UNIVERSITY OF CALIFORNIA, SAN DIEGO

Essays on Strategic Incentives for Information Revelation

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The dissertation of Ricardo Serrano-Padial is approved, and it is acceptable in quality and form for publication on microfilm:

Chair

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DEDICATION

A Ana y Lola.

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ABSTRACT OF THE DISSERTATION

Essays on Strategic Incentives for Information Revelation

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Information has strategic value in most economic environments because individuals have private information relevant to the decisions of others. The performance of economic institutions therefore depends on the extent to which they efficiently lead agents to reveal their information. My dissertation consists of three essays on information revelation (and aggregation) in two types of institutions: exchange environments in which traders have private information regarding a good whose value is commonly known to be the same for all ex post, as in common-value auctions, and decentralized provision of public goods. The first essay analyzes double auctions designed to model prediction markets. I identify conditions under which the equilibrium price in the model fully reveals the asset's value. The second essay deals with the lack of liquidity in trade institutions when agents have common values and are perfectly strategic. In the third essay, I look at the extent to which private information restricts the set of implicit agreements to provide public services over time that can be supported via the folk theorem with complete information.

I

Strategic Foundations of Prediction Markets and the Efficient Markets Hypothesis

Abstract

This paper studies information aggregation in pure common value double auctions with a continuum of traders. This trade environment captures some of the main features of prediction markets. The population includes both strategic traders and non-strategic (naïve) agents whose bidding behavior is not influenced by opponents' equilibrium strategies. Existence and uniqueness of monotone equilibrium prices is shown under mild conditions on the distribution of naïve bids. In any such equilibrium, the mapping from asset values to prices has a domain split into two distinct areas: a revealing region, where prices equal values, and a non-revealing region. There is a strictly positive lower bound on the share of naïve traders below which prices are always fully revealing and an upper bound beyond which prices are almost nowhere revealing. This indicates that, contrary to prevailing views, non-negligible levels of noise or liquidity trade are compatible with perfect information aggregation, although even moderate levels of noise can lead to nowhere revealing prices. An empirical method to distinguish between the revealing and non-revealing regions is suggested.

JEL Classification: *C72, D44, D82.*

Keywords: *Information aggregation, double auction, common values, private information, prediction markets, efficient markets hypothesis.*

I.A Introduction

Markets have long been touted not only for their role in allocating goods, but also for their properties as information processors. The *efficient markets hypothesis* (Fama (1970)) postulates that prices in competitive markets "fully reflect" all the available information, which

> ... never exists in concentrated or integrated form, but solely as the dispersed bits of incomplete and frequently contradictory knowledge which all the separate individuals possess.
>
> Hayek (1945, p. 519).

Based on this conjecture, exchange institutions designed with the sole purpose of forecasting future events, commonly referred to as *prediction markets*, have emerged in the last two decades.[1] Although empirical evidence suggests that market prices seem to perform well as information aggregators,[2] the mechanism by which such aggregation takes place is not yet clearly identified. A major reason for this gap is that economics lacks a theory of price formation in markets. Most models either leave the price setting mechanism unspecified (rational expectations equilibrium (REE) models) or assume the existence of a market maker, who sets prices by making inferences on the amount of information individual traders possess (market microstructure models). Existing research on information aggregation in auctions provides an explicit mechanism that links individual trader actions to market prices. However, it has focused primarily on single auctions, which do not account for the two-sided nature of most asset markets.[3]

[1] Some examples are the Iowa Electronic Markets (IEM) for presidential elections, the Hewlett-Packard internal market to predict future sales and Hollywood exchange, a virtual currency market aimed at forecasting movie ticket sales. I refer the reader to Wolfers and Zitzewitz (2004) for a more comprehensive list of existing markets.

[2] See, for instance, Forsythe, Nelson, Neumann, and Wright (1992), Berg, Forsythe, Nelson, and Rietz (2005), or Berg, Nelson, and Rietz (2003) for evidence from the IEM, and Chen and Plott (2002) for a study of the internal market at Hewlett-Packard.

[3] An important exception is the double auction model of Reny and Perry (2006), which provides theoretical support to the existence of fully revealing REE.

I study price formation in markets by modeling them as common value double auctions (CVDA) in which risk-neutral traders receive a private signal stochastically related to the value of the security traded.[4] The reasons behind this choice are threefold. First, double auctions provide an explicit mechanism by which individual trade decisions translate into prices. Moreover, this mechanism resembles existing markets, given that both buyers and sellers post offers to respectively buy and sell units of the asset. Second, having a common value component is essential to study information aggregation, if we understand it as the process of aggregating, through market prices, individual pieces of information about the unknown value of the asset.[5] Finally, if the efficient markets hypothesis is correct, pure common values plus risk neutrality imply that security prices in prediction markets can be directly interpreted as estimates of some parameter of the probability distribution of the events to be forecasted.

In order to test how robust information aggregation is to the presence of boundedly rational agents, I introduce heterogeneity in the traders' population by having both strategic traders and naïve traders. These naïve traders can be seen as the analogue of noise in REE models. I provide a very general definition of naïve bidding behavior, which includes the noise or liquidity traders used in market microstructure models as a special case.[6] Thus, their presence allows the comparison of existing results on information aggregation with the predictions of my model. In addition, naïve traders rule out no trade equilibria, which always exist in double auctions in which all traders are strategic.

[4] Common values means that the (unknown) value of the asset is the same for all traders, although their expectations do not coincide when they receive distinct signals. In contrast, private values imply that each agent values the asset differently and knows exactly her own valuation.

[5] Recent models of prediction markets proposed by Manski (2004), Gjerstad (2005) and Wolfers and Zitzewitz (2006) assume that agents have pure private values (referred to as beliefs). In this setting, each agent knows exactly her valuation of the asset, which differs across agents. Since there is no individual uncertainty, aggregation of beliefs cannot take place and, instead, these analyses look at how close prices are to the mean belief.

[6] The defining feature of naïve traders is that they follow a fixed bidding strategy, regardless of what the other traders do in equilibrium. That is, unlike strategic traders, they do not best respond in equilibrium. Moreover, since the analysis presented below applies to a wide class of naïve bid distributions, I do not require the naïve population to be homogeneous. For instance, the results of my model would hold for a bid distribution that is arbitrarily close to the distribution of bids generated by any mix of level-k agents.

A known issue in double auctions is that traders' ability to affect prices in finite agent environments makes equilibrium analysis quite intractable. However, since it vanishes as the market grows, I look at a limit case with a continuum of agents.[7, 8]

I characterize (increasing) monotone equilibrium prices in this continuum economy and show that they exist and are unique among the class of monotone prices when the distribution of naïve bids satisfies a mild monotonicity condition. Furthermore, I show that in any monotone equilibrium strategic traders place their bids in regions of the bidding space where prices are equal to asset values or outside the range of prices. Accordingly, prices are characterized by having its range partitioned into two distinct regions: a revealing region where prices equal asset values and a non-revealing area where prices differ from values and are completely determined by naïve bids. There are three distinct scenarios: for small shares of naïve traders, prices are fully revealing (i.e. they equal asset values). When there is a moderate presence of naïve bidders, the equilibrium price function has both a revealing and a non-revealing region. Finally, if the share of naïve traders surpasses some threshold they always determine the price.

This result represents a middle ground between two opposing views about the relationship between liquidity or noise trade and the informational content of prices. According to some models (Kyle (1985, 1989)), the introduction of noise prevents the market from collapsing by precluding prices from fully revealing asset values. On the other hand, REE models predict that, as long as there is a positive mass of risk-neutral traders, prices will be perfectly informative (Grossman (1976), Hellwig (1980)). In the double auction setting, perfectly revealing equilibria are compatible with non-negligible levels of noise trade. However, prices can be quite uninformative for moderate shares of naïve traders, which is at odds with the

[7] This paper should be regarded as part of a project aimed at analyzing information aggregation in large markets. The study of the limit economy provides insight into the informational content of prices as a function of the degree of noise trade, while subsequent research would explore the convergence of prices in finite markets to prices in the limit economy.

[8] It is important to note that when agents are price takers, the population of naïve traders can include pure private value strategic traders, given that bidding their own valuation is a weakly dominant strategy.

idea that a small presence of sophisticated traders suffices to get full information aggregation.

It is worth mentioning that, although I restrict my analysis to risk-neutral agents, having risk-averse or risk-loving sophisticated traders would lead to qualitatively similar prices. This is so because the key feature of their bidding behavior, i.e. that they avoid placing bids in regions of the bidding space where prices differ from values, does not depend on their risk attitudes. Hence, the forecasting properties of prediction markets may not depend on eliciting risk neutrality.

The characterization of equilibrium prices provided here lends itself to the development of an empirical test aimed at assessing how accurately prices reflect asset values. Since strategic traders place bids only in the revealing region while naïve traders generally bid in both regions, a simple method to identify revealing prices can be devised, based solely on data on prices and the density of bids in the neighborhood of prices.

This paper is organized as follows. First I describe a typical prediction market, highlighting its relevant features. I then look at existing theories that address information aggregation through the price mechanism. The common value double auction model is laid out in section two. Section three provides the characterization of equilibrium prices in a continuum agent economy. An empirical test of information aggregation is then suggested. Before concluding, I briefly discuss convergence issues regarding finite double auctions.

I.A.1 Morphology of a Prediction Market

Assume our goal is to predict the outcome of a U.S. presidential election, to be held at time T. Primaries are over and there are two candidates left, George and John. We would like to estimate at any time $t < T$ the probability of each candidate winning the popular vote. Denote such probabilities by $\mathbb{P}_t^G$ and $\mathbb{P}_t^J = 1-\mathbb{P}_t^G$, respectively. Each individual agent has some information about $\mathbb{P}_t^G$, denoted $\mathscr{F}_{i,t}$.

For this purpose, we set up a futures market in which agents can post and accept offers to trade two futures contracts: the "George security" (denoted by G), which pays \$1 at time T if George wins the popular vote and \$0 otherwise; and the "John security" (J), which pays \$1 if John wins and \$0 otherwise. According to the efficient markets hypothesis (henceforth EMH), the equilibrium price of each security at any time t (p_t^j, $j = G, J$) is such that the expected future price conditional on all the existing information $\mathscr{F}_t = \bigcup_i \mathscr{F}_{i,t}$ satisfies

$$[1 + \mathbb{E}(r_{t+1}^j | \mathscr{F}_t)] p_t^j = \mathbb{E}(p_{t+1}^j | \mathscr{F}_t), \tag{I.1}$$

where r_{t+1}^j is the one period return to security j at time $t+1$, which will depend on agents' (identical) preferences. Under risk neutrality and pure common values, $r_t^j = 0$ for all t and $p_T^j = 1_{\{j \text{ wins the popular vote}\}}$,[9] given that neither security pays dividends but instead has a liquidation value at T of \$1 or \$0, depending on the election outcome. This implies that (I.1) reduces to

$$p_t^j = \mathbb{E}(p_T^j | \mathscr{F}_t) = \mathbb{E}(1_{\{j \text{ wins the popular vote}\}} | \mathscr{F}_t) =: \hat{\mathbb{P}}_t^j. \tag{I.2}$$

Therefore, if the efficient market hypothesis is correct and traders are money-maximizers with no insurance motives, the price of each security provides the "best" estimate of the winning probability of its associated candidate, given the information available at that period.

Guided by this conjecture, prediction markets are usually set up as on-line futures markets with specific features so as to induce risk-neutral behavior among participants.[10] Trading rules resemble those of existing stock exchanges, which are in essence continuous-time double auctions: traders can either post offers to sell or buy each of the securities (known as *asks* and *bids*, respectively) or accept outstanding offers. An ask (bid) specifies the maximum number of units of the security to sell (buy) and the minimum (maximum) price to be accepted. All valid

[9] $1_{\{.\}}$ is the indicator function.

[10] In some of these markets there are tight limits on the amount of money that can be invested. For instance, the investment limit at the IEM is \$500. In addition, transaction costs in these markets are negligible.

asks are ranked from lowest to highest price, whereas all valid bids are ranked from highest to lowest. The two outstanding offers are the ask with the lowest price and the bid with the highest price. If an outstanding ask (bid) is accepted, the next highest ranked ask (bid) becomes outstanding. When an outstanding offer is accepted, the price of the ensuing transaction is the price specified in this offer. At each moment, traders are informed of the outstanding offers and the price of the last transaction. Additional information such as trade volume, price and bid/ask spread histories is also available.

The empirical evidence seems to suggest that prices perform well as forecasts. Regardless of the specific characteristics of the prediction market (size, virtual vs. real currency, type of event to be forecasted), the short and long-run forecasting properties of prices appear to be better than existing benchmarks, mainly experts' forecasts and opinion polls. For instance, Forsythe, Nelson, Neumann, and Wright (1992), Berg, Forsythe, Nelson, and Rietz (2005) and Berg, Nelson, and Rietz (2003) find that market prices were consistently closer than opinion polls to actual vote shares at the IEM election markets. With regards to experts' forecasts, Chen and Plott (2002) show that price forecasts in the (small) internal market set up by Hewlett-Packard were closer to actual sales of the company than the official company forecast.[11] Finally, experimental evidence in oral common value double auctions suggests that prices aggregate information, thus backing the EMH (Plott and Sunder (1988), Forsythe and Lundholm (1990), Guarnaschelli, Kwasnica, and Plott (2003)).

I.A.2 Theoretical Foundations of the EMH

Despite these empirical findings, REE models, on which the EMH is based, do not describe price formation as a function of traders' actions. Therefore, how information may be reflected in market prices is left unexplained.[12]

[11] See Sunstein (2004) and Wolfers and Zitzewitz (2004) for a more general discussion about the empirical evidence regarding prediction markets.

[12] In REE models such as those of Hellwig (1980), and Grossman (1976, 1978), traders observe prices before choosing their demands. Thus, the mechanism by which traders' actions translate into prices is

There are two strands of economic theory that analyze price informativeness by linking prices to individual trading behavior: market microstructure models and auction theory. In market microstructure models such as the canonical models of Glosten and Milgrom (1985) and Kyle (1985) prices typically arise as a result of the interaction between competitive market makers and individual traders. In this setting, uninformed market makers set prices according to a zero profit rule by making inferences about the information traders may have.[13] Two salient features are the sequential nature of these models and the heterogeneity of the trader population. The latter usually consists of informed, strategic traders and uninformed traders, who can be strategic or not (noise traders) depending on the model. This literature captures important aspects of the price formation process in some capital markets in which specialists operate the market. In addition, they provide insights into how traders' information gets reflected into prices and the timing of this process. Specifically, they show that information is incorporated into prices gradually, allowing informed traders to profit from their privileged information, contrary to the EMH. However, the presence of a non-trivial market maker renders these models unsuited for the analysis of two-sided decentralized institutions in which no specialists are present, where all the strategic interaction takes place between individual traders operating without the constraint of a zero-profit rule.

On the other hand, auction theory looks at information aggregation by modeling markets as static common value auctions, which have a very simple institutional structure and pricing rules are predetermined before the auction. Although the static nature of these trade environments precludes the study of information aggregation dynamics, this approach has the advantage of providing an explicit mechanism that links market prices to individual bids. Most research has focused on one-sided common value auctions, starting with the first price auction

left out of the model.

[13]There is an extensive literature emerging from these two models. Important examples are Easley and O'Hara (1992) and the analysis of Back and Baruch (2004), which shows that Glosten and Milgrom (1985) and Kyle (1985) merge into the same model under some conditions.

of Wilson (1977). The main finding is that equilibrium prices converge to the true value of the asset as the number of bidders gets large as long as either the upper bound of the asset value support grows (Wilson (1977) and Milgrom (1979, 1981)) or as the units at auction increase (this is the *double largeness* condition in Pesendorfer and Swinkels (1997)).[14] Information aggregation is caused by agents' inferences about prices based on their private information and on the equilibrium behavior of the other agents. These inferences influence bidding behavior which, in turn, determines prices.

The main drawback of using one-sided auctions to analyze information aggregation in markets is that there is an implicit non-strategic market maker (the seller) in charge of the supply. Double auctions models solve this issue by having both strategic buyers and sellers. However, common value double auctions have proven quite intractable and very little research exists in this area. A remarkable exception is the paper by Reny and Perry (2006), who study the existence of fully revealing equilibrium prices in large mixed value double auctions. They show the existence of approximately fully revealing prices in large double auctions when agents' utility is strictly increasing in the signals agents privately receive.[15] Since the private value component is non-negligible prices do not converge to values as the market grows.

I.B The model

There is a continuum of agents, denoted by $\mathcal{T}$. A fraction $\gamma \in (0,1)$ of them are sellers, each of them owning one unit of a security, with the remaining fraction being buyers, willing to buy at most one unit. The value of the security $V \in [0,1]$ is unknown with probability distribution $G(V)$. Each agent receives a private signal $S \in [0,1]$ stochastically related to V. Signals are independent and

[14] Kremer (2002) summarizes existing results and extends them to the English auction, while Hong and Shum (2004) study the rates of convergence under both scenarios.

[15] They prove this result for double auctions with finite bid grids.

identically distributed conditional on v, with probability distribution $F(S|v)$.[16]

Assumption 1 *$G(.)$ has a C^1 density $g(.)$ bounded away from 0 in $[0,1]$. $F(\,.\,|.)$ has a C^1 density $f(\,.\,|.)$ bounded away from 0 in $[0,1]^2$.*

Assumption 2 *$f(\,.\,|.)$ satisfies the strict monotone likelihood ratio property.*

The first assumption implies that the distribution of signals has full support for all values of the asset. That is, a trader receiving a signal $s \in [0,1]$ cannot rule out any asset value in $[0,1]$. The second assumption means that higher signals are more likely than lower signals when the asset value is high.

Buyers and sellers simultaneously submit bids and asks to buy and sell specifying, respectively, the maximum price willing to pay and the minimum price willing to accept. Bids are restricted to be in $[0,1]$.[17] The price p is given by the $(1-\gamma)$-th percentile of the bid distribution. Buyers with bids above p and sellers with asks below p get to trade.[18] If there is a positive mass of bids at p there is the possibility of rationing, i.e. some traders bidding exactly p may not trade. In this case, the traders bidding p who end up with the object are chosen randomly.[19]

A fraction $\eta \in [0,1]$ of both buyers and sellers are naïve traders who do not best respond in equilibrium but rather use a fixed bidding rule. The remaining mass of agents are risk-neutral, strategic traders, i.e. they best respond in equilibrium. The bidding behavior of naïve traders is summarized by the probability distribution of their bids, $H(.|v)$. I assume that $H(.|v)$ is continuous, weakly monotonic with respect to asset values and has the same connected support for all v.

Assumption 3 *$H(.|v)$ has full support in $[\underline{b}^H, \overline{b}^H] \subseteq [0,1]$ for all $v \in [0,1]$, with $\underline{b}^H < \overline{b}^H$. $H(.|.)$ is C^1 in $(\underline{b}^H, \overline{b}^H) \times [0,1]$ and absolutely continuous in $[0,1]^2$.*

[16] In what follows, I use uppercase letters to denote random variables (V, S) or cumulative distribution functions (G, F) and lowercase to denote realizations of random variables (v, s_i) or density functions (g, f). In addition boldface letters (e.g. $\mathbf{s}, \mathbf{S}$) denote vectors.

