Title
Categorification and Combinatorics on Oligomorphic Permutation Groups

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Publication Date
2026-04-01

Undergraduate

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Spring 2026

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Acknowledgements

Mathematics is often written and read with a disconnect between the author and the mathematics in the paper. Not so in other disciplines. Anthropology, critical theory, philosophy, gender studies, ethnic studies, and many more disciplines explicitly recognize the connection between the author and their written work. In mathematics, this connection is still there, despite our willful ignorance of it.

This thesis contains details about a topic in permutation group theory and the category-theoretic and model-theoretic implications. In another way, it contains the totality of my mathematical experience as an undergraduate. Everything I've learned as an undergraduate, *in mathematics and outside of it*, has in some small way, made it into this paper. So, it stands to reason to recognize the contribution of other people, where they are in my life and I'm in theirs.

My advisor, Ilia Nekrasov, has been relentlessly supportive and kind in the time spent completing this thesis and the directed reading that led up to it. I knew next to nothing about the landscape and structure of pure mathematics when we started working together, yet he always met me where I was at in the course of our research. I owe nearly the entirety of my mathematical world to our research meetings.

Thank you Jayadev Athreya, Kai Marshall, Rishi Nath, Owen Norwalk, Enmanuel Felix-Peña, Ather Zia, for comments, edits, readings. I am indebted to Paolo Mancosu and Jonathan Tanaka for teaching me logic and philosophy, and setting me on the path to self-actualizing myself as a mathematician.

Thank you to my many housemates over the years, including Alexander, Toby, Zenji, Mateo, Kai, Perez, Ronan, Catney, John, Barrett, Evan, Shiven, Jesus, Jayden, Guhan, Benny, Diego, Cameron, and Enmanuel. The people you live with, if you're lucky, also become your closest friends. I count myself lucky.

Thanks to Ryan, Gianna, Julius, Leslie, Minya, Keren, Clara, Rhea, Mehana, Liz, Anna, Vanness, and the cello section in the UC Berkeley Symphony Orchestra for being fantastic musicial partners, teachers, and sounding boards over the years.

I was lucky to benefit from stellar teachers inside and outside of the IB Program at Greeley West High School. Thank you to David Falter, Leah Sanford, and Elizabeth Dent.

Thank you to Margot and Kiyoko and the stewards of the Charlene Conrad Liebeau Library Prize for recognizing the research and work that has gone into this paper.

Thank you to Dr. Angana Chatterji for our countless readings, studies, and research meetings over the course of my undergraduate studies.

Thank you to my family, who are my largest intellectual, spiritual, and mental influences.

1 Introduction

In this text, we are primarily concerned with categorifying combinatorics on oligomorphic groups. A permutation group is said to be *oligomorphic* if it has but finitely many orbits on all n-tuples of the associated G-set. The theory of oligomorphic groups began with work done by Peter Cameron in the 80s and 90s. For example, see [Cam90], [Cam99], and [Cam09]. One of the central questions of Cameron's work is:

Question 1.1

Which sequences $(s_n)_{n\geq 0}$ of natural numbers can be realized as the number of orbits $|\Omega^n/G|$ for some oligomorphic group G.

Many early questions in Cameron's work, specifically those concerned with orbit-counting, were connected to combinatorial enumeration problems. That is, given a set G acting on X, we consider the cardinality of $G \cdot X$ and show that this cardinality can be counted by combinatorial means. Indeed, there is a deep connection between oligomorphic groups and countability; while not to be discussed further in this thesis, it is worth mentioning nonetheless. A theorem of Ryll-Nardzewski, Engeler, and Svevonius, proved independently in 1959, provides a direct link between oligomorphicity and model theory. A proof of the theorem can be found in [Hod97].

Theorem 1.1: Ryll-Nardzewski, Engeler, and Svevonius

Let T be a countable, complete first-order theory. Then T is ω-categorical if and only if the automorphism group of T, $\mathrm{Aut}(T)$, is oligomorphic.

Current literature concerning oligomorphic groups is within representation theory, category theory, and model theory. In this thesis, we will "lift" Cameron's work on oligomorphic permutation groups to the level of categories. We do this in two ways: viewing a group action as a pair (G, Ω), which is an object of the category G-$\underline{\text{Set}}$ and considering maps between G-spaces, viewed as sets. Equivalences between numbers become isomorphisms between objects of G-set, in the first case, and bijections between sets, in the second.

Oligomorphic groups are a physicalist object– they are permutation groups that satisfy a finiteness condition. Under the lens of categorification, then, it is interesting that the study of oligomorphic groups goes hand in hand with adding layers of mathematical abstraction.

2 Generalities on Oligomorphic Groups

We will present two definitions of oligomorphic groups from two different perspectives– combinatorial and categorical.

We say that a n-tuple $(a_1, \ldots, a_n)$, where $a_1, \ldots, a_n \in \Omega$ is **distinct** if, for all $i, j \leq n$, $a_i \neq a_j$. Otherwise, the n-tuple $(a_1, \ldots, a_n)$ is called **arbitrary** (but we will usually just say n-tuple). Let G be a group acting on a set Ω. There is an extension of the G-action onto three kinds of objects arising from Ω. The action is defined "componentwise" and is analogous for each of the three. If $g \in G$ and $(a_1, \ldots, a_n)$ is an n-tuple with $a_1, \ldots, a_n \in \Omega$, then

$$g \cdot (a_1, \ldots, a_n) = (g \cdot a_1, \ldots, g \cdot a_n)$$

Definition 2.1: Oligomorphic permutation group, combinatorial

Let G be a group acting on a set Ω. We say that G is an **oligomorphic permutation group** if and only if G has but finitely many orbits on n-tuples (equivalently, n-sets or (n)-tuples) of Ω for all $n \in \mathbb{N}$.

Remark 2.1

That the above definition is indeed valid for n-sets and (n)-tuples will be justified by the bounds given in Propositions 2.1, 2.2, and 2.3.

Since the definition of oligomorphic groups implicitly assumes a group action, we may drop the word "permutation". We may also speak of an action being oligomorphic if the associated group is oligomorphic. Note is that, if G is a finite group, them it is trivially oligomorphic. In section 1.2.1 we will show, using the *downward Lowenheim-Skolem theorem*, that we can restrict our study of oligomorphic groups to those which are at most of infinitely countable cardinality.

The above definition is implicitly combinatorial as we are thinking about the number of G-orbits. We can *categorify* the above definition by thinking about G-orbits as sets. Recall that G induces an equivalence relation $\sim_G$ on Ω where elements $a, b \in \Omega$ are considered equivalent if there is an element g such that $ga = b$. Let Ω/G denote the set of equivalence classes of Ω under $\sim_G$. Then Ω/G is Ω as a G-set and is the collection of all orbits of G on Ω. Let $P_n(X)$ denote the set of n-sets of Ω, $\Omega^{(n)}$ denote the set of distinct n-tuples of Ω, and, as usual, Ω^n denote the set of arbitrary n-tuples of Ω. Considering the set of orbits Ω^n/G (and $\Omega^{(n)}/G$ and $P_n(\Omega)/G$) allows us to express the condition for oligomorphicity in terms of it. This will lead to slightly hefty notation, but it sets the stage for Chapter 3, where we begin our discussion of the category G-Set.

Definition 2.2: Oligomorphic permutation group, categorified

Let G be a group acting on a set Ω. G is oligomorphic if and only if $|\Omega^n/G| = k$ (equivalently, $|\Omega^{(n)}/G| = k$ or $|P_n(\Omega)/G| = k$) for some $k \in \mathbb{N}$, for all $n \in N$.

