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Classification of contractively complemented Hilbertian operator spaces

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Abstract

We construct some separable infinite-dimensional homogeneous Hilbertian operator spaces $H_\infty^{m,R}$ and $H_\infty^{m,L}$, which generalize the row and column spaces R and C (the case $m=0$). We show that a separable infinite-dimensional Hilbertian JC^*-triple is completely isometric to one of $H_\infty^{m,R}$, $H_\infty^{m,L}$, $H_\infty^{m,R} \cap H_\infty^{m,L}$, or the space Φ spanned by creation operators on the full anti-symmetric Fock space. In fact, we show that $H_\infty^{m,L}$ (respectively $H_\infty^{m,R}$) is completely isometric to the space of creation (respectively annihilation) operators on the m (respectively $m+1$) anti-symmetric tensors of the Hilbert space. Together with the finite-dimensional case studied in [M. Neal, B. Russo, Representation of contractively complemented Hilbertian operator spaces on the Fock space, Proc. Amer. Math. Soc. 134 (2006) 475–485], this gives a full operator space classification of all rank-one JC^*-triples in terms of creation and annihilation operator spaces.

We use the above structural result for Hilbertian JC^*-triples to show that all contractive projections on a C^*-algebra A with infinite-dimensional Hilbertian range are "expansions" (which we define precisely) of normal contractive projections from A^{**} onto a Hilbertian space which is completely isometric to R, C, $R\cap C$, or Φ. This generalizes the well-known result, first proved for $B(H)$ by Robertson in [A.G. Robertson, Injective matricial Hilbert spaces, Math. Proc. Cambridge Philos. Soc. 110 (1991) 183–190], that all Hilbertian operator spaces that are *completely* contractively complemented in a C^*-algebra are completely isometric to R or C. We use the above representation on the Fock space to compute various completely bounded Banach–Mazur distances between these spaces, or Φ.

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1. Preliminaries

The goals of the present paper are to classify all infinite-dimensional rank 1 JC^*-triples up to complete isometry (Theorem 1 in Section 2) and then use that result to give a suitable "classification" of all Hilbertian operator spaces which are contractively complemented in a C^*-algebra or normally contractively complemented in a W^*-algebra (Theorems 2 and 3 in Section 3). In particular, we show that these latter spaces are "essentially" R, C, $R \cap C$, or Φ modulo a "degenerate" piece.

In Section 4, we show that the space $H_\infty^{m,L}$ (respectively $H_\infty^{m,L}$) can be represented completely isometrically as creation (respectively annihilation) operator spaces on pieces of the anti-symmetric Fock space. In Theorems 4 and 5 in Section 4 we compute the completely bounded Banach–Mazur distances between the spaces discussed in this paper.

In the rest of this section, we give some background on operator space theory and on JC^*-triples.

1.1. Operator spaces

Operator space theory is a non-commutative or quantized theory of Banach spaces. By definition, an operator space is a Banach space together with an isometric linear embedding into $B(H)$, the bounded linear operators on a complex Hilbert space. While the objects are obviously the Banach spaces themselves, the more interesting aspects concern the morphisms, namely, the completely bounded maps. These are defined by considering an operator space as a subspace X of $B(H)$. Its *operator space structure* is then given by the sequence of norms on the set of matrices $M_n(X)$ with entries from X, determined by the identification $M_n(X) \subset M_n(B(H)) = B(H \oplus H \oplus \cdots \oplus H)$. A linear mapping $\varphi : X \to Y$ between two operator spaces is *completely bounded* if the induced mappings $\varphi_n : M_n(X) \to M_n(Y)$ defined by $\varphi_n([x_{ij}]) = [\varphi(x_{ij})]$ satisfy $\|\varphi\|_{cb} := \sup_n \|\varphi_n\| < \infty$.

Operator space theory has its origins in the work of Stinespring in the 1950s, and Arveson in the 1960s. Many tools were developed in the 1970s and 1980s by a number of operator algebraists, and an abstract framework was developed in 1988 in the thesis of Ruan. All definitions, notation, and results used in this paper can be found in recent accounts of the subject, namely (in chronological order) [3,6,15,16]. Let us just recall that a completely bounded map is a *complete isomorphism* if its inverse exists and is completely bounded. Two operator spaces are *completely isometric* if there is a linear isomorphism T between them with $\|T\|_{cb} = \|T^{-1}\|_{cb} = 1$. We call T a *complete isometry* in this case. Other important types of morphisms in this category are complete contractions ($\|\varphi\|_{cb} \leqslant 1$) and complete semi-isometries (:= isometric complete contraction).

Examples of completely bounded maps are the restriction to a subspace of a C^*-algebra of a $*$-homomorphism and multiplication by a fixed element. It is a fact that every completely bounded map is essentially a product of these two examples, [16, Theorem 1.6]. The space $cb(X, Y)$ of completely bounded maps between operator spaces X and Y is a Banach space with the completely bounded norm $\|\cdot\|_{cb}$.

Analogous to the Banach–Mazur distance for Banach spaces, the class of all operator spaces can be made into a metric space by using the logarithm of the *completely bounded* Banach–Mazur distance:

$$d_{cb}(E,F)=\inf\bigl\{\|u\|_{cb}\cdot\|u^{-1}\|_{cb};\ u:E\to F \text{ complete isomorphism}\bigr\}.$$

Two important examples of Hilbertian operator spaces (:= operator spaces isometric to Hilbert space) are the row and column spaces R, C, and their finite-dimensional versions R_n, C_n. These are defined as follows. In the matrix representation for $B(\ell_2)$, *column Hilbert space* $C := \overline{\mathrm{sp}}\{e_{i1}\colon\ i \geqslant 1\}$ and *row Hilbert space* $R := \overline{\mathrm{sp}}\{e_{1j}\colon\ j \geqslant 1\}$. Their finite-dimensional versions are $C_n = \mathrm{sp}\{e_{i1}\colon\ 1 \leqslant i \leqslant n\}$ and $R_n = \mathrm{sp}\{e_{1j}\colon\ 1 \leqslant j \leqslant n\}$. Here of course e_{ij} is the operator defined by the matrix with a 1 in the (i,j)-entry and zeros elsewhere. Although R and C are Banach isometric, they are not completely isomorphic ($d_{cb}(R,C)=\infty$); and R_n and C_n, while completely isomorphic, are not completely isometric. In fact, it is known that $\mathrm{d}_{cb}(R_n,C_n)=n$.

R, C, R_n, C_n are examples of *homogeneous* operator spaces, that is, operator spaces E for which $\forall u:E\to E$, $\|u\|_{cb}=\|u\|$. Another important example of an Hilbertian homogeneous operator space is $\Phi(I)$. The space $\Phi(I)$ is defined by $\Phi(I)=\overline{\mathrm{sp}}\{V_i\colon\ i\in I\}$, where the V_i are bounded operators on a Hilbert space satisfying the canonical anti-commutation relations. In some special cases, the notations $\Phi_n := \Phi(\{1,2,\dots,n\})$, and $\Phi=\Phi(\{1,2,\dots\})$ are used. For more properties of this space and related constructs, see [16, 9.3].

Two more examples of homogeneous operator spaces are $\min(E)$, $\max(E)$, where E is any Banach space. For any such E, the operator space structure of $\min(E)$ is defined by the embedding of E into the continuous functions on the unit ball of E^* in the weak*-topology, namely, $\|(a_{ij})\|_{M_n(\min(E))}=\sup_{\xi\in B_{E^*}}\|(\xi(a_{ij}))\|_{M_n}$. The operator space structure of $\max(E)$ is given by

$$\bigl\|(a_{ij})\bigr\|_{M_n(\max(E))}=\sup\bigl\{\bigl\|\bigl(u(a_{ij})\bigr)\bigr\|_{M_n(B(H_u))}\colon\ u:E\to B(H_u),\ \|u\|\leqslant 1\bigr\}.$$

More generally, if F and G are operator spaces, then in $F\xrightarrow{u}\min(E)$, $\|u\|_{cb}=\|u\|$, and in $\max(E)\xrightarrow{v}G$, $\|v\|_{cb}=\|v\|$. The notations $\min(E)$ and $\max(E)$ are justified by the fact that for any operator space structure $\tilde{E}$ on a Banach space E, the identity map on E is completely contractive in $\max(E)\to\tilde{E}\to\min(E)$.

By analogy with the classical Banach spaces ℓ_p, c_0, L_p, $C(K)$ (as well as their "second generation," Orlicz, Sobolev, Hardy, Disc algebra, Schatten p-classes), we can consider the (Hilbertian) operator spaces R, C, $\min(\ell_2)$, $\max(\ell_2)$, OH, Φ, as well as their finite-dimensional versions R_n, C_n, $\min(\ell_2^n)$, $\max(\ell_2^n)$, OH_n, Φ_n, as "classical operator spaces." Among these spaces, only the spaces R, C, and Φ play important roles in this paper. (For the definition and properties of the space called OH, see [16, Chapter 7].) The classical operator spaces are mutually completely non-isomorphic. If E_n, F_n are n-dimensional versions, then $d_{cb}(E_n,F_n)\to\infty$, [16, Chapter 10].

We propose to add to this list of classical operator spaces the Hilbertian operator spaces $H_\infty^{m,R}$ and $H_\infty^{m,L}$ constructed here, as well as their finite-dimensional versions H_n^k studied in [13,14]. Like the space Φ, the spaces $H_\infty^{m,L}$, $H_\infty^{m,R}$ and H_n^k can be represented up to complete isometry as spaces of creation operators or annihilation operators on anti-symmetric Fock spaces ([14, Lemma 2.1] and Remark 4.2).

This paper relies significantly for the proofs and notations on previous work by two of the authors in [13,14]. In particular, a review of [13] may considerably help the reader to understand the proofs in this paper. More precisely, from [13], we quote Lemmas 5.1 and 5.4, Corollaries 5.3 and 7.3, and Theorem 3(b). In addition, we note that the proofs from [13] for Lemmas 5.8, 5.9 and 6.7, and Propositions 6.3 and 6.10 remain valid without change in the context of the present paper.

Let us recall from [13, Sections 6, 7] the construction of the spaces H_n^k, $1 \leqslant k \leqslant n$. Let I denote a subset of $\{1, 2, \ldots, n\}$ of cardinality $|I| = k-1$. The number of such I is $q := \binom{n}{k-1}$. Let J denote a subset of $\{1, 2, \ldots, n\}$ of cardinality $|J| = n-k$. The number of such J is $p := \binom{n}{n-k}$. We assume that each $I = \{i_1, \ldots, i_{k-1}\}$ is such that $i_1 < \cdots < i_{k-1}$, and that if $J = \{j_1, \ldots, j_{n-k}\}$, then $j_1 < \cdots < j_{n-k}$.

The space H_n^k is the linear span of matrices $b_i^{n,k}$, $1 \leqslant i \leqslant n$, given by

$$b_i^{n,k} = \sum_{I \cap J = \emptyset,\, (I \cup J)^c = \{i\}} \epsilon(I, i, J) E_{J,I}, \tag{1}$$

where $E_{J,I} = e_J \otimes e_I = e_J e_I^t \in M_{p,q}(\mathbf{C}) = B(\mathbf{C}^q, \mathbf{C}^p)$, and $\epsilon(I, i, J)$ is the signature of the permutation taking $(i_1, \ldots, i_{k-1}, i, j_1, \ldots, j_{n-k})$ to $(1, \ldots, n)$. (e_I denotes the basis vector consisting of a 1 in the "Ith" position.) Since the $b_i^{n,k}$ are the image under a triple isomorphism (actually ternary isomorphism) of a rectangular grid in a JW^*-triple of rank one, they form an orthonormal basis for H_n^k (cf. [13, Sections 5.3 and 7]).

The following definition from [16, 2.7] plays a key role in this paper. If $E_0 \subset B(H_0)$ and $E_1 \subset B(H_1)$ are operator spaces whose underlying Banach spaces form a compatible pair in the sense of interpolation theory, then the Banach space $E_0 \cap E_1$ (with the norm $\|x\|_{E_0 \cap E_1} = \max(\|x\|_{E_0}, \|x\|_{E_1})$) equipped with the operator space structure given by the embedding $E_0 \cap E_1 \ni x \mapsto (x, x) \in E_0 \oplus E_1 \subset B(H_0 \oplus H_1)$ is called the *intersection* of E_0 and E_1 and is denoted by $E_0 \cap E_1$. We note, for examples, that $\bigcap_{k=1}^n H_n^k = \Phi_n$ [14] and the space $R \cap C$ is defined relative to the embedding of C into itself and R into C given by the transpose map [16, p. 184]. The definition of intersection extends easily to arbitrary families of compatible operator spaces (cf. the proof of Theorem 1).

Lemma 1.1. *Let H be an Hilbertian operator space, and suppose that every finite-dimensional subspace of H is homogeneous. Then H itself is homogeneous.*

Proof. Let ϕ be any unitary operator on H. According to the first statement of [16, Proposition 9.2.1], it suffices to prove that ϕ is a complete isometry.

Let F be any finite-dimensional subspace of H and let G be the subspace spanned by $F \cup \phi(F)$. By the second statement of [16, Proposition 9.2.1], F and $\phi(F)$, being of the same dimension as subspaces of the homogeneous space G, are completely isometric, and $\phi|F$ is a complete isometry.