[17] This assumption is without loss of generality since bids outside the unit interval are weakly dominated by bidding either zero or one.

[18] In the remainder of the paper I use the term "bid" to refer both to seller asks and buyer bids.

[19] Reny and Perry (2006) use the same tie-breaking rule.

This assumption implies that the distribution of naïve bids is atomless. The full support assumption implies that $H(.|v)$ is strictly increasing in $(\underline{b}^H, \overline{b}^H)$ for all $v \in [0, 1]$, i.e. there are no intervals between the lowest and highest naïve bids where the mass of bids is zero.

Assumption 4 $H(b|.)$ *is non-increasing in* $[0, 1]$ *for all* $b \in [0, 1]$.

Assumption 4 implies that, for all v, v' such that $v > v'$, $H(.|v)$ first order stochastically dominates $H(.|v')$. It means that naïve traders tend to bid higher when the value of the asset is higher.

Examples of naïve bidding satisfying the above assumptions include the typical random or noise traders commonly used in finance models, who bid according to the bidding rule $\beta^n(s) \sim U[0, 1]\ \forall s$,[20] and traders bidding according to their interim private beliefs, $\beta^n(s) = \mathbb{E}(V|s)$. The latter are similar to traders in the prediction market models of Manski (2004), Gjerstad (2005) and Wolfers and Zitzewitz (2006). In those models, traders fail to consider the common value nature of the asset. Accordingly, one could interpret their behavior as strategic, in the sense that, if they deem their interim beliefs as their true valuation of the asset, they would be best responding by bidding $\mathbb{E}(V|s)$.[21] Thus, a double auction in which the trader population is divided into pure private value and pure common value strategic agents constitutes a special case of the double auction environment presented above. Also, the above definition of naïve traders can include any continuous approximation of the distribution of bids generated by a population consisting of any mix of level-k agents (see Crawford and Iriberri (2006) for a definition of level-k thinking in auctions).[22]

Given a profile of bidding (pure) strategies $\beta : [0, 1] \times \mathcal{T} \to [0, 1]$ with $\beta(s, t)$ denoting the bid of strategic trader $t \in \mathcal{T}$ when she receives signal s, let

[20] Their bid distribution is $H(b|v) = b$ for all v, which weakly satisfies *Assumption 4*.

[21] This is true given that in a continuum economy agents are price takers. Thus, it is optimal for a buyer (seller) to bid her private value in order to maximize her gains from trade.

[22] The distribution of bids generated by such population can include atoms and therefore violate *Assumption 3*. However, such distribution can be approximated by an atomless distribution satisfying the above assumptions.

$B(.|V,\eta)$ be the cumulative distribution function of bids when the share of naïve traders is η and $B\_(p|V,\eta)$ the mass of bids strictly less than p. Accordingly,

$$B(p|v,\eta) := \eta H(p|v) + (1-\eta)\int_{\mathcal{T}}\int_0^1 1_{\{\beta(s,t)\leq p\}} f(s|v)ds d\mu, \tag{I.3}$$

and

$$B\_(p|v,\eta) := \eta H(p|v) + (1-\eta)\int_{\mathcal{T}}\int_0^1 1_{\{\beta(s,t)< p\}} f(s|v)ds d\mu, \tag{I.4}$$

where μ is a suitable (atomless) measure on $\mathcal{T}$.

The asset value V determines the distribution of signals $F(.|V)$. Given that there is a continuum of traders receiving i.i.d. signals, by the law of large numbers, the profile of signals received by traders coincides with the whole distribution of signals conditional on V.[23] Accordingly, given strategy profile $\beta(.,.)$, the market clearing price is completely determined by V. Hence, for all $v \in [0,1]$ the market price is given by the function $\rho : [0,1]^2 \to [0,1]$ that satisfies

$$(1-\gamma) \in [B\_(\rho(v,\eta)|v,\eta), B(\rho(v,\eta)|v,\eta)]. \tag{I.5}$$

The payoff functions for a (strategic) buyer t and a seller t' are, respectively,

$$\begin{aligned}\pi^{buy}(s,t) &:= \mathbb{E}((V-\rho(V,\eta))1_{\{\beta(s,t)>\rho(V,\eta)\}}|s) \\ &+ \mathbb{E}((V-\beta(s,t))\lambda(\beta(s,t),V)1_{\{\beta(s,t)=\rho(V,\eta)\}}|s),\end{aligned} \tag{I.6}$$

and

$$\begin{aligned}\pi^{sell}(s,t') &:= \mathbb{E}((\rho(V,\eta)-V)1_{\{\beta(s,t')<\rho(V,\eta)\}}|s) \\ &+ \mathbb{E}((\beta(s,t')-V)(1-\lambda(\beta(s,t'),V))1_{\{\beta(s,t')=\rho(V,\eta)\}}|s),\end{aligned} \tag{I.7}$$

[23] As pointed out by Judd (1985) there are measurability problems when dealing with a continuum of random variables. While acknowledging those issues, I do not address them in the analysis presented here. Hammond and Sun (2006) propose extending the usual product probability space to one that retains the Fubini property so that measurability is restored. For instance, as shown by Sun (1996), we obtain the exact law of large numbers if we assume that $\mathcal{T}$ is a hyperfinite set (thus having cardinality continuum) and extend the standard product measure space defined on $[0,1]\times\mathcal{T}$ to the corresponding nonstandard Loeb space.

where $\lambda(b, v)$ represents the probability of getting the object given bid b and asset value v when $\rho(v, \eta) = b$.

I.C Equilibrium Prices

In this section I investigate how well equilibrium prices forecast asset values as a function of the presence of naïve traders (η) and of their specific bidding behavior ($H(.|.)$). Accordingly, I restrict my attention to equilibria with prices $\rho(v, \eta)$ that are increasing in v (henceforth monotone equilibria). The two main results are stated in *Propositions 1* and *2*. The first provides a characterization of monotone equilibrium prices, whereas the second shows existence and uniqueness of such prices. All proofs are relegated to the *Appendix*.

The characterization of equilibrium prices provided below is driven by the inability of a single trader to affect prices when there is a continuum of agents. Price taking behavior induces two key features of payoff functions (I.6)-(I.7): (i) buyers and sellers have the same preference ranking over bids (*Lemma 1*), and (ii) bidding behavior is oriented to maximize the probability of trading in favorable conditions while avoiding undesired trades, considering prices fixed. The latter, coupled with increasing prices, leads strategic traders to avoid bidding in areas of the price range where prices are not equal to asset values, i.e. where prices are non-revealing.

Lemma 1 (Symmetric Preferences) *Buyers and sellers receiving the same signal $s \in [0, 1]$ have the same ranking over bids in $[0, 1]$.*

To get some intuition on both the symmetry of preferences and the incentive to avoid bidding in non-revealing regions, consider the price function depicted in *Figure I.1*. The range of prices consists of a revealing interval $[0, p_1]$ and a non-revealing interval $(p_1, p_2]$. Prices are greater than values whenever $\rho(v, \eta)$ is above the diagonal and viceversa. Assuming there is no rationing, the payoff for a buyer bidding b when she receives signal s is the expected value of the difference

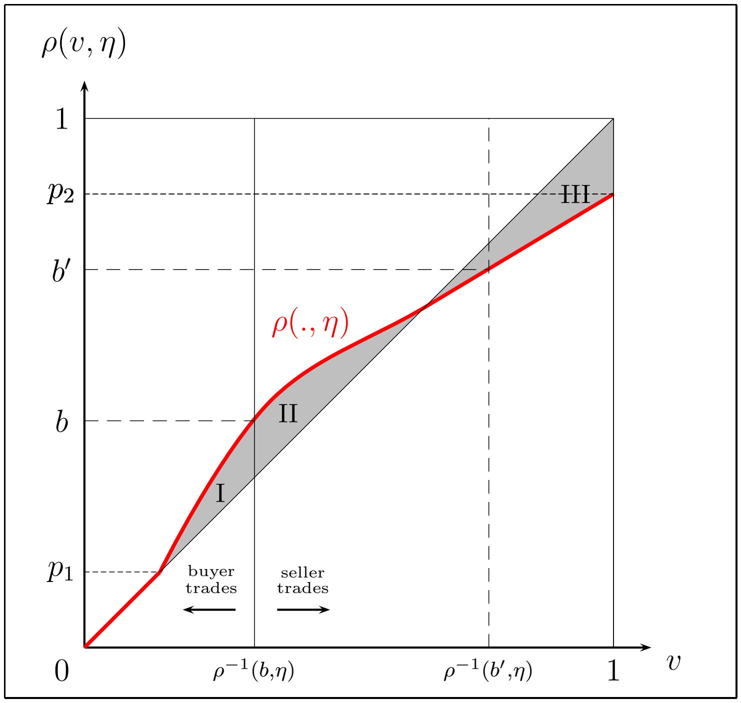

Figure I.1: Strategic Bidding

$v - \rho(v, \eta)$, conditional on s, for all prices below b. That is, it is the expected value of shaded area I. For a seller with signal s and bid b, the payoff is obtained by integrating the difference $\rho(v, \eta) - v$ for prices above b, i.e. the expected difference between transactions involving prices above values (area II) and transactions for which $\rho(v, \eta) < v$ (area III).

The rationale behind symmetric preferences is the following: for a seller with signal s, the payoff for trading the object when prices fall within two alternative bids b and b' is the negative of the payoff for a buyer receiving the same signal. In addition, when a seller bidding b trades (because $\rho(v, \eta) > b$) a buyer bidding b does not trade. If a seller with signal s strictly prefers bid b to bid $b' > b$ it is because the expected payoff, conditional on s, of trading at prices between b and b' is positive.[24] But then, a buyer with the same signal would rather avoid trading at those prices by also bidding b.

Consider now a seller who places a bid in a non-revealing area, e.g. by

[24] That is, for values between $\rho^{-1}(b, \eta)$ and $\rho^{-1}(b', \eta)$, where ρ^{-1} denotes the inverse image of the price function.

bidding b. She has an incentive to deviate and bid either in $[0, p_1]$ since all transactions in area I involve $\rho(v, \eta) > v$, or to bid in $[p_2, 1]$ if, conditional on her signal, the expected difference between the gains in areas I and II and the losses in area III is negative. In the latter case, by bidding above p_2 she abstains from trading and gets zero payoffs. On the other hand, if a seller bids b' she is engaging in negative transactions, which can be avoided by bidding above the area where $\rho(v, \eta) < v$ (i.e. in $[p_2, 1]$) or can be compensated with gains from areas I and II (by bidding in $[0, p_1]$). By the symmetry of preferences, no buyer would bid b or b'. This also indicates that no strategic trader would bid below a non-revealing interval that starts with $\rho(v, \eta) < v$.

Proposition 1 and *Corollary 1* are a direct consequence of these two key characteristics of traders' payoffs. Let $H(v) = H(v|v)$.

Proposition 1 (Equilibrium Prices) *In any monotone equilibrium of a CVDA satisfying Assumptions 1-4, there is a set* $\mathcal{V} = \bigcup_k [\underline{v}_k, \overline{v}_k]$ *with* $\overline{v}_k < \underline{v}_{k+1}$ *for all* $k = 1, ..., K \leq \infty$ *and a collection of signals* $\{s_k^*\}$ *with* $s_k^* < s_{k+1}^*$ *such that prices are given by*

$$\rho(v, \eta) = \begin{cases} v & \text{if } v \in [0, 1] \setminus \mathcal{V} \\ p \text{ s.t. } H(p|v) = \frac{1-\gamma-(1-\eta)F(s_k^*|v)}{\eta} & \text{if } v \in [\underline{v}_k, \overline{v}_k], \end{cases} \tag{I.8}$$

where all $\underline{v}_k, \overline{v}_k \in (0, 1)$ *and* $s_k^* \in [0, 1]$ *satisfy:*

$$1 - \gamma = \eta H(\underline{v}_k) + (1 - \eta) F(s_k^* | \underline{v}_k), \tag{I.9}$$

$$1 - \gamma = \eta H(\overline{v}_k) + (1 - \eta) F(s_k^* | \overline{v}_k), \tag{I.10}$$

and, for all $s \leq s_k^*$ $(s \geq s_k^*)$,

$$\mathbb{E}((V - \rho(V, \eta)) 1_{\{V \in (\underline{v}_k, \overline{v}_k)\}} | s) \leq 0 \;\; (\geq 0). \tag{I.11}$$

This result essentially describes monotone equilibria as the succession of non-revealing intervals ($[\underline{v}_k, \overline{v}_k]$), where prices are determined by the naïve bids, and revealing intervals ($[\overline{v}_k, \underline{v}_{k+1}]$) in which all strategic bids within the price range are concentrated. It also establishes that the allocation of strategic bids across the latter intervals is block-monotonic, i.e. lower signal traders bid in lower intervals. Specifically, traders with signals in (s^*_k, s^*_{k+1}) bid in $[\overline{v}_k, \underline{v}_{k+1}]$.[25] It is important to point out that prices may be fully revealing, i.e. $\mathcal{V}$ is the empty set, or completely determined by the distribution of naïve bids, i.e. $\mathcal{V} = [0, 1]$. The next corollary further requires that, in any non-revealing interval, prices are above values in the lower portion of the interval and below values in the upper part.

Corollary 1 *In any monotone equilibrium with $\mathcal{V}$ non-empty, given $(\underline{v}_k, \overline{v}_k)$, either $\rho(v, \eta) = v$ a.e. in $(\underline{v}_k, \overline{v}_k)$ or there exist v'_k, v''_k with $\underline{v}_k < v'_k \leq v''_k < \overline{v}_k$ such that $\rho(v, \eta) \geq v$ a.e. in $[\underline{v}_k, v'_k]$ with strict inequality in a non-null subset, and $\rho(v, \eta) \leq v$ a.e. in $[v''_k, \overline{v}_k]$ with strict inequality in a non-null subset. Moreover, if this is true for intervals $(\underline{v}_k, \hat{v}]$ and $(\hat{v}, \overline{v}_k)$, then $\mathbb{E}((V - \rho(V, \eta))1_{\{V \in (\hat{v}, \overline{v}_k)\}} | s^*_k) \geq 0$.*

Corollary 1 states that prices in intervals where no strategic bids are placed need to begin with prices above values and end with prices below values. When faced with the prospect of bidding just below an interval where values are always above prices, a seller would rather deviate and bid just above that region to avoid trading at those prices. A symmetric reasoning applies when the seller bids just above an interval in which $\rho(v, \eta) > v$. It also says that, if a non-revealing interval consists of two or more disjoint subintervals, each of them beginning with prices above values and ending with prices below values, no strategic bidder bidding below such interval has an incentive to deviate and bid in between two of those subintervals.

Examples of prices that can and cannot be an equilibrium are shown in *Figure I.2*.[26]

[25] When $\underline{v}_1 > 0$ traders with signals below s^*_1 bid in $[0, \underline{v}_1]$ and in $[0, \rho^{-1}(0, \eta)]$ if $\underline{v}_1 = 0$, i.e. outside the price range. Similarly, traders with signals above s^*_K either bid in $[\overline{v}_K, 1]$ (when $\overline{v}_K < 1$) or in $[\rho^{-1}(1, \eta), 1]$ (when $\overline{v}_K = 1$).

[26] To correctly interpret these figures assume that strategic bids are placed in the intervals where $\rho(v, \eta) = v$.

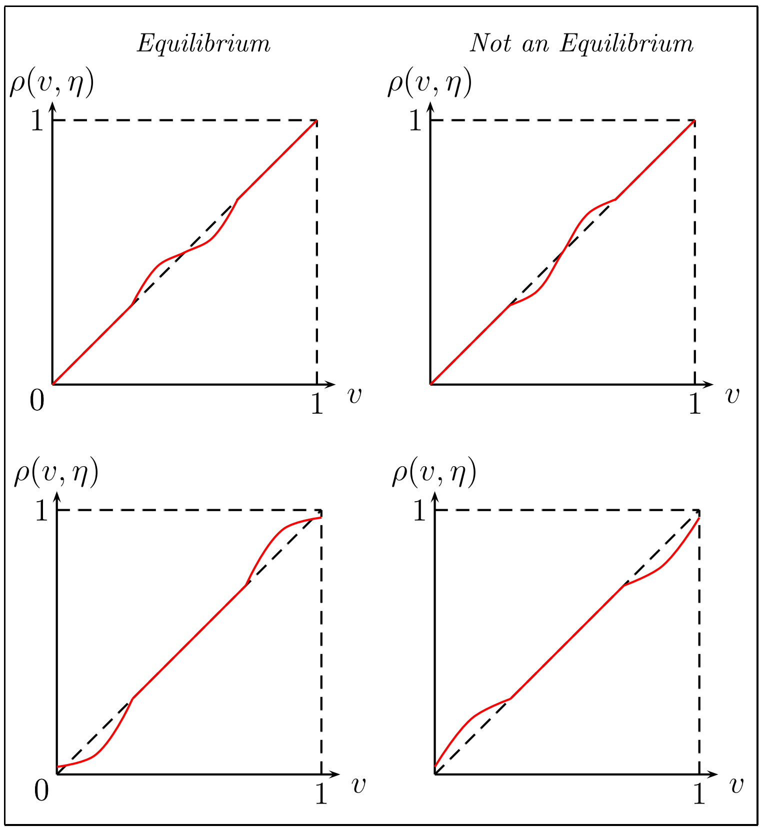

Figure I.2: Candidates for Equilibrium Prices

The proofs of *Proposition 1* and *Corollary 1* hinge upon a series of lemmas in the *Appendix*, which formalize the intuition about strategic bidding presented above. In addition to having symmetric preferences (*Lemma 1*), I show that no strategic bidder would place bids just below a non-revealing interval that starts with prices below values (*Lemma 2*). *Lemma 3* states that strategic bidders avoid placing bids in non-revealing intervals. Finally, as assumed above, no rationing takes place in equilibrium, since any atom is solely created by naïve traders and can only happen in very special cases (*Lemma 4*). In addition, the block-monotonicity of the distribution of strategic bids is a direct consequence of the MLRP property and of *Lemma 2*: if bidding just below a non-revealing interval is profitable for a trader with signal s, it is also profitable for all traders with signals below s.

The next result states that monotone equilibria exist in any CVDA with a continuum of agents satisfying *Assumptions 1-4*. Furthermore, monotone equilibrium prices are unique. Finally, it sheds light on how the presence of naïve traders affect the informational content of prices: there is a strictly positive lower bound on the share of naïve bidders below which prices are fully revealing and there is an upper bound above which prices are always set by naïve bidders and strategic bidders bid outside the price range.