In the remainder of this section we will prove three propositions showing that the three quantities $|P_n(\Omega)/G|, |\Omega^{(n)}/G|$, and $|\Omega^n/G|$ can be finitely bounded in terms of each other. Therefore, if one is finite, then the other two are. This justifies our definition of an oligomorphic group as one with finitely many orbits on any one of the three objects.

Proposition 2.1

For any pair (G, Ω), $|P_n(\Omega)/G| \leq |\Omega^{(n)}/G| \leq n!|P_n(\Omega)/G|$.

Proof. Recall that $P_n(\Omega)/G$ and $\Omega^{(n)}/G$ denote the set of orbits on n-sets and distinct n-tuples, respectively. Define an equivalence relation $\sim$ on the set $\Omega^{(n)}$ where $(\alpha_1, ..., \alpha_n) \sim (\beta_1, ..., \beta_n)$ if $\{\alpha_1, ..., \alpha_n\} = \{\beta_1, ..., \beta_n\}$; i.e. they are set-equivalent. Elements of $\Omega^{(n)}/\sim$ are of the form $[(\alpha_1, ..., \alpha_n)]$. Since any given n-tuple has n distinct orderings, the equivalence classes contain $n!$ elements of $\Omega^{(n)}$. Consider the function $h : P_n(\Omega)/G \to (\Omega^{(n)}/\sim)/G$ where $h(\{\alpha_1, ..., \alpha_n\}) = [(\alpha_1, ..., \alpha_n)]$. This function is clearly bijective, since each set determines a single equivalence class. But since the equivalence classes correspond to $n!$ elements of $\Omega^{(n)}$, they all either share the same orbit or not. If they share one orbit, then $|P_n(\Omega)| = |\Omega^{(n)}|$. If they all have different orbits, then there are $n!$ orbits corresponding to a single n-set, and so $F_n \leq n! f_n$. Then we arrive at the bound $|P_n(\Omega)| \leq |\Omega^{(n)}| \leq n! |P_n(\Omega)|$ □

Proposition 2.2

Let G be an oligomorphic group acting on Ω. Then $|P_{n+1}(\Omega)/G| = |\Omega^{(n+1)}/G|$ implies $|\Omega^{(n)}/G| = 1$

Proof. That $P_{n+1}(\Omega) = \Omega^{(n+1)}/G$ holds implies that any permutation of $n+1$ points can be induced by an element of G. Why? Because if our orbits on $(n+1)$-sets correspond exactly to orbits on distinct $(n+1)$-tuples, then all permutation of the points in the $(n+1)$-set are mapped to the same orbit on $n-$tuples. Then, if two n-tuples happen to share $n-1$ points, we can one to the other by a permutation that fixes their setwise union. The fact that such a permutation exists is provided for us because we can induce any permutation we like on $n+1$ points. □

The next proposition involves the *Stirling numbers of the second kind*, denoted by $S(n,k)$. They count the number of partitions of an n-set into k parts. A good reference is [Bru09].

Proposition 2.3

Let G be an oligomorphic group acting on Ω. Then $|\Omega^n/G| = \sum_{k=0}^{n} S(n,k)|\Omega^{(k)}/G|$.

2.1 Examples of Oligomorphic Groups

In this section we will cover two classic examples of oligomorphic groups are the infinite symmetric group, S_∞, and the automorphism group of the rationals $\mathrm{Aut}(\mathbb{Q})$.

Example 2.1

Let S_∞ denote the group of all permutations of $\mathbb{N}$ with *finite support*, that is, all permutations $\sigma = (\alpha_1, \ldots, \alpha_n)$ such that $n \in \mathbb{N}$, where $\alpha_1, \ldots, \alpha_n \in \Omega$, a countable set. We call S_∞ the **infinite symmetric group**.

Consider the action of S_∞ on $\Omega = \mathbb{N}$. Notice that for 1-tuples, which are just single elements, there is only one orbit. This is because, for any two elements $\alpha, \beta \in \Omega$, there exists $\sigma = (\alpha\beta) \in S_\infty$, and $\sigma(\alpha) = \beta$. Since our choice of α, β was arbitrary, we conclude that any two elements in Ω can be mapped to each other under the action of S_∞, so they all must sit in the same orbit. We have a definition for this.

Definition 2.3

Let G be a group acting on a set X. We say that G is **transitive** on X, if there is only one orbit. We say that G is n**-transitive** on X, if G has only one orbit on X^n.

If we continue this line of thinking for 2-tuples of Ω. We now claim there are two orbits, which correspond to tuples of the form (α, α) and (α, β), where $\alpha \neq \beta$. In this case, (α, α) can be mapped to any (α, α) and (α, β) can be mapped to any (γ, δ). We can extend our above definition reactively.

Proposition 2.4

The infinite symmetric group S_∞ acts transitively on $\Omega^{(n)}$, the set of distinct $n-$tuples of elements in Ω.

Proof. Let $(\alpha_1...\alpha_n)$ be an arbitrary element of $\Omega^{(n)}$. S_∞ is transitive if there is only one orbit on $\Omega^{(n)}$. Equivalently, for any $(\beta_1,...,\beta_n)\in\Omega^{(n)}$, there exists a $g\in S_\infty$ such that $(\alpha_1...\alpha_n)g=(\beta_1...\beta_n)$. Since $\Omega^{(n)}$ consists of distinct n-tuples, we know that for $(\alpha_1,...,\alpha_n)$, $\alpha_i=\alpha_j$ if and only if $i=j$, for $1\le i\le j\le n$. For $\alpha,\beta\in\Omega$, we know that there exists a $g\in S_\infty$ such that $\alpha g=\beta$. We can say more, that $g=(\alpha\beta)$. Then for $(\alpha_1...\alpha_n)$ and $(\beta_1,...,\beta_n)$, there exists a pairwise permutation mapping α_i to β_i, the 2-cycle $(\alpha_i\beta_i)$. Then let the permutation $g\in S_\infty$ be $g=(\alpha_1\beta_1)...(\alpha_n\beta_n)$. Clearly g maps $(\alpha_1...\alpha_n)$ to $(\beta_1...\beta_n)$. Since our choice of n-tuples was arbitrary, we conclude such a g exists for any pair of n-tuples in Ω^n. Equivalently, for any two elements of Ω^n, they lie in the same orbit induced by an element of S_∞. Then S_∞ acts transitively on $\Omega^{(n)}$.

Example 2.2

Let $\mathbb{Q}=(\mathbb{Q},<)$ denote the structure with models $T_{\mathrm{Aut}(\mathbb{Q})}$, the first-order theory of the rationals as a dense linear order.

Recall that an order relation $<$ on a set x is a dense linear order (DLO) if it is a partial order with the condition that for any two elements $x,y\in X$ such that $x<y$, there exists an element $z\in X$ such that $x<z<y$. We can show that $\mathrm{Aut}(\mathbb{Q})$ is oligomorphic using a combinatorial proof. In Section 3 we will be able to categorify this proof to take place within G-Set.

The group $\mathrm{Aut}(\mathbb{Q})$ is oligomorphic; moreover, we can place a tight bound of its orbits on $\mathbb{Q}^n$ for all $n\in\mathbb{N}$. We break these into two cases: for (n)-tuples and for n-tuples. The proof strategy will show some examples for small n to build intuition before proceeding to the general proof.

Theorem 2.1

The group $\mathrm{Aut}(\mathbb{Q})$, with the natural order-preserving action on $\mathbb{Q}$ is oligomorphic.

