23]. In some cases, the embedded tropicalization contains enough information to determine the structure of the skeleton of X^{an}.

Theorem 1.3.29 ([14, Corollary 5.38] and [16, Theorem 3.3])**.** *Let X be a smooth curve in $\mathbb{P}^n_{\mathbb{k}}$ of genus g. Further, suppose that* $\dim(H_1(\operatorname{trop}(X), \mathbb{R})) = g$*, all vertices of* $\operatorname{trop}(X) \subset \mathbb{R}^{n+1}/\mathbb{R}\mathbf{1}$ *are trivalent, and all edges have multiplicity 1. Then the minimal skeletons of* $\operatorname{trop}(X)$ *and X^{an} are isometric. In particular, if X is a smooth curve in $\mathbb{P}^2_{\mathbb{k}}$ whose Newton polygon and subdivision form a unimodular triangulation, then the minimal skeletons of* $\operatorname{trop}(X)$ *and X^{an} are isometric.*

Example 1.3.30. Consider the curve X in $\mathbb{P}^2$ over $\mathbb{C}\{\{t\}\}$ defined by

$$xyz^2 + x^2y^2 + 29t(xz^3 + yz^3) + 17t^2(x^3y + xy^3) = 0.$$

Tropicalizing X with this embedding, we obtain the embedded tropicalization in Figure 1.18. Since this is not a unimodular triangulation, Theorem 1.3.29 does not allow us to draw any

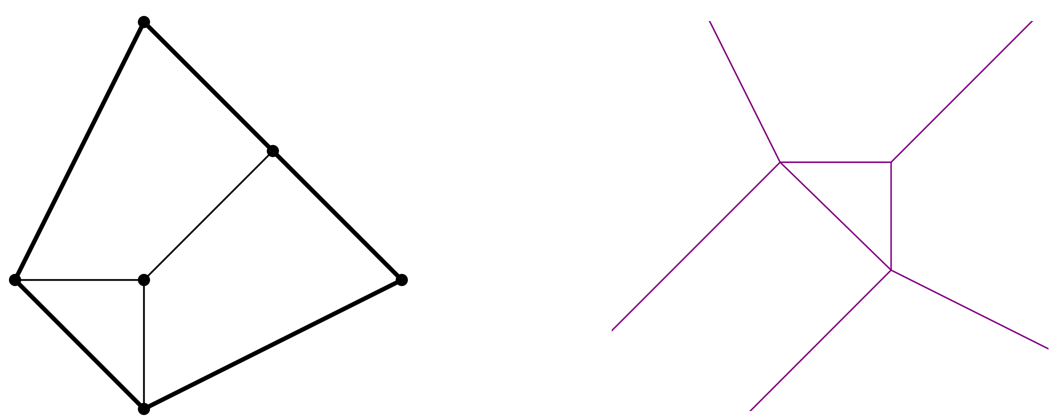


Figure 1.18: The Newton polygon from Example 1.3.30 with its regular subdivision on the left, and the embedded tropicalization on the right.

conclusions. However we know from having computed the skeleton of this curve in Example 1.1.13 that this is in fact a faithful tropicalization. The two points with valence four in Figure 1.18 each receive weight 1 in the abstract tropicalization.

1.4 Contributions in this Dissertation

Now that we have the language of tropical geometry and non-Archimedean geometry, we are prepared to state the main new results in this dissertation.

In *Chapter 2: Divisors*, I develop divisor theory on tropical curves and tropicalize the space of effective degree d divisors on a classical curve. The original work in this chapter appears in the following places. The subsections "Tropical Jacobians," "Tropical Abel-Jacobi Map," and "Tropical Schottky Problem" are based on [22], which is joint work with with Barbara Bolognese and Lynn Chua. It was published in the book *Combinatorial Algebraic Geometry: Selected Papers from the 2016 Apprenticeship Program.* The section "Tropicalizing Divisors" is joint work with Paul Helminck [26] and it will be published in the journal Advances in Geometry. The section "Tropicalizing the Symmetric Power" is joint work with Martin Ulirsch [28] and has been submitted to Transactions of the AMS.

The main results of this chapter are on tropicalizing the symmetric power of a curve [28]. Symmetric powers are moduli spaces for effective divisors on algebraic and tropical curves. Celebrated results in tropical geometry involve identifying non-Archimedean skeletons of classical moduli spaces with their combinatorial counterparts [1, 119, 38, 99]. The following theorem, which is joint work with Martin Ulirsch, gives an incarnation of this principle.

Theorem 2.3.1. [28] Let C be a smooth, projective curve over a non-Archimedean field. The symmetric power of the skeleton of C is the skeleton of the symmetric power of C.

This result allows us to prove a new version of the Riemann-Roch theorem on tropical curves. In the proof, we construct a nice model of the symmetric power of a curve by considering the fibered product

$$\mathcal{D}\mathrm{iv}_d^+(C) := \mathrm{Spec}(R) \times_{\overline{\mathcal{M}}_g} \overline{\mathcal{D}\mathrm{iv}}_{g,d},$$

where R is the valuation ring of K, the map $\mathrm{Spec}(R) \to \overline{\mathcal{M}}_g^{ss} \to \overline{\mathcal{M}}_g$ gives a strictly semistable model of X, and $\overline{\mathcal{D}\mathrm{iv}}_{g,d}$ is the Deligne-Mumford stack defined in [99] which compactifies the d-th symmetric power of the universal curve over $\mathcal{M}_g$. The proof of this result involves carefully describing the combinatorial structure of the strata of $\mathcal{D}\mathrm{iv}_d^+(C)$. The strata are the loci where the corresponding dual graph and divisor are constant.

Example 1.4.1. Consider a curve with two smooth components each having genus larger than one meeting in a node. Then our model has seven strata, corresponding to the combinatorial types of stable curves and divisors in Figure 1.19.

In *Chapter 3: Moduli Spaces*, I use this construction to develop a divisorial motivic zeta function. The original work in this chapter appears in the section "Divisorial Motivic Zeta Functions," which is joint work with Martin Ulirsch [27]. It will appear in the Michigan Mathematics Journal.

Our divisorial motivic zeta function $Z_{\mathrm{div}}(X, \vec{p}; t)$ is for marked stable curves $(X, \vec{p})$. In order to define it, we introduce a new moduli space $\overline{\mathcal{D}\mathrm{iv}}_{g,n,d}$ which takes in to account the

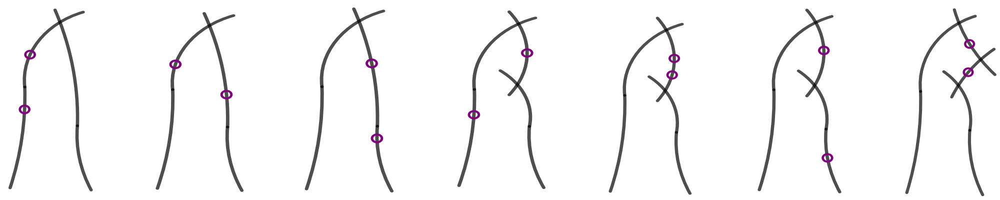


Figure 1.19: The strata in $\mathcal{D}\mathrm{iv}_d^+(C)$ in Example 1.4.1.

marked points. We prove $Z_{\text{div}}(X, \vec{p}; t)$ is rational over the Grothendieck ring of varieties and give a formula in terms of the dual graph of the curve [27].

Theorem 3.3.2. [27] Let $(X, \vec{p})$ be a stable quasiprojective curve over a field k with n marked points $\vec{p}$. Then $Z_{\text{div}}(X, \vec{p}; t)$ is rational over the Grothendieck ring $K_0(\mathrm{Var}/k)$. If $G = (E, V)$ is the dual graph of X, then

$$Z_{\text{div}}\big(X, \vec{p}; t\big) = \left(\frac{1 - \mathbb{L}t}{1 - \mathbb{L}t - t + t^2}\right)^{|E|+n} (1-t)^{2|E|+n} \prod_{v \in V} Z_{\text{mot}}\big(\widetilde{X}_v; t\big),$$

where $\widetilde{X}_v$ is the normalization of the component of X corresponding to the vertex $v \in V$, the function Z_{mot} is Kapranov's motivic zeta function, and $\mathbb{L}$ is the class of $\mathbb{A}^1$ in the Grothendieck ring.

In *Chapter 4: Skeletons*, I compute skeletons of algebraic curves. The original work in this chapter appears in the following places. The section "Hyperelliptic Curves" and the subsection "Existence of Faithful Tropicalizations" are based on [22], which is joint work with with Barbara Bolognese and Lynn Chua. It was published in the book *Combinatorial Algebraic Geometry: Selected Papers from the 2016 Apprenticeship Program*. The section "Superelliptic Curves" is joint work with Paul Helminck [26] and it will be published in Advances in Geometry.

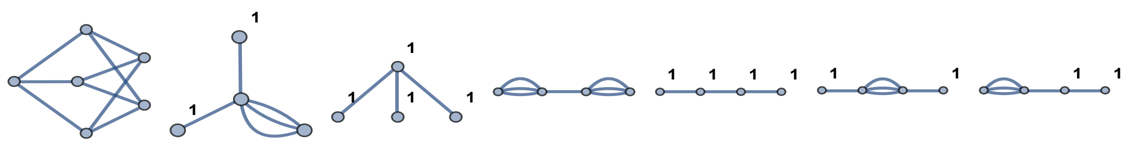
Despite the importance of skeletons to research in tropical geometry, it is relatively unknown how to compute them explicitly. There is a moderate body of work in computing skeletons in genus one [46, 77] and two [51, 70, 111] using many different approaches. My main results in this area are algorithms which compute the tropicalizations of hyperelliptic [22] curves $y^2 = f(x)$ and superelliptic [26] curves $y^m = f(x)$. The algorithms are combinatorial and use the valuations of the differences of the roots of f, together with the degree m as input. In genus higher than 3, there is no known general algorithm, and my result for the superelliptic case is the only algorithmic method computing skeletons in this domain.

Theorems 4.4.5 and 4.5.17. [22, 26] Given a curve with equation $y^m = f(x)$ over a field of characteristic 0 that is complete with respect to a non-Archimedean valuation, its skeleton can be computed algorithmically.

Together with Paul Helminck, we show that using our definition of superelliptic weighted tropical curves, each one of prime degree can be realized as the skeleton of an algebraic superelliptic curve. We also have computational results about the locus of superelliptic tropical curves inside the moduli space of tropical curves. For instance, for odd primes p there are 21447 maximal-dimensional types of tropical curves admitting a degree p map to a tree with 14 leaves.

Example 1.4.2. The tropical superelliptic curves of degree 3 and genus 4 of maximal dimensional type are pictured below. The first one is the skeleton of the curve over $\mathbb{C}\{\{t\}\}$ defined by

$$y^3 = x^3(x-t)(x-1)^2(x-1-t)(x-2)^2(x-2-t).$$



In *Chapter 5: Enumeration*, I count binodal cubic surfaces using tropical methods. The original material in this chapter appears in the section "Counting Surfaces in Space," and is joint work with Alheydis Geiger [25] and will be published in Le Matematiche.

The Gromov-Witten invariants of the plane count the complex algebraic curves of a given degree and genus passing through a given number of points. It is a breakthrough result of Mikhalkin that this question can be rephrased tropically, and that these numbers agree [96]. For plane curves, it is much simpler to perform the tropical count because singularities of tropical plane curves are well understood. Given the success of tropical methods in this case, we may ask if it is possible to count surfaces tropically. In [91], the authors determine the combinatorial types of nodes that a one-nodal tropical surface can have. It is possible to combine these nodes, as long as they are not too close together. With Alheydis Geiger, we obtain the following preliminary result.

Theorem 5.2.13 ([25])**.** Of the 280 binodal cubic surfaces through 17 fixed general points, 214 of them come from tropical surfaces with separated nodes.

2

Divisors

Theorems and constructions from classical algebraic geometry often carry over to tropical geometry. In this chapter, we develop divisor theory on tropical curves. This will include a tropical Jacobian, Riemann-Roch Theorem, Abel-Jacobi map, and more. That so much of the algebraic theory holds in the tropical world is remarkable in its own right. Additionally, this tropical divisor theory provides combinatorial insight into the classical theory. However, this translation is not always seamless; subtle and interesting realizability issues can arise.

The original work in this chapter appears in the following places. The subsections "Tropical Jacobians," "Tropical Abel-Jacobi Map," and "Tropical Schottky Problem" are based on [22], which is joint work with with Barbara Bolognese and Lynn Chua. It was published in the book *Combinatorial Algebraic Geometry: Selected Papers from the 2016 Apprenticeship Program*. The section "Tropicalizing Divisors" is joint with Paul Helminck [26] and it will be published in Advances in Geometry. The section "Tropicalizing the Symmetric Power" is joint work with Martin Ulirsch [28] and has been submitted to Transactions of the AMS.

2.1 Divisors on Tropical Curves

In this section, we develop divisors on tropical curves, computing many examples throughout.

Tropical Riemann-Roch

Definition 2.1.1. Let Γ be a tropical curve or metric graph, and let G be a model for Γ. A *divisor* D on G is a $\mathbb{Z}$-linear combination

$$\sum_{x\in V(G)} D(x)x.$$

A *divisor* D on Γ is a $\mathbb{Z}$-linear combination

$$\sum_{x\in \Gamma} D(x)x.$$

The *support* $\mathrm{Supp}(D)$ of a divisor D on Γ is the set of $x \in \Gamma$ for which $D(x) \neq 0$. The divisor D on Γ is said to be *supported on* G if G is a model for Γ and D is supported on the vertices of G. The divisor D is said to be *effective* if $D(x) \geq 0$ for all $x \in \Gamma$.

Example 2.1.2 (Canonical Divisor)**.** Given a metric graph Γ, its *canonical divisor* is

$$K_\Gamma = \sum_{p \in \Gamma} (\text{valence}(p) - 2) \cdot p,$$

where valence(p) is the valence of p in any model G for Γ with $p \in V(G)$.

Just as in algebraic geometry, in tropical geometry there is a notion of a principal divisor. A *tropical rational function* is a continuous function from Γ to $\mathbb{R}$ which is piecewise linear. For some tropical rational function f, let (f) be the divisor with degree n_x at the vertex $x \in \Gamma$, where n_x is the sum of the outgoing slopes of f at x. Two divisors D, D' are *linearly equivalent* if there is a tropical rational function f such that $D = D' + (f)$. In [66], it is shown that all tropical rational functions are sums of *chip firing moves*. This implies that two divisors are linearly equivalent if and only if one can be attained from the other via chip firing moves. Chip firing on graphs is a rich and extensively studied topic, see [49].

Given a divisor D on Γ, we denote by $R(D)$ the set of all rational functions f such that $D + (f)$ is effective. Let

$$|D| = \{D' \geq 0 \mid D' \sim D\}.$$

This is called the *complete linear system of* D. The map from $R(D)/\vec{1}$ to $|D|$ which sends f to $D + (f)$ is a homeomorphism.

Example 2.1.3. Consider the metric graph on the left in Figure 2.1. Its canonical divisor K is the divisor $v_1 + v_2$, where v_1 and v_2 are the two points of valence 3. Then, $|K|$ is displayed on the right. The colored points on the right hand side correspond to the colored divisors on the left hand side. Only the points in $|K|$ along the blue line are realizable, see [99].

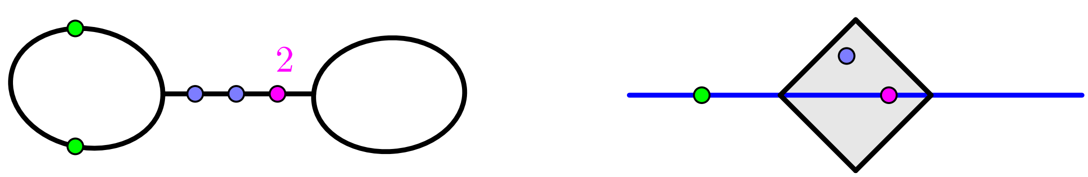


Figure 2.1: A graph and the complete linear system of its canonical divisor.

The Riemann-Roch Theorem for graphs and tropical curves is a celebrated result. The graph case was first proved by [13] and this result was later extended to tropical curves by [57] and [97]. This gives an analogue of the Riemann-Roch theorem for curves. Before stating the theorem, we must define the *rank* or a divisor.

Definition 2.1.4. Let D be a divisor of degree n on a tropical curve Γ. Then the *rank* of D, denoted $r(D)$, is the largest integer k such that any divisor D' of degree k has the property that $D - D'$ is equivalent to an effective divisor.

Remark 2.1.5. We remark here that the dimension of $R(D)$ is not equal to the rank $r(D)$. The space $R(D)$ is a polyhedral complex, but it is not of pure dimension and is often too large, in the sense that it contains points corresponding to divisors on Γ which do not lift to divisors on the curve X. For an example, see Example 2.1.3. This realizability issue is studied extensively in [99].

Theorem 2.1.6 ([13, 57, 97])**.** *Let D be a divisor on a tropical curve Γ. Then*

$$r(D) - R(K_\Gamma - D) = \deg D + 1 - g,$$

where g is the first Betti number of Γ.

Tropical Jacobians

In this subsection we define the tropical Jacobian. Jacobians of curves are principally polarized abelian varieties in a natural way; they are the most well known and extensively studied among abelian varieties. Both algebraic curves and abelian varieties have extremely rich geometries. Jacobians provide a link between such geometries, and they often reveal hidden features of algebraic curves which cannot be uncovered otherwise. In order to associate a tropical Jacobian to a complex algebraic curve A, one first constructs its classical Jacobian

$$J(X) := H^0(X, \omega_X)^*/H_1(X, \mathbb{Z}),$$

where we denote by ω_X the cotangent bundle of the curve. This complex torus admits a natural principal polarization Θ, called the *theta divisor*, such that the pair $(J(X), \Theta)$ is a principally polarized abelian variety. We can then obtain the tropical Jacobian by taking the Berkovich skeleton of the classical Jacobian.

There is another way to compute the tropical Jacobian. Instead, one could first tropicalize the curve X to obtain a tropical curve Γ, and then compute the Jacobian of Γ in a purely combinatorial fashion. In this section, we will compute the tropical Jacobian in this way.

Baker-Rabinoff, and independently Viviani, proved that both paths give the same result [16, 122]. However, the classical process hides more difficulties on the computational level and proves much more challenging to carry out in practice. Methods have been implemented in the *Maple* package *algcurves* for computing Jacobians numerically over $\mathbb{C}$ [52].

Given Γ, we describe a procedure to compute its tropical Jacobian via its *period matrix* Q_Γ, following [29, 42, 97]. Fix an orientation of the edges of G. For any $e \in E(G)$, denote the source vertex by e^+ and the target vertex by e^-. Let A be either $\mathbb{R}$ or $\mathbb{Z}$.

Definition 2.1.7. The free A-module generated by the vertex set $V(G)$ is called the *module of 0-chains with coefficients in* A. This is denoted $C_0(G, A)$:

$$C_0(G, A) = \left\{ \sum_{v \in V(G)} av \mid a \in A \right\}.$$

The free A-module generated by the edge set $E(G)$ is called the *module of 1-chains with coefficients in* A. This is denoted $C_1(G, A)$:

$$C_1(G, A) = \left\{ \sum_{v \in V(G)} av \mid a \in A \right\}.$$

Observe that the modules $C_0(G, A)$ and $C_1(G, A)$ are each isomorphic to their duals $\text{Hom}(C_0(G, A), A), \text{Hom}(C_0(G, A), A)$ via the map which sends $\sum_x a_x x$ to the function f, where $f(x) = a_x$. The module $C_1(G, A)$ is equipped with the inner product

$$\left\langle \sum_{e \in E(G)} a_e e, \sum_{e \in E(G)} b_e e \right\rangle = \sum_{e \in E(G)} a_e b_e l(e). \tag{2.1}$$

Definition 2.1.8. The *boundary map* $\partial : C_1(G, A) \to C_0(G, A)$ acts linearly on 1-chains by mapping an edge e to $t(e) - s(e)$. A *1-cycle* is an element of $\text{Ker}(\partial)$. The kernel of ∂ is the first homology group $H_1(G, A)$ of G, whose rank is

$$g(G) = |E(G)| - |V(G)| + 1.$$

Let $|w| = \sum_{v \in V(G)} w(v)$, and let g be the genus of Γ, defined as $g(G) + |w|$. Consider the positive semidefinite form Q_Γ on $H_1(G, \mathbb{R}) \oplus \mathbb{R}^{|w|}$, which vanishes on the second summand $\mathbb{R}^{|w|}$ and is defined on $H_1(G, \mathbb{R})$ by

$$Q_\Gamma \left(\sum_{e \in E(G)} \alpha_e e \right) = \sum_{e \in E(G)} \alpha_e^2 l(e).$$

Definition 2.1.9. Let $\omega_1, \ldots, \omega_{g(G)}$ be a basis of $H_1(G, \mathbb{Z})$. Then, we obtain an identification of the lattice $H_1(G, \mathbb{R}) \oplus \mathbb{R}^{|w|}$ with $\mathbb{R}^g$. Hence, we may express Q_Γ as a positive semidefinite $g \times g$ matrix, called the *period matrix* of Γ. Choosing a different basis gives another matrix related by an action of $GL_g(\mathbb{Z})$.

We now describe how to compute the period matrix. First, fix an arbitrary orientation of the edges of G and pick a spanning tree T of G. Label the edges such that $e_1, \ldots, e_{g(G)}$ are not in T, and $e_{g(G)+1}, \ldots, e_m$ are in T, where $m = |E(G)|$. Then $T \cup \{e_i\}$ for $1 \leq i \leq g(G)$ contains a unique cycle ω_i of G. The cycles $\omega_1, \ldots, \omega_{g(G)}$ form a cycle basis of G.

We traverse each cycle ω_i according to the direction specified by e_i. We compute a row vector b_i of length m, representing the direction of edges of G in this traversal. For each edge e_j in $E(G)$, let the j-th entry of b_i be 1 if e_j is in the correct orientation in the cycle, -1 if it is in the wrong orientation, and 0 if it is not in the cycle. Let B be the $g(G) \times m$ matrix whose i-th row is b_i. The matrix B has an interpretation in matroid theory as a totally unimodular matrix representing the cographic matroid of G [103].

Suppose that all vertices have weight zero, such that $g(G) = g$. Let D be the $m \times m$ diagonal matrix with entries $l(e_1), \ldots, l(e_m)$. Then the period matrix is given by $Q_\Gamma = BDB^T$. If we label the columns of B by $v_1, \ldots, v_m$, the period matrix equals

$$Q_\Gamma = l(e_1)v_1v_1^T + \cdots + l(e_m)v_mv_m^T\,.$$

Thus the cone of all matrices that are period matrices of G, allowing the edge lengths to vary, is the rational open polyhedral cone

$$\sigma_G = \mathbb{R}_{>0}\langle v_1v_1^T, \ldots, v_mv_m^T\rangle\,. \tag{2.2}$$

If Γ has vertices of nonzero weight, the period matrix is given by the construction above with $g - g(G)$ additional rows and columns with zero entries.

Example 2.1.10. Consider the complete graph on 4 vertices in Figure 2.2.

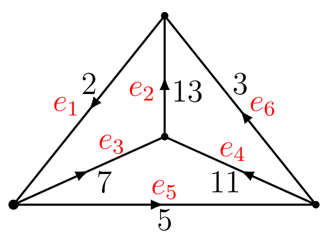


Figure 2.2: The metric graph and edge orientation used in Example 2.1.10.

We indicate in the figure an arbitrary choice of the edge orientations, and we choose the spanning tree consisting of the edges $T = \{e_2, e_3, e_4\}$. This corresponds to the cycle basis $\omega_1 = e_1 + e_3 + e_2$, $\omega_2 = -e_3 + e_5 + e_4$, and $\omega_3 = -e_2 - e_4 + e_6$. Next, we compute the matrix

$$B = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 & 0 & 1 \end{pmatrix}.$$

Let D be the 6×6 diagonal matrix with entries $13, 7, 11, 2, 5, 3$. The period matrix is then

$$Q_\Gamma = BDB^T = \begin{pmatrix} 22 & -7 & -13 \\ -7 & 23 & -11 \\ -13 & -11 & 27 \end{pmatrix}. \tag{2.3}$$

We now use period matrices to define and study tropical Jacobians of curves as principally polarized tropical abelian varieties. Let $\tilde{S}^g_{\geq 0}$ be the set of $g \times g$ symmetric positive semidefinite matrices with rational nullspace, meaning that their kernels have bases defined over $\mathbb{Q}$. The group $GL_g(\mathbb{Z})$ acts on $\tilde{S}^g_{\geq 0}$ by $Q \cdot X = X^T Q X$ for all $X \in GL_g(\mathbb{Z})$, $Q \in \tilde{S}^g_{\geq 0}$.

A *tropical torus* of dimension g as a quotient $X = \mathbb{R}^g/\Lambda$, where Λ is a lattice of rank g in $\mathbb{R}^g$. A *polarization* on X is given by a quadratic form Q on $\mathbb{R}^g$. Following [29, 42], we call the pair $(\mathbb{R}^g/\Lambda, Q)$ a *principally polarized tropical abelian variety (pptav)*, when $Q \in \tilde{S}^g_{\geq 0}$.

Two pptavs are isomorphic if there is some $X \in GL_g(\mathbb{R})$ that maps one lattice to the other, and acts on one quadratic form to give the other. We can choose a representative of each isomorphism class in the form $(\mathbb{R}^g/\mathbb{Z}^g, Q)$, where Q is an element of the quotient of $\tilde{S}^g_{\geq 0}$ by the action of $GL_g(\mathbb{Z})$. The points of this space are in bijection with the points of the *moduli space of principally polarized tropical abelian varieties*, which we denote by A_g^{trop}. We will describe the structure of A_g^{trop} in Section 2.1. The tropical Jacobian is our primary example of a principally polarized tropical abelian variety.

Definition 2.1.11. The *tropical Jacobian* of a tropical curve Γ is $(\mathbb{R}^g/\mathbb{Z}^g, Q)$, where Q is the period matrix of Γ.

Tropical Abel-Jacobi Map

The period matrix induces a Delaunay subdivision of $\mathbb{R}^g$, which has an associated Voronoi decomposition giving the *tropical theta divisor* of the tropical Jacobian. The tropical theta divisor has an alternate description, given in Theorem 2.1.14, as the image of all degree $d-1$ effective divisors under the *Abel-Jacobi map*. We describe this in more detail below.

Given a matrix $Q \in \tilde{S}^g_{\geq 0}$, consider the map

$$l_Q : \mathbb{Z}^g \longrightarrow \mathbb{Z}^g \times \mathbb{R}, \quad x \mapsto (x, x^T Q x).$$

We obtain a regular subdivision of $\mathbb{R}^g$ by weighting the points of $\mathbb{Z}^g$ by l_Q as follows. Take the convex hull of image of l_Q in $\mathbb{R}^g \times \mathbb{R} \cong \mathbb{R}^{g+1}$. By projecting down the lower faces via the morphism $\mathbb{R}^{g+1} \longrightarrow \mathbb{R}^g$ which forgets the last coordinate, we obtain a periodic dicing of the lattice $\mathbb{Z}^g \subset \mathbb{R}^g$, called the *Delaunay subdivision* $\text{Del}(Q)$ of Q. This is an infinite and periodic analogue of the regular subdivision of a polytope induced by weights on the vertices.

Example 2.1.12. Consider the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. The function $l_Q : \mathbb{Z}^2 \to \mathbb{Z}^2 \times \mathbb{R}$ is given by $(x, y) \mapsto (x, y, x^2)$. The convex hull of the points in the image of l_Q is pictured in Figure 2.3, together with the Delaunay subdivision.

Given a Delaunay decomposition, one can consider the dual decomposition, called the *Voronoi decomposition*. This is illustrated in Figure 2.4 for $g = 2$. The Voronoi decomposition gives the *tropical theta divisor* associated to a pptav $(\mathbb{R}^g/\Lambda, Q)$, which is the tropical

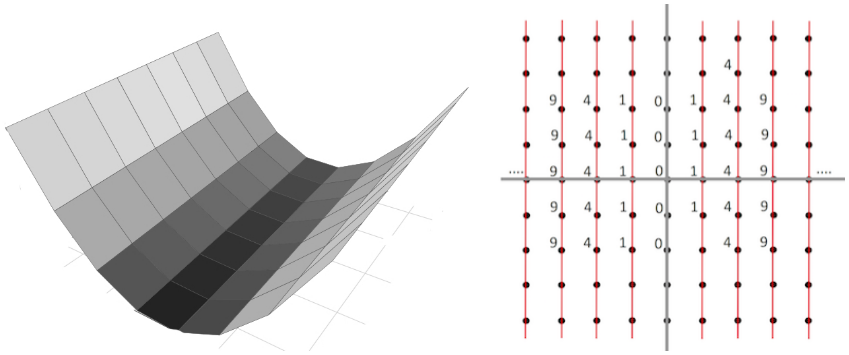


Figure 2.3: The weight function induced by the quadratic form in Example 2.1.12 on the left, and the corresponding Delaunay subdivision on the right.

hypersurface in $\mathbb{R}^g$ defined by the *theta function*

$$\Theta(x) = \max_{\lambda \in \Lambda} \left\{ \lambda^T Q x - \frac{1}{2} \lambda^T Q \lambda \right\}, \quad x \in \mathbb{R}^g .$$

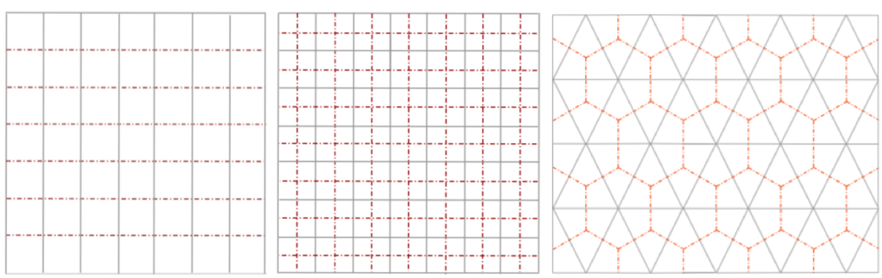


Figure 2.4: Delaunay decompositions of $\mathbb{R}^2$ (solid lines) and their associated Voronoi decompositions (dotted lines).

It is possible to give an alternate description of the tropical theta divisor using Theorem 2.1.14, as we are about to explain. Let $\Gamma = (G, w, l)$ be a tropical curve, and let $p_0 \in \Gamma$ be a fixed basepoint. Let $\omega_1, \ldots, \omega_g$ be a basis of $H_1(G, \mathbb{Z})$. For any point p in Γ, let $c(p) = \sum_i a_i e_i$ describe any path from p_0 to p.

Definition 2.1.13. For any $p \in \Gamma$, let

$$\mu(p) = (\langle c(p), \omega_1 \rangle, \ldots, \langle c(p), \omega_g \rangle) \in \mathbb{R}^g / \Lambda$$

where the inner products are as defined in Equation 2.1. By the identification of $\mathbb{R}^g/\Lambda$ with $\mathbb{R}^g/\mathbb{Z}^g$ induced by the choice of cycle basis, $\mu(p)$ is a point of the tropical Jacobian. The *tropical Abel-Jacobi map* $\mu : \mathrm{Div}(\Gamma) \to \mathrm{Jac}(\Gamma)$ is obtained by extending μ as above linearly to all divisors on Γ. This does not depend on the choice of path from p_0 to p [12].

Theorem 2.1.14 (Corollary 8.6, [97])**.** *Let W_{g-1} be the image of degree $g-1$ effective divisors under the tropical Abel-Jacobi map. The set W_{g-1} is the tropical theta divisor up to translation.*

Example 2.1.15 (Example 2.1.10, Continued)**.** Using the `GAP` package `polyhedral`, we compute that the Delaunay subdivision of the quadratic form in Equation 2.3 is given by six tetrahedra in the unit cube, all of which share the great diagonal as an edge [55]. We also compute the Voronoi decomposition dual to this Delaunay subdivision, which gives a tiling of $\mathbb{R}^3$ by permutohedra as illustrated in Figure 2.5. This is the tropical theta divisor, with f-vector $(6, 12, 7)$. In Figure 2.6, we illustrate the correspondence described by Theorem 2.1.14 between W_2 and the tropical theta divisor.

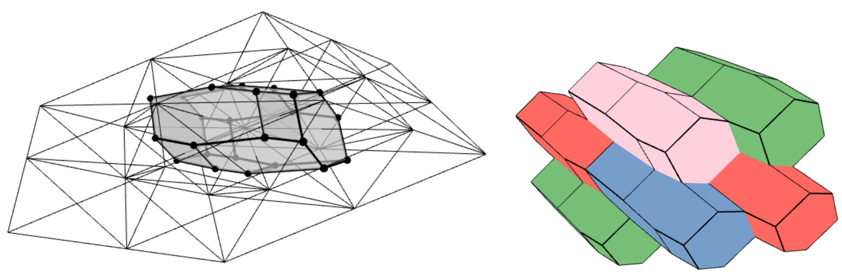


Figure 2.5: The left figure shows the Delaunay subdivision by tetrahedra and a dual permutohedron in grey. The right figure illustrates a tiling of $\mathbb{R}^3$ by permutohedra.

Break Divisors

Let $\mathrm{Pic}^d(\Gamma)$ be the set of divisor classes under rational equivalence of degree d on Γ.

Definition 2.1.16. A *break divisor* on Γ is a divisor which is supported on the closure of the complement of a spanning tree in Γ.

Theorem 2.1.17 ([5])**.** *Every degree g divisor on Γ is linearly equivalent to a unique break divisor. Every degree g divisor supported on $V(G)$ is equivalent to a unique break divisor supported on $V(G)$.*

We have that $\mathrm{Pic}^d(\Gamma) \approx J(G)$ via the Abel-Jacobi map which sends a break divisor representative D in $\mathrm{Pic}^d(\Gamma)$ to $\mu(D)$.

Given any spanning tree T of G let $\Sigma_T = \prod_{e \notin T} \bar{e}$ be the set break divisors supported on $\overline{\Gamma \backslash T}$. Any break divisor supported on $\Gamma \backslash V(G)$ uniquely determines a spanning tree T, so for two distinct spanning trees T, T', we have that $\Sigma_T^0 \cap \Sigma_{T'}^0 = \emptyset$. Then the mapping sending $\mathrm{Pic}^g(\Gamma)$ to $J(\Gamma)$ maps Σ_T^0 to an open parallelotope. These parallelotopes give the *canonical cell decomposition of* $J(G)$.

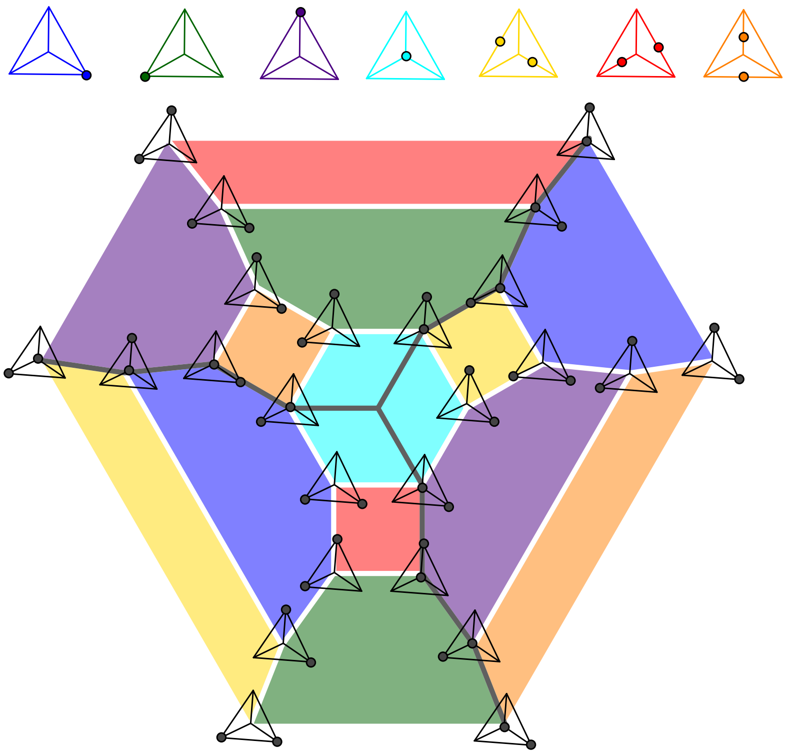


Figure 2.6: Each vertex of the permutohedron corresponds to a divisor supported on the vertices of Γ. The square faces correspond to divisors supported on the interiors of edges of Γ which do not meet in a vertex. Each hexagonal face corresponds to divisors which are supported on edges of Γ which are adjacent to a fixed vertex. Then, the edges correspond to keeping one point of the divisor fixed, and moving the other point along an edge of Γ. The grey curve depicted above represents the embedding of Γ into its Jacobian under the Abel-Jacobi map, which, under the identifications, is again K_4.

Example 2.1.18. We now give an example of the canonical cell decomposition of $J(G)$. Let Γ be the metric graph in Figure 2.7, where e_1, e_3 each have length 2 and e_2 has length 1. Let G be the model for Γ on the right, where each edge has length 1.

First, we will compute $J(\Gamma)$. As before, we first compute the period matrix $Q = BDB^T$:

$$Q = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 2 & 4 \end{pmatrix}.$$

So,

$$J(\Gamma) = \mathbb{R}^2/\langle (3,2), (2,4) \rangle = \mathbb{R}^2/\langle (4,0), (-1,2) \rangle,$$

Where we choose the second (equivalent) basis for $H_1(\Gamma, \mathbb{Z})$ to obtain Figure 2.8.

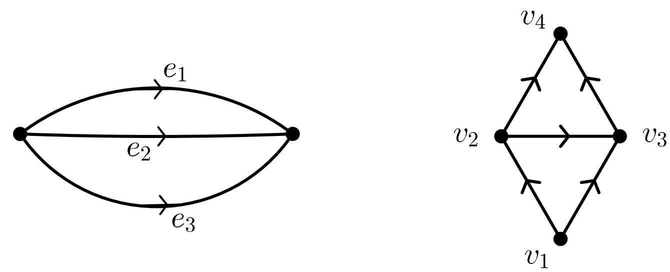


Figure 2.7: The graph Γ and the model G for Example 2.1.18.

We can also find the image of Γ under the Abel-Jacobi map. If we fix the vertex p to be the point halfway along e_3, then for a point te_i, for $t \in [0,1]$, we have

$$\mu(te_1) = (0,1) + t(2,2),$$
$$\mu(te_2) = (0,1) + t(-1,0),$$
$$\mu(te_3) = (0,1) + t(0,-2).$$

Now, consider the model G for Γ. Let the basis for $H_1(G,\mathbb{Z})$ be the cycles as before: $\{-\overline{23}+\overline{24}-\overline{34}, \overline{24}-\overline{13}+\overline{24}-\overline{34}\}$. We now compute the parallelotope for one tree in G, and the rest of the cell decomposition follows. Let T be the tree given by $T = \{\overline{24}, \overline{23}, \overline{12}\}$. Fix the base point p for the Abel-Jacobi map to be v_1. The edges not in the tree T are $\overline{12}$ and $\overline{34}$. Then a break divisor $a\overline{12} + b\overline{34}$ in Σ_T^0 maps to

$$\mu(a\overline{12} + b\overline{34}) = a(0,1) + (0,-1) + b(-1,-1).$$

Now we take the span of this region for $a, b \in [0,1]$. When performing this calculation, one must be careful to mind the orientation of the edge.

We obtain the cell decomposition of $J(\Gamma)$ shown in Figure 2.8. The image of Γ under the Abel-Jacobi map is represented by the thick black lines, and each vertex is labeled with the corresponding number.

Tropical Schottky Problem

From the classical perspective, the association $X \mapsto (J(X), \Theta)$, where $J(X)$ is the Jacobian of and algebraic curve X, gives the *Torelli map*

$$t_g : \mathcal{M}_g \to \mathcal{A}_g\,.$$

Here $\mathcal{M}_g$ denotes the moduli space of smooth genus g curves, while $\mathcal{A}_g$ denotes the moduli space of g-dimensional abelian varieties with a principal polarization. For more on moduli spaces, see Chapter 3. The content of Torelli's theorem is precisely the injectivity of

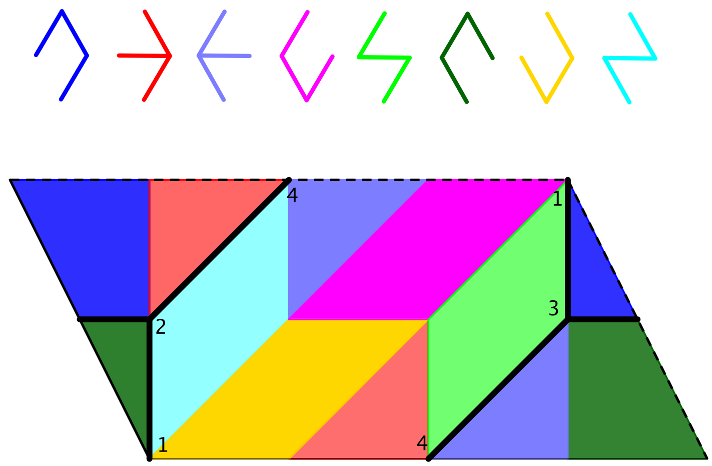


Figure 2.8: The cell decomposition for $J(\Gamma)$ in Example 2.1.18.

the Torelli map, which in fact can be proved to be dominant for $g = 2, 3$. Its image, i.e. the locus inside the moduli space of principally polarized abelian varieties, is called the *Schottky locus* and its complete description required several decades of work by many. The injectivity of the Torelli morphism implies, in particular, that we can always reconstruct an algebraic curve from its principally polarized Jacobian. The characterization of the Schottky locus was first worked out in genus 4 by Schottky himself and Jung (cf. [112]) via theta characteristics. Many other approaches followed for higher genus: the work by Andreotti and Mayer ([6]), Matsusaka and Ran [93, 107]) and Shiota ([115]) are worth mentioning. For extensive surveys see e.g. [7, 61].

In this section, we discuss the tropical Schottky problem, following [22]. The process of associating a period matrix to a metric graph can also be inverted. The *tropical Torelli map* $t_g^{\text{trop}} : M_g^{\text{trop}} \to A_g^{\text{trop}}$ sends a tropical curve of genus g to its tropical Jacobian, which is the element of A_g^{trop} corresponding to its period matrix. The image of this map is called the *tropical Schottky locus*, and these are the set of period matrices that arise as the tropical Jacobian of a curve. Starting with a principally polarized tropical abelian variety whose period matrix Q is known to lie in the tropical Schottky locus, we give a procedure to compute a curve whose tropical Jacobian corresponds to Q.

We describe the structure of the moduli space A_g^{trop} in this section, using Voronoi reduction theory. Given a Delaunay subdivision D, define the set of matrices that have D as their Delaunay subdivision to be

$$\sigma_D = \{Q \in \tilde{S}^g_{\geq 0} \mid \operatorname{Del}(Q) = D\}\,. \tag{2.4}$$

The *secondary cone* of D is the Euclidean closure $\overline{\sigma_D}$ of σ_D in $\mathbb{R}^{\binom{g+1}{2}}$, and is a closed rational polyhedral cone. There is an action of $GL_g(\mathbb{Z})$ on the set of secondary cones, induced by its action on $\tilde{S}^g_{\geq 0}$.

Theorem 2.1.19 ([123]). *The set of secondary cones forms an infinite polyhedral fan whose support is $\tilde{S}^g_{\geq 0}$, known as the* second Voronoi decomposition. *There are only finitely many $GL_g(\mathbb{Z})$-orbits of this set of secondary cones.*

By this theorem, we can choose Delaunay subdivisions $D_1, \dots, D_k$ of $\mathbb{R}^g$, such that the corresponding secondary cones are representatives for $GL_g(\mathbb{Z})$-equivalence classes of secondary cones. The moduli space A_g^{trop} is a *stacky fan* whose cells correspond to these classes [29, 42]. More precisely, for each Delaunay subdivision D, consider the stabilizer

$$\operatorname{Stab}(\sigma_D) = \{X \in GL_g(\mathbb{Z}) \mid \sigma_D \cdot X = \sigma_D\}\,. \tag{2.5}$$

Define the cell

$$C(D) = \overline{\sigma_D} / \operatorname{Stab}(\sigma_D) \tag{2.6}$$

as the quotient of the secondary cone by the stabilizer. Then we have

$$A_g^{\text{trop}} = \bigsqcup_{i=1}^{k} C(D_i)/\sim, \tag{2.7}$$

where we take the disjoint union of the cells $C(D_1), \dots, C(D_k)$ and quotient by the equivalence relation $\sim$ induced by $GL_g(\mathbb{Z})$-equivalence of matrices in $\tilde{S}^g_{\geq 0}$, which corresponds to gluing the cones.

Example 2.1.20. In genus two, there are four Delaunay subdivisions $D_1, \dots, D_4$ as in Figure 2.9. Their secondary cones give representatives for $GL_g(\mathbb{Z})$-equivalence classes of secondary cones. The corresponding secondary cones are as follows.

$$\overline{\sigma_{D_1}} = \left\{ \begin{bmatrix} a+c & -c \\ -c & b+c \end{bmatrix} \,:\, a, b, c \in \mathbb{R} \right\}, \tag{2.8}$$

$$\overline{\sigma_{D_2}} = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \,:\, a, b \in \mathbb{R} \right\}, \tag{2.9}$$

$$\overline{\sigma_{D_3}} = \left\{ \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \,:\, a \in \mathbb{R} \right\}, \tag{2.10}$$

$$\overline{\sigma_{D_4}} = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}. \tag{2.11}$$

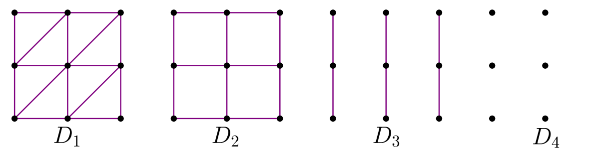


Figure 2.9: Delaunay subdivisions for $g = 2$.

The tropical Schottky locus has the following characterization using matroid theory. Given a graph G, we can define a *cographic matroid* $M^*(G)$ (see [103] for an introduction to matroid theory). The matroid $M^*(G)$ is representable by a totally unimodular matrix, constructed as the matrix B in Section 2.1. The cone σ_G defined in Equation 2.2 is a secondary cone in $\tilde{S}^g_{\geq 0}$. The $GL_g(\mathbb{Z})$-equivalence class of σ_G is independent of the choice of totally unimodular matrix representing $M^*(G)$. Hence we can associate to $M^*(G)$ a unique cell $C(M^*(G))$ of A_g^{trop}, corresponding to this equivalence class of secondary cones.

A matroid is *simple* if it has no loops and no parallel elements. We define the following stacky subfan of A_g^{trop} corresponding to simple cographic matroids,

$$A_g^{\text{cogr}} = \{C(M) \, : \, M \text{ simple cographic matroid of rank } \leq g\} \, . \tag{2.12}$$

The image of the tropical Torelli map t_g^{trop} is A_g^{cogr} [29, 42], and we call A_g^{cogr} the *tropical Schottky locus*. When $g \leq 3$, $A_g^{\text{cogr}} = A_g^{\text{trop}}$, hence every element of $\tilde{S}^g_{\geq 0}$ is a period matrix of a tropical curve. But when $g \geq 4$, this inclusion is proper. For example, A_4^{cogr} has 25 cells while A_4^{trop} has 61 cells, and A_5^{cogr} has 92 cells while A_5^{trop} has 179433 cells, according to the computations in [42].

2.2 Tropicalizing Divisors

In this section we specialize divisors on an algebraic curve X and obtain divisors on the associated tropical curve Γ.

Given an algebraic curve X and a strongly semistsable model $\mathcal{X}$ for X, we can *specialize* divisors on X to divisors on Γ or G. This is given by a map $\rho : \text{Div}(X) \to \text{Div}(\Gamma)$. Let $\text{Div}(\mathcal{X})$ be the group of Cartier divisors on $\mathcal{X}$ and let $\text{Div}_k(\mathcal{X})$ be the subgroup of Cartier divisors with support on the special fiber $\mathcal{X}_k$. Let $\{C_1, ..., C_r\}$ be the set of irreducible components in $\mathcal{X}_k$. Let G be the dual graph of $\mathcal{X}$. For any vertex $v \in G$, we let C_v be the associated irreducible component of $\mathcal{X}_k$.

Let $\mathcal{M}(G)$ be the group of $\mathbb{Z}$-valued functions on $V(G)$. Recall that we may associate an element $f \in \mathcal{M}(G)$ to a divisor (f) on G by

$$(f) = \sum_{v \in V(G)} \sum_{e = vw \in E(G)} (f(v) - f(w))(v).$$

Any divisor in the image of this map is called a *principal divisor*:

$$\mathrm{Prin}(G) := \{(f) \in \mathrm{Div}(G) \mid f \in \mathcal{M}(G)\}.$$

We have that $\mathrm{Prin}(G) \subset \mathrm{Div}^0(G)$, the set of degree 0 divisors on G [12, Corollary 1].

We will now transport algebraic divisors in $\mathrm{Div}(X)$ to divisors on the graph G. To do this, we will use the intersection pairing on $\mathcal{X}$. This is a bilinear pairing

$$\mathrm{Div}(\mathcal{X}) \times \mathrm{Div}_k(\mathcal{X}) \to \mathbb{Z},$$

which we will denote (see [85, Chapter 9, Theorem 1.12]) by $(\mathcal{D}_1 \cdot \mathcal{D}_2) \in \mathbb{Z}$. This then gives rise to the *specialization* homomorphism $\rho : \mathrm{Div}(\mathcal{X}) \to \mathrm{Div}(G)$ defined by

$$\rho(\mathcal{D}) = \sum_{v \in G} (\mathcal{D} \cdot C_v)(v), \tag{2.13}$$

see [10, Section 2] and [71, Section 2.2.1].

