Title
ELEMENTARY DERIVATION OF POISSON STRUCTURES FOR FLUID DYNAMICS AND ELECTRODYNAMICS

Author
Kaufman, A.N.

Publication Date
1982-02-01

LBL-14112
Preprint

Lawrence Berkeley Laboratory

UNIVERSITY OF CALIFORNIA

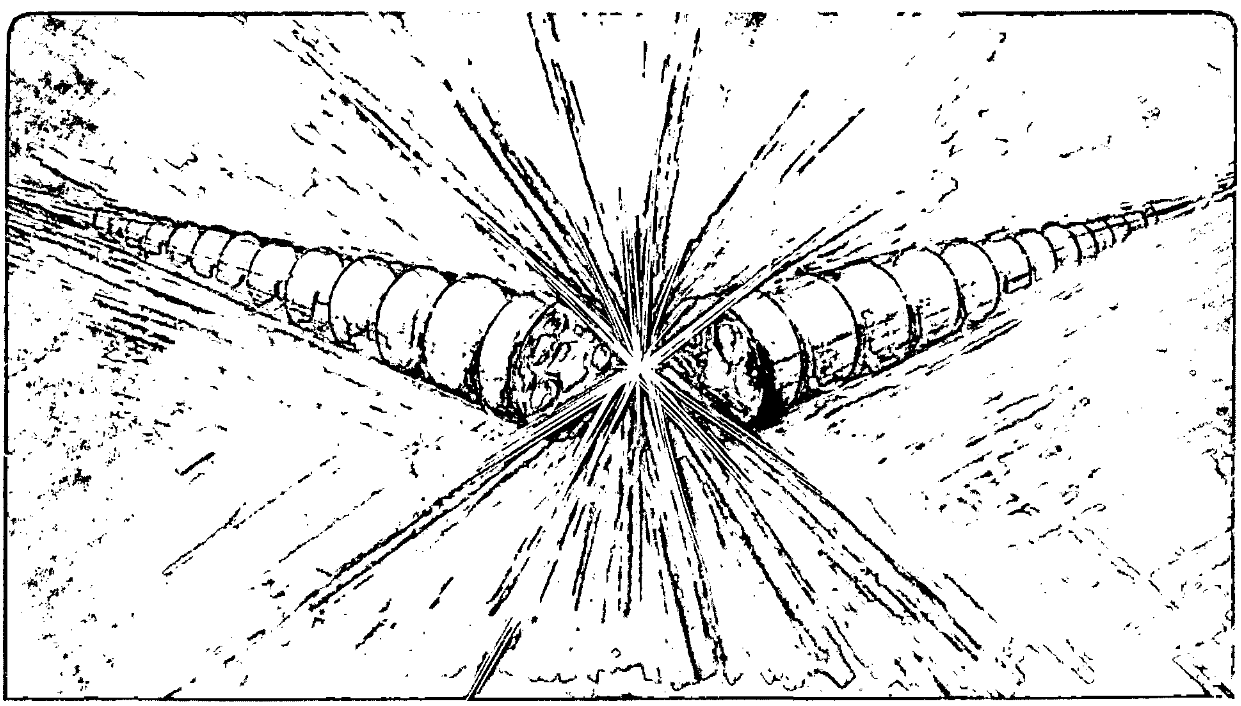
Accelerator & Fusion Research Division

Submitted to Physics of Fluids

ELEMENTARY DERIVATION OF POISSON STRUCTURES
FOR FLUID DYNAMICS AND ELECTRODYNAMICS

Allan N. Kaufman

February 1982



Prepared for the U.S. Department of Energy under Contract DE-AC03-76SF00098

DISCLAIMER

LBL-14112

Elementary Derivation of Poisson Structures for Fluid Dynamics and Electrodynamics

Allan N. Kaufman

Physics Department
and
Lawrence Berkeley Laboratory
University of California
Berkeley, CA 94720

February 1982

Abstract

The canonical Poisson structure of the microscopic Lagrangian is used to deduce the noncanonical Poisson structure for the macroscopic Hamiltonian dynamics of a compressible neutral fluid and of fluid electrodynamics.

---

* This work was supported by the Office of Energy Research and the Office of Basic Energy Sciences of the U. S. Department of Energy under Contract No. DE-AC03-76SF00098.

I. INTRODUCTION

It has only recently become apparent that Hamiltonian structures exist for several common non-dissipative fluid models. This recognition had been delayed because of the near-universal belief among physicists that canonical variables were required for Hamiltonian formalisms. For example, in Zakharov's review (1), Hamiltonian fluid models were introduced in terms of canonical but unphysical fields.

Great advantages accrue when a model can be formulated in terms of a Hamiltonian with physical variables. This was demonstrated by Littlejohn (2) for the single-particle guiding-center problem. The current interest in fluid Hamiltonians was stimulated by Morrison and Greene's discovery of a Hamiltonian structure for MHD (3), followed by Morrison's formulation of such a structure for the Vlasov-Coulomb and Vlasov-Maxwell systems (4).

