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Decay Spectra of Particles and Resonances Produced in a Central Plateau

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We consider the two-body decay of the spinless resonance or particle produced in a central plateau, with an arbitrary transverse-momentum spectrum. The spectrum of the decay products is calculated exactly as an integral over the spectrum of the centrally produced resonance or particle. Special forms applicable to large and small momentum transfer are presented along with an accurate inversion formula. We show how the large-transverse-momentum behavior of the resonance production is replicated in the decay products. The decay $\pi^0 \to \gamma + \gamma$ is considered in detail.

I. INTRODUCTION

The recent verification of the existence of a central plateau in the CERN intersecting Storage Rings (ISR) experiments has allowed us to probe deeper into the detailed mechanism of pionization. In previous papers we have investigated the properties of the pionization spectrum in $q_0$, resulting from the internal-damping structure. As originally discussed by Amati, Stanghellini, and Fubini (ASF), the pions in the central plateau arise from fireballs or resonances produced in a chain of peripheral pion exchanges. In this paper we calculate the inclusive spectrum of a particle resulting from decay of a spinless two-particle resonance which is peripherally produced in a central-plateau region. The generality of the calculation allows it to be applied also to the case of a $\pi^0$ produced in the central plateau, which then decays into two photons. It can then be used to infer the $\pi^0$ spectrum from the $\gamma$ spectrum.

Our work is an extension of the treatment of these problems as recently considered by others. Our formulation (1) includes an exact treatment of the kinematics and integrations; (2) is applicable to any $q_0$ spectrum of produced resonances or $\pi^0$'s; (3) applies to both large and small $q_0$; (4) has the integrations performed analytically, not numerically; (5) gives a unified treatment of massive and massless final particles. The formulation includes many of the earlier results as limiting or special cases.

The calculation proceeds by considering a resonance of momentum $q$ and mass $q^2 = m^2$ being produced in a central plateau with a spectrum $\rho(q^2, q_0^2)$ independent of longitudinal momentum. This then decays into two particles of masses $\mu_1$ and $\mu_2$ so that $q = q_1 + q_2$. Since only one particle $q_1$ is observed in the single-particle spectrum, we must integrate over the momentum of $q_2$. It is convenient to work with

$$\eta = (q_1^+ + q_2^+)^2 / m^2, \quad \eta_1 = (q_1^+)^2 / \mu_1^2, \quad \eta_2 = (q_2^+)^2 / \mu_2^2, \quad m_0^2 = m^2 - \mu_1^2 - \mu_2^2,$$

where the $\perp$ denotes two-dimensional transverse vectors. The integration over $q_1$ is performed by converting to integrals over $\eta$, $\eta_1$, and the rapidity $y_2 = \sinh^{-1} (q_2^+ / \eta_2^{1/2})$. The $\eta$ and $\eta_1$ integrations are performed exactly for infinite energy, and the integral over $\eta_1$ with the general function $\rho(\eta) = \rho (m^2, q_0^2)$ remains.
In Sec. II we formulate the problem and calculate the decay pionization spectrum for \( \eta_1 \gg m_\pi^2/4m^2 \).

In Sec. III we study the large-\( \eta_1 \) approximation and show how the large-\( \eta \) behavior of \( \rho(\eta) \) replicates itself in the large-\( \eta_1 \) behavior. In Sec. IV the decay spectrum is calculated for \( \eta_1 \ll m_\pi^2/4m^2 \). The \( \pi^0 \rightarrow 2\gamma \) decay is presented in Sec. V.

II. RESONANCE DECAY SPECTRUM

We assume that a spinless resonance or particle \( R \) is produced in the central-plateau region and calculate the transverse-momentum distribution of its decay products. For notation we call the decay products \( \pi_1 \) and \( \pi_2 \) of mass \( \mu_1 \) and \( \mu_2 \) and consider them as distinguishable. In Sec. V, however, they are considered as massless photons.