Proposition 2 (Existence of Monotone Equilibria) *Let Assumptions 1-4 be satisfied. Then a monotone Bayesian Nash equilibrium in pure strategies exists for all* $\eta \in [0,1]$ *and the resulting price function* $\rho(.,\eta)$ *is the unique monotone price function that can be supported in equilibrium. Furthermore, there exists* $\underline{\eta} \in (0, \min\{\gamma, 1-\gamma\})$ *such that* $\mathcal{V}$ *is the empty set (fully revealing prices) for all* $\eta < \underline{\eta}$*, and there exists* $\overline{\eta} \leq 1$ *such that* $\mathcal{V} = [0,1]$ *for all* $\eta > \overline{\eta}$ *(*$\overline{\eta} < 1$ *if* $H'(v) \geq 0$ *whenever* $H(v) = 1-\gamma$*).*

Existence of equilibrium is given by the continuity of distributions H, F and expectations, and by the monotonicity of H, F with respect to v. The former guarantees the existence, for each η, of a block-monotonic distribution of strategic bids satisfying the equilibrium conditions of *Proposition 1*. The latter leads to increasing prices when the distribution of strategic bids is block-monotonic. Uniqueness is based on the fact that, due to the strict MLRP, each triplet $(s_k^*, \underline{v}_k, \overline{v}_k)$ satisfying (I.9)-(I.11) and *Corollary 1* is unique. An algorithm to obtain the collection $\{(s_k^*, \underline{v}_k, \overline{v}_k)\}_{k=1}^K$ that characterizes equilibrium price $\rho(v, \eta)$ is provided in the proof of *Proposition 2*.

The last part of *Proposition 2* establishes the existence of three types of equilibrium prices depending on the proportion of naïve traders: fully revealing, partially revealing and nowhere revealing prices. To provide some intuition on this result, let the quantile function $\alpha(v, \eta)$ represent the highest signal corresponding to bids at or below v such that $\rho(v, \eta) = v$, assuming block-monotonicity. That is, for all $v \in [0, 1]$ such that $\frac{1-\gamma-\eta H(v)}{1-\eta} \in [0, 1]$, $\alpha(v, \eta)$ is given by

$$F(\alpha(v, \eta)|v) = \frac{1 - \gamma - \eta H(v)}{1 - \eta}. \tag{I.12}$$

For most distributions of naïve bids, $\alpha(.,.)$ has three distinct regions, depending on the value of η (see *Lemma 5* in the *Appendix*).[27] For $\eta \in [0, \underline{\eta}]$, it is increasing with respect to v in $[0, 1]$. It is non-monotonic (whenever it is well-defined) in v for $\eta \in (\underline{\eta}, \overline{\eta})$. Finally, it is decreasing for all $\eta \geq \underline{\eta}$. I show that, when $\alpha(., \eta)$ is increasing everywhere, prices must be fully revealing. This is true because the proportion of naïve traders is too small to create non-revealing intervals starting with prices above values, as required by *Corollary 1*, when the distribution of strategic bids is block-monotonic. On the other hand, prices cannot be revealing in intervals of values where $\alpha(., \eta)$ is not well-defined or decreasing (*Lemma 6*). For prices to be revealing in some interval $[v_1, v_2]$ in which $\alpha(., \eta)$ is decreasing, it would be necessary to have a mass of bids below v_2 that is strictly smaller than the mass of bids below $v_1 < v_2$. Moreover, that reduction of mass needs to be greater than

[27] Distributions $H(.|.)$ such that $H'(v) < 0$ for some v are typically multi-modal distributions, with most of the mass concentrated in a small subset of the support. These can be generated, for instance, by almost-jump bidding functions, which imply bidding in a small neighborhood of a finite set of bids for most signals.

$F(\alpha(v_1,\eta)|v_1) - F(\alpha(v_1,\eta)|v_2)$, given that $\alpha(v_2,\eta) < \alpha(v_1,\eta)$. However, this is not possible under the strict MLRP, because the highest possible reduction of mass is obtained by having bid distributions equal to $F(\alpha(.,\eta)|.)$.

Before presenting an example that illustrates how information aggregation can be quite sensitive to the presence of naïve traders, it is worth mentioning two aspects about trade activity in monotone equilibria. Block-monotonicity means that the strategic traders more active in the market are the low signal sellers and the high signal buyers. Since the former tend to bid relatively low and the latter bid relatively high, they engage in trade more often, compared to high signal sellers and low signal buyers. In addition, the density of bids will generally be higher around fully revealing prices, given that strategic traders avoid bidding in non-revealing regions. As suggested in *Section I.D*, this could be the basis to develop an empirical test of information aggregation.

I now provide an example to illustrate how the informational content of prices varies with the share and bidding behavior of naïve traders and to provide some intuition for *Proposition 2*.

Example 1 *Consider a CVDA with the following characteristics. V is distributed uniformly in $[0,1]$; the conditional distribution of signals is Beta$(1+v,1)$ (i.e. $F(s|v) = s^{1+v}$);*[28] *each naïve trader bids according to $\beta^n(s) := \frac{3}{5}s^{1/5}$, which is a rough approximation of bidding $\mathbb{E}(V|s)$.*[29]

Given $\beta^n(.)$, the distribution of naïve bids is given by

$$H(p|v) = \begin{cases} \left(\frac{5}{3}p\right)^{5(1+v)} & \text{if } v \leq \frac{3}{5} \\ 1 & \text{if } v > \frac{3}{5} \end{cases} \tag{I.13}$$

[28] *This distribution satisfies all assumptions except the full support, provided it has positive density in $(0,1)$ rather than in $[0,1]$.*

[29] *This approximation makes equilibrium computations more tractable without changing any substantive aspect of the analysis.*

By *Proposition 2*, there exist cutoff points $\underline{\eta}$, $\overline{\eta}$ that determine whether prices would be fully, partially or non-revealing as a function of η. Since $H'(v) \geq 0$ for all v, $\overline{\eta}$ is strictly less than one.

The first thing to note is that, given η, a necessary condition for a partially revealing equilibrium with $\mathcal{V} = [\underline{v}_1, \overline{v}_1]$ is that there exist a signal s_1^* satisfying (I.9) at three distinct values, namely $\underline{v}_1$, $\overline{v}_1$ and $v_1' \in (\underline{v}_1, \overline{v}_1)$, the latter being the point at which $\rho(v, \eta)$ goes from being above to go below v. Therefore, the function $\alpha(v, \eta)$ given by

$$\alpha(v,\eta) = F^{-1}(\tfrac{1-\gamma-\eta H(v)}{1-\eta}|v) = \left[\frac{1-\gamma-\eta(1_{\{v>3/5\}} + 1_{\{v\leq 3/5\}}\left(\frac{5}{3}v\right)^{5(1+v)})}{1-\eta}\right]^{\frac{1}{1+v}}$$

needs to be three-to-one in some subset of its range. If it is strictly increasing in $[0, 1]$, then equilibrium prices will necessarily be fully revealing. Otherwise, if the mass of bids in some interval $[v_1, v_2]$ is reallocated to $[0, 1] \setminus [v_1, v_2]$ prices would no longer be fully revealing in the latter set, and a positive mass of strategic bidders would rather deviate. On the other hand, if for some η there exists a signal s such that $\mathbb{E}(v - \rho(v, \eta)|s) = 0$ where $\rho(v, \eta)$ given by $1 - \gamma = \eta H(p|v) + (1 - \eta)F(s|v)$ for all $v \in [0, 1)$ satisfies $\rho(0, \eta) > 0$ and $\rho(1, \eta) < 1$, then $[v_1, v_2] = [0, 1]$ fulfils *Corollary 1*, and strategic bids will be confined to $[0, 1] \setminus (\rho(0, \eta), \rho(1, \eta))$.[30]

In a symmetric market ($\gamma = 0.5$), I find that $\underline{\eta} \approx 0.016$ and $\overline{\eta} \approx 0.214$. This shows that the range of η compatible with fully informative prices can be quite small. As an illustration, *Figure I.3* shows equilibrium prices when 10% of traders are naïve.

The first thing to note is that even with such a low proportion of naïve traders, prices can be substantially different from values in about half of the domain of values. Roughly speaking, the probability that prices reflect the true asset value is about one half in this example.

The mass of strategic bids is split between those with signals below s_1^*, who place bids in $[0, \underline{v}_1]$ and traders with signals above s_1^*, who bid in $[\overline{v}_1, 1]$.

[30]If $\mathbb{E}(V - \rho(V, \eta)|0) \geq 0$ all the mass of risk-neutral bids would be placed above $\rho(1, \eta)$ whereas it would be placed below $\rho(0, \eta)$ when $\mathbb{E}(V - \rho(V, \eta)|1) \leq 0$.

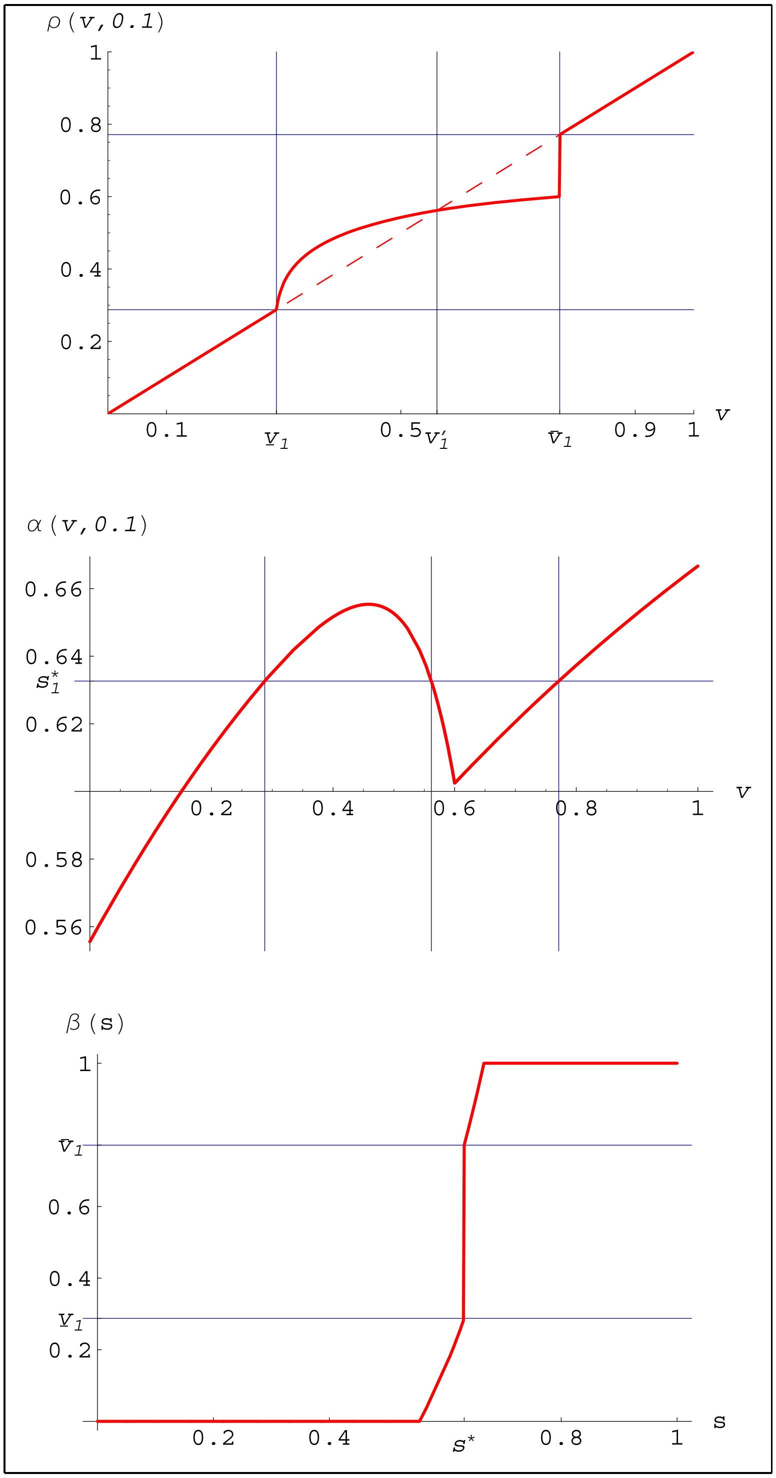

Figure I.3: $\alpha(v, \eta)$ and $\rho(v, \eta)$ for $\gamma = 0.5$, $\eta = 0.1$

This and the fact that buyers and sellers have the same preference ranking over bids implies that sellers with signal $s < s_1^*$ are "active" in the market (i.e. their probability of trading is positive) and get positive expected payoffs while buyers with the same signal are indifferent between trading or not provided they only trade when $\rho(v, \eta) = v$.

The graph of $\alpha(v, 0.1)$ (middle graph of *Figure I.3*) provides some intuition on the existence and uniqueness of prices. As mentioned above, $(s_1^*, \underline{v}_1, \overline{v}_1)$ are given by (I.9)-(I.11), that is $\rho(\underline{v}_1, 0.1) = \underline{v}_1$, $\rho(\overline{v}_1, 0.1) = \overline{v}_1$ and $\mathbb{E}((V - \rho(V, \eta)1_{\{v \in [\underline{v}_1, \overline{v}_1]\}} | s_1^*) = 0$. The latter implies that the expected gain a seller with signal s_1^* makes when she trades at $\rho(v, \eta) > v$ is exactly offset by trades at $\rho(v, \eta) < v$: these two regions are given by $[\underline{v}_1, v_1')$ and $(v_1', \overline{v}_1]$, respectively. Looking at the graph of $\alpha(v, 0.1)$ we can see that, as s_1^* increases, the distance between $\underline{v}_1$ and v_1' goes to zero implying that the set of trades with positive payoff shrinks to zero. Similarly, the distance between v_1' and $\overline{v}_1$ goes to zero when s_1^* decreases. Therefore, by the continuity of $\mathbb{E}(.|.)$ and $\alpha(., 0.1)$, we can find a unique triplet $(s_1^*, \underline{v}_1, \overline{v}_1)$ satisfying the conditions of *Proposition 1* and *Corollary 1.*

Before finishing the example, it is important to point out that, although $\rho(V, \eta)$ is the unique monotone equilibrium price, there are many possible equilibria associated with $\rho(V, \eta)$. Accordingly, I complete this example by characterizing one of the possible equilibrium bidding strategies (see bottom of *Figure I.3*), namely the symmetric equilibrium $\beta(s, t) = \beta(s)$ given by

$$\beta(s) = \begin{cases} 0 & \text{if } s \in [0, \frac{1-\gamma}{1-\eta}] \\ \min v \text{ s.t. } \alpha(v, \eta) = s & \text{if } s \in (\frac{1-\gamma}{1-\eta}, s^*] \\ \max v \text{ s.t. } \alpha(v, \eta) = s & \text{if } s \in (s^*, \frac{1-\gamma-\eta}{1-\eta}] \\ 1 & \text{if } s \in (\frac{1-\gamma-\eta}{1-\eta}, 1] \end{cases} \tag{I.14}$$

I.D An Empirical Test of Information Aggregation

Assuming the model laid out in the previous sections is a reasonable approximation of some existing asset markets,[31] *Proposition 1* suggests a way to identify empirically the intervals of asset values where $\rho(v, \eta) = v$. Furthermore, this identification should not require information on any of the parameters of the market, namely $\gamma, \eta, H(.|.), F(.|)$ and $G(.)$. The only identification restrictions (other than *Assumptions 1-4*) would be the monotonicity of equilibrium prices and the distribution of naïve bids having full support on the set of possible asset values.[32]

The heuristics of how to distinguish the set where prices are fully revealing from the set where $\rho(v, \eta) \neq v$ are rather simple. Recall that strategic bids are placed only in intervals of the price range where $\rho(v, \eta) = v$. Accordingly, the density of bids is higher in a small neighborhood of the observed price when it equals the unknown asset value than when value and price differ. That is, we should observe a discontinuous change in the density of bids at the boundaries $\{\underline{v}_k\}$ and $\{\overline{v}_k\}$ of non-revealing intervals. Specifically, as v increases, the density drops at $\{\underline{v}_k\}$ and jumps at $\{\overline{v}_k\}$, respectively. Hence, using only a series on prices and on a suitable measure of the size of the order flow around market prices one could statistically distinguish the two informational regimes: revealing versus non-revealing prices. Note also that by identifying the sets where prices differ from values we can establish a (not necessarily tight) upper bound on $|\rho(v, \eta) - v|$. Assume $[\rho(\underline{v}_k, \eta), \rho(\overline{v}_k, \eta)]$ is a non-revealing interval of prices. Then, $\rho(\underline{v}_k, \eta) = \underline{v}_k$ and $\rho(\overline{v}_k, \eta) = \overline{v}_k$. By monotonicity of prices, $|\rho(v, \eta) - v| < \rho(\overline{v}_k, \eta) - \rho(\underline{v}_k, \eta)$ for all $p \in [\rho(\underline{v}_k, \eta), \rho(\overline{v}_k, \eta)]$.

This approach can be of special relevance in some markets where the true value of the asset is never observed, and therefore the forecasting properties of

[31] Apart from prediction markets, other trading institutions with a double-auction format such as futures markets or stock exchanges could be suitable for this empirical approach as long as participants in those markets can reasonably characterized as either nonstrategic traders, who transact in these markets primarily driven by liquidity or similar considerations, and arbitrageurs (i.e. strategic agents), whose primary motive is to engage in speculative trading.

[32] Obviously, since this is an static model and most actual markets are dynamic, any empirical analysis would need some stationarity assumptions.

prices are hard to assess. For instance, in markets where Arrow-Debreu securities are traded, the true value of the security (i.e. the probability of the state of nature in which the security pays a dividend) is never observed. Only the realization of the state is observed.

I.E A note on Equilibria in large CVDA

In the continuum agent economy analyzed in the previous sections agents use their private information to decide at which prices they are willing to trade. However, in markets with a finite number of agents, individual decisions have a non-negligible effect on prices. Therefore, it is natural to ask whether the characterization of prices given in *Proposition 1* is a good approximation of what happens in large but finite common value double auctions with naïve traders. That is, whether we can find a sequence of finite economies such that, as the number of traders goes to infinity, (i) equilibrium prices exist and, (ii) they "converge" to monotone equilibrium prices of the continuum agent CVDA.[33]

Although this analysis is beyond the scope of this paper it is worth discussing what might happen when agents can affect the price. The first thing to note is that when agents have the ability to affect prices zero (interim) expected payoffs may not be possible in equilibrium, because traders may have an incentive to deviate in order to push prices in their favor if the missed transactions for doing so involve zero profits. Hence, for equilibrium prices to converge to fully revealing prices they would need to balance vanishing positive expected payoffs with vanishing pivotal probabilities (i.e. the probability of a given agent to affect equilibrium prices). However, I conjecture that this is not possible, because the shares of naïve traders compatible with fully revealing prices ($\eta \leq \underline{\eta}$) are too low to create such

[33]The appropriate notion of convergence would depend on the particular technique used in the asymptotic analysis. For instance, if the finite economies have a continuous bid space as in most analyses of private value double auctions, given the conditional independence of agents' signals, a natural notion is almost sure convergence in the probability space generated by the random vector of asset values and signal profiles. If an analysis à la Reny and Perry (2006) is required, where bids are restricted to be chosen from a discrete grid, a different notion of convergence would be needed.

arbitrage opportunities.[34] On the other hand, convergence in the partially and non-revealing regions should hold, given that in such scenario positive payoffs do not vanish in the limit.[35]

To provide some intuition, imagine that equilibrium prices in a large market look like those in the top graph of *Figure I.4*. Unlike in the continuum agent market, there is a fundamental asymmetry in the preferences of buyers and sellers due to their incentive to affect prices in opposite directions. As a consequence, buyers and sellers bid in different regions of the bidding space, i.e. now buyers (sellers) do not bid where expected prices lie above (below) values. Accordingly, prices below p_1 are slightly above expected values reflecting the overbidding of active sellers trying to increase the price while prices above p_2 are below expected values due to the underbidding of buyers. Note that no strategic seller (buyer) would bid very close to v_1, since he will only trade the object when prices are below (above) expected values. Therefore, only naïve bidders would bid in a neighborhood (p_1, p_2) of v_1. However, if η is too small, the mass of bids in (p_1, v_1) (respectively (v_1, p_2)) is not enough to bring expected prices above (below) values and prices would instead be like those in the bottom of *Figure I.4*, which cannot be an equilibrium since buyers (sellers) bidding close to p_2 (p_1) would like to revise their bids.