For any $D \in \mathrm{Div}(X)$, the closure inside $\mathcal{X}$ defines a Cartier divisor which we will denote by $\overline{D}$. This gives a homomorphism $\mathrm{Div}(X) \to \mathrm{Div}(\mathcal{X})$. Composing it with the specialization homomorphism, we obtain a homomorphism $\mathrm{Div}(X) \to \mathrm{Div}(G)$ which we will again denote by ρ. We then have the following lemma.

Lemma 2.2.1 ([26])**.** *Let $\mathcal{X}$ be a strongly semistable model over R for a smooth, proper, geometrically connected curve X. Let G be its intersection graph and let $\rho : \mathrm{Div}(\mathcal{X}) \to \mathrm{Div}(G)$ be the specialization map from Equation 2.13. Then $\rho(\mathrm{Prin}(X)) \subset \mathrm{Prin}(G)$.*

Proof. For any $f \in K(X)^*$, let $\mathrm{div}_\eta(f)$ be the induced divisor on X and let $\mathrm{div}(f)$ be the induced divisor on $\mathcal{X}$. Write $D \in \mathrm{Prin}(X)\backslash\{0\}$ as $D = \mathrm{div}_\eta(f)$ for some $f \in K(X)^*$. We then have

$$\mathrm{div}(f) = \mathrm{div}_\eta(f) + V_f,$$

where V_f is the vertical divisor associated to f. By [85, Chapter 9, Theorem 1.12.c] we have that $\rho(\mathrm{div}(f)) = 0$. By [71, Lemma 3.3.1] we then have that $\rho(C_v) \in \mathrm{Prin}(G)$ for every $v \in V(G)$, so $\rho(V_f) \in \mathrm{Prin}(G)$. This then gives $\rho(\mathrm{div}_\eta(f)) \in \mathrm{Prin}(G)$, as desired. □

In other words, for every $f \in K(X)^*$, there is a $g \in \mathcal{M}(G)$ such that $\rho(\mathrm{div}_\eta(f)) = (g)$. We now consider this specialization homomorphism for K-rational points $P \in X(K)$. By [85, Chapter 9, Corollary 1.32], we find that $\rho(P) = (v_\Gamma)$ for a unique irreducible component $\Gamma \subset \mathcal{X}_s$. Note that such a component need not exist for non-regular models.

Example 2.2.2. Let $\mathcal{X} := \mathrm{Proj}(R[X, Y, W]/\langle XY - t^2W^2\rangle)$. Let $U := \mathrm{Spec}(R[x, s]/\langle xs - t^2\rangle)$ be the affine open of $\mathcal{X}$ given by dehomogenizing with respect to W. Let $P = \langle x - t, s - t\rangle$ be a K-rational point of U. The closure of P in $\mathcal{X}$ is then given by $\{P, \overline{P}\}$, where $\overline{P} = \langle x - t, s - t, t\rangle$. Note that $\overline{P}$ lies on both irreducible components of the special fiber. We thus do not directly have an associated divisor on the intersection graph of $\mathcal{X}$.

The intersection graph G consists of two vertices $\{v_1, v_2\}$ with an edge e of length two. Taking a desingularization above this edge, we obtain a regular model $\mathcal{X}'$ with intersection graph consisting of three vertices $\{v_1, v', v_2\}$ and two edges $\{e_1, e_2\}$. Here v_1 and v' are connected by e_1, and v' and v_2 are connected by e_2. The original edge e has been subdivided into two edges $\{e_1, e_2\}$ and the vertex v' in this new intersection graph G'. The point P now specializes to the vertex v' in the middle. For explicit equations defining this model $\mathcal{X}'$, see [85, Chapter 8, Example 3.53].

2.3 Tropicalizing the Symmetric Power

For the remainder of the chapter, we tropicalize the space of degree d effective divisors on a curve. We now give an overview, summarizing the main results. The constructions, proofs, and consequences come in the subsections that follow.

Let K be a non-Archimedean field with valuation ring R whose residue field k is algebraically closed and contained in K. Let X be a smooth projective curve over K of genus $g \geq 1$ and let $d \geq 0$. The *d-th symmetric power* X_d of X is defined to be the quotient

$$X_d = X^d / S_d$$

of the d-fold product $X^d = X \times \cdots \times X$ by the action of the symmetric group S_d that permutes the entries. The symmetric power X_d is again a smooth and projective algebraic variety and functions as the fine moduli space of effective divisors of degree d on X (see [98, Section 3] for details).

Let $\Gamma = \Gamma_X$ be the dual tropical curve of X, i.e. the minimal skeleton of X^{an}. As a set, the *d-th symmetric power* Γ_d of Γ is defined to be the quotient

$$\Gamma_d = \Gamma^d / S_d$$

of the d-fold product by the S_d-action. We will see in Section 2.3 that, once we choose a strictly semistable model $(G, |.|)$ for Γ, the symmetric power Γ_d naturally carries the structure of a *colored polysimplicial complex* and it naturally functions as a moduli space of effective divisors of degree d on Γ.

Let $\mathcal{X}$ be a strictly semistable model of X over R. The special fiber of $\mathcal{X}$ is a strictly semistable curve whose weighted dual graph (together with the edge lengths given by the valuations of the deformation parameters at every node) provides us with a natural choice of a model $\big(G, |.|\big)$ of Γ. There is a natural tropicalization map

$$\mathrm{trop}_{X_d} \colon X_d^{an} \longrightarrow \Gamma_d$$

given by pushing forward an effective Cartier divisor D on X_L, for a non-Archimedean extension L of K, to the dual tropical curve $\Gamma_{X_L} = \Gamma_X$, which is essentially a version of Baker's specialization map for divisors in [11].

On the other hand, using the compactification of the moduli space of effective divisors over $\mathcal{M}_g$ constructed in [99, Section 2], a special case of the moduli space of stable quotients in [89], we find a natural strictly polystable model $\mathcal{X}_d$ of X_d over $\operatorname{Spec} R$ that has a natural modular interpretation. Despite its suggestive notation, $\mathcal{X}_d$ is not the S_d-quotient of the fibered product $\mathcal{X} \times_{\operatorname{Spec} R} \cdots \times_{\operatorname{Spec} R} \mathcal{X}$ but rather a resolution thereof with good moduli-theoretic properties.

By [18], associated to the model $\mathcal{X}_d$ there is a strong deformation retraction

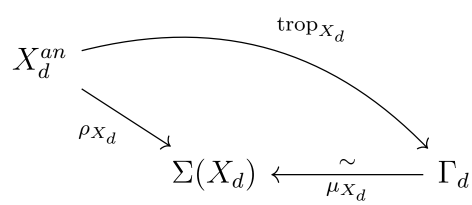
$$\rho_{X_d} \colon X_d^{an} \longrightarrow \Sigma(X_d)$$

onto the *non-Archimedean skeleton* $\Sigma(X_d)$ of X_d^{an}, which naturally carries the structure of a colored polysimplicial complex. We refer the reader to the subsection "Skeletons of Polystable Models" for a guide to this construction. The main result is Theorem 2.3.1.

Theorem 2.3.1 ([28])**.** *Let X be a smooth and projective algebraic curve over K and let $\mathcal{X}$ be a fixed strictly semistable model of X over the valuation ring R of K. Denote by Γ the dual tropical curve of X. There is a natural isomorphism*

$$\mu_{X_d} : \Gamma_d \xrightarrow{\sim} \Sigma(X_d)$$

of colored polysimplicial complexes that makes the following diagram commute:



In other words, the symmetric power Γ_d of the skeleton $\Gamma = \Gamma_X$ of X^{an} is isomorphic to the skeleton $\Sigma(X_d)$ of the symmetric power X_d^{an} of X^{an}. This is a first example of a phenomenon we will explore in depth in Chapter 3: that tropicalizations (skeletons) of classical moduli spaces are often themselves moduli spaces of tropical objects. The main idea of our proof is to carefully describe the combinatorial structure stratification of the polystable model of X_d coming from [99] and to identify it with the discrete data of Γ_d (thought of as the space $\operatorname{Div}_d^+(\Gamma)$ of effective divisors on Γ).

A slightly different version of Theorem 2.3.1 has appeared in [114] en route to the proof of a non-Archimedean Lefschetz hyperplane theorem for the locus of effective divisors in the Picard group. In [114, Section 6] the author first identifies the non-Archimedean skeleton of the d-fold product X^d with the d-fold product using Berkovich's skeleton construction in [18] and then shows that the deformation retraction is naturally S_n-invariant. This implies that the skeleton of the quotient $X_d^{an} = (X^d)^{an}/S_n$ is equal to Γ_d as a set. We refer the reader to [31] for more details on skeletons associated to products of degenerations.

The resulting polyhedral structure on the skeleton constructed in [114], however, is not the one we introduce in the subsection "Colored Polysimplicial Complexes" below, since it would have to take into account self-gluings of the polyhedra (quite like a generalized cone complex in [1] or a cone stack in [39]). We prefer our approach via the explicit model $\mathcal{X}_d$, since its modular interpretation simplifies the construction of the tropicalization map and the combinatorial stratification of its special fiber explains where the a priori only intrinsically defined polyhedral structure on Γ_d is coming from.

Symmetric powers of tropical curves have already appeared in [57, 66, 97], where they form one of the key ingredients to understand the polyhedral structure of tropical linear series. While the process of tropicalization on the level of divisor classes, e.g. as a tropicalization map of Picard groups, has been studied widely (e.g. in [11, 16]), the purpose here is to provide non-Archimedean foundations for the tropicalization of symmetric powers.

In [99, Theorem 3], Möller, Ulirsch, and Werner prove that the skeleton of the moduli space $\mathcal{D}iv_{g,d}$ of effective divisors over $\mathcal{M}_g$ is equal to the moduli space of effective tropical divisors $\mathrm{Div}_{g,d}^{trop}$ over $\mathcal{M}_g^{trop}$. It is tempting to speculate that this result would imply our Theorem 2.3.1. Unfortunately our current understanding of the functoriality of skeleton constructions does not seem to allow us to formally deduce such a result. The main obstacle to overcome here lies in the fact that the functor that associates to a K-analytic space its underlying topological space does not preserve fibered products.

Here, we restrict ourselves to the case of a strictly semistable model $\mathcal{X}$ of X. Suppose that $\mathcal{X}$ is only semistable, i.e. that we allow the components of the special fiber to not be smooth and have self-intersection. Then we may still construct a polystable model $\mathcal{X}_d$, which may also not be strict anymore, and, by the results of [18], there still is a strong deformation retraction map to its skeleton. We expect the analogue of Theorem 2.3.1 to be true; its proof, however, would require us to introduce significantly more technical background to describe self-gluings of polysimplicial complexes similar to what happens when moving from *cone complexes* to *generalized cone complexes* in [1] or to *cone spaces* in [39]. Taking care of these self-gluings is unavoidable when working universally over $\mathcal{M}_g$ as in [99, Theorem 3], but for a fixed X, as in our situation, we may simply blow up the points of self-intersection in a semistable model $\mathcal{X}$ of X in order to make it strictly semistable.

Colored Polysimplicial Complexes

We first set up the language of *colored polysimplicial complexes* so that we can speak about the symmetric power of a tropical curve Γ.

An *abstract n-simplex* is the power set of a finite set $S = \{0, \ldots, n\}$. An *abstract polysimplex* Δ is the cartesian product of a finite collection of simplices $\mathcal{P}(S_1), \ldots, \mathcal{P}(S_k)$. A *face* of Δ is a subset which is itself an abstract polysimplex. A *morphism* of abstract polysimplices $\phi : \Delta_1 \to \Delta_2$ is a map such that for all faces F_1 of Δ_1 we have $\phi(F_1) \subseteq F_2$ for some face F_2 of Δ_2. A morphism $\phi : \Delta_1 \to \Delta_2$ is a *face morphism* if it induces an isomorphism onto a face of Δ_2. Denote by **Poly** the category of polysimplices and by $\mathbf{Poly}^f$ the subcategory of polysimplices with face morphisms.

Let $M \subseteq \mathbb{R}_{\geq 0} \cup \{\infty\}$ be a submonoid. An *M-colored polysimplex* will be a polysimplex $\mathcal{P}(S_1) \times \cdots \times \mathcal{P}(S_k)$ together with a tuple $a \in M^k$ such that $a_i = 0$ whenever $S_i = \{0\}$ and

$a_i > 0$ whenever $|S_i| > 1$. A morphism of colored polysimplices

$$\phi \colon \big(\mathcal{P}(R_1) \times \cdots \times \mathcal{P}(R_l), a\big) \to \big(\mathcal{P}(S_1) \times \cdots \times \mathcal{P}(S_k), b\big)$$

is a morphism of abstract polysimplices such that whenever $|R_i| > 1$ and $\phi(R_i) \subset S_j$, then $a_i = b_j$. We denote by **cPoly** the category of colored polysimplices and by $\mathbf{cPoly}^f$ the subcategory of colored polysimplices with face morphisms.

In the following Definition 2.3.2 we give a new perspective on the notion of a colored polysimplicial complex, originally described by Berkovich in [18, Section 3 and 4].

Definition 2.3.2. A *colored polysimplicial complex* indexed by a poset $(\Xi, \preceq)$ is a functor

$$\Delta \colon (\Xi, \preceq) \longrightarrow \mathbf{cPoly}^f$$

to the category $\mathbf{cPoly}^f$ of polysimplices with face morphisms such that for every element $E \in \Xi$, we have that every face of $\Delta(E)$ is the image of exactly one morphism $\Delta(E' \prec E)$: $\Delta(E') \to \Delta(E)$.

We now discuss how to associate to a colored polysimplicial complex Σ a topological space $|\Sigma|$, called its *geometric realization*. Let $\vec{n} = (n_1, \dots, n_k) \in \mathbb{N}^k$ and let $\vec{a} = (a_1, \dots, a_k) \in \mathbb{R}_{>0}$. We define the *standard* $(\vec{n}, \vec{a})$*-polysimplex* $\Delta(\vec{n}, \vec{a}) := \Delta(n_1, a_1) \times \cdots \times \Delta(n_k, a_k)$ to be

$$\Delta(\vec{n}, \vec{a}) := \left\{ (x_{ij})_{1 \le i \le k, 0 \le j \le n_i} \in \mathbb{R}^{n_1 + \cdots + n_k} \,\middle|\, \sum_{j=0}^{n_i} x_{ij} = a_i \text{ for all } i \right\}. \tag{2.14}$$

For example, the standard $((1,1),(1,1))$-polysimplex is the unit square. The $((1,1),(2,3))$-polysimplex is a rectangle with side lengths 2 and 3.

Given a polysimplicial complex Σ indexed by a poset Ξ, its geometric realization $|\Sigma|$ is obtained by gluing the disjoint union of the standard polysimplices associated to each polysimplex in $\Delta(\Xi)$ along the images of the face morphisms. In other words, $|\Sigma|$ is the colimit of the functor F which takes each element in Ξ to the associated standard polysimplex, and takes each face morphism $\phi_{\alpha\beta} : \Delta_\alpha \to \Delta_\beta$ to the unique affine linear embedding taking $F(\Delta_\alpha)$ to the corresponding face of $F(\Delta_\beta)$.

Given a tropical curve Γ, we denote by $\mathrm{Div}_d^+(\Gamma)$ the set of effective divisors of degree d on Γ. Define the d-th symmetric product Γ_d of Γ to be the quotient of the d-fold product Γ^d of Γ by the action of S_d which permutes the factors.

Lemma 2.3.3 ([28])**.** *We have* $\mathrm{Div}_d^+(\Gamma) = \Gamma_d$.

Proof. Let $p \in \Gamma_d$. Then there is a representative $(p_1, \dots, p_d) \in \Gamma^d$ for p. Consider the map $\phi : \Gamma_d \to \mathrm{Div}_d^+(\Gamma)$, where

$$(p_1, \dots, p_d) \mapsto \sum_{i=1}^{d} p_i.$$

Then ϕ is well defined, because for any permutation $\sigma \in S_d$, we have $\sum_{i=1}^d p_i = \sum_{i=1}^d p_{\sigma(i)}$. The map ϕ is surjective, because given any effective divisor D of degree d, we may write it in the form $\sum_{i=1}^d p_i$. Then, we see that ϕ is injective because if $p_1 + \cdots + p_d = q_1 + \cdots + q_d$, then there is a permutation $\sigma \in S_d$ such that $p_i = q_{\sigma(i)}$. □

We now show that we can give $\mathrm{Div}_d^+(\Gamma)$ the structure of a colored polysimplicial complex. Fix a loop-free model $(G, |.|)$ of Γ. We show that, associated to this data, there is a natural polysimplical complex $\Delta(G, d)$ whose geometric realization is equal to $\mathrm{Div}_d^+(\Gamma) = \Gamma_d$.

Definition 2.3.4. A *stable pair* of degree d over G is a tuple $(\phi\colon G' \to G, D)$ consisting of a finite subdivision $\phi\colon G' \to G$ together with and an effective divisor $D \in \mathrm{Div}_d(G')$ such that $D(v) > 0$ for all exceptional vertices of G'.

In our notation we typically suppress the reference to ϕ and only write (G', D) instead of $(\phi\colon G' \to G, D)$. Denote by $\Xi(G)$ the set of stable pairs over G. It naturally carries the structure of a poset: We have $(G_1', D_1) \preceq (G_2', D_2)$ if and only if there is a finite subdivision $\phi_{12}\colon G_2' \to G_1'$ such that $\phi_1 \circ \phi_{12} = \phi_2$ and $\phi_{12,*} D_2 = D_1$. Consider the map

$$\Delta_{(G,d)}\colon \Xi(G) \longrightarrow \mathbf{cPoly}^f$$

that associates to (G', D) the colored polysimplex

$$\Delta_{(G,d)}(G', D) := \Delta(G', D) := \Delta\big(k_1, |e_1|\big) \times \cdots \times \Delta\big(k_l, |e_l|\big) \tag{2.15}$$

where $e_1, \ldots, e_l$ are the edges of G that G' is subdividing and k_i denotes the number of vertices in G' that live above e_i.

Proposition 2.3.5 ([28])**.** The map $\Delta_{(G,d)}$ defines a polysimplicial complex whose geometric realization is in natural bijection with $\mathrm{Div}_d^+(\Gamma)$.

From now on we always implicitly fix a strictly semistable model $(G, |.|)$ of Γ and, in a slight abuse of notation, denote $\Delta(G, d)$ by Γ_d.

Proof of Proposition 2.3.5. An inequality $(G_1', D_1) \preceq (G_2', D_2)$ naturally induces the face morphism $\Delta(G_1', D_1) \hookrightarrow \Delta(G_2', D_1)$ that is given by setting the edge lengths of those edges in G_2' to zero that are contracted in G_1. Conversely, a face of $F \subseteq \Delta(G_2', D_2)$ is determined by setting the length of certain edges of G_2' to zero. Let $\phi\colon G_2' \to G_1'$ be the graph given by contracting exactly those edges and set $D_1 = \phi_* D_2$. Then there is a unique edge contraction $\phi_1\colon G_1 \to G$ such that $\phi_1 \circ \phi_{12} = \phi_2$ and the induced face morphism $\Delta(G_1', D_1) \hookrightarrow \Delta(G_2, D')$ has image F. This shows that $\Delta_{(G,d)}$ defines a colored polysimplicial complex.

Let D be an effective divisor of degree d on Γ. Since Γ is semistable, there is a unique model (G', D) of Γ that arises as a (possibly not unique) subdivision of G and such that for all vertices v of G' we have that either $v \in V(G)$ or $D(v) > 0$. Then (G', D) naturally defines a point in the relative interior of $\Delta(G', D)$. Conversely, given a point in the relative interior of the geometric realization of $\Delta(G', D)$, the geometric realization of (G', d) is equal to Γ and D naturally defines an effective divisor of degree d on Γ. Thus the geometric realization of $\Delta_{(G,d)}$ is in natural bijection with $\mathrm{Div}_d^+(\Gamma)$. □

Example 2.3.6. Consider the graph G in Figure 2.10. The spaces G^2 and G_2 are displayed

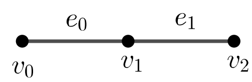


Figure 2.10: The graph G for Example 2.3.6

in Figures 2.11 and 2.12 respectively. In Figure 2.12, G_2 is displayed with the polysimplicial complex structure described in the proof of Proposition 2.3.5.

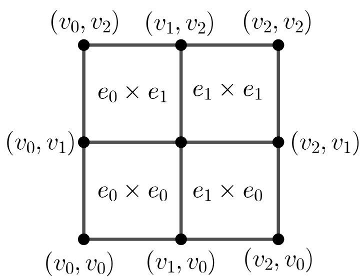


Figure 2.11: The space G^2 in Example 2.3.6

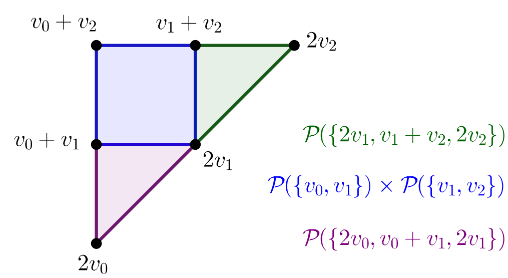
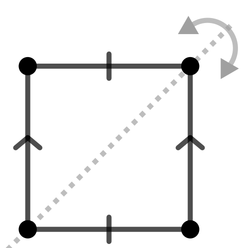
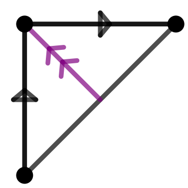


Figure 2.12: The space G_2 in Example 2.3.6

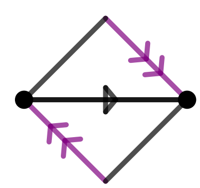
Example 2.3.7. Consider the metric graph S^1, the unit circle. Then $(S^1)_2$ is the Möbius band, as we see in Figure 2.13.



(a) $(S^1)^2$



(b) $(S^1)_2$



(c) Möbius band

Figure 2.13: We see $(S^1)_2$ is a Möbius strip through cutting and rearranging the pieces.

Example 2.3.8. Let G be the dumbell graph, or the chain of two loops. In Figure 2.14 we give the poset $\Xi(G)$. The polysimplicial complex $\Delta(G, 2)$ has 15 maximal cells, five of which are triangles and 10 of which are squares. It has 25 edges and 10 vertices.

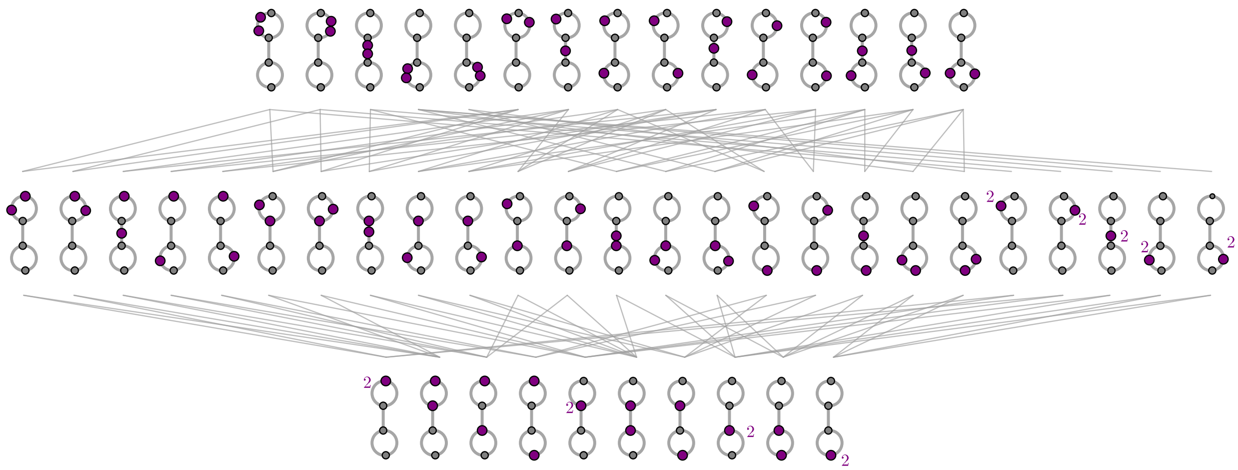


Figure 2.14: The poset $\Xi(G)$ for the dumbell graph G and $d = 2$.

Skeletons of Polystable Models

We now define polystable models of varieties and show how to take their Berkovich skeletons. It will turn out that the skeletons of polystable models of varieties will naturally have the structure of a colored polysimplicial complex. This may be viewed as a higher dimensional analogue of the theory developed in Section 1.3. The strategy is the same here as it was in the curve case: make a combinatorially tractable model, and then the Berkovich skeleton will be the dual complex of the special fiber.

Definition 2.3.9. Let X be a smooth variety over K. A *strictly polystable model of* X is a flat and separated scheme $\mathcal{X}$ over R, whose generic fiber is isomorphic to X and such that for every point x in the special fiber of $\mathcal{X}$ there is an open neighborhood $\mathcal{U}$ of x in $\mathcal{X}$ as well as an étale morphism $\gamma\colon \mathcal{U} \to \operatorname{Spec} A_1 \otimes_R \cdots \otimes_R A_r$ over R where A_i is of the form $R[t_1, \ldots, t_n]/(t_1 \cdots t_k - a)$ for $a \in R$.

Suppose that X is proper over K and that $\mathcal{X}$ is a proper polystable model of X. In [18] Berkovich constructed a strong deformation retraction $\rho_{\mathcal{X}}\colon \mathcal{X}^{an} \to \Sigma(\mathcal{X})$ onto a closed subset of $\mathcal{X}^{an}$ that naturally carries the structure of a colored polysimplicial complex, the *non-Archimedean skeleton* associated to $\mathcal{X}$. In this section we shall recall the basic properties of this construction. Our presentation is inspired by [64, Section 4] and [117].

Let $\mathcal{X}$ be a flat and separated model of X. Following [63, Section 4.9], we denote by $\mathcal{X}^\circ$ the analytic domain in X^{an} consisting of those points that naturally extend to the model $\mathcal{X}$.

If $\mathcal{X} = \operatorname{Spec} \mathcal{A}$ is affine, this means we consider only those seminorms on $A = \mathcal{A} \otimes K$ that are induced by multiplicative seminorms $|.|$ on $\mathcal{A}$ that are *bounded*, i.e. for which we have $|f|_x \leq 1$ for all $f \in \mathcal{A}$. Note that, if $\mathcal{X}$ is proper over R, then the valuative criterion for properness implies $\mathcal{X}^\circ = X^{an}$.

Tropicalization of a stable standard model

Let $n \geq 0$, $k \leq n$, and $a \in R$. We refer to the affine R-scheme $\mathcal{Z}(n,k,a) = \operatorname{Spec} A$ with

$$A = R[t_1, \ldots, t_n]/(t_1 \cdots t_k - a)$$

as a *stable standard model.* Consider the standard simplex (as in Equation 2.14)

$$\Delta(k,a) = \left\{ v \in \mathbb{R}^k_{\geq 0} \middle| v_1 + \ldots v_k = \operatorname{val}(a) \right\} .$$

There is a natural continuous tropicalization map

$$\operatorname{trop}_{n,k,a} \colon \mathcal{Z}(k,n,a)^\circ \longrightarrow \Delta(k,a)$$

given by

$$x \longmapsto \big(-\log |t_1|_x, \ldots, -\log |t_k|_x \big) .$$

This map is well-defined, since

$$-\log |t_1|_x + \ldots + \log |t_k|_x = -\log |t_1 \cdots t_{k_i}| = -\log |a| = \operatorname{val}(a) .$$

The skeleton of a stable standard model

The tropicalization map $\operatorname{trop}_{n,k,a}$ has a natural section $J_{n,k,a} \colon \Delta(k,a) \to \mathcal{Z}(n,k,a)^\circ$ given by sending $v \in \Delta(k,a)$ to the multiplicative seminorm given by

$$\sum_{\vec{l} \in \mathbb{N}^n} a_{\vec{l}} \big(t_1^{l_1} \cdots t_n^{l_n}\big) \longmapsto \max_{\vec{l} \in \mathbb{N}^n} \left(|a_{\vec{l}}| \cdot e^{-(l_1 v_1 + \cdots + l_k v_k)} \right) .$$

The section is well-defined, since

$$J_{n,k,a}(v)(t_1 \cdots t_n) = e^{-(v_1 + \cdots + v_k)} = e^{-\operatorname{val}(a)} = |a| .$$

The composition $\rho_{n,k,a} = J_{n,k,a} \circ \operatorname{trop}_{n,k,a}$ defines a retraction map

$$\rho_{n,k,a} \colon \mathcal{Z}(n,k,a)^\circ \longrightarrow \mathcal{Z}(n,k,a)^\circ$$

onto a closed subset of $\mathcal{Z}(n,k,a)^\circ$, the *non-Archimedean skeleton* $\Sigma(n,k,a)$ of $\mathcal{Z}(n,k,a)^\circ$.

Tropicalization of a polystable standard model

Write $\vec{n} = (n_1, \ldots, n_r)$, $\vec{k} = (k_1, \ldots, k_r)$ as well as $\vec{a} = (a_1, \ldots, a_r)$ so that $k_i \leq n_i$. A *polystable standard model* is an affine R-scheme of the form

$$\mathcal{Z}(\vec{n}, \vec{k}, \vec{a}) = \mathcal{Z}(n_1, k_1, a_1) \times_R \cdots \times_R \mathcal{Z}(n_r, k_r, a_r)$$

where each $\mathcal{Z}(n_i, k_i, a_i) = \operatorname{Spec} A_i$ with $A_i = R[t_1^{(i)}, \dots, t_{n_i}^{(i)}]/(t_1^{(i)} \cdots t_{k_i}^{(i)} - a_i)$ for $a_i \in R$. The colored polysimplex associated to $\mathcal{Z}(\vec{n}, \vec{k}, \vec{a})$ is defined to be (as in Equation 2.14)

$$\Delta(\vec{k}, \vec{a}) = \left\{ v \in \mathbb{R}_{\geq 0}^{k_1 + \dots + k_r} \middle| v_1^{(i)} + \ldots + v_{k_i}^{(i)} = \operatorname{val}(a_i) \text{ for all } i = 1, \dots r \right\} .$$

There is a natural continuous tropicalization map

$$\operatorname{trop}_{\vec{n},\vec{k},\vec{a}} \colon \mathcal{Z}(\vec{n}, \vec{k}, \vec{a})^\circ \longrightarrow \Delta(\vec{k}, \vec{a})$$

given by

$$x \longmapsto \big(-\log |t_1^{(i)}|_x, \dots, -\log |t_{k_i}^{(i)}|_x \big)_{i=1,\dots,r} .$$

It is well-defined, since

$$-\log |t_1^{(i)}|_x + \ldots + \log |t_{k_i}^{(i)}|_x = -\log |t_1^{(i)} \cdots t_{k_i}^{(i)}| = -\log |a_i| = \operatorname{val}(a_i) .$$

The skeleton of a polystable standard model

The tropicalization map has a natural section $J_{\vec{n},\vec{k},\vec{a}} \colon \Delta(\vec{k}, \vec{a}) \to \mathcal{Z}(\vec{n}, \vec{k}, a)^\circ$. This is given by associating to $v \in \Delta(\vec{k}, \vec{n})$ the bounded seminorm on $A_1 \otimes_R \cdots \otimes_R A_r$ given by

$$\sum_l a_l \big(f_1^{(l)} \otimes \cdots \otimes f_r^{(l)} \big) \longmapsto \max_l \Big(|a_l| \cdot J_{n_1,k_1,a_1}(v^{(1)})(f_1^{(l)}) \cdots J_{n_r,k_d,a_r}(v^{(r)})(f_r^{(l)}) \Big)$$

where $\sum_l a_l \big(f_1^{(l)} \otimes \cdots \otimes f_r^{(l)} \big)$ for $f_i \in A_i$ denotes a general element in $A_1 \otimes_R \cdots \otimes_R A_r$. The composition $\rho_{\vec{n},\vec{k},\vec{a}} := J_{\vec{n},\vec{k},\vec{a}} \circ \operatorname{trop}_{\vec{n},\vec{k},\vec{a}}$ defines a retraction map

$$\rho_{\vec{n},\vec{k},\vec{a}} \colon \mathcal{Z}(\vec{n}, \vec{k}, \vec{a})^\circ \longrightarrow \mathcal{Z}(\vec{n}, \vec{k}, \vec{a})^\circ$$

whose image is a closed subset in $\mathcal{Z}(\vec{n}, \vec{k}, \vec{a})^\circ$, the *non-Archimedean skeleton* $\Sigma(\vec{n}, \vec{k}, \vec{a})$ of $\mathcal{Z}(\vec{n}, \vec{k}, \vec{a})^\circ$.

Stratification of a polystable model

Given a strict polystable model $\mathcal{X}$ of X, the special fiber $\mathcal{X}_0$ admits a natural stratification by locally closed subsets, defined inductively as follows: We first write $\mathcal{X}_0$ as a disjoint union

$$\mathcal{X}_0 = \bigsqcup_{i=0}^{n} \mathcal{X}_0^{(i)} .$$

Let $\mathcal{X}_0^{(0)}$ be the open locus of regular points of $\mathcal{X}_0$ and let $\mathcal{X}_0^{(1)}$ be the open locus of regular points in $\mathcal{X}_0 - \mathcal{X}_0^{(0)}$. In general, given $\mathcal{X}_0^{(i)}$ for $i = 1, \dots, n$, we define $\mathcal{X}_0^{(i+1)}$ to be the open locus of regular points in

$$\mathcal{X}_0 - \big(\mathcal{X}_0^{(0)} \cup \cdots \cup \mathcal{X}_0^{(i)} \big) .$$

The subsets $\mathcal{X}_0^{(i)}$ are locally closed and smooth. We refer to the connected components of $\mathcal{X}_0^{(i)}$ as *the strata* of $\mathcal{X}_0$.

The skeleton of small open neighborhood

Let $\mathcal{X}$ be a strictly polystable model of X. For each stratum E, we set $\mathcal{X}_0(E)$ to be the union of all strata E' with $E \subseteq \overline{E'}$, i.e. if E is contained in the closure $\overline{E'}$ of E'. An open neighborhood $\mathcal{U}$ of a point $x \in E$ is said to be *small* if the special fiber of $\mathcal{U}$ is contained in $\mathcal{X}_0(E)$. We refer to a chart $\gamma\colon \mathcal{U} \to \mathcal{Z}(\vec{n}, \vec{k}, \vec{a})$ as in Definition 2.3.9 as *small* with respect to a stratum E if $\mathcal{U}$ is a small open neighborhood of a point in E and the image of E is contained in the closed stratum of $\mathcal{Z}(\vec{n}, \vec{k}, \vec{a})$.

Let $\mathcal{U}$ be a small open neighborhood in $\mathcal{X}$. In [18] it is shown that there is a retraction $\rho_{\mathcal{U}}\colon \mathcal{U}^\circ \to \mathcal{U}^\circ$ onto a closed subset $\Sigma(\mathcal{U})$ of $\mathcal{U}^\circ$ such that, whenever $\gamma\colon \mathcal{U} \to \mathcal{Z}(\vec{n}, \vec{k}, \vec{a})$ is small chart, the diagram

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{\rho_{\mathcal{U}}} & \mathcal{U}^\circ \\ \downarrow{\scriptstyle \gamma^\circ} & & \downarrow{\scriptstyle \gamma^\circ} \\ \mathcal{Z}(\vec{n}, \vec{k}, \vec{a})^\circ & \xrightarrow{\rho_{\vec{n},\vec{k},\vec{a}}} & \mathcal{Z}(\vec{n}, \vec{k}, \vec{a})^\circ \end{array}$$

commutes and the restriction of γ° to $\Sigma(\mathcal{U})$ induces a homeomorphism $\Sigma(\mathcal{U}) \xrightarrow{\sim} \Sigma(\vec{n}, \vec{k}, \vec{a})$.

The polysimplicial complex of a strict polystable model

Let X be a smooth variety over K and let $\mathcal{X}$ be a strictly polystable model of X over the valuation ring R. Denote by $\Xi(\mathcal{X})$ the set of all strata. It naturally carries a partial order $\prec$ that is given by $E \prec E'$ if and only $E' \subseteq \overline{E}$, i.e. if E' is contained in the closure $\overline{E}$ of E. Given two small charts $\gamma\colon \mathcal{U} \to \mathcal{Z}(\vec{n}, \vec{k}, \vec{a})$ and $\gamma'\colon \mathcal{U}' \to \mathcal{Z}(\vec{n}', \vec{k}', \vec{a}')$ around a stratum $E \in \Xi(\mathcal{X})$, note that we have $\vec{n} = \vec{n}'$, $\vec{k} = \vec{k}'$, and $\mathrm{val}(a_i) = \mathrm{val}(a_i')$ for all $i = 1, \ldots, r$. Therefore we may define the *colored polysimplex* $\Delta(E)$ associated to a stratum $E \in \Xi(\mathcal{X})$ to be (as in Equation 2.14)

$$\Delta(E) := \Delta(\vec{k}, \vec{a})$$

for a small chart $\gamma\colon \mathcal{U} \to \mathcal{Z}(\vec{n}, \vec{k}, \vec{a})$ around E.

Proposition 2.3.10 ([28])**.** The association $E \mapsto \Delta(E)$ defines a colored polysimplicial complex $\Delta(\mathcal{X})$ indexed by the poset $\Xi(\mathcal{X})$.

Proof. We show that there is a one-to-one correspondence between the faces of $\Delta(E)$ and the strata $E' \prec E$ of $\mathcal{X}_0$. In fact, let $\gamma\colon \mathcal{U} \to \mathcal{Z}(\vec{n}, \vec{k}, \vec{a})$ be a small chart around E. A face of $\Delta(E)$ is given by equations $v_j^{(i)} = 0$. Denote by $x_j^{(i)}$ the pullback of the corresponding $t_j^{(i)}$ in A_i. Then the corresponding stratum is the unique closed stratum of the open subset $\mathcal{U}_{x_j^{(i)}}$ of $\mathcal{U}$ on which the $x_j^{(i)}$ are invertible. Conversely, a stratum $E' \prec E$ as the unique closed stratum of $\mathcal{X}(E')$ and $\mathcal{U}(E')$ is determined by a collection of $x_j^{(i)}$. This, in turn, determines a face of $\Delta(E)$. $\square$

The skeleton of a polystable model

Let X be a smooth variety over K and let $\mathcal{X}$ be a strictly polystable model of X over the valuation ring R. In [18] Berkovich has shown that the retraction maps $\rho_{\mathcal{U}}$ on small open subsets naturally descend to a retraction map $\rho_{\mathcal{X}}\colon \mathcal{X}^\circ \to \mathcal{X}^\circ$ such that the following diagram commutes.

$$\begin{array}{ccc} \mathcal{U}^\circ & \xrightarrow{\rho_{\mathcal{U}}} & \mathcal{U}^\circ \\ \subseteq\uparrow & & \uparrow\subseteq \\ \mathcal{X}^\circ & \xrightarrow{\rho_{\mathcal{X}}} & \mathcal{X}^\circ \end{array}$$

The image of $\rho_{\mathcal{X}}$ in $\mathcal{X}^\circ$ is the *non-Archimedean skeleton* $\Sigma(\mathcal{X})$ of $\mathcal{X}^\circ$. From this construction we immediately obtain the following combinatorial characterization of $\Sigma(\mathcal{X})$.

Proposition 2.3.11 ([28])**.** The skeleton $\Sigma(\mathcal{X})$ is naturally homeomorphic to the geometric realization of the colored polysimplicial complex $\Delta(\mathcal{X})$.

In fact, the retraction map $\rho_{\mathcal{X}}$ is actually a strong deformation retraction onto $\Sigma(\mathcal{X})$ using the natural torus operation on a polystable standard model and formally lifting them to $\mathcal{X}$. Since this aspect of the construction will play no further role in the remainder of this article, we refer the avid reader to [18] for details.

Skeletons of Symmetric Powers

In order to tropicalize X_d, we must first make a polystable model of X_d. Then, we will apply the methods from the previous section to prove Theorem 2.3.1.

A polystable model of X_d

Let $X \to S$ be a scheme over k. A *relative effective Cartier divisor* D on X over S is a closed subscheme of X that is flat over S for which the ideal sheaf $I(D)$ is a line bundle.

Let $g \geq 2$ and $d \geq 0$. In [99, Section 2], the authors have introduced a smooth and proper Deligne-Mumford stack $\overline{\mathcal{D}iv}_{g,d}$ with (stack-theoretically) normal crossings boundary that compactifies the d-th symmetric power of the universal curve $\mathcal{X}_g$ over $\mathcal{M}_g$ (is a special instance of the moduli space of stable quotients, as explained in [89, Section 4]). Its objects are pairs consisting of a proper and flat family $\pi\colon X \to S$ of semistable curves of genus g over a scheme S together with a relative effective Cartier divisor D on X over S such that

- the support of D does not meet the nodes of X_s in every fiber X_s over $s \in S$; and
- the twisted canonical divisor $K_X + D$ is relatively ample.

Notice that, by Kleiman's criterion for ampleness, the twisted canonical divisor $K_X + D$ is relatively ample over S if and only if, for every point s of S, the support of D_s has non-empty intersection with every exceptional component of $\mathcal{X}$.

There is a natural morphism $\overline{\mathcal{D}iv}_{g,d} \to \overline{\mathcal{M}}_g$ given by associating to $(X \to S, D)$ the stabilization $X^{st} \to S$ of $X \to S$. Notice that the restriction $\mathcal{D}iv_{g,d}$ of $\overline{\mathcal{D}iv}_g$ to the locus $\mathcal{M}_g$ of smooth curves is nothing but the d-th symmetric power of the universal curve over $\mathcal{M}_g$.

Let X be a smooth projective curve of genus $g \geq 1$ over K and suppose that $\mathcal{X}$ is a strictly semistable model of X over $\operatorname{Spec} R$. Define $\mathcal{X}_d$ to be the category (fibered in sets over $\mathbf{Sch}_R$) whose objects are tuples $(\mathcal{X}', \mathcal{D})$ consisting of a semistable model $\mathcal{X}' \to S$ of $\mathcal{X}_S$ and a relative effective Cartier divisor $\mathcal{D}$ on $\mathcal{X}'_S = \mathcal{X}' \times_R S$ such that for every point $s \in S$

- the fiber $\mathcal{D}_s$ does not meet the singularities of $\mathcal{X}'_s$; and
- the support of D intersects every exceptional component of $\mathcal{X}'_s$.

Proposition 2.3.12 ([28])**.** The functor $\mathcal{X}_d$ is representable by an R-scheme (also denoted by $\mathcal{X}_d$) that is a proper polystable model of X_d over R.

Proof. Suppose first that $g \geq 2$. In this case, the datum of a strictly semistable model $\mathcal{X}$ over R corresponds to a morphism $\operatorname{Spec} R \to \overline{\mathcal{M}}_g^{ss} \to \overline{\mathcal{M}}_g$, where $\overline{\mathcal{M}}_g^{ss}$ is the moduli stack of semistable curves of genus g and the second arrow is the stabilization map. Observe that $\mathcal{X}_d$ is the fibered product

$$\mathcal{X}_d := \operatorname{Spec} R \times_{\overline{\mathcal{M}}_g} \overline{\mathcal{D}iv}_{g,d} \ .$$

Thus $\mathcal{X}_d$ is representable by a scheme that is flat and proper over R.

For a K-scheme S the set $\mathcal{X}_d(S)$ consists precisely of the relative effective Cartier divisors of degree d on $X_S = X \times_K S$ and, by [98, Theorem 3.13], the generic fiber of $\mathcal{X}_d$ is nothing but the d-th symmetric power X_d of X. So the generic fiber of $\mathcal{X}_d$ is isomorphic X_d.

Consider the nodes of $\mathcal{X}$ given by local equations $x_i y_i = a_i$ for $a_i \in R$ (for $i = 1, \dots, r$). Write the nodes in $\mathcal{X}'$ above $x_i y_i = a_i$ as $x_j^{(i)} y_j^{(i)} = t_j^{(i)}$ for $j = 1, \dots, k_i$ and coordinates $t_j^{(i)}$ on $\mathcal{X}_d$. In this case we have $a_i = t_1^{(i)} \cdots t_{k_i}^{(i)}$ on $\mathcal{X}_d$. Since the components of $\mathcal{X}_0$ do not have self-intersection, these coordinates can be chosen on a Zariski open neighborhood $\mathcal{U}$ and we may add further coordinates $t_{k_i+1}^{(i)} \dots, t_{n_i}^{(i)}$ (for $i = 1, \dots, r$) so that, possibly after shrinking $\mathcal{U}$, they define a small étale chart $\gamma \colon \mathcal{U} \to \mathcal{Z}(\vec{n}, \vec{k}, \vec{a})$.

Suppose now that $g = 1$. If $\mathcal{X}$ has a section, we may construct $\mathcal{X}_d$ as the fibered product

$$\operatorname{Spec} R \times_{\overline{\mathcal{M}}_{g,1}} \overline{\mathcal{D}iv}_{g,1,d}$$

where $\overline{\mathcal{D}iv}_{g,n,d}$ is a generalization of $\overline{\mathcal{D}iv}_{g,d}$ to $\overline{\mathcal{M}}_{g,n}$, as introduced in [89]. The morphism $\operatorname{Spec} R \to \overline{\mathcal{M}}_{g,1}$ is determined by a choice of a section that does not influence the construction of $\mathcal{X}_d$. The rest of the above proof goes through verbatim. If $\mathcal{X}$ does not have a section, we may choose a finite extension K' of K such that $\mathcal{X}_{R'}$ does have a section. We apply the above construction and show that it naturally descends to K. □

We explicitly point out that, despite its suggestive notation, $\mathcal{X}_d$ is not the quotient of $\mathcal{X} \times_R \cdots \times_R \mathcal{X}$ by the operation of S_d. While not being smooth over R, the R-scheme $\mathcal{X}_d$ only admits at most toroidal singularities over R, since we are allowed to perform a weighted blow-up of the special fiber of $\mathcal{X}$ whenever the support of D is at risk of meeting the singularities in $\mathcal{X}_s$.

Stratification by dual graphs

Consider a point in the special fiber of $\mathcal{X}_d$. It is given by a pair (X_0', D) where X_0' is a strictly semistable curve together with a morphism $X_0' \to X_0$ and D is an effective divisor on X_0' whose support is contained in the non-singular locus of X_0' and which has positive degree on every exceptional component of X_0'.

We associate to (X_0', D) a dual stable pair $\big(G', \mathrm{mdeg}(D)\big)$ over G as follows: The graph G' is the weighted dual graph of X_0'. Its vertices v correspond to the components X_v' of X_0' and it contains an edge e emanating from two vertices v and v' for every node connecting the two components X_v' and $X_{v'}'$. It is endowed with a natural vertex weight $h\colon V(G') \to \mathbb{Z}_{\geq 0}$ given by $h(v) = g(X_v)$, the genus of the component X_v. Finally the degree of the restriction of D to every component X_v defines a divisor

$$\mathrm{mdeg}(D) = \sum_{v \in V(G')} \deg\big(D|_{X_v}\big) \cdot v$$

on G' supported on the vertices of G', the *multidegree of* D.

The graph G' is naturally a subdivision of G, the dual graph of $\mathcal{X}_0$. The condition that $K_{X_0'} + D$ has non-empty intersection with every exceptional component of X_0' over X_0 is equivalent to the condition that $D(v) > 0$ for every exceptional vertex v of G' over G, i.e. to the condition that $(G', \mathrm{mdeg}(D))$ is a stable pair over G.

Proposition 2.3.13 ([28])**.** The colored polysimplicial complex Γ_d is isomorphic to $\Delta(\mathcal{X}_d)$.

Proof. The strata of the special fiber of $\mathcal{X}_d$ are precisely the locally closed subsets on which the dual graphs are constant. In fact, the smooth locus $\mathcal{X}_{d,0}^{(0)}$ of $\mathcal{X}_{d,0}$ is the exactly locus of stable pairs (X_0', D) for which X_0' is isomorphic to X_0, which translates into the dual graph G' of X_0' being isomorphic to the dual graph G of X_0. The different strata in $\mathcal{X}_{d,0}^{(0)}$ are distinguished by the multidegree of D.

Similarly, for $i = 0, \ldots, d-1$ the regular locus of $\mathcal{X}_{d,0}^{(i+1)}$ of

$$\mathcal{X}_{d,0} - \big(\mathcal{X}_{d,0}^{(0)} \cup \cdots \cup \mathcal{X}_{d,0}^{(i)}\big)$$

corresponds exactly to the locus of stable pairs (X_0', D) that contain $i+1$ exceptional components. This translates into the condition that the dual graph contains exactly $i+1$ exceptional vertices over G'. The different strata, again, are distinguished by the multidegree of D.

Moreover, notice that for every stable pair (G', D) the locus $E_{(G',D)}$ of points in $\mathcal{X}_{d,0}$ whose dual pair is (G', D) is non-empty. A stratum $E_{(G'',D)}$ is in the closure of a stratum $E_{(G',D')}$ if and only if there is a weighted edge contraction $\pi\colon G'' \to G'$ over G for which $\pi_* D' = D$. So there is an order-preserving one-to-one correspondence between the set $\Xi(\mathcal{X}_d)$ of strata of $\mathcal{X}_{d,0}$ and the set $\Xi(G)$ of stable pairs (G', D) over G.

Finally, consider the nodes of $\mathcal{X}$ given by by local equations $x_i y_i = a_i$ for $a_i \in R$ (for $i = 1, \ldots, r$). Write the nodes in $\mathcal{X}'$ above $x_i y_i = a_i$ as $x_j^{(i)} y_j^{(i)} = t_j^{(i)}$ for $j = 1, \ldots, k_i$ and coordinates $t_j^{(i)}$ on $\mathcal{X}_d$ and recall that in this case we have $a_i = t_1^{(i)} \cdots t_{k_i}^{(i)}$ on $\mathcal{X}_d$. From this description we see that the colored polysimplex (as in Equation 2.15) $\Delta(E_{(G',D)})$ of the stratum $E_{(G',D)}$ is equal to $\Delta(\vec{k}, \vec{a})$.