Proof. We are trying to determine the cardinality of $\mathbb{Q}^{(n)}/\mathrm{Aut}(\mathbb{Q})$. Consider the action of $\mathrm{Aut}(\mathbb{Q})$ on $\mathbb{Q}^{(n)}$. Let $(a_1,\dots a_n)\in(\mathbb{Q}^{(n)},<)$, where $a_i\ne a_j$ for all $i,j\le n$. Notice that an automorphism of $\mathbb{Q}$ is only required to preserve the order of the tuples. Thus, counting the orbits for (n)-tuples is equivalent to counting the different orderings of $(a_1,\dots,a_n)\in\mathbb{Q}$. Since our tuple is distinct, we can say WLOG $a_1<a_2<\dots<a_n$. The different orbits then correspond to all the possible orderings, which is $n!$. Then $F_n(\mathrm{Aut}(\mathbb{Q},<))=n!$ for all $n\in\mathbb{N}$.

Since $\mathrm{Aut}(\mathbb{Q})$ is transitive on $\mathbb{Q}$, there there is only one orbit, and $|\mathbb{Q}/\mathrm{Aut}(\mathbb{Q})|=1$. There are 3 orbits of $\mathrm{Aut}(\mathbb{Q},<)$ on $\mathbb{Q}^2$, which correspond to (a,a), (a,b) and (b,a), where $a<b$. If we move to $n=3$ a larger pattern emerges. Our strategy is to count all the different ways to relate the elements in a 3-tuple. First we ask what kinds of tuples (up to equality) we can form in $\mathbb{Q}^3$. They are listed here: $a=a=a, a\ne b=b, a=a\ne b, a\ne b\ne a, a\ne b\ne c,$

Notice that what we have just done is count the number of partitions of $[3]=\{1,2,3\}$. We now count the ways to order the elements in each partition. Once we have done so, we will have counted all possible relations on 3-tuples, and thus counted all possible orbits of $\mathrm{Aut}(\mathbb{Q},<)$ on $\mathbb{Q}^3$.

$$
\begin{aligned}
&a = a = a \Rightarrow 1 \text{ orbit}\\
&a \neq b = b \Rightarrow 2 \text{ orbits, } a < b, b < a\\
&a = a \neq b \Rightarrow 2 \text{ orbits; ditto}\\
&a \neq b \neq a \Rightarrow 2 \text{ orbits; ditto}\\
&a \neq b \neq c \Rightarrow 3! \text{ orbits, same as for } \mathbb{Q}^{(3)}
\end{aligned}
$$

We have $1 + (2 + 2 + 2) + 3! = 13$ total orbits, so we may write $\mathbb{Q}^3$ as a G-space as follows:

$$\mathbb{Q}^3 \cong \mathbb{Q}^{(1)} \sqcup \mathbb{Q}^{(2)} \sqcup \mathbb{Q}^{(3)}$$

. □

2.2 Model-theoretic connections

In this section we consider applications of model theory to the study of oligomorphic groups.

2.2.1 The Canonical Relational Structure

We first present a result about a structure which can be associated with a group action.

Theorem 2.2

If G is a permutation group acting on a countable set Ω, then there is a relational structure $\mathcal{M}$, where $|\mathcal{M}| = \Omega$, over a countable language, such that

1. $G \leq \mathrm{Aut}(\mathcal{M})$
2. G and $\mathrm{Aut}(\mathcal{M})$ have the same orbits on Ω^n for every n.

In [Cam90], Cameron gives a description of a structure satisfying the above, called the **canonical relational structure**.

Decompose $\bigcup_{n\geq 1} \Omega^n$ into orbits $O_1, O_2, \ldots$ and with each orbit associate a relation symbol R_i of arity n. The interpretation of R_i, which is $R_i^{\mathcal{M}}$, is taken to be O_i as a set.

For the first condition, we want G to be a subgroup of the automorphism group of $\mathcal{M}$. Up till this point $\mathcal{M}$ is the structure with Ω as its domain and the relations of $\mathcal{M}$ that need to be preserved under any automorphism are simply the orbits. We can indeed decompose the G-set in this way since orbits induce an equivalence class. It is also clear that the elements of G can be thought of as automorphisms of $\mathcal{M}$ in the natural way, because we think of them as acting on the domain of $\mathcal{M}$, which again is Ω. They preserve all the relations because the action of G on Ω is already defined.

Question 2.1

Now we see G is definitely some of the automorphisms of $\mathcal{M}$. But what automorphisms could exist that are not elements of G or somehow equivalent to them?

The next thing we need to check is that G and $\mathrm{Aut}(\mathcal{M})$ have the same orbits on Ω^n for every n. Well, if we have some tuple of elements of Ω^n, we can be sure it satisfies a unique R_i, meaning the i it holds for is unique. Then if two tuples lie in the same $\mathrm{Aut}(\mathcal{M})$ orbit, they satisfy the same relation R_i, so they both lie in the same "orbit" O_i (it's not really an orbit because at this point its just a set that holds for the relation R_i), but the elements in the set O_i are only there if they are in the same G-orbit O_i. Then this construction should be more or less clear.

Cameron makes a closing comment that G might be the automorphism group of a structure over a different language. What he means by this is a group G (an abstract group or permutation group? I think both) is a structure over the signature of groups $\sigma = (e, ^{-1}, \cdot)$, but we have seen examples where G is the group of permutations of $(\mathbb{Q}, <)$, but the automorphism group of that structure $\mathrm{Aut}(\mathbb{Q}, <)$, is a structure over the relational language $\sigma = (<)$.

Question 2.2

What are some examples of G being the automorphism group of a L-structure $\mathcal{A}$, where L is a language that contains constant symbols? Function symbols? A mixture of both?

2.2.2 Countability

The goal of our section is a "metatheorem" which says the following.

Theorem 2.3

It is sufficient to study oligomorphic groups of countably infinite degree.

We shall see it is a consequence of the downward Lowenheim-Skolem theorem

Theorem 2.4

(Lowenheim Skolem): If a set of sentences in a first order language L has a model, then it has a countable model.

To show our result, we will describe how we can express all the information concerning permutation groups of the oligomorphic type in a set of sentences. More specifically we will write up a theory which will clearly have a model and use LS to deduce it will have countable models.

We proceed with a rough sketch constructing a model and axiomatisable theory that will model all our enterprises in oligomorphic groups. Let L be the usual language of groups with two extra unary predicates P and R, where $P = X$ and $R = G$. So these unary predicates are intended to distinguish between elements of X and elements of G. Our theory consists of the three axioms for abstract groups and one for group action.

$$\forall g(e \cdot g = g \cdot e = e)$$
$$\forall g_1, g_2, g_3\big((g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_2)\big)$$
$$\forall g \exists g^{-1}(g \cdot g^{-1} = g^{-1} \cdot g = e)$$
$$\forall g, h, x\big(g \cdot (hx) = g \cdot h(x)\big)$$

where we use g and x to suggestively indicate group elements and G-set elements, respectively. To be absolutely precise we would also have to add some conjunctions inside of the scope of the quantifier specifying which relations the variables belong to, but we skip over that detail here.

We also need infinitely (countably) many sentence asserting that for each n, there are only finitely many orbits on Ω^b. As Cameron notes, this basically can be summed up as "there exists n elements of G such that none of them are mapped to any other".

$$\exists x_1, x_2, \dots, x_n \; \forall g \left(\bigvee_{i=1}^{n \quad (\mathrm{mod}\ n)} \left(g \cdot x_i \neq x_{i+1} \right) \right)$$

From the downward LS theorem, we can assume the structure we have given satisfies our theory of oligomorphic permutation groups and so it has a countable model.