The inclusive \( R \) production \( a + b \rightarrow R(\pi_1 \pi_2) + X \)

\[
\rho_1(\eta) = \frac{1}{2\pi} \int d\phi_2 \int d\Phi_{X}(2\pi)^4 \delta^4(q_1 + q_2 + q_X - \mu_1 - \mu_2) \left| T(ab \rightarrow RX) T(R \rightarrow \pi_1 \pi_2) \right|^2,
\]

where

\[
d\phi_2 = \frac{d^2q_2^+ dy_2}{2(2\pi)^3}
\]

and

\[
T(R \rightarrow \pi_1 \pi_2) = (16\pi \Gamma m_R)^{1/2}.
\]

We assume that the inclusive integration and summation over \( X \) produces the spinless resonances \( R \) in a central-plateau region constant in the \( R \)'s rapidity:

\[
\rho_1(\eta) = \frac{1}{2\pi} \int d\eta \int \frac{d\phi_{X}(2\pi)^4 \delta^4(q_1 + q_2 + q_X - \mu_1 - \mu_2)}{m_R^2 - m_R^2 + i\Gamma m_R^3} \left| T(ab \rightarrow RX) T(R \rightarrow \pi_1 \pi_2) \right|^2.
\]

In the narrow-width limit \( \Gamma \ll m_R \) the integral in brackets becomes 1, and we evaluate the remaining integral at \( m_R^2 = m_\pi^2 \). For the remainder of this paper we will suppress this integral over the resonance virtual mass as well as the dependence of \( \rho \) on \( m_R^2 \).

We now proceed to convert the integral over \( d\phi_2 \) to \( d\eta d\eta_2 dy_2 \) and perform the integrations over \( d\eta_2 dy_2 \), leaving only the \( d\eta \) integral over the unspecified resonance spectrum

\[
\rho(\eta) = \rho(m^2, (q_1 + q_2)^2).
\]

Using the variables:

\[
q_1^2 = \eta_1^{1/2} \cosh y_1,
q_1' = \eta_1^{1/2} \sinh y_1,
q_2^2 = \eta_2^{1/2} \cosh y_2,
q_2' = \eta_2^{1/2} \sinh y_2,
\eta_1 = (q_1^2 + \mu_2^2),
\eta_2 = (q_2^2 + \mu_1^2),
\]

we compute the argument of the \( \delta \) function:

\[
(q_1 + q_2)^2 - m^2 - \mu_1^2 + \mu_2^2 + (q_1')^2 + (q_2')^2
\]

\[
+ 2\eta_1 \eta_2^{1/2} \cosh(y_1 - y_2).
\]

The \( \delta \) function is then satisfied at two values of \( y_2 \) given by

\[
q_1' + q_2' - m^2 + \mu_1^2 - \mu_2^2
\]

\[
+ 2\eta_1 \eta_2^{1/2} \cosh(y_1 - y_2).
\]
Integrating over all \( y_2 \) for infinite energy will then give a \( \theta \) function when (2.9) can be satisfied:

\[
\rho_1(\eta_1) = \frac{2}{\pi} \int d^2q_2 \frac{\rho(\eta_1) \theta(\eta_1^{1/2} - \eta_2^{1/2} - \eta_{1/2}^{1/2})}{\Delta^{1/2}(\eta_1, \eta_2)},
\]

where

\[
\Delta(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2yz - 2zx.
\]

We now convert

\[
\rho_1(\eta_1) = \frac{2}{\pi} \int d\eta \int d\eta_2 \frac{\rho(\eta) \theta(\eta^{1/2} - \eta_2^{1/2} - \eta_{1/2}^{1/2})}{\Delta^{1/2}(\eta_1, \eta_2)},
\]

where the latter \( \theta \) function guarantees a physical angle between \( q_1 \) and \( q_2 \).

