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ZEROS in ππ SCATTERING

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ZEROS IN $\pi\pi$ SCATTERING

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ABSTRACT

We discuss how zeros in $\pi\pi$ scattering relate various dynamical aspects of the scattering amplitude. In particular, we consider the way in which many differing roles, such as the Legendre zeros of resonances, double-pole-killing zeros, the high energy crossover zero and the Adler zero, can be played by the same zero contour. We emphasize that the study of the scattering amplitude is essentially one of looking at zeros travelling across the Mandelstam plane.

INTRODUCTION TO ZEROS IN $\pi\pi$ SCATTERING

The purpose of this talk is to advertise a particularly useful tool for studying the scattering amplitude and that is the investigation of the zeros of the amplitude. We will illustrate, in a fairly straightforward way, the close relationship between the zero structure of the amplitude and various dynamical mechanisms. We shall, of course, only consider zeros in $\pi\pi$ scattering, but many of the properties we discuss are features of more general scattering processes.[1]

Before we can begin this discussion of what role zeros play we must introduce zeros into the Mandelstam plane. This we do in the simplest possible way. Whilst we cannot determine the scattering amplitude, $F(s,t)$, everywhere, we know that in certain limited energy regions we can make a reasonable approximation to it. For example, at high energies, near the forward or backward directions, we can supposedly represent $F(s,t)$ by a Regge expansion, involving only the rightmost Regge singularities. Similarly, in the low energy region, in the neighborhood of a narrow resonance, we can reasonably well describe the amplitude by a simple Breit-Wigner form. So for $s\epsilon(m^2 - m\Gamma,\ m^2 + m\Gamma)$, where m,Γ are the resonance mass and total width respectively, we have:

$$F(s,t) = (2\ell + 1)\,\frac{m\Gamma x}{m^2 - s - im\Gamma}\,P_\ell\left(1 + \frac{2t}{s - 4\mu^2}\right) \qquad (1)$$

$$+ \text{ background}$$

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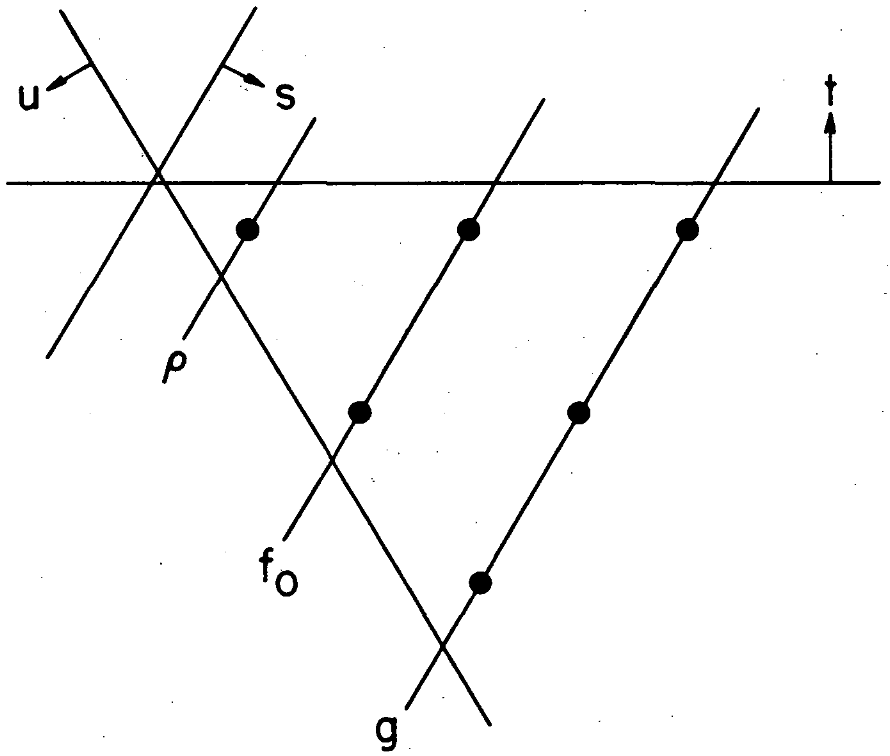

Fig. 1. The three most important low energy $\pi\pi$ resonances with spin are shown in the s-channel of the Mandelstam plane. The black dots mark the Legendre zeros of these resonances.

(ℓ is the spin of the resonance and $x = \Gamma_{e\ell}/\Gamma$). Such a simple form is a good description of the amplitude at least for narrow resonances, e.g., for the ρ, f_0, and g. We deliberately omit the ϵ or σ and S^* resonances, as neither is well-described by Eq. (1).

Let us notice a very important feature of the resonant part of Eq. (1), and that is the Legendre polynomial $P_\ell(\cos \Theta_s)$. This function has ℓ zeros inside the physical region, i.e., for $\cos \Theta_s \in [-1,+1]$, where ℓ is the spin of the resonance. Indeed we can regard these zeros as dynamical objects which give the spin to the resonance. We see that even though we are looking at the amplitude close to a pole of the S-matrix the zero structure is equally important in determining the amplitude. If the background in Eq. (1) is negligible, the whole amplitude will have ℓ zeros in the neighborhood of the spin ℓ resonance. In practice, of course, the background is not negligible and the zeros are shifted from their simple Legendre zero positions. However, as we shall see, unitarity generally constrains these zeros not to move far from where $P_\ell(\cos \Theta) = 0$.

Let us look where these zeros occur in the Mandelstam plane, which is where all the action takes place (see Fig. 1). The ρ has one zero, the f_0 two, the g three and so on. In Fig. 1 we have marked, for simplicity, the position of the Legendre zeros of these resonances, as if the background were negligible. So we now have zeros sitting in the Mandelstam plane, what do they do?

Well, the usefulness of zeros rests in the remark that the zeros of an analytic function of two variables are not isolated, as shown in Fig. 1, but are continuous. Thus we have zero contours (to be defined below), which traverse the Mandelstam plane relating various dynamical aspects of the scattering amplitude. As we shall see, not only are the Legendre zeros of different resonances related to each other, but are also related to high energy Regge zeros. In this way zero contours provide a natural vehicle for duality. We shall also discuss how zero contours enable us to relate low energy resonance dynamics to chiral dynamics.

Before we can connect up the isolated zeros plotted in Fig. 1, we have to define what we mean by a zero contour and this is what we do next.

ZERO CONTOUR DEFINED

Consider the scattering amplitude, $F(s,t)$, in the s-channel with s real (since this is where we have its value experimentally), then $F(s,t) = 0$ for complex values of $t = t_0(s)$. We then refer to the projection of this complex curve on the real t axis as a zero contour, i.e.,

$$t = \operatorname{Re} t_0(s) \qquad (2)$$

is a zero contour. In some cases it will be more useful to treat the amplitude as a function of s and $z = \cos\theta_s = 1 + 2t/(s-4\mu^2)$ rather than s and t. Then the same zero contour will be at $z = \mathrm{Re}\, z_0(s)$ where $z_0 = 1 + 2t_0/(s - 4\mu^2)$.

Since experimentally we look not only at real s but also real z, we next ask what does a zero at $z = \mathrm{Re}\, z_0 + i\, \mathrm{Im}\, z_0$ mean in terms of measurable quantities, in particular for $|\mathrm{Re}\, z_0| < 1$. Experimentally we measure $d\sigma/d\Omega$ or $|F|^2$, so let us look at that. Well, under certain conditions, we will see a minimum in the angular distribution at $z = \mathrm{Re}\, z_0(s)$, so that the zero contours defined above are just the paths of dips in the angular distribution. Now we know what $\mathrm{Re}\, z_0$ means, what about $\mathrm{Im}\, z_0$? This also has a well-defined interpretation. $\mathrm{Im}\, z_0$ is related to the magnitude of $|F|$ at $z = \mathrm{Re}\, z_0$. This is easy to see, since the smaller $|\mathrm{Im}\, z_0|$ is, the closer the zero is to a real zero and so the closer $|F|$ at the minimum is to zero. Now it is only nearby zeros, (with $|\mathrm{Re}\, z_0| < 1$), which have small imaginary parts, that produce marked dips in the differential cross sections. However, these are not the only zeros which play a role in understanding the scattering amplitude as we shall see, so we will, in fact, need our definition of zero contours given by Eq. (2) which is more general than that of just paths of minima of $d\sigma/d\Omega$.

ZEROS NEAR RESONANCES

Before we actually plot zero contours in the Mandelstam plane of Fig. 1, we discuss briefly what we might expect these contours to do when they approach resonance structures. We consider two examples of physical interest, for the purposes of illustration:
(i) a narrow p-wave resonance with a smooth s-wave background,
(ii) a narrow s-wave structure with a smooth p-wave background.