On the other hand, the k_i are precisely the number of exceptional vertices over an edge e_i of G and the edge length $|e_i|$ of e_i is equal to $\mathrm{val}(a_i)$. So the colored polysimplex $\Delta(G, D') = \Delta\big(\vec{k}, \mathrm{val}(\vec{a})\big)$ of a stable pair (G', D) is equal to $\Delta(\vec{k}, \vec{a}) = \Delta(E_{(G',D)})$. This identification naturally commutes with the face morphisms induced by $E_{(G',D)} \prec E_{(G'',D)}$ and $(G'', D') \to (G', D)$ respectively and so we have found a canonical isomorphism between Γ_d and $\Delta(\mathcal{X}_d)$. $\square$

The process of tropicalization

Let X be a smooth projective curve over K and suppose there is a fixed semistable model $\mathcal{X}$ of X over R. By the semistable reduction theorem (Theorem 1.1.12), we can always find $\mathcal{X}$ if we are willing to replace X by its base change to a finite extension of K.

Denote by Γ the dual tropical curve of $\mathcal{X}$. We now define the tropicalization map

$$\mathrm{trop}_{X_d} \colon X_d^{an} \longrightarrow \Gamma_d \ .$$

A point x in X_d^{an} can be represented by a morphism $\operatorname{Spec} L \to X_d$ for a non-Archimedean extension L of K. This, in turn, corresponds to an effective Cartier divisor D on X_L. Since $\mathcal{X}_d$ is proper over $\operatorname{Spec} R$, there is a unique semistable model $\mathcal{X}' \to \mathcal{X}$ and a relative effective Cartier divisor $\mathcal{D}$ in $\mathcal{X}'$ such that

- the generic fiber of $\mathcal{D}$ is equal to D,
- the support $\mathrm{Supp}(\mathcal{D}_0)$ in the special fiber does not meet the nodes of $\mathcal{X}_0'$, and
- $\mathrm{Supp}(\mathcal{D}_0) \cap E \neq \emptyset$ for every exceptional component E of $\mathcal{X}_0'$ over $\mathcal{X}_0$.

We may now define $\mathrm{trop}_{X_d}(x)$ to be the divisor that arises as the multidegree of $\mathcal{D}_0$ on Γ. It is naturally supported on the model G' of Γ given by the dual graph of $\mathcal{X}_0'$.

A posteriori, Theorem 2.3.1 implies that the construction of trop_{X_d} does not depend on any of the above choices and that the tropicalization map is invariant under base change. In other words, given a non-Archimedean extension K' of K the dual tropical curve $\Gamma_{X_{K'}}$ is naturally isometric to Γ_X and the natural diagram commutes:

$$\begin{array}{ccc} X_{K'}^{an} & \xrightarrow{\ \mathrm{trop}_{X_{K',d}}\ } & \big(\Gamma_{X_{K'}}\big)_d \\ \big\downarrow & & \big\downarrow \simeq \\ X^{an} & \xrightarrow[\ \mathrm{trop}_{X_d}\ ]{} & \Gamma_d \end{array}$$

The specialization map

Denote by $\mathrm{Div}_{K,d}(X)$ the group of K-*split* divisors on X, which may be written $\sum_{i=1}^{n} a_i p_i$ for points $p_i \in X(K)$. Write $\rho_X\colon X^{an} \to \Gamma$ the retraction of X^{an} to $\Gamma_X \simeq \Gamma$, which can be thought of as the skeleton of X^{an}. In [11], Baker constructs a *specialization homomorphism*

$$\rho_{X,*}\colon\ \mathrm{Div}_{K,d}(X) \longrightarrow \mathrm{Div}_d(\Gamma)\ .$$

It is defined by sending a K-split divisor $D = \sum_{i=1}^{n} a_i p_i$ on X to the following divisor on Γ:

$$\rho_{X,*}(D) = \sum_{i=1}^{n} a_i \rho_X(p_i).$$

Recall that, if Y is a scheme locally of finite type over K, then there is a natural injective map $i\colon Y(K) \hookrightarrow Y^{an}$, whose image is dense if K is algebraically closed. On an affine patch $U = \operatorname{Spec} A$ it is given by associating to a K-rational point the multiplicative seminorm

$$A \longrightarrow K \xrightarrow{|.|_K} \mathbb{R}_{\geq 0}$$

on A. So, in particular, there is a natural injective map

$$i\colon\ \mathrm{Div}^+_{K,d}(X) \hookrightarrow X_d^{an}$$

whose image is dense in X_d^{an} if K is algebraically closed.

Proposition 2.3.14 ([28])**.** Given a K-split effective divisor D on X of degree d, we have

$$\mathrm{trop}_{X_d}\big(i(D)\big) = \rho_{X,*}(D)\ .$$

In other words, the natural diagram commutes.

$$\begin{array}{ccc} \mathrm{Div}^+_{K,d}(X) & \xrightarrow{\rho_{X,*}} & \mathrm{Div}^+_d(\Gamma) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \simeq} \\ X_d^{an} & \xrightarrow{\mathrm{trop}_{X_d}} & \Gamma_d \end{array}$$

Proof of Proposition 2.3.14. Suppose first that $K = \overline{K}$ is algebraically closed. Let $D = \sum_{i=1}^{n} a_i p_i$ be an effective K-split divisor on X. Since $\mathcal{X}_d$ is proper over R, we find a unique semistable model $\mathcal{X}'$ over $\mathcal{X}$ as well as a relative effective Cartier divisor $\mathcal{D}$ on $\mathcal{X}'$ such that

- the generic fiber of $\mathcal{D}$ is equal to D,
- the support $\mathrm{Supp}(\mathcal{D}_0)$ in the special fiber does not meet the nodes of $\mathcal{X}'_0$, and
- $\mathrm{Supp}(\mathcal{D}_0) \cap E \neq \emptyset$ for every exceptional component E of $\mathcal{X}'_0$ over $\mathcal{X}_0$.

By [15, Theorem 4.11] (also see [19, Theorem 4.3.1]), the semistable model $\mathcal{X}'$ gives rise to a semistable vertex set V in X^{an}, i.e. a set of points v in X^{an} whose complement is a collection of closed pointed discs and annuli. The vertices v are precisely the vertices in the dual graph of $\mathcal{X}_0$ and the edges of the dual graph correspond to the annuli in $X^{an} - V$.

Let $r\colon X^{an} \to X_k$ be the reduction map. Then the pointed discs in X^{an} are given by $B(v) = r^{-1}(U_v) - V$ where U_v is the open subset of a component in $\mathcal{X}'_0$ given by removing all of its nodes, and the annuli are given by $B(e) = r^{-1}(x_e)$, where the x_e are the nodes of $\mathcal{X}'_0$. The restriction of the retraction map ρ_X to a pointed disc $B(v)$ shrinks all points $B(v)$ to the corresponding point $v \in \Gamma$ and its restriction to an annulus $B(e)$ is given by the retraction of the annulus to its skeleton which is isometric to e.

So, if the point p_i extends to a component X'_{0,v_i} of $\mathcal{X}'_0$ via $\mathcal{D}$, its reduction is an element of U_{v_i}. Therefore p_i is a point of $B(v_i)$ and thus $\rho_X(p_i) = v_i$. So, by linearity, we have:

$$\rho_{X,*}(D) = \sum_{i=1}^{n} a_i v_i = \sum_{v \in V(G')} \deg\left(D|_{X_v}\right) \cdot v = \operatorname{mdeg}_{\mathcal{X}'_0}(\mathcal{D}_0)\ .$$

The general case, when K may not be algebraically closed, follows from the invariance of $\operatorname{trop}_{X_d}$ and ρ_X under base change by a non-Archimedean extension K' of K. □

Proof of Theorem 2.3.1

We conclude this section with the proof of Theorem 2.3.1.

Proof of Theorem 2.3.1. By Propositions 2.3.5 and 2.3.13, there is a natural isomorphism $\mu_{X_d}\colon \Gamma_d \xrightarrow{\sim} \Sigma(X_d)$. What remains to show is that the following diagram commutes.

$$\begin{array}{ccc} X_d^{an} & \xrightarrow{\operatorname{trop}_{X_d}} & \Gamma_d \\ & \searrow^{\rho_{X_d}} & \downarrow{\scriptstyle \mu_{X_d}}\ \wr \\ & & \Sigma(X_d) \end{array}$$

Consider the nodes of $\mathcal{X}$ given by by local equations $x_i y_i = a_i$ for $a_i \in R$ (for $i = 1, \ldots, r$). Write the nodes in $\mathcal{X}'$ above $x_i y_i = a_i$ as $x_j^{(i)} y_j^{(i)} = t_j^{(i)}$ for $j = 1, \ldots, k_i$ and coordinates $t_j^{(i)}$ on $\mathcal{X}_d$. In this case we have $a_i = t_1^{(i)} \cdots t_{k_i}^{(i)}$ on $\mathcal{X}_d$. By the subsections "Tropicalization of a polystable standard model," "The skeleton of a polystable standard model," and "The skeleton of small open neighborhood," the retraction to the skeleton is given by sending $x \in X_d^{an}$ to $\left(-\log|t_1^{(i)}|_x, \ldots, -\log|t_{k_i}^{(i)}|_x\right)_{i=1}^{r}$ in $\Delta(\vec{k}, \vec{a})$. These are precisely the edge lengths of the dual graph of $\mathcal{X}'$ that subdivides the edges of Γ_X and this shows that the above diagram commutes. □

Functoriality

A surprisingly useful consequence of Theorem 2.3.1 is that the continuity of ρ_{X_d} implies the continuity of the tropicalization trop_{X_d}. This allows us to deduce a collection of functoriality results from the linearity of Baker's specialization map in [11] and from the compatibility of the process of tropicalization with the Abel-Jacobi map proved in [16, Theorem 1.3]. Moreover, the usual arguments from the proof of [56, Theorem 2.2.7] (also see [63, Proposition 3.5]) immediately imply Corollary 2.3.15.

Corollary 2.3.15 ([28]). *If $Y \subseteq X_d$ is connected, then the tropicalization $\mathrm{trop}_{X_d}(Y)$ is connected as well.*

There are two classes of tautological maps associated to symmetric powers:

(i) For $\mu = (m_1, \ldots, m_n) \in \mathbb{Z}^n_{\geq 0}$ and $\delta = (d_1, \ldots, d_n) \in \mathbb{Z}^n_{\geq 0}$ such that $m_1 d_1 + \ldots + m_n d_n = d$, we have the *diagonal morphism*

$$\begin{aligned} \phi_{\mu,\delta} \colon X_{d_1} \times \cdots \times X_{d_n} &\longrightarrow X_d \\ (D_1, \ldots, D_n) &\longmapsto m_1 D_1 + \ldots m_n D_n \ . \end{aligned}$$

(ii) For $d \geq 0$ we have the *Abel-Jacobi map*

$$\begin{aligned} \alpha_d \colon X_d &\longrightarrow \mathrm{Pic}_d(X) \\ D &\longmapsto \mathcal{O}_X(D) \ . \end{aligned}$$

In this section, we show that the process of tropicalization naturally commutes with both classes of morphisms.

Diagonal morphisms

Let $\mu = (m_1, \ldots, m_n) \in \mathbb{Z}^n_{\geq 0}$ and $\delta = (d_1, \ldots, d_n) \in \mathbb{Z}^n_{\geq 0}$ such that $m_1 d_1 + \ldots + m_n d_n = d$. Define the *tropical diagonal map* $\phi^{trop}_{\mu,\delta} \colon \Gamma_{d_1} \times \cdots \times \Gamma_{d_n} \to \Gamma_d$ by the association

$$(D_1, \ldots, D_n) \longmapsto m_1 D_1 + \ldots + m_n D_n \ .$$

Proposition 2.3.16 ([28]). The tropical diagonal map is a morphism of colored polysimplicial complexes that makes the diagram commute.

$$\begin{array}{ccc} X^{an}_{d_1} \times \cdots \times X^{an}_{d_n} & \xrightarrow{\mathrm{trop}_{X_{d_1}} \times \cdots \times \mathrm{trop}_{X_{d_n}}} & \Gamma_{d_1} \times \cdots \times \Gamma_{d_n} \\ \downarrow \phi^{an}_{\mu,\delta} & & \downarrow \phi^{trop}_{\mu,\delta} \\ X^{an}_d & \xrightarrow{\mathrm{trop}_{X_d}} & \Gamma_d \end{array}$$

Proof. The linearity of the specialization map implies the commutativity of

$$\begin{array}{ccc} \mathrm{Div}^+_{K,d_1}(X) \times \cdots \times \mathrm{Div}^+_{K,d_n}(X) & \xrightarrow{\rho_{X,*}\times\cdots\times\rho_{X,*}} & \mathrm{Div}^+_{d_1}(\Gamma) \times \cdots \times \mathrm{Div}^+_{d_n}(\Gamma) \\ \big\downarrow \phi_{\mu,\delta} & & \big\downarrow \phi^{trop}_{\mu,\delta} \\ \mathrm{Div}^+_{K,d}(X) & \xrightarrow{\rho_{X,*}} & \mathrm{Div}^*_d(\Gamma)\ . \end{array}$$

Suppose first that $K = \overline{K}$ is algebraically closed. Then the monoids $\mathrm{Div}^+_{K,d_1}(X) \times \cdots \times \mathrm{Div}^+_{K,d_n}(X)$ and $\mathrm{Div}^+_{K,d}(X)$ of effective divisors are dense in $X^{an}_{d_1} \times \cdots \times X^{an}_{d_n}$ and X^{an}_d respectively. Therefore the continuity of $\mathrm{trop}_{X_{d_1}} \times \cdots \times \mathrm{trop}_{X_{d_n}}$ and trop_{X_d} (coming from Theorem 2.3.1) together with Proposition 2.3.14 implies the claim.

The general case, when K may not be algebraically closed, follows from the compatibility of the tropicalization map with base changes by non-Archimedean extensions K' of K. □

Abel-Jacobi map

Denote by $\mathrm{Rat}(\Gamma)$ the abelian group of *rational functions* on Γ, i.e. the group of continuous piecewise integer linear functions on Γ. There is a natural homomorphism

$$\begin{aligned} \mathrm{Div}\colon\ \mathrm{Rat}(\Gamma) &\longrightarrow \mathrm{Div}(\Gamma) \\ f &\longmapsto \sum_{p\in\Gamma} \mathrm{ord}_p(f)\cdot p \end{aligned}$$

where $\mathrm{ord}_p(f)$ denotes the sum of all outgoing slopes at the point p. Its image is the subgroup $\mathrm{PDiv}(\Gamma)$ of principal divisors in $\mathrm{Div}(\Gamma)$. One can verify that $\mathrm{PDiv}(\Gamma)$ is, in fact, a subgroup of $\mathrm{Div}_0(\Gamma)$. The *Picard group* $\mathrm{Pic}(\Gamma)$ is defined to be the quotient $\mathrm{Div}(\Gamma)/\,\mathrm{PDiv}(\Gamma)$. Denote the image of a divisor D on Γ in $\mathrm{Pic}(\Gamma)$ by $[D]$.

Since $\mathrm{PDiv}(\Gamma)$ is actually a subgroup of $\mathrm{Div}_0(\Gamma)$, the quotient respects degrees and $\mathrm{Pic}(\Gamma)$ naturally decomposes into a disjoint union of union of $\mathrm{Pic}_d(\Gamma)$, each of which is naturally a torsor over $\mathrm{Pic}_0(\Gamma)$. By the tropical Abel-Jacobi Theorem [97, Theorem 6.2] and [12, Theorem 3.4], the Picard group naturally carries the structure of a principally polarized tropical abelian variety.

Let X be a smooth projective curve over K. In [16, Theorem 1.3], Baker and Rabinoff show that the non-Archimedean skeleton $\Sigma(\mathrm{Pic}_d(X))$ of $\mathrm{Pic}_d(X)^{an}$ is naturally isomorphic (as a principally polarized tropical abelian variety) to the Picard variety $\mathrm{Pic}_d(\Gamma)$ and that the continuous retraction $\rho_{\mathrm{Pic}_d(X)}\colon\ \mathrm{Pic}_d(X)^{an} \to \Sigma(\mathrm{Pic}_d(X))$ to the skeleton naturally commutes with the tropical Abel-Jacobi map (see Definition 2.1.13) $\alpha_q\colon X \to \mathrm{Pic}_0(X)$ given by $p \mapsto [p-q]$ for a fixed point q of X. We expand on their result in the following Theorem 2.3.17.

Theorem 2.3.17 ([28])**.** *For $d \geq 0$, the* tropical Abel-Jacobi map $\alpha^{trop}_d\colon \Gamma_d \to \mathrm{Pic}_d(\Gamma)$ *given by the association $D \mapsto [D]$ naturally makes the following diagram commute:*

$$\begin{array}{ccccc} X_d^{an} & & \xrightarrow{\mathrm{trop}_{X_d}} & & \Gamma_d \\ \downarrow{\scriptstyle \alpha_d^{an}} & & & & \downarrow{\scriptstyle \alpha_d^{trop}} \\ \mathrm{Pic}_d(X)^{an} & \xrightarrow{\rho_{\mathrm{Pic}_d(X)}} & \Sigma(\mathrm{Pic}_d(X)) & \xrightarrow{\sim} & \mathrm{Pic}_d(\Gamma) \end{array}$$

We remark that a version of Theorem 2.3.17 has also appeared in [114, Section 7]. We include the proof here for completeness.

Proof of Theorem 2.3.17. Suppose first that $K = \overline{K}$ is algebraically closed. There is a natural homomorphism

$$\mathrm{trop}\colon \mathrm{Rat}(X)^* \longrightarrow \mathrm{Rat}(\Gamma)$$

that is given by sending a non-zero rational function $f \in \mathrm{Rat}(X)^*$ to the map $x \mapsto -\log|f|_x$ on Γ, thought of as the non-Archimedean skeleton of X^{an}. Since K is algebraically closed, we have $\mathrm{Div}(X) = \mathrm{Div}_K(X)$. By the slope formula [14, Theorem 5.14], we have $\mathrm{Div}(\mathrm{trop}(f)) = \rho_{X,*}(\mathrm{Div}(f))$ where $\rho_{X,*}$ is the specialization map $\rho_{X,*}\colon \mathrm{Div}(X) \simeq \mathrm{Div}_K(X) \to \mathrm{Div}(\Gamma)$.Therefore the specialization map descends to a homomorphism

$$\rho_{X,*}\colon \mathrm{Pic}_d(X) \longrightarrow \mathrm{Pic}_d(\Gamma)$$

and this immediately implies that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Div}_d^+(X) & \xrightarrow{\rho_{X,*}} & \mathrm{Div}_d^+(\Gamma) \\ \downarrow{\scriptstyle \alpha_d} & & \downarrow{\scriptstyle \alpha_d^{trop}} \\ \mathrm{Pic}_d(X) & \xrightarrow{\rho_{X,*}} & \mathrm{Pic}_d(\Gamma) \end{array} \tag{2.16}$$

In [16, Proposition 5.3], the authors show that the Picard group $\mathrm{Pic}_d(\Gamma)$ is naturally isomorphic (as a principally polarized tropical abelian variety) to the non-Archimedean skeleton $\Sigma(\mathrm{Pic}_d(X))$ of $\mathrm{Pic}_d(X)^{an}$ such that the induced diagram commutes.

$$\begin{array}{ccc} \mathrm{Pic}_d(X) & \xrightarrow{\rho_{X,*}} & \mathrm{Pic}_d(\Gamma) \\ \downarrow{\scriptstyle \subseteq} & & \downarrow{\scriptstyle \simeq} \\ \mathrm{Pic}_d(X)^{an} & \xrightarrow{\rho_{\mathrm{Pic}_d(X)}} & \Sigma(\mathrm{Pic}_d(X)) \end{array}$$

In fact, Baker and Rabinoff only show this statement for $\mathrm{Pic}_0(X)$, but since K is algebraically closed, we may choose a point $p \in X(K)$ and identify $\mathrm{Pic}_d(X)$ with $\mathrm{Pic}_0(X)$.

Since K is algebraically closed, both $\mathrm{Div}_d^+(X)$ and $\mathrm{Pic}_d(X)$ are dense in X_d^{an} and $\mathrm{Pic}_d(X)^{an}$. Therefore, by Proposition 2.3.14 above, since the maps $\rho_{\mathcal{X}_d}$, $\rho_{\mathrm{Pic}_d(X)}$, and α_d are all continuous, the commutativity of diagram (2.16) implies that the following commutes.

$$\begin{array}{ccccc} X_d^{an} & & \xrightarrow{\mathrm{trop}_{X_d}} & & \Gamma_d \\ \downarrow{\scriptstyle \alpha_d^{an}} & & & & \downarrow{\scriptstyle \alpha_d^{trop}} \\ \mathrm{Pic}_d(X)^{an} & \xrightarrow{\rho_{\mathrm{Pic}_d(X)}} & \Sigma(\mathrm{Pic}_d(X)) & \xrightarrow{\sim} & \mathrm{Pic}_d(\Gamma) \end{array}$$

The general case for arbitrary K, again follows from the invariance of the projection to the skeleton under base change by non-Archimedean field extensions. □

A Bieri-Groves-Theorem

Let $Y \subseteq X_d$ be a closed subvariety. We define the *tropicalization* $\mathrm{trop}_{X_d}(Y)$ of Y to be the projection of Y^{an} to Γ_d via trop_{X_d}, i.e. essentially via the specialization of effective divisors from X to Γ_X from [11]. By Theorem 2.3.1 this is nothing but the projection of Y^{an} to the skeleton of X_d^{an} via ρ_{X_d}.

In this section we deduce the following Theorem 2.3.18 from the Bieri-Groves-Theorem (see [21, Theorem A], [56, Theorem 2.2.3], and Theorem 1.2.15) for projections to the skeleton associated to a polystable model, which for X_d can be stated as follows.

Theorem 2.3.18 ([28])**.** *Let X be a smooth and proper variety over K and let $\mathcal{X}$ be a proper polystable model of X. Suppose that Y is a closed subvariety of X (defined over K) that is equidimensional of dimension δ. Then the tropicalization*

$$\mathrm{trop}_{\mathcal{X}}(Y) := \rho_{\mathcal{X}}(Y^{an}) \subseteq \Sigma(\mathcal{X})$$

of Y (as a subspace of X) is a Λ-rational polyhedral complex in $\Sigma(\mathcal{X})$ of dimension $\leq \delta$. If $\mathcal{X}$ has a deepest stratum E that is a point and $\mathrm{trop}_{\mathcal{X}}(Y)$ contains a point in the interior of $\Delta(E)$, then the dimension of $\mathrm{trop}_{\mathcal{X}}(Y)$ is equal to δ.

Let $\mathbb{G}_m^n = \mathrm{Spec}\, K\big[t_1^{\pm 1}, \ldots, t_n^{\pm 1}\big]$ be a split algebraic torus over K. Recall e.g. from [63] that there is a natural proper and continuous tropicalization map

$$\mathrm{trop}_{\mathbb{G}_m^n} : \big(\mathbb{G}_m^n\big)^{an} \longrightarrow \mathbb{R}^n \ .$$

It is given by sending a point $x \in \big(\mathbb{G}_m^n\big)^{an}$, which corresponds to a multiplicative seminorm $|.|_x$ on $K\big[t_1^{\pm 1}, \ldots, t_n^{\pm 1}\big]$ extending the absolute value on K, to the point $\big(|t_1|_x, \ldots, |t_n|_x\big) \in \mathbb{R}^n$. Using this map, the *tropicalization* of a subvariety $Y \subseteq \mathbb{G}_m^n$ may be defined to be the projection of $Y^{an} \subseteq \big(\mathbb{G}_m^n\big)^{an}$ to $\mathbb{R}^n$ given by

$$\mathrm{trop}_{\mathbb{G}_m^n}(Y) := \mathrm{trop}_{\mathbb{G}_m^n}(Y^{an}).$$

Lemma 2.3.19 ([28])**.** *Let $\mathcal{Z}_{\vec{n},\vec{k},\vec{a}}$ be a standard polystable model over R. Then the generic fiber is the algebraic torus $\mathbb{G}_m^{|\vec{n}|}$ and the natural diagram commutes.*

$$\begin{array}{ccc} \mathcal{Z}^{\circ}_{\vec{n},\vec{k},\vec{a}} & \xrightarrow{\mathrm{trop}_{\vec{n},\vec{k},\vec{a}}} & \Delta(\vec{k},\vec{a}) \\ \subseteq \big\downarrow & & \big\downarrow \subseteq \\ \big(\mathbb{G}_m^{|\vec{n}|}\big)^{an} & \xrightarrow{\mathrm{trop}_{\mathbb{G}_m^{|\vec{n}|}}} & \mathbb{R}^{|\vec{n}|} \end{array}$$

Proof. Given a point $x \in \mathcal{Z}^\circ_{\vec{n},\vec{k},\vec{a}}$, we have:

$$\mathrm{trop}_{\vec{n},\vec{k},\vec{a}}(x) = \Big(-\log \big|t_1^{(1)}\big|_x, \dots, -\log \big|t_{k_1}^{(1)}\big|_x, \dots, -\log \big|t_1^{(r)}\big|, \dots, -\log \big|t_{k_r}^{(r)}\big|_x \Big) \ .$$

Let $j = k_i + 1, \dots, n_i$ and $i = 1, \dots, r$. Since $|.|_x$ is multiplicative, we have

$$\big|(t_j^{(i)})\big|_x \cdot \big|(t_j^{(i)})^{-1}\big|_x = \big|1\big|_x = 1$$

and since it is bounded, i.e. $|f| \leq 1$ for all $f \in K[\mathbb{N}^{|\vec{n}|}]$, we have $\big|t_j^{(i)}\big|_x \leq 1$ as well as $\big|(t_j^{(i)})^{-1}\big|_x \leq 1$. This implies $\big|t_j^{(i)}\big|_x = 1$. Therefore we have

$$\begin{aligned}
\mathrm{trop}_{\mathbb{G}_m^{|\vec{n}|}}(x) =& \big(-\log \big|t_1^{(1)}\big|_x, \dots, -\log \big|t_{n_1}^{(1)}\big|_x, \dots, -\log \big|t_1^{(r)}\big|_x, \dots, -\log \big|t_{n_r}^{(r)}\big|_x \big) \\
=& \Big(-\log \big|t_1^{(1)}\big|, \dots, -\log \big|t_{k_1}^{(1)}\big|_x, 0, \dots, 0, \\
& -\log \big|t_1^{(2)}\big|_x, \dots, -\log \big|t_{k_2}^{(2)}\big|_x, 0, \dots, 0, \\
& \dots \\
& -\log \big|t_1^{(r-1)}\big|, \dots, -\log \big|t_{k_{r-1}}^{(r-1)}\big|_x, 0, \dots, 0, \\
& -\log \big|t_1^{(r)}\big|, \dots, -\log \big|t_{k_r}^{(r)}\big|_x, 0, \dots, 0 \Big)
\end{aligned}$$

which is precisely the image of $\mathrm{trop}_{\vec{n},\vec{k},\vec{a}}(x)$ under the embedding $\mathbb{R}^{k_1+\dots+k_r} \hookrightarrow \mathbb{R}^{n_1+\dots+n_r}$. □

The proof of Theorem 2.3.18 closely follows along the lines of the proof of [118, Theorem 1.1], the Bieri-Groves-Theorem for subspaces of log-regular varieties.

Proof of Theorem 2.3.18. We need to show that that $\mathrm{trop}_{\mathcal{U}}(Y) = \mathrm{trop}_{\mathcal{X}}(Y) \cap \Sigma(\mathcal{U})$ is a Λ-rational polyhedral complex for a small open subset $\mathcal{U}$ around every stratum E of $\mathcal{X}_0$. We may choose $\mathcal{U}$ so that there is a small chart $\gamma\colon \mathcal{U} \to \mathcal{Z}(\vec{n}, \vec{k}, \vec{a})$. By the local description of $\rho_{\mathcal{U}}$ in terms of the tropicalization map $\mathrm{trop}_{\vec{n},\vec{k},\vec{a}}\colon \mathcal{Z}(\vec{n}, \vec{k}, \vec{a})^\circ \to \Delta(\vec{k}, \vec{a})$ in the subsection "Skeletons of Polystable Models," we may identify $\mathrm{trop}_{\mathcal{U}}(Y)$ with the projection $\mathrm{trop}_{\vec{n},\vec{k},\vec{a}}(\gamma^\circ(Y^{an} \cap \mathcal{U}^\circ))$ and, by Lemma 2.3.19, with $\mathrm{trop}_{\mathbb{G}_m^{|\vec{n}|}}(\gamma^\circ(Y^{an} \cap \mathcal{U}^\circ))$.

Since γ is étale, the image $\gamma(Y \cap U)$ is locally closed in $\mathbb{G}_m^{|\vec{n}|}$. Denote by $\overline{Y}_\gamma$ its closure in $\mathbb{G}_m^{|\vec{n}|}$. By a generalization of Draisma's tropical lifting lemma [54, Lemma 4.4] (see [63, Proposition 11.5] and [118, Lemma 3.10]), the tropicalization of a locally closed subset is equal to the tropicalization of its closure and so we have

$$\mathrm{trop}_{\mathcal{U}}(Y^{an}) = \mathrm{trop}(\overline{Y}_\gamma) \cap \Delta(\vec{k}, \vec{a}) \ .$$

The tropicalization $\mathrm{trop}_{\mathcal{U}}(Y^{an})$ is a Λ-rational polyhedral complex of dimension δ by the classical Bieri-Groves-Theorem [21, Theorem A] and [56, Theorem 2.2.3]. Due to the intersection with $\Delta(\vec{k}, \vec{a})$, the tropicalization $\mathrm{trop}_{\mathcal{U}}(Y^{an})$ might have dimension $\leq \delta$. If E is a point, then the cell $\Delta(E)$ will be $|\vec{n}|$-dimensional and, if $\mathrm{trop}_{\mathcal{U}}(Y^{an})$ has a point in the interior of a cell, it will be part of an δ-dimensional cell of $\mathrm{trop}(\overline{Y}_\gamma)$ whose intersection with $\Delta(\vec{k}, \vec{a}) = \Delta(E)$ is δ-dimensional, since $\mathrm{trop}_{\mathbb{G}_m^{|\vec{n}|}}(\overline{Y}_\gamma)$ fulfills the balancing condition. □

A Realizable Riemann-Roch Theorem

We can now study the tropical geometry of linear series by directly tropicalizing them as subvarieties of X_d. For example, Theorem 2.3.18 immediately implies the following *realizable Riemann-Roch Theorem.*

Corollary 2.3.20 ([28])**.** *Let X be a Mumford curve of genus g and let D be a divisor on X of degree d such that both $\operatorname{trop}_{X_d}\left|D\right|$ and $\operatorname{trop}_{X_d}\left|K_X - D\right|$ contain a point in the interior of a maximal cell of Γ_d. Then we have:*

$$\dim \operatorname{trop}_{X_d}\left|D\right| - \dim \operatorname{trop}_{X_d}\left|K_X - D\right| = d - g + 1 \ .$$

By [11], the dimension of $\operatorname{trop}_{X_d}\left|D\right|$ is not always equal to the rank of the specialization of D to Γ. So, this realizable Riemann-Roch Theorem does not in general imply the well-known intrinsic tropical Riemann-Roch Theorem from [13, 57, 3]. In the special case when Γ is a generic chain of loops, however, the lifting results of [36] allow us to say more.

Let Γ be a chain of g loops, where each loop consists of two edges having lengths l_i and m_i (see Figure 2.15). Suppose that Γ is a *generic*, i.e. suppose that none of the ratios l_i/m_i is equal to the ratio of two positive integers whose sum is less than or equal to $2g-2$ (see [48, Definition 4.1]). The results of [36] show that the algebraic Riemann-Roch-Theorem implies the tropical Riemann-Roch-Theorem.

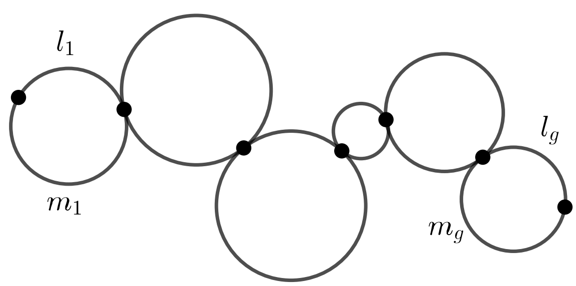


Figure 2.15: A chain of loops with edge lengths labeled.

Let D a divisor on Γ of degree d and rank r supported on Λ-rational points of Γ. Let X be a Mumford curve whose dual tropical curve is Γ. By [36] there is a line bundle L of degree d and rank r on X such that the specialization of L is equal to the divisor class $[D]$. The construction in [36] is naturally compatible with residue duality and thus the specialization of $\omega_X \otimes L^{-1}$ is equal to the class $[K_\Gamma - D]$ and the rank of $\omega_X \otimes L^{-1}$ is equal to the rank of $K_\gamma - D$. Therefore we have

$$r(D) - r(K_\Gamma - D) = \dim\left|L\right| - \dim\left|\omega \otimes L^{-1}\right| = d - g + 1$$

and, in this situation, the algebraic Riemann-Roch Theorem implies its tropical counterpart. If both $\operatorname{trop}_{X_d}|L|$ and $\operatorname{trop}_{X_d}|L \otimes \omega_X|$ contain an interior point of a maximal cell of Γ_d, then we have $r(D) = \dim \mathrm{trop}_{X_d} |L|$ as well as $r(K_\Gamma - D) = \dim \mathrm{trop}_{X_d} |\omega \otimes L^{-1}|$ and the realizable Riemann-Roch-Theorem from above is equivalent to the tropical Riemann-Roch-Theorem. One may think of $\mathrm{trop}_{X_d} |L|$ as the realizability locus in the tropical linear system $|D|$ and the Baker-Norine rank is equal to the polyhedral dimension of $\mathrm{trop}_{X_d} \big|L\big|$.

Faithful Tropicalization of Polystable Skeletons

The classical approach to the process of tropicalization goes by choosing an embedding into a suitable toric variety and then applying coordinate-wise valuations to the embedded variety. For symmetric powers, however, Theorem 2.3.1 suggests that it might be more natural to think of tropicalization as a projection to the non-Archimedean skeleton. The principle of *faithful tropicalization*, as pioneered in [14] and further developed e.g. in [50, 64, 65], seeks to realign these two perspectives.

Expanding on [64] we now prove a faithful tropicalization result for skeletons associated to polystable models. This is a generalization of [64, Theorem 9.5] to the polystable case (when no extra divisor at infinity is present).

Theorem 2.3.21 ([28])**.** *Let X be a smooth and proper variety over K and let $\mathcal{X}$ be a proper strictly polystable model of X over R. Then there is an open subset $U \subseteq X$ as well as a morphism $f\colon U \to \mathbb{G}_m^n$ such that the induced tropicalization map*

$$\mathrm{trop}_f\colon U^{an} \xrightarrow{f^{an}} \mathbb{G}_m^{n,an} \xrightarrow{\mathrm{trop}} \mathbb{R}^n$$

is faithful, i.e. the restriction of $\mathrm{trop} \circ f^{an}$ to $\Sigma(\mathcal{X}) \subseteq U^{an}$ is a homeomorphism onto its image in $\mathbb{R}^n$ and is unimodular on each cell of $\Sigma(\mathcal{X})$.

Proof. Fix a stratum E of $\mathcal{X}_0$ and choose a small chart $\gamma\colon \mathcal{U} \to \mathcal{Z}(\vec{n}, \vec{k}, \vec{a})$ around E. Let $U_{(E)} := \mathcal{U}_K$ be the generic fiber and let $f_{(E)}\colon U^E \to \mathbb{G}_m^{|\vec{n}|}$ be the base change of γ to the generic fiber. By Lemma 2.3.19 the tropicalization map

$$\mathrm{trop}_{f_{(E)}}\colon U_{(E)}^{an} \xrightarrow{f_{(E)}^{an}} \mathbb{G}_m^{|\vec{n}|,an} \xrightarrow{\mathrm{trop}} \mathbb{R}^{|\vec{n}|}$$

naturally restricts to the projection to $\Delta(\vec{n}, \vec{k}, \vec{a})$ on $\mathcal{U}^\circ \subseteq U_{(E)}^{an}$. Therefore, by the subsection "The skeleton of small open neighborhood," the restriction to the skeleton $\Sigma(\mathcal{U}) \subseteq \mathcal{U}^\circ \subseteq U_{(E)}^{an}$ is a unimodular homeomorphism onto its image in $\mathbb{R}^{|\vec{n}|}$.

In general, set $U = \bigcap_E U_{(E)}$, where E is passing through all the strata of $\mathcal{X}_0$. We take $f = (f_{(E)})\colon U \longrightarrow \mathbb{G}_m^n$, where $n = \sum_E |\vec{n}_e|$ and f is given by $f(x) = \big(f_{(E)}(x)\big)_E \in \mathbb{G}_m^n$. The above reasoning shows that the restriction to the skeleton $\Sigma(\mathcal{X})$ is a unimodular homeomorphism on every polysimplex in $\Sigma(\mathcal{X})$.

We finish by showing that trop_f is injective: Consider two points $x, x' \in \Sigma(\mathcal{X})$ such that $\mathrm{trop}_f(x) = \mathrm{trop}_f(x')$. Then $\mathrm{trop}_{f_E}(x) = \mathrm{trop}_{f_E}(x')$ for all strata E of $\mathcal{X}_0$. Suppose that x is in the relative interior of $\Delta(E)$ and x' is in the relative interior of $\Delta(E')$. By Lemma 2.3.19 and the subsection "The skeleton of small open neighborhood," we then have

$$x = \rho_{\mathcal{X}}(x) = \rho_{\mathcal{X}}(x') \ .$$

Since x is in the relative interior of $\Delta(E)$, the point x' is in the relative interior of $\Delta(E)$ as well, by the construction of $\Sigma(\mathcal{X})$ as a geometric realization of the polysimplicial complex associated to $\mathcal{X}$ (see Proposition 2.3.11). In particular, we have $E = E'$ and $x = x'$, since the restriction of $\mathrm{trop}_{f_{(E)}}$ to $\Delta(E) \subseteq \Sigma(\mathcal{X})$ is injective. □

Unfortunately the construction of the map $f\colon U \to \mathbb{G}_m^n$ is by no means effective. In the subsection "Effective faithful tropicalization via linear series" we speculate how the recent work of Kawaguchi and Yamaki [78] that uses linear series to find effective faithful tropicalizations of curves may be generalized to find effective faithful tropicalization of symmetric powers.

Open Questions

Effective faithful tropicalization via linear series

Let L be a line bundle. For $g \geq 0$ set

$$t(g) = \begin{cases} 1 & \text{if } g = 0 \\ 3 & \text{if } g = 1 \\ 3g-1 & \text{if } g \geq 2 \ . \end{cases}$$

In [78], Kawaguchi and Yamaki show that, if $\deg L \geq t(g)$, then there are sections $s_0, \ldots, s_r \in H^0(X, L)$ such that the associated map

$$\begin{aligned} X &\longrightarrow \mathbb{P}^r \\ x &\longmapsto \big(s_0(x), \ldots, s_r(x)\big) \end{aligned}$$

induces a tropicalization map

$$\begin{aligned} \mathrm{trop}_{(s_0,\ldots,s_r)}\colon X^{an} &\longrightarrow \mathbb{TP}^r \\ x &\longmapsto \big(-\log|s_0|_x, \ldots, -\log|s_r|_x\big) \end{aligned}$$

that is faithful on the skeleton Γ of X, i.e. that restricts to a piecewise integer linear and unimodular map on Γ.

Let L be an $(d-1)$-ample line bundle on X. There is a natural map

$$X_d \longrightarrow \mathrm{Gr}\big(d, H^0(X,L)^*\big)$$

into the Grassmannian of d-dimensional quotients of $H^0(X,L)$ that is given by associating to an effective divisor D on X the surjective restriction map

$$H^0(X,L) \twoheadrightarrow H^0(X, L\otimes \mathcal{O}_D)\ .$$

If L is d-ample, this map is an injection.

Choosing global sections $s_0,\ldots,s_r \in H^0(X,L)$ we find a map

$$X_d \longrightarrow \mathrm{Gr}(d,r)$$

that on the open locus X_d° (parametrizing reduced divisors on X) is given by sending a split reduced effective divisor $D=\sum_{i=1}^d p_i$ to the linear space spanned by the p_i. If we compose this with the Plücker embedding, we obtain a map $X_d \to \mathbb{P}^N$ with $N=\binom{r}{k}-1$ such that the vanishing of the Plücker coordinates precisely describes the locus of non-reduced divisors. In other words, we have $X_d \cap \mathbb{G}_m^N = X_d^\circ$.

Expanding on the work of Kawaguchi and Yamaki [78], we ask the following.

Question 2.3.22 ([28])**.** Suppose that L is a line bundle on X that is d-ample. Under which conditions is there a basis $s_0,\ldots,s_r$ of $H^0(X,L)$ such that the induced tropicalization map

$$X_d^{\circ,an} \hookrightarrow \mathrm{Gr}^\circ\left(d,r\right)^{an} \hookrightarrow \mathbb{G}_m^{N,an} \xrightarrow{\mathrm{trop}_{\mathbb{G}_m^N}} \mathbb{R}^N$$

is faithful on the skeleton $\Sigma(X_d)\simeq \Gamma_d$?

One can think of the desired condition as a tropical analogue of d-ampleness.

de Jonquiéres divisors

We now discuss de Jonquiéres divisors. Let X be a smooth projective curve with genus g. Consider a fixed complete linear series $l=(L,V)$ of degree d and dimension r. Then a *de Jonquiéres divisor of length* N is a divisor $a_1D_1+\cdots+a_kD_k \in X_d$ contained in $\mathbb{P}V$ that fulfills $\sum_{i=1}^k \deg(D_i)=N$. These are studied extensively in [120]. If $\mu_1=(a_1,\ldots,a_k)$ and $\mu_2=(d_1,\ldots,d_k)$ are positive partitions such that $\sum_{i=1}^k a_id_i=d$, then we denote the set of de Jonquiéres divisors of length N determined by μ_1 and μ_2 by $DJ_{k,N}^{r,d}(\mu_1,\mu_2,C,l)$. In [120], the author proves that for general curves, if $N-d+r\geq 0$, then $DJ_{k,N}^{r,d}(\mu_1,\mu_2,C,l)$ has the expected dimension $N-d+r$. In particular, when $N-d+r<0$, the variety $DJ_{k,N}^{r,d}(\mu_1,\mu_2,C,l)$ is empty.

One may wonder whether this result remains true tropically. The following proposition addresses the emptiness result and would imply its algebraic counterpart.

Proposition 2.3.23 ([28])**.** Let Γ be a generic chain of loops and K its canonical divisor, so $d=2g-2$ and $r=g-1$. If n is such that $n+d-r<0$, then $|K|$ does not contain a divisor of the form $d_1p_1+\cdots d_np_n$.

Proof. The canonical divisor has degree $2g-2$ and rank $g-1$. Therefore, in order for $n+d-r<0$ to hold, we must have $n<g-1$. The canonical divisor is supported on $g-1$ vertices, and because of the genericity condition, any divisor equivalent to the canonical divisor will have at least that many vertices in its support. Therefore, there is no divisor of the form $d_1p_1+\cdots d_np_n$ when $n<g-1$. □

However, unlike in the classical case, the result does not hold for all divisors. We give the following example.

Example 2.3.24. Consider the length 2 generic chain of loops, and let p be the middle vertex. Let $D=K+p$. Then the rank of D is 1 (because there is a divisor p_1+p_2, with each point coming from a separate loop, such that $D-D'$ is not effective), and so if $n=1$ then $n-d+1=-1<0$. So, in the classical case we would expect there to be no divisor in $|D|$ of the form $3q$ for $q\in\Gamma$. However, in this case $D=3p$.

Conclusion

In this chapter, we proved that the symmetric power, which is a moduli space for the set of degree d effective divisors on a curve, tropicalizes to the symmetric power of the tropicalization of the curve. Since the symmetric power of a tropical curve is also a moduli space for the degree d effective divisors on that tropical curve, this theorem provides an example of the principle that the non-Archimedean skeleton of a classical moduli space is a tropical moduli space. We will see many more examples of this in the next chapter.

3

Moduli Spaces

In this chapter we tropicalize moduli spaces and develop tropical moduli spaces. These are spaces whose points correspond to equivalence classes of some object of interest. Frequently, moduli spaces will not be compact and so we lightly relax our restrictions on the types of objects we are interested in order to compactify the space. Tropicalizations of algebraic moduli spaces often give tropical moduli spaces. The boundary of the compactified moduli space is stratified, and this is dual to the tropicalization of the moduli space.

The original work in this chapter appears in the section "Divisorial Motivic Zeta Functions," which is joint work with Martin Ulirsch [27]. It will be published in the Michigan Mathematics Journal.

We begin with an overview of moduli spaces, following [37]. Abstractly, to make a moduli problem, we need several ingredients:

1. A class P of geometric objects in some category $\mathcal{C}$,
2. the notion of a family of objects in P, and
3. a notion of what it means for two objects in P to be equivalent.

From this, we can sometimes create a moduli space $\mathcal{M}$, which is an object in the category $\mathcal{C}$ such that there is a bijection between points of $\mathcal{M}$ and equivalence classes of objects in P. We do this because the geometry of $\mathcal{M}$ will reflect the geometry of the objects it parameterizes.

Definition 3.0.1. Let $\mathcal{C}$ be the category of topological spaces, let P be a class of objects in $\mathcal{C}$, and let $B \in \mathcal{C}$. A *family* of objects in P over B is a topological space X together with a continuous function $\pi : X \to B$ such that for all $b \in B$, we have $\pi^{-1}(b) \in P$.

Example 3.0.2. Take the category of schemes over K, and let P be semistable curves. Then a family $\mathcal{X}$ over $B = \operatorname{Spec}(R)$ is a semistable model for $X = \pi^{-1}((0))$.

A family of objects over B will give a function $f_\pi : B \to \mathcal{M}$. This map is given by sending $b \in B$ to $\pi^{-1}(b) \in \mathcal{M}$. In order for this to work, we need that f_π is a morphism of

$\mathcal{C}$, that no two distinct families give the same function, and that all functions $f : B \to \mathcal{M}$ correspond to some family.

Definition 3.0.3. Two families $\pi : X \to B$ and $\pi' : X' \to B$ are *isomorphic* if there exists a map $\phi : X \to X'$ such that for all $x \in X$, $\pi'(\phi(x)) = \pi(x)$.

Definition 3.0.4. A *moduli functor* $\mathcal{M}$ *for equivalence classes of families of objects in* P is a functor $\mathcal{M}$ from schemes to sets sending:

1. a scheme B to the set of families over B, and
2. a map of schemes $f : B \to B'$ to the map from families over B' to families over B given by sending $\pi' : X' \to B'$ to the pullback $B \times_{B'} X' \to B$.

Definition 3.0.5. We say the moduli functor $\mathcal{M}$ is *represented by the scheme* X if $\mathcal{M}$ is isomorphic to the functor of points of X. More concretely, there is a natural transformation

$$\mathcal{M} \to \mathrm{Hom}(-, X).$$

In this case, we call X a *fine moduli space for* $\mathcal{M}$. A fine moduli space can fail to exist, and often this is because the objects have nontrivial automorphisms.

Example 3.0.6 ($\mathcal{M}_{0,n}$)**.** Consider the functor $\mathcal{M}_{0,n}$ from schemes to sets, which sends a scheme B to the set of tuples $(\pi : X \to B, \sigma_1, \ldots, \sigma_n : B \to X)$ where:

1. the map π is flat and proper,
2. for all $b \in B$, we have that $\pi^{-1}(b) \cong \mathbb{P}^1$,
3. the composition $\pi \circ \sigma_i$ is the identity,
4. the image of σ_i is disjoint from the image of σ_j for $i \neq j$.

Any individual fiber $\pi^{-1}(b)$ is a copy of $\mathbb{P}^1$ with n distinguished marked points, given by the images of $\sigma_1, \ldots, \sigma_n$ inside $\pi^{-1}(b)$. When $n \leq 3$, this functor is represented by a point.

Definition 3.0.7. Let X be a fine moduli space for $\mathcal{M}$. Then there is a *universal family* $\pi : U \to X$, where the fiber $\pi^{-1}(x)$ is the object represented by x. This universal family is $\mathcal{M}(X) \cong \mathrm{Hom}(X, X)$.

Example 3.0.8 (Universal family over $\mathcal{M}_{0,3}$)**.** The universal family over $\mathcal{M}_{0,3}$ is pictured in Figure 3.1. This marked $\mathbb{P}^1$ is the universal family for $\mathcal{M}_{0,3}$ because given a $\mathbb{P}^1$ with 3 marked points, there is an automorphism of $\mathbb{P}^1$ taking those three points to 0, 1, and ∞.

Example 3.0.9. (Universal family over $\mathcal{M}_{0,4}$) The moduli space $\mathcal{M}_{0,4}$ is represented by $\mathbb{P}^1 \backslash \{0, 1, \infty\}$. A universal family is given in Figure 3.2 by $\mathcal{M}_{0,4} \times \mathbb{P}^1$. The fourth section σ_4 is given by the diagonal.

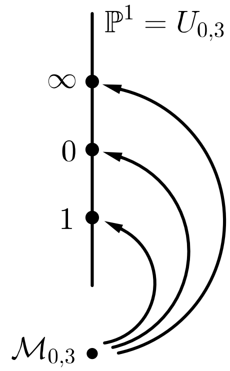


Figure 3.1: The universal family over $\mathcal{M}_{0,3}$.