I.F Discussion and Concluding Remarks

Unlike REE models where the use of Walrasian equilibrium leaves price formation unspecified, double auctions provide insight into how individual trading decisions determine market prices so that information aggregation can be explicitly analyzed. In the pure common value market presented here, agents use their private information in their bidding strategies, despite (i) being price takers and (ii) their private information not adding any new information to the market. This stems

[34] This is why for very low η only fully revealing prices are possible in any monotone equilibrium of the continuum agent CVDA.

[35] A way to get convergence to fully revealing prices could be to make the share of naïve traders converge from above to $\underline{\eta}$ at a lower rate than the rate at which pivotal probabilities vanish, instead of holding η constant.

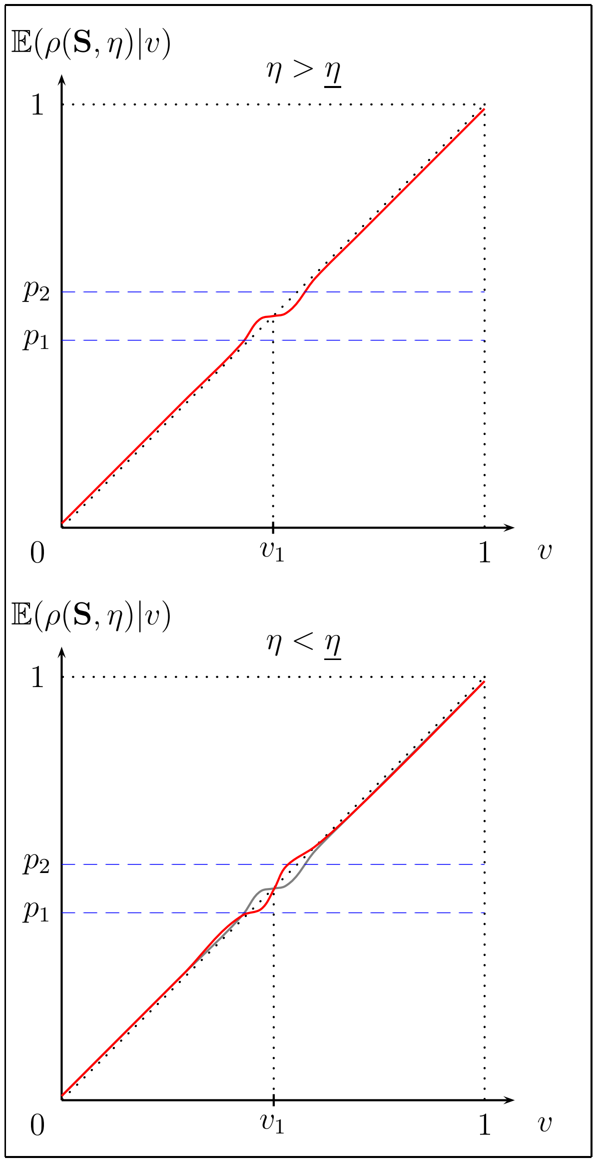

Figure I.4: Prices in a finite common value double auction from the fact that agents do not observe the price before deciding whether to trade or not. In contrast, in REE models the use of private information in large markets is often linked to risk aversion. This leads to a well-establish paradox associated to REE models (Hellwig (1980), Diamond and Verrecchia (1981), Grossman (1976, 1978)): if agents are risk-neutral, prices become fully revealing regardless the level of noise trade (modeled as random demand/supply).[36] Accordingly, agents dismiss their private information, which raises the question of how such information ends up reflected in the market price.

Hence, the analysis presented here provides an alternative theory of how and when prices aggregate information: even if no single agent can affect the price, a trader can make more accurate inferences about the market price by using his private signal. One could argue that the fact that information aggregation is driven by the use of private information to forecast prices is a more appealing explanation than information aggregation being driven solely by attitudes toward risk. In fact, the introduction of risk aversion should not change the implications of the model. Specifically, the key feature of strategic bidding behavior, namely that they do not bid in non-revealing areas of the equilibrium price function, still holds for risk averse agents since it is just a consequence of maximizing expected gains from trade when trading at prices different from values. Accordingly, one should be able to prove results similar to those presented above for a population of strictly risk averse agents.[37] This would imply that risk aversion does not substantively affect the informational properties of prices. With regards to prediction markets, my motivating example, this means that their performance should not hinge upon eliciting risk neutral behavior.

[36]For instance, in Hellwig (1980), as long as there is a non-vanishing share of risk-neutral agents, the unknown realization of the returns from the risky asset can be fully inferred from the price, for any finite *per capita* level of noise in the asset supply.

[37]Risk aversion breaks the symmetry of buyers and sellers' preferences and induces some strategic traders to abstain from trading. However, one should be able to get equilibrium prices similar to those given by *Proposition 1* by having two block-monotonic bid distributions, one for buyers and one for sellers. This reasoning applies to any population of traders with heterogeneous degrees of risk aversion, as long as the set of distinct attitudes toward risk is finite, so that the number of different block-monotonic distributions is also finite.

It is important to point out that the assumption of common values is key to analyze information aggregation, given that they trigger the inference process that drives information into prices.[38] Thus, recent models analyzing prediction markets where agents fail to understand the existence of a common value component (Manski (2004), Gjerstad (2005), Wolfers and Zitzewitz (2006)) do not regard prices as information aggregators. Instead, they look at how close prices are to some statistic (e.g. the mean) of the distribution of agents' *interim* beliefs (i.e. beliefs based only on their privately held information). Their approach would be equivalent to having a share of naïve traders $\eta = 1$ when naïve traders bids are equal to the expected value of the asset conditional on their private signals. Not surprisingly, in such a case, prices are almost nowhere equal to asset values.[39]

From the analysis of the common value double auction, the main conclusion to extract is that, with price-taking behavior, prices aggregate information as long as the level of noise or liquidity trade, embodied by the presence of naïve traders, is small.[40] Otherwise, prices can be far from fully revealing. As the above example shows, the market price can be quite uninformative even for shares of naïve traders as low as 10%. This conclusion is at odds with some well-established views about the informational efficiency of markets. One such view is that a non-negligible share of *marginal* traders is enough to induce fully revealing prices (Wolfers and Zitzewitz (2004), Forsythe, Nelson, Neumann, and Wright (1992), Hellwig (1980)).[41] As it is shown here, this is not true in the double auction. In the example provided, the minimum share of marginal traders compatible with fully revealing prices is about 98.5%. An alternative view is that, as long as there

[38]With private values individual beliefs depend only on the individual's private information, thus learning the market price does not provide any additional insight to the trader about the value of the asset.

[39]One could argue that private information already incorporates common information, and markets simply attract well-informed individuals. In such a case, it would be interesting to model entry decisions as a function of the quality of private information. Neither the double auction model presented here nor existing models aimed at analyzing prediction markets deal with this issue.

[40]In the absence of price-taking behavior, one should expect prices being less informative.

[41]Marginal traders refer to those traders who react to their information and to changes in prices so as to induce prices to reflect available information. The definition of "marginal" trader varies across models. For some authors it means traders not suffering from rationality "biases" (Forsythe, Nelson, Neumann, and Wright (1992)), whereas in REE models it means risk-neutral agents (Hellwig (1980)).

is a positive amount or noise or liquidity trade, prices cannot be fully revealing (Black (1986), Reny and Perry (2006), Kyle (1985, 1989)). In the double auction, this is not true either. When we account for the strategic incentives of traders, information aggregation is possible although it is quite sensitive to the amount of non-strategic traders in the market.

The analysis also provides insight into the qualitative behavior of strategic traders in double auctions. First, not surprisingly, low signal sellers and high signal buyers are more active in the market. That is, sellers with lower interim valuations and buyers with more positive views about the asset value will trade in a wider range of prices, compared to low signal buyers and high signal sellers. More interesting is that strategic traders restrict their asks and bids to areas where prices are fully revealing and, if that is not possible, they will bid outside the range of prices. This means that strategic traders are *perfect* price setters whenever possible, given that they set prices equal to values in the range of prices where they place their bids. This feature of the model provides the possibility to empirically distinguish between revealing and non-revealing areas without knowledge of the amount of naïve traders, their bidding behavior, or the distribution of private information.

I conclude with a word of caution. To induce price taking behavior I have analyzed a continuum agent market. Therefore, it remains to be seen if the results obtained carry through large but finite economies.

I.G Appendix: Proofs

I.G.1 Proofs of *Proposition 1* and *Corollary 1*

Proof of *Lemma 1.* Let $\rho(V,\eta)$ be the price function resulting from strategy profile $\beta(.,.)$, and assume buyer t and seller t' bid b when they receive signal s, i.e. $\beta(s,t) = \beta(s,t') = b$. If we subtract (I.7) from (I.6) we get

$$\pi^{buy}(s,t) = \pi^{sel}(s,t') + \mathbb{E}((V - \rho(V,\eta))|s). \tag{I.15}$$

Since the last term does not depend on b, a buyer and a seller receiving the same signal will have the same preference ranking over bids. ■

Let $\rho_+^{-1}(b,\eta) := \max\{v : \rho(v,\eta) = b\}$, $\rho_-^{-1}(b,\eta) := \min\{v : \rho(v,\eta) = b\}$.

Lemma 2 (Strategic Bidding in monotone equilibria (I)) *Let $\beta(.,.)$ be the sophisticated traders' strategy profile in a monotone equilibrium and (v_1, v_2) be a non-degenerate set of asset values.*

(i) If $\rho(v,\eta) < v$ for all $v \in (v_1, v_2)$ and there is some trader t such that $\beta(s,t) < \rho(v_1,\eta)$ for some s, then there exists $v' < v_1$ such that $\rho(v,\eta) \geq v$ for all $v \in (\rho_-^{-1}(\beta(s,t),\eta), v')$, with strict inequality in a non-null subset.

(ii) If $\rho(v,\eta) > v$ for all $v \in (v_1, v_2)$ and there is some trader t such that $\beta(s,t) > \rho(v_2,\eta)$ for some s, then there exists $v' > v_2$ such that $\rho(v,\eta) \leq v$ for all $v \in (v', \rho_+^{-1}(\beta(s,t),\eta))$, with strict inequality in a non-null subset.

Proof of *Lemma 2.* Part (i): assume that $\rho(v,\eta) \leq v$ holds for all $v \in (\rho_-^{-1}(\beta(s,t),\eta), v_1)$. Then, given that $\mathbb{E}((V - \rho(V,\eta))1_{\{\rho(V,\eta)<v_2\}}|s) > 0$ for all s, a buyer would strictly prefer to bid v_2 than $\beta(s,t)$. Since preferences are symmetric, a seller would also prefer to bid v_2.[42] A symmetric argument applies to (ii). ■

[42]Note that if b is an atom of the price distribution, $v' \leq \rho_+^{-1}(\beta(s,t),\eta)$ can only happen if the probability of getting the object is zero, i.e. $\lambda(\beta(s,t),v) = 0$ for all $v \in (\rho_-^{-1}(\beta(s,t),\eta), \rho_+^{-1}(\beta(s,t),\eta))$. Otherwise, a trader bidding at b would rather bid lower.

The following fact is used in the proofs of *Lemma 3* and *Proposition 1*.

Fact 1 *Let Assumptions 1 and 2 be satisfied. If* $\rho(v,\eta) > v$ *for all* $v \in [v_1, v_2)$ *and* $\rho(v,\eta) < v$ *for all* $v \in (v_2, v_3]$ *with* $\rho(.,\eta)$ *increasing, then for all* $s \in (0,1)$*,*

(i) If $\mathbb{E}((V-\rho(V,\eta))1_{\{V\in[v_1,v_3]\}}|s) \leq 0$*, then* $\mathbb{E}((V-\rho(V,\eta))1_{\{V\in[v_1,v_3]\}}|s') < 0$ *for all* $s' < s$*;*

(ii) If $\mathbb{E}((V-\rho(V,\eta))1_{\{V\in[v_1,v_3]\}}|s) \geq 0$*, then* $\mathbb{E}((V-\rho(V,\eta))1_{\{V\in[v_1,v_3]\}}|s') > 0$ *for all* $s' > s$*.*

Proof of *Fact 1*. Let $\mathbb{E}((V-\rho(V,\eta))1_{\{V\in[v_1,v_3]\}}|s) \leq 0$. Thus,

$$\frac{1}{f(s)}\int_{v_1}^{v_2}(\rho(V,\eta)-V)f(s|v)g(v)dv \geq \frac{1}{f(s)}\int_{v_2}^{v_3}(V-\rho(V,\eta))f(s|v)g(v)dv. \quad \text{(I.16)}$$

By the strict monotone likelihood ratio of F (*Assumption 2*) we have that for all $s' < s$ and all $v' \in [v_1, v_2)$ and $v \in [v_2, v_3)$, $\frac{f(s'|v')}{f(s|v')} > \frac{f(s'|v)}{f(s|v)}$. Given this and the above inequality, we have that

$$\int_{v_1}^{v_2}(\rho(V,\eta)-V)f(s|v)\frac{f(s'|v)}{f(s|v)}g(v)dv > \int_{v_2}^{v_3}(V-\rho(V,\eta))f(s|v)\frac{f(s'|v)}{f(s|v)}g(v)dv. \quad \text{(I.17)}$$

Given that $f(s') > 0$ for all s' by the full support of F and G (*Assumption 1*), (I.17) implies that $\mathbb{E}((\rho(V,\eta)-V)1_{\{V\in[v_1,v_2]\}}|s') > \mathbb{E}((V-\rho(V,\eta))1_{\{V\in[v_2,v_3]\}}|s')$. A symmetric argument applies to part (ii). ∎

Lemma 3 (Strategic Bidding in monotone equilibria (II)) *In any monotone equilibrium, the mass of strategic traders submitting bids in* $\{\rho(v,\eta) : v - \rho(v,\eta) \neq 0\}$ *is zero, except perhaps when there is a positive mass at the boundaries of the price range,* $\rho(0,\eta)$ *and* $\rho(1,\eta)$*, and there is complete rationing* ($1-\gamma = B(\rho(0,\eta)|v)$ *for all* $v \in [0, \rho_+^{-1}(0,\eta)]$) *and no rationing* ($1-\gamma = B\_(\rho(1,\eta)|v)$ *for all* $v \in [\rho_-^{-1}(1,\eta), 1]$)*, respectively.*

Proof of *Lemma 3.* By *Lemma 1*, I only need to look at a buyer's incentives. The proof goes along the following lines. First, I show that no strategic buyer is best-responding by bidding the interior of an interval of prices in which $\rho(v,\eta) \neq v$. Second, I show that if $\rho(.,\eta)$ is constant in an interval of values (i.e. the distribution of prices has an atom) a strategic buyer will only bid at the atom if she gets the object with probability zero or one, depending on whether the expected value of $\rho(V,\eta) - V$ at the atom is positive or negative, respectively. Otherwise, she would bid slightly above or below to avoid to either avoid trading or being rationed. Finally, I prove that these conditions cannot be satisfied at an atom in the interior of the price range. Therefore, the only possibility left for a strategic buyer to bid in $\{\rho(v,\eta) : v - \rho(v,\eta) \neq 0\}$ is to bid at the boundaries, with the condition that she does not trade almost surely when she bids $\rho(0,\eta)$ and that she trades with probability one when bidding $\rho(1,\eta)$.

Assume, that a buyer bids in an interval $(\rho(v_1,\eta), \rho(v_2,\eta))$ where $v > \rho(v,\eta)$ and $\rho(v,\eta)$ is strictly increasing a.e. in (v_1,v_2), i.e. there is no atom in (v_1,v_2). In this case, she prefers to bid v_2 to any bid $b \in (\rho(v_1,\eta), \rho(v_2,\eta))$, given that her payoff increases by $\mathbb{E}((V - \rho(V,\eta))1_{\{\rho(V,\eta)\in(b,\rho(v_2,\eta))\}}|s)$, which is strictly positive for all s. Now assume that $v < \rho(v,\eta)$ in (v_1,v_2). Then, a buyer would prefer to bid below $\rho(v_1,\eta)$ given that $\mathbb{E}((V - \rho(V,\eta))1_{\{\rho(V,\eta)\in(\rho(v_1,\eta),b)\}}|s) < 0$ for all s.

Now assume there is an atom at b. If there is a positive mass of strategic bids at b, with $\rho(v,\eta) = b$ on some interval (v_1,v_2), a buyer with signal s might bid at $b \in (0,1)$ under one of these conditions: (i) $\mathbb{E}((V-b)1_{\{\rho(V,\eta)=b\}}|s) = 0$; (ii) $\mathbb{E}((V-b)1_{\{\rho(V,\eta)=b\}}|s) > 0$ with $\lambda(b,v) = 1$ for all $v \in (v_1,v_2)$ (i.e. no rationing); (iii) $\mathbb{E}((V-b)1_{\{\rho(V,\eta)=b\}}|s) < 0$ with $\lambda(b,v) = 0$ for all $v \in (v_1,v_2)$ (i.e. no trading when $\rho(.,\eta) = b$).

In case (i), she is indifferent between bidding slightly above or below b. However, *Fact 1* implies that there can be at most one signal satisfying (i).[43]

[43] For (i) to hold $v < b$ in the lower part of (v_1,v_2) and $v > b$ in the upper part of the interval, so that *Fact 1* applies.

Therefore the mass of bids at b due to (i) is zero. In (ii), unless she gets the object with probability one ($\lambda(b,v)=1$) she would rather bid above b. Finally, in (iii) she may bid at b only if the probability of getting the object is zero ($\lambda(b,v)=0$). Since in each of the latter two cases $\lambda(b,.)$ is required to be zero or one in the whole interval (v_1,v_2), there cannot be two traders bidding at b with distinct signals satisfying (ii) and (iii), respectively. Accordingly, all the mass of strategic bidders bidding at b either satisfy (ii) or (iii).