We define the four roots in \( \eta_2 \) of the denominator:

\begin{align*}
\alpha &= (\eta^{1/2} + \eta_1^{1/2})^2, \\
\beta &= \mu_2^2 + (\eta - \mu_2^2)^{1/2} + (\eta_1 - \mu_2^2)^{1/2}, \\
\gamma &= (\eta^{1/2} - \eta_1^{1/2})^2, \\
\delta &= \mu_2^2 + (\eta - \mu_2^2)^{1/2} - (\eta_1 - \mu_2^2)^{1/2}.
\end{align*}

This gives

\[
\rho_1(\eta_1) = \frac{2}{\pi} \int d\eta \int d\eta_2 \frac{\theta(-\Delta(\eta, \eta_1, \eta_2, \eta_3, \eta_4))}{\Delta^{1/2}(\eta_1, \eta_2)}.
\]

This is equivalent to

\[
\eta = (q_1^2)^{1/2} + (q_2^2)^{1/2} + 2 | q_1^2 | | q_2^2 | \cos \theta + m^2,
\]

we have

\[
d^2q_2 = (2)^{1/2} \frac{d\eta_1 d\eta_2}{[-\Delta(\eta - m^2, \eta_1, \eta_2, \eta_3, \eta_4)]^{1/2}},
\]

where the extra (2) is included for the double-valued mapping \( \eta = \pm \theta_2 \). Rewriting this in terms of \( \eta_1, \eta_2 \) we have from (2.11)

\[
\eta_2 \leq c \leq a,
\]

\[
d \leq \eta_1 < b.
\]

Therefore, for the upper limit on \( \eta_2 \) we must know when \( d \) is greater or less than \( c \). The solutions in \( \eta \) to the equations \( b = c \) and \( c = d \) are formally

\[
\eta_1 = \frac{m_0^2}{4 \mu_1^2} \left[ \eta_1^{1/2} + (\eta_1 - \mu_1^2)^{1/2} \left( 1 - \frac{4 \mu_1^2 m_0^2}{m_0^4} \right) \right]^2.
\]

By substituting \( \eta = \eta_1 \) into (2.17), we find that \( \eta_1 \) corresponds to \( c = d \) but not to \( b = c \). It is therefore the upper boundary of the \( \eta_1 \) integration region (see Fig. 2).

In substituting \( \eta = \eta_1 \) in (2.17), we find that for \( \eta_1 > m_0^2/4m_0^2 \)

\[
(\eta_1 - m_0^2)^{1/2} = \frac{m_0^2}{2 \mu_1^2} \left[ \eta_1^{1/2} \left( 1 - \frac{4 \mu_1^2 m_0^2}{m_0^4} \right) \right],
\]

while for \( \eta_1 < m_0^2/4m_0^2 \), the right-hand side has the opposite sign. Then for \( \eta_1 > m_0^2/4m_0^2 \), the substitution \( \eta = \eta_1 \) corresponds to \( c = d \) but not to...
In this section we treat the case \( \eta_1 > m_0^4/4m^2 \) which for a \( \rho \) resonance is \( |q_1^2| > 0.36 \text{ GeV/c} \) and for a \( \pi^0 \) decay is \( |q_1^2| > 0.07 \text{ GeV/c} \). For \( \eta_1 > m_0^4/4m^2 \) we find from (2.17) that \( b > c \) always holds and, therefore, the limits are

\[
d \leq \eta_2 < c. \tag{2.22}
\]

Also, for \( \eta_1 > m_0^4/4m^2 \) we have shown that

\[
\eta_-. < \eta < \eta_. \tag{2.23}
\]

with \( a > b > c > d \) we define

\[
r^2 = \frac{(a - b)(c - d)}{(a - c)(b - d)},
\]

\[
r^2 = \frac{m_0^4 - 4 \left( (\eta_1 m_0^2 - (\eta - m^2)^{1/2} (\eta - m_0^2)^{1/2})^2 \right)}{16 \left( \eta_1 (\eta - m^2) (\eta - m_0^2) \right)^{1/2}}. \tag{2.24}
\]

Then the exact result for the single-particle spectrum from (2.18) is,\(^{12}\) for \( \eta_1 > m_0^4/4m^2 \),

\[
\rho_l(\eta_1) = \int_{\eta_-}^{\eta_+} d\eta \frac{F(1/2, 1/2; 1; r^2 \rho(\eta))}{2 \left[ (\eta_1 (\eta - m^2) (\eta - m_0^2) \right)^{1/2}}. \tag{2.25}
\]