Now let $[x_{ij}] \in M_n(H)$. Then $\{x_{ij}, \phi(x_{ij})\colon 1 \leqslant i, j \leqslant n\}$ spans a finite-dimensional subspace F of H, and

$$\|\phi_n([x_{ij}])\|_{M_n(H)} = \|\phi_n([x_{ij}])\|_{M_n(F)} = \|[x_{ij}]\|_{M_n(F)} = \|[x_{ij}]\|_{M_n(H)}. \qquad \square$$

1.2. Rank one JC^*-triples

A JC^*-triple is a norm closed complex linear subspace of $B(H, K)$ (equivalently, of a C^*-algebra) which is closed under the operation $a \mapsto aa^*a$. JC^*-triples were defined and studied (using the name J^*-algebra) as a generalization of C^*-algebras by Harris [10] in connection with function theory on infinite-dimensional bounded symmetric domains. By a polarization identity, any JC^*-triple is closed under the triple product

$$(a,b,c) \mapsto \{abc\} := \frac{1}{2}\big(ab^*c + cb^*a\big), \tag{2}$$

under which it becomes a Jordan triple system. A linear map which preserves the triple product (2) will be called a *triple homomorphism*. Cartan factors are examples of JC^*-triples, as are C^*-algebras, and Jordan C^*-algebras. Cartan factors are defined for example in [13, Section 1]. We shall only make use of Cartan factors of type 1, that is, spaces of the form $B(H,K)$ where H and K are complex Hilbert spaces.

A special case of a JC^*-triple is a *ternary algebra*, that is, a subspace of $B(H,K)$ closed under the *ternary product* $(a,b,c) \mapsto ab^*c$. A *ternary homomorphism* is a linear map ϕ satisfying $\phi(ab^*c) = \phi(a)\phi(b)^*\phi(c)$. These spaces are also called ternary rings of operators and abbreviated TRO. They have been studied both concretely in [11] and abstractly in [21]. Given a TRO M, its left (respectively right) linking C^*-algebra is defined to be the norm closed span of the elements ab^* (respectively a^*b) with $a,b \in M$. Ternary isomorphic TROs have isomorphic left and right linking algebras.

TROs have come to play a key role in operator space theory, serving as the algebraic model in the category. Recall that the algebraic models for the categories of order-unit spaces, operator systems, and Banach spaces, are respectively Jordan C^*-algebras, C^*-algebras, and JB^*-triples. Indeed, for TROs, a ternary isomorphism is the same as a complete isometry.

If v is a partial isometry in a JC^*-triple $M \subset B(H,K)$, then the projections $l = vv^* \in B(K)$ and $r = v^*v \in B(H)$ give rise to (Peirce) projections $P_k(v): M \to M$, $k = 2,1,0$, as follows; for $x \in M$,

$$P_2(v)x = lxr, \qquad P_1(v)x = lx(1-r) + (1-l)xr, \qquad P_0(v)x = (1-l)x(1-r).$$

The projections $P_k(v)$ are contractive, and their ranges, called Peirce spaces and denoted by $M_k(v)$, are JC^*-subtriples of M satisfying $M = M_2(v) \oplus M_1(v) \oplus M_0(v)$.

A partial isometry v is said to be *minimal* in M if $M_2(v) = \mathbf{C}v$. This is equivalent to v not being the sum of two non-zero orthogonal partial isometries. Recall that two partial isometries v and w (or any two Hilbert space operators) are orthogonal if $v^*w = vw^* = 0$. Orthogonality of partial isometries v and w is equivalent to $v \in M_0(w)$ and will be denoted by $v \perp w$. Each finite-dimensional JC^*-triple is the linear span of its minimal partial isometries. More generally, a JC^*-triple is defined to be *atomic* if it is the weak closure of the span of its minimal partial isometries. In this case, it has a predual and is called a *JW^*-triple*. The *rank* of a JC^*-triple is the maximum number of mutually orthogonal minimal partial isometries. For example, the rank of the Cartan factor $B(H,K)$ of type 1 is the minimum of the dimensions of H and K; and the rank of the Cartan factor of type 4 (spin factor) is 2.

In a JC^*-triple, there is a natural ordering on partial isometries. We write $v \leqslant w$ if $vw^*v = v$; this is equivalent to $vv^* \leqslant ww^*$ and $v^*v \leqslant w^*w$. Moreover, if $v \leqslant w$, then there exists a partial isometry v' orthogonal to v with $w = v + v'$.

Another relation between two partial isometries that we shall need is defined in terms of the Peirce spaces as follows. Two partial isometries v and w are said to be *collinear* if $v \in M_1(w)$ and $w \in M_1(v)$, notation $v \top w$. Let u, v, w be partial isometries. The following is part of [13, Lemma 5.4], and is referred to as "*hopping*": if v and w are each collinear with u, then $uu^*vw^* = vw^*uu^*$ and $u^*uv^*w = v^*wu^*u$; if u, v, w are mutually collinear partial isometries, then $\{uvw\} = 0$.

JC*-triples of arbitrary dimension occur naturally in functional analysis and in holomorphy. A special case of a theorem of Friedman and Russo [8, Theorem 2] states that if P is a contractive projection on a C^*-algebra A, then there is a linear isometry of the range $P(A)$ of P onto a *JC**-subtriple of A^{**}. A special case of a theorem of Kaup [12] gives a bijective correspondence between Cartan factors and irreducible bounded symmetric domains in complex Banach spaces.

Contractive projections play a ubiquitous role in the structure theory of the abstract analog of *JC**-triples (called *JB**-triples). Of use to us will be both of the following two conditional expectation formulas for a contractive projection P on a *JC**-triple M (which are valid for *JB**-triples) [7, Corollary 1]:

$$P\{Px, Py, Pz\} = P\{Px, Py, z\} = P\{Px, y, Pz\} \quad (x, y, z \in M). \tag{3}$$

By a special case of [4, Corollary, p. 308], every *JW**-triple of rank one is isometric to a Hilbert space and every maximal collinear family of partial isometries corresponds to an orthonormal basis. Conversely, every Hilbert space with the abstract triple product $\{xyz\} := ((x|y)z + (z|y)x)/2$ can be realized as a *JC**-triple of rank one in which every orthonormal basis forms a maximal family of mutually collinear minimal partial isometries. Recall from [13, p. 2245] that minimality implies that for all $i \neq j$, $\{u_i u_j u_i\} = 0$, and collinearity implies that for all $i \neq j$, $\{u_i u_i u_j\} = u_j/2$.

2. Operator space structure of Hilbertian *JC**-triples

2.1. Hilbertian *JC**-triples: The spaces $H_\infty^{m,R}$ and $H_\infty^{m,L}$

The general setting for this section will be the following: Y is a *JC**-subtriple of $B(H)$ which is Hilbertian in the operator space structure arising from $B(H)$, and $\{u_i\colon i \in \Omega\}$ is an orthonormal basis consisting of a maximal family of mutually collinear partial isometries of Y.

We let T and A denote the TRO and the C^*-algebra respectively generated by Y. For any subset $G \subset \Omega$, $(uu^*)_G := \prod_{i\in G} u_i u_i^*$ and $(u^*u)_G := \prod_{i\in G} u_i^* u_i$. By [13, Lemma 5.4(a)], these products do not depend on the order. The elements $(uu^*)_G$ and $(u^*u)_G$ lie in the weak closure of A and more generally in the left and right linking von Neumann algebras of T.

In the following lemma, parts (a) and (a′) justify the definitions of the integers m_R and m_L in parts (b) and (b′). Here and throughout the rest of this paper, $|F|$ denotes the cardinality of the finite set F.

Lemma 2.1. *Let Y be an Hilbertian operator space which is a JC*-subtriple of $B(H)$ and let $\{u_i\colon i \in \Omega\}$ be an orthonormal basis consisting of a maximal family of mutually collinear partial isometries of Y.*

(a) *If $(uu^*)_{\Omega-F} = 0$ for some finite set $F \subset \Omega$, then $(uu^*)_{\Omega-G} = 0$ for every finite set G with the same cardinality as F.*
(a′) *If $(u^*u)_{\Omega-F} = 0$ for some finite set $F \subset \Omega$, then $(u^*u)_{\Omega-G} = 0$ for every finite set G with the same cardinality as F.*

(b) *Assume* $(uu^*)_{\Omega-F} \neq 0$ *for some finite set* F. *Let* m_R *be the smallest nonnegative integer with* $(uu^*)_{\Omega-F} \neq 0$ *for every* F *with cardinality* m_R. *Define* $p_R = \sum_{|F|=m_R}(uu^*)_{\Omega-F}$. *Then the maps* $y \mapsto p_R y$ *and* $y \mapsto (1-p_R)y$ *are completely contractive triple isomorphisms* (*hence isometries*) *of* Y *onto rank one subtriples of the weak closure of* T *in* $B(H)$. *Moreover,* $p_R Y \perp (1-p_R)Y$.

(b′) *Assume* $(u^*u)_{\Omega-F} \neq 0$ *for some finite set* F. *Let* m_L *be the smallest nonnegative integer with* $(u^*u)_{\Omega-F} \neq 0$ *for every* F *with cardinality* m_L. *Define* $p_L = \sum_{|F|=m_L}(u^*u)_{\Omega-F}$. *Then the maps* $y \mapsto yp_L$ *and* $y \mapsto y(1-p_L)$ *are completely contractive triple isomorphisms* (*hence isometries*) *of* Y *onto rank one subtriples of the weak closure of* T *in* $B(H)$. *Moreover,* $Yp_L \perp Y(1-p_L)$.

(c) *In case* (b), *let* $w_i = p_R u_i$ *and let* m'_R *be the smallest nonnegative integer with* $(ww^*)_{\Omega-F} \neq 0$ *for all* F *with cardinality* m'_R. *Then* m'_R *exists, and* $m'_R = m_R$. *If* $v_i := (1-p_R)u_i$, *then* $(vv^*)_{\Omega-G} = 0$ *if* $|G| = m_R$. *Furthermore,* $(w^*w)_G \neq 0$ *for* $|G| = m_R + 1$ *and* $(w^*w)_G = 0$ *for* $|G| = m_R + 2$. *Thus, if we define* k_R *to be the largest integer* k *such that* $(w^*w)_G \neq 0$ *for* $|G| = k$, *then* $k_R = m_R + 1$.

(c′) *In case* (b′), *let* $w_i = u_i p_L$ *and let* m'_L *be the smallest nonnegative integer with* $(w^*w)_{\Omega-F} \neq 0$ *for all* F *with cardinality* m'_L. *Then* m'_L *exists, and* $m'_L = m_L$. *If* $v_i := u_i(1-p_L)$, *then* $(v^*v)_{\Omega-G} = 0$ *if* $|G| = m_L$. *Furthermore,* $(ww^*)_G \neq 0$ *for* $|G| = m_L + 1$ *and* $(ww^*)_G = 0$ *for* $|G| = m_L + 2$. *Thus, if we define* k_L *to be the largest integer* k *such that* $(ww^*)_G \neq 0$ *for* $|G| = k$, *then* $k_L = m_L + 1$.

Proof. The proofs of (a) and (a′) are identical to the proof given in [13, Lemma 5.8]. The fact that the set $\Omega - F$ is infinite has no effect on the proof in [13].

The proofs of (b) and (b′) are identical to the proof given in [13, Lemma 5.9]. The facts that the set $\Omega - F$ is infinite and that the sums defining the projections p_R and p_L are infinite have no effect on the proof in [13].

We now prove (c), the proof of (c′) being entirely similar. For any finite set $F \subset \Omega$,

$$(ww^*)_{\Omega-F} = \prod_{i\in\Omega-F}\Bigl(\sum_{|G|=m_R}(uu^*)_{\Omega-G}\Bigr)u_iu_i^*\Bigl(\sum_{|H|=m_R}(uu^*)_{\Omega-H}\Bigr)$$
$$= \prod_{i\in\Omega-F}\Bigl(\sum_{|G|=m_R, i\in\Omega-G}(uu^*)_{\Omega-G}\Bigr) = \sum_{G\subset F, |G|=m_R}(uu^*)_{\Omega-G}.$$

From this it follows that $(ww^*)_{\Omega-F} = 0$ if $|F| < m_R$ and that $(ww^*)_{\Omega-F} = (uu^*)_{\Omega-F} \neq 0$ if $|F| = m_R$. This proves that $m'_R = m_R$, that is $(ww^*)_{\Omega-F} = 0 \Leftrightarrow |F| < m_R$.

Next, we show that $(vv^*)_{\Omega-G} = 0$ if $|G| = m_R$. Since $u_iu_i^*$ commutes with p_R, $(vv^*)_{\Omega-G} = \prod_{i\in\Omega-G}(1-p_R)u_iu_i^*(1-p_R) = (1-p_R)(uu^*)_{\Omega-G}$. However, since $p_R(uu^*)_{\Omega-G} = \sum_{|F|=m_R}(uu^*)_{\Omega-(F\cap G)} = (uu^*)_{\Omega-G}$, $(vv^*)_{\Omega-G} = 0$ as desired.