Cameron makes the following observation: given a permutation group G acting on a countable set Ω, there is a countable subgroup with the same orbits on tuples as G. I was wondering if you get this from Lowenheim-Skolem but you don't in general because you would probably only get subgroups (elementary substructures) which go down to countable degree, but maybe not further. The way Cameron actually gets it is as follows: We can actually enumerate the pairs of tuples which lie in the same orbit, and thus choose an element of G which maps one to the other. Then it definitely preserves all the orbits since no element (in our new group H, the collection of elements of G found in this way) is mapped to an element that was not in its orbit, but since we did it pairwise and those pairs are countable, it must also be countable.

3 Category of G-Sets

In this section we are concerned with G-Set, the category of G-Sets. This will play a role in Section 3 because we will be lifting equivalences on orbits from numbers to sets. For now it suffices to say that a set X acted on by a group G can be lifted to the its G-set, which lives in the category of G-sets, by the mapping $G \cdot X$.

3.1 Category-theoretic preliminaries

This section primarily draws from [BL09] We proceed with some quick category theoretic definitions. For readers unfamiliar with category theory, one motivation is that categories seek to generalize a common situation in mathematics: studying some kinds of objects and structure-preserving maps between them. An example is a collection of groups and the homomorphisms between them.

Definition 3.1

A category $\mathcal{C}$ consists of two things:

- A collection of objects $Ob(\mathcal{C})$
- For any pair of objects $x, y \in Ob(\mathcal{C})$, a set $Mor(\mathcal{C})$ of morphisms $f : x \to y$

In addition to these things, a category is equipped with the following.

- For each object x, an identity morphism $1_x : x \to x$.
- For any pair of morphisms $f : x \to y$ and $g : y \to z$, a composite morphism $gf : x \to z$

The following laws should also be satisfied.

- For any morphism $f : x \to y$, the left and right unit laws hold: $1_x f = f$ and $f 1_x = f$.
- For any triple of morphisms $f : w \to x, g : x \to y, h : y \to z$, the associative law holds: $(hg)f = h(gf)$.

We use lowercase latin letters to denote both objects and morphisms in a category. However, when a letter denoting an object is clear from context, we will write $x \in \mathcal{C}$ as shorthand for $x \in Ob(\mathcal{C})$

In a category $\mathcal{C}$, morphisms serve as the maps between objects. We can also have maps between categories, called functors.

Definition 3.2

For categories $\mathcal{C}$ and $\mathcal{D}$, a functor consists of the following things:

- A function $F : Ob(\mathcal{C}) \to Ob(\mathcal{D})$
- For any pair of objects $x, y \in Ob(\mathcal{C})$, a function $F : Mor(x, y) \to Mor(F(x), F(y))$

The following laws should also be satisfied.

- F preserves identities- that is, $F(\mathrm{id}_{\mathcal{C}}) = \mathrm{id}_{\mathcal{D}}$
- F preserves composition- that is, for any pair of morphisms $f : x \to y$ and $g : y \to z$ $F(gf) = F(g)F(g)$

A bifunctor is a functor $f : A \to B$ where the source is a product category, i.e. $A = C \times D$ for some categories C and D.

A common theme in category theory is increasing levels of abstractions. We described maps between categories, and will now describe maps between functors.

Definition 3.3

Given functors $F, G : \mathcal{C} \to \mathcal{D}$, a natural transformation $\alpha : F \Rightarrow G$ consists of the following things:

- A function α associating each object $x \in \mathcal{C}$
- For any morphism $f : x \to y$ in $\mathcal{C}$, this diagram commutes.

$$\begin{tikzcd}
F(x) \arrow[r, "\eta_x"] \arrow[d, "F(f)"'] & G(x) \arrow[d, "G(f)"] \\
F(y) \arrow[r, "\eta_y"'] & G(y)
\end{tikzcd}$$

3.2 Monoidal categories

Another word for a monoidal category is "tensor category". In short, a monoidal category is a category equipped with a binary functor $\otimes : M \times M \to M$, called a *tensor product* which satisfies some structure-preserving conditions. In particular, we capture left and right unit laws along with associativity using natural transformations and identities governing them.

Definition 3.4

A monoidal category consists of the following things.

- A category $\mathcal{M}$
- A binary functor, called the **tensor product** $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$
- An **identity object**, denoted $1 \in \mathcal{M}$
- Three natural isomorphisms: the **associator**,

$$a_{x,y,x} : x \otimes (y \otimes z) \to (x \otimes y) \otimes z$$

the **left unit law**,

$$l_x : 1 \otimes x \to x$$

and the **right unit law**,

$$r_x : x \otimes 1 \to x$$

For M to truly be monoidal, the natural isomorphisms we declare the natural isomorphisms above to be such that the following diagrams commute. The **pentagon identity**,

$$\begin{tikzcd}
 & ((w \otimes x) \otimes y) \otimes z \arrow[dl, "a_{w\otimes x,y,z}"'] \arrow[dr, "a_{w,x,y} \otimes \mathrm{id}_z"] & \\
(w \otimes x) \otimes (y \otimes z) \arrow[d, "a_{w,x,y\otimes z}"'] & & (w \otimes (x \otimes y)) \otimes z \arrow[d, "a_{w,x\otimes y,z}"] \\
w \otimes (x \otimes (y \otimes z)) & & w \otimes ((x \otimes y) \otimes z) \arrow[ll, "\mathrm{id}_w \otimes a_{x,y,z}"]
\end{tikzcd}$$

and the **triangle identity**,

$$\begin{tikzcd}
 & (x \otimes \mathbf{1}) \otimes y \arrow[dl, "r_x \otimes \mathrm{id}_y"'] \arrow[dr, "a_{x,\mathbf{1},y}"] & \\
x \otimes y & & x \otimes (\mathbf{1} \otimes y) \arrow[ll, "\mathrm{id}_x \otimes l_y"]
\end{tikzcd}$$

The simplest example of a monoidal category is the category of sets where the cartesian product serves as the tensor product. It is fairly straightforward, but instructive to prove that this indeed forms a monoidal category. All we need to do is identify what the identity morphisms are and prove that the left and right unit laws as well as the pentagon identity.

$$\begin{tikzcd}
 & & ((w \otimes x) \otimes y) \otimes z \arrow[dll, "a_{w\otimes x,y,z}"'] \arrow[drr, "a_{w,x,y} \otimes \mathrm{id}_z"] & & \\
(w \otimes x) \otimes (y \otimes z) \arrow[dr, "a_{w,x,y\otimes z}"'] & & & & (w \otimes (x \otimes y)) \otimes z \arrow[dl, "a_{w,x\otimes y,z}"] \\
 & w \otimes (x \otimes (y \otimes z)) & & w \otimes ((x \otimes y) \otimes z) \arrow[ll, "\mathrm{id}_w \otimes a_{x,y,z}"] &
\end{tikzcd}$$

The cartesian product is essentially just juxtaposing elements and makes the proofs look rather vacuous as a result. The point is that at each step of proving an arrow relies on some choice of the associator, which we only have because we're working in a tensor category.

A fair question to ask is whether the pentagon identity, which only concerns all possible parenthesizations of four objects, holds for five. With the Coherence Theorem, Mac Lane proved that pentagon identity implies associativity for five objects, and in fact, any number. However there is an easier combinatorial proof. Right now we will prove it for five objects using brute force since the proof is instructive.

Insert proof here (a pain to latex, write out cleanly first)

Proof details: write out all the possible parenthesizings of five elements, arrange the arrows in the proper way, then use the pentagon identity to prove the whole thing commutes by identifying the little pentagons that exists inside to resulting associahedron. (Include note about saturation. The possible parenthesizations of n objects are basically the saturated ones.)

Note: about how associativity is really easy for groups and sets but its not so easy for categories and with categories there are all these geometric realizations of associativity laws.

