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J.L. Mañes

January 1984

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DIFFERENTIAL GEOMETRIC CONSTRUCTION OF THE GAUGED WESS-ZUMINO ACTION

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Abstract

A general expression for the gauged Wess-Zumino effective action is obtained by differential geometry methods. This expression is evaluated without the use of a "trial and error" method, and its dependence on the gauge fields is in the form of a finite polynomial. Our result is compared with that previously obtained by Witten.

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I. Introduction

Some new developments have taken place recently concerning the mathematical structure of chiral anomalies. The use of differential geometry methods has made possible the explicit calculation of both abelian and non-abelian chiral anomalies without evaluating Feynman diagrams,^{1,2} and the form of the Wess-Zumino³ term as well as its new *a priori* quantization has been derived by Witten⁴ on the basis of topological considerations. Furthermore, Witten has used a "trial and error" method to gauge the Wess-Zumino term, and has found that the dependence of the resulting action on the gauge fields is in the form of a finite polynomial.

In this paper, a general expression for the Wess-Zumino action is obtained by differential geometry techniques. This expression, which is evaluated directly without the use of a "trial and error" method, yields the action in the form given by Witten,⁴ except for some minor discrepancies that we discuss in detail.

We consider a theory of fermions interacting with external gauge fields. The spinor field has internal degrees of freedom and transforms under a representation of a (flavor) group G with generators $\{\lambda_k\}$. The Lagrangian includes vector and axial vector gauge fields V_μ^k and A_μ^k :

$$\mathcal{L} = i\bar{\psi}\gamma^\mu(\partial_\mu - i\lambda_k V_\mu^k - i\gamma_5 \lambda_k A_\mu^k)\psi \quad (1)$$

We will use the symbol γ_5 for the equivalent matrix in $2n$ -dimensional

(Minkowski) space-time, i.e. $\gamma_5 \equiv -i^{n+1} \prod_{\mu=0}^{2n-1} \gamma^\mu$, and $\gamma_5^2 = 1$. The fields V_μ^k and A_μ^k will be combined in the Lie Algebra-valued 1-forms V , A and V_+ :

$$\begin{aligned} V &\equiv -i \lambda_k V_\mu^k dx^\mu \equiv V_\mu dx^\mu \equiv -i \lambda_k V^k \\ A &\equiv -i \lambda_k A_\mu^k dx^\mu \equiv A_\mu dx^\mu \equiv -i \lambda_k A^k \\ V_+ &\equiv V + \gamma_5 A \end{aligned} \quad (2)$$

A general local transformation on the spinor fields is written:

$$\psi \longrightarrow \exp(i\beta^k \lambda_k + i\alpha^k \lambda_k \gamma_5) \psi \quad (3)$$

and if we define the O-forms:

$$\beta \equiv -i\beta^k \lambda_k, \quad \alpha \equiv -i\alpha^k \lambda_k \quad \text{and} \quad v \equiv \beta + \alpha \gamma_5 \equiv -i v^k \lambda_k \quad (4)$$

the transformation of the gauge fields can be written in a compact way:

$$V_+ \longrightarrow g^{-1} V_+ g + g^{-1} dg \quad (5)$$

where $g = e^{v} \in G \times G$.

It is well known that the vector and axial vector currents:

$$J_{\mu k}^V = \bar{\psi} \gamma_\mu \lambda_k \psi \quad \text{and} \quad J_{\mu k}^A = \bar{\psi} \gamma_\mu \gamma_5 \lambda_k \psi \quad (6)$$

satisfy anomalous non-conservation laws:

$$D^\mu J_{\mu k}^V = -G_k^V(V, A) \quad \text{and} \quad D^\mu J_{\mu k}^A = -G_k^A(V, A) \quad (7)$$

The functions G_k^V and G_k^A were obtained by Gross and Jackiw⁶ and Bardeen,⁵ who was able to remove the anomaly from the vector current by adding a suitable counterterm to the Lagrangian.

Wess and Zumino³ showed that the functions $G_k^{V,A}$ could be obtained from a functional $W(A, V, \xi)$ containing the gauge fields and an additional field ξ (a meson octet in the case $G \equiv SU(3)$), such that $e^{i\xi} \in G$, as:

$$G_k^V = \left. \frac{\delta W(V_g, A_g, \xi_g)}{\delta \beta^k} \right|_{\alpha=\beta=0} \quad \text{and} \quad G_k^A = \left. \frac{\delta W(V_g, A_g, \xi_g)}{\delta \alpha^k} \right|_{\alpha=\beta=0} \quad (8)$$

where V_g, A_g, ξ_g are the transformed of V, A, ξ by the local action of $g \in G \times G$. In their original form the Wess-Zumino effective action $W(V, A, \xi)$ was not apparently a finite polynomial expression in the gauge fields. As it was already noted, this fact only became clear with the explicit evaluation of the action by Witten,⁴ who obtained it as a functional $W(R, L, U)$ depending on the left and right gauge fields and $U = e^{i\xi}$.

More recently, it has been possible to obtain the functions $G_k^{V,A}$ by differential geometry methods^{1,2} based on the use of the Wess-Zumino consistency conditions that these functions must satisfy. Also, Zumino² has given an explicit expression for the action in the form $W(R, L, g_R, g_L) = W(R, g_R) - W(L, g_L)$, where g_L and g_R are elements of G . This expression is obtained directly as a polynomial in the gauge fields, but differs from the one given by Witten in that here we need two octets ξ_R and ξ_L of meson fields.