Finally, we prove the property stated for $(w^*w)_G$. Let $|F| = r$ and for convenience, suppose that $F = \{1, 2, \ldots, r\}$. Then

$$(w^*w)_F = (w^*w)_{\{1,2,\ldots,r\}} = \sum u_1^*(uu^*)_{\Omega-F_1}u_1u_2^*(uu^*)_{\Omega-F_2}u_2u_3^*\ldots u_r^*(uu^*)_{\Omega-F_r}u_r,$$

where the sum is over all $|F_j| = m_R$, $j \in \Omega - F_j$, $F - \{j\} \subset F_j$ (by "hopping"), and $j = 1, 2, \ldots, r$. Every term in this sum is zero if $r - 1 > m_R$, that is $r \geqslant m_R + 2$. Further, if $r = m_R + 1$, there is only one term, namely, writing m for m_R,

$$x := (w^*w)_{\{1,2,\dots,m+1\}} = u_1^*(uu^*)_{\Omega-\{2,3,\dots,m+1\}}u_1u_2^*(uu^*)_{\Omega-\{1,3,4,\dots,m+1\}}u_2u_3^* \times \cdots$$
$$\times (uu^*)_{\Omega-\{1,2,\dots,m-1,m+1\}}u_mu_{m+1}^*(uu^*)_{\Omega-\{1,2,\dots,m\}}u_{m+1},$$

which by a sequence of "hoppings" becomes

$$x = (u^*u)_{\{1,2,\dots,m\}}u_{m+1}^*(uu^*)_{\Omega-\{1,2,\dots,m+1\}}u_{m+1}.$$

In turn, using the collinearity of the u_k ($1 \leqslant k \leqslant m$) with u_{m+1}, we have

$$\begin{aligned}
x &= (uu^*)_{1,\dots,m-1}(u_{m+1}^* - u_{m+1}^*u_mu_m^*)(uu^*)_{\Omega-\{1,2,\dots,m+1\}}u_{m+1}\\
&= (uu^*)_{1,\dots,m-1}u_{m+1}^*(uu^*)_{\Omega-\{1,2,\dots,m+1\}}u_{m+1}\\
&\quad - (uu^*)_{1,\dots,m-1}u_{m+1}^*(uu^*)_{\Omega-\{1,2,\dots,m-1\}}u_{m+1}\\
&= (uu^*)_{1,\dots,m-1}u_{m+1}^*(uu^*)_{\Omega-\{1,2,\dots,m+1\}}u_{m+1} + 0\\
&\ \ \vdots\\
&= u_{m+1}^*(uu^*)_{\Omega-\{1,2,\dots,m+1\}}u_{m+1}.
\end{aligned}$$

Thus, if $x = 0$, then $0 = u_{m+1}xu_{m+1}^* = (uu^*)_{\Omega-\{1,2,\dots,m\}}$, a contradiction. □

Our goal for the remainder of this section is to give a completely isometric representation for the spaces p_RY and Yp_L in parts (b) and (b′) of Lemma 2.1. This will be achieved via a coordinatization procedure which we now describe.

In the following, let us restrict to the special case that Y is a Hilbertian JC^*-triple which satisfies the properties of p_RY in Lemma 2.1 part (c). For notational convenience, let $m = m_R$. Thus $(u^*u)_G \neq 0$ for $|G| \leqslant m+1$ and $(u^*u)_G = 0$ for $|G| \geqslant m+2$.

Analogous to [13, Definition 6.1], where Ω was a finite set, we are going to define elements which are indexed by an arbitrary pair of subsets I, J of Ω satisfying

$$|\Omega - I| = m+1, \qquad |J| = m. \tag{4}$$

The set $I \cap J$ is finite, and if $|I \cap J| = s \geqslant 0$, then $|(I \cup J)^c| = s+1$. Let us write $I \cap J = \{d_1, \dots, d_s\}$ and $(I \cup J)^c = \{c_1, \dots, c_{s+1}\}$, and let us agree (for the moment) that there is a natural linear ordering on Ω such that $c_1 < c_2 < \cdots < c_{s+1}$ and $d_1 < d_2 < \cdots < d_s$.

With the above notation, we define

$$u_{IJ} = u_{I,J} = (uu^*)_{I-J}u_{c_1}u_{d_1}^*u_{c_2}u_{d_2}^* \dots u_{c_s}u_{d_s}^*u_{c_{s+1}}(u^*u)_{J-I}. \tag{5}$$

Note that in general $I - J$ is infinite and $J - I$ is finite so that $u_{I,J}$ lies in the weak closure of T. In Definition 1, we shall consider similar elements in which I is finite and J is infinite.

In the special case of (5) where $I \cap J = \emptyset$, we have $s = 0$ and $u_{I,J}$ has the form

$$u_{I,J} = (uu^*)_I u_c (u^*u)_J, \tag{6}$$

where $I \cup J \cup \{c\} = \Omega$ is a partition of Ω. As in [13], we call such an element a "one," and denote it sometimes for emphasis by $u_{I,c,J}$.

The proof of the following lemma, which is the analog of [13, Lemma 6.6], is complicated by the fact that the sets I are infinite if Ω is infinite.

Lemma 2.2. *Let Y be an Hilbertian operator space which is a JC^*-subtriple of $B(H)$ and let $\{u_i\colon i \in \Omega\}$ be an orthonormal basis consisting of a maximal family of mutually collinear partial isometries of Y. Assume that Y satisfies the properties of $p_R Y$ in Lemma* 2.1 *part* (c) *with $m = m_R$, that is, $(u^*u)_G \neq 0 \Leftrightarrow |G| \leqslant m+1$, and $(uu^*)_{\Omega-F} = 0 \Leftrightarrow |F| < m$. For any $c \in \Omega$,*

$$u_c = \sum_{I,J} u_{I,c,J}, \tag{7}$$

where the sum is taken over all disjoint I, J satisfying (4) *and not containing c. The sum converges weakly in the weak closure of T. (Since $\{I, \{c\}, J\}$ is a partition of Ω, I is determined by J, so the sum is over all sets J of cardinality m and not containing c.)*

Proof. The proof of [13, Proposition 6.3] remains valid in our context insofar as $\{u_{I,J}\}$ (where $u_{I,J}$ denotes $u_{I,c,J}$) is a collection of pairwise orthogonal partial isometries in the weak closure of the ternary envelope T of Y. Since $u_c^*u_c$ commutes with $(u^*u)_J$, $u_{I,J}^*u_{I,J}u_c^*u_c = u_{I,J}^*u_{I,J}$, so that $u_{I,J}^*u_{I,J} \leqslant u_c^*u_c$ and similarly $u_{I,J}u_{I,J}^* \leqslant u_cu_c^*$ so that $\sum u_{I,J} \leqslant u_c$. To prove (7), we proceed as follows.

By the property of the ordering for partial isometries (see Section 1.2), we can write $u_c = v_c + w_c$, where $w_c = \sum_{I,J} Iu_cJ$ and, for example, Iu_cJ is shorthand for $u_{I,c,J} = (uu^*)_I u_c (u^*u)_J$, and v_c is a partial isometry orthogonal to w_c. We shall show that $v_c = 0$ which will prove the lemma. From collinearity and the definition of m, for any $k \in I$, $(uu^*)_I u_c (u^*u)_J = (uu^*)_{I-\{k\}}(u_c - u_cu_k^*u_k)(u^*u)_J = (uu^*)_{I-\{k\}}u_c(u^*u)_J$. By repeating this and taking a limit, we have $Iu_cJ = u_cJ$. Also, $w_cJ = (\sum_{I',J'} I'u_cJ')J = Iu_cJ$. Thus $v_cJ = u_cJ - w_cJ = 0$ for any $c \notin J$. Similarly $Iv_c = 0$ for any $c \notin I$.

From this, it follows that v_i is orthogonal to w_j for every i and j. For $i = j$ this is by the definition. For $i \neq j$, we have $v_i(Ju_j^*I) = (v_iJ)u_j^*I = 0$ if $i \notin J$. However, if $i \in J$, then

$$v_iJu_j^*I = v_i\big(J - \{i\}\big)u_i^*u_iu_j^*I = v_i\big(\big(J - \{i\}\big) \cup \{j\}\big)u_i^*u_iu_j^*I = 0$$

since $i \notin (J - \{i\}) \cup \{j\}$. Similarly, $Ju_j^*Iv_i = 0$ so that $v_iw_j^* = w_j^*v_i = 0$.

Next, we observe that for each pair $i \neq j$, $v_i \in M_1(u_j)$. Indeed,

$$\begin{aligned} v_i + w_i = u_i &= u_iu_j^*u_j + u_ju_j^*u_i \\ &= v_iu_j^*u_j + w_iu_j^*u_j + u_ju_j^*v_i + u_ju_j^*w_i \\ &= v_iu_j^*u_j + \sum_{J,\, j\in J} w_iJ + u_ju_j^*v_i + \sum_{J,\, j\notin J} w_iJ \\ &= v_iu_j^*u_j + u_ju_j^*v_i + w_i. \end{aligned}$$

Hence $v_i = v_iu_j^*u_j + u_ju_j^*v_i$, so $v_i \in M_1(u_j)$.

It now follows that $v_i \top v_j$ for $i \neq j$. Indeed,

$$v_j = v_j u_i^* u_i + u_i u_i^* v_j = v_j v_i^* v_i + v_j w_i^* w_i + v_i v_i^* v_j + w_i w_i^* v_j = v_j v_i^* v_i + v_i v_i^* v_j.$$

We are now ready to show that $v_j = 0$ for every $j \in \Omega$. Since $v_j \top v_i$, $v_j = v_i v_i^* v_j + v_j v_i^* v_i$ so it suffices to show that $v_i \perp v_j$ for $i \neq j$.

Let us adopt the notation J_v for $(v^*v)_J = \prod_{j\in J} v_j^* v_j$. (What we previously denoted by J, namely $(u^*u)_J$, would now be denoted by J_u.) We know that for $i \notin J$, $v_i J_u = 0$. But $v_i J_v = v_i J_u$, so that $v_i J_v = 0$; indeed, for any $k \neq i$, $v_i v_k^* v_k = v_i(u_k^* u_k - w_k^* u_k - u_k^* w_k + w_k^* w_k) = v_i(u_k^* u_k - u_k^* w_k)$ so it suffices to observe that $v_i u_k^* w_k = v_i(v_k^* - w_k^*) w_k = 0$. Thus $v_i J_v = 0$ for all sets J of cardinality m and all $i \notin J$. We now show that in fact $v_i J'_v = 0$ whenever $J' \subset \Omega$, $1 \leqslant |J'| < m$ and $i \notin J'$. Then taking $|J'| = 1$ yields the desired orthogonality. Suppose first that $|J'| = m-1$ and $i \notin J'$. In the first place, $v_i J'_v = I'_v v_i J'_v$, where, of course, $I' = \Omega - (J' \cup \{i\})$. Indeed, writing $I' = \{i_\alpha\colon \alpha \in \Lambda\}$, we have

$$v_i J'_v = v_i\big(v^*v\big)_{J'} = \big(v_{i_\alpha} v_{i_\alpha}^* v_i + v_i v_{i_\alpha}^* v_{i_\alpha}\big)\big(v^*v\big)_{J'} = v_{i_\alpha} v_{i_\alpha}^* v_i \big(v^*v\big)_{J'} = \cdots = I'_v v_i J'_v.$$

In the second place, by the orthogonality of v_j and w_j for any j,

$$I'_u u_j = \big(uu^*\big)_{I'} u_j = \left[\prod_{i\in I'} (v_i + w_i)\big(v_i^* + w_i^*\big)\right](v_j + w_j) = I'_v v_j + I'_w w_j.$$

In particular, for $j = i \notin J'$, $I'_u u_i = (uu^*)_{I'\cup\{i\}} u_i = 0$ since $|\Omega - (I' \cup \{i\})| < m$. On the other hand, $I'_v v_j$ and $I'_w w_j$ are easily seen to be orthogonal partial isometries. Thus $I'_v v_i = 0$ and hence $v_i J'_v = 0$. The same argument leads successively to $v_i J'_v = 0$ for $|J'|$ equal to $m-2, m-3, \ldots, 1$ as long as $i \notin J'$. As noted above, with $|J'| = 1$, we have $v_i v_j^* = 0$ for all $i \neq j$, that is, $v_i \perp v_j$. As also noted above, this implies $v_j = 0$ for all j. $\square$

We shall now assume that our set Ω is countable and for convenience set $\Omega = \mathbb{N} = \{1, 2, 3, \ldots\}$ with its natural order. Note that in this case, the number of possible sets I in (4) is $\aleph_0$ and the number of such J is also $\aleph_0$.

Again as in [13, Definition 6.8], we are going to assign a *signature* to each "one" $u_{I,k,J}$ (see (6)) as follows. Let the elements of I be $i_1 < i_2 < \cdots$ and the elements of J be $j_1 < j_2 < \cdots < j_m$. The permutation taking the infinite tuple $(j_1, \ldots, j_m, k, i_1, i_2, \ldots)$ onto $(1, 2, \ldots)$ actually moves only finitely many elements. Then $\epsilon(I, k, J)$ is defined to be the signature of this permutation. Note that, unlike in the context of [13], one of our sets here is infinite, and this infinite set must be placed at the end of the permutation in order to have a well-defined signature. This adjustment is not necessary in the definition of the spaces $H_\infty^{m,L}$ (see Definition 1) since the set I in that case is finite.