(i) For definiteness, consider $\pi^-\pi^0$ elastic scattering in the s-channel. Then we ask: where does $F^{s1}(s,z) + F^{s2}(s,z) = 0$ in the region of the ρ resonance? In this region we have just s and p waves and so consider*

* In terms of partial waves it is clear why our definition of zero contours of Eq. (2) considers the amplitude for real s and complex t or z. Experimentally we know the partial waves only for real s, but we can use the partial wave expansion to continue the amplitude to complex z. More general definitions of zero contours with both complex s and t are not useful phenomenologically although formally well defined in model amplitudes.[2]

$$3f_1^{\ 1}(s)z + f_0^{\ 2}(s) = 0 \quad . \tag{3}$$

In the ρ region we can represent $f_1^{\ 1}(s)$ by a unitary Breit-Wigner form and use a phase shift representation for $f_0^{\ 2}(s)$, so that

$$\text{Re } z_0(s) = -\frac{1}{3}\frac{(m^2 - s)}{m\Gamma}\frac{1}{2}\sin 2\delta_0^{\ 2}(s) - \frac{1}{3}\sin^2 \delta_0^{\ 2}(s). \tag{4}$$

We know that if the background were negligible the amplitude vanishes at $z = 0$, so we first ask how far is the zero shifted at $s = m^2$ by a nonzero background? From Eq. (4) we have

$$\text{Re } z_0(m^2) = -\frac{1}{3}\sin^2 \delta_0^{\ 2}(m^2) \tag{5}$$

which implies

$$0 \geq \text{Re } z_0(m^2) \geq -1/3 \quad . \tag{6}$$

We see that even if the background were as large as possible, unitarity constrains the zero contour to cross $s = m^2$ not far from $z = 0$. Indeed in this particular case the background is exotic, so $\sin^2 \delta_0^{\ 2}$ is small and $\text{Re } z_0(m^2) \simeq -1/20$.

Next we consider the energy dependence of the zero away from $s = m^2$, e.g., $s \epsilon (m^2 - m\Gamma,\ m^2 + m\Gamma)$. If $\delta_0^{\ 2}$ is slowly varying, the zero contour is close to linear [Eq. (4)] as depicted in Fig. 2a. If we know nothing about $\delta_0^{\ 2}$, other than it is between 0 and $-\pi/2$, the zero contour is constrained by unitarity to pass between the hatched boundaries of Fig. 2a.

For $\pi^+\pi^-$ elastic scattering the background in the ρ region has both $I = 0$ and $I = 2$ s-waves. Nonetheless unitarity still requires

$$0 \geq \text{Re } z_0(m^2) \geq -1/3$$

so that the zero cannot be shifted too far away from $z = 0$, in this case too.

In general for resonances with spin $\ell \geq 1$, zero contours will cross the resonance positions smoothly and close to where the appropriate Legendre polynomial vanishes. However, when we have an s-wave resonance this need not be the case and this is what we next consider.

(ii) The situation of a smooth p-wave background with a rapidly varying s-wave occurs between 900 and 1000 MeV in $\pi^+\pi^-$ elastic

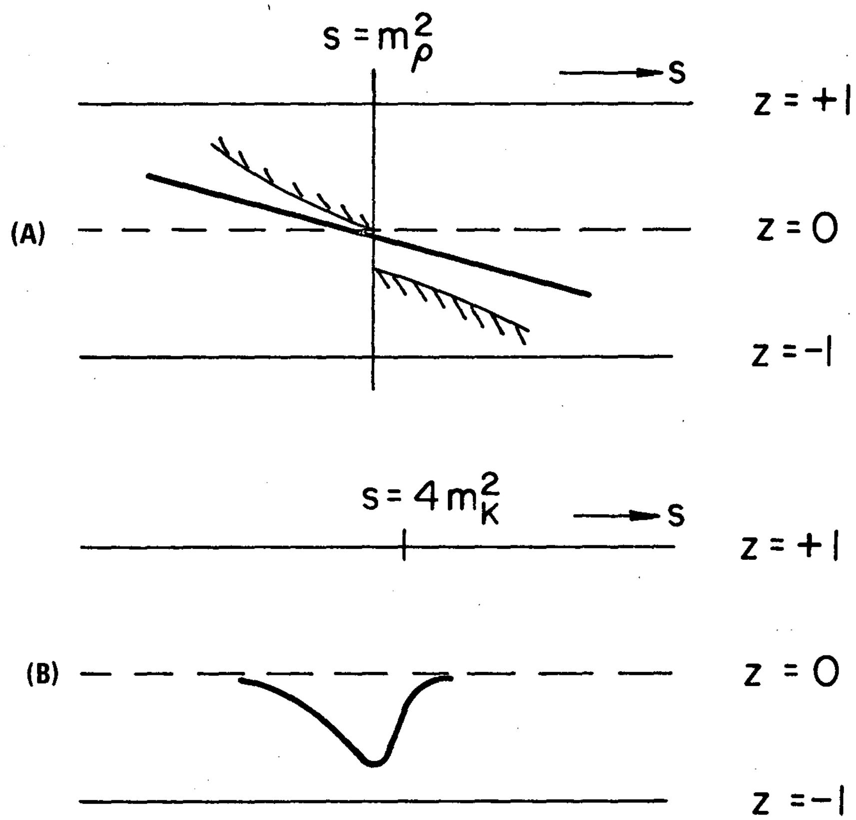

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Fig. 2. (a) The solid line marks the zero contour in $\pi^-\pi^0$ scattering at it crosses the ρ resonance, with δ_0^2 assumed to be a slowly varying function of energy for $s \simeq m_\rho^2$. With $\delta_0^2 \epsilon(-\frac{\pi}{2},0)$, but otherwise unknown, the zero is constrained by unitarity to pass between the hatched boundaries.

(b) An example of how a zero contour behaves when crossing a narrow s-wave structure, with a smooth p-wave background, as occurs in $\pi^+\pi^-$ elastic scattering near $K\bar{K}$ threshold.

scattering. Then the zero contour varies rapidly as shown in Fig. 2b, using $\delta_1^1 \sim 150^\circ$ and δ_0^0 varying from 80°-200°. We should therefore expect to see such a wiggle in the zero contour near $K\bar{K}$ threshold where such a rapidly varying δ_0^0 has been found.[3]

We are now almost in a position to plot zero contours on Fig. 1. However, a remark is called for about the number of zeros increasing with increasing energy. Near threshold the amplitude is just s-wave. As the p-wave begins to grow, a zero enters the physical region from outside, either through the backward or forward direction. It steadily moves towards the center of the physical region when the p-wave resonates. Then as the d-wave begins to grow, a second zero enters the physical region. If the d-wave is $I = 0$ this zero, together with the Legendre zero of the ρ, will become the two Legendre zeros of the f_0. As higher and higher partial waves grow, more and more zeros enter the physical region producing higher and higher spin resonances. Whether the zero contour enters the physical region through the forward or backward direction depends on the relative sizes and signs of the contributing phase shifts in that energy region. However, the ℓth zero's entrance is mainly controlled by the ℓth and $(\ell - 1)$th partial waves. As the energy increases we should therefore expect to see increasing numbers of zeros entering the physical region.

ZERO CONTOURS FROM THE DATA

Let us look at the actual pattern of zeros[4] for the two reactions $\pi^+\pi^-$ and $\pi^-\pi^0$ elastic scattering as determined from the s-, p-, and d-wave phase shifts of Protopopescu et al.[3] up to 1100 MeV (Fig. 3).

$\pi^+\pi^-$ We see the first zero enters the physical region through the backward direction and moves towards the center of the physical region, crosses $s = m_\rho^2$ close to $z = 0$, then wiggles near $K\bar{K}$ threshold as expected because of the rapid variation of δ_0^0. The zero then lines up ready to become one of the Legendre zeros of the f_0 resonance. As the $I = 0$ d-wave grows a zero approaches the physical region and enters through the forward direction* and moves ready to become the other Legendre zero of the f_0. Note that since the s and t channels are identical, the first zero looks as though it may well pass through the Mandelstam triangle as suggested by the dotted line — the possible significance of this will be discussed later.

* We cannot determine the zero contour far outside the physical region as a finite number of partial waves provides a poor representation there.