Since these differential geometry techniques are relatively unknown among physicists, we find it necessary to make this paper somewhat self-contained and to explain them in some detail. This is done in Section II, where proofs have been omitted in general, trying

(5)

instead to emphasize the notational conventions, definitions and main equations used in the rest of this work. The reader is referred to the original works^{1,2,7,8} for a more complete presentation of the subject. These techniques are extended in Sections III and IV, which contain the main results of this paper, so as to give a functional $W(R, L, U)$ depending on a single octet of meson fields. For the case of four-dimensional space-time this functional is actually constructed and compared with Witten's result in Section V.

(6)

II. Differential Geometry and the Effective

Functional $W(R, L, g_R, g_L)$

In what follows it will be more convenient to write the equations in terms of the left and right gauge fields L and R :

$$V_{\pm} \equiv V \pm \gamma_5 A \equiv \hat{r} R + \hat{l} L \quad (9)$$

with $R = V + A$ and $L = V - A$. $\hat{r}$ and $\hat{l}$ are the right and left projection operators:

$$\hat{r} = \frac{1 + \gamma_5}{2}, \quad \hat{l} = \frac{1 - \gamma_5}{2}$$

Defining left and right spinors $\psi_L = \hat{l}\psi$ and $\psi_R = \hat{r}\psi$ the Lagrangian (1) can be split:

$$\mathcal{L} = i \bar{\psi}_R \gamma^\mu (\partial_\mu - i \lambda R_\mu^\kappa) \psi_R + (R \leftrightarrow L) \quad (10)$$

A general transformation $g(x) \in G \times G$ will be written:

$$g = e^{\beta + \gamma_5 \alpha} = e^{\hat{r}(\beta + \alpha) + \hat{l}(\beta - \alpha)} = \hat{r} g_R + \hat{l} g_L \quad (11)$$

with $g_R = e^{\beta + \alpha} = e^{\beta + \alpha}$ and $g_L = e^{\beta - \alpha} = e^{\beta - \alpha}$. From (5), (9), and (11) it follows that the gauge fields $R \equiv A_R$ and $L \equiv A_L$ transform independently of each other:

$$\begin{aligned} A_H &\rightarrow g_H^{-1} A_H g_H + g_H^{-1} d g_H, \quad H = L, R \\ F_H &\rightarrow g_H^{-1} F_H g_H \end{aligned} \quad (12)$$

where $F_H = dA_H + A_H^2$ (d is the exterior differential operator, and it is understood that A_H^2 stands for $A_H \wedge A_H$).

The left and right currents have anomalous covariant divergences:

$$D^\mu J_{\mu\kappa}^H = -G_\kappa^H(A_H) \quad \text{with} \quad J_{\mu\kappa}^H = \bar{\psi}_H \gamma_\mu \lambda_\kappa \psi_H \quad (13)$$

The functions G_κ^H were obtained explicitly by Gross and Jackiw,⁶ and in fact:

$$G_\kappa^H \equiv \eta_H G_\kappa, \quad \eta_R = -\eta_L = 1 \quad (14)$$

These functions can also be obtained by differential geometry methods without the use of Feynman diagrams.¹ To this end we have to define a number of differential geometric objects. Given the left-right symmetry expressed in Equations (12) and (14), in what follows the index H will not be written. The first object is the n -th Chern character Ω_{2n} :

$$\Omega_{2n}(A) = \text{tr} F^n \quad (15)$$

The form Ω_{2n} is closed ($d\Omega_{2n} = 0$), and as a consequence can be locally expressed as the total differential of a form $\omega_{2n-1}^0(A)$. An explicit expression for ω_{2n-1}^0 is obtained by considering a 1-parameter family of gauge fields A_t . Under an ordinary variation δt , Ω_{2n} experiences a variation given by:

$$\delta \Omega_{2n}(A_t) = n \, d \, \text{tr} (\delta A_t F_t^{n-1}) \quad (16)$$

Choosing the parametrization $A_t = tA$, (16) can be readily integrated and we get for ω_{2n-1}^0 :

$$\omega_{2n-1}^0(A) = n \int_0^1 \delta t \, \text{tr} (A F_t^{n-1}) \quad (17)$$

with $F_t = dA_t + A_t^2 = t dA + t^2 A^2 = tF + t(t-1)A^2$.

Since ω_{2n-1}^0 is defined by $d\omega_{2n-1}^0 = \Omega_{2n}$, we could add a total differential to (17). This fact will be exploited later in Sections III and IV.

Ω_{2n} is invariant under gauge transformations. This allows the definition of a form $\omega_{2n-2}^1(A, v)$ ($g = e^v$) as follows. From the definition of ω_{2n-1}^0 :

$$\delta_v \Omega_{2n}(A) = \delta_v d \omega_{2n-1}^0(A) = d(\delta_v \omega_{2n-1}^0(A)) = 0 \quad (18)$$

This means that $\delta_v \omega_{2n-1}^0$ is closed and can be locally expressed as a total differential:

$$\delta_v \omega_{2n-1}^0(A) = d \omega_{2n-2}^1(A, v) \quad (19)$$

The form ω_{2n-2}^1 is important because it gives directly a solution to the Wess-Zumino consistency conditions, and contains the anomalous divergences G_κ in $(2n-2)$ -dimensional space-time:

$$K_n \omega_{2n-2}^1(A, v) = v^\kappa G_\kappa, \quad \text{with} \quad K_n = \frac{1}{n!} \frac{i^n}{(2\pi)^{n-1}} \quad (20)$$