The proof of [13, Lemma 6.7] shows that every element $u_{I,J}$ decomposes uniquely into a product of "ones." The signature $\epsilon(I, J)$ (also denoted by $\epsilon(IJ)$) of $u_{I,J}$ is then defined, as in [13, Definition 6.8], to be the product of the signatures of the factors in this decomposition. The proof of [13, Proposition 6.10] shows that the family $\{\epsilon(IJ)u_{I,J}\}$ forms a rectangular grid (see [13, p. 2239] for the definition) which satisfies the extra property

$$\epsilon(IJ)u_{IJ}\big[\epsilon(IJ')u_{IJ'}\big]^* \epsilon(I'J')u_{I'J'} = \epsilon(I'J)u_{I'J}. \tag{8}$$

It follows as in [13] that the map $\epsilon(IJ)u_{IJ} \to E_{JI}$ is a ternary isomorphism (and hence complete isometry) from the norm closure of $\mathrm{sp}_C u_{IJ}$ to the norm closure of $\mathrm{sp}_C\{E_{JI}\}$, where E_{JI} denotes an elementary matrix, whose rows and columns are indexed by the sets J and I, with a 1 in the (J, I)-position. By [4, Lemma 1.14], this map can be extended to a ternary isomorphism from the w*-closure of $\mathrm{sp}_C u_{IJ}$ onto the factor of type I consisting of all $\aleph_0$ by $\aleph_0$ complex matrices which act as bounded operators on ℓ_2. By restriction to Y and (7), Y is completely isometric to a subtriple $\tilde{Y}$, of this Cartan factor of type 1.

Definition 1. We shall denote the space $\tilde{Y}$ above by $H_\infty^{m,R}$. An entirely symmetric argument (with J infinite and I finite) under the assumption that Y satisfies the conditions of Yp_L in Lemma 2.1 part (c′) with $m = m_L$ defines the space $H_\infty^{m,L}$.

Explicitly,

$$H_\infty^{m,R} = \overline{\mathrm{sp}}_C\left\{b_i^{m,R} = \sum_{I\cap J=\emptyset,\,(I\cup J)^c=\{i\},\,|J|=m} \epsilon(I,i,J)E_{J,I}\colon i\in\mathbb{N}\right\}$$

and

$$H_\infty^{m,L} = \overline{\mathrm{sp}}_C\left\{b_i^{m,L} = \sum_{I\cap J=\emptyset,\,(I\cup J)^c=\{i\},\,|I|=m} \epsilon(I,i,J)E_{J,I}\colon i\in\mathbb{N}\right\}.$$

As noted above, in the case of $H_\infty^{m,L}$, the signature $\epsilon(I,i,J)$ is defined to be the signature of the permutation taking $(i_1,\ldots,i_m,k,j_1,j_2,\ldots)$ onto $(1,2,\ldots)$.

This discussion has proved the following lemma.

Lemma 2.3. *The spaces p_RY and Yp_L in Lemma* 2.1 *parts* (c) *and* (c′) *are completely isometric to $H_\infty^{m_R,R}$ and $H_\infty^{m_L,L}$, respectively.*

Remark 2.4. It is immediate from [13, Corollary 5.3] that $H_\infty^{0,R} = R$ and $H_\infty^{0,L} = C$. Also note that $H_\infty^{m,R}$ and $H_\infty^{m,L}$ are homogeneous Hilbertian operator spaces by Lemma 1.1 and [14, Theorem 1]. The results of Section 4 show that these spaces are all distinct from each other and from Φ.

2.2. The coordinatization of Hilbertian JC-triples*

Let Y satisfy the hypothesis of Lemma 2.1. Our analysis will consider the following three mutually exhaustive and (by the results of Section 4) mutually exclusive possibilities (in each case, the set F is allowed to be empty):

Case 1. $(uu^*)_{\Omega-F} \neq 0$ for some finite set $F \subset \Omega$.
Case 2. $(u^*u)_{\Omega-F} \neq 0$ for some finite set $F \subset \Omega$.
Case 3. $(uu^*)_{\Omega-F} = (u^*u)_{\Omega-F} = 0$ for all finite subsets F of Ω.

We will first address cases 1 and 2. The following proposition summarizes these cases and is stated here for easy reference in the proof of Theorem 1. Part (a) follows from Lemma 2.1(b) and (c), the coordinatization procedure outlined in Section 2.1, and Lemma 2.3. Part (b) follows by symmetry using Lemma 2.1(b′) and (c′), and Lemma 2.3.

Proposition 2.5. *Let Y be a separable infinite-dimensional Hilbertian operator space which is a JC^*-subtriple of $B(H)$ and let $\{u_i: i \in \Omega\}$ $(\Omega = \mathbb{N})$ be an orthonormal basis consisting of a maximal family of mutually collinear partial isometries of Y.*

(a) *Suppose there exists a finite subset F of Ω such that $(uu^*)_{\Omega-F} \neq 0$. Then Y is an intersection $Y_1 \cap Y_2$ such that Y_1 is completely isometric to a space $H_\infty^{m,R}$ (that is, in the notation of Lemma 2.1, $m'_R = m \geqslant 1$ and $k'_R = m+1$), or C, and Y_2 is a Hilbertian JC^*-triple containing an orthonormal basis of mutually collinear minimal partial isometries $\{v_i\}$ with $(vv^*)_{\Omega-F} = 0$ for all sets F of cardinality m.*
(b) *Suppose there exists a finite subset F such that $(u^*u)_{\Omega-F} \neq 0$. Then Y is an intersection $Y_1 \cap Y_2$ such that Y_1 is completely isometric to a space $H_\infty^{m,L}$ (that is, $m'_L = m \geqslant 1$ and $k'_L = m+1$), or R, and Y_2 is an Hilbertian JC^*-triple containing an orthonormal basis of mutually collinear minimal partial isometries $\{v_i\}$ with $(v^*v)_{\Omega-F} = 0$ for all sets F of cardinality m.*

It is worth emphasizing that the space $H_\infty^{m,R}$ (respectively $H_\infty^{m,L}$) is a Hilbertian JC^*-triple satisfying

$$\begin{aligned}&(uu^*)_{\Omega-F} = 0 \quad\Leftrightarrow\quad |F| < m, \qquad (u^*u)_G = 0 \quad\Leftrightarrow\quad |G| > m+1\\&\big(\text{respectively}\quad (u^*u)_{\Omega-F} = 0 \quad\Leftrightarrow\quad |F| < m, \qquad (uu^*)_G = 0 \quad\Leftrightarrow\quad |G| > m+1\big),\end{aligned} \tag{9}$$

and that $H_\infty^{0,R} = R$ and $H_\infty^{0,L} = C$.

To handle the remaining case 3, we shall need the following lemma.

Lemma 2.6. *Let Y be a separable infinite-dimensional Hilbertian operator space which is a JC^*-subtriple of a C^*-algebra A and let $\{u_i: i \in \Omega\}$ be an orthonormal basis consisting of a maximal family of mutually collinear partial isometries of Y.*

Let S and T be finite subsets of Ω and let $k \in \Omega - (S \cup T)$. If $(uu^*)_S u_k (u^*u)_T = 0$, then $(uu^*)_{S'} u_{k'} (u^*u)_{T'} = 0$ for all sets S', T' with $|S'| = |S|$, $|T'| = |T|$ and for all $k' \in \Omega - (S' \cup T')$.

Proof. It suffices to prove this with (S,k,T) replaced in turn by $(S \cup \{l\} - \{j\}, k, T)$ (with $l \notin S$ and $j \in S$); by (S,l,T) (with $l \neq k$); and by $(S,k,T \cup \{l\} - \{i\})$ (with $l \notin T$ and $i \in T$).

In the first case,

$$\begin{aligned}u_l u_l^* (uu^*)_{S-j} u_k (u^*u)_T &= (u_j u_j^* u_l + u_l u_j^* u_j) u_l^* (uu^*)_{S-j} u_k (u^*u)_T\\&= 0 + u_l u_j^* u_j u_l^* (uu^*)_{S-j} u_k (u^*u)_T\\&= u_l u_j^* (uu^*)_{S-j} u_j u_l^* u_k (u^*u)_T \quad \text{(by hopping)}\\&= -u_l u_j^* (uu^*)_{S-j} u_k u_l^* u_j (u^*u)_T\\&= -u_l u_j^* (uu^*)_{S-j} u_k (u^*u)_T u_l^* u_j = 0.\end{aligned}$$

By symmetry, $(uu^*)_S u_k (u^*u)_{T-i} u_l^* u_l = 0$, proving the second case.

Finally,

$$\begin{aligned}(uu^*)_S u_l (u^*u)_T &= (uu^*)_S (u_l u_k^* u_k + u_k u_k^* u_l)(u^*u)_T \\ &= (uu^*)_S u_l u_k^* u_k (u^*u)_T + (uu^*)_S u_k u_k^* u_l (u^*u)_T \\ &= u_l u_k^* (uu^*)_S u_k (u^*u)_T + (uu^*)_S u_k (u^*u)_T u_k^* u_l = 0. \qquad \square\end{aligned}$$

We can now handle the final case 3.

Proposition 2.7. *Let Y be a separable infinite-dimensional Hilbertian JC^*-triple and let $\{u_i\colon i \in \Omega\}$ be an orthonormal basis consisting of a maximal family of mutually collinear partial isometries of Y. Suppose that for all finite subsets $G \subset \Omega$, $(uu^*)_{\Omega-G} = 0$ and $(u^*u)_{\Omega-G} = 0$. Then Y is completely isometric to Φ.*

Proof. We show first that all finite products $(uu^*)_F u_i (u^*u)_G$ with $F, G, \{i\}$ pairwise disjoint (and F, G not both empty), are not zero. Suppose, on the contrary, that $(uu^*)_F u_i (u^*u)_G = 0$ for some F, G, i. If F and G are both non-empty, pick a subset $F' \subset F$ of maximal cardinality such that $(uu^*)_{F'} u_i (u^*u)_G \neq 0$ (F' could be empty). Then by repeated use of collinearity and passing to the limit, we arrive at $(uu^*)_{F'} u_i (u^*u)_G = (uu^*)_{F'} u_i (u^*u)_{\Omega-(\{i\}\cup F')} = 0$, a contradiction. So either $F = \emptyset$ and $u_i(u^*u)_G = 0$, or $G = \emptyset$ and $(u^*u)_F u_i = 0$. In the first case, picking a subset $G' \subset G$ of maximal cardinality such that $u_i(u^*u)_{G'} \neq 0$, then by collinearity $u_i(u^*u)_{G'} = (uu^*)_{\Omega-(\{i\}\cup G')} u_i (u^*u)_{G'} = 0$, a contradiction, and similarly in the second case. We have now shown that all finite products $(uu^*)_F u_i (u^*u)_G$ with $F, G, \{i\}$ pairwise disjoint, are not zero.

Now consider the space $Y_n := \mathrm{sp}\{u_1, \ldots, u_n\}$. By [13, Theorem 3(b)], Y_n is completely isometric to a space $H_n^{k_1} \cap \cdots \cap H_n^{k_m}$, where $n \geqslant k_1 > \cdots > k_m \geqslant 1$. We claim that $m = n$ and $k_j = n - j + 1$ for $j = 1, \ldots, n$. By way of contradiction, suppose that there is a k, $1 \leqslant k \leqslant n$, such that the space H_n^k is not among the spaces $H_n^{k_j}$, $1 \leqslant j \leqslant m$. Let $\psi : x \mapsto (x^{(k_1)}, \ldots, x^{(k_m)})$ denote the ternary isomorphism of the ternary envelope of Y_n whose restriction to Y_n implements the complete isometry of Y_n with $H_n^{k_1} \cap \cdots \cap H_n^{k_m}$, and consider the element $x := (uu^*)_{\{1,\ldots,k-1\}} u_k (u^*u)_{\{k+1,\ldots,n\}}$. As shown above, $x \neq 0$. However, $x^{(k_j)} = 0$ for each j, a contradiction. To see that $x^{(k_j)} = 0$, suppose first that $k_j < k$. Since ψ is a ternary isomorphism, $x^{(k_j)} = (u^{(k_j)} u^{(k_j)*})_{\{1,\ldots,k-1\}} u_k^{(k_j)} (u^{(k_j)*} u^{(k_j)})_{\{k+1,\ldots,n\}} = 0$ since $(u^{(k_j)} u^{(k_j)*})_{\{1,\ldots,k-1\}} u_k^{(k_j)}$ is zero in $H_n^{k_j}$. Similarly, if $k_j > k$, then $n - k_j + 1 < n - k + 1$, $u_k^{(k_j)} (u^{(k_j)*} u^{(k_j)})_{\{k+1,\ldots,n\}} = 0$ so that $x^{(k_j)} = 0$ in this case as well.

We now have for each n that, completely isometrically, $Y_n = \bigcap_{k=1}^n H_n^k$ and the latter space is completely isometric to Φ_n by [14, Lemma 2.1]. Since $Y = \overline{\bigcup Y_n}$ and $\Phi = \overline{\bigcup \Phi_n}$, it follows that $Y = \Phi$ completely isometrically. $\square$

We come now to the first main result of this paper.

Theorem 1. *Let Y be a JC^*-subtriple of $B(H)$ which is a separable infinite-dimensional Hilbertian operator space. Then Y is completely isometric to one of the following spaces*:

$$\Phi, \qquad H_\infty^{m,R}, \qquad H_\infty^{m,L}, \qquad H_\infty^{m,R} \cap H_\infty^{n,L}.$$

Proof. Let $\{u_i\}$ be an orthonormal basis for Y consisting of a maximal family of mutually collinear minimal partial isometries. By Propositions 2.7 and 2.5, either Y is completely isometric to Φ, in which case the theorem is proved, or Y is an intersection $Y_1 \cap Z$, where Y_1 is completely isometric to either $H_\infty^{m,R}$ or $H_\infty^{k,L}$, with $m, k \geqslant 0$.