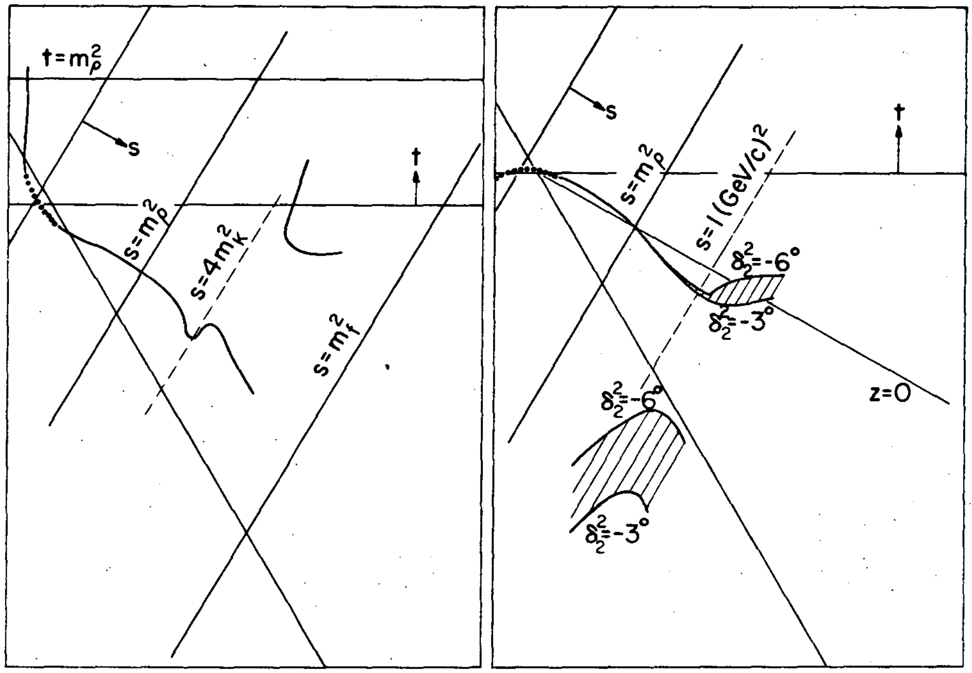

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Fig. 3. The zero contours for $\pi^+\pi^-$ and $\pi^-\pi^0$ elastic scattering as determined in Ref. 4 from the s-, p-, and d-wave phase shifts of Protopopescu et al. and Baton et al. The dotted line is shown to illustrate a possible continuation of the first zero contour through the Mandelstam triangle. Since in $\pi^-\pi^0$ elastic scattering the $I = 2$ d-wave is not well determined we have a range of positions of the zero contours, marked by the shaded regions. These were obtained by varying δ_2^2 at 1 GeV from $-3°$ to $-6°$.

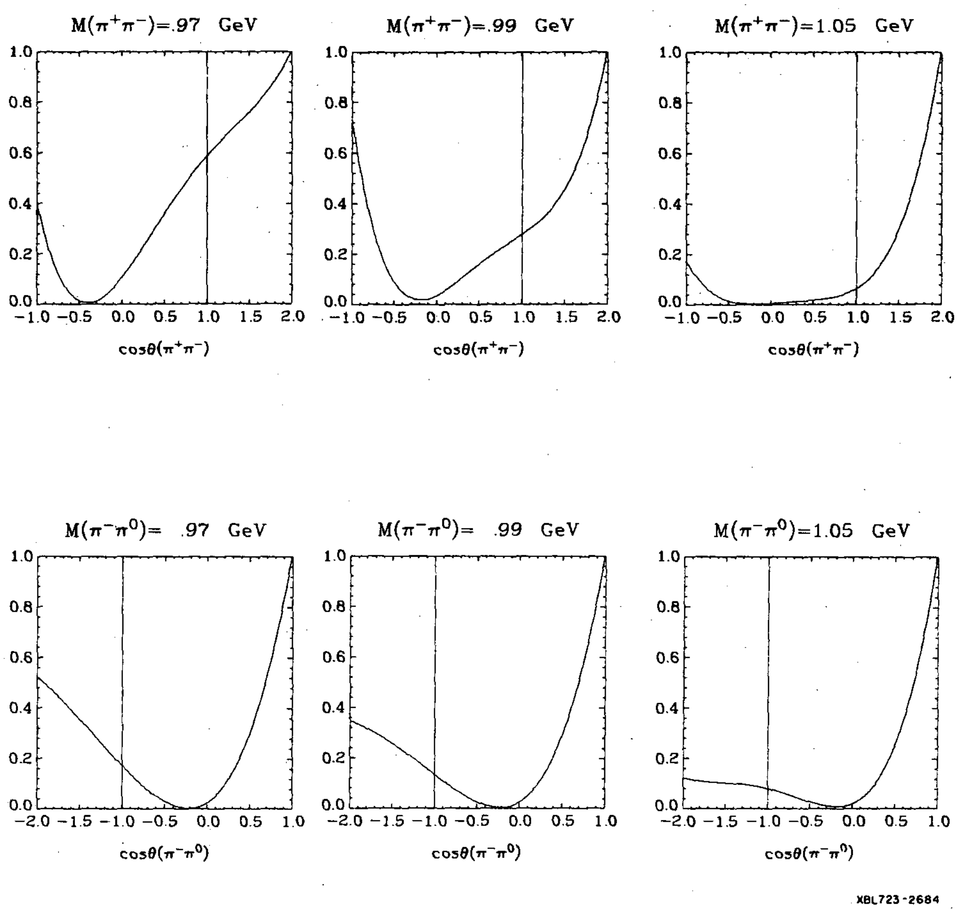

Fig. 4. The angular distributions (in arbitrary units) for $\pi^+\pi^- \to \pi^+\pi^-$ and $\pi^-\pi^0 \to \pi^-\pi^0$ scattering at 970, 990, and 1050 MeV.

$\underline{\pi^-\pi^0}$ In $\pi^-\pi^0$ elastic scattering the s and u channels are the same. The first zero leaves the region of the Mandelstam triangle enters the physical region through the forward direction crosses the line $z = 0$, this time very close to $s = m_\rho^2$ since the s-wave background is exotic (as in Fig. 2a). Then as the zero contour moves on there is some latitude because δ_2^2 is rather uncertain. The shaded area represents the possibility for δ_2^2 at 1 GeV being between $-3°$ and $-6°$. As can be seen, a second zero approaches the backward direction, because of the growing $|f_2^2|$, but has not yet entered the physical region by 1100 MeV.

The fact that the first zero enters the physical region through the backward direction in $\pi^+\pi^-$ scattering, whilst through the forward direction for $\pi^-\pi^0$ scattering is determined by $\mathrm{sgn}(2\delta_0^0 + \delta_0^2) = \mathrm{sgn}(\delta_1^1)$ and $\mathrm{sgn}(\delta_0^2) = -\mathrm{sgn}(\delta_1^1)$, respectively. Similarly, the second zero approaches the forward direction for $\pi^+\pi^-$ scattering as $\delta_2^0 > 0$, whilst for $\pi^-\pi^0$ scattering the d-wave is exotic and negative and the zero approaches the backward direction.

So far we have only considered the zero contours, i.e., the lines $z = \mathrm{Re}\, z_0(s)$, and we do not know if the zeros produce marked dips in the angular distributions or not. This depends on how large $\mathrm{Im}\, z_0(s)$ is.

In Fig. 4 we show the angular distributions[4] in the region where the second zero nears and enters the physical region, i.e., for $\sqrt{s}$ between 970 and 1050 MeV. As can be seen for both $\pi^+\pi^-$ and $\pi^-\pi^0$ elastic scattering, the first zero--the Legendre zero of the ρ--is always a very nearby zero producing a marked dip in the angular distribution (not just in the energy region shown). However, the second zero in $\pi^+\pi^-$ scattering has a large imaginary part and produces no marked dip in the differential cross-section until it is right inside the physical region, when its imaginary part decreases rapidly ready to become the Legendre zero of the f_0. In $\pi^-\pi^0$ scattering the second zero is not nearby even at 1.05 GeV.

WHERE DO THE ZERO COME FROM?

As mentioned above, when the energy increases, higher partial waves grow and an increasing number of zeros enter the physical region, resulting in higher and higher spin resonances where do these zeros come from? It appears that all the zeros except the first, the Legendre zero of the ρ, may have a common origin, which we next discuss, leaving consideration of the first zero till later.