(See appendix A for an explicit expression for ω_{2n-2}^1). We can now proceed to the construction of an effective functional. We define formally:

$$W^1(A) = K_n \int_{D_{2n-1}} \omega_{2n-1}^0(A) \quad (21)$$

(9)

where D_{2n-1} is a $(2n-1)$ -dimensional disk which has as boundary space-time compactified to a sphere S^{2n-2} . Under a gauge transformation:

$$\begin{aligned}\delta_v W' &= K_n \int_{D_{2n-1}} \delta_v \omega_{2n-1}^0(A) = K_n \int_{D_{2n-1}} d \omega_{2n-2}^1(A, v) = \\ &= K_n \int_{S^{2n-2}} \omega_{2n-2}^1(A, v) = \int_{S^{2n-2}} v^k G_k\end{aligned}\quad (22)$$

and W' satisfies:

$$\left. \frac{\delta W'}{\delta v^k} \right|_{v=0} = G_k \quad (23)$$

This definition of W' is not totally satisfactory, since although ω_{2n-1}^0 is a finite polynomial in the gauge fields, is not a total differential, and when we try to express it as an integral over $2n-2$ dimensional space-time, the integrand becomes non local, i.e., contains interactions of arbitrary high order. This problem can be fixed by considering the following property of ω_{2n-1}^0 :

$$T(g) \omega_{2n-1}^0(A, F) \equiv \omega_{2n-1}^0(A_g, F_g) = \quad (24)$$

$$= \omega_{2n-1}^0(A, F) + \omega_{2n-1}^0(g^{-1}dg, 0) + d\alpha_{2n-2}(A, g)$$

with $A_g \equiv T(g)A = g^{-1}Ag + g^{-1}dg$ and $F_g = g^{-1}Fg$.

In this expression $\omega_{2n-1}^0(g^{-1}dg, 0)$ has a topological significance.⁴

We will refer to it as $\Lambda_{2n-1}(g)$. An explicit expression follows easily from (17):

$$\omega_{2n-1}^0(g^{-1}dg, 0) \equiv \Lambda_{2n-1}(g) = (-1)^{n+1} \beta(n, n) \text{tr} (g^{-1}dg)^{2n-1} \quad (25)$$

where $\beta(n, n)$ is the β -function:

(10)

$$\beta(n, n) = \frac{\Gamma(n) \Gamma(n)}{\Gamma(2n)} = \frac{(n-1)!(n-1)!}{(2n-1)!}$$

An expression for α_{2n-2} in which it appears as a finite polynomial in A and $V \equiv dgg^{-1}$ can be found in Appendix A. Since F is a function of A , we will write $\omega_{2n-1}^0(A, F) \equiv \omega_{2n-1}^0(A)$. From (12) we have the group property:

$$T(g) T(h) = T(hg) \quad (26)$$

and we get the identity:

$$T(g) \omega_{2n-1}^0(A) = T(h^{-1}g) T(h) \omega_{2n-1}^0(A) = T(h^{-1}g) \omega_{2n-1}^0(A_h) \quad (27)$$

In other words, $T(g) \omega_{2n-1}^0(A)$ is invariant under the transformation $A \rightarrow A_h, g \rightarrow h^{-1}g$. If we define:

$$W(A, g) = K_n \int_{D_{2n-1}} \omega_{2n-1}^0(A) - T(g) \omega_{2n-1}^0(A) \quad (28)$$

by (27) we have:

$$\left. \frac{\delta W}{\delta v^k} \right|_{v=0} = \left. \frac{\delta W'}{\delta v^k} \right|_{v=0} = G_k(A) \quad (29)$$

and by (24) $W(A, g)$ can be written:

$$W(A, g) = -K_n \int_{D_{2n-1}} \Lambda_{2n-1}(g) + d\alpha_{2n-2}(A, g) = \quad (30)$$

$$= -K_n \int_{D_{2n-1}} \Lambda_{2n-1}(g) - K_n \int_{S^{2n-2}} \alpha_{2n-2}(A, g)$$

Now the gauge fields appear only in the second integral, which is explicitly a finite polynomial in A and $V \equiv dgg^{-1}$ as given by (A.3).

There exists only a finite number of vertices containing the gauge fields

and g . Since $\Lambda_{2n-1}(g)$ cannot be written as a total differential, we still have an infinite number of vertices, but only in the field g .

Taking into account the signs given by (14), the complete functional is:

$$W(R, L, g_R, g_L) = W(R, g_R) - W(L, g_L) \quad (31)$$

where $g_R(x) = e^{i\xi_R(x)}$ and $g_L(x) = e^{i\xi_L(x)}$ are elements of G .

III. The Functional $W(R, L, U)$:

In this section we modify the functional $W(R, L, g_R, g_L)$ so as to make it depend on a single field $U(x)$ ($U \in G$) instead of g_R and g_L . To this end we define:

$$\omega_{2n-1}^0(R, L) \equiv \omega_{2n-1}^0(R) - \omega_{2n-1}^0(L) \quad (32)$$

and notice that by (28), (31) and (32) the effective functional can be written:

$$W(R, L, g_R, g_L) = K_n \int_{D_{2n-1}} \omega_{2n-1}^0(R, L) - T(g_R, g_L) \omega_{2n-1}^0(R, L) \quad (33)$$

Here $T(g_R, g_L)$ defines the action of an element $g = g_R \hat{r} + g_L \hat{l} \in G \times G$:

$$T(g_R, g_L) \omega_{2n-1}^0(R, L) \equiv \omega_{2n-1}^0(R g_R, L g_L) \quad (34)$$