\( F \) is a hypergeometric function which is 1 at \( r^2 = 0 \). It is related to elliptic integrals\(^{13}\):

\[
F(1/2, 1/2; 1; r^2) = \frac{2}{\pi} K(r). \tag{2.26}
\]

### III. SPECTRA FOR \( \eta_1 \) LARGE COMPARED WITH \( m^2 \)

The relation between the resonance spectrum \( \rho(\eta) \) and the decay spectrum \( \rho_l(\eta_1) \) becomes even simpler for large \( \eta_1 \).

For \( \eta_1 > \mu_+^2 \), the limits in (2.20) become

\[
\eta_+ = \frac{m_0^4}{4\mu_+^2} \eta_1 \left[ 1 + \left( 1 - \frac{4\mu_+^2 m^2}{m_0^4} \right)^{1/2} \right] = \eta_1 \eta_+. \tag{3.1}
\]

\[
\eta_+ \gg \mu_+^2, \mu^2 \Rightarrow \frac{m_0^4}{\mu_+^2} \eta_1, \tag{3.2}
\]

\[
\eta_-. \gg \mu_+^2, \mu^2 \Rightarrow \frac{m_0^4}{4\mu_+^2} \eta_1 \left[ 1 - \left( 1 - \frac{4\mu_+^2 m^2}{m_0^4} \right)^{1/2} \right] = \eta_1 \eta_. \tag{3.3}
\]

For \( \eta_1 \gg m^2 \), \( r^2 \) approaches zero as

\[
r^2 = \frac{m_0^4}{16 \eta_1 \eta} - \frac{m_0^4}{16 \eta_1 \eta}. \tag{3.3}
\]

In fact, for \( \eta_1 > 1.5 m^2 \) we have to better than 1% accuracy

\[
F(1/2, 1/2; 1; r^2) \approx 1. \tag{2.27}
\]

In this region,

\[
\rho_l(\eta_1) \approx \frac{1}{2\eta_1^{1/2}} \int_{\eta_1}^{\eta_+} d\eta \frac{\rho(\eta)}{[\eta (\eta_1 - m^2)]^{1/2}}. \tag{3.4}
\]

We may invert this by differentiation and ignore \( \rho \) at the upper limit if it is rapidly falling to obtain

\[
\rho(\eta_1 \eta_.) = - \frac{2}{c_3} \left[ \frac{1}{\eta_1 (\eta_1 - m^2)} \right]^{1/2} \frac{d}{d\eta_1} [\eta_1^{1/2} \rho_l(\eta_1)]. \tag{3.5}
\]

We are particularly interested in the connection between the asymptotic behavior of \( \rho(\eta) \) and that of \( \rho_l(\eta_1) \). For the case \( \eta \gg m^2 \),\( \gg \mu_+^2 \) we take the upper limit effectively infinite:

\[
\rho_l(\eta_1) = \frac{1}{2\eta_1^{1/2}} \int_{\eta_1}^{\infty} d\eta \frac{\rho(\eta)}{\eta_1^{1/2}}. \tag{3.6}
\]

The three cases for large-\( \eta \) behavior,

\[
\rho(\eta) \sim [\eta^{-\alpha}, e^{-\alpha \eta}, e^{-\beta \eta^{1/2}}], \tag{3.7}
\]

become after integration at large \( \eta_1 (\alpha > 0) \)

\[
\rho_l(\eta_1) \sim \left[ \eta_1^{-\alpha}, e^{-\alpha \eta_1^{1/2}}, e^{-\beta \eta_1^{1/2}} \right]. \tag{3.8}
\]

A simple ASF model for resonance production with peripheral pion exchange gives \( \rho(\eta) \propto \eta^{-4} \) due to the pion propagators alone.\(^3\) The resulting \( \eta_1 \) spectrum is inconsistent with data. In fact, internal form factors must also be included\(^4\) to give \( \rho(\eta) \propto \eta^{-4} \) and therefore \( \rho_l(\eta_1) \propto \eta_1^{-4} \) to fit the data.