Consider $\pi^+\pi^-$ elastic scattering in the s and t channels and note that the u-channel is exotic having $I_u = 2$. Imagine we already have the first zero inside the physical region and we have the ρ resonance (Fig. 5). Now we look in the unphysical region near $s = m_\rho^2$, $t = m_\rho^2$. There the amplitude is approximately given by

$$F(s,t) \simeq \frac{a}{m^2 - s} + \frac{a}{m^2 - t} \tag{7}$$

where we can consider m^2 to be complex to include the width. Rewriting Eq. (7) as

$$F(s,t) = \frac{a(2m^2 - s - t)}{(m^2 - s)(m^2 - t)} \tag{8}$$

we see F has a double pole at $s = m^2$, $t = m^2$, but with a zero intersecting this double pole at $s + t = 2m^2$, i.e., $F = 0$ for $u = 4\mu^2 - 2m^2$. The occurrence of such a zero which "kills" the double pole in Eq. (8) is completely general. Let us forget, for the moment, the zero contours we saw above from the data and let us pretend that the fixed u zero, we have just seen occurs at the intersection of the ρ poles in the s and t channels, continues to traverse the Mandelstam plane at fixed u.[5] This zero contour then crosses $s = m_f^2$ remarkably close to $z = +1/\sqrt{3}$ (i.e., where $P_2(z) = 0$) and then crosses $s = m_g^2$ close to $z = 0$ (where $P_3(z) = 0$). Recall we already have the ρ's Legendre zero in the physical region, so we now have two zeros, which takes care of the f_0, but to have a spin 3 g resonance we need another zero to enter the physical region between $s = m_f^2$ and $s = m_g^2$. Where does this come from? Well, we now have double poles in the unphysical region where the ρ and f_0 poles in the s and t channels intersect. Again we must have a double pole killing zero, this time at $\mathrm{Re}\, u = 4\mu^2 - m_\rho^2 - m_f^2$ which crosses $s = m_g^2$ very close to where $P_3(z) = 0$. As each resonance occurs we have more and more double poles in the unphysical region, which are killed by more and more zeros, which enter the physical region producing higher and higher spin resonances and so the cycle continues. This structure of fixed u zeros is epitomized by the single term Veneziano amplitude.[6]

This amplitude, you recall, for $\pi^+\pi^-$ elastic scattering involves the product of poles in the s and t channels

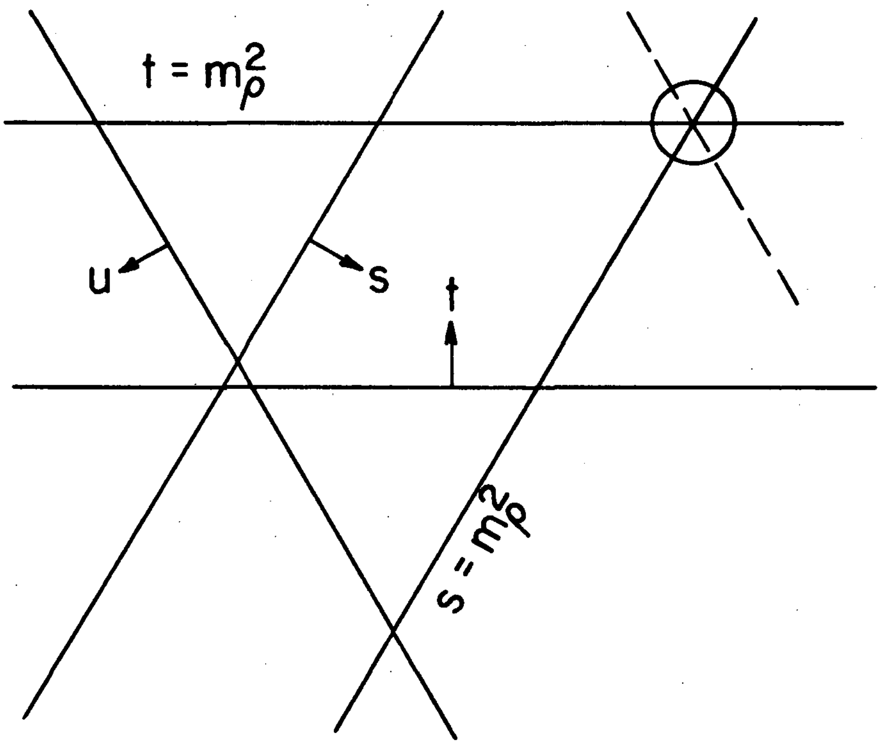

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Fig. 5. The Mandelstam plane for $\pi^+\pi^-$ elastic scattering showing the ρ poles in the s and t channels. The circle, centered on the double pole in the unphysical region, indicates where Eqs. (7,8) may be a reasonable approximation. The dashed line marks the fixed u zero which "kills" the double pole.

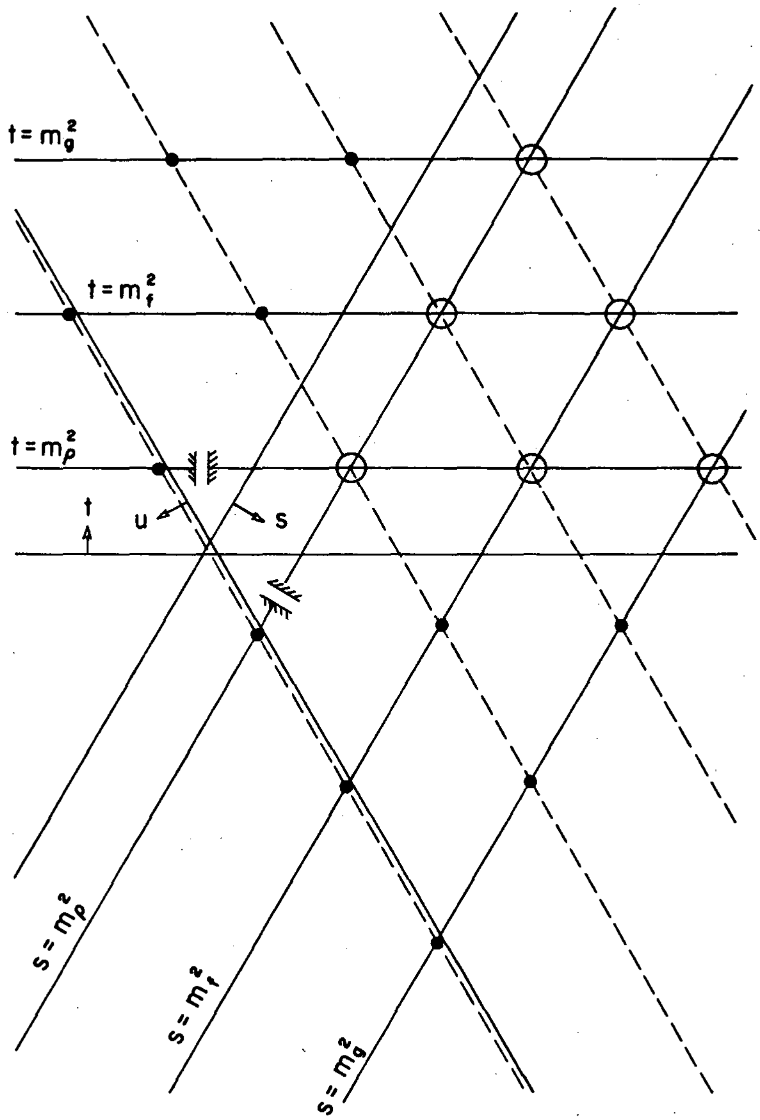

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Fig. 6. The zeros (dashed lines) and poles (solid lines) of the Lovelace-Veneziano model, Eq. (10). The open circles mark the double poles and the black dots the Legendre zeros of resonances. The hatched band at $s = m_\rho^2$ and at $t = m_\rho^2$ indicates where unitarity would constrain the Legendre zero of the ρ to be.

$$V_{st} \sim \Gamma(1-\alpha_s)\,\Gamma(1-\alpha_t)$$

where $\alpha_x = \alpha_0 + \alpha' x$. To kill the forbidden double poles the above form is divided by[6] $\Gamma(2-\alpha_s-\alpha_t)$. Let us look at the zero and resonance structure of this amplitude

$$V_{st} = \frac{\Gamma(1-\alpha_s)\,\Gamma(1-\alpha_t)}{\Gamma(2-\alpha_s-\alpha_t)} . \tag{9}$$

This is shown in Fig. 6. Ignoring for the moment the zero close to $u = 0$, we have a zero structure exactly as described before. However, the ρ has no Legendre zero, the f_0 one, the g two, and so on. This amplitude therefore has the wrong spin structure for $\pi\pi$ scattering. This was remedied by Lovelace[7] by multiplying Eq. (9) by $(1-\alpha_s-\alpha_t)$ to give

$$F_{LV}(s,t) = \frac{\Gamma(1-\alpha_s)\,\Gamma(1-\alpha_t)}{\Gamma(1-\alpha_s-\alpha_t)} . \tag{10}$$

This introduces an extra zero, not a double-pole-killing zero, at $u = 4\mu^2 - \dfrac{(1-2\alpha_0)}{\alpha'}$ which Lovelace fixes at $u = 2\mu^2$. As can be seen in Fig. 6, this zero crosses through the Mandelstam triangle (the significance of which will be discussed later) and gives the correct spin structure to the ρ, f_0, g etc. However, this zero is outside the physical region and clearly violates unitarity, since at the ρ, for example, we have already seen it should pass through the marked bands shown in Fig. 6. This violation occurs because the s-wave amplitude with its ϵ resonance, degenerate in mass with the ρ, grossly exceeds the unitarity bound.