3.3 G-Set

We are now ready to introduce the category G-Set. It is a category where the objects are ordered pairs (X, Φ), where X is a set and Φ is an action of G on X. All the actions are implicitly known to be G-actions so there is no missing data. To see what the morphisms should be, notice that our objects are just sets with some structure. So morphisms between our objects should be morphisms between sets, which are just functions between sets. This justifies our choice of notation since we are essentially working in the category Set while simultaneously paying mind to G-actions on its objects. For morphisms in G-Set we demand that these morphisms respect the action of G on X. We collect this ideas in a definition.

Definition 3.5

Fix a group G. Let G-Set denote the **category of G-sets**, consisting of the following things.

- The objects are ordered pairs $(X, \Phi : G \times X \to X)$, where Φ is a group action. When the action is clearly defined or arbitrary we will abbreviate as (X, Φ).
- For any pair of objects $(X, \Phi), (Y, \Psi)$, the set of morphisms $Mor((X, \Phi), (Y, \Psi))$ consists of all functions $f : X \to Y$ such that the following diagram commutes.

$$\begin{tikzcd} G \times X \arrow[r, "\Phi"] \arrow[d, "\mathrm{id} \times f"'] & X \arrow[d, "f"] \\ G \times Y \arrow[r, "\Psi"'] & Y \end{tikzcd}$$

Intuitively, we think of the above diagram as meaning that f preserves the actions of G on X and Y, respectively.

4 Bounds on Orbits via G-Set

4.1 Categorification

A common theme in this paper is lifting equivalences from the level of numbers to the level of categories. In this setting either two things can happen. The identity we are lifting can be implied by the commutativity of a certain diagram isomorphism of some kind. Consider the following identity, which might be proved in an undergraduate abstract algebra class.

Proposition 4.1

Let $\Phi : G \times X \to X$ be an action of G on X. The following identity holds.

$$|\mathcal{O}_x| = [G : \mathrm{Stab}(x)]$$

The strategy for the proof is as follows: We are trying to establish an identity between numbers. For our "categorified" proof, we will lift the equality from Set to G-Set. Informally, this will look like finding objects in G-Set that "play the role" of their counterparts. In this case, all we need to do is find an action Φ of G on the sets in question and our rephrasing is the object (X, Φ). At the level of categories, isomorphism plays the role of equality, so we will construct an isomorphism between the objects, which will establish the result.

Proof. We will rephrase the identity in terms of G-Set. $\mathcal{O}_x$ is a set with a G-action, so we rephrase it as $(\mathcal{O}_x, \Phi|_{\mathcal{O}_x})$. The action of G on $\mathcal{O}_x$ is the induced action of G on X, since $\mathcal{O}_x$ is a subset of X. For $\mathrm{Stab}(x)$, there is a G-action on it defined by left multiplication. Let Ψ be this action, giving us the object $(G/\mathrm{Stab}(x), \Psi)$. We now seek to prove the following identity.

$$(\mathcal{O}_x, \Phi|_{\mathcal{O}_x}) \cong (G/\mathrm{Stab}(x), \Psi)$$

An isomorphism between objects in a category is morphism that has a well-defined inverse. So all we need to do is find a map $\varphi : \mathcal{O}_x \to G/\mathrm{Stab}(x)$ that has an inverse. We claim that this map is

$x_i \mapsto g_i \operatorname{Stab}(x)$ and the converse map is $g_i \operatorname{Stab}(x) \mapsto g_i x_i$. We need to prove that $\varphi \circ \psi = \mathrm{id}_{G/\operatorname{Stab}(x)}$ and $\psi \circ \varphi = \mathrm{id}_{\mathcal{O}_x}$.

We have the following series of equivalences.

If $g(x) \mapsto g \operatorname{Stab}(x)$ and $h(x) \mapsto h \operatorname{Stab}(x)$ then $(h^{-1})g(x) = x)$ and so $h^{-1}g \in \operatorname{Stab}(x)$. Actually everything in the previous statement is an if and only if so we have shown well definedness. □

4.2 Bounds on orbits

Proposition 4.2

For a group G acting on Ω, we have the following:

$$|P_n(\Omega)/G| \leq |\Omega^{(n)}/G| \leq n!|P_n(\Omega)/G|$$

Remark 4.1

Note that an n-set gives rise to $n!$ distinct n-tuples corresponding to the possible orderings on the n-set. This proposition essentially says that there are no more orbits on n-sets than their possible orderings allow.

Proof. Let π_1 and π_2 be the canonical maps mapping $\Omega^{(n)}$ and $P_n(\Omega)$ to their G-sets, $\Omega^{(n)}/G$ and $P_n(\Omega)/G$, respectively. These maps are defined as

$$\pi_1(a_1, \dots, a_n) = G \cdot (a_1, \dots, a_n)$$
$$\pi_1\{a_1, \dots, a_n\} = G \cdot \{a_1, \dots, a_n\}$$

Define a map $\omega : \Omega^{(n)} \to P_n(\Omega)$ by $\omega(\bar{a}) = \{\bar{a}\}$. This map is clearly surjective; given an n-set $\{a_1, \dots, a_n\}$ the n-tuple $(a_1, \dots, a_n)$ maps to it under ω. Since ω is surjective, we can speak of the *fiber* of an element $\{\bar{a}\}$ under ω, that being the set

$$\omega^{-1}\{a_1, \dots, a_n\} = \left\{(a_1, \dots, a_n) : \omega(a_1, \dots, a_n) = \{a_1, \dots, a_n\}\right\}$$

Clearly, $|w^{-1}\{a_1, \dots, a_n\}| = n!$, since there are $n!$ orderings of an n-set corresponding to distinct n-tuples.

We then want to define a map $\overline{\omega} : \Omega^{(n)}/G \to P_n(\Omega)$. Our claim is that $\overline{\omega}(G \cdot (a_1, \dots, a_n)) := G \cdot \omega(a_1, \dots, a_n)$ is a well-defined map. To see this it suffices to check the following property about ω. $\omega(G\bar{a}) = G\omega(a)$

$$\omega(g \cdot (a_1, \dots, a_n)) = \omega(ga_1, \dots, ga_n) = \{ga_1, \dots, ga_n\} = g\{a_1, \dots, a_n\} = g \cdot \omega(a_1, \dots, a_n)$$

In light of this we see that the following diagram commutes.

$$\begin{tikzcd}
\Omega^{(n)} \arrow[r, "\omega"] \arrow[d, "\pi_1"'] & P_n(\Omega) \arrow[d, "\pi_2"] \\
\Omega^{(n)}/G \arrow[r, "\overline{\omega}"'] & P_n(\Omega)/G
\end{tikzcd}$$

We now move towards establishing the bounds themselves.

We first show that $\bar{\omega}$ is surjective. Consider some $G \cdot \{a_1, ..., a_n\} \in P_n(\Omega)/G$. We want to show that there is some $G \cdot (a_1, .., a_n) \in \Omega^{(n)}/G$ such that $\omega(G \cdot (a_1, .., a_n)) = G \cdot \{a_1, ..., a_n\}$. Notice that $\{a_1, ..., a_n\}$ is the preimage of $G \cdot \{a_1, .., a_n\}$ under π_2. The map π_2 is clearly surjective, though the preimage may not be unique. Since, ω is surjective, there must be some $(a_1, ..., a_n)$ such that $\omega(a_1, ..., a_n) = \{a_1, ..., a_n\}$. Then $\pi_2(a_1, ..., a_n) = G \cdot (a_1, ..., a_n)$, so we have found an element in a well-defined way that maps to $G \cdot \{a_1, ..., a_n\}$. We then conclude that $|P_n(\Omega)/G| \leq |\Omega^{(n)}/G|$.