Now I show that (ii) and (iii) can only happen when $b=\rho(1,\eta)$ and $b=\rho(0,\eta)$, respectively.

Assume (ii) is satisfied for all bidders bidding at b and let $\underline{s}$ be the lowest signal associated to b. Accordingly, a trader receiving $\underline{s}$ bids optimally at b if

$$\mathbb{E}((V-\rho(V,\eta))1_{\{p\leq b\}}|\underline{s})\geq 0, \tag{I.18}$$

and

$$\mathbb{E}((V-\rho(V,\eta))1_{\{p>b\}}|\underline{s})\leq 0. \tag{I.19}$$

By *Lemma 2*, we can apply *Fact 1* to (I.18) and (I.19).[44] Hence, all strategic traders with signals above $\underline{s}$ will bid at or above b (assuming $\lambda(b,v)=1$). Likewise, given (I.19) and the fact that $\mathbb{E}((V-b)1_{\{\rho(V,\eta)=b\}}|\underline{s})\geq 0$, all strategic traders with $s<\underline{s}$ will bid strictly below b.

For $\lambda(b,v)=1$ we need the mass of sellers bidding strictly less than b be equal to the mass of buyers bidding at b or above. That is, for all $v\in(v_1,v_2)$,

$$\gamma[\eta H(b|v)+(1-\eta)F(\underline{s}|v)]=(1-\gamma)[\eta(1-H(b|v))+(1-\eta)(1-F(\underline{s}|v))]. \tag{I.20}$$

Given that $B\_(b|v)=\eta H(b|v)+(1-\eta)F(\underline{s}|v)$, the above expression is satisfied when $B\_(b|v)=1-\gamma$ for all $v\in(v_1,v_2)$.

Now assume that $b<\rho(1,\eta)$, i.e. $v_2<1$ and $\rho(v,\eta)>b$ for all $v>v_2$. For that to happen we need $B(b|v)<1-\gamma$ for all $v>v_2$. But this implies, by

[44] By the linearity of expectations, the conclusions of *Fact 1* also apply to a succession of intervals satisfying the conditions in the lemma.

the continuity of $B(b|v)$ and $B\_(b|v)$, that there exists $v' < v_2$ such that $B\_(b|v) < B(b|v) \leq 1 - \gamma$ for all $v \geq v'$, which contradicts that $\lambda(b, v) = 1$ for all $v \in (v_1, v_2)$. Hence, (ii) is only possible in equilibrium if $b = \rho(1, \eta)$ and $v_2 = 1$.

A symmetric argument applies when (iii) is satisfied for almost all strategic traders bidding at b. In this case, the mass of sellers bidding at or below b needs to be equal to the mass of buyers bidding strictly above b for $\lambda(b, v) = 0$. This requires that $B(b|v) = 1 - \gamma$ for all $v \in (v_1, v_2)$. By the continuity of $B(b|v)$ and $B\_(b|v)$, $b = \rho(0, \eta)$ and $v_1 = 0$, otherwise there would be a subset of (v_1, v_2) for which $1 - \gamma \leq B\_(b|v) < B(b|v)$, contradicting that $\lambda(b, v) = 0$ for all $v \in (v_1, v_2)$.

Finally, if $\rho(1, \eta)$ is not an atom, the probability of rationing at $\rho(1, \eta)$ is zero and a buyer bidding $\rho(1, \eta)$ will always trade. In this case there can exist a positive mass of strategic bids at $\rho(1, \eta) < 1$, since any buyer bidding $\rho(1, \eta) < 1$ is indifferent between any two bids in $[\rho(1, \eta), 1]$. A symmetric argument can be made for bids at $\rho(0, \eta) > 0$. ■

Lemma 3 allows for the possibility of having strategic bids placed at an atom, at $\rho(0, \eta)$ or at $\rho(1, \eta)$, of the price distribution if either sellers or buyers bidding at the atom trade with probability one, respectively. However, as the next lemma shows, atoms can only occur for very particular naïve share and bid distributions.

Lemma 4 (No atoms) *In any monotone equilibrium if there exists $v_1 < v_2$ such that $\rho(v, \eta) = b$ on (v_1, v_2) then*

(a) $\mathbb{E}((V - \rho(v, \eta))1_{\{V<v_2\}}|s) \geq 0$ *for all s, and* $H(\rho(v, \eta)|v) = \frac{1-\gamma}{\eta}$ *for all $v \leq v_2$;*

(b) $\mathbb{E}((V - \rho(v, \eta))1_{\{V<v_2\}}|s) \leq 0$ *for all s, and* $H(\rho(v, \eta)|v) = \frac{\eta-\gamma}{\eta}$ *for all $v \geq v_1$.*

Lemma 4 basically states that atoms in the price distribution are solely created by naïve traders, and that very special circumstances need to occur: the share of naïve bids is very high compared to γ (or to $1-\gamma$); naïve traders completely determine prices at the low (high) end of the price range, with those prices being low (high) enough so that they do not encourage strategic traders to bid below (above) the atom; and the distribution of naïve bids is independent of asset values in the interval of values associated with the atom.[45]

Proof of *Lemma 4*. Assume there is an interval (v_1, v_2) such that $\rho(v, \eta) = b$ for all $v \in (v_1, v_2)$. By *Lemma 2* and *Fact 1*, if there exists a trader with signal s bidding below (above) b then it is optimal for all traders with signals below (above) s to also bid below (above) b. Accordingly, let $\underline{s} \in [0,1]$ be the highest signal associated with bids below b, and $\overline{s} \geq \underline{s}$ the lowest signal associated with bids above b. Since the distribution of naïve bids is atomless (*Assumption 3*), this implies that

$$B\_(b|v) = \eta H(b|v) + (1-\eta)F(\underline{s}|v),$$

and

$$B(b|v) = \eta H(b|v) + (1-\eta)F(\overline{s}|v).$$

There are two possible cases, depending on whether a positive mass of strategic bids is placed at b or not, i.e. whether $\underline{s} < \overline{s}$ or $\underline{s} = \overline{s}$.

If there is no positive mass of strategic bids at b, we have that $B(b|v) = B\_(b|v) = 1-\gamma$ for all $v \in (v_1, v_2)$. Since $F(s|v)$ is strictly decreasing in v for all $s \in (0,1)$ and $H(b|v)$ is non-increasing in v for all $b \in [0,1]$, $B\_(b|v) = 1-\gamma$ for all $v \in (v_1, v_2)$ only if $\underline{s} = 0$ or $\underline{s} = 1$.

a) $\underline{s} = 0$: in this case $H(b|v) = \frac{1-\gamma}{\eta}$ for all $v \in (v_1, v_2)$. But then, we need $\mathbb{E}((V - \rho(V,\eta))1_{\{\rho(V,\eta)\leq b\}}|s) = \mathbb{E}((V-\rho(V,\eta))1_{\{V<v_2\}}|s) \geq 0$ for all s, otherwise some strategic traders would rather bid below b. Finally, prices below b are

[45] An example of equilibrium prices being constant in some interval of values is given by a high enough presence of random traders bidding uniformly in $[0,1]$. In this case, $H(b|v) = b$ for all v. Hence, if η is high enough so that $\mathbb{E}(V|0) \geq b = \frac{1-\gamma}{\eta}$, then $\rho(v,\eta) = \frac{1-\gamma}{\eta}$ for all v, with all strategic traders bidding at or above $\frac{1-\gamma}{\eta}$. In this case, all strategic buyers and no strategic seller engage in trade.

completely determined by naïve bids, since no strategic trader bids below b, i.e. $H(\rho(v,\eta)|v) = \frac{1-\gamma}{\eta}$ for all $v \leq v_1$.[46]

b) $\underline{s} = 1$: in this case $H(b|v) = \frac{\eta-\gamma}{\eta}$ for all $v \in (v_1, v_2)$. In addition, we need $\mathbb{E}((V - \rho(V,\eta))1_{\{V<v_2\}}|s) \leq 0$ for all s. Since no strategic trader bids above b, prices above b are given by $H(\rho(v,\eta)|v) = \frac{\eta-\gamma}{\eta}$ for all $v \geq v_2$.

If there is a positive mass of strategic bids at b, *Lemma 3* applies, requiring either that $B\_(b|v) = 1-\gamma$ or $B(b|v) = 1-\gamma$. The former requires $\underline{s} = 0$ or $\underline{s} = 1$, while the latter can be possible only if $\overline{s} = 0$ or $\overline{s} = 1$. Therefore, they reduce to the same conditions on $H(.|.)$ and $\mathbb{E}((V - \rho(V,\eta))1_{\{\rho(V,\eta)\leq b\}}|s)$. ■

Proof of *Proposition 1.* By the monotonicity of $\rho(.,\eta)$ and *Lemma 3*, all the mass of strategic bids in $(\rho(0,\eta), \rho(1,\eta))$ is placed in a countable collection of disjoint intervals in which $\rho(v,\eta) = v$. Let $\mathcal{V}$ be the complement of such set in $[0,1]$. Thus, $\mathcal{V} \supseteq \{v : \rho(v,\eta) \neq v\}$ by *Lemma 3*. Assume $\mathcal{V}$ is non-empty, otherwise *Proposition 1* holds trivially.

Denote $B^*(.|.)$ the cdf of strategic bids and assume that $B^*(.|.)$ is atomless.[47] Accordingly, $B^*_-(.|v) = B^*(.|v)$ for all $v \in [\rho(0,\eta), \rho(1,\eta)]$ and $\mathcal{V}$ can be expressed, without loss of generality, as the countable union of disjoint closed intervals $[\underline{v}_k, \overline{v}_k]$, such that $\rho(v,\eta)$ is given by

$$H(\rho(v,\eta)|v) = \frac{1-\gamma-(1-\eta)B^*(\rho(\underline{v}_k,\eta)|v)}{\eta} \quad \text{for all } v \in [\underline{v}_k, \overline{v}_k]. \tag{I.21}$$

Further assume that prices are not a.e. equal to values in $[\underline{v}_k, \overline{v}_k]$. Otherwise, redefine $\mathcal{V}$ not to include such interval.

Notice that $B^*(\rho(\underline{v}_k,\eta)|v) = B^*(\rho(\overline{v}_k,\eta)|v)$ by *Lemma* 3 for all k, including non-revealing intervals with $\underline{v}_1 = 0$ (i.e. when $\rho(0,\eta) > 0$) and $\overline{v}_K = 1$

[46] This is possible in principle given that $H(.|.)$ is increasing in its first argument and decreasing in its second argument.

[47] This implies that $\rho(0,\eta) > 0$ and $\rho(1,\eta) < 1$, given that $H(0) = 0 < 1-\gamma$ and $H(1) = 1 > 1-\gamma$ by *Assumption 3.*

$(\rho(1,\eta) < 1)$. Hence, we just need to show that $B^*(\rho(\underline{v}_k,\eta)|v) = F(s_k^*|v)$, for all k and all $v \in [\underline{v}_k, \overline{v}_k]$, with s_k^* satisfying (I.9)-(I.11).

By *Lemma 2* and the fact that $\rho(0,\eta) > 0$ and $\rho(1,\eta) < 1$, there exist v_k', v_k'' with $\underline{v}_k < v_k' \leq v_k'' < \overline{v}_k$ such that $\rho(v,\eta) \geq v$ a.e. in $[\underline{v}_k, v_k']$ with strict inequality in a non-null subset, and $\rho(v,\eta) \leq v$ a.e. in $[v_k'', \overline{v}_k]$ with strict inequality in a non-null subset.[48] But this implies that, if bidding in $[\overline{v}_{k-1}, \rho(\underline{v}_k)]$ is optimal for a seller with signal s, i.e. $\mathbb{E}((V - \rho(V,\eta))1_{\{V\in(\underline{v}_k,\overline{v}_k)\}}|s) \leq 0$, then $\mathbb{E}((V - \rho(V,\eta))1_{\{V\in(\underline{v}_k,\overline{v}_k)\}}|s') < 0$ for all $s' < s$ by *Fact 1*.[49] Hence, bidding above $\rho(\overline{v}_k)$ is strictly dominated by bidding in $[\overline{v}_{k-1}, \rho(\underline{v}_k)]$ for all sellers with $s' < s$. A symmetric argument can be used for all $s' > s$ when it is optimal for a seller with signal s to bid in $[\rho(\overline{v}_k), \underline{v}_{k+1}]$. By *Lemma 1* the same applies for a buyer. Therefore, $B^*(\rho(\underline{v}_k)|v) = F(s_k^*|v)$ for some signal $s_k^* > 0$. Moreover, s_k^* needs to satisfy (I.9) if $\underline{v}_k > 0$ and (I.10) whenever $\overline{v}_k < 1$, given that $\rho(v,\eta) = v$ in $(\overline{v}_{k-1}, \underline{v}_k)$ and in $(\overline{v}_k, \underline{v}_{k+1})$ and that $H(.|.)$, $F(.|.)$ are atomless distributions. Finally, condition (I.11) is just the equilibrium condition for a seller with $s \leq s_k^*$ $(s > s_k^*)$ to optimally bid below $\rho(\underline{v}_k)$ (above $\rho(\overline{v}_k)$), which also implies that $s_{k-1}^* < s_k^*$ for all $k > 1$.

Now assume that $B^*(.|.)$ has an atom. Since $H(.|.)$ does not have atoms in $(\rho(0,\eta), \rho(1,\eta))$, neither can $B^*(.|.)$. An atom of $B^*(.|.)$ at $b \in (\rho(0,\eta), \rho(1,\eta))$ would imply that $B_{\_}(b|b) < B(b|b)$, creating an atom of the price distribution at b, which leads to a non-revealing interval where strategic bids are placed, a contradiction of *Lemma 3*. Therefore, $B^*(.|.)$ can have an atom only in $\{\rho(0,\eta), \rho(1,\eta)\}$.

If there is an atom in $B^*(.|.)$ at $\rho(0,\eta)$, the price distribution may or may not have an atom at $\rho(0,\eta)$. If the price distribution has an atom at $\rho(0,\eta)$, by *Lemma 4*, $B^*(\rho(0,\eta)|v) = 1$ for all v and $H(\rho(0,\eta)|v) = \frac{\eta-\gamma}{\eta}$ for all $v \leq \rho_+^{-1}(\rho(0,\eta),\eta)$. Accordingly, $B^*(\rho(0,\eta)|v) = F(1|v)$ and (I.8) is satisfied. If the

[48] In what follows, I use the convention, $\overline{v}_0 = 0$ and $\underline{v}_{K+1} = 1$.

[49] By the linearity of expectations, this is also true when there are non-revealing intervals above $\overline{v}_k$ satisfying *Lemma 2*, or subintervals in $[v_k', v_k'']$ with $\rho(v,\eta) > (<) v$. For instance, if $\mathbb{E}((V - \rho(V,\eta))1_{\{V\in(\underline{v}_k,\overline{v}_k)\}}|s) + \sum_{k'>k} \mathbb{E}((V - \rho(V,\eta))1_{\{V\in(\underline{v}_k',\overline{v}_k')\}}|s) \leq 0$ for some s, with $\mathbb{E}((V - \rho(V,\eta))1_{\{V\in(\underline{v}_k,\overline{v}_k)\}}|s) < 0$, then these inequalities hold strictly for all $s' < s$.

price distribution does not have an atom at $\rho(0,\eta)$, $\rho(0,\eta)$ is given by (I.21), i.e.

$$H(\rho(0,\eta)|0) = \frac{1-\gamma-(1-\eta)B^*(\rho(0,\eta)|0)}{\eta}. \tag{I.22}$$

Hence, if a non-revealing interval starts at $\rho(0,\eta)$ (i.e. $\underline{v}_1 = 0$), *Lemma 2* applies to the interval $[0,\overline{v}_1]$ and, by *Fact 1*, there exists a signal $s_1^* > 0$ satisfying (I.11) such that $B^*(\rho(0,\eta)|v) = F(s_1^*|v)$.

Finally, assume $B^*(.|.)$ has an atom at $\rho(1,\eta)$. If the price distribution has an atom at $\rho(1,\eta)$, $B_-^*(\rho(1,\eta)|v) = 0$ for all v and $H(\rho(1,\eta)|v) = \frac{1-\gamma}{\eta}$ for all $v \geq \rho_-^{-1}(\rho(0,\eta))$ by *Lemma 4*. Thus, $B_-^*(\rho(1,\eta)|v) = F(0|v)$ and (I.8) is also satisfied. If the price distribution does not have an atom at $\rho(1,\eta)$, $\rho(1,\eta)$ is given by

$$H(\rho(1,\eta)|1) = \frac{1-\gamma-(1-\eta)B_-^*(\rho(1,\eta)|1)}{\eta}. \tag{I.23}$$

Therefore, if a non-revealing interval ends at $\rho(1,\eta)$, *Lemma 2* applies to the interval $[\overline{v}_K, 1]$ and, by *Fact 1*, there exists a signal $s_K^* < 1$ satisfying (I.11) such that $B^*(\rho(0,\eta)|v) = F(s_1^*|v)$.

■

Proof of *Corollary 1*. First, notice that by *Lemma 2*, all price intervals $(\rho(\underline{v}_k,\eta), \rho(\overline{v}_k,\eta))$ with strategic bids below and above them satisfy the first part of *Corollary 1*, i.e. prices are above values in the lower portion of the interval $(\underline{v}_k, \overline{v}_k)$ and below values in the upper portion. When there are no strategic bids placed below $(\rho(\underline{v}_k,\eta), \rho(\overline{v}_k,\eta))$ we have that $\underline{v}_k = 0$ given that $H(0) = 0 < 1-\gamma$ implies $\rho(0,\eta) > 0$. If there are strategic bids above this interval part (ii) of *Lemma 2* applies. Finally, no strategic bids above $(\rho(\underline{v}_k,\eta), \rho(\overline{v}_k,\eta))$ lead to $\overline{v}_k = 1$ and $\rho(\overline{v}_k,\eta) < 1$. In addition, if there are strategic bids below part (i) of *Lemma 2* applies. Hence, these two cases also satisfy the first part of the corollary.

This also applies when there is an atom at $(\underline{v}_k, \overline{v}_k)$. Since atoms can only happen when there are no bids either below or above the atom (*Lemma 4*), we are in one of the above cases.

Regarding the second part of *Corollary 1*, assume there is a non-revealing interval $(\underline{v}_k, \overline{v}_k)$ that can be partitioned into two subintervals, $(\underline{v}_k, \hat{v}]$ and $(\hat{v}, \overline{v}_k)$, such that, in each of them, prices are above values in the lower portion of the subinterval and below values in the upper portion with $\mathbb{E}((V-\rho(V,\eta))1_{\{V\in(\hat{v},\overline{v}_k)\}}|s_k^*) < 0$. In such case, $\mathbb{E}((V-\rho(V,\eta))1_{\{V\in(\overline{v}_k,\hat{v}]\}}|s_k^*) > 0$ and a strategic seller bidding below $\rho(\underline{v}_k,\eta)$ with signal close to s_k^* would rather deviate and bid $\hat{v}$ since, by doing so, he can avoid negative payoffs from trading when $v \in (\overline{v}_k, \hat{v}]$. By symmetry of preferences a buyer with the same signal would also deviate. ■

I.G.2 Proof of ***Proposition 2***

I provide a series of technical lemmas and facts related to the quantile function $\alpha(.,\eta)$, which are used in the proof of *Proposition 2*. They show that $\alpha(.,\eta)$ is increasing for small values of η, non-monotonic for intermediate values and not well-defined and decreasing when η is high enough. As shown below, prices must equal values everywhere in the first case, and cannot be revealing in areas where $\alpha(.,\eta)$ is either not defined or decreasing.