ODORICO ZEROS INTRODUCED

Although the Lovelace-Veneziano model[7] violates unitarity it provides a beautiful example of how an amplitude, which is essentially real, is determined by knowing its poles and its zeros. However, the physical world is supposedly just the unitarization of some dual model and perhaps some features of the simple Veneziano model, for example, are preserved by this unitarization. Certainly the real parts of the pole positions are unchanged, perhaps then the real parts of the zero positions are still approximately at fixed u. Indeed this has been suggested by Odorico.[5] However, his hypothesis of straight line zeros does not just apply to amplitudes represented by single Veneziano terms, but also to amplitudes with resonances in all three channels, when the single term

Virasoro model[8] is an explicit example having straight line zeros (SLZ). Using this SLZ hypothesis we can build up the structure of zeros and poles in the Mandelstam plane for the "true" amplitude without recourse to a specific (nonunitary) model. Of course, to make any sense of this prescription Odorico avoids the most blatant violation of unitarity in $\pi^+\pi^-$ elastic scattering by shifting the first zero, which Lovelace fixes at $u = 2\mu^2$, to $u = 4\mu^2 - \frac{1}{2} m_\rho^2$. This zero, which you recall is not a double-pole-killing zero, then becomes the Legendre zero of the rho with an s-wave background consistent with unitarity. It is rather remarkable, as we shall mention again later, that this fixed u zero of Odorico's now also crosses both the f_0 and g resonances very close to where $P_2(z) = 0$ and $P_3(z) = 0$, respectively.

Let us look at Odorico's SLZ hypothesis in the two cases of $\pi^+\pi^-$ and $\pi^-\pi^0$ elastic scattering[4] (see Fig. 7) and compare it with the data of Fig. 3.

$\underline{\pi^+\pi^-}$ The first zero certainly follows an approximate straight line from $s = m_\rho^2$ to $s = m_f^2$, just as suggested by Odorico, aside from the wiggle at $K\bar{K}$ threshold. This is not surprising since no such coupled channel effect is built into the Veneziano model. However, closer to $\pi\pi$ threshold the data suggests that this zero leaves through the backward direction, while Odorico's zero leaves through the forward direction (corresponding to $\delta_0^0 < 0$) and passes far from the Mandelstam triangle.

The second zero also does not follow a very straight line, however, it is very likely that it is in fact related to the double-pole-killing zero in the unphysical region near $s = m_\rho^2$, $t = m_\rho^2$.

$\underline{\pi^-\pi^0}$ Here the pattern of Odorico zeros is more complicated, and here too the first zero does not follow Odorico's lines very closely. Nonetheless, it is very likely that the second zero is once again related to the double-pole-killing zero. However, Odorico's lines suggest that, in the physical region, this second zero becomes the Legendre zero of the g at $z = 0$, while the rho's Legendre zero becomes that of the g at $z = -\sqrt{3/5}$. From the data it seems that these zero contours do not cross, but that the double-pole-killing zero becomes the g's Legendre zero in the backward hemisphere (more on this later).

COMMENTS ON STRAIGHT LINE ZEROS

It is appropriate here to make a few general remarks about Odorico zeros. It is very likely that the increasing number of zeros which enter the physical region from outside are the double-pole-killing zeros in the unphysical region (as also emphasized by A. D. Martin and W. J. Ochs in these proceedings). This

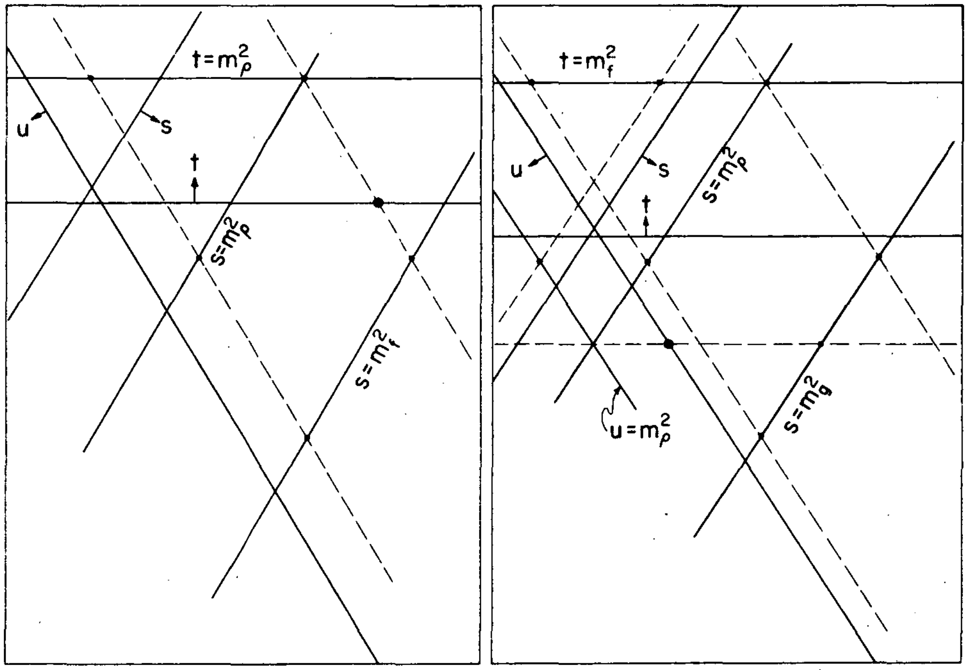

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Fig. 7. Structure of Odorico zeros for $\pi^+\pi^-$ and $\pi^-\pi^0$ elastic scattering. The dashed lines denote the zeros.

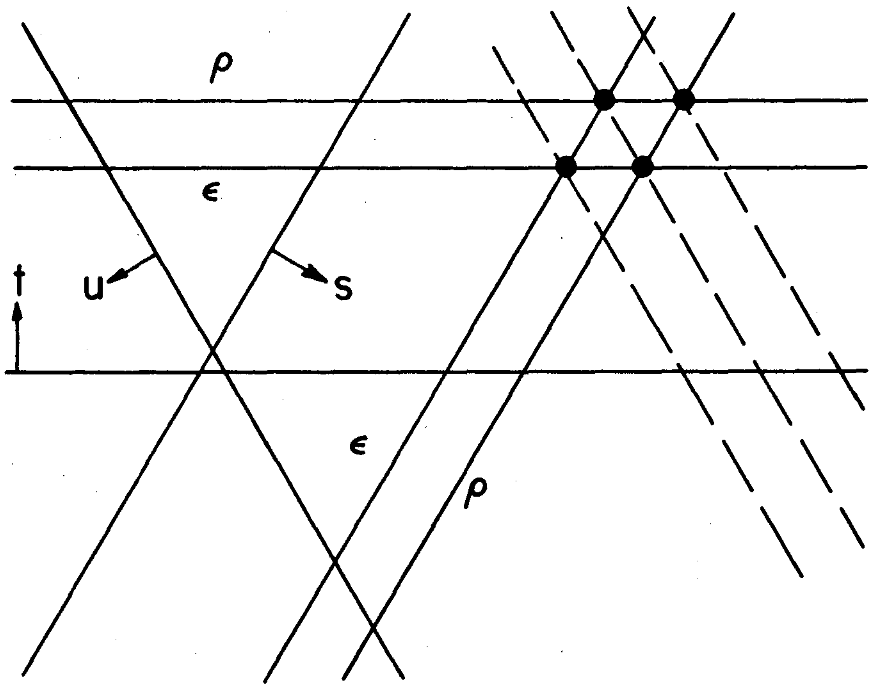

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Fig. 8. The Mandelstam plane for $\pi^+\pi^-$ elastic scattering showing the ϵ and ρ poles in the s and t channels. (There is no significance to the ϵ resonance being drawn at lower mass than the ρ: they are interchangeable). The black dots mark the four double poles in the unphysical region which must be killed by three different zeros (dashed lines) if only straight line zeros are allowed.

provides us with a simple way of keeping track of the zeros and poles of the scattering amplitude, at least in a somewhat idealized world, e.g., one with the mass spectrum of the Lovelace-Veneziano model. The idea of straight line zeros is a further idealization which need not occur in reality. However, it may well be that if we look at the Mandelstam plane as a whole and not in any local region, particularly near threshold, that straight line zeros are a reasonable first, or perhaps zeroth, approximation to reality.