We will then place a bound on the cardinality of $\Omega^{(n)}/G$ using the fibers of $\bar{\omega}$. We claim that the fibers of $G \cdot \{a_1, ..., a_n\} \in P_n(\Omega)/G$ is the set $\{G \cdot (a_1, ..., a_n) | \sigma \in S_n\}$. We can place a bound of $n!$ on the cardinality of this set since some permutations may map different tuples to the same orbit (Compare this to how the size of the set was completely determined by permutations in the case of $\Omega^{(n)}$).

Then suppose $G \cdot (a_1, ..., a_n) \xrightarrow{\bar{\omega}} G \cdot (\beta_1, ..., \beta_n)$ Since the orbit representatives differ up to an element of G, there exists some element $g \in G$ such that $g \cdot \{a_1, ..., a_n\} = \{ga_1, ..., ga_n\} = \{\beta_1, ..., \beta_n\}$. Since these are setwise equivalent, we have one of the following situations.

$$
\begin{aligned}
&ga_1 = b_1, ga_2 = b_2, ..., ga_n = b_n \\
&ga_1 = b_2, ga_2 = b_3, ..., ga_n = b_1 \\
&... \\
&ga_1 = b_{i_1}, ga_2 = b_{i_2}, ..., ga_n = b_{i_n}
\end{aligned}
$$

where the i_j are elements of some index set I. Then the permutations of I determine the ordering in which the elements $ga_1, ..., ga_n$ are sent to $b_1, ..., b_n$, which are precisely the elements of S_n, since $|I| = n$. Since $(ga_1, ..., ga_n) \xrightarrow{\bar{\omega}} (b_1, ..., b_n)$, we can say that $\{\sigma_1 ga_1, ..., \sigma ga_1\} = (b_1, ..., b_n)$. Since our choice of $G \cdot (a_1, ..., a_n)$ was arbitrary, we conclude that the preimage of $G \cdot \{b_1, ..., b_n\}$ is $\{G \cdot (\sigma a_1, ..., \sigma a_n) | \sigma \in S_n\}$ which, as we previously observed, has cardinality bounded by $n!$. We then conclude that $|\Omega^{(n)}/G| \leq n! |P_n(\Omega)|$ □

4.3 Bounds using wreath products

Recall that an oligomorphic group is a permutation group G acting on a set X such that G has but finitely many orbits on X^n for all n.

In [Cam90], Peter Cameron proposes the following question.

Proposition 4.3

For any oligomorphic permutation group G, $F_n^*(G) = F_n(S_\infty \operatorname{Wr} G)$.

Recall that F_n^* and F_N denote orbits on arbitrary n-tuples and distinct n-tuples, respectively. Now, we can rephrase the question above by *categorification*, where we view the equality between numbers as an equality between G-sets.

Proposition 4.4

Suppose $G \curvearrowright \Omega$ is an oligomorphic permutation group. Let $W = S_\infty \operatorname{Wr} G$, the wreath product of the infinite symmetric group and G. Then we have that $|\Omega^n/G| = |(\mathbb{N} \times \Omega)^{(n)}/W|$

Proof. For the moment we will consider the special case where $G = S_\infty$ and $\Omega = \mathbb{N} \times \mathbb{N}$, since S_∞ acts on $\mathbb{N}$.

$$\begin{tikzcd}
(\mathbb{N}\times\mathbb{N})^{(n)} \arrow[r, two heads, "\varphi"] \arrow[d, "\pi_1"'] & \mathbb{N}^n \arrow[d, "\pi_2"] \\
(\mathbb{N}\times\mathbb{N})^{(n)}/W \arrow[r, "1:1", "\psi"'] & \mathbb{N}^n/G
\end{tikzcd}$$

This proposition is then equal to this diagram being commutative and the sign "1:1" being valid.

The set $(\mathbb{N}\times\mathbb{N})^{(n)}$ consists of $n-$tuples of duples. The first coordinates of each duple correspond to the elements of the fibers and the second coordinates index the fibers themselves. Moreover, each duple is distinct.

Recall an n-tuple, $(a_1, ..., a_n)$ induces a partition of the n-set $\{1, ..., n\}$ where for some $i, j \leq n$, i and j lie in the same part if and only if $a_i = a_j$. We know that the orbits of n-tuples of $\mathbb{N}$ under S_∞ are determined up to the partition they induce. Then our strategy will be to consider certain "canonical" partitions and their φ-preimages. Because we are transitioning from between arbitrary and distinct tuples, we can be judicious about the preimages that we choose in order to reveral a larger pattern. As we will see, classifying all possible preimages will be enough to show equivalency of the orbits.

As an example, consider the element $(1, ..., 1) \in \mathbb{N}^n$. Now, we can characterize orbits of $\mathbb{N}^n$-elements, which we can also think of as sets. These sets are determined by the group action; in the case of S_∞ acting on $\mathbb{N}$, the orbit an an arbitrary element is determined by the partition it characterizes. (To see what we mean, note the general fact that that an arbitrary n-tuple $(a_1, \dots, a_n)$ determines a partition of $\{1, \dots, n\}$ where $i, j \in \{1, \dots, n\}$ lie in the same part iff $a_i = a_j$. In our example, we just skip a step). It is clear that $(1\dots1) \in \{(k\dots k) \mid k \in \mathbb{N}\} = \mathcal{O}_{(1\dots1)}$. Equivalently, S_∞ is transitive on $\mathcal{O}_{(1\dots1))}$, and the element $(1\dots1)$ happens to be in that very set. Our next move will be to find a φ-preimage of $(1\dots1)$, and then find an arbitrary φ-preimage of $(1\dots1)$, where it will then become clear that the orbit of $(1\dots1)$ and the orbit of a φ-preimage of $(1\dots1)$ are one and the same. This will show that we have a 1:1 correspondence between orbits. (The process is straightforward, but we choose to show clearly all the steps). Then let Let $1, ..., n \in N$ be a list of elements such that $1 \neq \dots \neq n$. Then $((1,1), ..., (n,1)) \in (\mathbb{N}\times\mathbb{N})^{(n)}$ is a suitable φ-preimage of $(1\dots1)$. Moreover, the orbit of this element is the set $\{(m_1, k), \dots, (m_n, k) \mid m_1, \dots, m_n, k \in \mathbb{N} \wedge m_1 \neq \dots \neq m_n\}$. But, an arbitrary φ-preimage of $(k\dots k)$ is precisely $(m_1, k), \dots, (m_n, k)$ where $m_1 \neq \dots \neq m_n$. So we have shown the orbits are equal, and we are done.

We will now move towards proving the general case (of the special case where $G = S_\infty$). Our strategy is this: We will prove the proposition once more for a judicious choice of $\mathbb{N}^n$-element, then argue that this proposition proves the general case. The element in question is $(1\dots12\dots2\dots p\dots p) \in \mathbb{N}^n$ where we have b_1 1s, b_2 2s, and so on, and finally b_p ps, such that $b_1 + b_2 + \dots + b_p = n$. It should be easy for us to see that the arbitrary φ-preimage of this element, and also the orbit is the set

$$\left\{(m_{b_{1_1}}, 1), \dots, (m_{b_{1_{b_1}}}, 1),\ (m_{b_{2_1}}, 2), \dots, (m_{b_{2_{b_2}}}, 2),\ \dots,\ (m_{b_{p_1}}, p), \dots, (m_{b_{p_{b_p}}}, p) \;\middle|\; \substack{m_{b_{i_j}} \in \mathbb{N} \text{ for all } i,j \\ m_{b_{i_1}} \neq \dots \neq m_{b_{i_{b_i}}} \text{ for all } i \leq p}\right\}$$

Our next move is to show that the 1:1 correspondence between orbits is preserved by an arbitrary permutation (or perhaps, reordering) of $(1\dots12\dots2\dots p\dots p)$, where we permute the 1s as one group, the 2s as one group, and so on. Now consider the element $(k_1\dots k_1 k_2\dots k_2 \dots k_p\dots k_p)$. If we can show that the orbit of an arbitrary φ-preimage of this is the same as our special element, **up to a relabeling** of the m's, then we will be done.