### IV. DECAY SPECTRUM FOR \( \eta_1 < m_0^4/4m^2 \)

For \( \eta_1 < m_0^4/4m^2 \), we find from Fig. 2 two regions of \( \eta_1, \eta_+ \) space which give different \( \eta_2 \) limits. \( \eta_1 \) is fixed and for region I (Fig. 2)

\[
\eta_1 < \eta_+; b > c \text{ so } d \leq \eta_2 < c. \tag{4.1}
\]

But for region II

\[
m^2 < \eta < \eta_+; d < b < c \text{ so } d \leq \eta_2 < b. \tag{4.2}
\]

From (2.16) we now include the additional contributions of region II:
\[ \rho_1(\eta_1) = \frac{2}{\pi} \int_{n_0}^{n_f} d\eta \rho(\eta) \int_{m_2}^{m_0} d\eta_2 \left( a - \eta_2 \right) \left( b - \eta_2 \right) \left( c - \eta_2 \right) \left( d - \eta_2 \right) \eta_2^{-1/2} \]
\[ + \frac{2}{\pi} \int_{m_2}^{n_f} d\eta \rho(\eta) \int_{c}^{b} d\eta_2 \left( a - \eta_2 \right) \left( b - \eta_2 \right) \left( c - \eta_2 \right) \left( d - \eta_2 \right) \eta_2^{-1/2}. \]

In the second integral \( a > c > b > d \), giving the result
\[ \rho_1(\eta_1) = \int_{n_0}^{n_f} d\eta \frac{F(\frac{1}{2}, \frac{3}{2}; 1; \eta^2)}{2[\eta \eta_1(\eta - m^2) - (\eta - \mu_1^2)]^{1/4}} + 2 \int_{m_2}^{n_0} d\eta \frac{F(\frac{1}{2}, \frac{3}{2}; 1; \eta^2) \rho(\eta)}{m_0^4 - 4[\eta^{1/2} \eta_1^{1/2} - (\eta - m^2)^{1/2} (\eta_1 - \mu_1^2)]^{1/2}}. \]

The \( 1/\eta^2 \) in the second integral resulted from the interchange of \( b \) and \( c \) in the ordering and in (2.24). In the first integral \( \eta^2 \ll 1 \), and in the second \( 1/\eta^2 < 1 \).

We note that for \( \eta_1 < m_4^4/4m_3^4 \) the curve \( \eta_2 = 1 \) or \( b = c \) always occurs in the integration region, Fig. 2, and it is no longer possible to approximate the hypergeometric function by 1 throughout the entire region.

We take the limiting case \( \eta_1 \to \mu_1^2 \) or \( (q^2_1)^2 \to 0 \). In this limit
\[ \eta_2 \ll \frac{1}{(\eta_1 - \mu_1^2)^{3/2}} \to 0, \]
\[ F(\frac{1}{2}, \frac{3}{2}; 1; \eta^2) = \frac{\ln \eta}{\sqrt{\eta}}, \]
\[ \alpha \sim (\eta_1 - \mu_1^2)^{3/2} \ln (\eta_1 - \mu_1^2), \]
\[ (\eta_2 - \eta_1) \sim (\eta_1 - \mu_1^2)^{1/2}. \]
Combining the above we find that the first integral in (4.4) vanishes as \( \eta_1 \to \mu_1^2 \), and in the second integral the hypergeometric function approaches 1:
\[ \rho_1(\eta_1 = \mu_1^2) = 2 \int_{m_2}^{m_0} \frac{d\eta}{m_0^4 - 4\eta\mu_1^2} \rho(\eta). \]

If \( m_2 \gg 4\mu_1^2 \) and \( \rho(\eta) \) is rapidly falling, we find that
\[ \rho_1(\eta_1 = \mu_1^2) \approx \frac{2}{m_2^2} \int_{m_2}^{m_0} d\eta \rho(\eta). \]