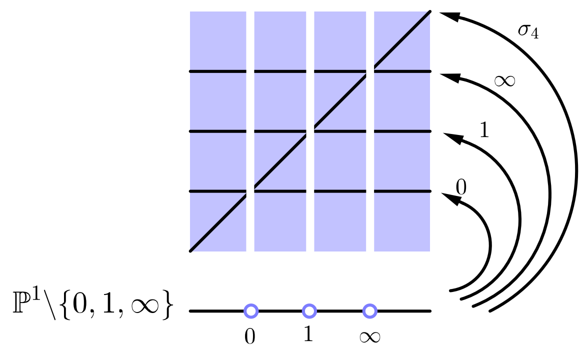


Figure 3.2: The universal family over $\mathcal{M}_{0,4}$.

3.1 Pointed Rational Curves

In this section, we compactify and tropicalize $\mathcal{M}_{0,n}$. The space $\mathcal{M}_{0,n}$ is a quasiprojective variety, meaning that it is the intersection inside some projective space of a Zariski-open and a Zariski-closed subset. So, it is not closed in the Zariski topology. However, it would be useful to have compact moduli space so that we can take limits. To that end, we will compactify $\mathcal{M}_{0,n}$. However, we would like to do this by adding some degenerate objects to the space such that compactification itself is a moduli space and the compactification is not too singular. This will ensure that our compactification is a meaningful one.

The degenerate objects we need are stable curves. We remind the reader of the definition.

Definition 3.1.1. A *rational n-pointed stable curve* $(C, p_1, \ldots, p_n)$ is a curve C with arithmetic genus 0 whose only singularities are nodes, together with distinct nonsingular points $p_1, \ldots, p_n$ on C. Additionally, we require that the only automorphism of $(C, p_1, \ldots, p_n)$ is the identity. Concretely, this stability condition means that on each component of C, there need to be at least 3 points that are either marked or singular.

Theorem 3.1.2 ([79])**.** *There exists an irreducible, smooth, projective, fine moduli space $\overline{\mathcal{M}}_{0,n}$ for n-pointed rational stable curves which compactifies $\mathcal{M}_{0,n}$. The universal family $\overline{U}_n$ is obtained from the universal family U_n of $\mathcal{M}_{0,n}$ via a finite sequence of blow-ups.*

Example 3.1.3. Some examples of curves we have added to $\mathcal{M}_{0,n}$ in order to make $\overline{\mathcal{M}}_{0,n}$ are pictured in Figure 3.3.

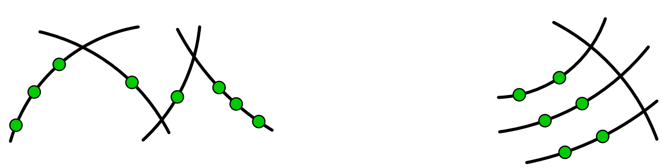


Figure 3.3: Stable pointed curves.

Boundary Strata

The *boundary* of $\overline{\mathcal{M}}_{0,n}$ is $\overline{\mathcal{M}}_{0,n} \setminus \mathcal{M}_{0,n}$ and consists of all points giving nodal stable curves.

Example 3.1.4. The boundary of $\overline{\mathcal{M}}_{0,4}$ consists of the three points in Figure 3.4.

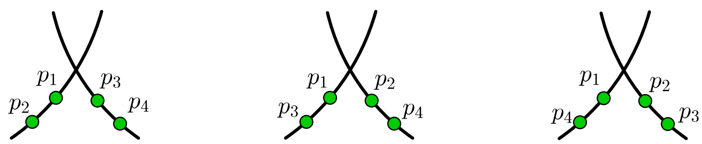


Figure 3.4: Points in the boundary of $\overline{\mathcal{M}}_{0,4}$.

Example 3.1.5. The boundary of $\overline{\mathcal{M}}_{0,5}$ consists of the 15 points with three irreducible components and 10 copies of $\mathcal{M}_{0,4}$ parameterizing rational pointed curves with two irreducible components. These are pictured in Figure 3.5. Combinatorially, they form a Petersen Graph (see [88, Example 4.3.2]).

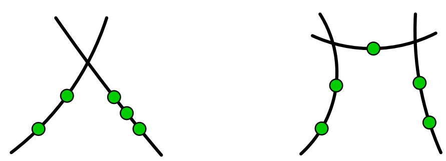


Figure 3.5: Combinatorial types of curves in the boundary of $\overline{\mathcal{M}}_{0,5}$.

There is a natural stratification of the boundary, which carries a poset structure. In order to understand this, we will look at the dual graph of the marked rational curve.

Definition 3.1.6. Given a rational, stable n-pointed curve $(C, p_1, \ldots, p_n)$, the *dual graph* is defined in the following way. There is a vertex for every irreducible component of C, and en edge for each node of C. Finally, we attach a labelled half-edge for each marked point at the vertex corresponding to the component containing the marked point.

Example 3.1.7. In Figure 3.6 is an example of a nodal labelled curve and its dual graph.

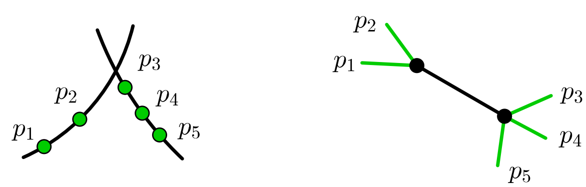


Figure 3.6: A nodal labelled curve and its dual graph.

A *boundary stratum* S is the set of all points in $\overline{\mathcal{M}}_{0,n}$ whose dual graph is a given graph Γ_S. The codimension of a boundary stratum S equals the number of nodes that any curve parameterized by S has. Equivalently, this is the number of edges in Γ_S. The poset structure is given in the following way. We say that $S_1 \prec S_2$ if Γ_{S_2} is obtained from Γ_{S_1} by contracting edges. Geometrically, this means that S_1 is in the closure of S_2. Each stratum is naturally isomorphic to a product of some copies of $\mathcal{M}_{0,j}$.

Tropicalization and Metric Trees

We now tropicalize $\mathcal{M}_{0,n}$. Observe that a point in $\mathcal{M}_{0,n}$ may be viewed as a $2 \times n$ matrix

$$\begin{bmatrix} a_1 & \cdots & a_n \\ b_1 & \cdots & b_n \end{bmatrix}.$$

Two matrices define equivalent points in $\mathcal{M}_{0,n}$ if one is obtained from the other by the action of $GL(2)$ or by column scaling. In general, two $2 \times n$ matrices define the same linear space if one is obtained from the other by the action of $GL(2)$. So, the tropicalization of $\mathcal{M}_{0,n}$ will be the tropicalization of the space of 2 dimensional linear subspaces of K^n modulo some lineality space. Thus, in order to study the tropicalization of $\mathcal{M}_{0,n}$, we will need to study the tropicalization of the *Grassmannian*.

Definition 3.1.8 ([94])**.** The *Grassmannian* $G(r, n)$ is the family of r-dimensional subspaces of K^n. It is a smooth projective variety of dimension $r(n - r)$.

We realize this as a subvariety of $\mathbb{P}^{\binom{n}{r}-1}$ in the following way. Elements of $\mathbb{P}^{\binom{n}{r}-1}$ are represented by vectors p in $K^{\binom{n}{r}}$ whose coordinates p_I are indexed by subsets of $I = \{1, \dots, n\}$, with $|I| = r$. Then $G(r, n)$ is defined by the prime ideal

$$I_{r,n} = \langle P_{I,J} : I, J \subset [n],\ |I| = r - 1, |J| = r + 1\rangle,$$

whose generators are the quadratic *Plücker relations*

$$P_{I,J} = \sum_{j \in J} \operatorname{sgn}(j; I, J) \cdot p_{I \cup j} p_{J \setminus j}.$$

We study $G^0(r, n) := G(r, n) \cap (K^*)^{\binom{n}{r}-1}$, where no Plücker coordinate vanishes.

Since we wish to understand $\mathcal{M}_{0,n}$, we must study the Grassmannian when $r = 2$. As we will see, the tropicalization of the Grassmannian $Gr(2, n)$ consists of *phylogenetic trees*.

Definition 3.1.9. A *phylogenetic tree* is a tree with n labelled leaves and no vertices of degree 2. It is also equipped with edge lengths l_e for each edge e in the tree.

Definition 3.1.10. A *tree distance* is a vector $d = (d_{ij}) \in \mathbb{R}^{\binom{n}{2}}$ where d_{ij} is the distance between leaves i and j in some phylogenetic tree. Then, this is the sum of the lengths l_e of the edges contained in the unique path from leaf i to leaf j. When all edge lengths l_e are non-negative, this is called a *tree metric*.

Definition 3.1.11. Let Δ denote the set of all tree distances in $\mathbb{R}^{\binom{n}{2}}$. This is called the *space of phylogenetic trees.*

We will now tropicalize the Grassmannian.

Theorem 3.1.12 ([88, Theorem 4.3.5, Corollary 4.3.12])**.** *The* $\binom{n}{4}$ *Plücker relations are a tropical basis (Definition 1.2.11) of* I_{2n}*. The tropical Grassmannian* $\mathrm{trop}(G^0(r,n))$ *is the support of a pure* $2m-4$ *dimensional fan with* $(2n-5)!!$ *maximal cones. Up to sign, the tropical Grassmannian* $\mathrm{trop}(G^0(2,n))$ *coincides with the space of phylogenetic trees:*

$$\mathrm{trop}(G^0(2,n)) = -\Delta.$$

Each cone corresponds to the collection of metrics on a fixed trivalent tree τ*.*

We note that $\mathrm{trop}(G^0(r,n))$ has lineality space $L = \mathrm{span}(\sum_{I:i\in I} e_I \mid 1 \le i \le m) \subset \mathbb{R}^{\binom{m}{r}}$. When $r = 2$, this lineality space corresponds to selecting the lengths of the leaf edges of the tree. Given any tree, one may fix the lengths of the leaf edges freely.

An *ultrametric* is a metric d such that $\max(d_{ij}, d_{ik}, d_{jk})$ is attained at least twice for all i, j, k. This is equivalent to saying that the metric is a tree metric for a rooted tree, where every leaf has the same distance to the root. This is given by $\mathrm{trop}(M_{K_n})$, the tropical linear space of the matroid of the complete graph. This is more or less immediate from observing the max condition given above, and considering what the cycles are in K_n.

Lemma 3.1.13 ([88, Lemma 4.3.9])**.** *Every tree distance is an ultrametric plus leaf-lengths, meaning we have the decomposition*

$$\Delta = -\mathrm{trop}(M_{K_n}) + L.$$

Now we return to the tropicalization of $\mathcal{M}_{0,n}$. We can see that $\mathcal{M}_{0,n} = \mathrm{Gr}(2,n)/(K^*)^{n-1}$ (as a GIT quotient) in the following way. Let

$$\begin{bmatrix} a_1 & \cdots & a_n \\ b_1 & \cdots & b_n \end{bmatrix}$$

be a collection of points marking $\mathbb{P}^1$. This defines a point in $\mathrm{Gr}(2,n)$ via the Plücker embedding. However, some points are equivalent. Let $\lambda_1 \in K^*$. Since $(a_1 : b_1) \sim (\lambda_1 a_1 : \lambda_1 b_1)$, we quotient out by the action of $(K^*)^n$ given by scaling each column. However, since $\mathrm{Gr}(2,n) \subset \mathbb{P}^{\binom{n}{2}-1}$ is already quotiented by simultaneous scaling, we only need to quotient by $(K^*)^{n-1} = (K^*)^n/(K^*)$.

The action of the torus $(K^*)^{n-1}$ can be extended naturally to all of $\mathbb{P}^{\binom{n}{2}-1}$. Then, the tropicalization of $\mathcal{M}_{0,n} = \mathrm{Gr}(2,n)/(K^*)^{n-1}$ is equal to the tropicalization of $\mathrm{Gr}(2,n)$ modulo its lineality space. So, the tropicalization of $\mathcal{M}_{0,n}$ is the space of metric trees with n leaves.

Example 3.1.14. The tropicalization of $\mathcal{M}_{0,5}$ is pictured in Figure 3.7. The vertices correspond to curves on the left of Figure 3.5 and the edges correspond to curves on the right.

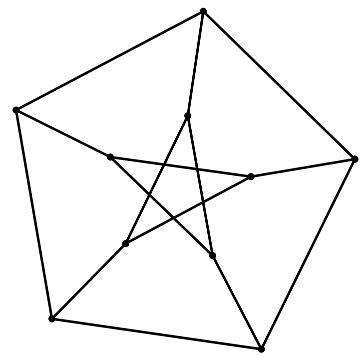


Figure 3.7: The tropicalization of $\mathcal{M}_{0,5}$.

In the next section, we will see that the tropicalization of $\overline{\mathcal{M}}_{0,n}$, viewed now as the non-Archimedean skeleton of $\overline{\mathcal{M}}_{0,n}$, gives the space of metric trees with n leaves as well.

3.2 Other Tropical Moduli Spaces

In the previous section, we saw that the tropicalization of the moduli space of genus 0 marked algebraic curves coincides with the moduli space of genus 0 marked tropical curves. Earlier, we saw that the tropicalization of the space of degree d divisors on a fixed algebraic curve C coincides with the space of degree d divisors on the tropicalization of C. Both of these examples illustrate a general phenomenon that the non-Archimedean skeleton of an algebraic moduli space is typically also a tropical moduli space. In what follows, we give more examples of this phenomenon. Other examples, which we do not outline below, include spin curves [35], Jacobians [16], and rational and elliptic stable maps [109, 110].

Marked Algebraic Curves

In [1], the authors tropicalize the moduli space of curves $\overline{\mathcal{M}}_{g,n}$. For completeness, we give a definition of the objects of interest here.

Definition 3.2.1. A *stable n pointed curve* $(C, p_1, \dots, p_n)$ is a complete connected curve C that has only nodes as singularities, together with an ordered collection $p_1, \dots, p_n \in C$ of distinct smooth points of C such that the $n+1$ tuple $(C, p_1, \dots, p_n)$ has only finitely many automorphisms. The coarse moduli space for stable n pointed curves of genus g is $\mathcal{M}_{g,n}$.

On the tropical side, we could construct a moduli space for marked tropical curves. A *marked tropical curve* is given by a tuple (G, w, l, L), where G is a graph, w is the weight function assigning genera to the vertices of G, l is the length function on the edges of G, and L is a set of labelled legs or infinite edges attached to the vertices of G. We will say that two marked tropical curves are *isomorphic* if one can be obtained from the other via the following operations:

1. Graph automorphisms,

2. Contracting a leaf of weight 0 or with fewer than 2 legs, together with the edge connected to it,

3. Removing a vertex of degree 2 and weight 0 with no legs, and replacing the corresponding edges by one edge whose length is the sum of the old lengths, or

4. Removing an edge of length 0 and adding the weights of the corresponding vertices and combining their legs.

In this way, every marked tropical curve has a *minimal skeleton*. This is a tropical curve with no vertices of weight 0 and degree less than or equal to two, or edges of length zero.

Given a fixed tuple (G, w, L), or *combinatorial type*, the moduli space of tropical curves of this type is given by $\mathbb{R}^{|E|}_{\geq 0}/\operatorname{Aut}(\Gamma)$. The coordinates in $\mathbb{R}^{|E|}_{\geq 0}$ give the edge lengths. The boundary of these cones corresponds to curves with at least one edge of length 0. Then, we glue the cones corresponding to curves of genus g and n legs along the boundaries of these cones to form $\overline{\mathcal{M}}^{\text{trop}}_{g,n}$. The moduli space $\overline{\mathcal{M}}^{\text{trop}}_{g,n}$ is a *stacky fan*. This is a fan together with some identifications, as described above.

Example 3.2.2. The stacky fan $\mathcal{M}^{\text{trop}}_{2,0}$ is displayed in Figure 3.8.

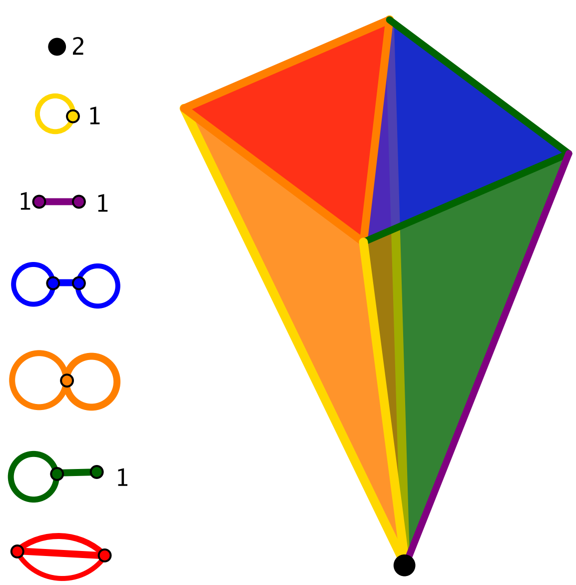


Figure 3.8: The moduli space M_2^{trop} of genus 2 tropical curves.

Let $\overline{\mathcal{M}}_{g,n}^{an}$ be the non-Archimedean analytification of $\overline{\mathcal{M}}_{g,n}$ as in Section 1.3, and let $\Sigma(\overline{\mathcal{M}}_{g,n})$ denote the non-Archimedean skeleton of $\overline{\mathcal{M}}_{g,n}$. Let $\rho : \overline{\mathcal{M}}_{g,n}^{an} \to \Sigma(\overline{\mathcal{M}}_{g,n})$ be the retraction map. Let trop : $\overline{\mathcal{M}}_{g,n}^{an} \to \overline{\mathcal{M}}_{g,n}^{\text{trop}}$ be the set-theoretic map descried as follows. A point in $\overline{\mathcal{M}}_{g,n}^{an}$ is represented (after a possible field extension) by a stable n pointed curve C of genus g over a valuation ring. Let Γ be the corresponding dual tropical curve. Then we define $\text{trop}(C) = \Gamma$.

Theorem 3.2.3 ([1])**.** *There is an isomorphism of generalized cone complexes between* $\Sigma(\overline{\mathcal{M}}_{g,n})$ *and* $\overline{\mathcal{M}}_{g,n}^{\text{trop}}$ *such that the following diagram commutes:*

$$\begin{array}{ccc} \overline{\mathcal{M}}_{g,n}^{an} & \xrightarrow{\rho} & \Sigma(\overline{\mathcal{M}}_{g,n}) \\ & \searrow^{\text{trop}} & \downarrow \sim \\ & & \overline{\mathcal{M}}_{g,n}^{\text{trop}} \end{array}$$

Weighted Stable Curves

In [69], Hassett introduces a moduli space of weighted stable curves. Let g be a non-negative integer, and let $A = (a_1, \ldots, a_n)$ be a vector of weights in $\mathbb{Q} \cap [0,1]$ such that $2g - 1 + a_1 + \cdots + a_n > 0$. Then a *weighted stable curve* $(C, p_1, \ldots, p_n)$ is a nodal curve of genus g such that a subset of the points p_i may coincide only if the sum of their weights is not greater than 1, and for every rational component of C, the number of singular points on C plus the weights of points p_i on C is greater than 2. Then, Hassett defines and constructs moduli spaces $\overline{M}_{g,A}$.

In [119], Ulirsch constructs a tropical moduli space $\overline{\mathcal{M}}_{g,A}^{\text{trop}}$ parameterizing isomorphism classes of tropical curves that are of stable type (g, A). For the complete definition, see [119]. There is a naive set-theoretic tropicalization map $\text{trop}_{g,A} : \overline{\mathcal{M}}_{g,A}^{\text{an}} \to \overline{\mathcal{M}}_{g,A}^{\text{trop}}$ sending a point $x \in \overline{\mathcal{M}}_{g,A}^{\text{an}}$ to (after possible field extension) the dual tropical curve of the curve corresponding to x. This tropical curve will also be stable of type (g, A). We now give Ulirsch's theorem identifying the skeleton of $\overline{\mathcal{M}}_{g,A}$ with the corresponding tropical moduli space. The genus 0 case of this was treated in [40].

Theorem 3.2.4 ([119])**.** *There is an isomorphism of generalized cone complexes between* $\Sigma(\overline{\mathcal{M}}_{g,A})$ *and* $\overline{\mathcal{M}}_{g,A}^{\text{trop}}$ *such that the following diagram commutes:*

$$\begin{array}{ccc} \overline{\mathcal{M}}_{g,A}^{an} & \xrightarrow{\rho} & \Sigma(\overline{\mathcal{M}}_{g,A}) \\ & \searrow^{\text{trop}} & \downarrow \sim \\ & & \overline{\mathcal{M}}_{g,A}^{\text{trop}} \end{array}$$

Remark 3.2.5. This theorem specializes to Theorem 3.2.3 when $A = (1, \ldots, 1)$.

Admissible Covers

In [38], the authors tropicalize the space of admissible covers.

Definition 3.2.6. An *admissible cover* $\pi : D \to C$ of degree d is a finite morphism of pointed curves such that

1. the branch locus of π is contained in the union of marked points and nodes of C,
2. All inverse images of marked points in C are marked points in D,
3. The set of nodes of D is the preimage under π of the set of nodes of C, and
4. Over a node, we have locally that D is given by $y_1y_2 = a$, C is given by $x_1x_2 = a^l$, and π is given by $x_1 = y_1^l$, $x_2 = y_2^l$ for some positive integer $l \leq d$.

Fix a vector of partitions $\vec{\mu} = (\mu^1, \ldots, \mu^r)$ of a non-negative integer d. Let $p_1, \ldots, p_r$, $q_1, \ldots, q_s$ be points on a smooth genus h curve C. Then the cover $\pi : D \to C$ is a *Hurwitz cover* if π is unramified over the complement of p_i and q_j, and over the point p_i it has ramification profile μ_i and over q_i it has simple ramification. Let $\mathcal{H}_{g\to h}(\vec{\mu})$ be the space of degree d Hurwitz covers $[D \to C]$ of smooth genus h curves C by genus g curves D with ramification μ^i over smooth marked points p_i of C, and simple ramification over smooth marked points $q_1, \ldots, q_s$. Let $\overline{\mathcal{H}}_{g\to h}(\vec{\mu})$ the space of admissible covers, which compactifies $\mathcal{H}_{g\to h}(\vec{\mu})$.

A *Hurwitz cover* of tropical curves is a harmonic map of tropical curves that satisfies the local Riemann-Hurwitz condition at every point. Denote by $\mathcal{H}^{\mathrm{trop}}_{g\to h}(\vec{\mu})$ the space of admissible covers of genus h tropical curves by genus g tropical curves with expansion factors along infinite edges prescribed by $\vec{\mu}$. For details on how to construct this moduli space, see [38].

We now describe a tropicalization map from from the Berkovich analytification (see Section 1.3) $\mathcal{H}^{\mathrm{an}}_{g\to h}(\vec{\mu})$ to $\mathcal{H}^{\mathrm{trop}}_{g\to h}(\vec{\mu})$. A point $[D \to C]$ of $\mathcal{H}^{\mathrm{an}}_{g\to h}(\vec{\mu})$ is represented by an admissible cover over $\mathrm{Spec}(K)$, where K is a valued field extension of $\mathbb{C}$. This map extends uniquely to a family of curves over $\mathrm{Spec}(R)$ where R is a rank 1 valuation ring. Let $[\Gamma_D \to \Gamma_C]$ be the associated morphism of the dual tropical curves of the special fibers. The ramification data of the admissible cover determines the expansion factors on the leaf edges. Thus we have a map

$$\begin{array}{rccc} \mathrm{trop}: & \mathcal{H}^{\mathrm{an}}_{g\to h}(\vec{\mu}) & \to & \mathcal{H}^{\mathrm{trop}}_{g\to h}(\vec{\mu}) \\ & [D \to C] & \mapsto & [\Gamma_D \to \Gamma_C]. \end{array}$$

We are now ready to state the main result of [38].

Theorem 3.2.7 ([38]). *The tropicalization map* trop : $\mathcal{H}^{an}_{g\to h}(\vec{\mu}) \to \mathcal{H}^{\text{trop}}_{g\to h}(\vec{\mu})$ *factors through the retraction to the skeleton* $\Sigma(\overline{\mathcal{H}}^{an}_{g\to h}(\vec{\mu}))$*:*

$$\begin{array}{ccc} \mathcal{H}^{an}_{g\to h}(\vec{\mu}) & \xrightarrow{\text{trop}} & \mathcal{H}^{\text{trop}}_{g\to h}(\vec{\mu}) \\ & \searrow^{\rho} & \uparrow{\scriptstyle \text{trop}_\Sigma} \\ & & \overline{\mathcal{M}}^{\text{trop}}_{g,A} \end{array}$$

The Space of Divisors

Denote by $\mathcal{D}iv_{g,d}$ the moduli space parameterizing pairs $(X, \tilde{D})$ consisting of a smooth algebraic curve X of genus g and an effective divisor $\tilde{D}$ on X of degree d. This is the symmetric product of the universal curve over $\mathcal{M}_g$. In [99, Theorem 3], the authors show that the non-Archimedean skeleton of this space gives a tropical moduli space of divisors.

Let $\mathcal{D}iv^{\text{trop}}_{g,d}$ denote the moduli space of pairs (Γ, D) consisting of a tropical curve Γ together with an effective divisor of degree d on Γ.

There is a tropicalization map

$$\text{trop} : \mathcal{D}iv^{an}_{g,d} \to \mathcal{D}iv^{\text{trop}}_{g,d}$$

that is defined in the following way. A point in $\mathcal{D}iv^{an}_{g,d}$ is represented by a tuple $(X, \tilde{D})$ consisting of a curve X over a non-Archimedean extension L of K and a divisor $\tilde{D}$ of degree d. We then send this to the dual tropical curve Γ associated to X together with the specialization of $\tilde{D}$ to Γ.

Theorem 3.2.8 ([99]). *The tropicalization map* trop : $\mathcal{D}iv^{an}_{g,d} \to \mathcal{D}iv^{\text{trop}}_{g,d}$ *has a continuous section J that induces an isomorphism between* $\mathcal{D}iv^{\text{trop}}_{g,d}$ *and the non-Archimedean skeleton* $\Sigma(\mathcal{D}iv_{g,d})$ *making the following diagram commute:*

$$\begin{array}{ccc} \mathcal{D}iv^{an}_{g,d} & \xrightarrow{\rho} & \Sigma(\mathcal{D}iv_{g,d}) \\ & \searrow^{\text{trop}} & \uparrow{\scriptstyle J} \\ & & \mathcal{D}iv^{\text{trop}}_{g,d} \end{array}$$

3.3 Divisorial Motivic Zeta Functions

In this section, we construct a new moduli space, and we apply our understanding of its boundary strata to define a divisorial motivic zeta function. A motivic zeta function can be thought of as a generating function over the *Grothendieck ring*. Here, the coefficients in the series are varieties of increasing size related to a variety of interest. The properties of these functions, such as their rationality, are of great interest. These form an analogue of the classical zeta function, which is the subject of the Weil conjectures.

Definition 3.3.1. Let k be an algebraically closed field. The *Grothendieck ring* $K_0(\mathrm{Var}/k)$ *of varieties over* k is the free abelian group on the set of isomorphism classes of varieties modulo the relations $[X] = \big[X \backslash Y\big] + [Y]$, where Y is a closed subvariety of X. It naturally carries a product given by taking the product of varieties: $[X] \cdot [Y] = \big[(X \times_k Y)\big]$.

For simplicity, we assume that k has characteristic zero. Otherwise, we must instead work with $\widetilde{K}_0(\mathrm{Var}/k)$, which is the quotient of $K_0(\mathrm{Var}/k)$ by the relations generated by $[X] - [Y]$ whenever there is a radical surjective morphism $X \to Y$ of varieties over k; the product in $\widetilde{K}_0(\mathrm{Var}/k)$ is given by the reduced product of algebraic varieties. We denote the class of $\mathbb{A}^1_k$ by $\mathbb{L}$ in $K_0(\mathrm{Var}/k)$.

Let X be a quasiprojective variety over k. For $d \geq 1$, the symmetric group S_d acts on X^d, and the quotient by this action gives X_d, the *d-th symmetric product* of X. By convention, we set $X_0 = \mathrm{Spec}(k)$. Kapranov [76] defines the *motivic zeta function of* X with coefficients in the Grothendieck ring:

$$Z_{\mathrm{mot}}\big(X; t\big) = \sum_{d \geq 0} \big[X_d\big] t^d \qquad \in \quad 1 + t \cdot K_0(\mathrm{Var}/k)[\![t]\!].$$

This generalizes Weil's zeta function for varieties over finite fields to the motivic setting. When X is a smooth projective curve, $Z_{\mathrm{mot}}(X, t)$ is rational (see e.g. [76] and [84]).

Here we propose a natural generalization $\mathrm{Z}_{\mathrm{div}}(X, \vec{p})$ (see Definition 3.3.5) of Kapranov's motivic zeta function for a stable curve X with n marked points $\vec{p}$ that takes into account the behavior at the nodes and the marked points. The basic idea is to replace the symmetric power X_d in the definition of Kapranov's zeta function by the fiber over $(X, \vec{p})$ in a quotient of Hassett's moduli space of weighted stable curves of type $(1^n, \epsilon^d)$ (as in [69]). In the case $n = 0$, this space functions as a natural desingularization of the moduli space of effective divisors on X (see [99, Section 2]). When X is smooth and does not have marked points, our coefficients equal the symmetric power, giving $Z_{\mathrm{div}}(X, t) = Z_{\mathrm{mot}}(X, t)$. Our main result is the following Theorem 3.3.2. In the subsections that follow, we give the necessary background and prove this result.

Theorem 3.3.2 ([27])**.** *Let $(X, \vec{p})$ be a stable quasiprojective curve over k with n marked points $\vec{p}$. Then $\mathrm{Z}_{\mathrm{div}}(X, \vec{p}; t)$ is rational over $K_0(\mathrm{Var}/k)$. Moreover, if $G = (E, V)$ is the dual graph of X, then*

$$\mathrm{Z}_{\mathrm{div}}\left(X, \vec{p}; t\right) = \left(\frac{1 - \mathbb{L}t}{1 - \mathbb{L}t - t + t^2}\right)^{|E|+n} (1-t)^{2|E|+n} \prod_{v \in V} \mathrm{Z}_{\mathrm{mot}}\left(\widetilde{X}_v; t\right),$$

where $\widetilde{X}_v$ is the normalization of the component of X corresponding to the vertex $v \in V$.

In [17] Bejleri, Ranganathan, and Vakil define a motivic Hilbert zeta function $Z_{\mathrm{Hilb}}(X; t)$, where the coefficients are given by Hilbert schemes of points on a variety X. Their zeta function is sensitive to the singularities of X, while also agreeing with the usual motivic zeta function when X is smooth. They show that the motivic Hilbert zeta function of a reduced curve is rational. In contrast to our divisorial zeta function, the motivic Hilbert zeta function in [17] does not take into account marked points. Using [17, Lemma 2.1, Corollary 2.2, and Proposition 6.1], one can calculate that, for a nodal quasiprojective curve X with dual graph $G=(V,E)$, we have

$$Z_{\mathrm{Hilb}}\big(X;t\big)=\big(1-t+\mathbb{L}t^2\big)^{|E|}\cdot\prod_{v\in V}Z_{\mathrm{mot}}\big(\widetilde{X}_v;t\big)\ . \tag{3.1}$$

It is instructive to compare our formula in Theorem 3.3.2 as well as formula (3.1) for the Hilbert motivic zeta function with the formula for the usual Kapranov motivic zeta function $Z_{\mathrm{mot}}(X;t)$. Using [41, Chapter 7, Proposition 1.1.7] (which is also stated as Lemma 3.3.12 below) one may calculate that

$$Z_{\mathrm{mot}}\big(X;t\big)=(1-t)^{|E|}\cdot\prod_{v\in V}Z_{\mathrm{mot}}\big(\widetilde{X}_v;t\big)\ .$$

While $Z_{\mathrm{mot}}\big(X;t\big)$ appears to be insensitive to the nodal singularities of X, both $\mathrm{Z}_{\mathrm{div}}\left(X;t\right)$ and $Z_{\mathrm{Hilb}}\big(X;t\big)$ see the nodes by adding extra components.

Kapranov's motivic zeta function is known to be irrational for many surfaces. Let X be a smooth projective connected surface. In [83] Larsen and Lunts prove that X is only pointwise rational when X has Kodaira dimension -1 (over $\mathbb{C}$) and in [82] they show (over any field) that $Z_{\mathrm{mot}}(X;t)$ is not pointwise rational when the Kodaira dimension of X is ≥ 2.

Effective Divisors on Pointed Stable Curves

Let k be an algebraically closed field of characteristic 0 and let $g,n\geq 0$ with $2g-2+n>0$.

Definition 3.3.3. Define a category $\overline{\mathcal{D}\mathrm{iv}}_{g,n,d}$ fibered in groupoids over schemes, whose objects are tuples $(\pi\colon X'\to S,\vec{p}',D)$ consisting of the following data:

(i) $\pi\colon X'\to S$ is a flat and proper morphism of connected nodal curves;

(ii) $\vec{p}'$ is an ordered collection of sections $p'_1,\ldots,p'_n\colon S\to X$ that do not meet the nodes in each fiber X'_s of π; and

(iii) D is a relative effective Cartier divisor of degree d on X' over S, whose support does not intersect the nodes and sections in each fiber X'_s of X' over S.

We also require that the twisted canonical divisor

$$K_\pi+\epsilon D+p'_1+\ldots+p'_n$$

is π-relatively ample, where $\epsilon=\frac{1}{d}>0$.

Denote by $\overline{\mathcal{M}}_{g,1^n,\epsilon^d}$, the moduli space of weighted stable curves of genus g with n marked points of weight one and d marked points of weight $\epsilon = \frac{1}{d} > 0$ in the sense of [69]. There is a natural operation of S_d on $\overline{\mathcal{M}}_{g,1^n,\epsilon^d}$ that permutes the d marked points of weight ϵ. Then $\overline{\mathcal{D}\mathrm{iv}}_{g,n,d}$ is naturally equivalent to the relative coarse moduli space of

$$\left[\overline{\mathcal{M}}_{g,1^n,\epsilon^d}/S_d\right]$$

over $\overline{\mathcal{M}}_{g,n}$ in the sense of [2, Theorem 3.1]. So, in particular, it is a smooth and proper Deligne-Mumford stack with a projective coarse moduli space. There is a natural forgetful morphism $\overline{\mathcal{D}\mathrm{iv}}_{g,n,d} \to \overline{\mathcal{M}}_{g,n}$ and we write $\mathcal{D}\mathrm{iv}_{g,n,d}$ for its restriction to $\mathcal{M}_{g,n}$. The complement of $\mathcal{D}\mathrm{iv}_{g,n,d}$ in $\overline{\mathcal{D}\mathrm{iv}}_{g,n,d}$ has (stack-theoretically) normal crossings.

Remark 3.3.4. For $n = 0$, the moduli space $\overline{\mathcal{D}\mathrm{iv}}_{g,d}$ was constructed in [99, Section 2]. It is also equal to a special case of the moduli space of stable quotients, as defined in [89, Section 4].

Let $(X, \vec{p}) = (X, p_1, \ldots, p_n)$ be a stable marked curve of genus g given by a morphism $\mathrm{Spec}(k) \to \overline{\mathcal{M}}_{g,n}$. The fiber over this point is given by

$$\mathrm{Div}_d^+\left(X, \vec{p}\right) := \overline{\mathcal{D}\mathrm{iv}}_{g,n,d} \times_{\overline{\mathcal{M}}_{g,n}} \mathrm{Spec}(k).$$

This describes tuples $(X', \vec{p}', D)$ consisting of

(i) a nodal curve X';

(ii) a collection of marked points $\vec{p}' = (p_1', \ldots, p_n')$ of X' such that $p_1', \ldots, p_n'$ do not meet the nodes of X' and the stabilization of $(X', \vec{p}')$ is isomorphic to $(X, \vec{p})$;

(iii) a relative effective Cartier divisor D of degree d on X' whose support does not intersect the nodes or marked points of X'.

We also require that the twisted canonical divisor

$$K + \epsilon D + p_1' + \cdots + p_n'$$

is ample, where $\epsilon = \frac{1}{d} > 0$. If X is smooth and has no marked points, the space $\mathrm{Div}_d^+(X)$ gives effective divisors on X and is the d-th symmetric power X_d (see [98, Theorem 3.13]).

If X is quasiprojective, we choose a compactification $\overline{X}$ of X by smooth points and define $\mathrm{Div}_d^+(X, \vec{p})$ to be the open locus in $\mathrm{Div}_d^+(\overline{X}, \vec{p})$ where the support of D does not intersect the preimage of the boundary $\overline{X} - X$ in X'. This does not depend on the choice of $\overline{X}$.

Now, we describe the strata of $\mathrm{Div}_d^+(X, \vec{p})$ as in [28]. We associate to $(X', \vec{p}', D)$ a dual stable pair $\left(G', \mathrm{mdeg}(D)\right)$ as follows: The graph G' is the dual graph of $(X', \vec{p}')$, where the vertices v of G' each correspond to a component X_v' of X'. For a node between components X_v' and $X_{v'}'$ of X, there is an edge between vertices v and v' of G'. For a marked point in a component X_v we add a leg at v. The restriction of D to each component X_v' defines a

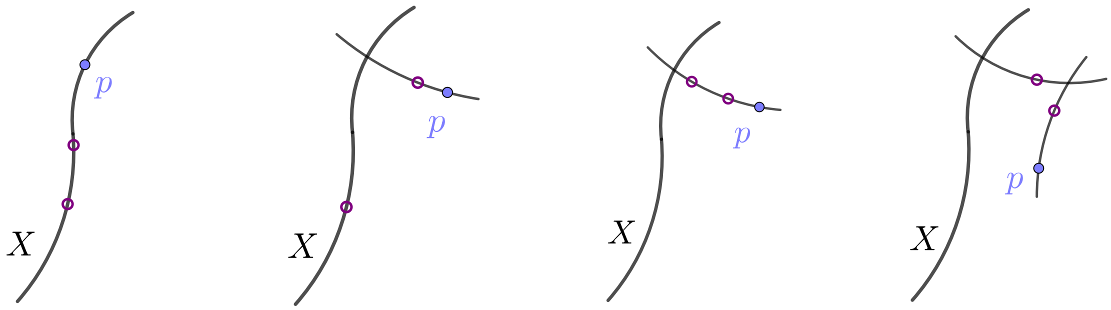


Figure 3.9: Let (X, p) be a smooth curve with genus $g \geq 1$ and one marked point p. In this case, $\mathrm{Div}_2^+(X, p)$ has four strata, corresponding to the pictured combinatorial types of marked stable curves and divisors.

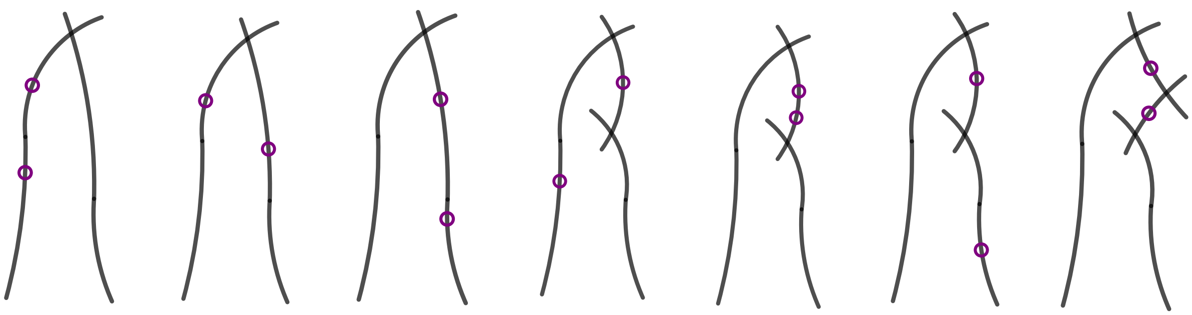


Figure 3.10: Let X be a curve with two smooth components each having genus larger than one meeting in a node. In this case, $\mathrm{Div}_2^+(X)$ has seven strata, corresponding to the pictured combinatorial types of stable curves and divisors.

divisor $\mathrm{mdeg}(D) = \sum_v \deg(D|_{X'_v}) \cdot v$ on G'. The graph G' is a subdivision of G, the dual graph of $(X, \vec{p})$. The pair (G', D) is a stable pair over G, meaning that the degree of D is at least 1 on all exceptional vertices of G'. Denote by $\Delta(G, d)$ the collection of all stable pairs of degree d over G.

One can generalize the results in [28, Section 3.2] to show that the strata of $\mathrm{Div}_d^+(X, \vec{p})$ are precisely the locally closed subsets on which the dual pairs are constant. Denote by $\mathrm{Div}^+_{(G',D)}(X, \vec{p})$ the locus of points in $\mathrm{Div}_d^+(X, \vec{p})$ whose dual pair is (G', D).

Divisorial Motivic Zeta Function

Definition 3.3.5. Let $(X, \vec{p})$ be a stable marked quasiprojective curve over k of genus g with n marked points. The *divisorial motivic zeta function of* $(X, \vec{p})$ is defined to be

$$Z_{\mathrm{div}}\big(X, \vec{p}; t\big) = \sum_{d \geq 0} \big[\,\mathrm{Div}_d^+(X, \vec{p})\big] t^d \qquad \in \quad 1 + t \cdot K_0(\mathrm{Var}/k)[\![t]\!].$$

We break up the classes $\big[\,\mathrm{Div}_d^+(X, \vec{p})\big]$ along their strata using the following lemma.

Lemma 3.3.6 ([41, Chapter 2, Lemma 1.3.3]). *Let Y be a variety over k and suppose we have a decomposition $Y = Y_1 \sqcup \cdots \sqcup Y_r$ where all Y_i are locally closed subvarieties of Y. Then:*

$$\big[Y\big] = \big[Y_1\big] + \cdots + \big[Y_r\big].$$

Lemma 3.3.7 ([27]). *Let $(X, \vec{p})$ be a stable marked quasiprojective curve over k.*

$$\mathrm{Z_{div}}\left(X, \vec{p}; t\right) = \sum_{d \geq 0} t^d \sum_{\Delta(G,d)} \left[\,\mathrm{Div}_{(G',D)}(X, \vec{p})\right],$$

where the second sum is over stable pairs (G', D) of degree d over the dual graph G of $(X, \vec{p})$.

Proof. This follows from the description of the strata of $\mathrm{Div}_d^+(X, \vec{p})$ and Lemma 3.3.6. □

Given a stable marked quasiprojective curve $(X, \vec{p})$ and points $q_1, \ldots, q_m \in X$, write

$$\big(X, \vec{p}, -\vec{q}\big) := \big(X \backslash \{q_1, \ldots, q_m\}, \{p_1, \ldots, p_n\} \backslash \{q_1, \ldots, q_m\}\big).$$

Given a connected component X_v of X, we denote the non-special locus of X_v by

$$X_v^{(0)} := \big(X_v, -\vec{p}, -\operatorname{Sing}(X_v)\big).$$

We now describe the class of $\mathrm{Div}^+_{(G',D)}(X, \vec{p})$ in the Grothendieck ring.

Lemma 3.3.8 ([27]). *Let $(X, \vec{p})$ be a stable marked quasiprojective curve, and let (G', D) be a stable pair such that the stabilization of G' is equal to the dual graph G of X. Then:*

$$\left[\,\mathrm{Div}_{(G',D)}(X, \vec{p})\right] = \prod_{v \in G} \left[(X_v^{(0)})_{D(v)}\right] \prod_{v' \in G' \backslash G} \left[\mathbb{G}_{D(v')-1}\right],$$

where $\mathbb{G}$ denotes the one-dimensional algebraic torus $\mathbb{A}^1 - \{0\}$ over k.

Proof. A point in $\mathrm{Div}^+_{(G',D)}(X, \vec{p})$ gives a divisor of degree $D(v)$ on $X_v^{(0)}$ for each $v \in G'$. On non-exceptional components, these are points in $(X_v^{(0)})_{D(v)}$. On exceptional components, these are points in $\mathbb{G}_{D(v')-1}$. □

We now prove a series of propositions which give us a way to iteratively relate the divisorial motivic zeta function of a stable curve to the divisorial motivic zeta functions of its components.

Proposition 3.3.9 ([27]). Let $(X, \vec{p}, q_1, q_2)$ be a stable quasiprojective curve with $n+2$ marked points and let $(X/_{q_1 \sim q_2}, \vec{p})$ be the curve with a nodal self-intersection obtained by gluing q_1 and q_2. Then:

$$\mathrm{Z_{div}}\left(X, \vec{p}, q_1, -q_2; t\right) = \mathrm{Z_{div}}\left(X/_{q_1 \sim q_2}, \vec{p}; t\right).$$

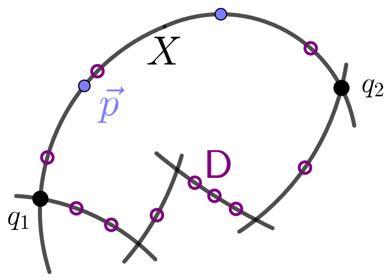


Figure 3.11: Stable pair in Proposition 3.3.9

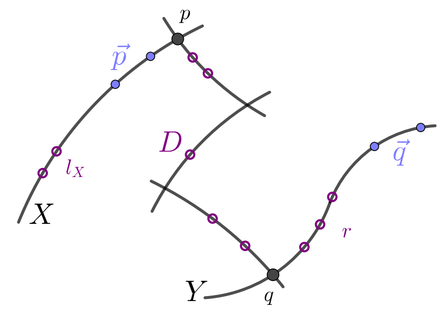


Figure 3.12: Stable pair in Proposition 3.3.10

Proof. Using Lemmas 3.3.7 and 3.3.8, we have that

$$\mathrm{Z}_{\mathrm{div}}\left(X/_{q_1\sim q_2},\vec{p};t\right)=\sum_{d\geq 0}t^d\sum_{j=0}^{d}\left[\mathrm{Div}_j^+(X,\vec{p},-p,-q)\right]\sum_{\alpha\in\mathrm{comp}(d-j)}\left[\mathbb{G}\right]_\alpha=\mathrm{Z}_{\mathrm{div}}\left(X,\vec{p},q_1,-q_2;t\right),$$

where we interpret j to be the degree of the divisor restricted to $(X,\vec{p},-q_1,-q_2)$. The last sum is taken over all ordered ways to write the integer $d-j$ as a sum of positive integers, and $\left[\mathbb{G}\right]_\alpha:=\left[\mathbb{G}_{\alpha_1-1}\right]\cdots\left[\mathbb{G}_{\alpha_l-1}\right]$. We see this because in both cases the strata are given by stable pairs whose exceptional components emanate from the point q_1 (see Figure 3.11). □

Proposition 3.3.10 ([27])**.** Let $(X,\vec{p},p)$ be a stable quasiprojective curve with $n+1$ marked points and let $(Y,\vec{q},q)$ be a stable quasiprojective curve with $m+1$ marked points. Denote by $X\sqcup_{p\sim q}Y$ the curve obtained by gluing X to Y along a node at the points p,q. Then:

$$\mathrm{Z}_{\mathrm{div}}\left(X\sqcup_{p\sim q}Y,\vec{p},\vec{q};t\right)=\mathrm{Z}_{\mathrm{div}}\left(X,\vec{p},p;t\right)\cdot\mathrm{Z}_{\mathrm{div}}\left(Y,\vec{q},-q;t\right).$$

Proof. By Lemmas 3.3.7 and 3.3.8, we have that the divisorial zeta function is

$$\mathrm{Z}_{\mathrm{div}}\left(X\sqcup_{p\sim q}Y,\vec{p},\vec{q};t\right)=\sum_{d\geq 0}t^d\sum_{r+l=d}\left[\mathrm{Div}_r^+(Y,\vec{q},-q)\right]\left(\sum_{l_x=0}^{l}\left[\mathrm{Div}_{l_x}^+(X,\vec{p},-p)\right]\sum_{\alpha\in\mathrm{comp}(l-l_x)}\left[\mathbb{G}\right]_\alpha\right).$$

where we interpret r to be the degree of the divisor restricted to Y, l to be the degree of the divisor restricted to X and the exceptional components, and l_x to be the degree of the divisor restricted to X (See Figure 3.12). We observe that this is in fact a product of series:

$$\left(\sum_{d\geq 0}t^d\left[\mathrm{Div}_d^+(Y,\vec{q},-q)\right]\right)\left(\sum_{d\geq 0}t^d\sum_{d_x=0}^{d}\left[\mathrm{Div}_{d_x}^+(X,\vec{p},-p)\right]\sum_{\alpha\in\,\mathrm{comp}(d-d_x)}\left[\mathbb{G}\right]_\alpha\right)$$

On the left we have $\mathrm{Z}_{\mathrm{div}}\left(Y,\vec{q},-q;t\right)$ and on the right we have $\mathrm{Z}_{\mathrm{div}}\left(X,\vec{p},p;t\right)$. □

Proposition 3.3.11 ([27])**.** Let X be a smooth quasiprojective curve and let $p \in X$ be a point. Then,

$$\mathrm{Z}_{\mathrm{div}}\big(X, -p; t\big) = \mathrm{Z}_{\mathrm{div}}(X; t) \cdot (1 - t).$$

First, we need the following lemma.