Remark 4.2

It is important to note these elements are not necessarily in the same orbit, but simply that we can characterize them the same way, and in turn they range over all possible elements of $\mathbb{N}^n$.

We conclude with a rough sketch of how to do the relabeling.

$$(m_{b_{1_1}}, k_1), \ldots (m_{b_{1_{b_1}}}, k_1), (m_{b_{2_1}}, k_2), \ldots (m_{b_{2_{b_2}}}, k_2), \ldots, (m_{b_{p_1}}, k_p), \ldots (m_{b_{p_{b_p}}}, k_p)$$

We do the relabeling in the following way. If $k_i = i$, then do nothing. If $k_i = j$, then switch the labels from m_i to m_j. If we repeat for each k_i, we will be done. □

Remark 4.3

We can characterize elements of $\mathbb{N}^n$ as partitions. We do this because we know that the partition of some n-tuple in $\mathbb{N}^n$ uniquely determines its orbit in $(\mathbb{N} \times \mathbb{N})^{(n)}$. But since the partition in $\mathbb{N}^n$ uniquely determines its orbit in $(\mathbb{N}^n) \cdot S_\infty$, we have a 1-1 correspondence between the orbits, and the proposition is proved (for this very special case).

A partition of $\bar{a} = (1 \ldots a)$ is a uniquely determined by the following two objects. First, for some $a \in \mathbb{N}$, an equality $\bar{a} \sim n_1 a_2 + n2a_2 + \ldots + n_k a_b$, where $\forall i \leq k, n_i \in \mathbb{N}$ and $\forall j \leq n, a_j \in \mathbb{N}$ $k, l \in \mathbb{N}$. Note that some of the a_j's may be equal to each other.

To motivate this definition, notice that an element of $\mathbb{N}^a$ has length a and contains an ordered string of integers with some repetition and multiplicity.

Then a is to the length of the string (in $\mathbb{N}^n$), the a_j's are the natural numbers (possibly with repetition) that make up the string, and the n_i's are the respective multiplicities (in a row, so to speak) assigned to each integer. As an example (112211) is written as $6 \sim 2 \cdot 1 + 2 \cdot 2 + 2 \cdot 3$

It is important that we consider which elements in the string are equal to each other up to not being in the same run. Let $A = \{a_1, a_2, \ldots, a_n\}$, and let $E = \{a_k = a_l, k, l \leq n, a_k, a_l \in A\}$. Now we have all we need to characterize elements of $\mathbb{N}^n$ by looking at them as partitions.

We proceed with a proof of the Proposition 3.4 where $G = \mathrm{Aut}(\mathbb{Q})$. We first need a lemma.

Lemma 4.1

The right action of f on the right-coordinate $(k_1, \ldots k_n)$ preserves the distinctness of the k's.

Proof. Note that the value of $f(a_i)$ is dependent on which coordinate we are in, i.e. which value of i. Note that, in a tuple $((k_1, a_1), \ldots, (k_n, a_n))$, k_i and k_j only need be distinct if $a_i = a_j$. Then the action of (f, q) on the tuple looks like $(...(k_i f(a_i), a_i q), (k_i f(a_i), a_i, q)...)$. Since the action restricted to k_i and k_j is the same, the distinctness of k_i and k_j is preserved. □

Lemma 4.2

Let $\mathrm{Aut}(\mathbb{Q})$ act on $\mathbb{Q}$ and S_∞ act on $\mathbb{N}$. Now let $W := S_\infty \mathrm{Wr}\, \mathrm{Aut}(\mathbb{Q})$ be the *wreath product* of these two groups. Then $|(\mathbb{N} \times \mathbb{Q})^{(n)} / W| = |\mathbb{Q}^n / \mathrm{Aut}(\mathbb{Q})|$.

Proof. The proof proceeds as follows. Let $\mathcal{O} = \mathrm{Aut}(\mathbb{Q}) \cdot (a_1, \dots, a_n)$ be an arbitrary orbit in $\mathbb{Q}^n / \mathrm{Aut}(\mathbb{Q})$. Firstly, $\mathcal{O}$ is a tuple which determines a partition. Then it has some order relation R on it which determines the orbit. The orbit is completely determined by the partition and the order relation on it. Then if we make a judicious choice of preimage of $\mathcal{O}$ in $(\mathbb{N} \times \mathbb{Q})^{(n)}/W$, we get the following.

$$\{((k_1, a_1), \dots (k_n, a_n)) \mid k_i \neq k_j \text{ if } a_i = a_j, \forall i, a_i \in \mathbb{Q}, k_i \in \mathbb{N}, \text{the order relation } R \text{ is satisfied}\}$$

If we then consider the action of W on the tuples of the following set, we get, for $(f, k) \in W$,

$$((k_1, a_1), \dots (k_n, a_n)) \cdot (f, q) = ((k_1 f(a_1), a_1 q), \dots (k_n f(a_n), a_n q))$$

The componentwise action of $f(a_i)$ on k_i will preserve the distinctness of the k_i's as we will show in the lemma below, and the componentwise action of q on a_i will preserve the order relation. Then our choice of preimage corresponds precisely with the orbit induced by an arbitrary element in it, and we are done. □

5 Appendix

In this section we present some preliminary details to understand the wreath product construction covered in the previous section.

5.1 Semidirect Products

The material in this section draws from [DM96], Chapter 2.

To understand wreath products we must look at the simpler construction of semidirect products, a generalization of the idea of direct products.

Let H and K be groups and suppose that we have an action of H on K which respects the group structure on K. If H respects the group structure on K that means that $h \cdot (k_1 k_2) = (hk_1)(hk_2)$, and also $h \cdot e_K = e_K$. Then for each $x \in H$ the map $u \mapsto u^x = u \cdot x$ is an automorphism of K. Then put $G := \{(u, x) | u \in K, x \in H\}$ (remember H acts on K), and define a product on G as $(u, x)(v, y) := (uv^{x^{-1}}, xy) = (uvx^{-1}, xy)$ for all $(u, x)(v, y) \in G$.

My first thought is this product is kind of weird and I don't really know why it's defined in this way. But I think the first thing to do is check that it's associative and that we'll be okay defining it in this way.

We have two properties because H acts on K by automorphisms. The first is that $(k^a)^b = k^{ab}$ and the second is that $(k_1 k_2)^a = k_1^a k_2^a$. So then perform the following calculation, to show that the product on G is associative

$$\begin{aligned}
(u, x)\big((v, y)(w, z)\big) &= (u, x)(vw^{y^{-1}}, yz) \\
&= \big((uvw^{y^{-1}})^{x^{-1}}, x(yz)\big) \\
&= \big(uv^{x^{-1}}(w^{y^{-1}})^{x^{-1}}, (xy)z\big) \\
&= \big(uv^{x^{-1}} w^{(xy)^{-1}}, (xy)z\big) \\
&= (uv^{x^{-1}}, xy)(w, z) \\
&= \big((u, x)(v, w)\big)(w, z)
\end{aligned}$$

The next thing to check is that G has an identity element. But it's obvious that is identity element is $(1,1)$. Finally, we need to know that each $(u,x) \in G$ has an inverse. We claim that this inverse is $\big((u^x)^{-1}, x^{-1}\big)$. Recall that because H acts on K by automorphisms, it also respects inverses, which explicitly means that $(u^x)^{-1} = (u^{-1})^x$. Then we can perform another calculation.

$$\begin{aligned}
&(u,x)\big((u^x)^{-1}, x^{-1}\big) \\
&= (u(u^x)^{-1}x^{-1}, xx^{-1}) \\
&= (u(u^{-1})^x x^{-1}, 1) \\
&= (uu^{-1}xx^{-1}, 1) = (1,1)
\end{aligned}$$

The fact that inverses commute can be seen by rearranging inside the product (you should check this).