In what follows, D_i represents the partial derivative with respect to the ith argument.

Lemma 5 *If Assumptions 1-4 are satisfied the following statements are true:*

(i) $\alpha(0,\eta)$ *is well-defined for* $\eta < \gamma$*, strictly positive and increasing in* η*;* $\alpha(1,\eta)$ *is well-defined for* $\eta < 1-\gamma$*, strictly less than one and decreasing in* η*.*

(ii) If $D_1\alpha(v,\eta) < 0$ *then* $D_1\alpha(v,\eta') < 0$ *for all* $\eta' > \eta$ *for which* $\alpha(v,\eta)$ *is well defined.*

(iii) There exists $\underline{\eta} \in (0, \min\{\gamma, 1-\gamma\})$ *such that* $\alpha(.,\eta)$ *is well-defined and strictly*

increasing for all $\eta < \underline{\eta}$*, and it is non-monotonic or decreasing for all* $\eta > \underline{\eta}$ *in the subset of values where it is well-defined.*

(iv) If $H'(v) > 0$ *for all* v *such that* $H(v) = 1 - \gamma$*, there exists* $\overline{\eta} \in [\underline{\eta}, 1)$ *such that* $\alpha(., \eta)$ *is decreasing whenever it is well-defined for all* $\eta > \overline{\eta}$*.*

Proof of Lemma 5.

Part (i): since $H(0) = 0$, $\alpha(0, \eta) = F^{-1}(\frac{1-\gamma}{1-\eta}|0)$, which is well-defined if $\eta < \gamma$. Since $F(.|v)$ has full support for all v and $\frac{1-\gamma}{1-\eta}$ is increasing in η, $\alpha(0, \eta)$ is increasing in η. Similarly, $H(1) = 1$ so $s(1, \eta) = F^{-1}(\frac{1-\gamma-\eta}{1-\eta}|0)$ is well-defined for $\eta < 1 - \gamma$ and decreasing in η.

Part (ii): by the a.e. smoothness of H and F (*Assumptions 1* and *3*), $\alpha(v, \eta)$ is a.e. differentiable. I differentiate both sides of (I.12) to obtain $D_1\alpha(v, \eta)$:

$$D_1\alpha(v, \eta) = -\frac{\frac{\eta}{1-\eta}H'(v) + D_2F(\alpha(v, \eta)|v)}{f(\alpha(v, \eta)|v)}. \tag{I.24}$$

Note that $f(.|.) > 0$ by the full support assumption and $D_2F(.|.) < 0$ by strict MLRP. Therefore, for $D_1\alpha(v, \eta) < 0$ we need the numerator of (I.24) to be negative, i.e.

$$\frac{\eta}{1-\eta}H'(v) + D_2F(\alpha(v, \eta)|v) > 0. \tag{I.25}$$

Thus, if we show that (I.25) implies

$$\frac{\partial}{\partial \eta}\left[\frac{\eta}{1-\eta}H'(v) + D_2F(\alpha(v, \eta)|v)\right] \geq 0,$$

which is equivalent to

$$\frac{H'(v)}{(1-\eta)^2} \geq -D_2f(\alpha(v, \eta)|v)D_2\alpha(v, \eta), \tag{I.26}$$

then we would have shown that if the numerator of (I.24) is negative, it becomes more negative as η grows. This will suffice to prove part (ii) of the lemma.

Given that $D_2\alpha(v,\eta) = \frac{1-\gamma-H(v)}{(1-\eta)^2 f(\alpha(v,\eta)|v)}$, (I.26) can be expressed as

$$H'(v) \geq -(1-\gamma-H(v))\frac{D_2 f(\alpha(v,\eta)|v)}{f(\alpha(v,\eta)|v)}. \tag{I.27}$$

Therefore, we need to prove that (I.25) implies (I.27). By the strict MLRP of f, $\frac{F(s|v)}{f(s|v)}$ is decreasing in v and $\frac{1-F(s|v)}{f(s|v)}$ is increasing in v for all s. Thus,

$$\frac{\partial}{\partial v}\left[\frac{F(s|v)}{f(s|v)}\right] = \frac{f(s|v)D_2F(s|v) - D_2 f(s|v)F(s|v)}{f^2(s|v)} \leq 0, \tag{I.28}$$

and

$$\frac{\partial}{\partial v}\left[\frac{1-F(s|v)}{f(s|v)}\right] = \frac{-f(s|v)D_2F(s|v) - D_2 f(s|v)(1-F(s|v))}{f^2(s|v)} \geq 0. \tag{I.29}$$

Consequently, (I.28) and (I.29) imply that $\frac{D_2 f(s|v)}{f(s|v)} \in \left[\frac{D_2F(s|v)}{F(s|v)}, \frac{-D_2F(s|v)}{1-F(s|v)}\right]$ for all $s \in (0,1)$ and all v.[50]

We need to consider two possible cases: $H(v) < 1-\gamma$ and $H(v) > 1-\gamma$.[51]

1. $H(v) < 1-\gamma$: if we divide both sides of (I.25) by $F(\alpha(v,\eta)|v)$,[52] we obtain

$$\frac{\eta}{1-\eta}\frac{H'(v)}{F(\alpha(v,\eta)|v)} > -\frac{D_2F(\alpha(v,\eta)|v)}{F(\alpha(v,\eta)|v)} \geq -\frac{D_2 f(\alpha(v,\eta)|v)}{f(\alpha(v,\eta)|v)}.$$

   Substituting $F(\alpha(v,\eta)|v) = \frac{1-\gamma-\eta H(v)}{1-\eta}$ in the above expression and multiplying both sides by $(1-\gamma-H(v)) > 0$ we get

$$H'(v)\frac{\eta(1-\gamma-H(v))}{1-\gamma-\eta H(v)} > -(1-\gamma-H(v))\frac{D_2 f(\alpha(v,\eta)|v)}{f(\alpha(v,\eta)|v)}. \tag{I.30}$$

[50]By the full support assumption $F(s|v) \in (0,1)$ for all $s \in (0,1)$ and the bounds on $\frac{D_2 f(s|v)}{f(s|v)}$ are well defined.

[51]If $H(v) = 1-\gamma$, (I.27) is satisfied given that $H'(v) > 0$ is needed for (I.25) to hold.

[52]We can do so since $F(\alpha(v,\eta)|v) > 0$ whenever $H(v) < 1-\gamma$. Assume $F(\alpha(v,\eta)|v) = 0$ otherwise. Then $1-\gamma = \eta H(v)$ and $H(v) > 1-\gamma$, a contradiction.

Since $H(v) < 1-\gamma$, $\gamma \in (0,1)$ and $\eta \in (0,1]$,[53] $\frac{\eta(1-\gamma-H(v))}{1-\gamma-\eta H(v)}$ is strictly positive and less than one. Hence, (I.30) implies (I.27) given that $H'(v) > 0$ by (I.25).

2. $H(v) > 1-\gamma$: two subcases need to be considered. If $D_2 f(\alpha(v,\eta)|v) \leq 0$ the right-hand side of (I.27) is non-positive. Thus, (I.27) is satisfied for all v such that $H'(v) > 0$ and all η. When $D_2 f(\alpha(v,\eta)|v) > 0$, by dividing both sides of (I.25) by $1 - F(\alpha(v,\eta)|v)$ we get[54]

$$\frac{\eta}{1-\eta}\frac{H'(v)}{1-F(\alpha(v,\eta)|v)} > -\frac{D_2 F(\alpha(v,\eta)|v)}{1-F(\alpha(v,\eta)|v)} \geq \frac{D_2 f(\alpha(v,\eta)|v)}{f(\alpha(v,\eta)|v)}.$$

Substituting $F(\alpha(v,\eta)|v)$ and rearranging terms, the above inequality becomes

$$\begin{aligned} \eta H'(v) &> (1-\eta+\eta H(v)-(1-\gamma))\frac{D_2 f(\alpha(v,\eta)|v)}{f(\alpha(v,\eta)|v)} \\ &\geq -(1-\gamma-H(v))\frac{D_2 f(\alpha(v,\eta)|v)}{f(\alpha(v,\eta)|v)}. \end{aligned} \tag{I.31}$$

The second inequality holds because $1-\eta+\eta H(v) \geq H(v)$ and, therefore, $(1-\eta+\eta H(v)-(1-\gamma)) \geq -(1-\gamma-H(v)) > 0$.[55] Since $\eta \in (0,1]$, (I.31) implies (I.27).

Part (iii): first, note that $\alpha(.,\eta)$ is well-defined in $[0,1]$ iff $\eta \leq \eta_\gamma := \min\{\gamma, 1-\gamma\}$, given that $\frac{1-\gamma-\eta H(v)}{1-\eta} \in [\frac{1-\gamma}{1-\eta}, \frac{1-\gamma-\eta}{1-\eta}]$ for all $v \in [0,1]$.

As $\eta \to 0$, $\frac{1-\gamma-\eta H(v)}{1-\eta} \to 1-\gamma$. Thus, $\lim_{\eta\to 0}\alpha(v,\eta) = F^{-1}(1-\gamma|v)\ \forall v$. By *Assumptions 1-2*, $F^{-1}(1-\gamma|.)$ is well-defined and strictly increasing in $[0,1]$. By the continuity of F and H, $\alpha(.,\eta)$ is continuous and strictly increasing in $[0,1]$ for

[53] If (I.25) holds then $\eta > 0$.

[54] Note that $F(\alpha(v,\eta)|v) < 1$ whenever $H(v) > 1-\gamma$. If $F(\alpha(v,\eta)|v) = 1$ then $1-\gamma-\eta H(v) = 1-\eta$, which can only hold if $H(v) < 1-\gamma$ given that $1-\eta+\eta H(v) = H(v)+(1-\eta)(1-H(v))$.

[55] To see why notice that $1-\eta+\eta H(v) = (1-\eta)(1-H(v))+H(v)$, which is at least $H(v)$ given that $\eta \in (0,1]$ and $H(v) \in (1-\gamma, 1]$.

all η sufficiently small. This takes care of $\underline{\eta}$ being strictly positive. We need to show that there exists $\underline{\eta} < \eta_\gamma$ such that for all $\eta < \underline{\eta}$, $\alpha(.,\eta)$ is everywhere increasing, and for all $\eta > \underline{\eta}$, there exists some $v \in [0,1]$ such that $\alpha(v,\eta)$ is well-defined with $D_1\alpha(v,\eta) < 0$.

Since $H(0) = 0$ and $H(1) = 1$, $H'(v) > 0$ in a set of asset values with positive Lebesgue measure, so for high enough η inequality (I.25) is satisfied for some v. Assume for the moment that such η is lower than η_γ. By part (ii) of the lemma, if (I.25) is satisfied for η and v it will also be satisfied for all $\eta' > \eta$. Accordingly, if $D_1\alpha(v,\eta) > 0$, then $D_1\alpha(v,\eta'') > 0$ for all $\eta'' < \eta$. Therefore, given that $D_1\alpha(v;.)$ is well-defined and continuous in $[0,\eta_\gamma]$ for all v, $\underline{\eta}$ exists and is given by the highest η such that $D_1\alpha(v,\eta) \geq 0$ for all v, i.e.

$$\underline{\eta} := \sup_{\eta}\left\{\eta \in (0,1) : \eta \leq -\frac{D_2F(F^{-1}(\frac{1-\gamma-\eta H(v)}{1-\eta}|v)|v)}{H'(v) - D_2F(F^{-1}(\frac{1-\gamma-\eta H(v)}{1-\eta}|v)|v)} \quad \forall v\right\}. \quad \text{(I.32)}$$

It remains to be shown that there is a $\eta < \eta_\gamma$ satisfying (I.25) for some v. Assume otherwise that $\underline{\eta}$, the highest η for which there is no v satisfying (I.25), is greater than η_γ.

1. If $\underline{\eta} \geq \gamma$, then $\frac{1-\gamma}{1-\underline{\eta}} > 1$. Since $\frac{1-\gamma-\underline{\eta}}{1-\underline{\eta}} < 1$ and $H(.)$ is continuous, there exists an interval of values $[\underline{v},\overline{v}]$ such that $\frac{1-\gamma-\underline{\eta}H(\underline{v})}{1-\underline{\eta}} = 1$ and $\frac{1-\gamma-\underline{\eta}H(v)}{1-\underline{\eta}}$ is strictly decreasing in v for all $v \in [\underline{v},\overline{v}]$. Hence, by the full support assumption, $\alpha(\underline{v},\underline{\eta}) = 1$ and $\alpha(v,\underline{\eta}) < 1$ for all $v \in (\underline{v},\overline{v}]$, which implies that $D_1\alpha(v,\underline{\eta}) < 0$ for some v, contradicting that (I.25) is not satisfied by $\underline{\eta}$.

2. If $\underline{\eta} \geq 1-\gamma$, we have that $\frac{1-\gamma-\underline{\eta}}{1-\underline{\eta}} < 0$. Since $\frac{1-\gamma}{1-\underline{\eta}} > 0$, there exists an interval of values $[\underline{v}',\overline{v}']$ such that $\frac{1-\gamma-\underline{\eta}H(\overline{v}')}{1-\underline{\eta}} = 0$ and $\frac{1-\gamma-\underline{\eta}H(v)}{1-\underline{\eta}}$ is strictly decreasing in v for all $v \in [\underline{v}',\overline{v}']$. Hence, by the full support assumption, $\alpha(\overline{v}',\underline{\eta}) = 0$ and $\alpha(v,\underline{\eta}) > 0$ for all $v \in [\underline{v}',\overline{v}')$, which implies that $D_1\alpha(v,\underline{\eta}) < 0$ for some v, a contradiction.

Part (iv): note that, as $\eta \to 1$, $\frac{1-\gamma-\eta H(v)}{1-\gamma} \to \infty$ for all v such that $H(v) < 1-\gamma$ and $\frac{1-\gamma-\eta H(v)}{1-\gamma} \to -\infty$ for all v such that $H(v) > 1-\gamma$. Therefore, for high enough η,

$\alpha(.,\eta)$ is only well-defined in a small neighborhood of all v such that $H(v) = 1-\gamma$. If for any such v we have that $H'(v) > 0$ then $\alpha(.,\eta)$ will be decreasing in such neighborhood.[56] By part (ii) of the lemma, if $\alpha(.,\eta)$ is decreasing, it is decreasing for all $\eta' > \eta$. ■

Fact 2 *Let* $\mathcal{S}$ *be a measurable subset of* $[0,1]$ *and* $s \in (0,1)$ *be such that* $\mathbb{P}(\mathcal{S}|v) = F(s|v)$ *for some* $v \in [0,1)$*. Then,* $D_2\mathbb{P}(\mathcal{S}|v) \geq D_2 F(s|v)$*.*

Proof of Fact 2. Assume $\mathcal{S} \cap [s,1]$ is a non-null set, otherwise $\mathbb{P}(\mathcal{S}|v) = F(s|v)$ for all v by the full support of $F(.|v)$ for all v. Since $\mathbb{P}(\mathcal{S}|v) = F(s|v) = \mathbb{P}([0,s]|v)$, we have that

$$\mathbb{P}([s,1] \cap \mathcal{S}|v) = \mathbb{P}([0,s] \setminus \mathcal{S}|v).$$

By the strict MLRP of $F(.|.)$, the left hand side is strictly greater than the right-hand side for all $v' > v$. Thus, $D_2\left[\mathbb{P}(\mathcal{S}|v) - \mathbb{P}([0,s]|v)\right] \geq 0$. ■

Lemma 6 *If* $\alpha(.,\eta)$ *is strictly decreasing in some interval* $[v_1, v_2]$ *then any monotone equilibrium price* $\rho(.,\eta)$ *satisfies* $\rho(v,\eta) \neq v$ *a.e. in* $[v_1, v_2]$*.*

Proof of Lemma 6. Assume $\rho(v,\eta) = v$ and $\rho(v',\eta) = v'$ for some $v' > v$ with $\alpha(v,\eta) > \alpha(v',\eta)$. Accordingly, if the mass of strategic bids below v is given by bidders with signals in $[0, \alpha(v,\eta)]$, then the mass of bids below $v' > v$ is strictly smaller than the mass of bids below v, a contradiction. Hence, it must be that there is an alternative, well-defined distribution of strategic bids $B^a(.|.)$ such that $B^a(v|v) = \frac{1-\gamma-\eta H(v)}{1-\eta}$ for all $v \in [v_1, v_2]$. Since $\alpha(.,\eta)$ is decreasing in that interval, we have that $\frac{d}{dv}B^a(v|v) = -\frac{\eta}{1-\eta}H'(v) < D_2 F(\alpha(v,\eta)|v)$ (see inequality (I.25)).

[56]In this case there is a unique v such that $H(v) = 1-\gamma$. Assume otherwise that there are two such values v, v' such that $H'(v), H'(v') > 0$. Since $H(0) = 0$ and $H(1) = 1$, by the continuity of $H(.)$, there would have to be a value $v'' \in$ such that $H(v'') = 1-\gamma$ and $H'(v'') < 0$.

Denoting $\beta^a(s,t)$ the bid of strategic trader t when she receives signal s, we have that

$$B^a(v|v) = \int_{\mathcal{T}}\int_0^1 1_{\{\beta^a(s,t)\leq v\}} f(s|v)dsd\mu = \int_{\mathcal{T}} \mathbb{P}(\mathcal{S}^a(v,t)|v)d\mu,$$

where $\mathcal{S}^a(v,t) = \{s \in [0,1] : \beta^a(s,t) < v\}$.

By *Fact 2*, $D_2\mathbb{P}(\mathcal{S}^a(v,t)|v) \geq D_2F(s^a(v,t)|v)$ with $s^a(v,t)$ being the signal such that $\mathbb{P}(\mathcal{S}^a(v,t)|v) = F(s^a(v,t)|v)$. Accordingly, given that $D_1B^a(v|v) \geq 0$,

$$\frac{d}{dv}B^a(v|v) = D_1B^a(v|v) + D_2B^a(v|v) \geq \int_{\mathcal{T}} D_2F(s^a(v,t)|v)d\mu.$$

Therefore, it is enough to show that $\int_{\mathcal{T}} D_2F(s^a(v,t)|v)d\mu \geq D_2F(\alpha(v,\eta)|v)$ whenever $\int_{\mathcal{T}} F(s^a(v,t)|v)d\mu = F(\alpha(v,\eta)|v)$ in order to prove that there is no $B^a(.|.)$ leading to revealing prices in $[v_1, v_2]$.