Let us suppose first that $Y_1 = H_\infty^{m,R}$ and denote m by m_1. (The other case is handled in the same way.) For convenience, we shall say that $m_R = \infty$ if $(uu^*)_{\Omega - F} = 0$ for every finite subset $F \subset \Omega$. Then by repeated application of Proposition 2.5, $Y = \mathcal{R} \cap Z$, where $\mathcal{R} = H_\infty^{m_1,R} \cap \cdots \cap H_\infty^{m_n,R}$ or $H_\infty^{m_1,R} \cap \cdots \cap H_\infty^{m_n,R} \cap \cdots$, where $m_1 < m_2 < \cdots$ and the m_R for Z is ∞. This is clear in the case that the intersection is finite. In the other case, by the construction in the proof of Proposition 2.5, $Z = \prod_{j=1}^\infty (1 - p_j)Y$ for an appropriate sequence of projections p_j (see Lemma 2.1).

Again by Propositions 2.7 and 2.5, either Z is completely isometric to Φ, or Z is an intersection $Z_1 \cap W$, where Z_1 is completely isometric to $H_\infty^{k,L}$, for some $k \geqslant 0$. Again, by repeated application of Proposition 2.5, $Z = \mathcal{L} \cap W$, where $\mathcal{L} = H_\infty^{k_1,L} \cap \cdots \cap H_\infty^{k_l,L}$ or $H_\infty^{k_1,L} \cap \cdots \cap H_\infty^{k_l,L} \cap \cdots$, where $k_1 < k_2 < \cdots$ and the m_L for W is ∞. Since W has $m_R = \infty$ and $m_L = \infty$, by Proposition 2.7, it is completely isometric to Φ. We now have $Y = \mathcal{R} \cap \mathcal{L} \cap \Phi$.

An argument similar to the one in the proof of Proposition 2.7 shows that an n-dimensional subspace $H_n^{m,L}$ of $H_\infty^{m,L}$ is completely isometric to $\bigcap_{k=1}^{m+1} H_n^k$ and hence the formal identity map $H_\infty^{m',L} \to H_\infty^{m,L}$ is a complete contraction when $m' > m$ (and similarly for $H_\infty^{k,R}$). On the other hand, from Lemma 4.1, Φ is completely isometric to $\bigcap_{m=0}^\infty H_\infty^{m,L}$ and to $\bigcap_{m=1}^\infty H_\infty^{m,R}$. Thus Φ is completely isometric to $\bigcap_{j=1}^\infty H_\infty^{n_j,L}$ for any sequence $n_j \to \infty$, and from this the theorem follows. □

Remark 2.8. As just noted, in Section 4 we will show that the spaces $H_\infty^{m,L}$ and $H_\infty^{k,R}$ are completely isometric to spaces of creation and annihilation operators on pieces of the anti-symmetric Fock space. Hence all separable rank 1 JC^*-triples are really spaces of creation and annihilation operators.

We close this section with a well-known lemma about Hilbertian TROs. Recall that TROs are operator subspaces of a C^*-algebra which are closed under the product xy^*z, and are fundamental in operator space theory. Indeed, every operator space has both a canonical injective envelope [18] and a canonical "Shilov boundary" [2] which are TROs. A proof of the following lemma can be found in [19], which classifies all W^*-TROs up to complete isometry. We include a quick alternate proof from the point of view of this section.

Lemma 2.9. *If X is a Hilbertian TRO, then X is completely isometric to R or C.*

Proof. Let $\{u_j\}$ be an orthonormal basis consisting of mutually collinear minimal partial isometries in X. For a fixed $i \neq j$, since $u_i u_i^* u_j$ is a partial isometry in X, $u_i u_i^* u_j = P_2(u_j)(u_i u_i^* u_j)$ is either equal to $e^{i\theta} u_j$ or 0. If the latter case holds, then by the calculation in [13, Lemma 5.1], $u_i u_i^* u_j = 0$ for all $i \neq j$, and X is ternary isomorphic and thus completely isometric to C. On the other hand, if $u_i u_i^* u_j = e^{i\theta} u_j$, then by collinearity, $e^{i\theta} = 1$, $u_j u_i^* u_i = 0$, and again by [13, Lemma 5.1], X is completely isometric to R. □

3. Contractively complemented Hilbertian operator spaces

Suppose that a Hilbert space H is complemented in a C^*-algebra A via a contractive projection P. Let L be a contractive linear map from H into A with the properties that $L(H) \perp H$ and $P(L(H)) = 0$. Then the space $K = \{h + L(h)\colon h \in H\}$ is clearly contractively complemented by $P + LP$. From this it follows that a classification of contractively complemented Hilbertian operator spaces is hopeless without some qualifications.

3.1. *Expansions of contractive projections*

The following definitions are crucial.

Definition 2. Consider a triple $\{K, A, P\}$ consisting of a Hilbertian operator space K, a C^*-algebra A, and a contractive projection P from A onto K. If there exists a Hilbertian subspace H of A which is contractively complemented by a projection Q and a contractive linear map L from H into A such that $P = Q + LQ$, $L(H) \perp H$ and $Q(L(H)) = 0$, we say that $\{K, A, P\}$ is an *expansion* of $\{H, A, Q\}$. (Note that this implies that $K = \{h + L(h)\colon h \in H\}$.)

The following is immediate.

Lemma 3.1. *If $\{K, A, P\}$ is an expansion of $\{H, A, Q\}$ then $Q|_K$ is a completely contractive isometry from K onto H.*

Suppose $X \subset A$ is a contractively complemented Hilbertian operator subspace by a projection Q. Further suppose that Y is a Hilbertian operator subspace of A which is isometric to X and which is orthogonal to X and lies in $\ker(Q)$. Then $\{x + Lx\colon x \in X\}$ is contractively complemented in A by the projection $P = Q + LQ$, where L is any isometry from X onto Y. It is clear that $\{x + Lx\colon x \in X\}$ is an expansion of X. Thus one cannot hope to classify contractively complemented Hilbertian operator spaces up to complete isometry. However, we will show in this section that all contractively complemented Hilbert spaces are expansions of a “minimal” 1-complemented Hilbert space which is a JC^*-triple.

Definition 3. The *support partial isometry* of a non-zero element ψ of the predual A_* of a JW^*-triple A is the smallest element of the set of partial isometries v such that $\psi(v) = \|\psi\|$, and is denoted by v_ψ. For each non-empty subset G of A_*, the *support space* $s(G)$ of G is the smallest weak*-closed subspace of A containing the support partial isometries of all elements of G.

The existence and uniqueness of the support partial isometry was proved and exploited in the more general case of a JB^*-triple (in which case the partial isometries are replaced by their abstract analog, the tripotents) in [9]. One of its important properties is that of “faithfulness”: if a non-zero partial isometry w satisfies $w \leqslant v_\psi$, then $\psi(w) > 0$.

We now give two examples of expansions which naturally occur and are relevant to our work.

Example 1. From [14, Theorem 2], if P is a contractive projection on a C^*-algebra A, with $X := P(A)$ which is isometric to a Hilbert space, then there are projections $p, q \in A^{**}$, such that, $X = P^{**}A^{**} = \{pxq + (1-p)x(1-q)\colon x \in X\}$. The space pXq is exactly the norm closed span of the support partial isometries of the elements of P^*A^* (see [8] for the construction).

The map $\mathbf{E_0}: x \mapsto pxq$ is an isometry of X onto a JC^*-subtriple $\mathbf{E_0}X$ of A^{**}, $\mathbf{E_0}P^{**}$ is a normal contractive projection on A^{**} with range $\mathbf{E_0}X$ and clearly $pXq \perp (1-p)X(1-q)$. It follows that

$$\{X, A^{**}, P^{**}\} \quad \text{is an expansion of} \quad \{\mathbf{E_0}X, A^{**}, \mathbf{E_0}P^{**}\}.$$

Specifically, let $L: \mathbf{E_0}X \to A^{**}$ be the map $pxq \mapsto (1-p)x(1-q)$. Then $P(A) = P^{**}A^{**} = \{pxq + (1-p)x(1-q)\colon x \in P(A)\}$ and $P^{**} = \mathbf{E_0}P^{**} + L\mathbf{E_0}P^{**}$, since if $a \in A^{**}$, there is $x \in A$ with $a = Px = p(Px)q + (1-p)Px(1-q)$ and $\mathbf{E_0}P^{**}a + L\mathbf{E_0}P^{**}a = \mathbf{E_0}Px + L\mathbf{E_0}Px = p(Px)q + (1-p)Px(1-q)$. Finally, if $x \in A$, then $L\mathbf{E_0}Px = (1-p)Px(1-q)$ and $\mathbf{E_0}P^{**}L\mathbf{E_0}Px = \mathbf{E_0}P^{**}((1-p)Px(1-q)) = 0$ by [14, Theorem 2(e)].

Definition 4. The triple $\{\mathbf{E_0}X, A^{**}, \mathbf{E_0}P^{**}\}$ (or simply $\mathbf{E_0}X$) will be called the *support* of $\{X, A^{**}, P^{**}\}$. It is also called the *enveloping support* of $\{X, A, P\}$.

Example 2. It follows from [5] that for a normal contractive projection P (with predual P_*) from a von Neumann algebra (or JW^*-triple) A onto a Hilbert space X, there is a similar projection $\mathbf{E}$ on A such that

$$\{X, A, P\} \quad \text{is an expansion of} \quad \{\mathbf{E}A, A, \mathbf{E}\}$$

and $\mathbf{E}A$ is the norm closure of the span of support partial isometries of elements of P_*A_*.

Indeed, as set forth in [5, Lemma 3.2], $P(A) \subset s(P_*(A_*)) \oplus s(P_*(A_*))^\perp \subset A$, and $\mathbf{E}: A \to A$ is a normal contractive projection onto $s(P_*(A_*))$ given by $\mathbf{E} = \phi \circ P$, where $\phi: P(A) \to s(P_*(A_*))$ is the restriction of the M-projection of $s(P_*(A_*)) \oplus s(P_*(A_*))^\perp$ onto $s(P_*(A_*))$. (Although we will not use these facts, ϕ is a triple isomorphism from $P(A)$ with the triple product $\{xyz\}_{P(A)} := P\{xyz\}$ onto the JW^*-subtriple $s(P_*(A_*))$ of A, and ϕ^{-1} coincides with P on $s(P_*(A_*))$.

The map $L: \mathbf{E}(A) \to \mathbf{E}(A)^\perp$ in this case is given by $L = \phi^\perp \circ \phi^{-1} = \phi^\perp \circ P$, where $\phi^\perp: P(A) \to s(P_*(A_*))^\perp$ is the restriction of the M-projection of $s(P_*(A_*)) \oplus s(P_*(A_*))^\perp$ onto $s(P_*(A_*))^\perp$. Then for $h \in s(P_*(A_*))$, say $h = \phi \circ P(x)$ for some $x \in A$, $h + Lh = \phi(Px) + \phi^\perp(Px) = Px$ so that $P(A) = \{h + Lh\colon h \in s(P_*(A_*))\}$. Furthermore, for $x \in A$, $\mathbf{E}x + L\mathbf{E}x = \phi(Px) + \phi^\perp(Px) = Px$. It is obvious that $L\mathbf{E}(A) \perp \mathbf{E}(A)$. Finally, for $x \in A$, $\mathbf{E}(L\mathbf{E}x) = \phi \circ P(\phi^\perp(Px)) = 0$ since $Px = PPx = P\phi Px + P\phi^\perp Px$ and $\phi \circ Px = (\phi \circ P)^2 x + \phi \circ P(\phi^\perp(Px))$.

Definition 5. By analogy with Example 1, we will call $\{\mathbf{E}A, A, \mathbf{E}\}$ (or simply $\mathbf{E}A$) the *support* of $\{X, A, P\}$ in this case. If $\{X, A, P\}$ is not the expansion of any tuple other than itself, we say that $\{X, A, P\}$ is *essential* and that X is *essentially normally complemented* in A.

A concrete instance of Example 2 is the projection of $B(H)$ onto R (or C). It is easy to see that R and C are essentially normally complemented in $B(H)$, as is $R \cap C$ in $B(H \oplus H)$. (See the paragraph preceding Theorem 3.)

Remark 3.2. If $\{P(A), A, P\}$ is as in Example 1, then $\{P^{**}(A^{**}), A^{**}, P^{**}\}$ is as in Example 2, and the enveloping support of P is the same as the support of P^{**}, since both $\mathbf{E_0}P(A)$ and $\mathbf{E}(A^{**})$ coincide with the norm closed linear span of A^{**} generated by $s(P^*(A^*))$.

Proposition 3.3. *Suppose X is Hilbertian and complemented in a von Neumann algebra A by a normal contractive projection P. Then $\{X, A, P\}$ is essential if and only if it equals its support.*

Proof. Suppose $\{X, A, P\}$ equals its support and is the expansion of $\{Y, A, Q\}$ given by a contractive map L. For each partial isometry $v \in X$, $v = w + z$ where w and z are orthogonal partial isometries, $w = Qv$, $QL = 0$, $z = L(w)$ and $P = Q + LQ$. Suppose v is the support partial isometry of $\psi \in P_*A_*$. Then

$$\psi(v) = \psi(Pv) = \psi\big((Q + LQ)(v)\big) = \psi\big((Q + QLQ)(v)\big) = \psi(Qv) = \psi(w),$$

and hence $w = v$, $L = 0$ and $\{X, A, P\} = \{Y, A, Q\}$. The converse is immediate. □

3.2. Operator space structure of 1-complemented Hilbert spaces

As noted at the beginning of the previous subsection, we cannot classify 1-complemented Hilbert spaces up to complete isometry. However, in Theorem 2, we are able to give a classification up to "trivial" expansion.