If we define the n -th Chern character for the whole group $G \times G$ as:

$$\Omega_{2n}(R, L) = \Omega_{2n}(R) - \Omega_{2n}(L) \quad (35)$$

then $\omega_{2n-1}^0(R, L)$ as given by (32) is a solution to the equation:

$$\Omega_{2n}(R, L) = d \omega_{2n-1}^0(R, L) \quad (36)$$

In the next section we will show that another solution $\tilde{\omega}_{2n-1}^0(R, L)$ to (36) can always be found, such that it is invariant under vector gauge transformations:

$$\delta_\beta \tilde{\omega}_{2n-1}^0(R, L) = 0 \quad (37)$$

By (36) it can only differ from (32) by a total differential:

$$\tilde{\omega}_{2n-1}^0(R, L) = \omega_{2n-1}^0(R, L) + dS_{2n-2}(R, L) \quad (38)$$

If we consider the functional:

$$\tilde{W}(R, L, g_R, g_L) = K_n \int_{D_{2n-1}} \omega_{2n-1}^0(R, L) - T(g_R, g_L) \tilde{\omega}_{2n-1}^0(R, L) \quad (39)$$

and use Stokes theorem ($\partial D_{2n-1} = S^{2n-2}$), we can see that $\tilde{W}$ is related to W by:

$$\tilde{W}(R, L, g_R, g_L) = W(R, L, g_R, g_L) - K_n \int_{S^{2n-2}} T(g_R, g_L) S_{2n-2}(R, L) \quad (40)$$

By the group property $T(g_R, g_L) T(h_R, h_L) = T(h_R g_R, h_L g_L)$, the last piece of (40) is invariant under the local transformation:

$$R \rightarrow R_{h_R} ; L \rightarrow L_{h_L} ; g_R \rightarrow h_R^{-1} g_R ; g_L \rightarrow h_L^{-1} g_L \quad (41)$$

This means that:

$$\frac{\delta \tilde{W}}{\delta v_H^a} = \frac{\delta W}{\delta v_H^a} = G_H^a ; H=L, R \quad (42)$$

Also, in the next section $S_{2n-2}(R, L)$ will be calculated explicitly as a finite polynomial in the gauge fields. $\tilde{W}$ is thus local in R and L , and by (42) is a satisfactory effective functional.

Now we use the group property to combine g_R and g_L into one single field:

$$\begin{aligned} T(g_R, g_L) \tilde{\omega}_{2n-1}^0(R, L) &= T(g_R, g_R) T(e, g_L g_R^{-1}) \tilde{\omega}_{2n-1}^0(R, L) = \\ &= T(g_R, g_R) \tilde{\omega}_{2n-1}^0(R, L g_L g_R^{-1}) = \tilde{\omega}_{2n-1}^0(R, L g_L g_R^{-1}) = \\ &= T(e, g_L g_R^{-1}) \tilde{\omega}_{2n-1}^0(R, L) \end{aligned} \quad (43)$$

since $T(g_R, g_R)$ is a vector gauge transformation and has no effect on

$\tilde{\omega}_{2n-1}^0$. If we define $U \equiv g_L g_R^{-1} \in G$, we have:

$$T(g_R, g_L) \tilde{\omega}_{2n-1}^0(R, L) = T(e, U) \tilde{\omega}_{2n-1}^0(R, L) \quad (44)$$

where e is the identity element of G , and from (41) U transforms as

$$U \rightarrow h_L^{-1} U h_R \quad (45)$$

We can now define the effective functional $W(R, L, U)$:

$$W(R, L, U) \equiv \tilde{W}(R, L, g_R, g_L) = K_n \int_{D_{2n-1}} \omega_{2n-1}^0(R, L) - T(e, U) \tilde{\omega}_{2n-1}^0(R, L) \quad (46)$$

By (38) and (24) the last piece can be written:

$$T(e, U) \tilde{\omega}_{2n-1}^0 = \omega_{2n-1}^0(R, L) - \Lambda_{2n-1}(U) - d\alpha_{2n-2}(L, U) + dS_{2n-2}(R, L, U) \quad (47)$$

and the final expression is:

$$W(R, L, U) = K_n \int_{D_{2n-1}} \Lambda_{2n-1}(U) + K_n \int_{S^{2n-2}} \alpha_{2n-2}(L, U) - S_{2n-2}(R, L, U) \quad (48)$$

where $L_U = U^{-1} L U + U^{-1} dU$, and $U(x) = \exp\{i\xi(x)\}$ is an element of G .

$W(R, L, U)$ is the Wess-Zumino effective action. The first integral contains the Wess-Zumino term and the second one is the result of its gauging. Notice that, by (37), S_{2n-2} has the following behavior under vector gauge transformations:

$$\delta_\beta \omega_{2n-1}^0(R, L) = -\delta_\beta dS_{2n-2} \quad (49)$$

and it follows that:

$$\frac{\delta}{\delta \beta^a} \left(W(R, L, U) + \int_{S^{2n-2}} S_{2n-2}(R, L) \right) \equiv \frac{\delta \bar{W}(R, L, U)}{\delta \beta^a} = \bar{G}_a^v(R, L) = 0 \quad (50)$$

i.e. the addition of $S_{2n-2}(R, L)$ to the Lagrangian removes the anomaly from the vector current. $S_{2n-2}(R, L)$ is thus equivalent, in four

dimensions, to the counterterm used by Bardeen⁵ to conserve the vector current and we will refer to it simply as "the counterterm". An expression for $S_{2n-2}(R, L)$ valid for any n is derived in the next section.