V. \( \gamma^0 \to 2\gamma \) DECAY SPECTRUM

The decay \( \gamma^0 \to 2\gamma \) is of course characterized by a spinless decaying particle with no width, \( \mu = 0 \), \( \eta_1 = (q^2_1)^2 \), \( m = m_\eta \), and \( \rho_\eta(\eta) \) is the \( \gamma^0 \) production spectrum. The kinematics and results are obtained from the general case above by taking the \( \mu \to 0 \) limit. In this limit we have from (2.20) (denoting variables for \( \mu = 0 \) with primes)
\[ \eta_1' = +\infty, \]
\[ \eta_1'' = \left( \eta_1' + \frac{m^2}{4\eta_1''} \right)^2. \]
The resulting kinematic region is shown in Fig. 3. Again there are two regions of integration for \( \eta_1 < m^2/4 \).

From (2.24) we now have
\[ r' = \frac{m^4 - 4\eta_1(\eta_1 - (\eta - m^2)^{1/2})^2}{16\eta_1^2(\eta - m^2)^{1/2}}. \]
For the photon spectrum we include an extra factor of two for identical photons. For \( \eta_1 > m^2/4 \) or \( |q_1^2| > 0.07 \) GeV we obtain from (2.25)
\[ \rho_\gamma(\eta_1) = 2 \int_{\eta_1}^{\infty} d\eta_1' \frac{F(\frac{1}{2}, \frac{3}{2}; 1; \eta^2) \rho_\eta(\eta_1)}{2\eta_1'^{1/2}[\eta_1' - (\eta_1 - m_0^2)]^{1/2}}. \]
Again, for \( \eta_1 > 1.5 m^2 \) or \( |q_1^2| > 0.1 \) GeV we can approximate the hypergeometric function by 1 and obtain an accuracy of 1% accuracy.
\[ \rho_+(\eta_1) = \frac{1}{\eta_1^{1/2}} \int_{-\infty}^{\eta_1} d\eta \frac{\rho_+(\eta)}{[\eta(\eta - m^2)]^{1/4}}. \]  

(5.4)

The inverse is obtained by differentiation:
\[ \rho_+(\eta_1) = -\left(1 - \frac{m^2}{16\eta_1^2}\right)^{3/2} \frac{d}{d\eta_1} \left[\eta_1^{1/2} \rho_+(\eta_1)\right]. \]

(5.5)

For \( \eta_1 \gg m^2 \) this becomes the approximation of Sternheimer:
\[ \rho_+(q_+^2) = -\frac{1}{2} \frac{d}{dq_+^2} \left[\rho_+(q_+^2)\right]. \]

(5.6)

For \( \eta_1 < m^2/4 \), or \( |q_+^2| < \frac{1}{2} m^2 \), we have from (4.4)

\[ \rho_+(\eta_1) = \int_{-\infty}^{\eta_1} d\eta \frac{F\left(\frac{1}{2}, \frac{3}{2}; \frac{1}{2}; r^2 \right) \rho_+(\eta)}{\eta_1^{1/2}[\eta(\eta - m^2)]^{1/4}}. \]

\[ + 4 \int_{m^2}^{\eta_1} d\eta \frac{F\left(\frac{1}{2}, \frac{3}{2}; \frac{1}{2}; \frac{1}{r^2}\right) \rho_+(\eta)}{[m^4 - 4\eta_1(\eta - m^2)^{1/2} - (\eta - m^2)^{1/2}]^{1/2}}. \]

(5.7)

The point \( |q_+^2| = 0 \) obtained from (4.6) is
\[ \rho_+(\eta_1 = 0) = \frac{4}{m^2} \int_{0}^{\infty} d\eta \rho_+(\eta). \]

(5.8)

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10S. Pinsky and P. R. Stevens, Phys. Rev. D (to be published).


13M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1965), Eq. 15.3.13.

14This result also follows from Ref. 3 [Eq. (30)\textendash Eq. (32)] by completing an integral contained there to obtain the \( F\left(\frac{1}{2}, \frac{3}{2}; \frac{1}{2}; r^2\right) \) function.