We assume in what follows that P is a normal contractive projection on a von Neumann algebra A, whose range $Y = P(A)$ is a JC^*-subtriple of A of rank one, and $\{u_i\}$ is an orthonormal basis for Y consisting of a maximal family of minimal (in Y) collinear partial isometries. We shall assume for convenience that Y is infinite-dimensional and separable. In Theorem 2, we shall also be able to handle the case of a contractive projection on a C^*-algebra.

We know from the proof of Theorem 1 that each element $y \in Y$ has an orthogonal decomposition $y = \sum_n y^{r_n} + \sum_k y^{l_k} + y^{\Phi}$ corresponding to the equality $Y = [\bigcap_n H_\infty^{r_n,R}] \cap [\bigcap_k H_\infty^{l_k,L}] \cap \Phi$. We shall prove the following lemma.

Lemma 3.4. *If $u_j = \sum_n u_j^{r_n} + \sum_k u_j^{l_k} + u_j^{\Phi}$ is the decomposition of u_j into orthogonal partial isometries, then for all j, $u_j^{r_n} = 0$ if $r_n \neq 0$ and $u_j^{l_k} = 0$ if $l_k \neq 0$.*

By this lemma and the proof of Theorem 1, $P(A)$ coincides with the intersection of at most the three spaces R, C, Φ. Together with Examples 1 and 2, Remark 3.2, and Proposition 3.3 in Section 3.1, this proves the second main theorem of this paper.

Theorem 2. *Suppose Y is a separable infinite-dimensional Hilbertian operator space which is contractively complemented (respectively normally contractively complemented) in a C^*-algebra A (respectively W^*-algebra A) by a projection P. Then*

(a) *$\{Y, A^{**}, P^{**}\}$ (respectively $\{Y, A, P\}$) is an expansion of its support $\{H, A^{**}, Q\}$ (respectively $\{H, A^{**}, Q\}$, which is essential).*
(b) *H is contractively complemented in A^{**} (respectively A) by Q and is completely isometric to either R, C, $R \cap C$, or Φ.*

This theorem says that, in A^{**}, Y is the diagonal of a contractively complemented space H which is completely isometric to $R, C, R \cap C$ or Φ and an orthogonal degenerate space K which is in the kernel of P. As pointed out at the beginning of Section 3.1, this is the best possible classification.

It remains to prove Lemma 3.4. We will show that $u_j^{r_n} = 0$ if $r_n \neq 0$ and a similar argument will show that $u_j^{l_k} = 0$ if $l_k \neq 0$.

We again adopt the more compact notation Iu_iJ, used in the proof of Lemma 2.2, for the "one" $(uu^*)_I u_i (u^*u)_J$, and recall that, for example,

$$I^{r_n} u_j^{r_n} J^{r_n} := \left(\prod_{\alpha\in I} u_\alpha^{r_n} u_\alpha^{r_n *}\right) u_j^{r_n} \left(\prod_{\beta\in J} u_\beta^{r_n *} u_\beta^{r_n}\right).$$

We note first that for $j \neq i$, $\{u_j, Iu_iJ, u_j\} = u_j(Iu_iJ)^* u_j = u_j J u_i^* I u_j = u_j(J \cup \{j\}) u_i^* (I \cup \{j\}) u_j = 0$ since $u_j u_i^* u_j = 0$. By one of the conditional expectation formulas in (3),

$$\{u_j, P(Iu_iJ), u_j\} = P(\{u_j, Iu_iJ, u_j\}) = 0.$$

Since every element of Y is in the closed linear span of the u_j, we may write $P(Iu_iJ) = \sum_k \lambda_k^{i,J} u_k$ and thus $0 = \sum_k \overline{\lambda_k^{i,J}} \{u_j u_k u_j\} = \overline{\lambda_j^{i,J}} u_j$. We conclude that $\lambda_j^{i,J} = 0$ for $j \neq i$ and hence $P(Iu_iJ) = \lambda_{i,J} u_i$ for each "one" Iu_iJ, where we have written $\lambda_{i,J}$ for $\lambda_i^{i,J}$.

Now suppose that i is fixed and $k \neq i$, say $k \in J$. Then

$$\begin{aligned}
2\{u_k, u_i, Iu_iJ\} &= u_k u_i^* I u_i J + I u_i J u_i^* u_k = u_k u_i^* I u_i J \quad \text{(by minimality as } k \in J) \\
&= I u_k u_i^* u_i J \quad \text{(by “hopping”)} \\
&= I u_k \big((J - \{k\}) \cup \{i\}\big) = I u_k L,
\end{aligned}$$

where we have written L for $(J - \{k\}) \cup \{i\}$. Then by another conditional expectation formula in (3),

$$\begin{aligned}
\lambda_{k,L} u_k = P(I u_k L) &= P\big(2\{u_k, u_i, Iu_iJ\}\big) = 2\big\{u_k, u_i, P(Iu_iJ)\big\} \\
&= 2\lambda_{i,J} \{u_k u_i u_i\} = \lambda_{i,J} u_k.
\end{aligned}$$

Thus $\lambda_{i,J} = \lambda_{k,(J-\{k\})\cup\{i\}}$ and so $\lambda_{i,J} = \lambda$ is independent of i, J (J of fixed cardinality) such that $i \notin J$. Put another way, $P(Iu_iJ) = \lambda u_i$ for all partitions $I \cup \{j\} \cup J$ of Ω, where λ is a complex number depending only on the cardinality of J.

It is easy to check that for any fixed integer $m \geqslant 0$, $\{Iu_iJ\}$, where $|J| = m$ and $\{I, i, J\}$ runs over the partitions of $\mathbb{N}$, is an orthogonal family of partial isometries with $\sum Iu_iJ \leqslant u_i$. On the other hand, for each n, by Lemma 2.2, $u_i^{r_n} = \sum_{|J|=r_n} I^{r_n} u_i^{r_n} J^{r_n}$ so that

$$u_i^{r_n} = \sum_{|J|=r_n} I^{r_n} u_i^{r_n} J^{r_n} \leqslant \sum_{|J|=r_n} I u_i J \leqslant u_i.$$

Let u_i be the support partial isometry of $\psi_i \in P_*(A_*)$. Then

$$\psi_i\left(\sum_{|J|=r_n} I u_i J\right) = \psi_i\left(\sum_{|J|=r_n} P(I u_i J)\right) = \psi_i\left(\sum_{|J|=r_n} \lambda u_i\right),$$

and so $\lambda = 0$ if $r_n \neq 0$. Thus $\psi_i(u_i^{r_n}) = 0$ and by the faithfulness of ψ_i, $u_i^{r_n} = 0$ if $r_n \neq 0$. □

By [20], the range Y of a completely contractive projection on a C^*-algebra is a TRO. By Lemma 2.9 it follows that, if Y is Hilbertian, Y is completely isometric to R or C. This gives an alternate proof of the result of Robertson [17], stated here for completely contractive projections on a C^*-algebra.

Although Theorem 2 is only a classification modulo expansions, the following lemma shows that it is the correct analogue for contractively complemented Hilbert spaces.

Lemma 3.5. *Suppose that $\{Y, A, P\}$ is an expansion of $\{H, A, Q\}$ and that P is a completely contractive projection. Then Y is completely isometric to H.*

Proof. By definition of expansion, in A^{**}, Y coincides with $\{h + L(h)\colon h \in H\}$, $Q + LQ = P$, $L(H) \perp H$ and $Q(L(H)) = 0$. Thus, $P^{**}|_H$ is a complete contraction from H onto Y with completely contractive inverse $Q|_Y$. Hence, Y is completely isometric to H. □

3.3. An essential contractive projection onto Φ

As noted earlier, the spaces R, C and $R \cap C$ are each essentially normally contractively complemented in a von Neumann algebra. We now proceed to show that the same holds for Φ.

We begin by taking a closer look at the contractive projection $P = P_n^k$ of the ternary envelope $T = T(H_n^k) = M_{p_k,q_k}(\mathbf{C})$ of H_n^k, onto H_n^k. This projection was constructed in [1] as follows:

$$P_n^k x = \frac{1}{\binom{n-1}{k-1}} \sum_{i=1}^{n} \operatorname{tr}\big(xU_i^*\big)U_i, \quad \text{for } x \in T.$$

In this formula, we let U_i denote orthonormal basis given in (1). Thus, $U_i = \sum \epsilon(I, i, J)E_{J,I}$ which is a sum of "ones." Recall that a "one" in this context is an element of the form $\epsilon(I, i, J)E_{J,I}$ with $I \cap J = \emptyset$. Similarly, a non-"one" is an element of the form $\epsilon(I, i, J)E_{J,I}$ with $I \cap J \neq \emptyset$. The "ones" and non-"ones" together form a basis for T.

Lemma 3.6. *The action of $P = P_n^k$ is as follows: if $x = \epsilon(I, J)E_{J,I} \in T$ is not a "one," then $Px = 0$. If $x = \epsilon(I, i, J)E_{J,I}$ is a "one," then*

$$P\big(\epsilon(I, i, J)E_{J,I}\big) = \frac{1}{\binom{n-1}{k-1}} U_i.$$

Proof. Suppose first that $x = \epsilon(I, J)E_{J,I}$ is not a "one," that is, $I \cap J \neq \emptyset$. Then

$$\begin{aligned} xU_i^* &= \epsilon(I, J)E_{J,I} \sum_{I',J'} \epsilon(I', i, J')E_{I',J'} = \epsilon(I, J) \sum_{I',J'} \epsilon(I', i, J')E_{J,I}E_{I',J'} \\ &= \epsilon(I, J) \sum_{J'} \epsilon(I, i, J')E_{J,J'}. \end{aligned}$$

Since $J' \cap I = \emptyset$ and $J \cap I \neq \emptyset$, J' is never equal to J and so $\operatorname{tr}(xU_i^*) = 0$ and $Px = 0$.

Suppose now that $x = \epsilon(I, i, J)E_{J,I} \in T$ is a "one." Then for $1 \leqslant j \leqslant n$,

$$U_j = \sum_{I' \cap J' = \emptyset} \epsilon(I', j, J')E_{J',I'},$$

and as above

$$xU_j^* = \epsilon(I,i,J)\sum_{J'}\epsilon(I,i,J')E_{J,J'}.$$

Thus, $\mathrm{tr}(xU_j^*) = 1$ if $j = i$ and $\mathrm{tr}(xU_j^*) = 0$ if $j \neq i$. It follows that

$$P(x) = \frac{1}{\binom{n-1}{k-1}}\sum_j \mathrm{tr}\bigl(xU_j^*\bigr)U_j = \frac{1}{\binom{n-1}{k-1}}U_i. \qquad \Box$$

We proceed to construct a contractive projection defined on a TRO A which has range Φ. Since every TRO is the corner of a C^*-algebra, we will have constructed a projection on a C^*-algebra with range Φ. Now, let u_i be an orthonormal basis for the Hilbertian operator space Φ and let $H_n = \mathrm{sp}\{u_1,\ldots,u_n\}$. As noted in the proof of Proposition 2.7, $H_n = \Phi_n$ is completely isometric to the intersection $\bigcap_{i=1}^n H_n^i \subset \bigoplus_{k=1}^n T(H_n^i) =$ the ternary envelope $T(H_n)$ of H_n in A.

We construct a contractive projection P^n on $T(H_n)$ with range H_n as follows. For $x = \bigoplus_{i=1}^n x_i \in T(H_n)$, write $x = \sum_{i=1}^n (0 \oplus \cdots \oplus x_i \oplus \cdots \oplus 0)$, ($x_i$ is in the ith-position). Then define

$$P^n(x) = \sum_{i=1}^n P^n(0 \oplus \cdots \oplus x_i \oplus \cdots \oplus 0) := \frac{1}{n}\sum_{i=1}^n \bigl(P_n^i(x_i),\ldots,P_n^i(x_i)\bigr).$$

Note that since $(P_n^i(x_i),\ldots,P_n^i(x_i))$ belongs to $H_n = \bigcap_{i=1}^n H_n^i$, we shall sometimes write it as $((P_n^i(x_i))^1,\ldots,(P_n^i(x_i))^n)$ and view $(P_n^i(x_i))^j$ as an element of H_n^j.