It is an interesting fact that the counterterm, which was first used by Bardeen to get the conserved vector current form of the anomaly, appears in our method as an essential ingredient in the construction of $W(R, L, U)$, which by (42) gives the anomaly in the left-right symmetric form

IV. The Counterterm

To find solutions to the equation

$$d\omega_{2n-1}^0 = \Omega_{2n}(R, L) \quad (36)$$

in a systematic way it is convenient to express Ω_{2n} as a function of $V_+ = \hat{F}R + \hat{L}L$:

$$\Omega_{2n}(V_+) = \frac{2}{D} \text{tr}(\gamma_5 F_+^n) \equiv \Omega_{2n}(R, L) \quad (51)$$

where $F_+ = dV_+ + V_+^2 = (dR + R^2)\hat{F} + (dL + L^2)\hat{L}$, and $D \times D$ is the dimension of the Dirac matrices. The equivalence of (35) and (51) follows from the fact that $\hat{F}$ and $\hat{L}$ are projection operators and $\gamma_5 = \hat{F} - \hat{L}$.

We consider a 2-parameters family of gauge fields:

$$A_{\lambda\mu} = \lambda V_+ + \mu V_- \quad (52)$$

with $V_- = V - \gamma_5 A = \hat{F}L + \hat{L}R$. An ordinary variation $(\delta\lambda, \delta\mu)$ of the parameter induces a variation in Ω_{2n} given by:

$$\delta\Omega_{2n}(A_{\lambda\mu}) = \frac{2n}{D} d \text{tr}(\gamma_5 \delta A_{\lambda\mu} F_{\lambda\mu}^{n-1}) \quad (53)$$

where $\delta A_{\lambda\mu} = \delta\lambda V_+ + \delta\mu V_-$ and $F_{\lambda\mu} = dA_{\lambda\mu} + A_{\lambda\mu}^2$.

Now we observe that $\Omega_{2n}(A_{00}) = 0$ and :

$$\Omega_{2n}(A_{10}) = \Omega_{2n}(V_+) \quad (54)$$

$$\Omega_{2n}(A_{01}) = \Omega_{2n}(V_-) = -\Omega_{2n}(V_+)$$

From this it follows that $\Omega_{2n}(V_+)$ can be expressed as the differential of a line integral:

$$\Omega_{2n}(V_+) = \frac{2n}{D} d \int_{(a) \text{ or } (b)} \text{tr}(\gamma_5 \delta A_{\lambda\mu} F_{\lambda\mu}^{n-1}) = \frac{n}{D} d \int_{(c)} \text{tr}(\gamma_5 \delta A_{\lambda\mu} F_{\lambda\mu}^{n-1}) \quad (55)$$

where the integrals are along the oriented paths (a), (b) or (c) shown in Fig. 1. Each path gives a different solution to Equation (36). In particular, if we integrate along the straight line $\mu = 0$ we get the solution (32):

$$\frac{2n}{D} \int_0^1 \delta \lambda \text{tr}(\gamma_5 V_+ F_+^{n-1}) = n \int_0^1 \delta \lambda \text{tr}(R F_R)^{n-1} - (R \leftrightarrow L) = \omega_{2n-1}^0(R) - \omega_{2n-1}^0(L) \quad (56)$$

The same result is obtained by integration along the line $\lambda = 0$. It is easy to show (see Appendix B) that the solution obtained when we integrate along $\mu + \lambda = 1$ is invariant under vector gauge transformations, i.e.:

$$\tilde{\omega}_{2n-1}^0(R, L) = \frac{n}{D} \int_{(a,1) \text{ along } \lambda+\mu=1}^{(1,0)} \text{tr}(\gamma_5 \delta A_{\lambda\mu} F_{\lambda\mu}^{n-1}) \quad (57)$$

From (56), (57), and (38) an expression in which $dS_{2n-2}(R, L)$ is represented by a closed line integral is readily obtained (see Fig. 2):

$$\frac{n}{D} \int_{\square} \text{tr}(\gamma_5 \delta A_{\lambda\mu} F_{\lambda\mu}^{n-1}) = \tilde{\omega}_{2n-1}^0 - \omega_{2n-1}^0 = dS_{2n-2}(R, L) \quad (58)$$

(Here all the integrals are along straight lines)

A more convenient expression giving directly $S_{2n-2}(R, L)$ is constructed from (58) in Appendix B. The result is:

$$S_{2n-2}(R, L) = \frac{n(n-1)}{D} \iint \text{Str}(\gamma_5 V_- V_+ F_{\lambda\mu}^{n-2}) \quad (59)$$

where the integral is over the interior of the triangle represented in Fig. 2. The following property is useful in the actual evaluation of (59): given the trace of a monomial containing γ_5 and the fields $V_{\pm}$ and/or $F_{\pm}$, the reversal of all the signs in $V_{\pm}$ and $F_{\pm}$ reproduces the original expression with opposite sign. For instance:

$$\text{tr}\{\gamma_5 V_+ V_- F_-^2\} = -\text{tr}\{\gamma_5 V_- V_+ F_+^2\}$$

V. An Application

In this section the functional $W(R, L, U)$ is actually constructed for the case $n = 3$, i.e. four-dimensional space-time. W contains three functions that have to be evaluated:

$$W(R, L, U) = K_3 \int_{D_5} \Lambda_5(U) + K_3 \int_{S^4} \alpha_4(L, U) - S_4(R, L, U) \quad (60)$$