Lemma 3.3.12 ([41, Chapter 7, Proposition 1.1.7])**.** *If X is a quasiprojective variety, and $Y \hookrightarrow X$ is a closed subvariety with complement U, then*

$$\big[X_n\big] = \sum_{i+j=n} \big[Y_i\big] \cdot \big[U_j\big].$$

Proof of Proposition 3.3.11. In our case, we take $Y = p$ and $U = X(-p)$. Then,

$$\big[X_n\big] = \sum_{i+j=n} \big[1\big] \cdot \big[(X, -p)_j\big] = \sum_{j=0}^{n} \big[(X, -p)_j\big].$$

Since X is smooth, we have $X_d = \mathrm{Div}_d^+(X)$. Applying the above equation, we find

$$\mathrm{Z}_{\mathrm{div}}(X; t) = \sum_{d=0}^{\infty} t^d \sum_{j=0}^{d} \big[X(-p)_j\big] = \mathrm{Z}_{\mathrm{div}}(X, -p; t)(1 + t + t^2 + t^3 + \cdots) = \frac{\mathrm{Z}_{\mathrm{div}}(X, -p; t)}{1 - t}.$$

□

Proposition 3.3.13 ([27])**.** Denote by $\mathbb{G}$ be the one-dimensional algebraic torus $\mathbb{A}^1 - \{0\}$ over k and by $\mathbb{L}$ the class of $\mathbb{A}^1$ in the Grothendieck ring. Then:

$$\mathrm{Z}_{\mathrm{div}}(\mathbb{G}; t) = \mathrm{Z}_{\mathrm{mot}}(\mathbb{G}; t) = \frac{1 - t}{1 - \mathbb{L}t}.$$

Proof. By Lemma 3.3.12 applied to $X = \mathbb{A}^1$ and $U = \mathbb{G}$, we find that $\big[\mathbb{A}^1_d\big] = \sum_{i=0}^{d} \big[\mathbb{G}_i\big]$. Using the fact that $(\mathbb{A}^1)_d = \mathbb{A}^d$, and therefore $\big[(\mathbb{A}^1)_d\big] = \mathbb{L}^d$, we have:

$$\mathrm{Z}_{\mathrm{div}}(\mathbb{G}; t) = \sum_{d \geq 0} t^d \big[\mathbb{G}_d\big] = \sum_{d \geq 0} t^d \left(\mathbb{L}^d - \sum_{i=0}^{d-1} \big[\mathbb{G}_i\big]\right) = \sum_{d \geq 0} t^d \mathbb{L}^d - \sum_{d \geq 0} t^d \sum_{i=0}^{d-1} \big[\mathbb{G}_i\big].$$

After re-indexing, we find:

$$\begin{aligned} \mathrm{Z}_{\mathrm{div}}(\mathbb{G}; t) &= \frac{1}{1 - \mathbb{L}t} - t \sum_{d \geq 0} t^d \sum_{i=0}^{d} \big[\mathbb{G}_i\big] \\ &= \frac{1}{1 - \mathbb{L}t} - t \cdot \mathrm{Z}_{\mathrm{div}}(\mathbb{G}; t) \cdot (1 + t + t^2 + \cdots) = \frac{1}{1 - \mathbb{L}t} - \frac{t \cdot \mathrm{Z}_{\mathrm{div}}(\mathbb{G}; t)}{1 - t}. \end{aligned}$$

Solving for $\mathrm{Z}_{\mathrm{div}}(\mathbb{G}, t)$, we find that $\mathrm{Z}_{\mathrm{div}}(\mathbb{G}, t) = \frac{1-t}{1-\mathbb{L}t}$, as claimed. The equality $\mathrm{Z}_{\mathrm{div}}(\mathbb{G}; t) = \mathrm{Z}_{\mathrm{mot}}(\mathbb{G}; t)$ holds, since $\mathbb{G}$ is a smooth curve without marked points. □

Proposition 3.3.14 ([27])**.** Let $(X, \vec{p}, q)$ be a quasiprojective stable marked curve. Then,

$$\mathrm{Z}_{\mathrm{div}}\left(X, \vec{p}, q; t\right) = \mathrm{Z}_{\mathrm{div}}\left(X, \vec{p}, -q; t\right) \cdot \frac{1 - \mathbb{L}t}{1 - \mathbb{L}t - t + t^2}.$$

Proof. We have

$$\begin{aligned}\mathrm{Z}_{\mathrm{div}}\left(X, \vec{p}, q; t\right) &= \sum_{d \geq 0} t^d \sum_{j=0}^{d} \left[\mathrm{Div}^{+}_{d-j}(X, \vec{p}, -q)\right] \sum_{\alpha \in \mathrm{comp}(j)} \left[\mathbb{G}\right]_{\alpha} \\ &= \left(\sum_{d \geq 0} \left[\mathrm{Div}^{+}_{d}(X, \vec{p}, -q)\right] t^d\right) \left(1 + \sum_{d \geq 1} t^d \sum_{\alpha \in \mathrm{comp}(d)} \left[\mathbb{G}\right]_{\alpha}\right),\end{aligned}$$

where we think of j as the degree of the divisor restricted to exceptional components. We evaluate the right term in this product. Re-organizing by the length of the composition,

$$\begin{aligned}1 + \sum_{d \geq 1} t^d \sum_{\alpha \in \mathrm{comp}(d)} \left[\mathbb{G}\right]_{\alpha} &= 1 + \sum_{k \geq 1} \sum_{d \geq k} t^d \sum_{\substack{\alpha \in \mathrm{comp}(d) \\ |\alpha| = k}} \left[\mathbb{G}\right]_{\alpha} \\ &= 1 + \sum_{k \geq 1} \left(t \cdot \mathrm{Z}_{\mathrm{div}}(\mathbb{G}; t)\right)^k = \frac{1}{1 - t \cdot \mathrm{Z}_{\mathrm{div}}(\mathbb{G}, t)}.\end{aligned}$$

Applying Proposition 3.3.13 and simplifying, we obtain the result. □

We are now ready to prove Theorem 3.3.2 from the introduction.

Proof of Theorem 3.3.2. Let $(X, \vec{p})$ be a pointed stable curve over k with dual graph G. We use Propositions 3.3.9 and 3.3.10 to break up X into its components. Each node in X yields a new marked point and a new hole. By Proposition 3.3.14, exchanging the $|E| + n$ marked points for a holes leads to factors of

$$\frac{1 - \mathbb{L}t}{1 - \mathbb{L}t - t + t^2}.$$

Stitching the $2|E| + n$ holes leads to factors of $1 - t$ by Proposition 3.3.11. So we obtain

$$\mathrm{Z}_{\mathrm{div}}(X, \vec{p}; t) = \left(\frac{1 - \mathbb{L}t}{1 - \mathbb{L}t - t + t^2}\right)^{|E|+n} (1 - t)^{2|E|+n} \prod_{v \in V} \mathrm{Z}_{\mathrm{div}}\left(\widetilde{X}_v, t\right).$$

Finally, we use that the motivic zeta function is equal to the divisorial zeta function for each $\widetilde{X}_v$ because $\widetilde{X}_v$ is smooth and does not have marked points. □

Conclusion

In this chapter, we studied many instances of tropical and classical moduli spaces. The tropicalization of a classical moduli space typically has cells corresponding to the boundary strata in the compactification. Points in these cells represent tropical objects which are dual to the classical object represented by points the corresponding stratum. Using this description of the boundary strata for a particular moduli space, we defined a divisorial motivic zeta function for marked stable curves, and proved that it is rational.

4

Skeletons

Given a curve X, we have seen how to associate to X its *abstract tropicalization*, which is the dual tropical curve Γ of the special fiber of a semistable model of X. In practice, finding the abstract tropicalization of a general curve is difficult and there is no known algorithm to do this [45, Remark 3]. In low genus, and for special types of curves, methods for computing abstract tropicalizations exist. In this chapter, we describe and compare the existing methods for computing abstract tropicalizations of curves in low genus. Then, we compute abstract tropicalizations of hyperelliptic and superelliptic curves algorithmically.

The original work in this chapter appears in the following places. The section "Hyperelliptic Curves" and the subsection "Existence of Faithful Tropicalizations" are based on [22], which is joint work with with Barbara Bolognese and Lynn Chua. It was published in the book *Combinatorial Algebraic Geometry: Selected Papers from the 2016 Apprenticeship Program.* The section "Superelliptic Curves" is joint work with Paul Helminck [26] and it will be published in Advances in Geometry.

We now highlight some of the difficulties of finding abstract tropicalizations of curves, offering the following example as motivation for why this is a difficult problem.

Example 4.0.1. ([116, Problem 9 on Abelian Combinatorics]) We begin with a curve in $\mathbb{P}^2$, given by the zero locus of

$$\begin{aligned} f(x,y,z) = {} & 41x^4 + 1530x^3y + 3508x^3z + 1424x^2y^2 + 2490x^2yz \\ & - 2274x^2z^2 + 470xy^3 + 680xy^2z - 930xyz^2 + 772xz^3 \\ & + 535y^4 - 350y^3z - 1960y^2z^2 - 3090yz^3 - 2047z^4, \end{aligned}$$

defined over $\mathbb{Q}_2$. We first compute its embedded tropicalization. The induced regular subdivision of the Newton polygon will be trivial, since the 2-adic valuation of the coefficients on the x^4, y^4, z^4 terms is each 0. Therefore, we can detect no information about the structure of the abstract tropicalization from this embedded tropicalization.

On the other hand, if we apply a change of coordinates

$$x = \frac{1}{12}X + \frac{1}{2}Y - \frac{1}{12}Z, \quad y = \frac{1}{2}X - \frac{1}{2}Y, \quad z = -\frac{5}{12}X - \frac{1}{12}Z,$$

we obtain

$$0 = -256X^3Y - 2X^2Y^2 - 256XY^3 - 8X^2YZ - 8XY^2Z - XYZ^2 - 2XZ^3 - 2YZ^3.$$

We may then calculate the regular subdivision of the Newton polygon in *Polymake 3.0* [58], weighted by the 2-adic valuations of the coefficients.

The embedded tropicalization and corresponding metric graph, with edge lengths, are depicted in Figure 4.1. Since all vertices are trivalent, all edges have multiplicity 1, and $\dim(H_1(\text{trop}(X), \mathbb{R})) = 3$, by Theorem 1.3.29 we conclude that this is the abstract tropicalization of the curve.

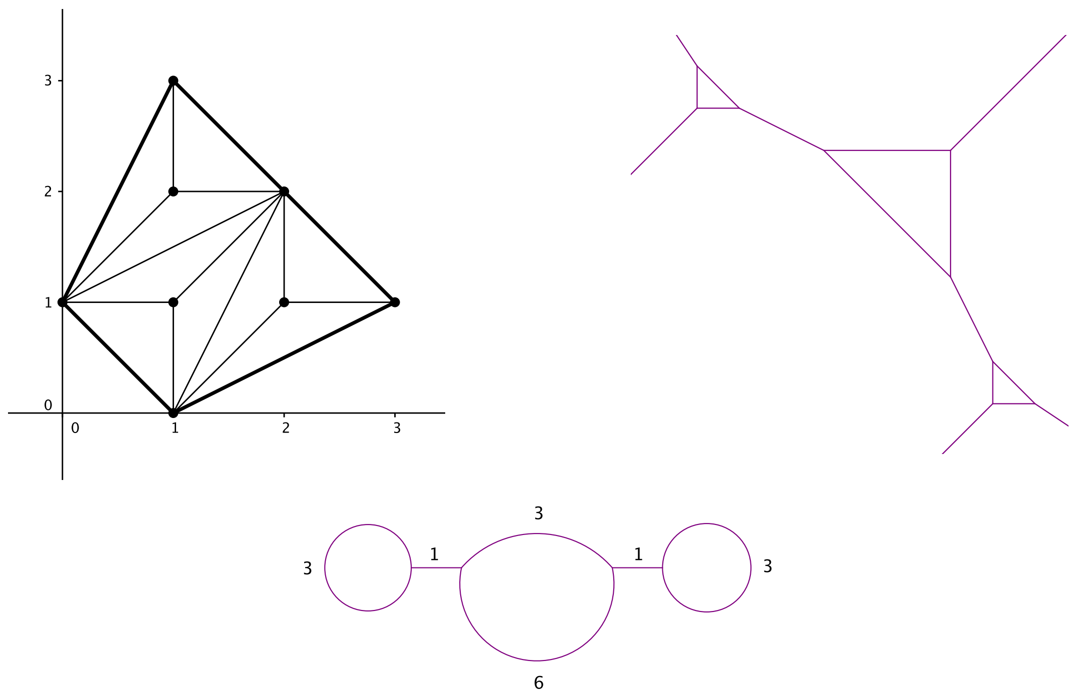


Figure 4.1: The Newton polygon from Example 4.0.1 together with its unimodular triangulation on the upper left, the embedded tropicalization on the upper right, and the metric graph at the bottom.

So, given an embedding of a curve, the corresponding embedded tropicalization may contain partial or no information about the structure of the abstract tropicalization. Much work has been done on the problem of computing abstract tropicalizations of curves. In what follows, we describe the known results in this active area of research.

4.1 Genus One

Let X be a genus one curve over K, meaning that it is an elliptic curve. The curve X can be embedded in $\mathbb{P}^2$ so that it is the zero locus of a single ternary cubic polynomial

$$f(x, y, z) = c_{300}x^3 + c_{210}x^2y + \cdots + c_{012}yz^2 + c_{003}z^3$$

whose discriminant does not vanish. Assume that the residue field of K has characteristic not equal to 2 or 3. There are two types of genus 1 tropical curves: the point of weight 1 and the cycle of length l. In this section we describe several methods for determining the tropical curve associated to X given an equation defining X. One method will compute the tropical curve by studying maps from X to $\mathbb{P}^1$. Another method will compute the tropical curve from the j-invariant of X. Finally, one method will compute a good embedding of the curve such that the embedded tropicalization contains the abstract tropicalization.

Covers of Trees with Four Leaves

Every elliptic curve has an embedding with its defining equation in *Weierstrass normal form*

$$g(x, y, z) = y^2z - x^3 - axz^2 - bz^3$$

for some $a, b \in K$. Tropicalizing curves with this embedding never yields a cycle in the tropicalization. In fact, the tropicalization can only ever be a tree as in Figure 4.2. So, we cannot use this embedding to distinguish what type of abstract tropicalization the curve has.

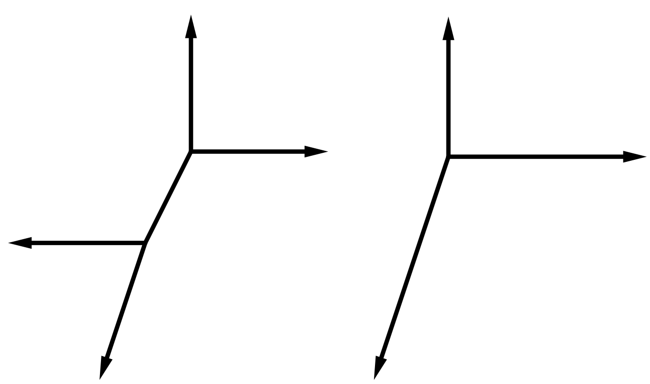


Figure 4.2: Tropicalizations of elliptic curves in normal form.

Consider the map $X \to \mathbb{P}^1$ given by sending $(x : y : z) \mapsto (x : z)$. This map will be ramified at the roots r_1, r_2, r_3 of $x^3 - ax - b$ and ∞. Up to automorphism of $\mathbb{P}^1$, the four ramification points give a point

$$\begin{bmatrix} r_1 & r_2 & r_3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in \mathrm{Gr}(2, 4).$$

Then, the valuation of the corresponding Plücker vector gives a metric on a tree T with 4 leaves as in Section 3.1. The metric graph Γ, the abstract tropicalization of X, double-covers T. If T has no interior edge, then Γ is a point of weight 1. If T has an interior edge of length l, then Γ is a cycle of length $2l$.

The j-Invariant

Elliptic curves are distinguished by their j-invariants. The j-invariant can be written explicitly in terms of the coefficients of f and is an element of the ground field K. The j-invariant also can be used to determine the tropical curve completely.

Theorem 4.1.1 ([77])**.** *Let X be an elliptic curve over K. If the j-invariant of X has non-negative valuation, then the abstract tropicalization of X is a vertex with weight one. Otherwise, it is a cycle with length minus the valuation of the j-invariant.*

Faithful Tropicalization

In [46], Chan and Sturmfels improve upon this result by providing an explicit embedding so that when the curve is tropicalized, it is in *honeycomb form* as in Figure 4.3.

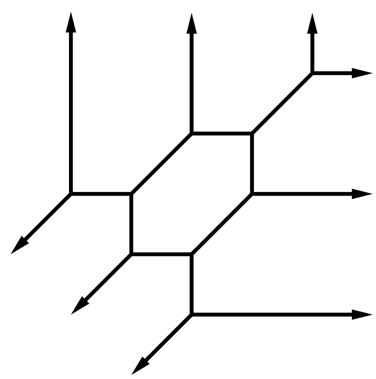


Figure 4.3: Tropicalization of an elliptic curve in honeycomb form.

Theorem 4.1.2 ([46])**.** *Every elliptic curve with* $\mathrm{val}_K(j) < 0$ *has an embedding such that the embedded tropicalization is a honeycomb (see Figure 4.3).*

The honeycomb reveals that the curve has genus one, and it remembers the valuation of j. Indeed, the lattice length of the cycle equals $-\mathrm{val}(j)$.

4.2 Genus Two

Let X be a curve of genus 2. There are now seven possibilities for the abstract tropicalization Γ of X, shown in Figure 4.4.

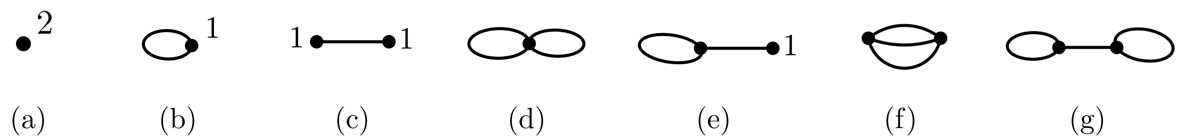


Figure 4.4: The seven types of genus two tropical curves.

Given a genus 2 curve X, we wish to compute which of these seven types the abstract tropicalization Γ is, and what its edge lengths are. We now describe three methods for doing this. These are analogous to the genus one case, but each will pose new complications. One method will study maps of the curve to $\mathbb{P}^1$. Another method will use invariants associated to the curve to compute the abstract tropicalization. The last method will find a good embedding of the curve to carry out the embedded tropicalization, so that the skeleton Γ can be detected from the embedded tropicalization.

Covers of Trees with Six Leaves

Every genus two curve is *hyperelliptic*, meaning that there is a two-to-one map from $X \to \mathbb{P}^1$. In [111], the authors give a method for tropicalizing genus 2 curves using this map. The main idea is that the map $X \to \mathbb{P}^1$ tropicalizes to a map $\Gamma \to T$, where T is the tree obtained by studying the tropicalization of $\mathbb{P}^1$ marked at the ramification points of the double cover. Here is the statement of their result.

Theorem 4.2.1 ([111]). *There is a commutative diagram*

$$\begin{array}{ccc} \mathcal{M}_{0,6} & \longrightarrow & \mathcal{M}_{0,6}^{trop} \\ \downarrow & & \downarrow \\ \mathcal{M}_2 & \longrightarrow & \mathcal{M}_2^{trop} \end{array}$$

The map we wish to understand is the bottom one. We may study this map by instead studying the top horizontal map and the right vertical map, which we understand well.

The map $\mathcal{M}_{0,6} \to \mathcal{M}_2$ sends six marked points on $\mathbb{P}^1$ to the genus 2 curve obtained from the hyperelliptic cover of $\mathbb{P}^1$ with the 6 marked ramification points. All genus 2 curves arise in this way, by specifying 6 points in $\mathbb{P}^1$. The curve X is then the double cover of $\mathbb{P}^1$ branched at the 6 points. Every genus 2 curve can be written in the form

$$y^2 = f(x).$$

for a polynomial f of degree 5 or 6. With this embedding, the cover to $\mathbb{P}^1$ is given by $(x, y) \mapsto x$, and the marked points are precisely the roots of f, plus possibly the point at infinity depending upon if f is degree 5.

The space $\mathcal{M}_{0,6}^{trop}$ is the space of trees with 6 taxa as in Section 3.1. Combinatorially, it agrees with the tropical Grassmannian trop($Gr(2,6)$) modulo its lineality space. So, $\mathcal{M}_{0,6}^{trop}$ has a tropical basis (Definition 1.2.11) given by the Plücker relations for $Gr(2,6)$. It is a fan in $\mathbb{TP}^{14}$ with 25 rays, 105 two dimensional cones, and 105 three dimensional cones. The dimension is the number of interior edges in the corresponding tree. The map

$$\mathcal{M}_{0,6} \to \mathcal{M}_{0,6}^{trop}$$

can now be described as follows. Denote the 6 points in $\mathbb{P}^1$ by $(a_i : b_i)$. Then, take the valuations of all 2×2 minors of the matrix

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_6 \\ b_1 & b_2 & \cdots & b_6 \end{bmatrix}.$$

This describes a point in $\mathbb{P}^{14}$, denoted

$$\Delta = (p_{12} : p_{13} : \cdots : p_{56}).$$

This point gives a tree metric on a tree with 6 taxa by taking $d(i,j) = -2p_{ij} + (n, \cdots, n)$ for a suitable constant n. From this tree metric, we may reconstruct the tree T using the Neighbor Joining Algorithm [104].

The map

$$\mathcal{M}_{0,6}^{trop} \to \mathcal{M}_{2}^{trop}$$

is a morphism of generalized cone complexes, and can be described as follows. Given a point in $\mathcal{M}_{0,6}^{trop}$, compute the tree associated to it including the interior edge lengths. This tree is one of the seven types in Figure 4.5. Then, the map is described by sending the tree to the corresponding tropical curve in Figure 4.4.

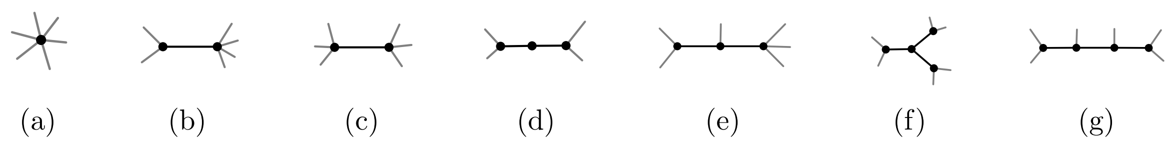


Figure 4.5: The seven types of trees with six leaves

For example, the caterpillar tree, Figure 4.5g, maps to the dumbell graph, Figure 4.4g. It remains to define the lengths of the edges in the corresponding tropical curve. If an interior edge in the tree has length l and is mapped to two-to-one by the corresponding edges in the tropical curve, then each edge in the preimage receives length l. Otherwise, the edge in the tropical curve receives length $l/2$. So, in the case of the dumbell, if all interior edges of the tree have length l, then the two loops of the dumbell obtain length $2l$, and the edge joining them obtains length $l/2$.

This method for finding the tropical curve takes advantage of the fact that all genus two curves are hyperelliptic. This allows us to look at the space of trees on six taxa by mapping valuations of the differences of the roots into the tropical Grassmannian. Then, the difficult task is to find the correct scaling factors for the edge lengths in the tropical curve. In Sections 4.4 and 4.5, we will extend this method to all hyperelliptic and superelliptic curves.

Example 4.2.2. Suppose exactly two roots of the polynomial f coincide in the residue field. Call these two points a_5, a_6, and set $a = v(a_5 - a_6) > 0$. This gives a point in $\mathcal{M}_{0,6}^{trop}$ the form

$$(0, \dots, 0, a) = (p_{1,2}, p_{1,3}, \dots, p_{5,6}).$$

This corresponds to the tree metric

$$(n, \dots, n, n - 2a) = (d_{1,2}, d_{1,3}, \dots, d_{5,6}).$$

So the tree is the type pictured in Figure 4.5b, with an interior edge length of a. This corresponds to the metric graph in Figure 4.4b. The length of the corresponding loop in the tropical curve is twice the length of the interior edge of the tree. So, the abstract tropicalization is a loop with length $2a$ and a vertex of weight 1.

Example 4.2.3. Consider the polynomial

$$y^2 = (x-1)(x-2)(x-3)(x-6)(x-7)(x-8),$$

with the 5-adic valuation. In $\mathcal{M}_{0,6}^{trop}$ we have the point

$$(0,0,1,0,0,0,0,1,0,0,0,1,0,0,0) = (p_{1,2}, p_{1,3}, \dots, p_{5,6}).$$

So that a possible tree metric is

$$(4,4,2,4,4,4,4,2,4,4,4,2,4,4,4) = (d_{1,2}, d_{1,3}, \dots, d_{5,6}).$$

This is a metric for the snowflake tree in Figure 4.5f with edge length 1 on each interior edge. So, the abstract tropicalization is given by the graph in Figure 4.4f with edges of length 2.

Igusa Invariants

As we saw with elliptic curves, the valuation of the j-invariant determines the semistable reduction type of the curve. Therefore, it is natural to wonder whether or not there are other invariants which play a similar role for higher genus. In [70], Helminck completely determines the semistable reduction type of a genus two curve using the *Igusa invariants* $\{J_2, J_4, J_6, J_8, J_{10}\}$ and $\{I_2, I_4, I_6, I_8, I_{12}\}$, first defined by Igusa in [72], as follows. Since X is hyperelliptic we have that X is given by an equation of the form $y^2 = f(x)$ where f has degree 5 or 6. The exact values of the invariants can then be written down in terms of the roots $x_1, \dots, x_6$ of the polynomial f. For instance,

$$J_2 = \frac{1}{8} \sum_{\text{fifteen}} (x_1 - x_2)^2 (x_3 - x_4)^2 (x_5 - x_6)^2,$$

where the sum is over all 15 ways of grouping 6 objects into pairs.

Definition 4.2.4. The *tropical Igusa invariants* are the valuations of J_i and I_i.

Using the tropical Igusa invariants, Helminck's theorem can determine the abstract tropicalization of X. The full theorem statement including all formulas may be found in [70]. Helminck also determines lengths of the edges in the tropical curve Γ.

Theorem 4.2.5 ([70])**.** *Let X be a curve of genus 2 over K. Then the edge lengths and reduction type of the abstract tropicalization Γ of X can be completely described in terms of the tropical Igusa invariants.*

Example 4.2.6. Suppose exactly two roots of the polynomial f coincide in the residue field. Call these two points a_5, a_6. Then the tropical Igusa invariants are

$$(0, 0, 0, 0, 2a, 0, 0, 0, 0, 0),$$

where $a = v(a_5 - a_6) > 0$. This tells us that the abstract tropicalization is a single loop of length $2a$ and a vertex of weight 1, as in Figure 4.4b.

Example 4.2.7. Consider the polynomial

$$y^2 = (x-1)(x-2)(x-3)(x-6)(x-7)(x-8),$$

with the 5-adic valuation. Its tropical Igusa invariants are

$$(0, 2, 2, 2, 6, 0, 0, 2, 2, 2).$$

From this, we may compute that the abstract tropicalization is the graph in Figure 4.4f with edges of length 2.

Tropical Modifications

Given the hyperelliptic equation $y^2 = f(x)$ defining a genus two curve, the embedded tropicalization is always a tree [51]. So, outside of the cases in Figures 4.4a and 4.4c, this embedded tropicalization will never be faithful. In [51], Cueto and Markwig give a method for re-embedding the curve in a simple way into three dimensions so that the corresponding embedded tropicalization is faithful. Their method is called *tropical modification*. Throughout this section we use the max convention.

Theorem 4.2.8 ([51])**.** *Tropical modifications can be used to explicitly re-embed the curve $y^2 = u\prod_{i=1}^{6}(x - r_i)$ into three dimensions so that the embedded tropicalization is faithful.*

We now give an overview of the method of tropical modifications and give an example.

Definition 4.2.9. Fix a tropical polynomial $F : \mathbb{R}^n \to \mathbb{T} = \mathbb{R} \cup \{\infty\}$. The graph of F is a rational polyhedral complex which is pure of dimension n. If F is not linear, the bend locus has codimension 1. At each cell σ in the bend locus, attach a new cell σ' spanned by σ and $-e_{n+1}$. The result is called the *tropical modification of* $\mathbb{R}^n$ *along* F.

Example 4.2.10. The tropical line in the plane is the tropical modification of $\mathbb{R}^1$ along $F = \max(0, x)$.

Let X be a curve in the plane defined by a polynomial $g(x, y) \in K[x^{\pm 1}, y^{\pm 1}]$, and let $f(x, y) \in K[x^{\pm 1}, y^{\pm 1}]$. Then the tropicalization of $I = \langle g, z - f\rangle \subset K[x^{\pm 1}, y^{\pm 1}, z^{\pm z}]$ is a tropical curve in the modification of $\mathbb{R}^2$ along the tropical polynomial $\operatorname{trop}(f)$. For special choices of f, this re-embedding can reveal hidden structure in the tropical curve.

In [51] it is shown that we may use a modification along a tropical polynomial of the form $\max(y, A + x, B + 2x)$ for $A, B \in \mathbb{R}$ to repair tropical curves coming from hyperelliptic embeddings. It turns out that the tropical modification is completely determined by the tropical plane curves in each coordinate direction.

Proposition 4.2.11 ([51])**.** Given a curve $X \subset (K^*)^2$ defined by $g(x, y)$ and a polynomial $f(x, y) = y - ax - bx^2$, the tropicalization of $V(g, z - f)$ is completely determined by $\operatorname{trop}(V(g))$, $\operatorname{trop}(V(\langle g, z - f\rangle \cap K[x^{\pm 1}, z^{\pm 1}]))$, and $\operatorname{trop}(V(\langle g, z - f\rangle \cap K[y^{\pm 1}, z^{\pm 1}]))$.

Example 4.2.12. Consider the genus 2 curve X over the Puiseaux series $\mathbb{C}\{\{t\}\}$ defined by the hyperelliptic equation

$$g = y^2 - x(x - (t^{14} + t^{16})^2)(x - (5 + t^9)^2)(x - (3 + t^8)^2)(x + (1 + t^5)^2).$$

Set f as above to be

$$f = y - (5 + t^9)(3 + t^8)(1 + t^5)x + (1 + t^5)x^2.$$

We will compute the tropical plane curve of the modification of $\operatorname{trop}(X)$ along $\operatorname{trop}(f)$ projected in the x-z direction. First, we compute the principal ideal

$$\langle g, z - f\rangle \cap K[x^{\pm 1}, z^{\pm 1}].$$

This ideal is generated by the equation

$$\begin{aligned}
&(t^{32} + 2t^{30} + t^{28} + t^{18} + t^{16} + 10t^9 + 6t^8 + 34)x^4 \\
&+ x^2(t^{66} + 2t^{64} + t^{62} - t^{60} + 3t^{58} + 10t^{57} + 9t^{56} + 18t^{55} + 5t^{54} + 4t^{53} - 16t^{51} + 2t^{50} \\
&+ 38t^{49} + 28t^{48} + 110t^{47} + 30t^{46} + 48t^{45} - 16t^{44} - 24t^{43} - 54t^{42} + 68t^{41} + 76t^{40} \\
&+ 160t^{39} + 254t^{38} + 12t^{37} + 144t^{36} - 136t^{35} - 68t^{33} + 191t^{32} + 382t^{30} + 191t^{28} + (-2t^5 - 2)z) \\
&+ (-t^{50} - 3t^{48} - 3t^{46} - t^{44} + t^{42} - 10t^{41} - 4t^{40} - 20t^{39} - 11t^{38} - 8t^{37} - 6t^{36} + 4t^{35} - t^{34} \\
&+ 2t^{33} - 33t^{32} - 66t^{30} - 32t^{28} - 2t^{27} - 5t^{26} - 10t^{25} + 2t^{23} - 4t^{22} + 2t^{21} + 4t^{19} - 12t^{18} \\
&- 62t^{17} - 24t^{16} + 8t^{14} - 8t^{13} + 4t^{10} - 86t^9 - 154t^8 + 8t^5 - 221)x^3 \\
&+ x(t^{76} + 2t^{74} + t^{72} + 2t^{71} + 4t^{69} + 6t^{68} + 12t^{67} + 13t^{66} + 20t^{65} + 8t^{64} \\
&+ 22t^{63} + 21t^{62} + 24t^{61} + 49t^{60} + 72t^{59} + 69t^{58} + 130t^{57} + 71t^{56} + 98t^{55} + 151t^{54} \\
&+ 96t^{53} + 240t^{52} + 208t^{51} + 279t^{50} + 290t^{49} + 343t^{48} + 210t^{47} + 389t^{46} + 360t^{45} \\
&+ 385t^{44} + 600t^{43} + 405t^{42} + 390t^{41} + 600t^{40} + 180t^{39} + 525t^{38} + 540t^{37} + 150t^{36} + 900t^{35} \\
&+ 450t^{33} + 225t^{32} + 450t^{30} + 225t^{28} + (2t^{22} + 2t^{17} + 6t^{14} + 10t^{13} + 6t^9 + 10t^8 + 30t^5 + 30)z) \\
&- x^5 + z^2.
\end{aligned}$$

Its tropicalization, pictured in Figure 4.6 is the tropical variety of the tropical polynomial

$$\max(5x, 4x, 3x, 2x + z, -28 + 2x, x + z, -28 + x, 2z).$$

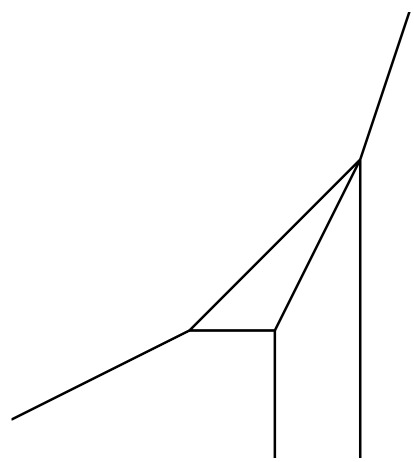


Figure 4.6: The tropicalization of the curve in Example 4.2.12

The vertices of the triangle in Figure 4.6 are $(0, 0)$, $(-14, -28)$, and $(-28, -28)$. So, the cycle has length 56. By [51], this is a faithful tropicalization, and so the abstract tropicalization of X is a cycle of length 56 with a vertex of weight 1.

As in Example 4.2.2, we could compute the abstract tropicalization using trees to verify this computation. Using this method, we find that the tree is the one pictured in Figure 4.5b. Its interior edge has length 28, so it is double-covered by a cycle of length 56 with a vertex of weight 1.

4.3 Higher Genus

We now summarize the work that has been done in higher genus on faithful tropicalizations and computing skeletons of curves. First, we discuss existence of faithful tropicalizations in general, and then we focus on faithful tropicalizations of plane curves.

Existence of Faithful Tropicalizations

It was proved in [14] that faithful tropicalizations of curves exist. The proof, however, is not algorithmic.

Theorem 4.3.1 ([14, Theorem 5.20]). *Let Γ be any finite subgraph of X^{an}. Then there exists a closed immersion of X into a toric variety such that the corresponding embedded tropicalization faithfully represents Γ.*

On the other hand, we know from [1, Theorem 1.2.1] that given any metric graph Γ there is an algebraic curve that tropicalizes to Γ. In [47], the authors give a method for producing curves over $\mathbb{C}((t^{1/l}))$ together with an embedding in a toric variety such that the corresponding tropicalization is faithful. We now describe their method. They start by defining a suitable nodal curve whose dual graph is a model for Γ, and use deformation theory to show that the nodal curve can be lifted to a proper, flat, semistable curve over R with the nodal curve as its special fiber, which tropicalizes to Γ.

Suppose that (G, w) is a stable weighted graph such that:

1. For each vertex $v \in V(G)$, the weight $w(v)$ is of the form
$$w(v) = \binom{d(v)-1}{2} \tag{4.1}$$
for some integer $d(v)$.

2. For every pair of vertices $v, w \in V(G)$, we have
$$|E(v,w)| \leq d(v)d(w), \tag{4.2}$$
where $|E(v,w)|$ denotes the number of edges between v and w.

We now describe a procedure for finding a nodal curve over $\mathbb{C}$ whose dual graph is G. The original idea for this procedure is due to Kollár, cf. [80].

1. Label the vertices as $\{v_1, ..., v_n\} = V(G)$. For each $i = 1, ..., n$, take a general smooth plane curve C_i of degree $d_i = d(v_i)$.

2. We have now a reducible plane curve C, whose irreducible components are the curves C_i of degree d_i (and hence, by the genus degree formula, of genus $w(v_i)$). Any two components C_i and C_j will intersect in $d_i d_j$ points, by Bézout's formula. We choose any $k_{ij} = d_i d_j - |E(v_i, v_j)|$ of those, and set $r = \sum_{i,j} k_{ij}$.

3. Take the blow up of $\mathbb{P}^2$ at all the points chosen for each i and j, which we will label by $p_1, ..., p_r$:
$$X = \mathrm{Bl}_{p_1,\dots,p_r}\mathbb{P}^2, \tag{4.3}$$
and consider the proper transform $\tilde{C}$ of C in X.

4. The curve is inside $\mathbb{P}^2 \times \mathbb{P}^1$. Embed X in $\mathbb{P}^{2+3-1} = \mathbb{P}^4$ via the Segre embedding, and take the image of $\tilde{C}$. This will now be a projective curve with components of the correct genera (as the genus is a birational invariant), and any two components will intersect precisely at the correct number of points. Hence its dual graph will be G.

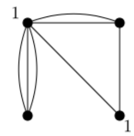


Figure 4.7: The weighted graph in Example 4.3.2.

Example 4.3.2. Consider the graph in Figure 4.7. It has two components of genus zero and two components of genus one, which we can realize as a pair of lines (respectively, of cubics) in general position in $\mathbb{P}^2$. The two lines will intersect in a point, the two cubics in nine points and each cubic will intersect each line in three points. The corresponding curve arrangement is as shown in Figure 4.8.

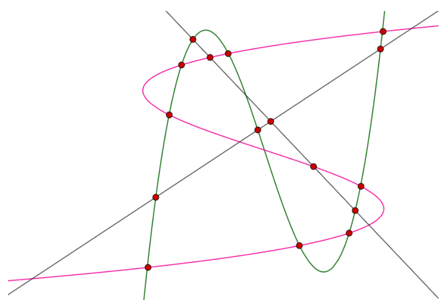


Figure 4.8: Arrangement of curves in $\mathbb{P}^2$.

We need to blow up eight of the nine intersection points between the two cubics, since they correspond to edges between the components of genus one in the graph. Moreover, the two components of genus zero do not share an edge, hence the unique intersection point between the two lines must be blown up, as well as the three intersection points of a chosen cubic with a line, two out of the three intersection points with the remaining line, and one of the three intersection points of the first cubic with the second line. In Figure 4.8, these points are marked in red. The result will be a curve in $\mathbb{P}^2 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^4$, whose components will have the correct genera and will intersect at the correct number of points, and whose equations can be explicitly computed.

Remark 4.3.3. The theory shows that in principle we may find a smooth curve over K with a prescribed metric graph as its tropicalization. For certain types of graphs, more work has been done in this direction [47].

Plane Quartics

Classically, every smooth projective curve of genus 3 is either hyperelliptic or a plane quartic. Is this true in the tropical setting?

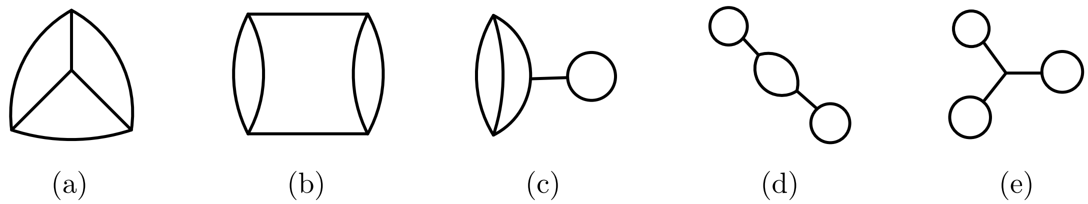


Figure 4.9: The types of trivalent genus 3 tropical curves.

The curves pictured in Figure 4.9 are trivalent, which means that they have the maximum number of edges for a genus 3 graph. So, in $\mathcal{M}_3^{\mathrm{trop}}$, these correspond to cones of maximal dimension. In [30], the authors show that not every genus 3 metric graph arises as the tropicalization of a plane curve.

Theorem 4.3.4 ([30, Theorem 5.1])**.** *Of the five types of trivalent genus 3 metric graphs pictured in Figure 4.9, only 4 can be realized as the tropicalization of a smooth quartic curve. In particular, the curve in Figure 4.9e cannot be realized. For the other curves, there are explicit restrictions on the edge lengths which make them realizable.*

However, another tropical version of the classical statement that every smooth projective curve of genus 3 is either hyperelliptic or a plane quartic can be made. In [67], the authors prove the following result.

Theorem 4.3.5 ([67, Theorem 1.1])**.** *Any maximal tropical curve of genus 3 which is not the tropicalization of a hyperelliptic curve can be embedded as a faithfully tropicalized quartic in a linear modification of the tropical plane.*

The proof of this result is constructive. For a given genus 3 tropical curve Γ, they construct a map from Γ to $\mathbb{R}^2$, as well as linear modifications. These give an embedding of Γ into the modified plane as a faithfully tropicalized quartic.

4.4 Hyperelliptic Curves

Let X be a nonsingular hyperelliptic curve of genus g over K, an algebraically closed field which is complete with respect to a nontrivial, non-archimedean valuation v. Our goal is to find Γ, the abstract tropicalization of X.

We denote by $\mathcal{M}_{g,n}$ the moduli space of genus g curves with n marked points. The space $\mathcal{M}_{0,2g+2}$ maps surjectively onto the hyperelliptic locus inside $\mathcal{M}_g$ by identifying each hyperelliptic curve of genus g with a double cover of $\mathbb{P}^1$ ramified at $2g+2$ marked points. When the characteristic of K is not 2, the normal form for the equation of a hyperelliptic curve is $y^2 = f(x)$, where $f(x)$ has degree $2g+2$, and the roots of f are distinct. The roots of f are precisely the ramification points of the double cover $X \to \mathbb{P}^1$.

The space $\mathcal{M}_{0,2g+2}^{trop}$, the tropicalization of $\mathcal{M}_{0,2g+2}$, parametrizes phylogenetic trees with $2g+2$ leaves. So, $\mathcal{M}_{0,2g+2}^{trop}$ is a $2g-1$ dimensional fan inside $\mathbb{TP}^{\binom{2g+2}{2}-1}$ (cf. [88, Section 2.5]). The space $\mathcal{M}_{0,2g+2}^{trop}$ can be computed as a tropical subvariety of $\mathbb{TP}^{\binom{2g+2}{2}-1}$, since it has a tropical basis (Definition 1.2.11) given by the Plücker relations for $Gr(2, K^{2g+2})$, see Section 3.1. Each cone corresponds to a combinatorial type of tree (see Figure 4.10), and the dimension of each cone corresponds to the number of interior edges in the tree.

To find the tropicalization of the hyperelliptic curve X, we must compute the corresponding point in $\mathcal{M}_{0,2g+2}^{trop}$, as a tree on $2g+2$ leaves, and then compute a tropical curve in $\mathcal{M}_g^{trop}$. Figure 4.10 gives this correspondence in the case $g = 3$.

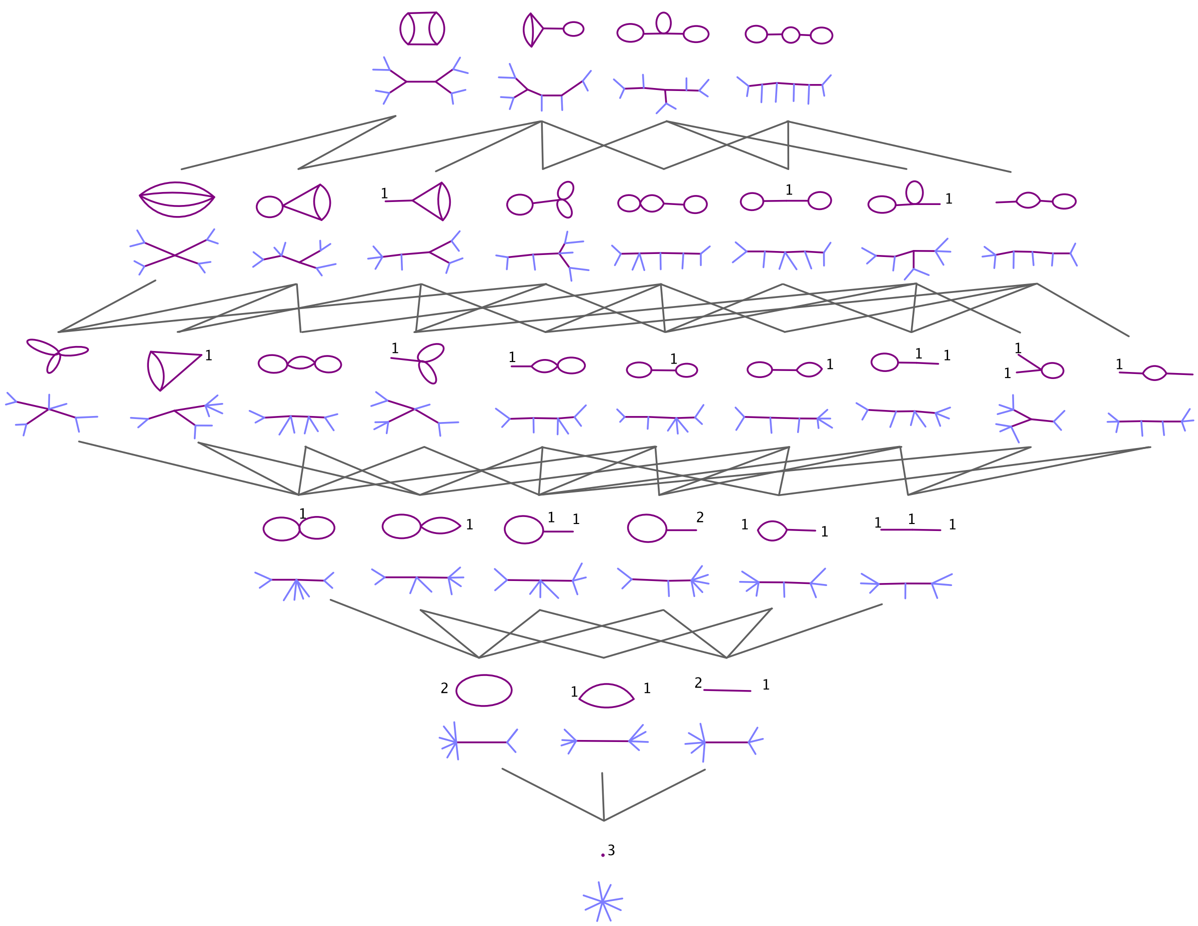


Figure 4.10: The poset of unlabeled trees with 8 leaves, and tropicalizations of hyperelliptic curves of genus 3. Both are ordered by the relation of contracting an edge.

The tropical curves that appear in this section will also be hyperelliptic.

Definition 4.4.1. [44, Theorem 1.3] Let Γ be a tropical curve, and let G be its canonical loopless model. We say that Γ is *hyperelliptic* if there exists a nondegenerate harmonic morphism of degree 2 from G to a tree.

Not every hyperelliptic tropical curve is the tropicalization of a hyperelliptic curve.

Theorem 4.4.2 ([4, Corollary 4.15]). *Let Γ be a tropical curve of genus $g \geq 2$. Then there is a smooth proper hyperelliptic curve X over K of genus g having Γ as its minimal skeleton if and only if Γ is hyperelliptic and for every $p \in \Gamma$ the number of bridge edges adjacent to p is at most $2w(p) + 2$.*

We now give a procedure for taking a tree with $2g + 2$ infinite leaves and obtaining a metric graph which is an admissible cover of the tree.

Lemma 4.4.3 ([22]). *Every tree T with $2g + 2$ infinite leaves has an admissible cover ϕ by a unique hyperelliptic metric graph Γ of genus g, and ϕ is harmonic of degree 2.*

Proof. Let T be a tree with $2g + 2$ infinite leaves. If all infinite leaves are deleted, then a finite tree T' remains. Let $v_1, \ldots, v_k$ be the vertices of T', ordered such that the distance from v_i to v_k is greater than or equal to the distance from v_j to v_k, for $i < j$.

We construct Γ iteratively by building the preimage of each vertex v_i, asserting along the way that the local Riemann-Hurwitz condition (Equation 1.5) holds. This gives an algorithm for finding Γ. We begin with v_1, which has a positive number n_1 of leaf edges in T. Since ϕ has degree 2, it is locally of degree 1 or 2 at every vertex of Γ. Since the preimage of each infinite leaf is an infinite leaf, we attach n_1 infinite leaves at the preimage $\phi^{-1}(v_1)$ in Γ.

At any vertex in Γ with infinite leaves, ϕ has local degree 2, hence we will attach to Γ an infinite leaf e such that $l(\phi(e))/l(e) = 2$. Then, there is a unique vertex in the preimage $\phi^{-1}(v_1)$. Otherwise, there would need to be another edge in the preimage of each leaf, so the degree of the morphism would be greater than 2.

Let e_1 be the edge connecting v_1 to some other v_i. There are two possibilities:

1. The preimage of e_1 is two edges in Γ, each with length $l(e_1)$. The local Riemann-Hurwitz equation reads
$$2 - 2w(\phi^{-1}(v_1)) = 2(2 - 0) - (n_1 + 0 + 0).$$
This is only possible if n_1 is even, and $\phi^{-1}(v_1)$ has weight $(n_1 - 2)/2$.

2. The preimage of e_1 is one edge in Γ, with length $l(e_1)/2$. The local Riemann-Hurwitz equation reads
$$2 - 2w(\phi^{-1}(v_1)) = 2(2 - 0) - (n_1 + 1).$$
This is only possible if n_1 is odd, and $\phi^{-1}(v_1)$ has weight $(n_1 - 1)/2$.