With $u_k = (u_k,\ldots,u_k) = (u_k^1,\ldots,u_k^n) = \sum_i (0,\ldots,u_k^i,\ldots,0)$, we have

$$\begin{aligned}
P^n(u_k) &= \sum_i P^n\bigl((0,\ldots,u_k^i,\ldots,0)\bigr) = \frac{1}{n}\sum_i \bigl(P_n^i\bigl(u_k^i\bigr),\ldots,P_n^i\bigl(u_k^i\bigr)\bigr)\\
&= \sum_i \bigl(u_k^i,\ldots,u_k^i\bigr)/n = \sum_i \bigl(u_k^1,\ldots,u_k^n\bigr)/n\\
&= \bigl(u_k^1,\ldots,u_k^n\bigr) = u_k.
\end{aligned}$$

By Lemma 3.6 and the fact that $T(H_n)$ is generated by the "ones" and non-"ones" of the H_n^i, the range of P^n is H_n. To calculate the action of P^n it suffices to consider its effect on elements of the form $x = Iu_kJ$, where $\{I,k,J\}$ is a partition of $\{1,\ldots,n\}$. We claim that for such x,

$$P^n(x) = \frac{u_k}{n\binom{n-1}{i-1}}, \tag{10}$$

where $|I| = i-1$. Let us illustrate this first in a specific example. Let $n = 3$, $x = u_2u_2^*u_1u_3^*u_3 = x_1 \oplus x_2 \oplus x_3 \in H_3^1 \cap H_3^2 \cap H_3^3$, so that $x_1 = 0$, $x_3 = 0$, and $i = 2$. By Lemma 3.6 again,

$$\begin{aligned}
P^3(x) &= P^3(x_1 \oplus 0 \oplus 0) + P^3(0 \oplus x_2 \oplus 0) + P^3(0 \oplus 0 \oplus x_3)\\
&= \frac{1}{3}\bigl[\bigl(P_1^3(x_1), P_1^3(x_1), P_1^3(x_1)\bigr) + \bigl(P_2^3(x_2), P_2^3(x_2), P_2^3(x_2)\bigr)
\end{aligned}$$

$$
\begin{aligned}
&+ \big(P_3^3(x_3), P_3^3(x_3), P_3^3(x_3)\big)\big] \\
&= \frac{1}{3}\bigg[(0,0,0) + \bigg(\frac{1}{2}u_1^2, \frac{1}{2}u_1^2, \frac{1}{2}u_1^2\bigg) + (0,0,0)\bigg] \\
&= \frac{1}{3}\frac{1}{2}\big(u_1^2, u_1^2, u_1^2\big) = \frac{1}{6}u_1.
\end{aligned}
$$

In general, for $x = \bigoplus x_i$ as above,

$$
P^n(x) = \frac{1}{n}\Big[\sum \big(P_n^i(x_i), \dots, P_n^i(x_i)\big)\Big] = \frac{1}{n}\bigg[\frac{1}{\binom{n-1}{i-1}}\big(u_k^1, \dots, u_k^n\big)\bigg],
$$

as required for (10).

Lemma 3.7. *Identifying $T(H_n)$ with a subspace of $T(H_{n+1})$ given by the injection $u_i \mapsto u_i$ of H_n into H_{n+1}, we have $P^{n+1}|T(H_n) = P^n$.*

Proof. This is obviously true for generators of $T(H_n)$ which are not composed of "ones" since all of the P_n^k and P_{n+1}^k vanish on non-"ones." On the other hand, if x is of the form Iu_kJ, where $\{I, k, J\}$ is a partition of $\{1, \dots, n\}$, then by collinearity $u_k = u_k u_{n+1}^* u_{n+1} + u_{n+1} u_{n+1}^* u_k$, and by (10),

$$
\begin{aligned}
P^{n+1}(Iu_kJ) &= P^{n+1}\big(\big(I \cup \{n+1\}\big)u_kJ + Iu_k\big(J \cup \{n+1\}\big)\big) \\
&= \frac{1}{n+1}\frac{1}{\binom{n}{i}}u_k + \frac{1}{n+1}\frac{1}{\binom{n}{i-1}}u_k \\
&= \frac{1}{n}\frac{1}{\binom{n-1}{i-1}}u_k = P^n(Iu_kJ). \qquad \square
\end{aligned}
$$

Lemma 3.7 enables the definition of a contractive projection P on a TRO A which is the norm closure of $\bigcup_{n=1}^{\infty} T(H_n)$ (in the ternary envelope of Φ) with $P(A) = \Phi$, namely $Px = P^n x$ if $x \in T(H_n)$. As noted earlier, we can assume that A is a C^*-algebra. By Example 1, $\{\Phi, A^{**}, P^{**}\}$ is an expansion of $\{\mathbf{E}_0\Phi, A^{**}, \mathbf{E}_0P^{**}\}$, so $\mathbf{E}_0P^{**}(A^{**}) = \mathbf{E}_0\Phi$. Thus $\mathbf{E}_0\Phi$ is a normally contractively complemented JC^*-subtriple of A^{**}. As in the proof of Theorem 2, $\mathbf{E}_0\Phi$ coincides with an intersection $R \cap C \cap \Phi$, with the understanding that one or two terms in this intersection may be missing. We claim in fact that R and C are both missing, that is, we have the following lemma.

Lemma 3.8. *The support $\mathbf{E}_0\Phi$ of $\{\Phi, A^{**}, P^{**}\}$ for the above construction coincides with Φ.*

Proof. Because of (10), for any partition $\{i_1, i_2, \dots\} \cup \{k\} \cup \{j_1, j_2, \dots, j_m\}$ of $\{1, 2, 3, \dots\}$,

$$
P^{**}(Iu_kJ) = \lim_{n\to\infty} P^{n+m+1}\big(\{i_1, \dots, i_n\}u_kJ\big) = \lim_{n\to\infty}\frac{1}{n+m+1}\frac{1}{\binom{n+m}{n}}u_k = 0. \tag{11}
$$

Let us write $u_j = u_j^R + u_j^C + u_j^\Phi + u_j^K$, where $\mathbf{E}_0 u_j = u_j^R + u_j^C + u_j^\Phi$, $u_j^K = (1-p)u_j(1-q) \in \ker P^{**}$, and $\mathbf{E}_0 = p \cdot q$. For $I = \mathbb{N} - \{j\}$, $I^C u_j^C = I^\Phi u_j^\Phi = 0$, so by (11), $0 = P^{**}(Iu_j) =$

$P^{**}(I^R u_j^R + I^K u_j^K)$. Since $\mathbf{E}_0(I^K u_j^K) = 0$, $0 = \mathbf{E}_0 P^{**}(I^R u_j^R + I^K u_j^K) = \mathbf{E}_0 P^{**}\mathbf{E}_0(I^R u_j^R + I^K u_j^K) = \mathbf{E}_0 P^{**}\mathbf{E}_0(I^R u_j^R) = \mathbf{E}_0 P^{**}(I^R u_j^R)$.

By collinearity and by the fact that $u_j^R u_\alpha^{R*} = 0$, $u_j^R = (\mathbb{N} - \{j\})^R u_j^R$ so that $\mathbf{E}_0 P^{**} u_j^R = 0$ and similarly $\mathbf{E}_0 P^{**} u_j^C = 0$. However, $P^{**}\mathbf{E}_0 P^{**} = P^{**}$ and so $P^{**}(u_j^R) = 0$. Similarly, $P^{**}(u_j^C) = 0$.

Now $\mathbf{E}_0 u_j$ is the support partial isometry of a norm 1 element ψ_j in $P^* A^*$, and $\psi_j(\mathbf{E}_0 u_j) = \psi_j(P^{**}\mathbf{E}_0 u_j) = \psi_j(P^{**}(u_j^R + u_j^C + u_j^\Phi)) = \psi_j(P^{**} u_j^\Phi) = \psi_j(u_j^\Phi)$. Thus $\mathbf{E}_0 u_j = u_j^\Phi$, so that $\mathbf{E}_0 \Phi$ coincides with Φ. $\square$

Since R, C and $R \cap C$ are trivially contractively complemented in $B(H)$ as spans of finite rank operators in such a way that they clearly equal their support spaces, this proves that each of the spaces occurring in Theorem 2(b) are essentially contractively complemented.

Theorem 3. *The operator spaces R, C, $R \cap C$, and Φ are each essentially normally contractively complemented in a von Neumann algebra.*

4. Completely bounded Banach–Mazur distance

4.1. Representation on the Fock space

Let H be any separable Hilbert space. For any $h \in H$, let $l_m(h)$ denote the wedge (or creation) operator from $H^{\wedge m}$ to $H^{\wedge m+1}$ given by $l_m(h)x = h \wedge x$. The space of creation operators $\overline{\mathrm{sp}}\{l_m(e_i)\}$, where $\{e_i\}$ is an orthonormal basis, will be denoted by $\mathcal{C}^m$. Its operator space structure is given by its embedding in $B(H^{\wedge m}, H^{\wedge m+1})$. $\mathcal{A}^m$ will denote the space of annihilation operators from $H^{\wedge m}$ to $H^{\wedge m-1}$, which consists of the adjoints of the creation operators on $H^{\wedge m-1}$.

It will be convenient to identify the space $H^{\wedge k}$ with $\ell_2(\{J \subset \mathbb{N}\colon |J| = k\})$ (via $e_{j_1} \wedge \cdots \wedge e_{j_k} \leftrightarrow e_J$ if $J = \{j_1, \ldots, j_k\}$ where $j_1 < \cdots < j_k$), or with $\ell_2(\{I \subset \mathbb{N}\colon |\mathbb{N} - I| = k\})$. We assume that each $I = \{i_1, i_2, \ldots\}$ is such that $i_1 < i_2 < \cdots$, and that the collection of all such subsets I is ordered lexicographically. Similarly, if I (or J) is finite.

Define the unitary operators V_k and W_k on $H^{\wedge k}$ by

$$V_k : \ell_2\big(\big\{J \subset \mathbb{N}\colon |\mathbb{N} - J| = k\big\}\big) \to \ell_2\big(\big\{K \subset \mathbb{N}\colon |K| = k\big\}\big)$$

and

$$W_k : \ell_2\big(\big\{I \subset \mathbb{N}\colon |I| = k\big\}\big) \to \ell_2\big(\big\{I \subset \mathbb{N}\colon |I| = k\big\}\big)$$

as follows:

- $V_k(e_J) = e_{i_1} \wedge \cdots \wedge e_{i_k}$, where $\mathbb{N} - J = \{i_1 < \cdots < i_k\}$.
- $W_k(e_I) = \epsilon(i, I)\epsilon(I, i, J)e_I$, where $I \cup \{i\} \cup J = \mathbb{N}$ is a disjoint union, and $\epsilon(i, I)$ is the sign of the permutation taking $(i, i_1, \ldots, i_k)$ to $(i_1, \ldots, i, \ldots, i_k)$.

It is easy to see, just as in [14, Section 2], that the definition of W_k is independent of the choice of $i \in \mathbb{N} - J$.

Recall that the space $H_\infty^{m,L}$ is the closed linear span of matrices $b_i^{m,L}$, $i \in \mathbb{N}$, given by

$$b_i^{m,L} = \sum_{I \cap J = \emptyset,\, (I \cup J)^c = \{i\},\, |I| = m} \epsilon(I, i, J) E_{J,I},$$

where $E_{J,I} = e_J \otimes e_I = e_J e_I^t \in M_{\aleph_0,\aleph_0}(\mathbf{C}) = B(\ell_2)$, and $\epsilon(I, i, J)$ is the signature of the permutation defined for disjoint I, J in Section 2.1.

Lemma 4.1. *$H_\infty^{m,R}$ is completely isometric to $\mathcal{A}^{m+1}$ and $H_\infty^{m,L}$ is completely isometric to $\mathcal{C}^m$.*

Proof. With $b_i = \sum \epsilon(I, i, J) E_{J,I}$ and $I_0 = \{i_1 < \cdots < i_m\}$, we have for $i \notin I_0$,

$$V_{m+1} b_i(e_{I_0}) = V_{m+1}\big(\epsilon(I_0, i, J_0) e_{J_0}\big) = \epsilon(I_0, i, J_0) e_{i_1 < \cdots < i < \cdots < i_m},$$

and

$$\begin{aligned} l_m(e_i) W_m(e_{I_0}) &= l_m(e_i)\big(\epsilon(i, I_0)\epsilon(I_0, i, J_0) e_{I_0}\big) = \epsilon(i, I_0)\epsilon(I_0, i, J_0) e_i \wedge e_{i_1} \wedge \cdots \wedge e_{i_m} \\ &= \epsilon(I_0, i, J_0) e_{i_1 < \cdots < i < \cdots < i_m}. \end{aligned}$$

In the case that $i \in I_0$, both $l_m(e_i)(e_{I_0})$ and $b_i(e_{I_0})$ are zero. □

We summarize this subsection in the following remark.

Remark 4.2. Every finite- or infinite-dimensional separable Hilbertian JC^*-subtriple Y is completely isometric to a finite or infinite intersection of spaces of creation and annihilation operators, as follows:

(a) if Y is infinite-dimensional, then by Lemma 4.1, Theorem 1, and [14, Lemma 2.1], it is completely isometric to one of

$$\mathcal{A}^m, \qquad \mathcal{C}^m, \qquad \mathcal{A}^m \cap \mathcal{C}^k, \qquad \bigcap_{k=1}^{\infty} \mathcal{C}^k;$$

(b) if Y is of dimension n, then by [13, Theorem 3(b)] and [14, Lemma 2.1], Y is completely isometric to $\bigcap_{j=1}^m \mathcal{C}^{k_j}$, where $n \geqslant k_1 > \cdots > k_m \geqslant 1$.

4.2. Completely bounded Banach–Mazur distance

In [14], the completely bounded Banach–Mazur distances from the space H_n^k studied in [13] to the row and column spaces were computed. Theorem 4 in this section generalizes this to the distance between any two of the spaces H_n^k, thereby answering Problem 1 in [13]. Then, in Theorem 5, we compute the completely bounded Banach–Mazur distances among the operator spaces $H_\infty^{m,R}$, $H_\infty^{k,L}$, and Φ.