By the strict MLRP we have that, for all $v \in [v_1, v_2]$ and all $v' > v$,

$$\begin{aligned}
0 &= \int_{\mathcal{T}} F(s^a(v,t)|v)d\mu - F(\alpha(v,\eta)|v) = \\
&= \int_{\mathcal{T}}\int_{\alpha(v,\eta)}^{s^a(v,t)\vee\alpha(v,\eta)} f(x|v)dxd\mu - \int_{\mathcal{T}}\int_{\alpha(v,\eta)\wedge s^a(v,t)}^{\alpha(v,\eta)} f(x|v)dxd\mu \\
&\leq \int_{\mathcal{T}}\int_{\alpha(v,\eta)}^{s^a(v,t)\vee\alpha(v,\eta)} f(x|v)\frac{f(x|v')}{f(x|v)}dxd\mu - \int_{\mathcal{T}}\int_{\alpha(v,\eta)\wedge s^a(v,t)}^{\alpha(v,\eta)} f(x|v)\frac{f(x|v')}{f(x|v)}dxd\mu \\
&= \int_{\mathcal{T}} F(s^a(v,t)|v')d\mu - F(\alpha(v,\eta)|v').
\end{aligned}$$

Therefore, $\int_{\mathcal{T}} D_2F(s^a(v,t)|v)d\mu \geq D_2F(\alpha(v,\eta)|v)$, which implies $\frac{d}{dv}B^a(v|v) \geq D_2F(\alpha(v,\eta)|v)$, a contradiction. ■

Proof of Proposition 2.

The proof is divided into two cases, depending on the value of η. For $\eta \in [0, \underline{\eta}]$, where $\underline{\eta} > 0$ is given by (I.32), I show that prices are necessarily fully revealing; whereas when $\eta > \underline{\eta}$ prices cannot be fully revealing. In the latter case, I provide an algorithm to find monotone equilibrium prices satisfying *Proposition 1* and *Corollary 1* and show that they exist and are unique. After that, I show that, except for a very particular class of naïve distributions, there exists $\overline{\eta} < 1$ such that for all $\eta \geq \overline{\eta}$ strategic bids are confined outside the range of equilibrium prices, implying that $\mathcal{V} = [0, 1]$.

Before turning into these cases, a prerequisite for existence is that any equilibrium prices satisfying *Proposition 1* and *Corollary 1* are in fact increasing. This is guaranteed if any block-monotonic distribution of strategic bids leads to market clearing prices that are increasing. According to *Proposition 1*, market prices in non-revealing intervals are given by

$$H(p|v) = \frac{1 - \gamma - (1 - \eta)F(s_k^*|v)}{\eta}. \tag{I.33}$$

Given any $\eta \in (0, 1)$, the right-hand side of this expression is constant for $s_k^* \in \{0, 1\}$ and strictly increasing in v for $s_k^* \in (0, 1)$. Hence, when $H(.|.)$ satisfies *Assumption 4*, the resulting price is increasing in v.

Now I turn into the two cases to be considered, $\eta \in [0, \underline{\eta}]$ and $\eta \in (\underline{\eta}, 1]$.

Case 1: $(\eta \in [0, \underline{\eta}])$. The function $\alpha(v, \eta)$ given by (I.12) is the quantile of the signal distribution leading to strategic bids at or below v such that $\rho(v, \eta) = v$. By *Lemma 5*, $\alpha(., \eta)$ is increasing for all $\eta < \underline{\eta}$. This implies that there exists a fully revealing equilibrium for all $\eta < \underline{\eta}$, with the distribution of strategic bids satisfying $B^*(v|v) = F(\alpha(v, \eta)|v)$. Since prices are fully revealing and agents cannot affect the price, no strategic trader has an incentive to deviate and, hence, any profile of bidding strategies yielding B^* constitutes a BNE. One such profile is given by $\beta(s, t) = \beta(s)$ for all t, with

$$\beta(s) = \begin{cases} 0 & \text{if } s \in [0, \alpha(0, \eta)] \\ v \text{ s.t. } \alpha(v, \eta) = s & \text{if } s \in (\alpha(0, \eta), \alpha(1, \eta)) \\ 1 & \text{if } s \in [\alpha(1, \eta), 1] \end{cases} \tag{I.34}$$

This takes care of existence of monotone equilibrium for $\eta \in [0, \underline{\eta}]$.

Regarding uniqueness of monotone equilibrium prices, assume there exists a monotone equilibrium with $\rho(v,\eta) \neq v$ a.e. in $(\underline{v}_1, \overline{v}_1)$ with $\underline{v}_1 < \overline{v}_1$ for some $\eta \leq \underline{\eta}$. If $\rho(\underline{v}_1,\eta) < \rho(\overline{v}_1,\eta)$, by *Lemma 3*, the mass of strategic bids placed in $[\rho(\underline{v}_1,\eta), \rho(\overline{v}_1,\eta)]$ is zero. Since the distribution of strategic bids is block-monotonic (*Proposition 1*), all strategic traders with signals below (above) some signal s_1^* bid below $\rho(\underline{v}_1,\eta)$ (above $\rho(\overline{v}_1,\eta)$). However, given that $\alpha(.,\eta)$ is increasing, we have that $s_1^* > \alpha(\underline{v}_1,\eta)$ and/or $s_1^* < \alpha(\overline{v}_1,\eta)$.[57] When $s_1^* > \alpha(\underline{v}_1,\eta)$ then $\rho(\underline{v}_1,\eta) < \underline{v}_1$ if $\underline{v}_1 > 0$ or $\rho(v,\eta) = 0$ in $[0,v')$ for some $v' > 0$ if $\underline{v}_1 = 0$, contradicting *Corollary 1*. On the other hand, if $s_1^* < \alpha(\overline{v}_1,\eta)$ then $\rho(\overline{v}_1,\eta) > \overline{v}_1$ if $\overline{v}_1 < 1$ or $\rho(v,\eta) = 1$ in $(v'',1]$ for some $v'' < 1$ if $\overline{v}_1 = 1$, which again violates *Corollary 1*. Therefore, the only possibility left is that $\rho(v_1,\eta) = \rho(v_2,\eta)$, i.e. there exist an atom in the distribution of prices. But, according to *Lemma 4*, this can only happens when $\eta \geq \min\{\gamma, 1-\gamma\}$, i.e. when $\eta > \underline{\eta}$.

Hence, when $\eta \in [0,\underline{\eta}]$ any monotone equilibrium price satisfies $\rho(v,\eta) = v$.

<u>Case 2</u>: ($\eta \in (\underline{\eta}, 1]$). By part (ii) of *Lemma 5* $\alpha(.,\eta)$ is either non-monotonic or decreasing. Hence, prices cannot be fully revealing, given *Lemma 6*. The following algorithm identifies the values $\{\underline{v}_k\}_{k=1}^K$, $\{\overline{v}_k\}_{k=1}^K$ and signals $\{s_k^*\}_{k=1}^K$ that satisfy the conditions of *Proposition 1* and *Corollary 1*, which characterize equilibrium prices. Then I show that these values and signals always exist and are unique. Finally, I provide bidding strategies that implement equilibrium prices.

The steps of the algorithm are:

1. Find asset values $\{v_i^m\}_{i=1}^I$ and $\{v_i^M\}_{i=1}^{I'}$ at which $\alpha(.,\eta)$ reaches a local minimum and a local maximum, respectively. If $\alpha(.,\eta)$ is not well-defined in an interval (v',v'') with $\alpha(v',\eta)$ or $\alpha(v'',\eta) \in \{0,1\}$, let v' be the "unique" local maximum in that interval when $\alpha(v'',\eta) = 1$ and v'' be the "unique" local

[57] Note that $\alpha(.,\eta)$ is strictly increasing for $\eta < \underline{\eta}$. If $\eta = \underline{\eta}$ and $\alpha(.,\underline{\eta})$ is constant in $[\underline{v}_1, \overline{v}_1]$ then $s_1^* = \alpha(\underline{v}_1, \underline{\eta})$ would involve $\rho(v,\underline{\eta}) = v$ in $[\underline{v}_1, \overline{v}_1]$. Hence, one of these inequalities still needs to hold for $\rho(v,\underline{\eta}) \neq v$ a.e. in $[\underline{v}_1, \overline{v}_1]$.

minimum when $\alpha(v', \eta) = 0$.[58] Let $v_0^m = 0$ and $v_{I'+1}^M = 1$.[59]

2. For each interval $\{[v_{i-j}^m, v_{i+1}^M]\}_{i=1}^{I-1+j}$, with $j = 0$ if $v_1^m = 0$ and $j = 1$ if $v_1^M = 0$, find signal values $\{s_i\}_{i=1}^{I-1+j}$ such that, when $\rho(v, \eta)$ satisfies $1 - \gamma = \eta H(\rho(v, \eta)|v) + (1 - \eta)F(s_i|v)$, are given by

$$s_i = \begin{cases} 0 & \textit{if } \mathbb{E}((V - \rho(V, \eta))1_{\{V \in [v_{i-j}^m, v_{i-j+1}^m]\}}|0) > 0, \\ 1 & \textit{if } \mathbb{E}((V - \rho(V, \eta))1_{\{V \in [v_i^M, v_{i+1}^M]\}}|1) < 0, \\ s & \textit{if } \mathbb{E}((V - \rho(V, \eta))1_{\{V \in [\underline{\nu}_i(s), \overline{\nu}_i(s)]\}}|s) = 0, \end{cases} \tag{I.35}$$

where $\underline{\nu}_i(s), \overline{\nu}_i(s)$ are respectively given by

$$\underline{\nu}_i(s) = \begin{cases} v_{i-j}^m & \textit{if } \alpha(v_{i-j}^m, \eta) > s, \\ v \in [v_{i-j}^m, v_i^M] \textit{ s.t. } \alpha(v, \eta) = s & \textit{otherwise}, \end{cases} \tag{I.36}$$

and

$$\overline{\nu}_i(s) = \begin{cases} v_{i+1}^M & \textit{if } \alpha(v_{i+1}^M, \eta) < s, \\ v \in [v_{i-j+1}^m, v_{i+1}^M] \textit{ s.t. } \alpha(v, \eta) = s & \textit{otherwise.} \end{cases} \tag{I.37}$$

3. If $s_i > s_{i+1}$ merge intervals $[v_{i-j}^m, v_{i+1}^M]$ and $[v_{i+1-j}^m, v_{i+2}^M]$ and redefine $s_i = s_i'$ and $\overline{\nu}_i(s_i') = \overline{\nu}_{i+1}(s_i')$, with s_i' given by

$$s_i' = \begin{cases} 0 & \textit{if } \mathbb{E}((V - \rho(V, \eta))1_{\{V \in [v_{i-j}^m, v_{i-j+2}^m]\}}|0) > 0, \\ 1 & \textit{if } \mathbb{E}((V - \rho(V, \eta))1_{\{V \in [v_i^M, v_{i+2}^M]\}}|1) < 0, \\ s & \textit{if } \mathbb{E}((V - \rho(V, \eta))1_{\{V \in [\underline{\nu}_i(s), \overline{\nu}_{i+1}(s)]\}}|s) = 0. \end{cases} \tag{I.38}$$

   Repeat this step until $s_i \leq s_{i+1}$ for $i = 1, ..., K$, with K being the new number of intervals.

[58] Note that when $\alpha(v'', \eta) = 1$ either $\alpha(v', \eta) = 1$ or it is not well-defined. Similarly, when $\alpha(v', \eta) = 0$, when $\alpha(v'', \eta) = 0$ or it is not well-defined.

[59] By the continuity of $\alpha(., \eta)$, $v_i^m < v_i^M$ for all i if $\alpha(0, \eta)$ is a local minimum, and $v_i^M < v_i^m$ for all i if $\alpha(0, \eta)$ is a local maximum.

4. Define $s_k^* = s_k$, $\underline{v}_k = \underline{\nu}_k(s_k)$ and $\overline{v}_k = \overline{\nu}_k(s_k)$, $k = 1, ..., K$.

Several things are worth noting. First, each interval $[v_{i-j}^m, v_{i+1}^M]$ contains v_{i-j+1}^m and v_i^M. Thus, $\alpha(., \eta)$ is increasing in $(v_{i-j}^m, v_i^M) \cup (v_{i-j+1}^m, v_{i+1}^M)$ and decreasing in (v_i^M, v_{i-j+1}^m).[60] This implies $s_i \in \left[\alpha(v_{i-j+1}^m, \eta), \max\{\alpha(v_i^M, \eta), \alpha(v_{i+1}^M, \eta)\}\right]$. Assume otherwise that $s_i < \alpha(v_{i-j+1}^m, \eta) < 1$. Then $\rho(v, \eta) > v$ in $[\underline{\nu}_i(s_i), \overline{\nu}_i(s_i)]$ since $\alpha(v, \eta)$ is above s_i in $[\underline{\nu}_i(s_i), v_{i+1}^M(s_i)]$. But this in turn leads to $\mathbb{E}((V - \rho(V, \eta))1_{\{V \in [\underline{\nu}_i(s_i), \overline{\nu}_i(s_i)]\}}|s_i) < 0$ when $s_i < 1$, which violates (I.35). Given these bounds on s_i, there is a unique value $\nu_i'(s_i) \in (v_i^M, v_{i-j+1}^m)$ such that $\alpha(\nu_i'(s_i), \eta) = s_i$. Accordingly, $\rho(v, \eta) > v$ in $(\underline{\nu}_i(s_i), \nu_i'(s_i))$ and $\rho(v, \eta) < v$ in $(\nu_i'(s_i), \overline{\nu}_i(s_i))$.[61]

Second, $\underline{\nu}_i(.)$ and $\overline{\nu}_i(.)$ are increasing, while $v_i'(.)$ is decreasing. By the continuity assumptions and *Fact 1*, each tuple $(s_i, \underline{\nu}_i(s_i), \overline{\nu}_i(s_i))$ exists and it is unique. To see why, note that as s_i grows the interval where prices are above values $(\underline{\nu}_i(s_i), \nu_i'(s_i))$ shrinks while $(\nu_i'(s_i), \overline{\nu}_i(s_i))$ grows. Furthermore, as s_i grows the probability mass (conditional on s_i) associated to $(\nu_i'(s_i), \overline{\nu}_i(s_i))$ grows relative to the mass associated to $(\underline{\nu}_i(s_i), \nu_i'(s_i))$, by the MLRP of $F(.|s_i)$. Therefore, there is a unique signal s_i (which in turn uniquely determines $\underline{\nu}_i(s_i)$ and $\overline{\nu}_i(s_i)$) satisfying (I.35).

Third, when two adjacent intervals with signals s_i, s_{i+1} are merged (step 3 of the algorithm), the new pivotal signal s_i' lies in (s_{i+1}, s_i). Thus, any subinterval of $[\underline{\nu}_i(s_i'), \overline{\nu}_{i+1}(s_i')]$ with $\rho(v, \eta) < v$ is preceded by a subinterval with $\rho(v, \eta) > v$, which means that we can apply the same existence and uniqueness argument to the tuple $(s_i', \underline{\nu}_i(s_i'), \overline{\nu}_{i+1}(s_i'))$.

Finally, $\alpha(., \eta)$ is increasing on $[0, \underline{\nu}_1(s_1)]$, $[\overline{\nu}_i(s_i), \underline{\nu}_{i+1}(s_i)]$ and also on $[\underline{\nu}_K(s_K), 1]$. That is, it is increasing in $[0, 1] \setminus \bigcup_k [\underline{v}_k, \overline{v}_k]$, which enables prices to be

[60] Note that for $i = 0$, $v_{i-j}^m = v_i^M$ when $v_1^M = 0$, and $v_{i-j+1}^m = v_{i+1}^M$ for $i = 1$ when $v_K^m = 1$.

[61] This is also true when $s_i \in \{0, 1\}$. Given (I.36)-(I.37), $s_i = 0$ implies that $\underline{\nu}_i = v_{i-j}^m$ and $\frac{1-\gamma-\eta H(v)}{1-\eta} < 0$ in some interval (v', v_i^m) (otherwise (I.35) would be violated), which leads to $\overline{\nu}_i = v_i^m$ (according to step 1 of the algorithm, v_i^m is the upper bound of the interval of values where $\alpha(., \eta)$ is not well-defined). The latter implies that $\rho(v, \eta) < v$ in $(v', \overline{\nu}_i)$. Since $\alpha(., \eta)$ is either increasing in (v_{i-j}^m, v_i^M) or above 0 when $v_{i-j}^m = 0$ (part (i) of *Lemma 5*), $\alpha(v, \eta) > 0$ (and thus $\rho(v, \eta) > v$) in $(\underline{\nu}_i, v')$. Similarly, $s_i = 1$ implies that $\underline{\nu}_i = v_i^M$ and $\frac{1-\gamma-\eta H(v)}{1-\eta} > 1$ in some interval (v_i^M, v'), which means that $\overline{\nu}_i = v_{i+1}^M$. Hence, $\rho(v, \eta) > v$ in (v_i^M, v') and $\rho(v, \eta) < v$ in (v', v_{i+1}^M).

fully revealing in such set (*Lemma 6*).

Given all these facts, (I.35)-(I.38) imply that $\{(s_k^*, \underline{v}_k, \overline{v}_k)\}$ satisfy (I.9)-(I.11). Moreover, prices given by (I.8) are monotonic and satisfy *Corollary 1*.

Since this algorithm provides a unique solution, we need to show that a collection $\{(s'_h, \underline{v}'_h, \overline{v}'_h)\}$ not satisfying (I.35)-(I.38) violates (I.9)-(I.11) or *Corollary 1*.