Now, we proceed to the other vertices. As long as the order of the vertices is respected, at each vertex v_i there will be at most one edge e_i whose preimage in Γ we do not know. Then, what happens at v_i can be completely determined by studying the local Riemann-Hurwitz data. For $i > 1$, let n_i be the number of infinite leaves at v_i plus the number of edges $e \in T$ such that $e = \{v_i, v_j\}$, $j < i$, and $\phi^{-1}(e)$ is a bridge in Γ. If $n_i > 0$, then either 1 or 2 holds. However, it is possible that $n_i = 0$, in which case we have a third possibility:

3. If $n_i = 0$, let $v_i' \in \phi^{-1}(v_i)$. The local Riemann-Hurwitz equation reads:

$$2 - 2w(v_i') = d_{v_i}(2 - 0) - (0).$$

   We have $d_{v_i} = 1$ and $w(v_i') = 0$, which implies that there are two vertices in $\phi^{-1}(v_i)$.

Finally, we glue the pieces of Γ as specified by T, and contract the leaf edges on Γ. The fact that Γ has genus g is a consequence of the local Riemann-Hurwitz condition. □

We remark that this process did not require the fact that the tree had an odd number of leaves. Indeed, if one repeats this procedure for such a tree, a hyperelliptic metric graph will be obtained. However, this graph is not the tropicalization of a hyperelliptic curve.

Example 4.4.4. In Figure 4.11, we have a tree with vertices labelled $v_1, \ldots, v_7$. Beginning with v_1, we observe that $n_1 = 2$, which means that the edge from v_1 to v_3 has two edges in its preimage. The same is true for v_2. Moving on to v_3, we see that $n_3 = 0$, which means that v_3 has two points in Γ which map to it. We can connect the edges from $\phi^{-1}(v_1)$ and $\phi^{-1}(v_2)$ to the two points in $\phi^{-1}(v_3)$. Since $\phi^{-1}(v_3)$ has two points, the edge from v_3 to v_4 corresponds to two edges in Γ, so $n_4 = 2$, which means that the edge from v_4 to v_5 also splits. Next, $n_5 = 1$, which means that the edge from v_5 to v_6 corresponds to a bridge in Γ. Then, $n_6 = 4$, which means that the edge v_6 to v_7 splits, and the vertex mapping to v_6 has genus 1. Lastly, since $n_7 = 2$, the point mapping to v_7 has genus 0. All edges depicted in the image have the same length as the corresponding edges in the tree, except for the bridge, which has length equal to half the length of the corresponding edge in the tree.

The following theorem shows that the tropical curve constructed in the proof of Lemma 4.4.3 is actually the tropicalization of a hyperelliptic curve.

Theorem 4.4.5 ([22])**.** *Let $g \geq 1$ be an integer. Let X be a hyperelliptic curve of genus g over K, given by taking the double cover of $\mathbb{P}^1$ ramified at $2g+2$ points $p_1, \ldots, p_{2g+2}$. If T is the tree which corresponds to the tropicalization of $\mathbb{P}^1$ with the marked points $p_1, \ldots, p_{2g+2}$ described above, and Γ is the unique hyperelliptic tropical curve which admits an admissible cover to T, then Γ is the abstract tropicalization of X.*

Proof. This follows from [38], Remark 20 and Theorem 4. Indeed, the hyperelliptic locus of $\mathcal{M}_g$ can be understood as the space $\overline{H}_{g\to 0,2}((2), \ldots, (2))$ of admissible covers with $2g+2$ ramification points of order 2. Its tropicalization is constructed and studied in [38]. The space $\overline{H}^{an}_{g\to 0,2}((2), \ldots, (2))$ is the Berkovich analytification of $\overline{H}_{g\to 0,2}((2), \ldots, (2))$, and thus a

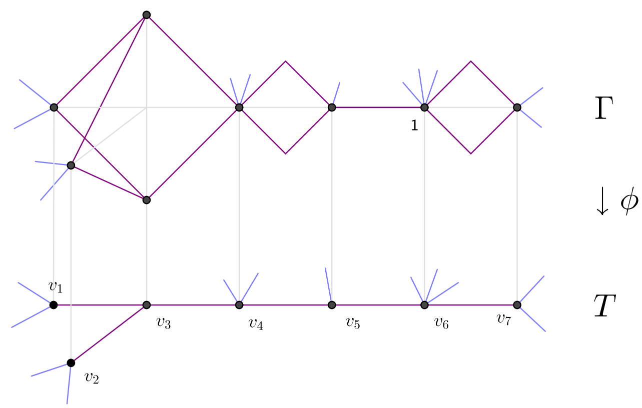


Figure 4.11: The tree T with 12 infinite leaves from Example 4.4.4 and the hyperelliptic tropical curve Γ of genus 5 which admissibly covers T by ϕ.

point X is represented by an admissible cover over $\mathrm{Spec}(K)$ with $2g+2$ ramification points of order 2. By Theorem 4 in [38], the following diagram commutes.

$$\begin{array}{ccccc}
 & & \overline{H}^{an}_{g\to 0,2}((2),\ldots,(2)) & & \\
 & \swarrow^{br^{an}} & & \searrow^{src^{an}} & \\
\overline{\mathcal{M}}^{an}_{0,2g+2} & & \downarrow \text{trop} & & \overline{\mathcal{M}}^{an}_{g} \\
\downarrow \text{trop} & & \overline{H}^{\text{trop}}_{g\to 0,2}((2),\ldots,(2)) & & \downarrow \text{trop} \\
 & \swarrow^{br^{\text{trop}}} & & \searrow^{src^{\text{trop}}} & \\
\overline{\mathcal{M}}^{\text{trop}}_{0,2g+2} & & & & \overline{\mathcal{M}}^{\text{trop}}_{g}
\end{array}$$

The morphisms src take a cover to its source curve, marked at the entire inverse image of the branch locus, and the morphisms br take a cover to its base curve, marked at its branch points. We start with an element X of $\overline{H}^{an}_{g\to 0,2}((2),\ldots,(2))$, and we wish to find $\mathrm{trop}(src^{an}(X)) \in \overline{\mathcal{M}_g}^{trop}$. The unicity in Lemma 4.4.3 enables us to find an inverse for br^{trop}. Then $T = \mathrm{trop}(br^{an}(X))$, and so by commutativity of the diagram, $\mathrm{trop}(src^{an}(X)) = src^{\text{trop}}((br^{\text{trop}})^{-1}(T)) = \Gamma$. $\square$

4.5 Superelliptic Curves

We now develop an algorithm for tropicalizing superelliptic curves. This will require much more work than in the hyperelliptic case, because trees T do not have unique metric graphs covering them. New techniques will be required in order to determine the correct graph.

Let K be a field of characteristic 0 that is complete with respect to a non-Archimedean discrete valuation. Let R be the valuation ring of K with maximal ideal m, let $k := R/m$ be the residue field, and let t be a uniformizer for K. A *superelliptic curve* over K is a curve X which admits a Galois covering $\phi : X \to \mathbb{P}^1$ such that the Galois group is cyclic of order n. We assume the characteristic of the residue field k is relatively prime to n. Assuming K contains n distinct primitive n-th roots of unity, Kummer theory [101, Proposition 3.2] tells us the covering comes as $y^n = f(x)$, where $f(x)$ is some rational function in $K(x)$. This normal form allows us to directly relate ramification data of the corresponding covering $(x, y) \mapsto x$ to the rational function f. We can in fact assume $f(x)$ is a polynomial by the following transformation. For $f(x)$ of the form $f(x) = g(x)/h(x)$, we multiply both sides of $y^n = f(x)$ by $h(x)^n$ and make a change of coordinates $\tilde{y} = h(x) \cdot y$ to obtain the integral equation $(\tilde{y})^n = g(x)h(x)^{n-1}$.

In this section we will study tropical superelliptic covers and give an algorithm for computing abstract tropicalizations of superelliptic curves. We will also study realizability of tropical superelliptic covers, and perform a computational study of the moduli space of tropical superelliptic curves.

Tropical Superelliptic Curves

Let Γ be a metric graph with model G. We now define what it means for Γ to be superelliptic.

Definition 4.5.1. An *automorphism* of Γ is a harmonic morphism $\theta : \Gamma \to \Gamma$ of degree 1. Given a subgroup H of $\mathrm{Aut}(\Gamma)$, the *quotient graph* Γ/H has a model G/H whose vertices are the H-orbits of $V(G)$ and whose edges are the H-orbits of edges defined by vertices lying in distinct H-orbits. If $\theta(e)$ is an edge in G/H, then $l(\theta(e)) = l(e) \cdot |\,\mathrm{Stab}(e)|$. The quotient map is a nondegenerate harmonic morphism. For any subgroup H of $\mathrm{Aut}(\Sigma)$, we call a nondegenerate harmonic morphism $\Gamma \to \Gamma/H$ a *Galois covering* of metric graphs if it satisfies the local Riemann-Hurwitz conditions at every v. The group H is the *Galois group* of the covering.

Definition 4.5.2. A nondegenerate harmonic morphism $\theta : \Gamma \to T$ is a *superelliptic* covering of metric graphs if θ is a Galois covering of metric graphs with Galois group $H := \mathbb{Z}/n\mathbb{Z}$ and the target T is a tree.

Later, when we present the algorithm for tropicalizing superelliptic curves, we will make use of the following lemma for computing slopes of rational functions on graphs.

Lemma 4.5.3 ([26])**.** *Let T be a tree, v_0 a vertex in T, and let $\psi : \Gamma \to \mathbb{R}$ be a rational function with principal divisor (ψ) on T. For a point $P \in T$, let $\sigma_P(\psi)$ be the sum of the slopes of ψ in all outgoing directions at P. If e is deleted from T, let T_e denote the connected component of T not containing v_0. Then the magnitude of the slope of the rational function ψ along e is equal to $\sum_{x \in T_e} (\psi)(x)$.*

Proof. Let x and y be the vertices of e and suppose first $x \neq v_0$ is a leaf. Let ψ_e be the slope of ψ along the edge e. Then, $(\psi)(x) = -\sigma_x(\psi)(x) = \psi_e$. Now, we proceed by induction on the number of edges in the tree T. Suppose y is on the path from x to v_0. We have $(\psi)(x) = -\sum_{e \ni x} \psi_{e'}$. Isolating ψ_e, we find

$$-\psi_e = (\psi)(x) + \sum_{e' \neq e, e' \ni x} \psi_{e'}.$$

Using the inductive assumption, we can solve for the $\psi_{e'}$ and arrive at the result. □

Galois Covers of Semistable Models

In this section, we show *disjointly branched morphisms* of semistable models yield Galois coverings of metric graphs. Furthermore, we discuss inertia groups and prove they are preserved on reduction to the special fiber (Proposition 4.5.11), allowing us to relate ramification degrees on a two dimensional scheme (that is, $\mathcal{C}$, a model for C) to those on a one dimensional scheme (a component in the special fiber $\mathcal{C}_s$). We use this equality in the subsection "Tropicalization Algorithm" to reconstruct the Berkovich skeleton for superelliptic covers.

Disjointly branched morphisms and inertia groups

Let $\phi : C \to D$ be a finite morphism of smooth, projective, geometrically connected curves over K. We say ϕ is *Galois* if the corresponding morphism on function fields $K(D) \to K(C)$ is Galois. That is, it is normal and separable. Let $\mathcal{C}$ be a model for C and $\mathcal{D}$ be a model for D. A *finite morphism of models for* ϕ is a finite morphism $\mathcal{C} \to \mathcal{D}$ such that the base change to $\mathrm{Spec}(K)$ gives $\phi : C \to D$.

Definition 4.5.4. Let $\phi : C \to D$ be a finite, Galois morphism of curves over K with Galois group G. Let $\phi_{\mathcal{C}} : \mathcal{C} \to \mathcal{D}$ be a finite morphism of models for ϕ. We say $\phi_{\mathcal{C}}$ is *disjointly branched* if the following hold:

1. The closure of the branch locus in $\mathcal{D}$ consists of disjoint, smooth sections over $\mathrm{Spec}(R)$.
2. The induced morphism $\mathcal{O}_{\mathcal{D},\phi(y)} \to \mathcal{O}_{\mathcal{C},y}$ is étale for every y a generic point of an irreducible component in the special fiber of $\mathcal{C}$.
3. $\mathcal{D}$ is strongly semistable, meaning that $\mathcal{D}$ is semistable and that the components in the special fiber are all smooth.

A theorem by Liu and Lorenzini [87, Theorem 2.3] says if $\phi_{\mathcal{C}}$ is disjointly branched then $\mathcal{C}$ is actually also *semistable* and [71, Proposition 3.1] shows $\mathcal{C}$ is also strongly semistable.

We now study the action of the Galois group G on $\Sigma(\mathcal{C})$, the intersection graph of $\mathcal{C}$.

Theorem 4.5.5 ([71, Lemma 4 and Theorem 3.1]). *Let $\phi_{\mathcal{C}} : \mathcal{C} \to \mathcal{D}$ be a disjointly branched morphism of models for a finite Galois morphism $\phi : C \to D$ of curves, with Galois group G. There is a natural action of G on $\Sigma(\mathcal{C})$ and the induced morphism of graphs $\Sigma(\mathcal{C}) \to \Sigma(\mathcal{D})$ coincides with the quotient map $\Sigma(\mathcal{C}) \to \Sigma(\mathcal{C})/G$.*

This is a statement about graphs. We give the result for metric graphs in the subsection "Tropicalizing Galois Covers."

We now define *inertia groups* and *decomposition groups* for a finite group G acting on a scheme X. For any point x of X, we define the decomposition group $D_{x,X}$ to be $\{\sigma \in G : \sigma(x) = x\}$, the stabilizer of x. Every element $\sigma \in D_{x,X}$ naturally acts on $\mathcal{O}_{X,x}$ and the residue field $k(x)$. We define the *inertia group* $I_{x,X}$ of x to be the elements of D_x reducing to the identity on $k(x)$. In other words, $\sigma \in I_{x,X}$ if and only if for every $z \in O_{X,x}$, we have $\sigma z \equiv z \mod m_x$, where m_x is the unique maximal ideal of $O_{X,x}$. When context is clear, we omit the X in $I_{x,X}$ and $D_{x,X}$.

Suppose we have a normal integral scheme Y with function field $K(Y)$ and a finite Galois extension L of $K(Y)$ with Galois group $G = \mathrm{Gal}(L/K(Y))$. We take the normalization X of Y in L (which we now write as $K(X)$) to obtain a morphism of normal integral schemes $X \to Y$. In fact, we have $Y := X/G$ (See [71, Proposition 3.5]). Locally, we have the following lemma.

Lemma 4.5.6 ([26]). *Let A be a normal domain with fraction field K, let L be a Galois extension of K, let $G := \mathrm{Gal}(L/K)$ be the Galois group, and let B be the integral closure of A in L. If $\mathfrak{q} \in \mathrm{Spec}(B)$ is a prime lying over $\mathfrak{p} \subset A$, then $k(\mathfrak{q})/k(\mathfrak{p})$ is an algebraic normal extension and the following sequence is exact:*

$$1 \to I_{\mathfrak{q}} \to D_{\mathfrak{q}} \to \mathrm{Aut}(k(\mathfrak{q})/k(\mathfrak{p})) \to 1.$$

Proof. The group I_q is the kernel of the surjective morphism described there. □

In our case, the extension of residue fields is always Galois; our assumption that the degree of the Galois extension is relatively prime to the characteristic of the residue field implies separability.

We now show inertia groups directly measure and control ramification. This is the content of Proposition 4.5.10 which we will use to relate inertia groups on $\mathcal{C}$ to inertia groups on the special fiber. Let us study the inertia group $I_{\mathfrak{q}}$ and the invariant ring $B^{I_{\mathfrak{q}}}$ a bit closer. We have that $B^{I_{\mathfrak{q}}}$ is normal and finite over A because B is normal and finite over A. Furthermore, there is only one prime lying above $\mathfrak{q} \cap B^{I_{\mathfrak{q}}}$.

Lemma 4.5.7 ([26]). *Let $j : B^{I_{\mathfrak{q}}} \to B$ be the natural inclusion map and let $j^* : \mathrm{Spec}(B) \to \mathrm{Spec}(B^{I_{\mathfrak{q}}})$ be the corresponding map on the spectra. Then $(j^*)^{-1}(j^*(\mathfrak{q})) = \{\mathfrak{q}\}$.*

Proof. The morphism j^* coincides with the quotient map $\mathrm{Spec}(B) \to \mathrm{Spec}(B)/I_{\mathfrak{q}}$ (See [60, Exposé V, Proposition 1.1, Page 88]). This means any other prime mapping to $\mathfrak{q} \cap B^{I_{\mathfrak{q}}}$ is of the form $\sigma(\mathfrak{q})$ for some σ in $I_{\mathfrak{q}}$. But for those σ, we have $\sigma(\mathfrak{q}) = \mathfrak{q}$. □

Lemma 4.5.8 ([26]). *Let $B \supset B^{I_{\mathfrak{q}}}$ be as before, and let $k(\mathfrak{q})^{\mathrm{sep}}$ be the separable closure of $k(\mathfrak{q} \cap A)$ in $k(\mathfrak{q})$. Then $(k(\mathfrak{q}))^{\mathrm{sep}} = k(\mathfrak{q} \cap B^{I_{\mathfrak{q}}})$.*

Proof. Consider the Galois extension $L \supset L^{I_{\mathfrak{q}}}$ with Galois group $I_{\mathfrak{q}}$. We find $I_{\mathfrak{q}} = D_{\mathfrak{q}}$ and so the automorphism group $\mathrm{Aut}(k(\mathfrak{q})/k(\mathfrak{q} \cap B^{I_{\mathfrak{q}}}))$ is trivial by Lemma 4.5.6. This automorphism group is isomorphic to the Galois group of the separable closure of $k(\mathfrak{q} \cap B^{I_{\mathfrak{q}}})$ in $k(\mathfrak{q})$. By Galois theory, $k(\mathfrak{q} \cap B^{I_{\mathfrak{q}}})$ is separably closed inside $k(\mathfrak{q})$. By [60, Proposition 2.2, page 92], the extension $B^{I_{\mathfrak{q}}} \supset A$ is étale at $\mathfrak{q} \cap B^{I_{\mathfrak{q}}}$, so the residue field extension $k(\mathfrak{q} \cap B^{I_{\mathfrak{q}}}) \supset k(\mathfrak{q} \cap A)$ is separable and the fact that étale morphisms are stable under base change. Thus $k(\mathfrak{q} \cap B^{I_{\mathfrak{q}}}) \subseteq (k(\mathfrak{q}))^{\mathrm{sep}}$, and every element of $k(\mathfrak{q})$ that is separable over $k(\mathfrak{q} \cap A)$ is also separable over the field $k(\mathfrak{q} \cap B^{I_{\mathfrak{q}}})$. We thus find $(k(\mathfrak{q}))^{\mathrm{sep}} = k(\mathfrak{q} \cap B^{I_{\mathfrak{q}}})$, as desired. □

Corollary 4.5.9 ([26]). *Suppose $\mathrm{char}(k(\mathfrak{q})) \nmid |G|$. Then $k(\mathfrak{q}) = k(\mathfrak{q} \cap B^{I_{\mathfrak{q}}})$.*

We study inertia groups because they measure ramification, as in the next proposition.

Proposition 4.5.10 ([26]). Let A be a normal domain with fraction field K and L a Galois extension of K. Let $G := \mathrm{Gal}(L/K)$ and B the integral closure of A in L. Let $\mathfrak{q}$ be any prime ideal in B.

1. Consider the subring $B \supset B^{I_{\mathfrak{q}}} \supset A = B^G$. Then $B^{I_{\mathfrak{q}}} \supset A$ is étale at $\mathfrak{q} \cap B^{I_{\mathfrak{q}}}$.
2. More generally, consider any subgroup H of G. Then $B^H \supset A$ is étale at $\mathfrak{q} \cap B^H$ if and only if $H \supseteq I_q$.

Proof. The fact that if $H \supseteq I_{\mathfrak{q}}$, then $B^H \supset A$ is étale at $\mathfrak{q} \cap B^H$ is [60, Proposition 2.2., page 92]. Taking $H = I_{\mathfrak{q}}$ also proves the first statement. So let us prove that $H \supseteq I_q$ if $B^H \supset A$ is étale at $\mathfrak{q} \cap B^H$. Since $B^H \supset A$ is finite étale, the base change $B^H \cdot B^{I_{\mathfrak{q}}}$ is finite étale over $B^{I_{\mathfrak{q}}}$. Let $\mathfrak{q}_I := \mathfrak{q} \cap B^{I_{\mathfrak{q}}}$, and $L^{I_{\mathfrak{q}}} := \mathrm{Quot}(B^{I_{\mathfrak{q}}})$. Since $B^{I_{\mathfrak{q}}} \cdot B^H$ is étale over $\mathfrak{q}_I$, it is also flat there by definition. This implies that the $B^{I_{\mathfrak{q}}}$ module $B^{I_{\mathfrak{q}}} \cdot B^H$ is locally free of some finite rank m. This rank m in fact has to be equal to $[L^{I_{\mathfrak{q}}} \cdot L^H : L^{I_{\mathfrak{q}}}]$. Taking the base change to the residue field $k(\mathfrak{q}_I)$, we find the $k(\mathfrak{q}_I)$-algebra $S := (B^{I_{\mathfrak{q}}} \cdot B^H) \otimes k(\mathfrak{q})$ is étale of dimension m over $k(\mathfrak{q}_I)$. It is a product of separable field extensions of $k(\mathfrak{q}_I)$, and it is local by Lemma 4.5.7. Since the residue field is separable over $k(\mathfrak{q}_I)$, it must be the same as $k(\mathfrak{q}_I)$ by Lemma 4.5.8, so $m = 1$. We thus have $L^H \subseteq L^{I_{\mathfrak{q}}} \cdot L^H = L^{I_{\mathfrak{q}}}$ and, and so $H \supseteq I_{\mathfrak{q}}$. □

For any subfield $K \subseteq K' \subseteq L$, we can write $K' = L^H$ for some subgroup H of G by Galois theory. By Proposition 4.5.10, $B^H \supseteq B^G = A$ is ramified at some point x' if and only if the inertia group I_x of some point x lying above is *not* contained in H. In other words, we can describe ramification in terms of Galois theory. This criterion is very useful in relating different inertia groups I_x and I_y for points x and y in $\mathrm{Spec}(B)$. For instance, if $B^{I_x} \supset A$ is étale at the image of y in B^{I_x}, and B^{I_y} is étale at the image of x in B^{I_y}, then $I_y = I_x$.

We make two further assumptions on the morphism $\phi_{\mathcal{C}} : \mathcal{C} \to \mathcal{D}$ of models over $\mathrm{Spec}(R)$. We assume the ramification points of $\phi : C \to D$ are rational over K. We assume the residue field k is large enough so that for every intersection point $x \in \mathcal{C}$, we have $D_x = I_x$.

Let us find out what these decomposition and inertia groups are for a disjointly branched morphism $\phi_{\mathcal{C}} : \mathcal{C} \to \mathcal{D}$ of models.

1. Let $x \in \mathcal{C}$ be an intersection point on the special fiber. By our second assumption, $D_x = I_x$. Since the action of G is transitive on the edges lying above $\phi_{\mathcal{C}}(x)$, there are $|G|/|I_x|$ edges lying above $\phi_{\mathcal{C}}(x)$.

2. Let x be the generic point of an irreducible component Γ_x in the special fiber $\mathcal{C}_s$. Let y and Γ_y be their respective images in $\mathcal{D}_s$. By our second assumption for disjointly branched morphisms, the inertia group I_x is trivial. Thus, the decomposition group can be identified with the automorphisms of the function field $k(x)$ of the component Γ_x fixing the function field $k(y)$ of the component Γ_y, by Lemma 4.5.6. This implies $\Gamma_x/D_x = \Gamma_y$ as curves over the residue field, since morphisms of smooth curves are determined by their corresponding inclusions of function fields. We have Γ_x and Γ_y are smooth, since $\mathcal{C}$ and $\mathcal{D}$ are strongly semistable.

3. Let $x \in C$ be a generic ramification point. Then $|I_x|$ is just the usual ramification degree. This follows from the fact that x is totally ramified in the extension $L \supset L^{I_x}$, which has degree $|I_x|$. Let us study an example where the decomposition group D_x for a generic branch point is bigger than I_x. Take the Galois covering $(x, y) \mapsto x$ for the curve C defined by $y^4 = x^2 \cdot (x + 2)$ over $\mathbb{Q}(i)$. This is Galois with Galois group $G = \mathbb{Z}/4\mathbb{Z}$, where the action on fields comes from multiplication by i on y and the identity on x. Let us find the normalization of the algebra $A := \mathbb{Q}(i)[x, y]/(y^4 - x^2 \cdot (x + 2))$. The integral element $z = y^2/x$ satisfies $z^2 = x + 2$. The maximal ideal $\mathfrak{m} = \langle x, y, z^2 - 2\rangle$ is then locally principal with generator y, as we can write
$$z^2 - 2 = x = \frac{y^2}{z}$$
in the localization of $A' := A[z]/(z^2 - x - 2)$ at $\mathfrak{m}$. Since A was already normal at the other primes (by the Jacobi criterion for instance), A' is the normalization. Here we use that a domain is normal if and only if it is normal at all its localizations. The Galois group $\mathbb{Z}/4\mathbb{Z}$ fixes $\mathfrak{m}$, so $D_{\mathfrak{m}} = G$. By inspecting the action on the residue field, we have $|I_{\mathfrak{m}}| = 2$. Indeed, the automorphism defined by $\sigma(y) = iy$ sends z to $-z$, which is nontrivial on the residue field. In this case the decomposition group is strictly larger than the inertia group. If we consider the curve over the field $\mathbb{Q}(i)(\sqrt{2})$, the situation changes. The above equations still define the normalization, but $\mathfrak{m} = (x, y, z^2 - 2)$ is no longer maximal. There are now two maximal ideals lying above $\mathfrak{m}_0 = (x)$, namely $\mathfrak{m}_{\pm} = (x, y, z \pm \sqrt{2})$. We have $|I_{\mathfrak{m}_{\pm}}| = |D_{\mathfrak{m}_{\pm}}| = 2$. For disjointly branched morphisms, we assume extensions of this form have already been made.

Comparing inertia groups

Consider a disjointly branched morphism $\phi_{\mathcal{C}} : \mathcal{C} \to \mathcal{D}$. Let $\Gamma \subset \mathcal{C}_s$ be any irreducible component in the special fiber. There is a Galois morphism of smooth curves $\phi_{\Gamma} : \Gamma \to \Gamma'$

where Γ' is an irreducible component in the special fiber $\mathcal{D}_s$. This uses the condition on the characteristic of k. To see this, one can use [71, Section 3.2.1] or Lemma 4.5.6 and Corollary 4.5.9. The morphism ϕ_Γ is induced by the Galois morphism of function fields $k(\Gamma) \to k(\Gamma')$, which is obtained from the map of discrete valuation rings $\mathcal{O}_{\mathcal{D},y_\Gamma} \to \mathcal{O}_{\mathcal{C},y_{\Gamma'}}$. Here y_Γ and $y_{\Gamma'}$ are the generic points of the irreducible components Γ and Γ'.

We recall the definition of the natural *reduction maps* associated to $\mathcal{C}$ and $\mathcal{D}$. Let us describe this map $r_{\mathcal{C}}$ for $\mathcal{C}$. Let C^0 be the set of closed points of the generic fiber C of $\mathcal{C}$. For every closed point x, we can consider its closure $\overline{\{x\}}$ inside $\mathcal{C}$. This closure is then an irreducible scheme, finite over Spec(R). Since R is Henselian, $\overline{\{x\}}$ is local, giving a unique closed point. We let $r_{\mathcal{C}}(x)$ be this closed point. We now also extend this map to intersection points for convenience. Let $x \in \mathcal{C}$ be an intersection point. We define $r_{\mathcal{C}}(x)$ to be $x \cap \mathcal{C}_s$, just as we did with closed points on the generic fiber. In other words, we consider this intersection point as a point on the special fiber.

Proposition 4.5.11 ([26])**.** Let $x \in \mathcal{C}$ be a generic ramification point or an intersection point of a disjointly branched morphism $\phi_{\mathcal{C}} : \mathcal{C} \to \mathcal{D}$. Let Γ be any component in the special fiber $\mathcal{C}_s$ containing $r_{\mathcal{C}}(x)$. Then

$$I_{x,\mathcal{C}} = I_{r_{\mathcal{C}}(x),\Gamma}$$

where the second inertia group is an inertia group of the Galois covering $\Gamma \to \Gamma'$ on the special fiber.

Proof. We first let $x \in \mathcal{C}_s$ be any closed point in the special fiber and y the generic point of Γ. We then have a natural injection $D_{x,\mathcal{C}} \to D_{y,\mathcal{C}}$. For x smooth this follows directly from the fact that y is the unique generic point under x. For x an intersection point, this follows from [71, Proposition 3.8.]. By Lemma 4.5.6, $D_{y,\mathcal{C}}$ can be identified with the Galois group of the function field extension $k(\Gamma) \supset k(\Gamma')$. The image of $D_{x,\mathcal{C}}$ in this Galois group is then in fact equal to $D_{r_{\mathcal{C}}(x),\Gamma}$. We thus see $D_{x,\mathcal{C}} = D_{r_{\mathcal{C}}(x),\Gamma}$ for any closed point x in the special fiber. By our assumption on the residue field, these decomposition groups are equal to their respective inertia groups and we have $I_{x,\mathcal{C}} = I_{r_{\mathcal{C}}(x),\Gamma}$.

Using this identification, the case where x is an intersection point immediately follows. We are thus left with the case of a generic ramification point x of the morphism $\phi : C \to D$. Let $z := r_{\mathcal{C}}(x)$. For any subgroup H of G, we let z_H be the image of z under the natural map $C \to C/H$. We show $I_{x,\mathcal{C}} = I_{z,\mathcal{C}}$. By our earlier considerations, we then see $I_{x,\mathcal{C}} = I_{z,\Gamma}$.

If $\sigma \in I_{x,\mathcal{C}}$, then $\sigma \in D_{x,\mathcal{C}}$. Then σ must fix z as well, because otherwise there would be at least two points in the closure of x lying above the special fiber. So $\sigma \in D_{z,\mathcal{C}}$. But by our earlier assumption on the residue field k, we have $D_{z,\mathcal{C}} = I_{z,\mathcal{C}}$. This yields $I_{x,\mathcal{C}} \subseteq I_{z,\mathcal{C}}$.

For the other inclusion, we use the following criterion. Let H be a subgroup of G. Let x_H be the image of x in $\mathcal{C}/H$. The induced map $\mathcal{C}/H \to \mathcal{D}$ is étale at x_H if and only if $H \supseteq I_x$. This is a consequence of the second part of Proposition 4.5.10 in the subsection "Disjointly branched morphisms and inertia groups".

We now only need to show $\mathcal{C}/I_x \to \mathcal{D}$ is unramified at z_{I_x}. Suppose it is ramified at z_{I_x}. Then z_G is a branch point of the covering $\mathcal{C}/I_x \to \mathcal{D}$. Since ϕ_C is disjointly branched, z_G

is in the smooth part of the special fiber. This implies $\mathcal{D}$ is regular at z_G. We use *purity of the branch locus* on some open subset U of $\mathcal{D}$ containing z_G to conclude there must be a generic branch point P such that z_G is in the closure of P. Indeed, purity of the branch locus tells us a point of codimension 1 has to be in the branch locus and this cannot be a vertical divisor by our second assumption for disjointly branched morphisms. We must have $P = x_G$ because the branch locus is disjoint. This contradicts the fact that the morphism $\mathcal{C}/I_x \to \mathcal{D}$ is unramified above x_G (it is the largest extension with this property), so we conclude $\mathcal{C}/I_x \to \mathcal{D}$ is unramified at z_{I_x}. In other words, we have $I_{z,\mathcal{C}} = I_{x,\mathcal{C}}$, as desired. □

We use Proposition 4.5.11 in the tropicalization algorithm to relate the inertia groups on the two-dimensional scheme $\mathcal{C}$ to inertia groups on the special fiber. This allows us to calculate inertia groups without calculating any normalizations. This in turn tells us how many edges and vertices there are in the pre-image of any edge e or vertex v in the dual graph of the special fiber of $\mathcal{D}$.

Tropicalizing Galois Covers

We now study the transition from algebraic Galois coverings (as in the Subsection "Disjointly branched morphisms and inertia groups") to Galois coverings of metric graphs (as in the subsection "Tropical Superelliptic Curves"). To do this, we modify our graphs to reflect the geometry further by assigning lengths to the edges, adding weights to the vertices to account for genera, and adding leaves to account for the ramification coming from generic ramification points.

Consider the quotient map of *graphs* $\Sigma(\mathcal{C}) \to \Sigma(\mathcal{D})$ coming from a disjointly branched morphism. We must add infinite *leaves* to the graphs $\Sigma(\mathcal{C})$ and $\Sigma(\mathcal{D})$. We first perform a base change to make all the ramification points rational over K. Consider the morphism of smooth curves $\phi : C \to D$. Let $P \in C(K)$ be a ramification point and let $Q = \phi(P) \in D(K)$ be the corresponding branch point. The points P and Q reduce to exactly one component on $\mathcal{C}$ and $\mathcal{D}$ respectively. We add a leaf E_P to the vertex V_P that P reduces to and a leaf E_Q to the vertex V_Q that Q reduces to. Doing this for every ramification point gives two loopless models $\tilde{\Sigma}(\mathcal{C})$ and $\tilde{\Sigma}(\mathcal{D})$. There is a natural map between the two, which is induced by the map $\Sigma(\mathcal{C}) \to \Sigma(\mathcal{D})$ and sends leaves E_P to E_Q. The integer $l'(E_Q)/l(E_P)$ we assign to these edges is $|I_P|$. There is a natural action of G on this loopless model, given as follows. On $\Sigma(\mathcal{C})$, this is the usual action. For leaves E_P, we define an action by $\sigma(E_P) = E_{\sigma(P)}$. This in accordance with the algebraic data, since $\sigma(P)$ reduces to $\sigma(V_P)$.

Lemma 4.5.12 ([26])**.** *For every edge $e \in \tilde{\Sigma}(\mathcal{C})$ corresponding to a point $x \in \mathcal{C}$, we have $l'(e')/l(e) = |I_x|$.*

Proof. For edges corresponding to generic ramification points, this is by definition. For edges corresponding to intersection points, this follows from [86, Chapter 10, Proposition 3.48, Page 526]. □

Proposition 4.5.13 ([26])**.** The natural quotient map $\Sigma(\mathcal{C}) \to \Sigma(\mathcal{D}) = \Sigma(\mathcal{C})/G$ yields a Galois covering of metric graphs $\tilde{\Sigma}(\mathcal{C}) \to \tilde{\Sigma}(\mathcal{D})$.

Proof. By construction every edge e in $\tilde{\Sigma}(\mathcal{C})$ corresponds to either a generic (geometric) ramification point or an intersection point. To every point $x \in \mathcal{C}$, we apply the Orbit-Stabilizer Theorem from group theory to obtain $|G| = |D_P| \cdot \#\{\sigma(P) : \sigma \in G\}$. For generic ramification points and intersection points, we have $I_P = D_P$, by our assumptions in the Subsection "Disjointly branched morphisms and inertia groups". Then the degree is just $|G|$ everywhere, so in particular it is independent of the edge e.

We now have to check the local Riemann-Hurwitz conditions at every vertex v mapping to a vertex w. By Lemma 4.5.12, the quantities $l'(e')/l(e)$ are just the ramification indices of the generic points reducing to v and the indices of the edges. But these indices correspond with the indices on the special fiber by Proposition 4.5.11. Furthermore, the ramification points of the morphism on the special fiber $\Gamma \to \Gamma'$ (corresponding to $v \mapsto w$) all arise from either the closure of a generic ramification point (using purity of the branch locus as before) or as an intersection point of $\mathcal{C}$. These are all accounted for, so the Riemann-Hurwitz conditions must be satisfied. This proves the proposition. □

Tropicalization Algorithm

Let K be a field with valuation and uniformizer t. In this section, we provide the algorithm producing the Berkovich skeleton for a superelliptic curve C. To do this, we first give a semistable model $\mathcal{D}$ separating the branch locus over R. We use Proposition 4.5.11 to reduce all the calculations to one dimensional schemes.

A separating semistable model

We describe a semistable model $\mathcal{D}$ of $\mathbb{P}^1$ separating the branch locus B of $\phi : C \to \mathbb{P}^1$. We start with the model $\mathcal{D}_0 := \operatorname{Proj} R[X, Y]$ where $R[x, y]$ has the usual grading. The reader can think of this as being obtained from gluing together the rings $R[x]$ and $R[1/x]$. We now have a canonical reduction map $r_{\mathcal{D}_0}$, which takes a closed point $P \in \mathbb{P}^1_K$ and sends it to the unique point in the closure of P lying over the special fiber $(\mathcal{D}_0)_s = \mathbb{P}^1_k$ as in [86, Section 10.1.3, Page 467]. Informally, this map is given as reducing modulo the maximal ideal of R. This reduction map depends on the choice of the model $\mathcal{D}_0$.

We now use this reduction map on the branch points B to obtain a collection of points in the special fiber. We group together all points having the same reduction under this reduction map. This provides a subdivision of B into subsets B_i. We consider the subsets with $|B_i| > 1$. For these subsets with their corresponding reduced points p_i we now *blow-up* the model $\mathcal{D}_0$ in the points p_i. This gives a new model $\mathcal{D}_1$. On this new model $\mathcal{D}_1$, we again have a canonical reduction map $r_{\mathcal{D}_1}$ and similarly consider the image of every subset B_i under this reduction map to obtain a new subdivision $B_{i,j}$. For every two points P_1 and P_2 in B, we have they are in the same $B_{i,j}$ if and only if their reductions in $\mathcal{D}_1$ are the same.

This gives a new set of points $p_{i,j}$ (the reduction of the points in $B_{i,j}$) in the special fiber of $\mathcal{D}_1$. We consider the points $p_{i,j}$ such that $|B_{i,j}| > 1$. Blowing up these points $p_{i,j}$ gives a new model $\mathcal{D}_2$.

This process terminates: at some point all the $B_{i_0,i_1,\dots,i_k}$ have cardinality 1, since the coordinates of the points on the special fiber of the blow-up are exactly the coefficients of the t-adic expansions of those points. The t-adic expansions of distinct P_i and P_j are different after a certain height k, giving different coordinates on the corresponding blow-up. The last semistable model $\mathcal{D}_k$ before the process above terminates is our separating semistable model. We simply call this model $\mathcal{D}$.

Ramification indices for superelliptic coverings

Let $C \to \mathbb{P}^1$ be a superelliptic covering of degree n given by $(x, y) \mapsto x$ for the curve C defined by $y^n = f(x)$, where we can assume $f(x)$ is a polynomial in $K[x]$. For every α a root of $f(x)$, we can consider the valuation v_α corresponding to $x - \alpha$ in the function field $K(x)$. Then, α is in the branch locus if and only if $n \nmid v_\alpha(f(x))$. Indeed, the Newton polygon of $y^n - f(x)$ with respect to this valuation has slope $-v_\alpha(f(x))/n$, which is integral if and only if $n|v_\alpha(f(x))$. Here, the Newton polygon of a polynomial $g \in K[y]$ for some valued field K is the lower convex hull of the points $(i, v(g_i)) \in \mathbb{R}^2$. The g_i satisfy $g = \sum_{i=0}^n g_i y^i$.

We now consider the canonical model $\mathcal{D}$ constructed in the previous subsection. We do not need to write the equations for this model, and we may instead work with the intersection graph, which is the tropical separating tree of these points minus the leaves at the end. For this canonical model $\mathcal{D}$, we take the corresponding disjointly branched morphism $\mathcal{C} \to \mathcal{D}$ obtained by normalization after a finite extension. That is, we take a finite extension of K to eliminate the ramification on the components of the special fiber of $\mathcal{D}$ and then we take the normalization $\mathcal{C}$ of $\mathcal{D}$ inside the function field $K(C)$ of C. By [87, Theorem 2.3] and [71, Proposition 3.1.], the morphism $\phi_{\mathcal{C}} : \mathcal{C} \to \mathcal{D}$ is then disjointly branched, as defined in Definition 4.5.4. We use this disjointly branched morphism $\phi_{\mathcal{C}}$ throughout this section.

Proposition 4.5.14 ([26])**.** Let $P \in \mathbb{P}^1(K)$ be a (generic) branch point of the superelliptic covering $\phi : C \to \mathbb{P}^1$ given by the equation $y^n = f(x)$ with a corresponding superelliptic morphism of metric graphs $\phi_\Sigma : C_\Sigma \to T$ induced from the morphism of semistable models $\mathcal{C} \to \mathcal{D}$. Let $c_P := v_P(f(x))$, where v_P is the valuation associated to P in the function field $K(x)$. For any point $x \in \mathcal{C}$, we let I_x be the inertia group of x, as defined in the subsection "Galois Covers of Semistable Models."

1. Let Q be any point in the preimage of P, and let $\tilde{Q} := r_{\mathcal{C}}(Q)$. Then

$$|I_Q| = n/\gcd(c_P, n) = |I_{\tilde{Q}}|.$$

2. Let ψ be a rational function on T satisfying $\Delta(\psi) = \rho(\mathrm{div}(f))$. Let ψ_e be the slope of ψ along the edge e of T. Let e' be any edge lying above e. Then,

$$|I_{e'}| = n/\gcd(\psi_e, n).$$

In other words, there are $\gcd(\psi_e, n)$ edges lying above e.

3. Let g_v be the number of vertices in Σ lying above $v \in T$. Then

$$g_v = \frac{n}{\operatorname{lcm}(|I_Q|)} = \gcd(n/|I_Q|),$$

where the least common multiple and greatest common divisor are taken over all ramification points Q reducing to components $\Gamma_{v'}$ for any vertex v' lying above v and edges adjacent to v'.

Proof.

1. Consider the polynomial $y^n - f(x) \in K(x)[y]$. The point P gives a natural valuation of the function field $K(x)$. The Newton polygon of $y^n - f(x)$ with respect to this valuation is a single line with slope $-c_P/n$. This means there are n roots with valuation $-c_P/n$. We clear the denominator and the numerator and obtain $n/\gcd(c_P, n)$ in the denominator. This denominator is exactly the ramification index for the extension of discrete valuation rings corresponding to Q and P, which proves the desired result. The second equality is Proposition 4.5.11.

   We remark that a simpler proof (which does not use Proposition 4.5.11) is possible here. One may calculate the order of the inertia group $I_{\tilde{Q}}$ directly using [71, Proposition 5.1.] and then conclude that it equals $|I_Q|$.

2. For the second statement, pick any vertex v with corresponding irreducible component Γ containing the edge e. We consider the Γ-modified form of f, defined as follows. The component Γ has a generic point y with discrete valuation ring $\mathcal{O}_{\mathcal{D},z}$, valuation v_Γ, and uniformizer t. We set $k := v_\Gamma(f)$ and define the Γ-modified form to be the element

$$f^\Gamma := \frac{f}{t^k}.$$

   The corresponding morphism of components is described by $(y')^n - f^\Gamma$, where $y' = \dfrac{y}{t^{k/n}}$. On the special fiber, the intersection point corresponds to a smooth point of Γ. By the Poincaré-Lelong formula, as presented in [71, Corollary 5.1], the valuation of f^Γ at this smooth point is exactly the slope of the function ψ on e. As in the previous statement, the ramification index on the special fiber is $n/\gcd(\psi_e, n)$. By Proposition 4.5.11, this is the order of the inertia group at e', as desired.

3. For the third statement, we consider as before the algebra

$$\mathcal{O}_{\mathcal{D},z}[y]/\langle y^n - f^\Gamma \rangle.$$

   The number of irreducible factors of $\overline{y^n - f^\Gamma}$ is the number of vertices lying above Γ. By considering the prime decomposition of n, we have $n = \operatorname{lcm}(|I_Q|) \cdot (\gcd(n/|I_Q|))$. Then, $n/|I_Q| = \gcd(n, v_\alpha(\overline{f^\Gamma}))$.

We have $\overline{f^\Gamma} = \overline{h}^r$ for some $h \in \mathcal{O}_{\mathcal{D},y}$. There must be distinct roots α_i and α_j of $\overline{h}$ such that the valuations of h at $x - \alpha_i$ and $x - \alpha_j$ are coprime. Otherwise, this would contradict the fact that r is the greatest common divisor of all the valuations. We have the factorization

$$\overline{y^n - f^\Gamma} = \prod_{i=1}^{r} (\overline{y^c - (\zeta_r)^i \cdot h})$$

for some primitive r-th root of unity ζ_r. We claim the factors $\overline{y^c - (\zeta_r)^i \cdot h}$ are irreducible. Indeed, there are two roots of h such that their valuations have no common factor. Any further factorization of h would contradict this fact. We thus conclude there are exactly r factors of $\overline{y^n - f^\Gamma}$. This implies the statement of the proposition.

□

The algorithm

We now give an algorithm producing the Berkovich skeleton of a curve C defined by an equation $y^n = f(x)$ for some $n \geq 2$ and $f(x) \in K(x)$. This algorithm generalizes the algorithm for finding the Berkovich skeleton of hyperelliptic curves as seen in Section 4.4. We take as input to the algorithm a polynomial $f(x) \in K[x]$, which we may do because for $f(x)$ of the form $f(x) = g(x)/h(x)$, we may multiply both sides of $y^n = f(x)$ by $h(x)^n$ and make a change of coordinates $\tilde{y} = h(x) \cdot y$ to obtain the integral equation $(\tilde{y})^n = g(x)h(x)^{n-1}$.

Algorithm 4.5.15 (Tropicalization Algorithm, [26]).

Input: A curve C defined by the equation $y^n = f(x) = \prod_{i=1}^{r}(x - \alpha_i)$.
Output: The Berkovich skeleton C_Σ of C.

1. *Compute the tree T.* This is the abstract tropicalization of $\mathbb{P}^1$ together with the marked ramification points $Q_1, \ldots, Q_s$. This is done in the following way (See [88, Section 4.3]).
   a) Let M be the $2 \times s$ matrix whose columns are the branch points $Q_1, \ldots, Q_s$. Let m_{ij} be the (i, j)-th minor of this matrix.
   b) Let $d_{ij} = N - 2v(m_{ij})$, where v is the valuation on K and N is an integer such that $d_{ij} \geq 0$.
   c) The number d_{ij} is the distance between leaf i and leaf j in the tree T. These distances uniquely specify the tree T, and one can use the Neighbor Joining Algorithm [104, Algorithm 2.41] to reconstruct the tree T from these distances.
2. *Compute the slopes ψ_e along each edge of T.* The divisor $\rho(\mathrm{div}(f))$ is a principal divisor on T, so there exists a rational function ψ on T with $\Delta(\psi) = \rho(\mathrm{div}(f))$ (as defined in [10, Page 4]). One can compute $\rho(\mathrm{div}(f))$ by observing where the zeros and poles of f specialize. Use this to compute the slopes ψ_e of ψ along edges e of T using Lemma 4.5.3.

3. *Compute the intersection graph of $\mathcal{C}_s$.*

   a) *Edges.* The number of preimages of each edge is $\gcd(\psi_e, n)$ by Proposition 4.5.14.2.

   b) *Vertices.* The number of preimages of each vertex v is $\gcd((n/(\psi_e))|e \ni v)$ by Proposition 4.5.14.3.

4. *Determine the edge lengths and vertex weights to find C_Σ.*

   a) *Edges.* If an edge e in T has length $l(e)$, then the length of each of its preimages in C_Σ is $\frac{l(e)\cdot\gcd(\psi_e,n)}{n}$, by Proposition 4.5.14 and [86, Chapter 10, Proposition 3.48, Page 526]. Remove any infinite leaf edges.

   b) *Vertices.* The weight on each vertex v is determined by the local Riemann-Hurwitz formula. The degree d at v can be determined from the edge lengths. The weight of v is determined by

$$2w(v) - 2 = -2\cdot d + \sum_{e\ni v}\left(\frac{n}{\gcd(n,\psi_e)} - 1\right).$$

Remark 4.5.16. For the input of this algorithm, we assume the function f has already been factored. Using the Newton-Puiseux Method [124], one can make a finite expansion for the roots. Since we are only interested in the valuations of the root differences, a finite expansion is sufficient. An explicit upper bound for the needed height of this expansion is given by $v(\Delta(f))$, where $\Delta(f)$ is the discriminant of f. Typically, this method is offered as a proof that the field of Puiseux series is algebraically closed, but it can also be used to actually find the roots of univariate polynomials over the Puiseux series. This method has been implemented in `Maple`[9, `algcurves`] and `Magma` [24].

Theorem 4.5.17 (Tropicalization Algorithm, [26])**.** *The Tropicalization Algorithm 4.5.15 terminates and is correct.*

Proof. The tree T created in the algorithm is the tree obtained from the canonical semistable model in the subsection "A separating semistable model" with the leaves attached. The formulas for the number of preimages of the edges and the vertices are given by Proposition 4.5.14, parts 2 and 3 respectively. There is only one graph up to a labeling of the vertices satisfying the covering data found in the algorithm. We thus obtain the intersection graph of the semistable model $\mathcal{C}$. Contracting any leaf edges yields the Berkovich skeleton. □

Example 4.5.18. We compute the abstract tropicalization of the curve defined by

$$y^3 = x^2(x-t)(x-1)^2(x-1-t)(x-2)^2(x-2-t).$$

1. The matrix M is

$$M = \begin{bmatrix} 0 & t & 1 & 1+t & 2 & 2+t \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix},$$

and so the vector m (organized lexicographically) is

$$m = (-t,\ -1,\ -1-t,\ -2,\ -2-t,\ t-1,\ -1,\ t-2,\ -2,\ -t,\ -1,\ -1-t, -1+t,\ -1,\ -t).$$

Taking $N = 2$, we have $m = (0,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 0,\ 2,\ 2,\ 2,\ 2,\ 0)$. Therefore, the tree is as displayed in Figure 4.12.