To see simply that the straight line zero hypothesis refers to an idealized world with Veneziano model mass spectrum, let us consider a simple example. Let us return to Fig. 5 and introduce the ϵ or σ resonance. This introduces new double poles in the unphysical region. In the Lovelace-Veneziano model, all poles are real and the ϵ and ρ poles are degenerate in mass. Then a single real fixed u zero can kill the $\rho\rho$, $\epsilon\epsilon$, and $\epsilon\rho$ double poles. However, it is clear that in nature, if the ϵ pole exists, it is not degenerate with the ρ. Then the complex double poles in the unphysical region are not degenerate and each double pole must be killed separately. Clearly, if we only allow fixed u zeros, we must have three distinct zeros, which will enter the physical region before 1200 MeV and produce a spin four resonance in the f_0 region (Fig. 8). Of course, no such zeros occur here. Such a difficulty with straight line zeros not only occurs here but everywhere the ϵ and S^*, or any other "daughter" resonance, in one channel cross resonances in the other.

If we abandon the SLZ hypothesis then one zero can be very clever and wiggle around to kill the four double poles in Fig. 8. Then there is certainly no need for this zero to enter the physical region along a fixed u line. Alternatively, it is possible that there are a number of zeros in the unphysical region killing the double poles, but with only one of them coming near to and entering the physical region.

HIGH ENERGY ZEROS

We next briefly discuss what happens to zeros near the forward or backward directions at high energies. It has been noted by Schmid,[9] Harari,[10] Cohen-Tannoudji et al.[11] and many others that if we look at the near forward Legendre zeros of the ρ, f_0, and g of Fig. 1, then they are at the same t-value: $t \simeq -0.3(\mathrm{GeV}/c)^2$. Similarly, their near backward Legendre zeros lie at fixed $u \simeq -0.3(\mathrm{GeV}/c)^2$. Perhaps such a fixed t or u zero persists at higher energies, as we may expect from duality considerations.

If the amplitude considered involves isospin one in the t-channel, then we expect to see a fixed t zero at high energy. This is the well-known crossover zero which arises from the vanishing of the imaginary part of the ρ exchange amplitude. In a model of just Regge poles and strong exchange degeneracy this vanishing occurs at $t \simeq -0.6(\mathrm{GeV/c})^2$, whereas absorption models with no strong exchange degeneracy place this zero at $t \simeq -0.2(\mathrm{GeV/c})^2$.

In the absorption model of Cohen-Tannoudji et al.[11] which includes duality, it is conjectured that in $\pi^+\pi^-$ scattering (see Figs. 3,7) that a zero leaves the region of the Mandelstam triangle becomes the rho's Legendre zero and then moves on at fixed t becoming the near forward Legendre zeros of the f_0 and g and then the cross over zero. Such a fixed t zero is to be contrasted with Odorico's fixed u zero. We see the data does not locally indicate such a behavior. However, it appears that instead of one zero producing this fixed t effect it is mocked by different zeros in different energy regions, and so exhibiting "average" but not "local" duality.

Similarly backward $\pi^-\pi^0$ scattering is controlled by ρ exchange. We might then expect a zero to leave the Mandelstam triangle become the rho's Legendre zero and move off at fixed u along the line of Odorico's zero. Again such a high energy zero is built up not by one zero but by different zeros in different energy regions. In both forward $\pi^+\pi^-$ and backward $\pi^-\pi^0$ scattering the high energy zero pattern may be distorted by the presence of $I = 0,2$ and $I = 2$ exchange amplitudes, respectively, in addition to ρ exchange, so let us isolate the amplitude with just isospin one in the t-channel to see what happens. The zero contours for this amplitude are shown in Fig. 9 from the data of Carroll et al.[12] as determined by Eguchi et al.[13] * Here the approximately fixed t zero of Cohen-Tannoudji et al. and others appears quite distinctly in this amplitude, which involves just ρ exchange in the t channel. Indeed, the zero contour (Fig. 9) is approximately at $t = -0.3(\mathrm{GeV/c})^2$. This dip in the differential cross section is usually thought of in terms of the vanishing of the imaginary part of the amplitude at a real value of t, rather than as a zero of the whole amplitude at a complex value of t. Tryon[14] has, in fact, determined the imaginary part of the $\rho\pi\pi$ Regge residue function $\gamma_\rho(t)$, from the same data[12] by performing a physical region crossing sum rule calculation. Tryon finds γ_ρ

* Eguchi et al.[13] also determine the zero contours for $\pi^+\pi^-$ and $\pi^-\pi^0$ elastic scattering from the data of Carroll et al. Apart from near $K\bar{K}$ threshold, their results are absolutely identical to those found by Pennington and Protopopescu,[4] using the data of Protopopescu et al. (and shown in Fig. 3) despite the comments made by Eguchi et al. implying the contrary.

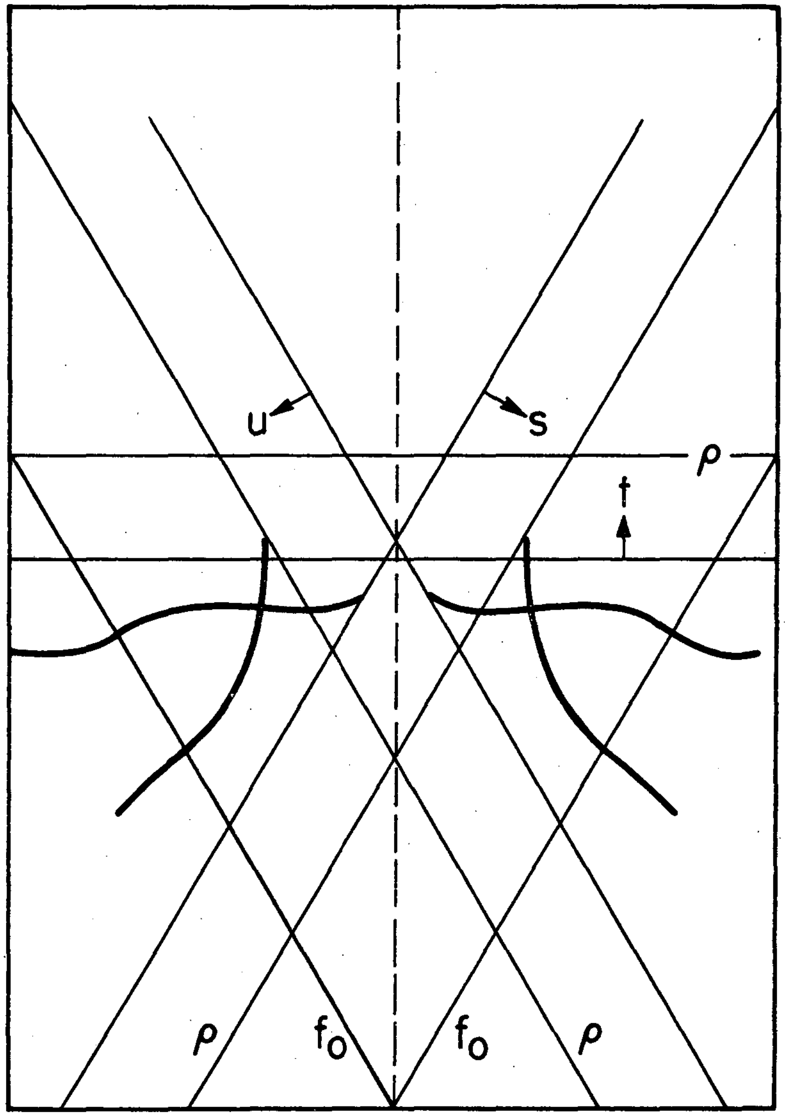

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Fig. 9. The zero contours for the $\pi\pi$ amplitude with isospin one in the t-channel from Eguchi et al. (Ref. 13) using the data of Carroll et al. (Ref. 12). This amplitude vanishes on the line $s = u$ (the vertical dashed line) by Bose symmetry.

vanishes at $t = -0.52(\mathrm{GeV}/c)^2$ which looks rather like the zero implied by strong exchange degeneracy in a pure pole model. However, in Ref. 15 it is reported that Basdevant and Schomblond[16] suggest that this zero is nearer -0.2 than $-0.6(\mathrm{GeV}/c)^2$. We will have to wait to see the details, before we can draw any conclusions.