We can ask about the subgroups of G. We have two, which are familiar to us from direct products. They are $H^* := \{(1,x)|x \in H\}$ and $K^* := \{(u,1)|u \in K\}$. These groups are isomorphic to H and K respectively, but more than that $G = K^*H^*$ and $K^* \cap H^* = 1$. I think I'll verify all these facts in a moment. Also K^* (the set that was acted upon) is normal in G, but also we have something interesting when H^* acts upon K^* by conjugation (which I think is an important thing in wreath products). To see this,

$$(1,x)^{-1}(u,1)(1,x) = (1,x)^{-1}(u1, 1x) = (u11^x, x^{-1}x) = (u^x, 1)$$

because $(1,x)^{-1} = (1,x^{-1})$.

Recall that $K^*H^* = \{(u,1)(1,x)|u \in K, x \in H\}$, all the products between the two. Actually there is something to check here for sure because the product here is not the same as that of direct products. But actually because of the form $(u,1)$ and 1 being self inverse, we just end up getting all the pairs (u,x) for $u \in K$ and $x \in H$. So then we can be sure $G = K^*H^*$. And that $K^* \cap H^* = 1$ should be clear. We'll leave K^* being normal in G as an exercise for now.

A big takeaway is that for H to respect the group structure of K is absolutely critical for all this stuff to happen.

Now we can say that we call G the **semidirect product** of K by H, and we denote G using the notation $K \rtimes H$. We say "by H" to call attention to the fact that H is the group that acts on K. The semidirect product, specifically its multiplication, is definitely dependent on the action of H on K, but its not explicit in the notation.

Lastly, it should be clear that $|G| = |H||K|$.

The following is Cameron's construction of the wreath product in [Cam99]. Let H and K be two permutation groups acting on the sets Γ and Δ, respectively. Then let $\Omega = \Gamma \times \Delta$. Cameron says we should think of this as a covering space Δ where the fibers are bijective to Γ.

The quick answer is that can say the wreath product of two groups is the semidirect product of the base group and the top group.

Next we let B, called the **base group**, be the Cartesian product of $|\Delta|$ copies of H, where each copy is associated with each fiber of Ω, which is to say each element of Δ.

Question 5.1

Why is it important that we have $|\Delta|$ copies of H? Why couldn't it be some other number?

One answer is because we have an action of K on Δ so we need the action of the top group K_1 on the bottom group B to model the action of K on Δ. In the original world, K acts on the $n = |\Delta|$ elements of Δ, and in the wreath product world we have the top group K_1 acting on the n fibers of the bottom group B.

Next we let K_1, called the **top group**, be the permutation group on Ω that permutes the fibers of Ω (the elements of Δ). It does this according to whatever action it has on Δ.

Wreath product. Given the above construction, the **wreath product** of H and K, denoted by $H \operatorname{Wr} K$, is the semi-direct product of B and K_1.

We have a result concerning the embedding of a group G into a wreath product.

Theorem 5.1

Let G be a a group which is transitive but imprimitive on Ω. Let Γ be a congruence class. Let H be the permutation group induced on Γ by its setwise stabilizer in G. Let Δ be the set of congruence classes. Let K be the group induced by Δ on G. Then G can be embedded in a natural way into $H \operatorname{Wr} K$, as a permutation group.

Remark 5.1

The setwise stabilizer of a set Γ is the set of all $x \in G$ such that $\Gamma^x = \Gamma$.

Any finite transitive permutation group can be broken down into its primitive "components" (he uses parentheses here to suggest they are not really components, perhaps it is because, as he suggests, some information is lost). In general we cannot do this for infinite permutation groups. However, if the group we are considering is oligomorphic, we can always do this. Actually, all we need is that the group has finitely many orbits on ordered pairs. Such a group with the lattermost condition can only have finitely many congruences. Any congruence can be thought of as a set of ordered pairs, which is the union of orbits of G in its action on $\Omega \times \Omega$.

The last thing we discuss is an important action of the wreath product $H \operatorname{Wr} K$ on the set of functions $\varphi : \Delta \to \Gamma$. An element f of the base group acts by $\varphi \cdot f(i) = \varphi(i) \cdot f(i)$ while the top group act on the arguments of the function by $\varphi \cdot k(i) = \varphi(ik^{-1})$.

5.2 Wreath Products

We begin with some material developed in [BMMN98].

Definition 5.1

Let C be a group and D a group acting on a set Δ. Define $K := C^{\Delta} = \{f | f : \Delta \to C\}$ and pointwise multiplication on K as follows $fg(\delta) = f(\delta)g(\delta)$. The identity element of K, denoted 1_K, maps every element of Δ to the identity of C.

We can define an action of D on K if we say, for $f \in K$ and $d \in D$, $f^d(\delta) := d(\delta d^{-1})$. To see that this defines an action, $f^{d_1 d_2}(\delta) = f(\delta(d_1 d_2)^{-1}) = f(\delta d_2^{-1} d_1^{-1}) = f^{d_1}(\delta d_2^{-1}) = (f^{d_1})^{d_2}(\delta)$ where the second to last inequality follows because D acts on Δ, I think.

Definition 5.2

The **wreath product** W of C by D is defined to be the semidirect product of K by D. This is the explicit definition.

$$W := K \rtimes D = C^{\Delta} \rtimes D$$

is defined on the set $\{(f,d) | f \in K, d \in D\}$ where multiplication is given by $(f_1,d_1)(f_2,d_2) = (f_1 f_2^{d_1^{-1}}, d_1 d_2)$

Let H and K be two permutation groups acting on sets Γ and Δ, respectively. Let $\Omega = \Gamma \times \Delta$, which we think of as a covering space of Δ with fibers bijective to Γ. In particular, for a fixed $\delta \in \Delta$, its fiber is $\Gamma_\delta = \{(\gamma,\delta) : \gamma \in \Gamma\}$.

We now describe the main ingredients of the wreath product. Let B be the direct product of $|\Delta|$ copies of H, i.e. $B = H^{|\Delta|}$. We think of each copy of H as permuting the fiber is it associated with, meaning we consider Δ to be a kind of indexing set. The group B is isomorphic to the group $\mathrm{Fun}(\Delta, H)$, which is the set of functions from Δ to H, where the group operation is pointwise multiplication. This provides a nice characterization of B where we think of its action as being "indexed" by the fiber in question.

Next, let T be a copy of K which permutes the fibers themselves. Informally, we can think of B as permuting Ω "up-and-down" and T as permuting Ω "left-and-right". Given B and T, we can form their (external) semidirect product BT, which is called the *wreath product* of H and K. We denote the semidirect product BT by $H \,\mathrm{Wr}\, K$.

The (right) action BT on Ω, sometimes called the *imprimitive action*, for $f \in B$ and $t \in T$, is given by $(\gamma,\delta)BT = (\gamma f(\delta), \delta t)$.

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