Morrison's approach (5) requires a considerable amount of ingenuity. An alternate approach, developed by Marsden and Weinstein (6), uses sophisticated group-theoretical techniques. A third method has now been introduced by Bialynicki-Birula and Hubbard (7), based on the corresponding microscopic model, which leads to the required Poisson brackets quite easily. They applied their method to the relativistic Vlasov-Maxwell system. In the present paper, their method is used to derive the known Poisson structures for non-relativistic compressible fluid dynamics (3) and electrodynamics (8). Our derivation is far more elementary than previous ones. As a slight generalization, curvilinear coordinates are employed.

II. FLUID DYNAMICS

Consider first a neutral fluid, one species for simplicity, composed of N interacting particles. The velocity-dependent part of the Lagrangian is

$$L = \sum_i \frac{1}{2} m\, g_{\mu\nu}(\underline{r}_i)\, v_i^\mu\, v_i^\nu \quad . \tag{1}$$

The canonical momenta,

$$p_\mu^i \equiv \partial L / \partial v_i^\mu = m\, g_{\mu\nu}(\underline{r}_i)\, v_i^\nu \quad , \tag{2}$$

satisfy the canonical Poisson brackets:

$$\{r_i^\mu,\, p_\nu^j\} = \delta_{ij}\, \delta_\nu^\mu \quad . \tag{3}$$

For a macroscopic model, introduce the mass and momentum densities:

$$\rho(\underline{x}) \equiv \sum_i m\delta(\underline{x}-\underline{r}_i); \quad g_\mu(\underline{x}) \equiv \sum_i p_\mu^i\, \delta(\underline{x}-\underline{r}_i) \quad . \tag{4}$$

Their brackets follow immediately from (3); for example,

$$\begin{aligned} \{\rho(\underline{x}),\, g_\mu(\underline{x}')\} &= \sum_i m\{\delta(\underline{x}-\underline{r}_i),\, p_\mu^i\}\, \delta(\underline{x}'-\underline{r}_i) \\ &= \sum_i m\, \delta(\underline{x}'-\underline{r}_i)(\partial/\partial r_i^\mu)\, \delta(\underline{x}-r_i) \\ &= -\, \partial_\mu\, \delta(\underline{x}-\underline{x}')\, \rho(\underline{x}') \ , \end{aligned} \tag{5}$$

where $\partial_\mu \equiv \partial/\partial x^\mu$. Similarly,

$$\{g_\mu(\underline{x}),\, g_\nu(\underline{x}')\} = [-g_\mu(\underline{x}')\, \partial_\nu + g_\nu(\underline{x})\, \partial_\mu']\, \delta(\underline{x}-\underline{x}'). \tag{6}$$

Next we consider functionals of $\rho(\underline{x})$ and $g_\mu(\underline{x})$. For two such functionals F, G, we have (by the chain rule)

$$\{F,G\} \equiv \int d^3x \int d^3x' \; [F^\mu(\underline{x})\; G^\nu(\underline{x}')\; \{g_\mu(\underline{x}),\; g_\nu(\underline{x}')\}$$

$$+ [F_\rho(\underline{x})\; G^\mu(\underline{x}') - F^\mu(\underline{x}')\; G_\rho(\underline{x})]\; \{\rho(\underline{x}),\; g_\mu(\underline{x}')\}], \qquad (7)$$

where $F^\mu(\underline{x}) \equiv \delta F/\delta g_\mu(\underline{x})$, $F_\rho(\underline{x}) \equiv \delta F/\delta\rho(\underline{x})$. Now substitute (5) and (6) into (7), obtaining

$$\{F,G\} = \int d^3x \; [- F^\mu(\partial_\mu G^\nu) g_\nu + G^\nu(\partial_\nu F^\mu) g_\mu$$

$$- F^\mu(\partial_\mu \; G_\rho)\rho + G^\mu(\partial_\mu F_\rho)\rho]. \qquad (8)$$

To generalize to several (interacting) species, simply add species labels and sum over species. This is the result of Morrison and Greene (3).

To include an advected quantity σ_i, define its density

$$\sigma(\underline{x}) \equiv \sum_i \sigma_i \; \delta(\underline{x}-\underline{r}_i).$$

By (3), we have

$$\{\sigma(\underline{x}),\; g_\mu(\underline{x}')\} = - \partial_\mu \; \delta(\underline{x}-\underline{x}')\sigma(\underline{x}') \qquad (9)$$

For functionals of ρ, g_μ, σ, we find an additional term for (8):

$$\int d^3x \, [- F^\mu(\partial_\mu \, G_\sigma)\sigma + G_\mu(\partial_\mu \, F_\sigma)\sigma] \, , \tag{10}$$

where $F_\sigma \equiv \delta F/\delta\sigma(\underline{x})$.