THE LEGENDRE ZERO OF THE RHO AND CURRENT ALGEBRA?

We next discuss the most important zero in very low energy $\pi\pi$ scattering and that is the ρ's Legendre zero. We have seen that this zero contour appears to pass close to threshold and near to the Mandelstam triangle. It is in this region that current algebra has some predictions to make and so let us see if they are relevant in any way.

Within and near the Mandelstam triangle the amplitude is controlled by just the s- and p-waves $f_0^{\,0}(s)$, $f_0^{\,2}(s)$, and $f_1^{\,1}(s)$. If the exotic channel is weak (in a sense defined below), these partial waves will be quasilinear given by

$$\begin{aligned} f_0^{\,0}(s) &\simeq \tfrac{3}{2}\, a_1^{\,1}(2\mu^2 + s - \tfrac{5}{2}\, s_A) \\ f_0^{\,2}(s) &\simeq \tfrac{3}{4}\, a_1^{\,1}(4\mu^2 - s - 2s_A) \\ f_1^{\,1}(s) &\simeq \tfrac{1}{4}\, a_1^{\,1}(s - 4\mu^2) \end{aligned} \tag{11}$$

on using crossing symmetry, so that

$$\frac{a_0^{\,0}}{a_0^{\,2}} \simeq \frac{5}{2} - \frac{6\mu^2}{s_A} . \tag{12}$$

The forms of Eq. (11) only apply for values of s in, or close to, the gap in the cut s-plane. For values of s_A in this same range, the parameter s_A of Eqs. (11, 12) is the value of s at which the s-wave of the Chew-Mandelstam invariant amplitude $A(s,t) = \frac{1}{3}(F^{s0} - F^{s2})$ vanishes. We see that the amplitude, in the neighborhood of the Mandelstam triangle, is determined essentially by two parameters: $a_1^{\,1}$ to set the scale and the dimensionless ratio $a_0^{\,0}/a_0^{\,2}$. It is for these two parameters that current algebra makes predictions, which we will shortly discuss.

Let us remark that by a weak exotic channel we mean that the contribution of the $I = 2$ absorptive part, both in the direct channel and in the crossed channel, to the Froissart-Gribov integrals for the p- and d-wave amplitudes below threshold, is very much less than the corresponding contributions of the $I = 0$ and 1 absorptive parts. These conditions are justified by the nonexistence of exotic resonances and hence of leading exotic Regge trajectories.

This weakness of the $I = 2$ channel has implied that the s and p waves of Eq. (11) are quasilinear below threshold with derivatives determined by the p-wave scattering length, $a_1^{\,1}$. This scattering length can also be simply calculated using the weakness of the exotic channel. From an unsubtracted fixed s dispersion relation for $F^{s1}(s,t)$, we have

$$a_1^{\,1} = \frac{4}{3\pi}\int_{4\mu^2}^{\infty} \frac{dt}{t^2}\, A^{s1}(t,4\mu^2) \tag{13}$$

where $A^{s1}(t,s)$ is the t-channel absorptive part with $I_s = 1$. Using the crossing matrix we have

$$a_1^{\,1} = \frac{4}{3\pi}\int_{4\mu^2}^{\infty} \frac{dt}{t^2}\, [A^{t1}(t,4\mu^2) - A^{t2}(t,4\mu^2) + A^{s2}(t,4\mu^2)]$$

$$\simeq \frac{4}{3\pi}\int_{4\mu^2}^{\infty} \frac{dt}{t^2}\, A^{t1}(t,4\mu^2) \tag{14}$$

by the weakness of the $I = 2$ absorptive part in both the direct and crossed channels. To a good approximation the integrand of Eq. (14) is dominated by the ρ resonance, so we have

$$a_1^{\,1} \simeq a_1^{\,1}(\rho) = \frac{4\Gamma_\rho}{m_\rho^{\,3}}\left(1 + \frac{10\mu^2}{m_\rho^{\,2} - 4\mu^2}\right) \tag{15}$$

using the narrow resonance approximation. Numerically this gives*

* This simple way of estimating $a_1^{\,1}$, which neglects all contributions except that of the rho, gives a value for $a_1^{\,1}$ which is 10-15% smaller than a more exact calculation[15,17] using the data and high energy estimates in Eq. (13). So our simple calculation is rather good.

$a_1^{\,1}(\rho) = 0.032$ in pion mass units. We see, from Eq. (15), how the ρ resonance essentially determines the scale of the near threshold $\pi\pi$ amplitude,[17] using the weakness of the $I = 2$ amplitude [see Eq. (11)].

We now discuss what current algebra has to say about the $\pi\pi$ amplitude in the neighborhood of the Mandelstam triangle. Current algebra and PCAC make definite predictions about the 4π amplitude when one or two pions have zero 4-momentum, that is for the amplitude at certain kinematic points on the planes $s + t + u = 3\mu^2$ or $2\mu^2$. To compare these predictions with experiment we have to know how to extrapolate to the plane $s + t + u = 4\mu^2$. Assuming this extrapolation is smooth and simple, two definite predictions from PCAC become relevant.

(1) The Adler-Weisberger relation expresses the derivative of the $I_s = 1$ amplitude at $s = 0$, $t = u = \mu^2$, in terms of the pion decay constant f_π (experimentally $f_\pi = 93$ MeV). The derivative of the $I = 1$ amplitude near threshold is just the p-wave scattering length, so that the Adler-Weisberger relation, together with smoothness, predicts[18]

$$a_1^{\,1} = \frac{1}{24\pi f_\pi^{\,2}} \qquad (= 0.030 \text{ in pion mass units}) . \tag{16}$$

Equating the algebraic conditions Eq. (15) and Eq. (16) we obtain

$$\frac{1}{f_\pi^{\,2}} = \frac{96\pi\Gamma_\rho}{m_\rho^{\,2}}\left(1 + \frac{10\mu^2}{m_\rho^{\,2} - 4\mu^2}\right) . \tag{17}$$

With $\mu = 0$ this condition is the well-known KSRF relation.[19] Since this condition is empirically well satisfied PCAC predicts the correct "scale" for low energy $\pi\pi$ dynamics. The second prediction of PCAC, we consider, concerns zeros of the amplitude:

(2) The Adler self-consistency condition[20] states that the $\pi\pi \to \pi\pi$ amplitude vanishes at the point

$$s = t = u = \mu^2 . \tag{18}$$

Weinberg has given an explicit extrapolation of this condition to the plane $s + t + u = 4\mu^2$:

$$A(s = \mu^2, t, u) = 0 \qquad \text{for all } t,u \in (0, 4\mu^2) \tag{19}$$

where

$$A(s,t) = \tfrac{1}{3}(F^{s0} - F^{s2}) = \tfrac{1}{2}(F^{t1} + F^{t2}) . \tag{20}$$

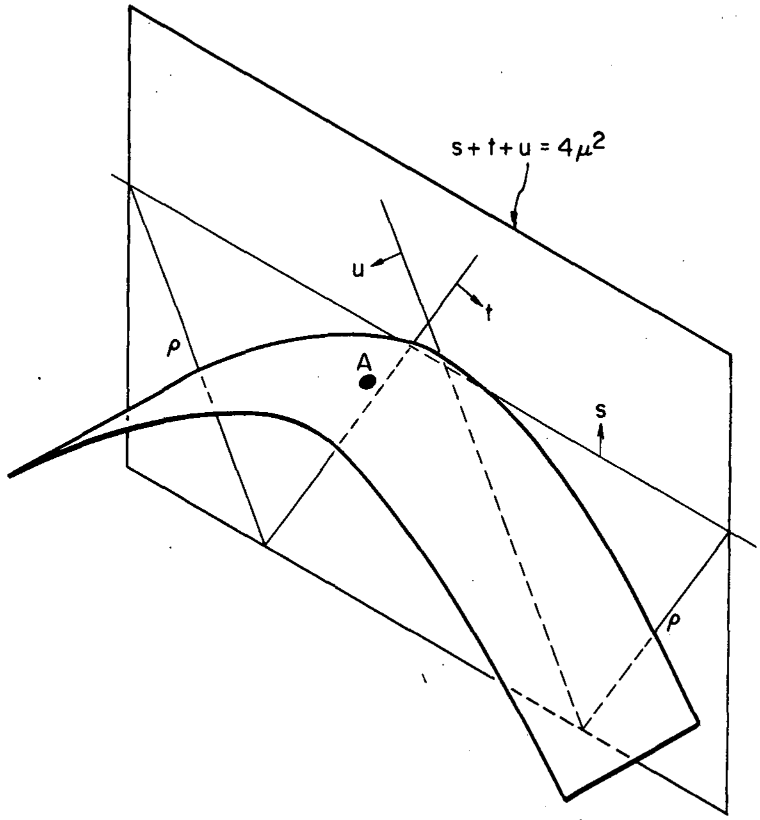

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Fig. 10. A simple three dimensional illustration of how a surface of zeros for the invariant amplitude $A(s,t)$ can relate the Adler zero at the point $s = t = u = \mu^2$ (marked by A) to the Legendre zero of the ρ. This zero extrapolates to the plane $s + t + u = 4\mu^2$ like Weinberg's amplitude, inside the Mandelstam triangle. The amplitude $A(s,t)$ is the $\pi^+\pi^- \rightarrow \pi^0\pi^0$ amplitude in the s-channel and the $\pi^-\pi^0$ elastic scattering amplitude in the t and u channels. On the physical plane this figure is to be compared with Fig. 3. The three dimensions are specified by the orthogonal coordinates $s, t - u$ and $s + t + u$. The latter variable is normal to the plane $s + t + u = 4\mu^2$ and represents varying one pion mass at a time from 140 MeV to zero.