2. We have $\mathrm{div}(f) = 2(0) + (t) + 2(1) + (t+1) + 2(2) + (2+t) - 9(\infty)$. Then, $\rho(\mathrm{div}(f)) = 3v_{12} + 3v_{34} + 3v_{56} - 9v$, and $\phi_{e_{12}} = \phi_{e_{34}} = \phi_{e_{56}} = 3$. On all leaf edges ϕ_e is 1 or 2.

3. Each of the edges e_{12}, e_{34}, e_{56} has 3 preimages, and all leaves have 1 preimage. We can contract these in the tropical curve, so we do not draw them in the graph, but we mention them here because they are necessary for bookkeeping the ramification in the formulas. The middle vertex v has 3 preimages, and the other vertices have 1 preimage. So, the graph is $K_{3,3}$.

4. The lengths of all interior edges in the tree T were 1. These lengths are preserved in $K_{3,3}$ because all edges were unramified. The weights on all vertices are 0. For example,

$$w(v_{12}) = -3 + 1 + (3(3/3 - 1) + 2(3/1) - 1)/2 = 0.$$

So, the abstract tropicalization of our curve is the metric graph in Figure 4.13. Each vertex is labeled with its image in the tree T.

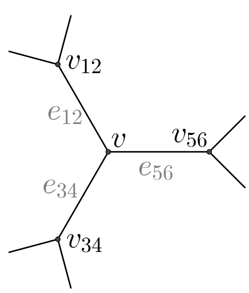


Figure 4.12: The tree T in Example 4.5.18.

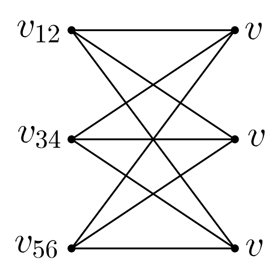


Figure 4.13: Tropicalization of the curve in Example 4.5.18.

Example 4.5.19. In [100] the author shows there is a unique Shimura-Teichmüller curve of genus three, X_3, defined by the equation $y^4 = x(x-1)(x-t)$, and there is a unique Shimura-Teichmüller curve of genus four, X_4, defined by the equation $y^6 = x(x-1)(x-t)$. In [62, Section 2], the authors compute the period matrix of X_4. We now use the Tropicalization Algorithm to compute their Berkovich skeletons.

1. The ramification points are $0, 1, t$, and ∞. The corresponding tree is in Figure 4.14, where the interior edge has length 1. We call the interior vertices v_1 and v_2.

2. The divisor of $f := x(x-1)(x+t)$ is $\text{div}(f) = (0) + (t) + (-1) - 3(\infty)$. The divisor specializes to $\rho(\text{div}(f)) = 2v_0 - 2v_1$. The corresponding rational function ϕ has slope 2 on the only edge in the tree.

3. We have $\gcd(\phi_e, n) = 2$ in both cases. Therefore, the edge has 2 preimages. Both vertices on the tree have leaves, so both v_0 and v_1 each have one preimage each in the graphs $X_{3,\Sigma}$ and $X_{4,\Sigma}$.

4. The length of the interior edge in the tree is 1, so in X_3 there are two edges of length 1/2 and in X_4 there are two edges of length 1/3. For the genera of the vertices, we apply the Riemann-Hurwitz formula to obtain the complete picture of the graphs; $X_{3,\Sigma}$ is in Figure 4.15 and $X_{4,\Sigma}$ is in Figure 4.16.

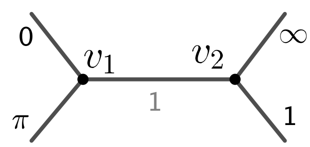


Figure 4.14: Tree for the Shimura-Teichmüller curves.

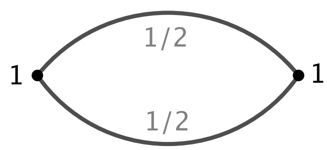


Figure 4.15: Tropical genus 3 Shimura-Teichmüller curve.

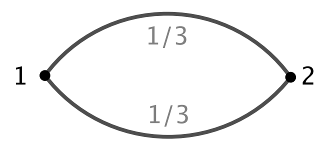


Figure 4.16: Tropical genus 4 Shimura-Teichmüller curve.

Realizability

We now show that every superelliptic cover of prime degree of metric graphs comes from an algebraic superelliptic cover. A similar result was proved for degree d admissible coverings in [38]: for every degree d admissible covering of metric graphs $C_\Sigma \to T$, there exists an algebraic covering $C \to \mathbb{P}^1$ tropicalizing to $C_\Sigma \to T$. We note however the covering obtained by this theorem is not necessarily Galois. Unlike in [38], our approach is constructive; the proof of our result presents a method for finding the defining equation of a curve C.

We first recall the set up: given a superelliptic covering of curves $C \to \mathbb{P}^1$, we obtain a superelliptic covering of metric graphs $\Gamma \to T$ by computing the tree T and the divisor $\rho(\text{div}(f)) = \sum a_i P_i$ on T. The main difficulty in reversing this process is finding a_i which give the graph Γ. We show inductively there are enough ways of assigning values to the a_i such that the desired tropicalization is obtained. We denote this collection of rational functions by

$$S_\psi = \{\phi : \text{the covering associated to the divisor } (\phi) \text{ is } \psi\}.$$

There is a natural faithful action of the group $\mathbb{F}_p^*$ on this set, given by multiplication. We would like to prove $|S_\psi| > 0$. By faithfulness, we immediately have at least $p-1$ solutions. The number of branch points of the covering ψ is denoted by R_ψ. To show there exists an algebraic covering tropicalizing to the given covering, we construct R_ψ points in $\mathbb{P}^1$, labeled P_i, which tropicalize to T, and a divisor $\sum_{i=1}^{R_\psi} a_i P_i$ inducing the desired covering. We already know the vertices which the points P_i reduce to; they are the leaves in the tree T. We write $v(P_i)$ for these vertices. Any choice of $a_i \in \mathbb{Z}$ gives a divisor

$$\rho(\mathrm{div}(f)) := \sum_{i=1}^{R_\psi} a_i v(P_i)$$

on the tree T. For the remainder of the section, we fix a target vertex v_0 with at least two branch points. For any edge e in T, consider the connected component T_e of $T\backslash\{e\}$ not containing v_0 as in Lemma 4.5.3. The slope of a rational function giving this divisor along an edge e is now given by the formula in Lemma 4.5.3:

$$\sum_{x\in T_e} (\psi)(x) = \sum_{P_i \in T_e} a_i.$$

Definition 4.5.20. Let s_e be the number of P_i reducing to the connected component T_e. The *total Laplacian* on the component T_e is a function $\Delta_e(\psi) : (\mathbb{F}_p)^{s_e} \to \mathbb{F}_p$, sending $(a_i) \mapsto \sum_{P_i \in T_e} a_i$. We consider these as elements of $\mathbb{F}_p$ because we are only interested in the value of the slopes and the exponents mod p.

This definition allows us to view the formula for the slope of the Laplacian on e as a function of the a_i laying on the connected component T_e. The covering ψ must satisfy the *total Laplacian equations*

$$\begin{cases} \Delta_e(\psi)(a_i) = \overline{0} & \text{if } p \text{ edges map to } e, \\ \Delta_e(\psi)(a_i) \in \mathbb{F}_p^* & \text{if only 1 edge maps to } e. \end{cases}$$

We adopt the following notation for the rest of this section.

1. We write $\Delta_e(\psi) \equiv 1$ if there exist a_i such that $\Delta_e(\psi) \in (\mathbb{F}_p)^*$.

2. Similarly, we write $\Delta_e(\psi) \equiv 0$ if there exist a_i such that $\Delta_e(\psi) = \overline{0}$.

3. Given a set of edges $E := \{e_i\}$, we write $\Delta_E(\psi) \equiv \delta_{e_i}$ for $\delta_{e_i} \in \{0,1\}$ if there exist a_i such that all conditions $\Delta_{e_i}(\psi) \equiv \delta_{e_i}$ are met simultaneously for this set of $\{a_i\}$.

4. Given an edge e with connected component T_e and numbers $\delta_{e_i} \in \{0,1\}$ for $e_i \in T_e$, we write $\Delta_{T_e}(\psi) = c$ for a $c \in \mathbb{F}_p$ if there exist a_i such that $\Delta_{e_j}(\psi) \equiv \delta_{e_j}$ for every e_j in T_e and such that $\Delta_e(\psi) = c$.

So, the covering $\psi : \Gamma \to T$ gives us a set of $\{\delta_{e_i}\}$ with $\delta_{e_i} \in \{0,1\}$: for every unramified edge e_i we obtain $\Delta_{e_i}(\psi) \equiv \delta_{e_i} = 0$ and for every ramified edge e_i we obtain $\Delta_{e_i}(\psi) \equiv \delta_{e_i} = 1$.

Lemma 4.5.21 ([26]). *Let $\Gamma \to T$ be a superelliptic covering of metric graphs. For every $e_i \in T_e$, let $\delta_{e_i} \in \{0,1\}$ be given by the covering, as above.*

1. *If e is ramified, then $\Delta_{T_e}(\psi) = c$ for any $c \in \mathbb{F}_p^*$.*
2. *If e is unramified, then $\Delta_{T_e}(\psi) = 0$.*

Proof. We prove the lemma by induction on $|E(T_e)|$. The inductive hypothesis is

I_n: For every e such that $|E(T_e)| \leq n$, we have $\Delta_{T_e}(\psi) = 0$ if e is unramified and $\Delta_{T_e}(\psi) = c$ for every $c \in \mathbb{F}_p^*$ if e is ramified.

For $n = 0$, T_e consists of a single vertex v. If e is ramified, there exists at least one branch point P. If it is the only branch point, then $\Delta_e(\psi) = v(P)$, which can be any $c \in \mathbb{F}_p^*$. If there exists another branch point Q, then any value c can be attained by a combination $aP + bQ$ for some a and b. If e is unramified, there exist at least two branch points at v. The valuations can be chosen to satisfy $\Delta_e(\psi) = 0$ as required.

Now suppose the statement is true for n. Let e be any edge such that $|E(T_e)| = n + 1$. Let v be the vertex in T_e connected to e. Then for every other edge connecting to v, we have $|E(T_{e_i})| \leq n$, so by the induction hypothesis we know the statement is true for T_{e_i}. Suppose e is ramified. Then v is branched over at least one other point, which can be a global branch point or an edge. If it is the only branch point, we are done because $\Delta_{T_e}(\psi)$ is equal to the valuation of this branch point. This valuation is equal to the slope along the branched edge (which we can control by the induction hypothesis) or the valuation of the global branch point reducing to it. If there is another branch point we can use its valuation to adjust the value of $\Delta_{T_e}(\psi)$. This can attain any value $c \in \mathbb{F}_p^*$. Suppose e is unramified. The argument is similar to the previous case. There are at least two branch points reducing to v, which can be edges or global branch points. In both cases we have complete control over them (as in the ramified case) and we can solve $\Delta_{T_e}(\psi) = 0$. By induction, we now conclude the statement holds for any n. □

We apply Lemma 4.5.21 to the edge connected to v_0 to obtain an assignment for all a_i.

Corollary 4.5.22 ([26]). *Given any superelliptic covering $\Gamma \to T$ with covering data δ_i for every edge, we have $\Delta_E(\psi) \equiv \delta_i$ for $E = E(T)$.*

Theorem 4.5.23 ([26]). *Let p be a prime number. A covering $\phi : \Gamma \to T$ is a superelliptic covering of degree p of weighted metric graphs if and only if there exists an algebraic superelliptic covering $\phi : X \to \mathbb{P}^1$ of degree p tropicalizing to it.*

Proof. Suppose we have a superelliptic admissible covering of graphs $\phi : \Gamma \to T$ of degree p. We present a procedure for constructing a polynomial f such that the covering from the curve $y^p = f(x)$ defined by $(x, y) \mapsto x$ tropicalizes to $\phi : \Gamma \to T$.

1. On each vertex $v_i \in T$, use the local Riemann-Hurwitz condition to determine the number of leaves $r(v_i)$ needed on each vertex.

2. Each vertex $v_i \in T$ corresponds to points $P_{i,1}, \ldots, P_{i,r(v_i)} \in \mathbb{P}^1(K)$, each corresponding to the leaves attached at v_i. The equation for f is $f(x) = \prod_{v_i \in T} \prod_{j=1}^{r(v_i)} (x - P_{i,j})^{a_{ij}}$.

3. Find the a_{ij} as follows. Select a target vertex v_0 with at least two leaves. For every edge in the graph, we solve the corresponding total Laplacian equation with respect to v_0. The fact that there is a solution follows from Corollary 4.5.22. Pick a solution to these equations. Consider the branch points $P_{v_0,1}, ..., P_{v_0,s}$ reducing to v_0. The valuations at these points satisfy
$$\sum_{i=1}^{s} a_{v_0,i} = \sum_{P \text{ not reducing to } v_0} -a_P,$$
Picking $a_{v_0,i}$ satisfying this equation concludes the algorithm for finding the a_{ij}.

4. To obtain the desired points P_i, we view these trees as describing t-adic expansions of elements in K. To be explicit, let S be a set of representatives for the residue field k. Let v_0 be an endpoint of T, and let v_1 be the vertex connected to v_0. For every leaf e (with end vertex not equal to v_0) attached to v_1, construct a point $P_e = c_e t$, with the $c_e \in S$ distinct. This might require a finite extension of the residue field k, which corresponds to a finite (unramified) extension of K. For every nonleaf e_i, take an element $c_i \in S$ that is not equal to the c_e. For such an edge e_i, consider the connecting vertex $v_{1,i}$. For every leaf e attached to $v_{1,i}$, find distinct $c_{i,e} \in S$ and construct $P_e = c_i t + c_{i,e} t^2$. For every nonleaf $e_{i,j}$ connected to $v_{1,i}$, repeat the procedure and construct elements $c_{i,j} \in S$ distinct from the $c_{i,e}$, where e is a leaf. We do one more step of the inductive procedure. Let $v_{1,i,j}$ be the other vertex connected to $e_{i,j}$. For every leaf e attached to $v_{1,i,j}$, find distinct $c_{i,j,e}$ and construct $P_e = c_i t + c_{i,j} t^2 + c_{i,j,e} t^3$. At some point, we reach vertices that only have leaves as neighboring edges. At this point, we stop the procedure and find a set of points $\{P_e\}$. The tree corresponding to this set of points is T. On the algebraic side, we take the canonical semistable $\mathcal{D}$ corresponding to this set (see the subsection "A separating semistable model"). Its intersection graph is T minus the leaves.

The other direction is obtained by combining [71, Theorem 3.1] and [86, Chapter 10, Proposition 3.48, Page 526] for the edge lengths. □

A natural question following from this is whether the same result holds for non-prime integers n. We conjecture that this is indeed the case and that a similar proof could be used.

Moduli space

The moduli space $\mathcal{M}_g^{\text{trop}}$ of weighted metric graphs of genus g was defined in [29], and has the structure of a $(3g-3)$-dimensional stacky fan. The cones in M_g^{trop} of dimension d correspond to *combinatorial types*, which are pairs consisting of a graph H with d edges and a weight function w on its vertices. A *constrained type* is a triple (H, w, r), where r is an equivalence relation on the edges of H. In a metric graph Σ corresponding to the constrained type (H, w, r), the equivalence relation r requires that edges in the same equivalence class have the same length. One can contract edges of a constrained type to arrive at a new constrained type. Contraction is discussed in [44, Section 4.1] and depicted in Figure 4.17.

Definition 4.5.24. The *moduli space of tropical superelliptic curves* $\mathcal{S}_{g,n}^{\mathrm{trop}}$ is the set of weighted metric graphs of genus g which have a degree n superelliptic covering to a tree. We will see below in Proposition 4.5.25 that it has the structure of a stacky polyhedral fan. Let $\mathcal{S}p_{g,n}^{\mathrm{trop}} \subset \mathcal{S}_{g,n}^{\mathrm{trop}}$ denote the image under tropicalization of superelliptic curves defined by equations of the form $y^n = f(x)$ with distinct roots.

By Theorem 4.5, when n is prime we have $\mathcal{S}_{g,n}^{\mathrm{trop}} \subset \mathcal{M}_g^{\mathrm{trop}}$ is equal to the image under tropicalization of the locus of superelliptic curves inside $\mathcal{M}_g$, the moduli space of genus g curves. We comment $\mathcal{S}p_{g,n}^{\mathrm{trop}} \subsetneq \mathcal{S}_{g,n}^{\mathrm{trop}}$ when $n > 2$. See Figure 4.17 for the combinatorial types of weighted metric graphs corresponding to cones inside $\mathcal{S}_{4,3}^{\mathrm{trop}}$ and $\mathcal{S}p_{4,3}^{\mathrm{trop}}$.

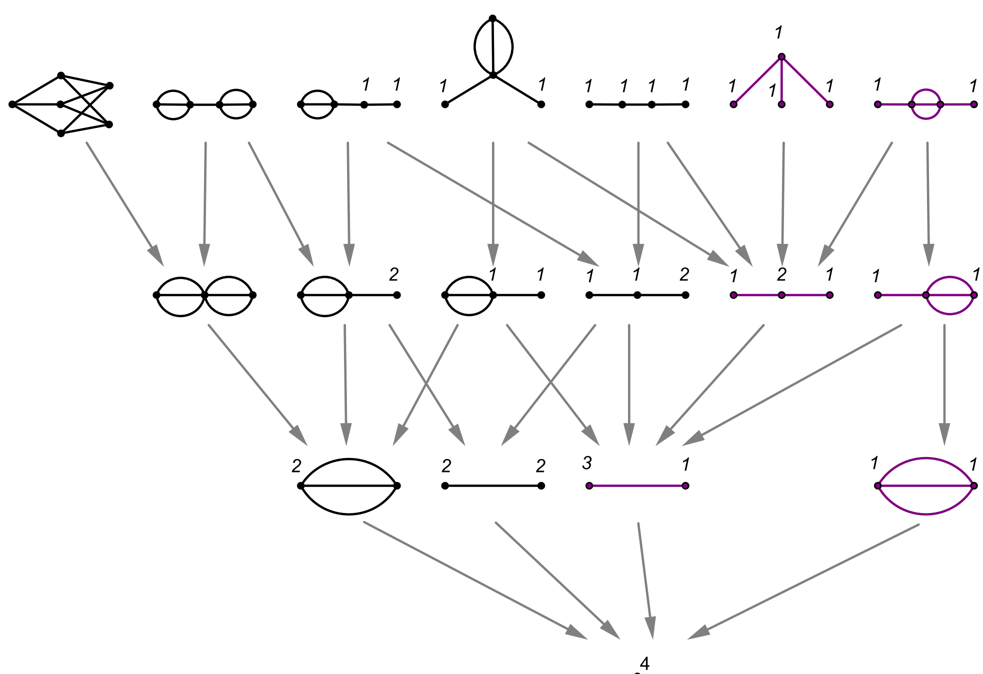


Figure 4.17: Weighted metric graphs corresponding to maximal cones in $\mathcal{S}_{4,3}^{\mathrm{trop}}$ and $\mathcal{S}p_{4,3}^{\mathrm{trop}}$ (shown in purple).

Proposition 4.5.25 ([26])**.** The locus $\mathcal{S}_{g,n}^{\mathrm{trop}}$ of weighted metric graphs of genus g which have a degree n superelliptic covering to a tree has the structure of a stacky polyhedral fan.

Proof. In [44, Section 4.1], Chan proves that given a collection S of constrained types which are closed under contraction, the space M_S they define is a stacky fan with cells in correspondence with the constrained types. We can obtain a constrained type from a combinatorial type of genus g graph with a degree n superelliptic covering to a tree by making the relation r to equate any edges which have the same image under the covering map. Let (H, w, r) be such a type, admitting a degree n superelliptic covering θ to the tree T which gives the relation r. If (H', w', r') is a contraction of (H, w, r) along the equivalence class of edges $[e]$, and T' is the contraction of T along the edge $\theta([e])$, one can see using the local Riemann-Hurwitz equations (H', w', r') admits a degree n superelliptic covering to the tree T'. □

Using the Riemann-Hurwitz equation, we compute the genus of a graph in the case when p is prime and the map has r ramification points. Let $g(p, r) := (p-1)(r/2-1)$.

Theorem 4.5.26 ([26])**.** *Let $r \geq 4$ be an integer number of ramification points. Given two odd primes p and p', the stacky polyhedral fan $\mathcal{S}^{\text{trop}}_{g(p,r),p}$ is the same as $\mathcal{S}^{\text{trop}}_{g(p',r),p'}$.*

Proof. Define a new stacky polyhedral fan $Ts(r)$ for $r \geq 4$ whose cones correspond to pairs (T, s), where T is a tree on r leaves and s is any subset of the edges of T which we call a *signature*. Each cone has dimension equal to the number of interior edges of T. We glue the cone (T, s) to the cone (T', s') when (T', s') is a pair in which an edge $e \in T$ has been contracted, and $s' = s \backslash \{e\}$.

Given a tree T with r leaves and m interior edges, one can compute all superelliptic graphs with a degree p map to T in the following way. A choice of signature on T corresponds to deciding which interior edges have p preimages or 1 preimage in a superelliptic graph mapping to T. This yields 2^m signatures, but some signatures do not yield admissible covers. On each interior vertex v, compute the weight using the local Riemann-Hurwitz equation. If the vertex has no leaves and all edges adjacent to it have multiple preimages, then the vertex has p preimages and weight 0. Otherwise $w(v) = (p-1)(r_v - 2)/2$, where r_v is the number of leaves at v plus the number of ramified edges. The graph is superelliptic if and only if this number is a positive integer for all vertices of the tree. Since p is odd, this is always be an integer. Then, we need that at each vertex, $r_v \geq 2$. Any signature on a tree satisfying this yields a superelliptic graph. So, graphs admitting a degree p superelliptic cover of T are in bijection with good choices of signatures on T. The space $S^{\text{trop}}_{g(p,r),p}$ naturally sits inside $Ts(r)$; each cone corresponding to a superelliptic curve $\Gamma \to T$ is mapped to the cone (T, s) where T is the tree corresponding to that curve and s is the set of ramified edges in the covering. Whether or not a signature is admissible does not depend on p, so for any odd primes p and p', the images $\mathcal{S}^{\text{trop}}_{g(p,r),p} \subset Ts(r)$ and $\mathcal{S}^{\text{trop}}_{g(p',r),p'} \subset Ts(r)$ are the same. □

Theorem 4.5.27 ([26])**.** *For primes $p \leq 17$ and number of ramification points $r \leq 14$, the number of maximal cones in $S^{\text{trop}}_{g(p,r),p}$ and $Sp^{\text{trop}}_{g(p,r),p}$ is given in Table 4.1.*

Proof. This was done by direct computation in `Mathematica`, as we describe below. We restrict to the case of prime $n = p$ to simplify the computation. First, we precompute all trivalent trees on r leaves.

For the computations for $\mathcal{S}^{\text{trop}}_{g(p,r),p}$, we create all graphs arising from assigning on each interior edge of the tree which ones are ramified and which are unramified (this gives 2^{r-3} possibilities). We check which of the resulting graphs have an assignment of non-negative integer weights on the vertices satisfying the local Riemann-Hurwitz condition. Then, we remove the isomorphic duplicates. For fourteen leaves, this took 2.6 days to compute.

For the computations on $\mathcal{S}p^{\text{trop}}_{g(p,r),p}$, the possible metric graphs arising from a fixed tree T depend only on the choice of where ∞ specializes. This is because the metric graph is determined by the divisor of f, and in the case with distinct roots, this is completely determined by where ∞ specializes on T. So, we make all possible choices and compute the resulting metric graphs. Then, we remove the isomorphic duplicates. The computations with twenty leaves took 16 hours each to compute. □

r	$S^{\text{trop}}_{g,2}$	$S^{\text{trop}}_{g,p>2}$	$Sp^{\text{trop}}_{g,3}$	$Sp^{\text{trop}}_{g,5}$	$Sp^{\text{trop}}_{g,7}$	$Sp^{\text{trop}}_{g,11}$	$Sp^{\text{trop}}_{g,13}$	$Sp^{\text{trop}}_{g,17}$
4	1	2	1	1	1	1	1	1
5	0	2	2	1	1	1	1	1
6	2	7	2	0	2	2	2	2
7	0	11	0	5	2	2	2	2
8	4	34	11	7	0	4	4	4
9	0	80	6	12	17	6	6	6
10	11	242	0	11	22	11	11	11
11	0	682	92	0	40	18	18	18
12	37	2146	37	160	70	0	37	37
13	0	6624	0	227	132	273	66	66
14	135	21447	916	457	135	342	0	135
15	0	-	265	265	0	679	1248	265
16	552	-	0	0	3167	1173	1535	552
17	0	-	10069	8011	4323	2374	3098	1132
18	2410	-	2410	12029	8913	4687	5359	0
19	0	-	0	24979	16398	9859	10996	29729
20	11020	-	117746	11020	34511	20542	21833	35651

Table 4.1: The number of maximal cones in $\mathcal{S}p^{\text{trop}}_{g(p,r),p}$ and $\mathcal{S}^{\text{trop}}_{g(p,r),p}$.

Conclusion

In this chapter, we algorithmically computed non-Archimedean skeletons, or abstract tropicalizations, of hyperelliptic and superelliptic curves. There is still much work to be done in computing skeletons of more general curves. The ideas used in the proofs of these algorithms could lead to computations of skeletons for other types of curves.

5

Enumeration

Tropical geometry has seen great success in enumerative geometry, which counts the number of varieties of a particular type satisfying chosen conditions. For example, the *Gromov-Witten invariants* of the plane count the complex algebraic curves of a given degree and genus passing through a given number of points. It is a breakthrough result of Mikhalkin that this question can be rephrased tropically, and that the counts agree [96].

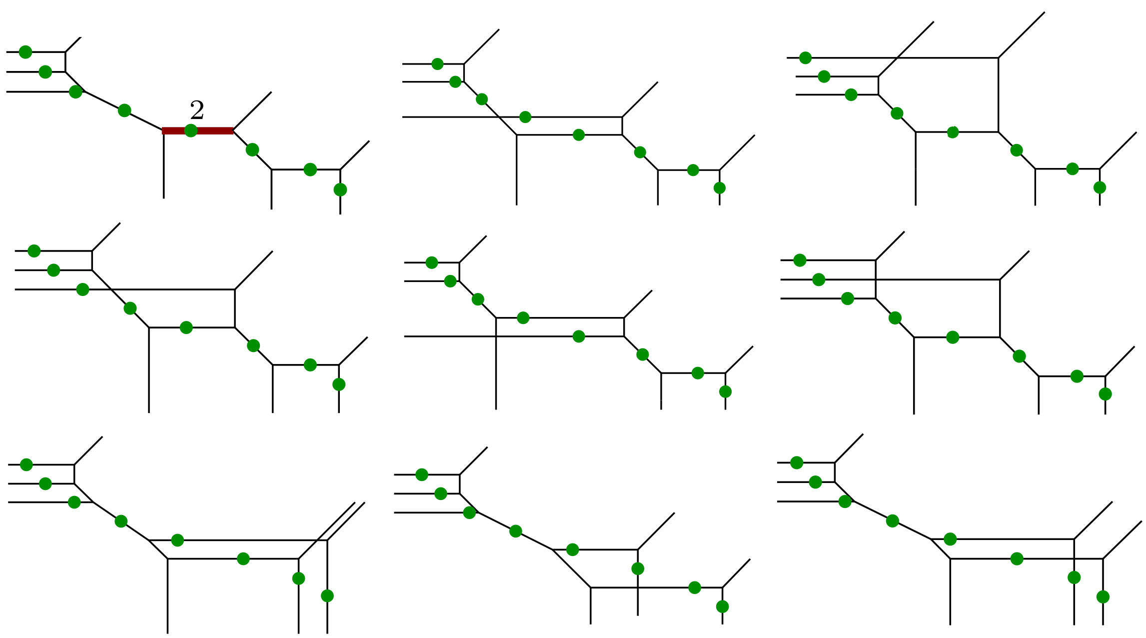


Figure 5.1: The nodal tropical plane cubic curves through 8 points (max convention). The first curve is counted with multiplicity 4. Images from [113].

This *correspondence principle* ([96], Theorem 5.1.8) allows us to instead count the number of tropical plane curves of a given degree and genus passing through points in a specific configuration called *Mikhalkin position*, together with their multiplicities. For plane curves, it is much simpler to perform the tropical count because singularities of tropical plane curves are well understood. Moreover, combinatorial tools like *floor diagrams* [32, 33] and the *lattice path algorithm* [96] have been developed in order to facilitate these counts. For example, there are 12 irreducible nodal plane curves of degree 3 through 8 points (Figure 5.1).

The efficacy of tropical geometry in curve counting led to initial steps towards counting higher dimensional varieties using tropical methods. In [91], the authors determine the combinatorial types of nodes that a one-nodal tropical surface can have. In [90], the authors give a correspondence theorem for singular tropical surfaces, and develop a three-dimensional version of the lattice path algorithm enumerating singular tropical surfaces passing through points in Mikhalkin position in $\mathbb{R}^3$. In [92] the authors extend the concept of floor diagrams to *floor plans* for surfaces. In this chapter, we develop counting techniques for tropical surfaces. Using these techniques, we count tropical binodal cubic surfaces.

The original material in this chapter appears in the section "Counting Surfaces in Space," and is joint work with Alheydis Geiger [25] and will be published in Le Matematiche.

5.1 Counting Curves in the Plane

Throughout this section we follow [73].

Gromov-Witten Invariants

We begin with two examples of Gromov-Witten invariants before defining them.

Example 5.1.1 ($N(d, 0)$)**.** Let d be a non-negative integer. Given a generic choice of $\frac{d(d+3)}{2}$ points in $\mathbb{P}^2$, how many curves of degree d pass through the points?

The collection of degree d curves in $\mathbb{P}^2_{\mathbb{C}}$ can be parameterized by the coefficients of their defining equation

$$a_{d00}x^d + \cdots + a_{00d}z^d.$$

This gives a projective space of dimension $\frac{d(d+3)}{2}$ in the a_{ijk} parameterizing the collection of curves of degree d in $\mathbb{P}^2_{\mathbb{C}}$. Given a point p in $\mathbb{P}^2$, the collection of all curves of degree d passing through p is a hyperplane in $\mathbb{P}^{\frac{d(d+3)}{2}}$. Therefore, asking for the curves that pass through $\frac{d(d+3)}{2}$ points just gives a single point in $\mathbb{P}^{\frac{d(d+3)}{2}}$. So, there is only one such curve.

Example 5.1.2 ($N(d, 1)$)**.** Given a generic choice of $\frac{d(d+3)}{2} - 1$ points in $\mathbb{P}^2$, how many 1-nodal curves of degree d pass through the points? Let $D \subset \mathbb{P}^{\frac{d(d+3)}{2}}$ denote the hypersurface of singular curves. This degree $3(d-1)^2$ variety is called the *discriminant*. The smooth locus of D consists of points whose only singularity is a node.

Our selection of $\frac{d(d+3)}{2} - 1$ points gives a line in $\mathbb{P}^{\frac{d(d+3)}{2}}$ of curves passing through those points. So, wherever this line intersects D we obtain a nodal curve of degree d passing through the $\frac{d(d+3)}{2} - 1$ points. This will consist of finitely many points, and their number will be the degree of D.

We may generalize Examples 5.1.1 and 5.1.2 with the following question.

Question 5.1.3. Let δ be an integer satisfying $0 \leq \delta \leq \frac{(m-1)(m-2)}{2}$. How many δ-nodal degree m curves in $\mathbb{P}^2_{\mathbb{C}}$ are there passing through a generic choice of $\frac{m(m+3)}{2} - \delta$ points?

Definition 5.1.4. Let $N(d, \delta)$ be answer to Question 5.1.3. Let $N^{irr}(d, \delta)$ count the irreducible such curves. The numbers $N^{irr}(d, \delta)$ are the *Gromov-Witten invariants of the plane.*

Let $\delta_{max} = \frac{(m-1)(m-2)}{2}$. The number $N^{irr}(d, \delta_{max})$ is the number of rational degree m curves passing through a generic collection of $3m-1$ points in $\mathbb{P}^2$. Kontsevich gave a recursive formula for the numbers $N^{irr}(d, \delta_{max})$ [81]. A recursive formula for arbitrary δ was given by Caporaso and Harris [34].

Mikhalkin Correspondence Principle

Mikhalkin proposed a new formula for $N(d, \delta)$ [96, 95]. His strategy was to reformulate the counting problem in to a problem about counting tropical curves, by showing that each tropical curve can be counted with a multiplicity so that the tropical and algebraic counts agree. Let $\Delta_d \subset \mathbb{R}^2$ be $\mathrm{conv}((0,0),(d,0),(0,d))$.

Definition 5.1.5. Let T be a tropical curve of degree d, and let Δ_d^T be the corresponding dual subdivision of Δ_d. Then T (or Δ_d^T) is called *simple* if the dual subdivision Δ_d^T satisfies the following criteria:

1. Every polygon in Δ_d^T is either a triangle or a parallelogram, and
2. Any lattice point in the boundary of Δ_d is a vertex of Δ_d^T.

Remark 5.1.6. If T is simple, it has a unique representation as the union of irreducible tropical curves.

Definition 5.1.7. The *rank* of T is

$$|\{\text{vertices of } \Delta_d^T\}| - |\{\text{parallelograms of } \Delta_d^T\}| - 1$$

The *multiplicity* $\mu(T) = \mu(\Delta_d^T)$ is

$$\prod_{\text{triangles } \Delta \in \Delta_d^T} \mathrm{area}(\Delta)$$

where the area is normalized so that $\mathrm{area}(\Delta_1) = 1$.

Let r be a positive integer, and let $\mathcal{U}$ be a generic collection of r points in $\mathbb{R}^2$. Let $C(\mathcal{U})$ be the set of simple tropical curves of degree d and rank r passing through all points of $\mathcal{U}$. Let $C^{irr}(\mathcal{U})$ be the irreducible such curves.

Theorem 5.1.8 ([96]). *Let $0 \leq \delta \leq \frac{(d-1)(d-2)}{2}$. Let $\mathcal{U}$ be a generic set of $r = \frac{d(d+3)}{2} - \delta$ points in $\mathbb{R}^2$. Then we have the following equalities:*

$$N(d,\delta) = \sum_{T \in C(\mathcal{U})} \mu(T)$$
$$N^{irr}(d,\delta) = \sum_{T \in C^{irr}(\mathcal{U})} \mu(T).$$

Lattice path algorithm

Mikhalkin gives a combinatorial algorithm to calculate the number of tropical curves in question in a simple way. Fix a linear function $\lambda : \mathbb{R}^2 \to \mathbb{R}$, such that λ is injective on the lattice points of Δ_d. Let p be the vertex of Δ_d where λ achieves its minimum, and let q be the vertex of Δ_d where λ achieves its maximum. Then p and q divide the boundary of Δ_d into two parts, which we call $\partial\Delta_d^+$ and $\partial\Delta_d^-$.

Definition 5.1.9. Let l be a natural number. A path $\gamma : [0,l] \to \Delta_d$ is λ-*admissible* if

1. We have $\lambda(0) = p$, $\lambda(l) = q$,
2. the composition $\lambda \circ \gamma$ is injective,
3. for any integer $0 \leq i \leq l-1$, we have that $\gamma(i)$ is an integer and $\gamma([i,i+1])$ is a line segment.

In this case, l is called the *length* of γ, and the integer points of $\gamma(i)$ for $0 \leq i \leq l$ are called the *vertices of* γ.

A λ-admissible path γ divides Δ_d into two parts:

1. The region bounded by γ and $\partial\Delta_d^+$, denoted by $\Delta_d^+(\gamma)$, and
2. the region bounded by γ and $\partial\Delta_d^-$, denoted by $\Delta_d^-(\gamma)$.

Definition 5.1.10. If it exists, let j be the smallest positive integer such that $1 \leq j \leq l-1$ and such that $\gamma(j)$ is the vertex of $\Delta_d^+(j)$ with angle less than π. A *compression* of $\Delta_d^+(\gamma)$ is $\Delta_d^+(\gamma')$, where γ' is either

1. the path defined by
$$\gamma'(i) = \begin{cases} \gamma(i) & i < j \\ \gamma(i+1) & i \geq j, \end{cases}$$
2. or the path defined by
$$\gamma'(i) = \begin{cases} \gamma(i) & i \neq j \\ \gamma(j-1) + \gamma(j+1) - \gamma(j) & i = j. \end{cases}$$

Definition 5.1.11. A sequence of compressions starting with $\Delta_d^+(\gamma)$ and ending with a path whose image coincides with $\partial\Delta_d^+$ defines a subdivision of $\Delta_d^+(\gamma)$ called a *compressing*.

Remark 5.1.12. A compression and a compressing subdivision of $\Delta_d^-(\gamma)$ are defined analogously.

Definition 5.1.13. A pair $(S_+(\gamma), S_-(\gamma))$ where $S_\pm$ is a compressing subdivision of $\Delta_d^\pm(\gamma)$ produces a subdivision of Δ_d, which is called γ-consistent. Denote by $\mathcal{N}_\lambda(\gamma)$ the collection of γ-consistent subdivisions of Δ_d.

Theorem 5.1.14 ([95, 96])**.** *Let $0 \leq \delta \leq \frac{(m-1)(m-2)}{2}$ be an integer. There exists a generic set $\mathcal{U}$ of $r = \frac{m(m+3)}{2} - \delta$ points in $\mathbb{R}^2$ such that the map associating a simple tropical curve T of degree m and rank r to the dual subdivision Δ_d^T of Δ_d establishes a one-to-one correspondence between the set $\mathcal{C}(\mathcal{U})$ and the disjoint union*

$$\bigsqcup_\gamma \mathcal{N}_\lambda(\gamma),$$

where γ runs over all λ-admissible paths in Δ_d of length r.

Example 5.1.15. Consider 1-nodal tropical cubic curves passing through 8 points. Let $\lambda(i,j) = i - \epsilon j$ for ϵ sufficiently small. Then $\delta = 1$ and $r = 8$. The corresponding subdivisions are pictured in Figure 5.2, where the lattice paths are pictured in green.

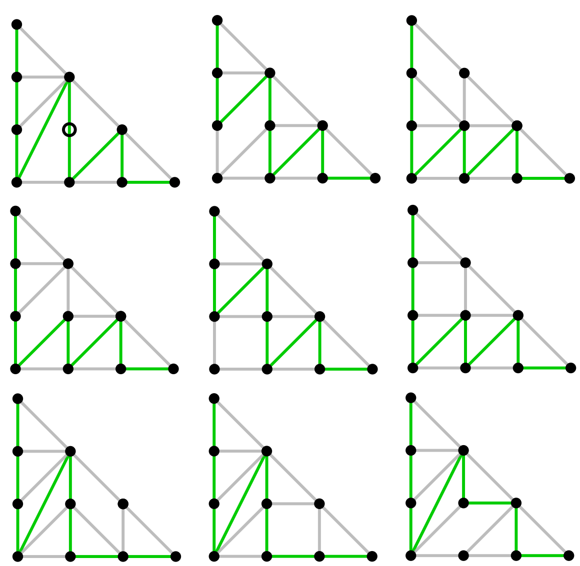


Figure 5.2: The dual subdivisions of the nodal tropical plane cubic curves through 8 points. The first curve is counted with multiplicity 4. This gives a total of 12 curves.

5.2 Counting Surfaces in Space

We can rephrase questions like the one posed in Question 5.1.3 in higher dimensions. Explicitly, what is the number of k-nodal degree d surfaces passing through $\binom{d+3}{3} - 1 - k$ points in $\mathbb{P}^3$? For the answer, see Theorem 5.2.11. In this section we count nodal surfaces in three space using tropical methods.

In order to do this, we must understand singular tropical hypersurfaces and how to construct them using a higher dimensional analogue of the lattice path algorithm. Some progress has been made towards counting nodal surfaces in three space using tropical methods. A surface with δ nodes as its only singularities is called *δ-nodal.* The tropicalization of a δ-nodal surface is called a *δ-nodal* tropical surface. A *binodal* tropical surfaces is therefore the tropicalization of a binodal surface. We say a δ-nodal surface is *real* if the polynomial defining the surface is real and the surface has real singularities.

Given a generic choice of points in $\mathbb{P}^3_K$, the δ-nodal cubic surfaces passing through the points will tropicalize to tropical surfaces passing through the tropicalizations of the points. However, an arbitrary choice of points might lead to the tropicalizations of the points not being distinct, or not being tropically generic. Furthermore, these surfaces would be difficult to characterize in general.

Luckily, we can choose points in *Mikhalkin position* (see Definition 5.2.1). This is a configuration of points in generic position such that their tropicalizations are tropically generic. Additionally, tropical surfaces passing through such points have a very nice form, and the combinatorics of the dual subdivision is well understood.

In what follows, we present preliminary techniques for counting surfaces tropically, and apply these techniques to give a partial count of binodal cubic surfaces.

Tropical Floor Plans

In this section, we give an overview of the current technology for counting surfaces using tropical geometry. Let $K = \cup_{m\geq 1}\mathbb{C}\{t^{1/m}\}$ and $K_{\mathbb{R}} = \cup_{m\geq 1}\mathbb{R}\{t^{1/m}\}$.

We now give the definition of points in Mikhalkin position. This will provide the choice of tropical points that the surfaces we count will pass through. The reason for this particular choice of points is that tropical surfaces passing through these points will have a particular structure. We can take advantage of this structure to easily list the surfaces.

Definition 5.2.1 ([90, Section 3.1])**.** Let $\omega = (p_1, ..., p_n)$ be a configuration of n points in $\mathbb{P}^3_K$ or $\mathbb{P}^3_{K_{\mathbb{R}}}$. Let $q_i \in \mathbb{R}^3$ be the tropicalization of p_i for $i = 1, \ldots, n$. We say ω is in *Mikhalkin position* if the q_i are distributed with growing distances along a line $\{\lambda\cdot(1, \eta, \eta^2)|\lambda \in \mathbb{R}\} \subset \mathbb{R}^3$, where $0 < \eta \ll 1$, and the p_i are generic.

This is possible by [96, Theorem 1]. From now on all tropical surfaces are assumed to satisfy point conditions from points in Mikhalkin position.

We now summarize the recipe for constructing δ-nodal tropical cubic surfaces through n points in Mikhalkin position. Given a singular tropical surface S passing through $\omega =$

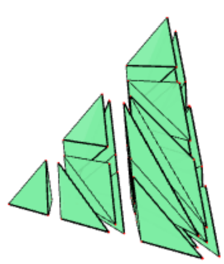
$(p_1, ..., p_n)$ in Mikhalkin position, each point p_i is contained in the relative interior of its own 2-cell of S [90, Remark 3.1]. Therefore, we can encode the positions of these points by their dual edges in the Newton subdivision. Marking these edges in the subdivision leads to a path through $n+1$ of the lattice points in the Newton polytope Δ (see Figure 5.3b). The path will miss δ lattice points in Δ. Due to our special configuration, this path is always connected for cubics [90, Section 3.4]. Moreover, satisfying point conditions in Mikhalkin position implies that the surface is floor decomposed [20].

Definition 5.2.2. A subdivision of $\Delta = \mathrm{conv}((0,0,0),(d,0,0),(0,d,0),(0,0,d))$ is *floor decomposed* if the subdivision is a union of the subdivided polytopes:

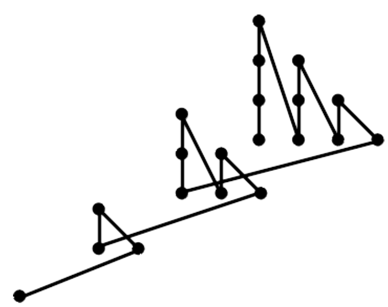
$$\begin{aligned}
&\mathrm{conv}\{(0,0,0),(0,d,0),(0,0,d),(1,0,0),(1,d-1,0),(1,0,d-1)\},\\
&\mathrm{conv}\{(1,0,0),(1,d-1,0),(1,0,d-1),(2,0,0),(2,d-2,0),(2,0,d-2)\},\\
&\vdots\\
&\mathrm{conv}\{(d-1,0,0),(d-1,1,0),(d-1,0,1),(3,0,0)\}.
\end{aligned}$$

These are the *slices*, see Figure 5.3a.

The edges dual to the 2-cells containing the points in Mikhalkin position give rise to a path. This path leads through the triangular faces of the boundary of the slices of Δ and connects each of them by one step, see Figure 5.3b, [90, Section 3.2]. By looking at the triangle faces of the slices independently, we obtain subdivisions of polytopes dual to tropical curves of degrees 1 up to d. These are the *floors* of our floor plans (see Definition 5.2.6).



(a) Floor decomposed dual subdivision of a cubic surface



(b) The lattice path through the points of Δ corresponding to a smooth tropical cubic surface

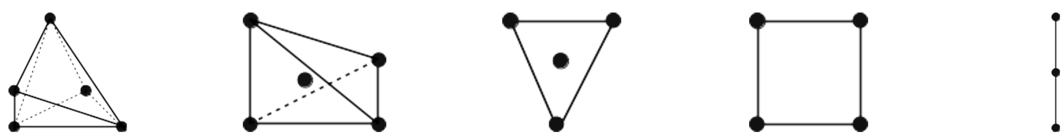
Figure 5.3: Subdivision and lattice path to a smooth tropical cubic surface through points in Mikhalkin position.

For tropical surfaces passing through points in Mikhalkin position this process is reversible. We start with a lattice path through n points in Δ that proceeds through the slices in the prescribed way. From this path we reconstruct the floors of the surface. Then, we extend this to a floor-decomposed subdivision of Δ by the smooth extension algorithm [90, Lemma 3.4], thus giving a tropical surface passing through points in Mikhalkin position.

Tropicalizations of singularities leave a mark in the dual subdivision [91]. By [91] a tropical surface is one-nodal if it contains one of the 5 circuits shown in Figure 5.4. A *circuit* is a set of affinely dependent lattice points such that each proper subset is affinely independent. We now give a proposition that will be used for counting cubic surfaces.

Proposition 5.2.3 ([25]). *In the tropicalization of a nodal cubic surface passing through points in Mikhalkin position only the circuits A, D and E can occur in the dual subdivision (see Figure 5.4).*

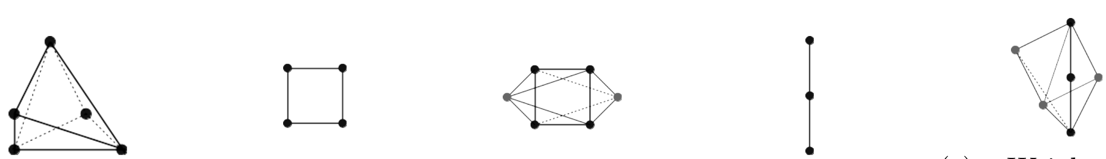
Proof. Since the Newton polytope to a cubic surface does not contain interior lattice points, circuit B is eliminated. The point conditions induce a lattice path in the dual subdivision, which eliminates the possibility of interior points in a triangle, so circuit C cannot occur. □



(a) Circuit A (b) Circuit B (c) Circuit C (d) Circuit D (e) Circuit E

Figure 5.4: Circuits in the dual subdivision inducing nodes in the surface.

The lower dimensional circuits have to satisfy some additional conditions so that their dual cell in the tropical surface contains a node. Circuit A is a pentatope, which is full dimensional. Its dual cell is a vertex and this vertex is the node. To encode a singularity, circuit D must be part of a bipyramid (see Figure 5.5c). The node is the midpoint of the edge dual to the parallelogram. Circuit E must have at least three neighboring points in special positions, forming at least two tetrahedra with the edge (see Figure 5.5e). The weighted barycenter of the 2-cell dual to the edge of length two is the node, where the weight is given by the choice of the three neighbors. We now introduce the definition of a node germ, which is a feature of a tropical curve appearing in a floor plan giving rise to one of these circuits in the subdivision dual to the tropical surface.



(a) Circuit A (b) Circuit D (c) Bipyramid (d) Circuit E (e) Weight two configuration

Figure 5.5: Circuits in the dual subdivision.

Definition 5.2.4 ([92], Definition 5.1)**.** Let C be a plane tropical curve. A *node germ* of C is one of the following:

1. a vertex dual to a parallelogram,
2. a horizontal or diagonal end of weight two,
3. a right or left *string* (see below).

If the lower right (resp. left) vertex of the Newton polytope has no point conditions on the two adjacent ends, we can prolong the adjacent bounded edge in direction $(1,0)$ (resp. $(-1,-1)$) and still pass through the points. The union of the two ends is called a *right* (resp. *left*) *string*. See Figures 5.6a and 5.6b.

(a) Left string (b) Right string

Figure 5.6: Right and left strings.

Definition 5.2.5. A tropical surface has *separated nodes* if the nodes arise from polytopes of the form in Figure 5.5a, 5.5c or 5.5e. Any such two complexes might intersect in a unimodular face.

In [92] tropical floor plans are introduced to count surfaces satisfying point conditions, similar to the concept of floor diagrams used to count tropical curves. Their definition of tropical floor plans requires node germs to be separated by a smooth floor. This neglects surfaces where the nodes are still separated but closer together.