Let H be an n-dimensional Hilbert space with orthonormal basis $\{e_i\}$. For $e \in H$, $l(e)$ will denote the creation operator by e on the antisymmetric Fock space, and $l_k(e)$ is its restriction to $H^{\wedge k-1}$. We recall [14, Lemma 2.1] that

$$H_n^k = \mathrm{sp}_{1 \leqslant i \leqslant n}\bigl\{l_k(e_i) : H^{\wedge k-1} \to H^{\wedge k}\bigr\}.$$

We have $H_n^1 = C_n$ and $H_n^n = R_n$. As is customary, $\{r_i\}$ and $\{c_i\}$ will denote orthonormal bases for R_n and C_n, respectively. Since the spaces H_n^k are homogeneous Hilbertian operator spaces [14, Theorem 1], as in [14], using [22, Theorem 3.1], one only needs to estimate the norms of the formal identity maps between them.

Theorem 4. *For* $1 \leqslant l \leqslant k \leqslant n$,

$$d_{cb}\bigl(H_n^k, H_n^l\bigr) = \sqrt{\frac{n-k+1}{n-l+1}\frac{l}{k}}.$$

Remark 4.3. Note that $H_n^{k,\mathrm{op}} = H_n^{n-k+1}$ completely isometrically, where E^{op} is the operator space obtained from E with the family of norms $\|(x_{i,j})\|_{M_n(E^{\mathrm{op}})} = \|(x_{j,i})\|_{M_n(E)}$ [16, 2.10]. It is also clear that for $u : E \to F$, $\|u\|_{cb(E,F)} = \|u\|_{cb(E^{\mathrm{op}},F^{\mathrm{op}})}$. These two observations will be referred to as symmetry.

The following lemma, which will be used in the proof of Theorem 4, provides an alternate and more illuminating proof of [14, Theorem 3].

Lemma 4.4. *If* $u : H \to H$ *is a linear map with singular values* $\{a_i\}_{i=1}^n$, *then*

$$\|u\|_{cb(C_n,H_n^k)} = \sup_{|J|=k} \sum_{i\in J} a_i^2, \qquad \textit{and}$$

$$\|u\|^2_{cb(H_n^k,R_n)} = \inf\Biggl\{ \sum_{\substack{J\\ \|J|=k}} \sup_{j\in J} a_{J,j} \colon a_i^2 = \sum_{\substack{K\\ i\in K}} a_{K,i},\ a_{K,i} \geqslant 0\Biggr\}.$$

In the last expression, the infimum is taken over all decompositions of a_i^2 *as a sum* (*indexed by subsets of* $\{1,\dots,n\}$ *with cardinality* k *and containing* i) *of nonnegative numbers* $a_{K,i}$.

Proof. The first part is easy as $cb(C_n, H_n^k) = R_n \otimes_{\min} H_n^k$ and $q_i = l_k(e_i)l_k(e_i)^* : H^{\wedge k} \to H^{\wedge k}$, is the projection onto the subspace generated by e_J with $i \in J$.

The second part is a bit more delicate. We will use a variant of Smith's lemma [6, Proposition 2.2.2]: since u has values in R_n, we have $\|u\|_{cb} = \|u \otimes \mathrm{Id}_{R_n}\|$. To prove this, one just needs to notice that for any map t with values in R one has that $\|t\| = \|t \otimes \mathrm{Id}_C\|$ as R is homogeneous. With $t = u \otimes \mathrm{Id}_{R_n}$, one gets that

$$\|u \otimes \mathrm{Id}_{R_n}\| = \|u \otimes \mathrm{Id}_{R_n} \otimes \mathrm{Id}_{C_n}\| = \|u \otimes \mathrm{Id}_{M_n}\| = \|u\|_{cb},$$

by Smith's lemma.

We have $R_n \otimes_{\min} H_n^k = cb(C_n, H_n^k)$, so if $x \in R_n \otimes_{\min} H_n^k$, it can be considered as a completely bounded map from C_n to H_n^k with $\|x\|_{cb} = \|x\|_{R_n \otimes_{\min} H_n^k}$. The previous observation leads to

$$\|u\|_{cb(H_n^k, R_n)} = \sup_{\substack{x \in cb(C_n, H_n^k) \\ \|x\|_{cb} \leqslant 1}} \|ux\|_{C_n \otimes_{\min} R_n} = \sup_{\|x\|_{cb} \leqslant 1} \|ux\|_{HS}$$
$$= \sup_{\|x\|_{cb} \leqslant 1} \left(\operatorname{Tr} u^* u x x^*\right)^{1/2}.$$

It is a classical result that when A, B are positive matrices chosen with a given distribution (i.e. eigenvalues) so that $\operatorname{tr} AB$ is maximal, then A and B commute. This means that we can assume that both u^*u and xx^* are diagonal. So one has that $\|u\|_{cb}$ is the best constant in the inequality

$$\sum_{i=1}^n (a_i x_i)^2 \leqslant C^2 \sup_{|J|=k} \sum_{i \in J} x_i^2.$$

The result now follows by a standard application of the Hahn–Banach theorem. The term on the right is a norm in $\ell_\infty(S, \ell_1^k)$ where S is the set of subsets of $1, \dots, n$ with cardinal k. Let $T : \mathbb{C}^n \to \ell_\infty(S, \ell_1^k)$ be given by $(t_i) \mapsto ((t_j)_{j \in J})_J$. The inequality says that the sequence a_i^2 defines a linear form on the range of T with norm C^2. Every bounded extension of this form to $\ell_\infty(S, \ell_1^k)$ is given by a vector $((a_{J,j})_{j \in J})_J$ in $\ell_1(S, \ell_\infty^k)$ and has norm equal to $\|(a_{J,j})_J\|_{\ell_1(S, \ell_\infty^k)} = \sum_{J \in S} \sup_{i \in J} a_{J,i}$. Hence $\sum_i (\sum_{J, i \in J} a_{J,i}) t_i = \sum_i a_i^2 t_i$ which yields $a_i^2 = \sum_{J, i \in J} a_{J,i}$. $\square$

We shall apply this lemma for $u = \mathrm{Id}$. We need to find the best constant C in the inequality

$$\sum_{i=1}^n x_i^2 \leqslant C^2 \sup_{|J|=k} \sum_{i \in J} x_i^2.$$

We can assume that x_i is decreasing, and $\sup_{|J|=k} \sum_{i \in J} x_i^2 = 1$. Then $k x_k^2 \leqslant \sum_{i=1}^k x_i^2 = 1$ and $x_i^2 \leqslant \frac{1}{k}$ for $i \geqslant k$. So

$$\sum_{i=1}^n x_i^2 = \sum_{l=1}^{[n/k]} \sum_{i=1}^k x_{lk+i}^2 + \sum_{i=[n/k]k+1}^n x_i^2 \leqslant \left[\frac{n}{k}\right] + \sum_{i=[n/k]k+1}^n \frac{1}{k} \leqslant \frac{n}{k}.$$

We get that $\|u\|_{cb} \leqslant \sqrt{n/k}$, but there is equality with $x_i^2 = 1/k$.

Finally using symmetry one recovers [14, Theorem 3]:

$$d_{cb}\big(R_n, H_n^k\big) = \sqrt{(n-k+1)\frac{n}{k}}, \qquad d_{cb}\big(C_n, H_n^k\big) = \sqrt{k \frac{n}{n-k+1}},$$

Proof of Theorem 4. We first compute the *cb* norm of $\mathrm{Id}: H_n^{k+1} \to H_n^k$. To this end, we will find a Stingspring dilation. We have the commutation relations

$$l_{k+1}(e_i)l_k(e_j) = -l_{k+1}(e_j)l_k(e_i).$$

Moreover, $\sum_s l_{k+1}(e_s)^* l_{k+1}(e_s) = (n-k)\mathrm{Id}_{H^{\wedge k}}$. Let $m_i \in \mathbb{M}_d$, and compute

$$\begin{aligned}
&\Big(\sum_j \mathrm{Id}_{\mathbb{M}_d} \otimes l_{k+1}(e_j)^* \otimes r_j\Big)\Big(\sum_i m_i \otimes l_{k+1}(e_i) \otimes \mathrm{Id}_{\mathbb{M}_n}\Big)\Big(\sum_s \mathrm{Id}_{\mathbb{M}_d} \otimes l_k(e_s) \otimes c_s\Big)\\
&\quad = \sum_i m_i \otimes \sum_s l_{k+1}(e_j)^* l_{k+1}(e_i) l_k(e_s) \otimes e_{1,1}\\
&\quad = (k-n)\sum_i m_i \otimes l_k(e_i) \otimes e_{1,1}.
\end{aligned}$$

In this way we have written a factorization $\mathrm{Id}(x) = U\pi(x)V$, where π is a representation and U and V are bounded maps. This is a Stingspring dilation for $\mathrm{Id}: H_n^{k+1} \to H_n^k$, and so

$$\big\|\mathrm{Id}: H_n^{k+1} \to H_n^k\big\|_{cb} \leqslant \Big\|\sum_j l_{k+1}(e_j)^* \otimes r_j\Big\| \Big\|\sum_s l_k(e_s) \otimes c_s\Big\| \Big/ (n-k).$$

By using the previous lemma, one gets

$$\big\|\mathrm{Id}: H_n^{k+1} \to H_n^k\big\|_{cb} \leqslant \sqrt{\frac{n-k+1}{n-k}}.$$

The other estimate for $\mathrm{Id}: H_n^k \to H_n^{k+1}$ follows by symmetry:

$$\big\|\mathrm{Id}: H_n^k \to H_n^{k+1}\big\|_{cb} \leqslant \sqrt{\frac{k+1}{k}}.$$

So

$$d_{cb}\big(H_n^k, H_n^{k+1}\big) \leqslant \sqrt{\frac{n-k+1}{n-k}\frac{k+1}{k}}.$$

But

$$d_{cb}\big(H_n^k, H_n^{k+1}\big) \geqslant d_{cb}\big(R_n, H_n^k\big) \big/ d_{cb}\big(R_n, H_n^{k+1}\big) = \sqrt{(n-k+1)\frac{n}{k}}\sqrt{\frac{1}{n-k}\frac{k+1}{n}}.$$

Now, if $l > k$,

$$d_{cb}\big(H_n^k, H_n^l\big) \leqslant d_{cb}\big(H_n^k, H_n^{k+1}\big) \dots d_{cb}\big(H_n^{l-1}, H_n^l\big) = \sqrt{\frac{l(n-k+1)}{k(n-l+1)}}.$$

If this inequality were strict, then from

$$d_{cb}(C_n, R_n) \leqslant d_{cb}\big(C_n, H_n^l\big) d_{cb}\big(H_n^l, H_n^k\big) d_{cb}\big(H_n^k, R_n\big)$$

it would follow that $n < n$. □

Remark 4.5. The spaces H_n^k form a segment in the space of operator spaces of dimension n equipped with the Banach–Mazur distance. This is also the case for the interpolated spaces $(\min X, \max X)_\theta$ (X a n-dimensional Banach space), which form a line between $(\min X, \max X)_0 = \min X$ and $(\min X, \max X)_1 = \max X$.

We come back to the main object of this paper, the spaces $H_\infty^{m,L}$ and $H_\infty^{m,R}$.

Theorem 5. *For $m, k \geqslant 1$,*

(a) $d_{cb}(H_\infty^{m,R}, H_\infty^{k,R}) = d_{cb}(H_\infty^{m,L}, H_\infty^{k,L}) = \sqrt{\frac{m+1}{k+1}}$ *when* $m \geqslant k$;
(b) $d_{cb}(H_\infty^{m,R}, H_\infty^{k,L}) = \infty$;
(c) $d_{cb}(H_\infty^{m,R}, \Phi) = d_{cb}(H_\infty^{m,L}, \Phi) = \infty$.

Proof. We start with (a). Fix $n > m, k$. We will look at $\tilde{H}_n^m$ for n-dimensional subspace of $H_\infty^{m,L}$. As noted in the proof of Theorem 1, for $m \geqslant k$, the formal identity $\mathrm{Id} : \tilde{H}_n^m \to \tilde{H}_n^k$ is a complete contraction. The estimation for the reverse map

$$\big\|\mathrm{Id} : \tilde{H}_n^k \to \tilde{H}_n^m\big\|_{cb} \leqslant \sqrt{\frac{m+1}{k+1}},$$

follows from the estimates for $\mathrm{Id} : H_n^d \to H_n^p$ obtained during the proof of the previous theorem.

As before, to show that

$$d_{cb}\big(\tilde{H}_n^m, \tilde{H}_n^k\big) = \sqrt{\frac{m+1}{k+1}},$$

it suffices to show that there is equality when $k = 0$. But, in this case $\tilde{H}_n^0 = C_n$ and we have

$$\sqrt{m+1} \geqslant \big\|\mathrm{Id} : \tilde{H}_n^0 \to \tilde{H}_n^m\big\|_{cb} \geqslant \big\|\mathrm{Id} : \tilde{C}_n \to H_n^m\big\|_{cb} = \sqrt{m+1}.$$

To conclude, one just needs to let n go to ∞, and notice that $H_\infty^{m,L,\mathrm{op}} = H_\infty^{m,R}$ completely isometrically.

Now (b) and (c) are direct consequences of (a) as $d_{cb}(C, H_\infty^{m,R}) d_{cb}(R, H_\infty^{k,L}) < \infty$ and $d_{cb}(C, \Phi) = d_{cb}(R, \Phi) = d_{cb}(C, R) = \infty$. □

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