This condition, Eq. (19), implies that the s-wave projection of A vanishes at $s_A = \mu^2$, which we recall from Eq. (12) predicts $a_0^0/a_0^2 = -7/2$. We refer the reader to Weinberg's paper[18] for a discussion of the assumptions involved in this particular extrapolation. This same extrapolation of the Adler zero implies that the $\pi^+\pi^-$ elastic amplitude vanishes on the line $u = 2\mu^2$ (in the Mandelstam triangle), which is where Lovelace imposes the zero in his single term Veneziano model [see Eq. (10) and Fig. 6] for all values of s.[7] Different on mass-shell extrapolations give different values for s_A, and hence for a_0^0/a_0^2, but most[21] give s_A not far from μ^2.

From the data shown in Fig. 3 we have seen that the zero contour, which is the Legendre zero of the ρ, may very well pass through or near the Mandelstam triangle, as suggested by the dotted curves. In Fig. 10 we show a simple extrapolation of the Adler zero on-mass-shell consistent with Weinberg's model for the Chew-Mandelstam invariant amplitude, Eq. (20), inside the Mandelstam triangle and consistent with unitarity in the region of the ρ resonance. (Note no zero at fixed s, t, or u can be consistent with both of these.) Whilst this simple zero contour agrees with the data beyond 500 MeV, we have to look in a different way to see that there is a zero closer to the Mandelstam triangle. Let us look at Re A(s,t) along the line $t = 0$ as a function of $(s - u)$ as found from the data. From Fig. 11, it is clear that a zero of the amplitude really must occur near the Mandelstam triangle, though, of course, not necessarily exactly where Weinberg's model places it. In fact, a zero will pass through the Mandelstam triangle for $1 > a_0^0/a_0^2 > -\infty$, and just below it (in the sense of Fig. 10) if the ratio a_0^0/a_0^2 is large and positive. From Eqs. (11) and (12), it is clear that an exact knowledge of s_A, an as yet poorly known parameter, is required before we can know the s-wave scattering lengths in any way but crudely.

By relating the on-mass-shell appearance of the Adler zero to the Legendre zero of the ρ we understand immediately why if we take a model like that of Weinberg's or any other having s-waves with near threshold zeros and unitarize it, we obtain a resonating p-wave.[22] The near-threshold zeros of the s-wave imply a zero in the invariant amplitude which unitarity forces to curve down into the t and u channel physical regions (Fig. 10), and when this zero contour crosses the line $z_t = 0$ we have to have a p-wave resonance,[23,2] at least if the exotic amplitude is weak. However, the actual parameters of this p-wave resonance are not quite so simply understood.

Having discussed most of the zeros of present physical interest all that remains is to summarize.

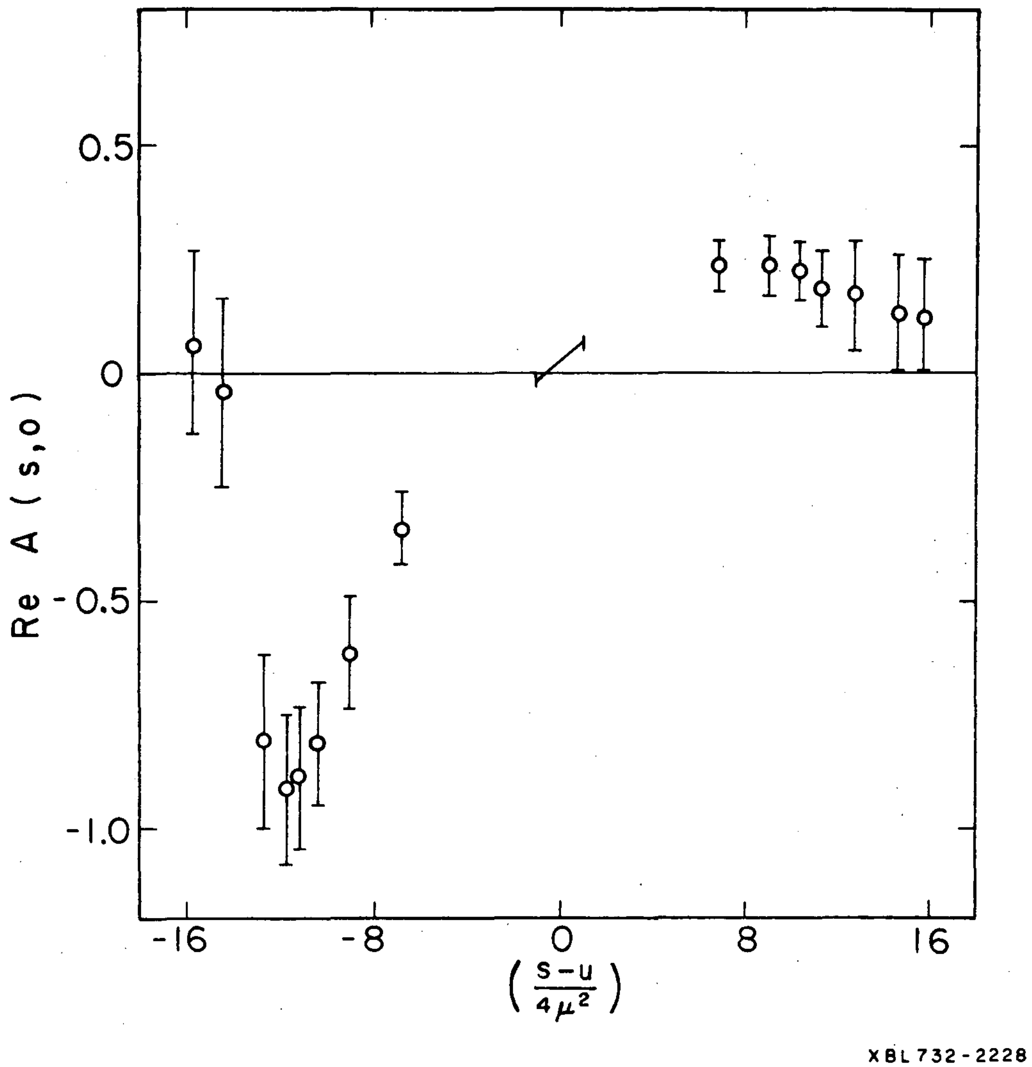

Fig. 11. A plot of the Re A(s,t = 0) against $(s - u)/4\mu^2$ from the data of Protopopescu et al. and Baton et al., illustrating that there must be a zero of this amplitude between $(s - u)/4\mu^2 = -5$ and $+5$. The solid line is Weinberg's prediction for this amplitude inside the Mandelstam triangle. The amplitude A(s,t) describes $\pi^+\pi^- \to \pi^0\pi^0$ scattering in the s-channel. For similar plots for other $\pi\pi$ amplitudes, see D. Morgan (Ref. 1).

CONCLUSIONS

Zeros provide us with a particularly beautiful, yet simple, way of relating various dynamical aspects of the scattering amplitude. We have discussed how double-pole-killing zeros become the Legendre zeros of resonances and how these zeros appear in the high energy Regge region. Finally we have discussed how the near-threshold zeros of the $\pi\pi$ s-wave amplitudes are related to the existence of a resonating p-wave and how this Legendre zero of the ρ may be thought of as the on-mass-shell manifestation of the Adler zero appearing in the physical scattering amplitude. In this way,[24] the investigation of zeros may help us to understand how unitarity, analyticity, and crossing build the dynamics of the scattering amplitude. Indeed, we can think of the study of the scattering amplitude, both theoretically and experimentally, as essentially one of looking at zeros travelling across the Mandelstam plane.

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