From (25):

$$\Lambda_5 = \frac{1}{10} \text{tr} (U^{-1} dU)^5 \quad (61)$$

and from (A.3)

$$\alpha_4(L, U) = -\frac{1}{2} \text{tr} \left\{ (dUU^{-1})(LdL + dLL + L^3) - \frac{1}{2} (dUU^{-1})L(dUU^{-1})L - (dUU^{-1})^3 L \right\} \quad (62)$$

S_4 is obtained from (59) as a function of the fields V_+ and V_- :

$$S_4 = \frac{1}{4} \text{tr} \left\{ \gamma_5 V_- (V_+ F_+ + F_+ V_+ - V_+^3) + \frac{\gamma_5}{4} V_- V_+ V_- V_+ \right\} \quad (63)$$

Using the fact that $\hat{r}$ and $\hat{l}$ are projection operators and that in four dimensions $\text{tr}(\hat{r}) = \text{tr}(\hat{l}) = 2$, S_4 may be written in terms of R and L :

$$S_4(R, L) = \frac{1}{2} \text{tr} \left\{ (LR - RL)(F_R + F_L) + R^3 L + RL^3 + \frac{1}{2} LRLR \right\} \quad (64)$$

In this last expression we have to make the substitution:

$$L \rightarrow L_U \equiv U^{-1} L U + U^{-1} dU \quad (65)$$

The normalization constant K_3 is given by (20):

$$K_3 = \frac{-i}{24\pi^2} \quad (66)$$

After collecting the terms arising from (61), (62), and (64) with (65), and making the substitution:

$$R dU^{-1} dL U - R dU^{-1} L dU \rightarrow -dR dU^{-1} L U \quad (67)$$

(the left and right hand side of (67) differ by the total differential of $R dU^{-1} L U$) to get a more symmetrical expression we obtain:

$$W(R, L, U) = \frac{-i}{240\pi^2} \int_{D_5} \text{tr} (U^{-1} dU)^5 - \frac{i}{48\pi^2} \int_{S^4} Z \quad (68)$$

where the 4-form Z is:

$$\begin{aligned} Z = & -\text{tr} \{ U_L (L dL + dLL + L^3) - (U_L)^3 L \} - \text{tr} \{ R \leftrightarrow L \} \\ & + \frac{1}{2} \text{tr} (U_L L U_L L) - \frac{1}{2} \text{tr} (R \leftrightarrow L) \\ & - \text{tr} (U^{-1} L U R^3) + \text{tr} (U R U^{-1} L^3) \\ & - \text{tr} \{ U^{-1} L U (R dR + dR R) \} + \text{tr} \{ U R U^{-1} (L dL + dL L) \} \\ & - \text{tr} \{ U R U^{-1} L U_L L \} - \text{tr} \{ U^{-1} L U R U_R R \} \\ & + \text{tr} \{ L dU U_R R U^{-1} \} + \text{tr} \{ R dU^{-1} U_L L U \} \\ & - \text{tr} \{ dL dU R U^{-1} \} + \text{tr} \{ dR dU^{-1} L U \} \\ & + \frac{1}{2} \text{tr} \{ R U^{-1} L U R U^{-1} L U \} \end{aligned} \quad (69)$$

(21)

Here U_L stands for dUU^{-1} and U_R for $U^{-1}dU$.

Equation (69) is our final result. To compare it with the expression obtained by Witten⁴ we must notice that $A_R \equiv -iR$ and $A_L \equiv -iL$ due to a different convention in the definition of the covariant derivative. We find some minor differences with Witten's effective functional, which he calls $\tilde{F}$, which doesn't seem to have the prescribed transformation properties. If one makes the following changes in $Z_{\mu\nu\alpha\beta}$ (of $\tilde{F}$):

- 1) Delete the coefficient $\frac{1}{2}$ in front of $A_{\mu L} A_{\nu L} U A_{\alpha R} A_{\beta R} U^{-1}$.
- 2) Interchange $R \leftrightarrow L$ in the second term of the third line:

$$A_{\mu L} U^{-1} (\partial_\nu A_{\alpha R}) \partial_\beta U \rightarrow A_{\mu R} U^{-1} (\partial_\nu A_{\alpha L}) \partial_\beta U$$

- 3) Add $iA_{\mu L} A_{\nu L} A_{\alpha L}$ to the expression $(\partial_\mu A_{\nu L}) A_{\alpha L} + A_{\mu L} (\partial_\nu A_{\alpha L})$ in the second line.

then the final expression transforms correctly, and the relation with (69) is:

$$\tilde{F} = W - \frac{i}{48\pi^2} \int_{S^4} \text{tr} (F_L \cup F_R U^{-1}) - d \text{tr} (L \cup dR U^{-1}) \quad (70)$$

The last term, being a total differential, vanishes upon integration by parts, and the first term is invariant under both vector and axial vector gauge transformations and is therefore irrelevant.

Witten has established the *a priori* quantization of the Wess-Zumino action in the sense that W has to be multiplied by an integer which he has found to be equal to the number of colors N_c .⁴ Taking this into account, the anomalous contribution of the field U to the baryon current can be calculated directly from (69). To this end, consider a

(22)

local transformation of the fundamental spinor fields in the Lagrangian (1) given by:

$$\psi(x) \rightarrow \exp \left(i \frac{\gamma^0(x)}{N_c} \right) \psi(x) \quad (71)$$

where $1/N_c$ is the baryon number of a fermion (quark). By (3) and (4) this corresponds to a generator λ_0 given by:

$$\lambda_0 = \frac{1}{N_c} \quad (72)$$

where 1 is the $f \times f$ unit matrix (f : number of flavors).