III. FLUID ELECTRODYNAMICS

To allow for interaction with the self-consistent electromagnetic field, we generalize (1) (for one species) to

$$L = \sum_i \frac{1}{2} m \, g_{\mu\nu}(\underline{r}_i) \, v_i^\mu \, v_i^\nu \; + \frac{1}{2} \int d^3x (\partial A_\mu/\partial t)(\partial A^\mu/\partial t)$$

$$+ \sum_i e v_i^\mu \, A_\mu(\underline{r}_i), \tag{11}$$

where $E_\mu(\underline{x}) = -\partial A_\mu/\partial t$ (in the radiation gauge). The particle canonical momenta are

$$\pi_\mu^i \equiv \partial L/\partial v_i^\mu = p_\mu^i + eA_\mu(\underline{r}_i), \tag{12}$$

while the canonical field conjugate to $\underline{A}$ is

$$\pi^\nu(\underline{x}) \equiv \partial L/\partial \dot{A}_\nu(\underline{x}) = \dot{A}^\nu(\underline{x}) = - E^\nu(\underline{x}) \, . \tag{13}$$

The canonical brackets are

$$\{r_i^\mu, \pi_\nu^j\} = \delta_{ij} \, \delta_\nu^\mu, \quad \{A_\mu(\underline{x}), \pi^\nu(\underline{x}')\} = \delta_\mu^\nu \, \delta(\underline{x}-\underline{x}'). \tag{14}$$

It is now straightforward to calculate the brackets connecting $\underline{p}$, $\underline{E}$, $\underline{B}$:

$$\{p_\mu^i,\ p_\nu^j\} = \delta^{ij}\ e\ B_{\mu\nu}\ (\underline{r}_i), \tag{15}$$

where $\quad B_{\mu\nu}(\underline{x}) = \partial_\mu A_\nu(\underline{x}) - \partial_\nu A_\mu(\underline{x}) = \varepsilon_{\mu\nu\lambda}\ B^\lambda(\underline{x});$

$$\{p_\mu^i,\ E^\nu(\underline{x})\} = \delta_\mu^\nu\ \delta(\underline{x}-\underline{r}_i); \tag{16}$$

$$\{E^\mu(\underline{x}),\ B^\nu(\underline{x}')\} = -\ \varepsilon^{\mu\nu\lambda}\ \partial_\lambda\ \delta(\underline{x}-\underline{x}'). \tag{17}$$

Moving on to the kinetic momentum density $\underline{g}$, we find

$$\{g_\mu(\underline{x}),\ g_\nu(\underline{x}')\} = \delta(\underline{x}-\underline{x}')(e/m)\rho(\underline{x})\ B_{\mu\nu}(\underline{x}) + \text{Eq. (6)}; \tag{18}$$

$$\{g_\mu(\underline{x}),\ E^\nu(\underline{x}')\} = \delta(\underline{x}-\underline{x}')(e/m)\rho(\underline{x})\delta_\mu^\nu. \tag{19}$$

Finally, for functionals of $\underline{g}$, ρ, σ, $\underline{E}$, $\underline{B}$, we obtain

$$\begin{aligned}\{F,\ G\} = \int d^3x[(&\underline{G}\cdot\nabla\underline{F} - \underline{F}\cdot\nabla\underline{G})\cdot\underline{g} + (\underline{G}\cdot\nabla F_\rho - \underline{F}\cdot\nabla G_\rho)\rho \\ &+ (\underline{G}\cdot\nabla F_\sigma - \underline{F}\cdot\nabla G_\sigma)\ \sigma + (\underline{G}^B\cdot\nabla\times\underline{F}^E - \underline{F}^B\cdot\nabla\times\underline{G}^E) \\ &+ (e/m)\rho\ (\underline{F}\times\underline{G}\cdot\underline{B} + \underline{F}\cdot\underline{G}^E - \underline{G}\cdot\underline{F}^E),\end{aligned} \tag{20}$$

where $\quad \underline{F} \equiv \delta F/\delta\underline{g}, \quad \underline{F}^E \equiv \delta F/\delta\underline{E}, \quad \underline{F}^B \equiv \delta F/\delta\underline{B}.$

To generalize to several species, simply sum (20) over species; there are no cross terms. This result agrees with that obtained by Spencer (8).

References

1. V. E. Zakharov, "Hamiltonian Formalism for Waves in Nonlinear Media having Dispersion," Radiophysics and Quantum Electronics 17, 326 (1975).
2. R. G. Littlejohn, J. Math. Phys. 20, 2445 (1979); Phys. Fluids 24, 1730 (1981).
3. P. J. Morrison and J. M. Greene, Phys. Rev. Lett. 45, 790 (1980); 48, 569 (1982).
4. P. J. Morrison, Physics Letters 80 A, 383 (1980).
5. P. J. Morrison, "Poisson Brackets for Fluids and Plasmas," in "Mathematical Methods in Hydrodynamics" (M. Tabor and Y. Treve, eds.) AIP Conf. Proceedings (1981).
6. J. E. Marsden and A. Weinstein, Physica D (in press).
7. I. Bialynicki-Birula and J. C. Hubbard, submitted to Phys. Rev. A.
8. R. G. Spencer and A. N. Kaufman, Phys. Rev. A (Apr. 1982).