Definition 5.2.6 ([92], Definition 5.2)**.** Let Q_i be the projection of q_i along the x-axis. A *δ-nodal floor plan F of degree d* is a tuple $(C_d, \dots, C_1)$ of plane tropical curves C_i of degree i together with a choice of indices $d \geq i_\delta \geq \cdots \geq i_1 \geq 1$, such that $i_{j+1} > i_j + 1$ for all j, satisfying:

1. The curve C_i passes through the following points (where we set $i_0 = 0$ and $i + \delta + 1 = d + 1$):

$$\begin{aligned} \text{if } i_\nu > i > i_{\nu-1} : \quad & Q_{\sum_{k=i+1}^{d}\binom{k+2}{2}-\delta+\nu}, \dots, Q_{\sum_{k=i}^{d}\binom{k+2}{2}-2-\delta+\nu} \\ \text{if } i = i_\nu : \quad & Q_{\sum_{k=i+1}^{d}\binom{k+2}{2}-\delta+\nu+1}, \dots, Q_{\sum_{k=i}^{d}\binom{k+2}{2}-2-\delta+\nu}. \end{aligned}$$

2. The plane curves C_{i_j} have a node germ for each $j = 1, \ldots, \delta$.

3. If the node germ of C_{i_j} is a left string, then its horizontal end aligns with a horizontal bounded edge of C_{i_j+1}.

4. If the node germ of C_{i_j} is a right string, then its diagonal end aligns either with a diagonal bounded edge of C_{i_j-1} or with a vertex of C_{i_j-1} which is not adjacent to a diagonal edge.

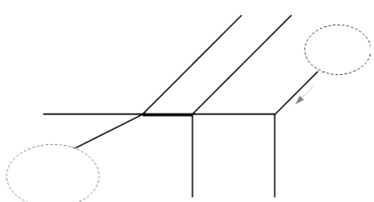
5. If $i_\delta = d$, then the node germ of C_d is either a right string or a diagonal end of weight two.

6. If $i_1 = 1$, then the node germ of C_1 is a left string.

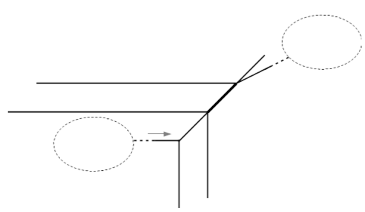
The information contained in a floor plan defines a unique tropical binodal cubic surface [92, Proposition 5.9].

Remark 5.2.7. This definition only allows node germs in floors that are separated by a smooth floor. To count all surfaces with separated singularities, we have to allow node germs in adjacent or the same floors and hence we need to extend this definition to the new cases, that cannot occur in the original setting.

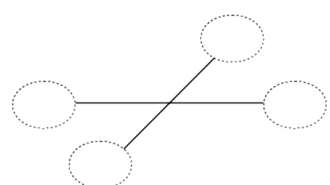
As soon as adjacent floors can contain node germs, a new alignment option for the left string is possible: analogous to the second alignment option for the right string, a left string in C_i can also align with a vertex of C_{i+1} not adjacent to a horizontal edge.



(a) Left string aligning with a horizontal bounded edge



(b) Right string aligning with a diagonal bounded edge



(c) Parallelogram in subdivision dual to floor

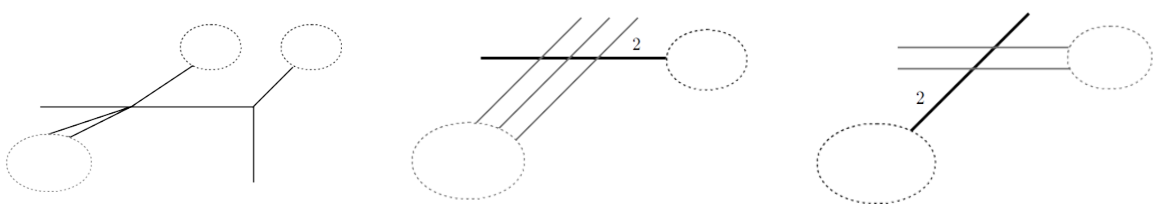
Figure 5.7: Node germs giving a circuit of type D.

We now describe how the node germs from Definition 5.2.4 together with the alignment conditions described in Definition 5.2.6 produce one of the circuits from Figure 5.5 inside the dual subdivision.

Figure 5.7 shows all node germs which lead to a parallelogram in the subdivision of the Newton polytope. If the node germ in a curve is dual to a parallelogram we have a picture as in Figure 5.7c. The right vertex of the floor of higher degree and the left vertex of the floor of lower degree form a bipyramid over the parallelogram as in Figure 5.5c. Figure 5.7a depicts the alignment of the horizontal end of the left string with a bounded horizontal edge of a curve of higher degree. In the floor plan, this translates to the dual vertical edges in the subdivisions forming a parallelogram. Since the string passes through the two vertices bounding the horizontal edge it aligns with, the dual polytope is a bipyramid over the parallelogram. The two top vertices of the pyramids are the vertices forming triangles with the vertical bounded edge in the dual subdivision to the floor of higher degree. Analogously, a right string aligning with a diagonal bounded edge (see Figure 5.7b) produces a bipyramid in the dual subdivision.

Figure 5.8a shows the alignment of a left string with a vertex not adjacent to a horizontal edge. The 5-valent vertex in this figure is dual to a type A circuit, as in Figure 5.5a. The occurring cases in our count are due to node germs in the conic and lead not to a pentatope as in Figure 5.5a, but to different complexes considered in the subsection "Dual Complexes of Unseparated Nodes."

Figures 5.8b and 5.8c show the node germs coming from an undivided edge of length two in the subdivision, as shown in Figure 5.5d. The node is contained in the dual 2-cell of the length two edge. Every intersection point of the weight two diagonal (resp. horizontal) end with the lower (resp. higher) degree curve of the floor plan can be selected to lift the node [90]. In the dual subdivision this corresponds to choosing three neighboring vertices which could form the polyhedral complex shown in Figure 5.5e. With our chosen point condition the neighboring vertex in the dual subdivision of the floor containing the undivided edge is always one of the three neighboring vertices. If the length two edge is diagonal (resp. vertical) the other two vertices form a vertical (resp. diagonal) length one edge in the boundary of the subdivision dual to the lower (resp. higher) degree curve of the floor plan.



(a) Intersection dual to a pentatope

(b) Horizontal end of weight two

(c) Diagonal end of weight two

Figure 5.8: Node germs leading to circuits of type A and type E.

The complex lifting multiplicity of the node germs in the floors can be determined combinatorially using [90]. Here, we only list the multiplicities for node germs which can occur in a degree 3 surface.

Definition 5.2.8 (Definition 5.4, [92])**.** Let F be a δ-nodal floor plan of degree d. For each node germ $C^*_{i_j}$ in C_{i_j}, we define the following local complex multiplicity $\mathrm{mult}_{\mathbb{C}}(C^*_{i_j})$:

1. If $C^*_{i_j}$ is dual to a parallelogram, then $\mathrm{mult}_{\mathbb{C}}(C^*_{i_j}) = 2$.
2. If $C^*_{i_j}$ is a horizontal end of weight two, then $\mathrm{mult}_{\mathbb{C}}(C^*_{i_j}) = 2(i_j + 1)$.

3. If $C^*_{i_j}$ is a diagonal end of weight two, then $\text{mult}_{\mathbb{C}}(C^*_{i_j}) = 2(i_j - 1)$.

4. If $C^*_{i_j}$ is a left string, then $\text{mult}_{\mathbb{C}}(C^*_{i_j}) = 2$.

5. If $C^*_{i_j}$ is a right string whose diagonal end aligns with a diagonal bounded edge, then $\text{mult}_{\mathbb{C}}(C^*_{i_j}) = 2$.

6. If $C^*_{i_j}$ is a right string whose diagonal end aligns with a vertex not adjacent to a diagonal edge, then $\text{mult}_{\mathbb{C}}(C^*_{i_j}) = 1$.

The multiplicity of a δ-nodal floor plan is $\text{mult}_{\mathbb{C}}(F) = \prod_{j=1}^{d} \text{mult}_{\mathbb{C}}(C^*_{i_j})$.

To determine the real multiplicity, we have to fix the signs of the coordinates of the points in ω, as they determine the existence of real solutions of the initial equations in [90]. The dependence on the signs of the coordinates of the points is shown by including s in the notation $\text{mult}_{\mathbb{R},s}$ for the real multiplicity. Here we only consider points where every coordinate is positive. In the following definition, we only list the multiplicities for node germs which can occur in a degree 3 surface.

Definition 5.2.9 ([92], Definition 5.10)**.** For a node germ $C^*_{i_j}$ in C_{i_j}, we define the local real multiplicity $\text{mult}_{\mathbb{R},s}(C^*_{i_j})$:

1. If $C^*_{i_j}$ is dual to a parallelogram, it depends on the position of the parallelogram in the Newton subdivision:

   - if the vertices are $(k, 0)$, $(k, 1)$, $(k-1, l)$ and $(k-1, l+1)$, then

$$\text{mult}_{\mathbb{R},s}(C^*_{i_j}) = \begin{cases} 2 \\ 0 \end{cases} \text{if } (\frac{3}{2}i_j + 1 + k + l)(i_j - 1) \equiv \begin{cases} 1 \\ 0 \end{cases} \text{modulo } 2.$$

   - if the vertices are $(k, 3 - i_j - k)$, $(k, 3 - i_j - k - 1)$, $(k+1, l)$ and $(k+1, l+1)$, then

$$\text{mult}_{\mathbb{R},s}(C^*_{i_j}) = \begin{cases} 2 \\ 0 \end{cases} \text{if } \frac{1}{2} \cdot (i_j + 2 + 2l)(i_j - 1) \equiv \begin{cases} 1 \\ 0 \end{cases} \text{modulo } 2.$$

2. If $C^*_{i_j}$ is a diagonal edge of weight two, $\text{mult}_{\mathbb{R},s}(C^*_{i_j}) = 2(i_j - 1)$.

3. If $C^*_{i_j}$ is a left string, then it depends on the position of the dual of the horizontal bounded edge of C_{i_j+1} with which it aligns. Assume it has the vertices (k, l) and $(k, l+1)$. Then

$$\text{mult}_{\mathbb{R},s}(C^*_{i_j}) = \begin{cases} 2 \\ 0 \end{cases} \quad \text{if } i_j - k \equiv \begin{cases} 0 \\ 1 \end{cases} \quad \text{modulo } 2.$$

4. If $C^*_{i_j}$ is a right string whose diagonal end aligns with a vertex not adjacent to a diagonal edge, then $\text{mult}_{\mathbb{R},s}(C^*_{i_j}) = 1$.

A tropical δ-nodal surface S of degree d given by a δ-nodal floor plan F has at least $\text{mult}_{\mathbb{R},s}(F) = \prod_{j=1}^{\delta} \text{mult}_{\mathbb{R},s}(C^*_{i_j})$ real lifts with all positive coordinates satisfying the point conditions [92, Proposition 5.12]. Several cases are left out of the above definition because the number of real solutions is hard to control. We address this in the subsection "Undetermined Real Multiplicities." This is why we can only give a lower bound of real binodal cubic surfaces where the tropicalization contains separated nodes.

We now count surfaces from the floor plans defined in [92, Definition 5.2], which have node germs in the linear and cubic floors. Since we adhere exactly to Definition 5.2.6 the nodes will always be separated.

Proposition 5.2.10 ([25])**.** *There are* 20 *cubic surfaces containing two nodes such that there is one node germ in the cubic floor and one in the linear floor. Of these binodal surfaces at least* 16 *are real.*

Proof. By Definition 5.2.6 a floor plan consists of a cubic curve C_3, a conic C_2, and a line C_1, where the tropical curves C_3 and C_1 contain node germs. Recall that the notation C^*_i stands for the node germ in C_i. By Definition 5.2.6 (6) the node germ of C_1 is a left string as in Figure 5.10a, which always aligns with the horizontal bounded edge in C_2, so $\text{mult}_{\mathbb{C}}(C^*_1) = 2$. The node germs in C_3 possible by Definition 5.2.6 (5) are depicted in Figures 5.9b-5.9d and each one gives a different floor plan.

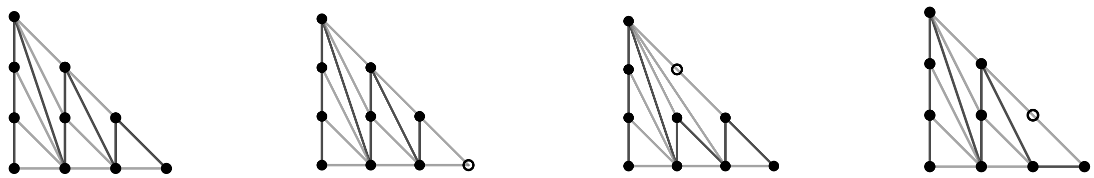
- (5.9b) There is a right string in the cubic floor. In the smooth conic, there is no vertex which is not adjacent to a diagonal edge. So, the right string of the cubic must align with the diagonal bounded edge. This gives $\text{mult}_{\mathbb{C}}(F) = \text{mult}_{\mathbb{C}}(C^*_3) \cdot \text{mult}_{\mathbb{C}}(C^*_1) = 2 \cdot 2 = 4$. In this case, $\text{mult}_{\mathbb{R},s}(F)$ is undetermined, see the subsection "Undetermined Real Multiplicities."

- (5.9c, 5.9d) The cubic has a weight two diagonal end. We have $2 \cdot \text{mult}_{\mathbb{C}}(F) = 2 \cdot \text{mult}_{\mathbb{C}}(C^*_3) \cdot \text{mult}_{\mathbb{C}}(C^*_1) = 2 \cdot (2(3-1) \cdot 2) = 16$. By Definition 5.2.9 (3) the real multiplicity of the left string depends on coordinates of the dual of the edge it aligns with: $(1, 0)$ and $(1, 1)$. This gives $2 \cdot \text{mult}_{\mathbb{R},s}(F) = 2 \cdot \text{mult}_{\mathbb{R},s}(C^*_3) \cdot \text{mult}_{\mathbb{R},s}(C^*_1) = 2 \cdot (2(3-1) \cdot 2) = 16$.

□

A Tropical Count of Binodal Cubic Surfaces

We now compute tropical counts of binodal cubic surfaces over $\mathbb{C}$ and $\mathbb{R}$. The space $\mathbb{P}^{19}$ parameterizes cubic surfaces by the coefficients of their defining polynomial. The singular cubic surfaces form a hypersurface of degree 32 called the *discriminant* in $\mathbb{P}^{19}$. The surfaces



(a) (b) (c) (d)

Figure 5.9: The triangulation dual to a smooth cubic floor and the three possible subdivisions dual to a tropical cubic curve with one node germ.

(a) Left string in C_1

(b) A triangulation dual to a smooth conic

Figure 5.10: The triangulations dual to linear and conic curves appearing as part of the floor plans to Proposition 5.2.10.

passing through a particular point in $\mathbb{P}^3$ are a hyperplane in $\mathbb{P}^{19}$. So, through 18 generic points there are 32 nodal surfaces. The reducible singular locus of the discriminant is the union of the two codimension 2 varieties in $\mathbb{P}^{19}$ given by cuspidal cubic surfaces and binodal cubic surfaces.

In [121, Section 7.1] Vainsencher gives formulas for the number of k-nodal degree m surfaces in a k-dimensional family in $\mathbb{P}^3$. That is, for $k = 2$ and $m = 3$ he determines the degree of the variety parameterizing 2-nodal cubics. For $k = 2$ nodes, there are $2(m-2)(4m^3 - 8m^2 + 8m - 25)(m-1)^2$ such surfaces. Setting $m = 3$, we have the following count.

Theorem 5.2.11 ([121, Section 7.1])**.** *There are 280 binodal complex cubic surfaces passing through 17 general points.*

In our setting, we ask:

Question 5.2.12 (Question 10 [108])**.** Can the number 280 of binodal cubic surfaces through 17 general points be derived tropically?

For points in Mikhalkin position, as introduced in Definition 5.2.1, tropical methods are useful because the dual subdivisions of the Newton polytope are very structured. This allows us to study only 39 subdivisions of $\Delta = \text{Conv}\{(0,0,0), (3,0,0), (0,3,0), (0,0,3)\}$, the Newton polytope of a cubic surface. This is minuscule compared to the 344,843,867 unimodular triangulations of this polytope [74, 75].

If we count with multiplicities all tropical binodal cubic surfaces through our points, we will recover the true count. We study tropical surfaces with *separated* nodes, in the sense that the topological closures of the cells in the tropical surface containing the nodes have empty intersection. To count them, we list the dual subdivisions of candidate binodal tropical cubic surfaces and count their multiplicities.

Theorem 5.2.13 ([25])**.** *There are* 39 *tropical binodal cubic surfaces through* 17 *points in Mikhalkin position (see Definition 5.2.1) containing separated nodes. They give rise to* 214 *complex binodal cubic surfaces through* 17 *points.*

Proof. We distinguish five cases based on which *floors* (see Definition 5.2.6) of the tropical cubic surface contain the nodes and count with complex multiplicities (see Definition 5.2.8).

$$214 = \underbrace{20}_{\text{Proposition 5.2.10}} + \underbrace{24}_{\text{Proposition 5.2.18}} + \underbrace{90}_{\text{Proposition 5.2.19}} + \underbrace{72}_{\text{Proposition 5.2.22}} + \underbrace{8}_{\text{Proposition 5.2.23}} .$$

□

Theorem 5.2.14 ([25])**.** *There exists a point configuration* ω *of* 17 *real points in* $\mathbb{P}^3$ *all with positive coordinates, such that there are at least* 58 *real binodal cubic surfaces through* ω*.*

Proof. We count the floor plans in Theorem 5.2.13 with real multiplicities. Since the real multiplicities are difficult to determine in some cases, the propositions only give us lower bounds. We obtain that there are at least 58 real binodal cubic surfaces passing through ω.

$$58 = \underbrace{16}_{\text{Proposition 5.2.10}} + \underbrace{4}_{\text{Proposition 5.2.18}} + \underbrace{34}_{\text{Proposition 5.2.19}} + \underbrace{4}_{\text{Proposition 5.2.22}} + \underbrace{0}_{\text{Proposition 5.2.23}} .$$

□

As we conduct the counts in Theorems 5.2.13 and 5.2.14, we encounter cases with *unseparated* nodes. Here, the two node germs (see Definition 5.2.4) are close together, and so the cells that would normally contain the nodes interact and their topological closures intersect. Thus, the node germs interfere with the conditions on producing nodes [90]. These cases account for the 66 surfaces missing from our count. Their dual subdivisions contain unclassified polytopes, which we list in the subsection "Dual Complexes of Unseparated Nodes."

Nodes in adjacent floors

We now extend Definition 5.2.6 to cases where node germs are in adjacent floors of the floor plan. Then, we check that the resulting nodes are separated.

Lemma 5.2.15 ([25])**.** *If a floor plan of a degree d surface in* $\mathbb{P}^3$ *contains a diagonal or horizontal end of weight two and a second node germ leading to a bipyramid in the subdivision, such that the bipyramid does not contain the weight two end, the nodes are separated.*

Proof. The bipyramid and the weight two end share at maximum one vertex. The neighboring points of the weight two end can be part of the bipyramid. This causes no obstructions to the conditions in [90] for the existence of a binodal surface tropicalizing to this. □

Lemma 5.2.16 ([25])**.** *If a floor plan of a degree d surface in $\mathbb{P}^3$ has separated nodes, C_2 cannot have a right string.*

Proof. By Definition 5.2.6 (4) a right string in C_2 would have to align with a diagonal bounded edge of C_1 or with a vertex of C_1 not adjacent to a diagonal edge. Since C_1 is a tropical line, both cases can never occur. □

We now give the lemma used to eliminate cases with polyhedral complexes in the Newton subdivision that cannot accommodate two nodes. We use obstructions arising from dimensional arguments, which are independent of the choice of generic points.

Lemma 5.2.17 ([25])**.** *Let $\Gamma \subset \mathbb{Z}^3$ be finite, and let B_Γ be the variety of binodal hypersurfaces with defining polynomial having support Γ. If the dimension of B_Γ is less than $|\Gamma| - 3$, then any tropical surface whose dual subdivision consists of unimodular tetrahedra away from* $\mathrm{Conv}(\Gamma)$ *is not the tropicalization of a complex binodal cubic surface.*

Proof. If a binodal cubic surface had such a triangulation and satisfied our point conditions, then we could obtain from it a binodal surface with support Γ satisfying $|\Gamma| - 3$ point conditions. However, if the dimension of B_Γ is less than $|\Gamma| - 3$ we do not expect any such surfaces to satisfy $|\Gamma| - 3$ generic point conditions. □

Typically the dimension of B_Γ is the expected dimension $|\Gamma| - 3$. For some special point configurations Γ the dimension is less than this, and these are the cases we want to eliminate.

To apply the lemma, suppose $\mathrm{conv}(\Gamma)$ is a subcomplex of the subdivision of Δ. If apart from $\mathrm{conv}(\Gamma)$ the subdivision of Δ only contains unimodular simplices, cutting Δ down to $\mathrm{conv}(\Gamma)$ corresponds to removing the lattice points of $\Delta \setminus \mathrm{conv}(\Gamma)$, loosing one point condition each. Thus, if $\mathrm{conv}(\Gamma)$ cannot accommodate 2 nodes, neither can Δ.

Proposition 5.2.18 ([25])**.** *There are 24 cubic surfaces containing two nodes such that the tropical cubic has two separated nodes and the corresponding node germs are contained in the conic and linear floors. Of these, at least 4 are real.*

Proof. Here a floor plan consists of a smooth cubic curve C_3 (see Figure 5.9a), a conic C_2 and a line C_1, both with a node germ. The node germ of C_1 is by Definition 5.2.6 (6) a left string, see Figure 5.10a. For C_2 all possibilities from Definition 5.2.4 are depicted in Figure 5.11. We examine all choices for the floor plan F and check whether the nodes are separated.

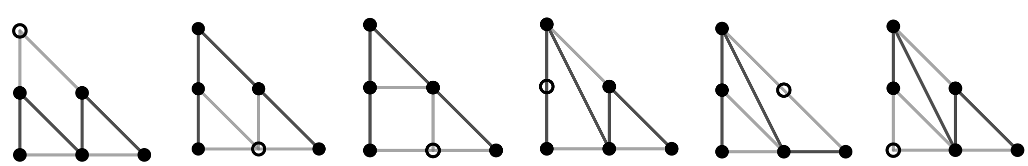
(5.11a)-(5.11c) By Definition 5.2.6 (3) the left string in C_1 must align with the horizontal bounded edge of C_2, which is dual to a face of the parallelogram in the subdivision. We obtain a prism polytope between the two floors, and by completion of the subdivision, we get two pyramids sitting over those two rectangle faces of the prism, that are not on the boundary of the Newton polytope. This complex may hold two nodes, see see the subsection "Dual Complexes of Unseparated Nodes."

(5.11d) By Definition 5.2.6 (3), the left string of C_1 aligns with the horizontal bounded edge of C_2, giving a bipyramid in the subdivision, with top vertices the neighbors to the dual of the bounded diagonal edge in C_2. The length two edge dual to the horizontal end of weight two is surrounded by tetrahedra that only intersect the bipyramid in a unimodular face. So, the nodes are separated and we count their multiplicities: $\text{mult}_{\mathbb{C}}(F) = \text{mult}_{\mathbb{C}}(C_1^*) \cdot \text{mult}_{\mathbb{C}}(C_2^*) = 2 \cdot 2(2+1) = 12$. In this case, $\text{mult}_{\mathbb{R},s}(F)$ is undetermined, see the subsection "Undetermined Real Multiplicities."

(5.11e) The left string in C_1 must align with the vertex in C_2 not adjacent to a horizontal edge, but this vertex is dual to the area two triangle in the subdivision. The resulting volume two pentatope contains the neighbors of the length two edge. This configuration is eliminated using Lemma 5.2.17.

(5.11f) The left strings in C_1 and C_2 lead to two bipyramids in the subdivision. For each of the 3 alignment possibilities of the left string in C_2, the resulting bipyramids are disjoint and the nodes separate. We get $3 \cdot \text{mult}_{\mathbb{C}}(F) = 3 \cdot \text{mult}_{\mathbb{C}}(C_1^*) \cdot \text{mult}_{\mathbb{C}}(C_2^*) = 3 \cdot (2 \cdot 2) = 12$. By Definition 5.2.9 (3) we need to consider the positions of the dual edges the left strings align with in order to compute the real multiplicities. The left string in C_1 aligns with the edge given by the vertices $(1,0), (1,1)$ in the conic floor, it has $\text{mult}_{\mathbb{R},s}(C_1^*) = 2$. For the conic, two of the three choices have x-coordinate $k = 1$ in the cubic floor, so $\text{mult}_{\mathbb{R},s}(C_2^*) = 0$. The last alignment is dual to x-coordinate $k = 2$, so we have $\text{mult}_{\mathbb{R},s}(C_2^*) = 2$. We obtain $\text{mult}_{\mathbb{R},s}(F) = 4$.

□



(a) (b) (c) (d) (e) (f)

Figure 5.11: The possible subdivisions dual to a tropical conic curve with one node germ appearing as part of a floor plan of a nodal cubic surface.

Proposition 5.2.19 ([25]). *There are* 90 *cubic surfaces containing two nodes such that the tropical binodal cubic has separated nodes and the node germs are contained in the cubic and conic floors. Of these, at least* 34 *are real.*

Proof. A floor plan consists of a cubic C_3 with a node germ (Figures 5.9b-5.9d), a conic C_2 with a node germ (Figure 5.11), and a smooth line C_1. There are 18 combinations.

(5.9b, 5.11a-5.11b) The cubic contains a right string, which must align with a diagonal bounded edge by Definition 5.2.6 (4). The resulting subdivision contains a triangular prism with two pyramids. This complex may contain two nodes, see see the subsection "Dual Complexes of Unseparated Nodes."

(5.9b, 5.11c) The right string in the cubic must align with the vertex of the conic dual to the square in the subdivision, giving rise to the polytopes shown in the subsection "Dual Complexes of Unseparated Nodes."

(5.9b, 5.11d) The right string in the cubic aligns with the vertex dual to the left triangle in the conic containing the weight two edge. The resulting complex is in the subsection "Dual Complexes of Unseparated Nodes."

(5.9b, 5.11e) The subdivision has a bipyramid and a weight two configuration only overlapping in vertices, so the nodes are separated. We have $\mathrm{mult}_{\mathbb{C}}(F) = \mathrm{mult}_{\mathbb{C}}(C_3^*) \cdot \mathrm{mult}_{\mathbb{C}}(C_2^*) = 2 \cdot 2(2-1) = 4$. In this case, $\mathrm{mult}_{\mathbb{R},s}(F)$ is undetermined, see the subsection "Undetermined Real Multiplicities."

(5.9b, 5.11f) The left string in C_2 has to align with a horizontal bounded edge of C_3 by Definition 5.2.6 (4). There are 3 possibilities. If it aligns with the bounded edge adjacent to the right string in the cubic, we obtain a prism with two pyramids as in (5.9b, 5.11a). See the subsection "Dual Complexes of Unseparated Nodes." If it aligns with either of the other two horizontal bounded edges, we obtain two bipyramids in the dual subdivision. Because the diagonal bounded edge of C_2 is part of the left sting aligning with a horizontal bounded end not adjacent to the right string of C_3, we cannot align the right string with the diagonal edge, such that the end of the right string contains the whole horizontal bounded edge of C_2. Instead the end meets the bounded edge somewhere in the middle and passes only through one vertex. Therefore, in the subdivision the second pyramid over the alignment parallelogram must have its vertex in C_3 instead of in the C_2, see Figure 5.12. In total, we get two bipyramids that only share an edge, so the node germs are separated. We have $2 \cdot \mathrm{mult}_{\mathbb{C}}(F) = 2 \cdot \mathrm{mult}_{\mathbb{C}}(C_3^*) \cdot \mathrm{mult}_{\mathbb{C}}(C_2^*) = 2 \cdot (2 \cdot 2) = 8$. In these two cases, the edge the string aligns with has x-coordinate $k = 1$ in the cubic floor and thus by Definition 5.2.9 they both give $\mathrm{mult}_{\mathbb{R},s}(F) = 0$.

(5.9c, 5.11a-5.11b) We obtain a bipyramid only overlapping with the configuration of the weight two end in vertices or edges. So the nodes are separated and $2 \cdot \mathrm{mult}_{\mathbb{C}}(F) = 2 \cdot \mathrm{mult}_{\mathbb{C}}(C_3^*) \cdot \mathrm{mult}_{\mathbb{C}}(C_2^*) = 2(2(3-1) \cdot 2) = 2 \cdot 8$. The parallelogram has vertices as in the first case of Definition 5.2.9 (1) with $k = 1, l = 1$ and $i_j = 2$, so $\mathrm{mult}_{\mathbb{R},s}(F) = 0$.

(5.9c, 5.11c) As in (5.9c, 5.11a) we have $\mathrm{mult}_{\mathbb{C}}(F) = 8$. For the real multiplicity we need the vertices of the parallelogram. They are as in the first case of Definition 5.2.9 (1) with $k = 1, l = 0$ and $i_j = 2$, so $\mathrm{mult}_{\mathbb{R},s}(C_2^*) = 2$. The weight 2 end in C_3 has $\mathrm{mult}_{\mathbb{R},s}(C_3^*) = 4$, so $\mathrm{mult}_{\mathbb{R},s}(F) = 8$.

(5.9c-5.9d, 5.11d) This subdivision contains a tetrahedron which is the convex hull of both weight two ends. We need a choice of the neighboring points of the two weight two edges. By their special position to each other, it only remains to add the two vertices neighboring the edges in the respective subdivisions dual to their floors. Whether it can contain 2 nodes is undetermined, see the subsection "Dual Complexes of Unseparated Nodes."

(5.9c-5.9d, 5.11e) The nodes are separated, since the weight two ends with any choice of their neighboring points intersect in one vertex. So $2 \cdot \mathrm{mult}_{\mathbb{C}}(F) = 2 \cdot \mathrm{mult}_{\mathbb{C}}(C_3^*) \cdot \mathrm{mult}_{\mathbb{C}}(C_2^*) = 2 \cdot (2(3-1) \cdot 2(2-1)) = 2 \cdot 8$ and $2 \cdot \mathrm{mult}_{\mathbb{R},s}(F) = 2 \cdot \mathrm{mult}_{\mathbb{R},s}(C_3^*) \cdot \mathrm{mult}_{\mathbb{R},s}(C_2^*) = 2 \cdot (2(3-1) \cdot 2(2-1)) = 2 \cdot 8$.

(5.9c, 5.11f) There are two possibilities to align the left string in C_2 with a horizontal bounded edge in C_3. If we select the left edge, we have a bipyramid, which does not contain the weight two end. By Lemma 5.2.15 the nodes are separate. However, we need to adjust the multiplicity formula from Definition 5.2.8 (3) to this case, because due to the alignment of the left string we obtain one intersection point less of the diagonal end of weight two with C_2. So instead of $3 - 1 = 2$ intersection points to chose from when lifting the node we have $3 - 2 = 1$. Thus, we obtain $\mathrm{mult}_{\mathbb{C}}(F) = \mathrm{mult}_{\mathbb{C}}(C_3^*) \cdot \mathrm{mult}_{\mathbb{C}}(C_2^*) = 2(3-2) \cdot 2 = 4$. Since the left edge has x-coordinate $k = 1$, we obtain $\mathrm{mult}_{\mathbb{R},s}(F) = 0$. If we select the right edge, then the bipyramid contains the weight two end. See the subsection "Dual Complexes of Unseparated Nodes." As the cubic floor contains a vertex of C_3 not adjacent to a horizontal edge, it is also possible to align the left string with this. In the dual subdivision this gives rise to a pentatope spanned by the triangle dual to the vertex in C_3 and the vertical edge in the conic floor dual to the horizontal end of the left string, see Figure 5.5a. The nodes dual to the length two edge and the pentatope are separated. By [90] we have $\mathrm{mult}_{\mathbb{C}}(C_2^*) = \mathrm{mult}_{\mathbb{R},s}(C_2^*) = 1$.

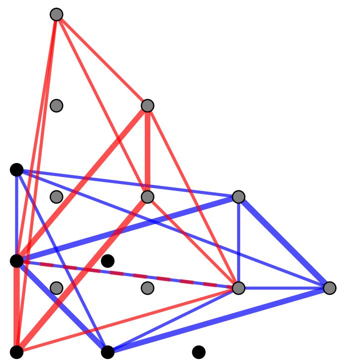


Figure 5.12: The two bipyramids for one alignment of (5.9b, 5.11f). The gray (resp. black) dots are the lattice points of the dual polytope to C_3 (resp. C_2). The shared edge of the bipyramids is marked blue and red.

We count: $\mathrm{mult}_{\mathbb{C}}(F) = \mathrm{mult}_{\mathbb{C}}(C_3^*) \cdot \mathrm{mult}_{\mathbb{C}}(C_2^*) = 2(3-2) \cdot 1 = 2$ and $\mathrm{mult}_{\mathbb{R},s}(F) = \mathrm{mult}_{\mathbb{R},s}(C_3^*) \cdot \mathrm{mult}_{\mathbb{R},s}(C_2^*) = 2(3-2) \cdot 1 = 2$.

(5.9d, 5.11a-5.11b) We obtain a bipyramid overlapping with the weight two configuration in one or two vertices, so the nodes are separated and $2 \cdot \mathrm{mult}_{\mathbb{C}}(F) = 2 \cdot \mathrm{mult}_{\mathbb{C}}(C_3^*) \cdot \mathrm{mult}_{\mathbb{C}}(C_2^*) = 2(2(3-1) \cdot 2(2-1)) = 2 \cdot 8$. With the parallelogram as in $(5.9c, 5.11a)$: $\mathrm{mult}_{\mathbb{R},s}(F) = 0$.

(5.9d, 5.11c) This follows (5.9d, 5.11a), and we have $\mathrm{mult}_{\mathbb{C}}(F) = 8$. The real multiplicity follows (5.9c, 5.11c), and we have $\mathrm{mult}_{\mathbb{R},s}(F) = 8$.

(5.9d, 5.11f) For each of the two choices for the alignment of the left string of the conic with a horizontal bounded edge of the cubic, we obtain a bipyramid which may share two vertices with the neighbors of the edge of weight two. As in (5.9c, 5.11f) we need to adjust the multiplicity formula for the weight two end to $\mathrm{mult}_{\mathbb{C}}(C_3^*) = 2(3-2) = 2$. We have $2 \cdot \mathrm{mult}_{\mathbb{C}}(F) = 2 \cdot 4$. For both alignments the dual edges have x-coordinate $k = 1$ in the cubic floor, giving $\mathrm{mult}_{\mathbb{R},s}(F) = 0$. As C_3 also contains a vertex not adjacent to a horizontal edge, this opens a third alignment possibility. However, this vertex is adjacent to the weight two, so the nodes are not separated. The polytope can be seen in Figure 5.16.

□

Nodes in the same floor

We now examine cases where both node germs are in the same floor of the floor plan. By Lemma 5.2.16 we cannot have a right string in the conic part of the floor plan, if the nodes are separated. A few more cases, depicted in Figure 5.13, can be eliminated with the following Lemma 5.2.20.

Lemma 5.2.20 ([25])**.** *The ways of omitting 2 points in the floor path in the conic floor shown in Figure 5.13 do not give separated nodes.*

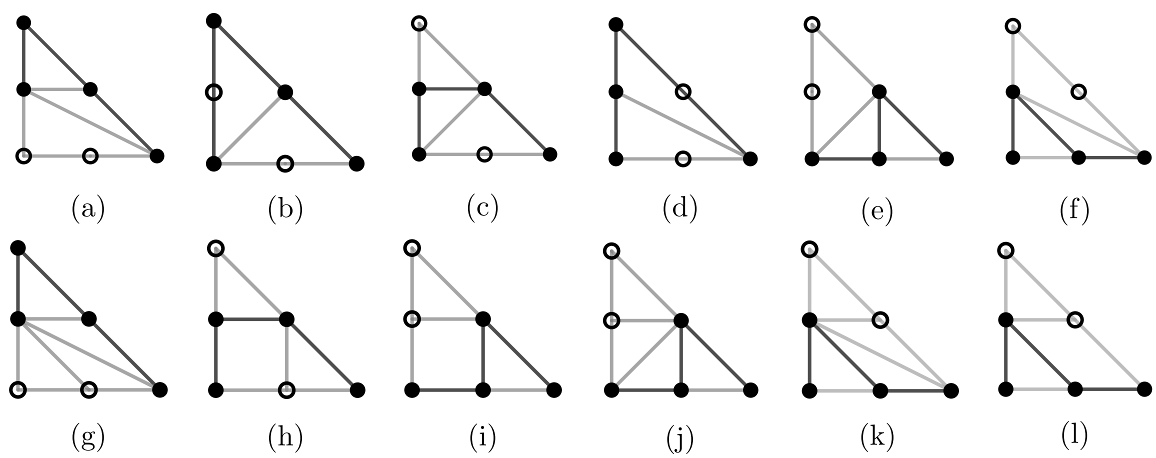


Figure 5.13: Conics through 3 points eliminated by Lemma 5.2.20.

Proof. If the conic in a floor plan has two node germs, it passes only through 3 points of the point configuration. In order to fix our cubic surface, every point we omit in the lattice path of the conic floor needs to compensate for the omitted point condition on our cubic surface.

A vertical weight two end does allow our conic to be fixed by fewer points. But our point configuration ensures the end has no interaction with the other floors and thus cannot give rise to a node-encoding circuit as in Figure 5.5. So, combined with a classical node germ this does not encode two separated nodes, dealing with 5.13a, 5.13b, 5.13c and 5.13d.

If the top vertex of the Newton polytope of C_2 is omitted in the floor path, we always obtain an upwards string. If the upwards string is to be pulled vertically upwards, it can never be aligned with any part of the other floors, thus not fixing the curve, eliminating 5.13h, 5.13j and 5.13l.

If the direction to pull the upwards string has some slope, as in 5.13e and 5.13f, or in the 2-dimensional strings in 5.13i and 5.13k, we still cannot align with any bounded edges of the other cubic, since we are above the line through the points due to our chosen point configuration. In 5.13g on the other hand we can align the vertical end of the string, but since we have two degrees of freedom this does not fix the curve, as we can still move the first vertical end. □

Remark 5.2.21. The last issue in the proof of Lemma 5.2.20 can be fixed, if we allow alignments with ends. These however do not give rise to separated nodes [90]. Therefore the cases 5.13a, 5.13e, 5.13f, 5.13g, 5.13i and 5.13k require further investigation, see the subsection "Dual Complexes of Unseparated Nodes." In this light the non-existence of right strings in the conic floor needs to be investigated.

Proposition 5.2.22 ([25])**.** *There are* 72 *cubic surfaces containing two nodes, such that the tropical binodal conic has separated nodes and the corresponding node germs are both contained in the conic floor. Of these, at least* 4 *are real.*

Proof. See Figure 5.14.

(5.14a) Since the end of the left string, which aligns with a bounded horizontal edge of the conic, is of weight two, we obtain a bipyramid over a trapezoid. We get two different complexes depending upon the alignment, see the subsection "Dual Complexes of Unseparated Nodes."

(5.14b) We have a string with two degrees of freedom, because we can pull on both horizontal ends and vary their distance. Hence, we can align them both with the horizontal bounded edges of the cubic. There are three ways to do this. In the dual subdivisions this gives rise to two bipyramids. In all three cases they intersect maximally in two 2-dimensional unimodular faces, and thus are separated. Since the bipyramids arise not from classical node germs, we check their multiplicities via the underlying circuit. By [90, Lemma 4.8] we obtain multiplicity 2 for each, and thus $3 \cdot \mathrm{mult}_{\mathbb{C}}(F) = 3 \cdot \mathrm{mult}_{\mathbb{C}}(C_2^*) = 3 \cdot 2 \cdot 2 = 12$. We get $\mathrm{mult}_{\mathbb{R},s}(F) = 0$, since one end has to align with a bounded edge in C_3 with dual edge of x-coordinate $k = 1$.

(5.14c-5.14d) The conic floor has a left string and a parallelogram. This gives two bipyramids in the subdivision which, depending on the choice of alignment for the left string, have a maximal intersection of an edge. We obtain $2 \cdot (3 \cdot \mathrm{mult}_{\mathbb{C}}(F)) = 2 \cdot 12$. The vertex positions of the parallelogram give $\mathrm{mult}_{\mathbb{R},s}(F) = 0$ as in Proposition 5.2.18 (5.11a).

(5.14e) As in (5.14c), we obtain $3 \cdot \mathrm{mult}_{\mathbb{C}}(F) = 12$. The formulas for real multiplicities in Definition 5.2.9 do not match this case, see the subsection "Undetermined Real Multiplicities."

(5.14f) The bipyramids arising from the different alignment options only intersect with the neighboring points of the weight two end in one vertex, so $3 \cdot \mathrm{mult}_{\mathbb{C}}(F) = 12$. Only the alignment with the horizontal bounded edge of C_3 dual to the vertical edge of x-coordinate $k = 2$ has non-zero real multiplicity, giving $\mathrm{mult}_{\mathbb{R},s}(F) = 4$.

(5.14g) The two sets of neighboring points to the two weight two ends intersect in one vertex. So the nodes are separated and $\mathrm{mult}_{\mathbb{C}}(F) = 6 \cdot 2 = 12$, while $\mathrm{mult}_{\mathbb{R},s}(F)$ is undetermined, see the subsection “Undetermined Real Multiplicities.”

□

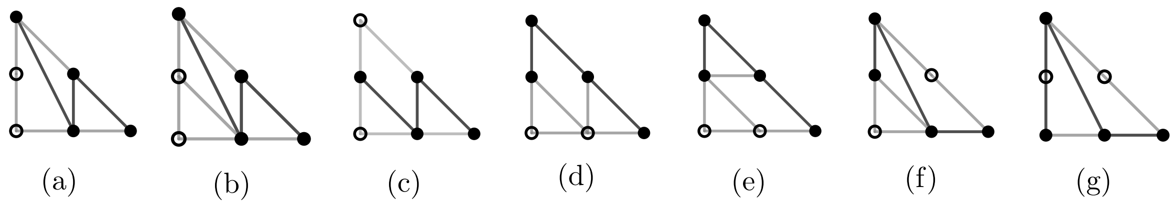


Figure 5.14: Dual subdivisions of conics with two node germs.

Proposition 5.2.23 ([25])**.** *There are* 8 *cubic surfaces containing two nodes, such that the tropical binodal cubic surface has separated nodes and the corresponding node germs are both contained in the cubic floor.*

So far the number of real surfaces is undetermined.

Proof. Only two types of node germs may occur in C_3, see Figure 5.15.

(5.15a) Since the weight two end is not contained in the bipyramid the two nodes are separated by Lemma 5.2.15, giving $\mathrm{mult}_{\mathbb{C}}(F) = 2 \cdot 4 = 8$. In this case, $\mathrm{mult}_{\mathbb{R},s}(F)$ is undetermined, see the subsection “Undetermined Real Multiplicities.”

(5.15b) The classical alignment condition of the right string with diagonal end of weight two can not be satisfied, since the direction vector of the variable edge has a too high slope. Due to the point conditions the diagonal end of weight two and the diagonal bounded edge of the conic curve never meet.

(5.15c) Here we have a two-dimensional string. By the same argument as in (5.15b) we cannot align the middle diagonal end with the diagonal bounded edge of the conic. Aligning the right string with the diagonal bounded edge of the conic does not fixate our floor plan, since we can still move the middle diagonal end of the cubic.

(5.15d) We have three tetrahedra in the subdivision with the weight three edge. This could contain two nodes, see the subsection “Dual Complexes of Unseparated Nodes.”

In (5.15b), (5.15c) alignments with ends are an option, see the subsection “Dual Complexes of Unseparated Nodes.” □

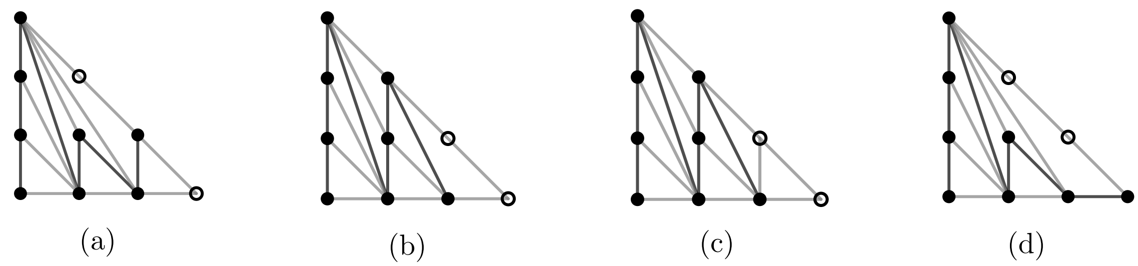


Figure 5.15: Cubics with two node germs.

Undetermined Real Multiplicities

In the previous sections, we encountered cases in which the real multiplicity was undefined. This happens when $C^*_{i_j}$ is a horizontal edge of weight two ((5.11d) and (5.14g)), and $C^*_{i_j}$ is a right string whose diagonal end aligns with a diagonal bounded edge ((5.9b), (5.9b, 5.11e), (5.14e), and (5.15a)). There might be real lifts satisfying the point conditions coming from floor plans containing these node germs, but the number of real solutions is hard to control. An investigation of these cases is beyond the scope of this paper, so we leave Theorem 5.2.14 as a lower bound under these assumptions.

We may compute the real multiplicity of (5.14e), as well as of right strings aligning with diagonal bounded edges as follows. Shift the parallelogram to a special position used to prove [90, Lemma 4.8]. The equations of the proof of [90, Lemma 4.8] applied to our exact example then need to be checked for the existence of real solutions.

Dual Complexes of Unseparated Nodes

In previous sections, we encountered cases where two distinct node germs did not give rise to separated nodes. The dual complexes arising from these cases are shown in Figure 5.16. We also encountered the floors which do not give separated nodes in Figure 5.13 and in the proof of Proposition 5.2.23. By new alignment conditions, they might encode unseparated nodes, see Remark 5.2.21. Alignment with ends is not allowed for separated nodes, because circuit D (Figure 5.5b) is then contained in the boundary of the Newton polytope and cannot encode a single node [90]. However, with one point condition less than for one-nodal surfaces, we can obtain strings with one degree of freedom more and this makes not only the alignment of two ends possible, but additionally the alignment of the vertices the ends are adjacent to. This leads to a triangular prism shape in the subdivision, which has at least one parallelogram shaped face in the interior of the Newton polytope. At this time, we do not yet know whether any of these cases can contain two nodes or with what multiplicity they should be counted with, but in total they ought to give the 66 missing surfaces from our count.

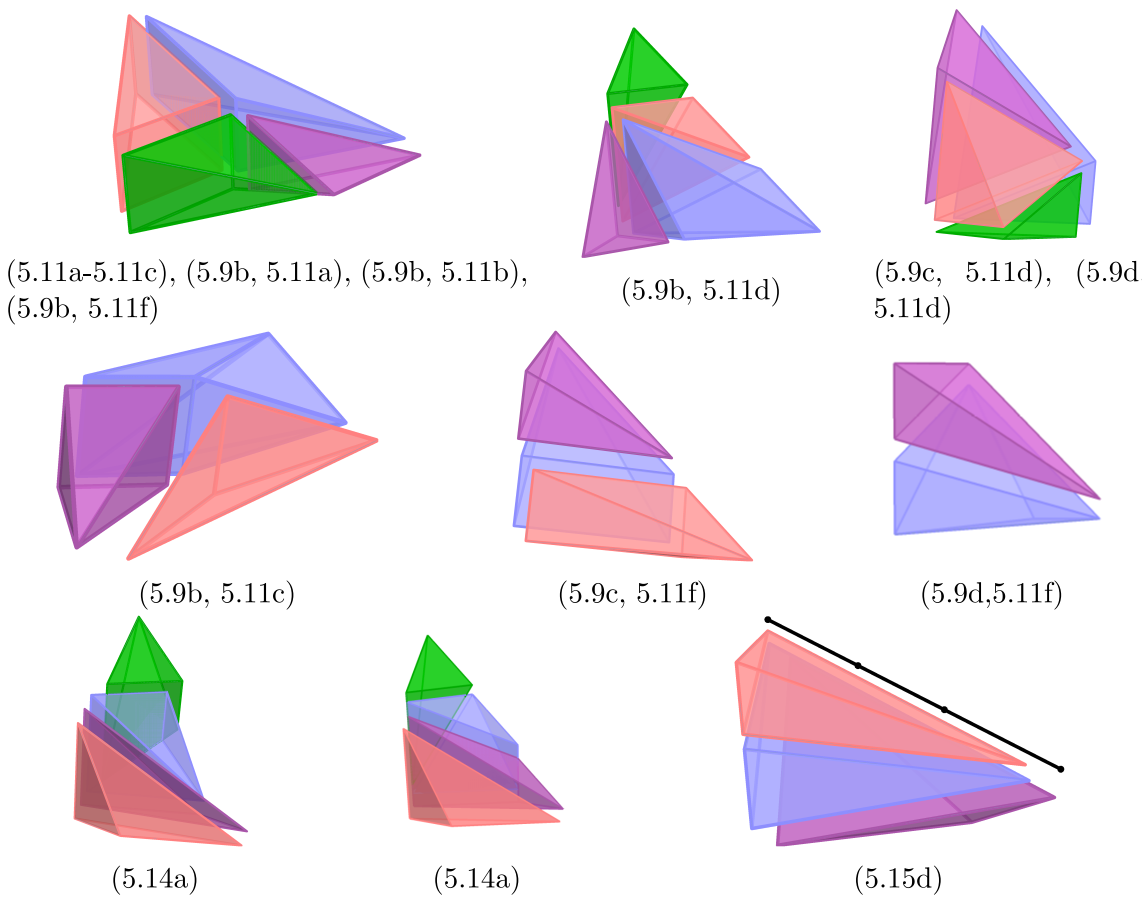


Figure 5.16: Complexes whose duals could have two nodes.

Conclusion

In this chapter we developed techniques for counting tropical surfaces in three dimensional space. Using these techniques, we counted tropical binodal cubic surfaces with separated tropical nodes. In order to provide a full count of tropical binodal cubic surfaces, polytopes which can accommodate two nodes must be classified. Extending these tools to multi-nodal tropical surfaces will be a challenging topic of future research.

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