Then the anomalous vector baryon current is easily computed:

$$\begin{aligned} *J_0^V &= *J_0^R + *J_0^L = N_c \left(\frac{\delta W}{\delta R^0} + \frac{\delta W}{\delta L^0} \right)_{R=L=0} = \\ &= -\frac{iN_c}{48\pi^2} \text{tr} \left\{ \frac{1}{N_c} (iU_L^3 + iU_R^3) \right\} = \frac{1}{24\pi^2} \text{tr} (U^{-1}dU)^3 \end{aligned} \quad (73)$$

since $\text{tr} (U_L^3) = \text{tr} (U_R^3) \equiv \text{tr} (U^{-1}dU)^3$. Notice that J_0^A (anomalous) vanishes identically.

Equation (73) coincides with the expression given by Witten⁴ by analogy with the electromagnetic anomalous current. The addition of the $U(1)$ generator λ_0 to the generators of $SU(3)$ increases the number of mesons to nine, the new one being a pseudoscalar (singlet). From the fact that $\Pi_3(SU(2)) = \mathbb{Z}$ it is clear that all meson fields but the pion triplet can be set equal to zero, and one still gets an anomalous baryon number contribution B_0 .⁹

$$B_0 = \frac{1}{24\pi^2} \int_{\text{space}} \text{tr} (U^{-1}dU)^3 \quad (74)$$

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Appendix A

We present here explicit expressions for some of the objects defined in Section II. These expressions are taken from [1] and [2] and adapted to the conventions used in this paper.

For $\omega_{2n-2}^{-1}(A, v)$ we have:

$$\omega_{2n-2}^{-1}(A, v) = n(n-1) \int_0^1 dt (1-t) \text{Str} (v d(A F_t^{n-1})) \quad (\text{A.1})$$

Here $F_t = dA_t + A_t^2 = tdA + t^2 A^2$, and Str is a "symmetrized trace". The symmetrized trace of a product of Lie algebra valued forms Λ_i is defined:

$$\text{Str}(\Lambda_1 \Lambda_2 \dots \Lambda_n) = \frac{1}{n!} \sum_{(i_1 \dots i_n)} \text{sgn}(i_1 \dots i_n) \text{tr}(\Lambda_{i_1} \Lambda_{i_2} \dots \Lambda_{i_n}) \quad (\text{A.2})$$

where the sum is over all permutations $(i_1 \dots i_n)$ of $(1, 2, \dots, n)$ and

$\text{sgn}(i_1 \dots i_n)$ is the sign arising from the commutative properties of Λ_i as forms.

α_{2n-2} is given by:

$$\alpha_{2n-2}(A, g) = -n(n-1) \iint \text{Str}(V A F_{\lambda\mu}^{n-2}) \quad (\text{A.3})$$

where $V \equiv dgg^{-1}$; $F_{\lambda\mu} = dA_{\lambda\mu} + A_{\lambda\mu}^2$ and $A_{\lambda\mu} = \lambda A - \mu V$. The integral is over a triangle in the parameter space (λ, μ) :

$$\iint = \int_0^1 d\lambda \int_0^{1-\lambda} d\mu \quad (\text{A.4})$$

In the evaluation of (A.3) all the integrals are of the type:

$$\iint \lambda^h \mu^k = \frac{h! k!}{(h+k+2)!} \quad (\text{A.5})$$

(25)

(26)

For $n = 3$ (A.1) gives:

$$\omega_4^1(R, v) = \text{tr} \left\{ v d \left(R dR + \frac{1}{2} R^3 \right) \right\} \quad (\text{A.6})$$

and we can express the variation of the functional $W(R, L, U)$ as follows:

$$\begin{aligned} \delta_v W(R, L, U) &= K_3 \int_{D_5} \delta_v \omega_5^0(R, L) = K_3 \int_{S^4} \omega_4^1(R, v_R) - \omega_4^1(L, v_L) = \\ &= \frac{-i}{24\pi^2} \int_{S^4} \text{tr} \left\{ v_R d \left(R dR + \frac{1}{2} R^3 \right) - (R \leftrightarrow L) \right\} \end{aligned} \quad (\text{A.7})$$

in agreement with Witten's results⁴ and computations at the quark level.⁶

Appendix B

Here we complete the proof of Equations (57) and (59) which give ω_{2n-1}^0 and S_{2n-2} . To show the invariance of (57) under a vector gauge transformation we have to notice that, for $g_R = g_L = g = e^{\beta}$, V_+ and V_- transform in the same way:

$$V_{\pm} \rightarrow g^{-1} V_{\pm} g + g^{-1} dg \quad (\text{B.1})$$

which can be written to first order in β :

$$\delta_{\beta} V_{\pm} = [V_{\pm}, \beta] + d\beta \quad (\text{B.2})$$

Also, under a global (space-time independent) vector gauge transformation the integrand is obviously invariant, and we have only to consider the part of the variation proportional to $d\beta$, which we indicate by writing $\sim$ instead of $=$:

$$\delta_{\beta} V_{+} \sim \delta_{\beta} V_{-} \sim d\beta \quad (\text{B.3})$$

Since $\delta_{\beta} F_{\pm} = [F_{\pm}, \beta]$, we have:

$$\delta_{\beta} F_{+} \sim \delta_{\beta} F_{-} \sim 0 \quad (\text{B.4})$$

On the line $\lambda + \mu = 1$, $\delta A_{\lambda\mu}$ is:

$$\delta A_{\lambda\mu} = \delta \lambda (V_{+} - V_{-}) \quad (\text{B.5})$$

and

$$\delta_{\beta} (\delta A_{\lambda\mu}) \sim \delta \lambda (d\beta - d\beta) = 0 \quad (\text{B.6})$$

$F_{\lambda\mu}$ can be written:

$$F_{\lambda\mu} = \lambda \{ F_{+} + (\lambda - 1) V_{+}^2 \} + \mu \{ F_{-} + (\mu - 1) V_{-}^2 \} + \lambda \mu (V_{+} V_{-} + V_{-} V_{+}) \quad (\text{B.7})$$

From (B.3), (B.4), and (B.7) it follows immediately that, on the line $\lambda + \mu = 1$, $\delta_\beta F_{\lambda\mu} \sim 0$.

This completes the proof of Equation (57). Let's consider now the expression (58) for dS_{2n-2} :

$$dS_{2n-2} = \frac{n}{D} \int_{\Delta} \text{Str}(\gamma_5 \delta A_{\lambda\mu} F_{\lambda\mu}^{n-1}) = \frac{n}{D} \int_{\Delta} \left(\delta\lambda \text{Str}(\gamma_5 V_+ F_{\lambda\mu}^{n-1}) + \delta\mu \text{Str}(\gamma_5 V_- F_{\lambda\mu}^{n-1}) \right) \quad (\text{B.8})$$

This expression is transformed by using Stokes theorem into an integral over the triangle of Fig. 2:

$$dS_{2n-2} = \frac{n}{D} \iint \text{Str} \gamma_5 \left(V_+ \frac{\partial}{\partial \mu} - V_- \frac{\partial}{\partial \lambda} \right) F_{\lambda\mu}^{n-1} \quad (\text{B.9})$$

Notice that in (B.8) we are using the symmetrized trace, which in this case coincides with the ordinary one, but is more convenient in what follows.

Using: $\frac{\partial F_{\lambda\mu}}{\partial \lambda} = dV_+ + \{A_{\lambda\mu}, V_+\}$ and $\frac{\partial F_{\lambda\mu}}{\partial \mu} = dV_- + \{A_{\lambda\mu}, V_-\}$ (B.10)

we can rewrite (B.9) as:

$$dS_{2n-2} = \frac{n(n-1)}{D} \iint \text{Str} \gamma_5 \left((V_+ dV_- - V_- dV_+) F_{\lambda\mu}^{n-2} + V_+ \{A_{\lambda\mu}, V_-\} F_{\lambda\mu}^{n-2} - V_- \{A_{\lambda\mu}, V_+\} F_{\lambda\mu}^{n-2} \right) \quad (\text{B.11})$$

Since the symmetrized trace is invariant under permutations (up to a sign due to the commutation properties of the forms involved) the last two terms can be combined:

$$\begin{aligned} \text{Str} \gamma_5 \left(\{A_{\lambda\mu}, V_-\} V_+ F_{\lambda\mu}^{n-2} - V_- \{A_{\lambda\mu}, V_+\} F_{\lambda\mu}^{n-2} \right) &= \\ &= -\text{Str}(\gamma_5 V_- V_+ [A_{\lambda\mu}, F_{\lambda\mu}^{n-1}]) = \text{Str}(\gamma_5 V_- V_+ dF_{\lambda\mu}^{n-1}) \end{aligned} \quad (\text{B.12})$$

In the last step the Bianchi identity for $F_{\lambda\mu}$ has been used. This allows (B.11) to be written as a total differential:

$$dS_{2n-2} = \frac{n(n-1)}{D} d \iint \text{Str}(\gamma_5 V_- V_+ F_{\lambda\mu}^{n-2}) \quad (\text{B.13})$$

and

$$S_{2n-2} = \frac{n(n-1)}{D} \iint \text{Str}(\gamma_5 V_- V_+ F_{\lambda\mu}^{n-2}) \quad (\text{B.14}) \quad (59)$$

The transformations leading from (B.8) to (B.14) are identical to the ones used by Zumino to prove Equation (A.3) for α_{2n-2} .²

Notice the similarity of Equations (A.3) and (B.24), which up to a factor differ only by the introduction of a γ_5 and the substitution $V_- \leftrightarrow -V_+$. Nevertheless, in the process of evaluating α_{2n-2} from (A.3) the equality $dV = V^2$ is used. This equality does not hold for V_- , and as a consequence the final expression for S_{2n-2} cannot be obtained simply from that for α_{2n-2} , but has to be evaluated independently, from (B.14).

Added Note

After completion of this work, we have received the paper by Ö. Kaymakçalan et al., COO-3533-278 SU-4222-278 preprint, in which the construction of the gauged Wess-Zumino action by the "trial and error" method is reconsidered, and we find complete agreement between their expression, Eq. (15), and ours, Eq. (69) (with the identifications $A_{L,R} \equiv iL, R$, and $\alpha, \beta = U_{L,R}$). In their paper they also give some applications of the W-Z action to the study of "unnatural parity" hadronic reactions.

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Figure Captions

- (1). The (λ, μ) parameter space and paths associated with different solutions to Eq. (36).
- (2). Domain of integration for S_{2n-2} .

(32)

Figure 1

Figure 2

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