Assume that there is a collection $\{(s'_h, \underline{v}'_h, \overline{v}'_h)\}$ satisfying *Proposition 1*. If $s'_h \in (0,1)$ then $\mathbb{E}((V - \rho(V,\eta))1_{\{V\in[\underline{v}'_h,\overline{v}'_h\}}]|s'_h) = 0$ by (I.11). In addition, (I.9)-(I.10) and *Corollary 1* require that $\alpha(\underline{v}'_h, \eta) \geq s'_h$ with equality when $\underline{v}'_h \in (0,1)$ and $\alpha(\overline{v}'_h, \eta) \leq s'_h$ with equality when $\underline{v}'_h \in (0,1)$. *Corollary 1* further requires $\alpha(v,\eta)$ to be increasing at $v = \underline{v}'_h, \overline{v}'_h$ whenever $\alpha(v,\eta) = s'_h$. All these conditions imply that $\underline{v}_h \in [v^m_{i-j}, v^M_i]$ and $\overline{v}_h \in [v^m_{l-j}, v^M_l]$ for some i, l with $i < l$. But then, if $i = l+1$, $(s'_h, \underline{v}'_h, \overline{v}'_h) = (s_i, \underline{\nu}_i, \overline{\nu}_i)$ given (I.35)-(I.38). On the other hand, if $i < l+1$ let s_k, $k = i, ..., l$, be the signals given by (I.35). If $s_k < s_{k+1}$ for all k then $s'_h \in (s_i, s_l)$ for $\mathbb{E}((V - \rho(V,\eta))1_{\{V\in[\underline{v}'_h,\overline{v}'_h\}}]|s'_h) = 0$ to hold. But this implies that $\mathbb{E}((V - \rho(V,\eta))1_{\{V\in[\underline{v}'_h,\overline{\nu}_i(s'_h)\}}]|s'_h) > 0$, which by *Fact 1* means that $\mathbb{E}((V - \rho(V,\eta))1_{\{V\in[\overline{\nu}_i(s'_h),\overline{v}'_h\}}]|s'_h) < 0$. Thus, a strategic trader receiving s'_h would rather bid $\overline{\nu}_i(s'_h)$ than bid below $\underline{v}'_h$, contradicting that $\{(s'_h, \underline{v}'_h, \overline{v}'_h)\}$ correspond to equilibrium prices. Assume then that there exists some $i \leq h \leq l$ such that $s_h \geq s_{h+1}$. In such case, abusing notation, let $\{s_{i'}\}$ denote the new collection of signals given by (I.38) after merging intervals $[v^m_{h-j}, v^M_{h+1}]$ and $[v^m_{h+1-j}, v^M_{h+2}]$. If $s_{i'} < s_{i'+1}$ for some i' in the new collection of signals, we again have that $\mathbb{E}((V - \rho(V,\eta))1_{\{V\in[\overline{\nu}_{i'}(s'_h),\overline{v}'_h\}}]|s'_h) < 0$, which leads to a profitable deviation by a trader receiving signal s'_h. By using this argument iteratively, we arrive at the conclusion that applying the algorithm to the subcollection of intervals $[v^m_{i-j}, v^M_{i+1}]$ that are included in $[\underline{v}'_h, \overline{v}'_h]$ we obtain a unique signal s_i such that $\mathbb{E}((V - \rho(V,\eta))1_{\{V\in[\underline{\nu}_i(s_i),\overline{\nu}_{i+1}(s_i)]\}}|s_i) = 0$, with $\underline{\nu}_i(s_i) \in [v^m_{i-j}, v^M_i]$ and $\overline{\nu}_i(s_i) \in [v^m_{l-j}, v^M_l]$. But then, as shown above, $(s_i, \underline{\nu}_i(s_i), \overline{\nu}_i(s_i))$ is the unique tuple satisfying (I.36)-(I.38), which are equivalent to (I.9)-(I.11), and that is compatible with equilibrium behavior by strategic traders. Hence, $\{(s'_h, \underline{v}'_h, \overline{v}'_h)\}$ cannot part of a characterization of equilibrium prices if $(s'_h, \underline{v}'_h, \overline{v}'_h) \neq (s_i, \underline{\nu}_i(s_i), \overline{\nu}_i(s_i))$.

If $s'_h = 0$ then $\underline{v}'_h = 0$ by part (i) of *Lemma 5*. We also have that $\mathbb{E}((V - \rho(V,\eta))1_{\{V \in [0,\overline{v}'_h\}}]|0) \geq 0$ by (I.11). In addition, *Corollary 1* requires that $\rho(v,\eta) < v$ in the upper part of $[0, \overline{v}'_h)$, which means that $\overline{v}'_h \in [v^m_{i-j}, v^m_{i-j+1}]$ for some $i = 1, ..., I-1+j$. But this can only happen if $\frac{1-\gamma-\eta H(v)}{1-\eta} < 0$ in some interval (v', v^m_{i-j+1}). Thus, $\overline{v}'_h = v^m_{i-j+1}$, otherwise $\overline{v}'_1$ would not satisfy (I.10). We need to consider two cases. If $i = 1$ we have that the unique triplet satisfying these conditions is $(s_1, \underline{v}_1, \overline{v}_1)$ as defined by the above algorithm. If $i > 1$ and $s_l < s_{l+1}$ for all $l < i$, then $\mathbb{E}((V - \rho(V,\eta))1_{\{V \in [\underline{\nu}_{l+1}(s_{l+1}), \overline{\nu}_{l+1}(s_{l+1})]\}}|s_{l+1}) = 0$ with $\underline{\nu}_{l+1}(s_{l+1}) < v^m_{i-j+1}$ (otherwise $\rho(v,\eta) < v$ a.e. in $[\underline{\nu}_{l+1}(s_{l+1}), \overline{\nu}_{l+1}(s_{l+1})]$) and a trader receiving a signal in (s_l, s_{l+1}) would rather deviate and bid $\underline{\nu}_i(s_{l+1})$. Therefore, $s_l > s_{l+1} = 0$ for some $l < i$. Using iterative merging we arrive at the conclusion that either $(s'_h, \underline{v}'_h, \overline{v}'_h) = (s^*_1, \underline{v}_1, \overline{v}_1)$ or that $(s'_h, \underline{v}'_h, \overline{v}'_h)$ violates *Proposition 1*.

Finally, when $s'_h = 1$ we have that $\overline{v}'_h = 1$ by part (i) of *Lemma 5* and $\mathbb{E}((V - \rho(V,\eta))1_{\{V \in [\underline{v}'_h\}}]|1) \leq 0$ by (I.11). The latter implies that $\frac{1-\gamma-\eta H(v)}{1-\eta} > 1$ in an interval $(v^M_{i-j'}, v')$ for some $i = 1, ..., I'$ with $j' = 0$ if $v^M_{I'} < 1$ and $j = 1$ otherwise, whereas *Corollary 1* requires that $\rho(v,\eta) > v$ in the lower part of $(\underline{v}'_h, 1]$. Hence, $\underline{v}'_h = v^M_i$ by (I.9). When $i = I'$, $(s_K, \underline{v}_K, \overline{v}_K)$ is the only triplet satisfying the above conditions. If $i < I'$ it has to be that $s_l > s_{l+1} = 0$ for some $l \geq i$, otherwise a trader receiving a signal in (s_l, s_{l+1}) would rather deviate and bid $\overline{\nu}_{l+1}(s_{l+1}) > v^M_i$, given *Fact 1* and that $\mathbb{E}((V - \rho(V,\eta))1_{\{V \in [\underline{\nu}_{l+1}(s_l), \overline{\nu}_l(s_l)]\}}|s_l) = 0$. By the usual merging argument it has to be that either $(s'_h, \underline{v}'_h, \overline{v}'_h) = (s^*_K, \underline{v}_K, \overline{v}_K)$ or that it violates *Proposition 1*.

This completes the proof that a collection $\{(s^*_k, \underline{v}_k, \overline{v}_k)\}$ satisfying (I.9)-(I.11) exists and it is unique. We just need to provide an example of bidding strategies yielding such equilibrium prices. The following symmetric strategies implement equilibrium prices characterized by $\{(s_k^*, \underline{v}_k, \overline{v}_k)\}$ and (I.8):

$$\beta(s) = \begin{cases} 0 & \text{if } s \in [0, s_1^*] \\ v \in [\overline{v}_k, \underline{v}_{k+1}] \text{ s.t. } \alpha(v,\eta) = s & \text{if } s \in (s_k^*, s_{k+1}^*] \\ 1 & \text{if } s \in (s_K^*, 1] \end{cases} \tag{I.39}$$

To complete the proof of *Proposition 2*, we need to show that there exists $\overline{\eta}$ such that $\mathcal{V} = [0,1]$ for all $\eta \geq \overline{\eta}$. By *Lemma 6*, revealing prices can only exist for values such that $\alpha(.,\eta)$ is increasing. In addition, by *Lemma 5*, once $\alpha(.,\eta)$ is decreasing at v it is decreasing for all $\eta' > \eta$. Therefore, if there exists a share $\overline{\eta}$ such that $\alpha(.,\eta)$ is either decreasing or not well-defined, it will also be so for all $\eta > \overline{\eta}$. In this context, the mass of bids at $[0, \rho(0,\eta)]$ (resp. $[\rho(1,\eta), 1]$) is given by the mass of signals $s \leq s_1^*$ ($s > s_1^*$), where

$$s_1^* = \begin{cases} 0 & \textit{if } \mathbb{E}(V - \rho(V,\eta)|s) > 0 \ \ \forall s, \\ 1 & \textit{if } \mathbb{E}(V - \rho(V,\eta)|s) < 0 \ \ \forall s, \\ s \ \textit{s.t.}\ \mathbb{E}(V - \rho(V,\eta)|s) = 0 & \textit{otherwise,} \end{cases} \tag{I.40}$$

with $\rho(v,\eta)$ satisfying $1-\gamma = \eta H(\rho(v,\eta)|v) + (1-\eta)F(s_i|v)$. The signal s_1^* exists and it is unique as shown above.

Note that, by *Lemma 5*, if for any v such that $H(v) = 1-\gamma$ we have that $H'(v) > 0$, then there exists $\overline{\eta} < 1$ such that $\alpha(.,\eta)$ is decreasing for all $\eta > \underline{\eta}$, leading to nowhere revealing prices (*Lemma 6*).

If, however, $H'(v) < 0$ for at least a value v satisfying $H(v) = 1-\gamma$, a revealing region may exist around v for all $\eta < 1$. To see why, note that for any such value there are two values v' and v'' with $v' < v < v''$ such that $H(v') = H(v'') = 1-\gamma$ and $H(v'), H(v'') > 0$. Accordingly, for η close to one, $\alpha(.,\eta)$ is decreasing in a neighborhood of v' and v'' and increasing in a neighborhood of v, and by its continuity, its range in these neighborhoods is the whole unit interval. Thus there are at least two intervals $\{[v_{i-j}^m, v_{i+1}^M]\}_{i=1}^{I-1+j}$, $i = 1, 2$ as defined in the above algorithm. If the two signals satisfying (I.35) for each interval are such that $s_1 < s_2$, there exists a revealing region in the interval $[\overline{\nu}(s_1), \underline{\nu}(s_2)]$ with $\underline{\nu}(s_2)$, $\overline{\nu}(s_1)$ given by (I.36) and (I.37), respectively.

Hence, if $H'(v) > 0$ for any value v such that $H(v) = 1 - \gamma$, then $\overline{\eta} < 1$ whereas $\overline{\eta}$ might be equal to one otherwise. ■

II

On the Possibility of Trade with Pure Common Values under Risk Neutrality

Abstract

This paper investigates the existence of bargaining mechanisms that induce trade with positive probability when agents are risk neutral, which constitutes a polar case not covered by existing no trade results. It is shown that a *quasi* no-trade theorem holds in the bilateral case: if the distributions of traders' private signals are continuous, no equilibrium with positive probability of trade exists in any trade environment with pure common values. With discrete distributions trade only occurs when the seller and the buyer receive their lowest and highest signals, respectively. A counterexample in which trade happens with probability one is provided to show that the result fails to hold when there are more than two traders. A property of multilateral mechanisms eliciting trade is that buyers' payments cannot equal expected conditional values almost everywhere. This implies that trade is incompatible with information aggregation in common value environments.

JEL Classification: *C72, C78, D82.*

Keywords: *no trade, common values, private information, bargaining mechanisms.*

II.A Introduction

Existing no-trade theorems assert that, under strict risk aversion, trade cannot happen with positive probability in pure common value environments when there is uncertainty about the value of the goods traded (Tirole (1982), Milgrom and Stokey (1982), Morris (1994)). However, it is an open question whether risk neutral agents may agree to trade. Under risk neutrality, these no-trade theorems only state that, if the initial allocation is Pareto optimal (i.e. any allocation in a common value setting) and agents receive private information about the value of the goods traded, there is no trade that strictly improves such initial allocation. Hence, they do not rule out the existence of weakly individually rational (IR) and incentive compatible (IC) bargaining mechanisms leading to trade with positive probability. In any such mechanism, agents would be indifferent between the initial and the final allocation, but they could nonetheless decide to trade.

I address this often overlooked indeterminacy by investigating whether trade can happen when (i) risk-neutral traders have common priors about the unknown value of the object to be traded and (ii) some of them receive private signals about such value.[1] I show that, if there is one buyer and one seller, there is no bargaining mechanism in which trade can happen with positive probability when the signal distributions are continuous. When the signal space is finite, trade can only occur when the seller receives her lowest possible signal and the buyer gets her highest signal.

This result does not extend to the multilateral case with more than two traders, as long as at least two of them receive private information. I provide a counterexample with two buyers and one seller in which trade happens with probability one. The key feature of multilateral mechanisms eliciting trade that is absent in any bilateral environment is the possibility to condition the transfer

[1] No trade results also apply to the case of heterogenous priors as long as players have concordant beliefs in the sense of Milgrom and Stokey (1982) and the initial allocation is Pareto optimal. By restricting attention to the common prior case, I look at a class of trade environments in which all the initial allocations are optimal.

between a buyer and a seller on the signals received by other agents. That is, even in the extreme case in which there are no gains from trade, adverse selection can be mitigated by using private information of agents not directly involved in a given transaction.[2]

An important restriction on multilateral mechanisms eliciting trade is that buyers' payments and expected values (conditional on the vector of traders' signals) cannot be equal almost everywhere.[3] This means that no multilateral bargaining mechanism is *ex post* incentive compatible, i.e. in some transactions one party will regret trading.[4] In addition, it implies that, when we do observe trade, there is a strong violation of the information aggregation properties of prices. Consequently, the *efficient markets hypothesis* (Fama (1970)) for risk neutral traders, which states that prices equal expected asset values conditional on all the available information, either violates incentive compatibility or requires zero trade volume.

II.B A Pure Common Value Environment with Private Information

There are n risk neutral sellers, each of them owning one unit of an indivisible object, and m risk neutral buyers, who can buy at most one unit. The unknown value of the object is given by V, with probability distribution G. The support of G is denoted by $\mathcal{V}$. Each individual i receives a private signal stochastically related to V, $S_i \sim F_i(.|v)$.[5] Let $\mathcal{S}_i$ be the support of $F_i(.|v)$.

I make the following assumptions.

Assumption 1 $\mathcal{V} \subset \mathbb{R}_+$ *is compact and has at least two elements.* $\mathcal{S}_i$ *are compact*

[2] I thank Joel Sobel for pointing out this fact by suggesting the use of a third party's signals to induce trade between a buyer and a seller.

[3] An exception to this rule exists when trade only takes place among lowest signal sellers and highest signal buyers.

[4] For instance, in the example provided, when all traders receive the highest possible signal, buyers pay the seller more than the conditional expected value of the object.

[5] In what follows, I use uppercase letters to denote random variables (V, S_i) or cumulative distribution functions(G, F) and lowercase to denote realizations of random variables (v, s_i) or probabilities and densities (g, f).

for $i = 1, ..., n+m$, and there exists at least an agent j for whom $\mathcal{S}_j$ has more than one element.

Compactness is not essential, while the minimum number of elements in the signal supports is necessary for the existence of at least two agents with distinct posterior probabilities about V. Notice that I allow for asymmetries in the quality of information by letting the signal distributions to differ across agents.

Assumption 2 *G and F_i are common knowledge, $i = 1, ..., n+m$.*

Assumption 3 *G has full support. In addition, for any $v \in \mathcal{V}$, the conditional distribution of $S = (S_1, S_2, ..., S_{n+m})$, denoted $F(.|v)$, has full support.*

This implies that the conditional distribution of S_i with respect to any vector of the other traders' signals $s_{\text{-}i} \in \mathcal{S}_{\text{-}i}$ also has full support. The next assumption establishes the stochastic relationship between values and signals. It roughly states that higher signals are more likely when the value is high and viceversa.

Assumption 4 (MLRP) *For all i, $F_i(.|v)$ satisfies the strict monotone likelihood ratio property: $\frac{f_i(s_i|v)}{f_i(s_i'|v)} > \frac{f_i(s_i|v')}{f_i(s_i'|v')}$ for all $s_i, s_i' \in \mathcal{S}_i$ such that $s_i > s_i'$ and all $v, v' \in \mathcal{V}$ such that $v > v'$.*

Overall, the above assumptions lead to the strict monotonicity of the expected value of the object conditional on agents' signals:

$$\mathbb{E}(v|s) > \mathbb{E}(v|s'), \tag{II.1}$$

for all $s, s' \in \mathcal{S} = \prod_i \mathcal{S}_i$ such that $s > s'$.

Assumption 5 (Common values) *Signals are payoff irrelevant, i.e. agents' utility is only a function of V.*

II.C Trade Mechanisms

Direct bargaining mechanisms specify, for every signal profile $s \in \mathcal{S}$ reported (truthfully) by the agents, both a payment vector and a vector of probabilities of trading the object.[6] The sum of payments is zero (balanced budget) and the sum of sellers' probabilities is equal to the sum of buyers' probabilities. I denote the payment function $x\colon \mathcal{S} \to \mathbb{R}^{n+m}$, with $\sum_i x_i(s) = 0$ for all $s \in \mathcal{S}$, and the vector of trade probabilities $q\colon \mathcal{S} \to [0,1]^{n+m}$, which satisfies $\sum_{i=1}^{n} q_i(s) = \sum_{i=n+1}^{n+m} q_i(s) \leq n$ for all $s \in \mathcal{S}$. Abusing notation, I use $v(s)$ to refer to $\mathbb{E}(V|s)$. Given the above assumptions, the expected (interim) payoffs in mechanism (q, x) for sellers and buyers are, respectively,[7]

$$\begin{aligned} \pi_i(s_i) &:= \mathbb{E}_{s_{\text{-}i}}(-x_i(s_i, S_{\text{-}i}) - q_i(s_i, S_{\text{-}i})v(s_i, S_{\text{-}i})|s_i) \\ &= \int_{s_{\text{-}i} \in \mathcal{S}_{\text{-}i}} \big\{-x_i(s_i, s_{\text{-}i}) - q_i(s_i, s_{\text{-}i})v(s_i, s_{\text{-}i})\big\} dF_{\text{-}i}(s_{\text{-}i}|s_i) \end{aligned} \tag{II.2}$$

and

$$\begin{aligned} \pi_j(s_j) &:= \mathbb{E}_{s_{\text{-}j}}(q_j(s_j, S_{\text{-}j})v(s_j, S_{\text{-}j}) - x_j(s_j, S_{\text{-}j})|s_j) \\ &= \int_{s_{\text{-}j} \in \mathcal{S}_{\text{-}j}} \big\{q_j(s_j, s_{\text{-}j})v(s_j, s_{\text{-}j}) - x_j(s_j, s_{\text{-}j})\big\} dF_{\text{-}j}(s_{\text{-}j}|s_j). \end{aligned} \tag{II.3}$$

The mechanism (q, x) is *individually rational* (IR) if $\pi_i^s(s_i) \geq 0$ for all $s_i \in \mathcal{S}_i$, $i = 1, ..., n$ and $\pi_j^b(s_j) \geq 0$ for all $s_j \in \mathcal{S}_j$, $j = n+1, ..., n+m$, i.e.

$$-\mathbb{E}_{s_{\text{-}i}}(x_i(s_i, S_{\text{-}i})|s_i) \geq \mathbb{E}_{s_{\text{-}i}}(q_i(s_i, S_{\text{-}i})v(s_i, S_{\text{-}i})|s_i), \tag{II.4}$$

and

$$\mathbb{E}_{s_{\text{-}j}}(x_j(s_j, S_{\text{-}j})|s_j) \leq \mathbb{E}_{s_{\text{-}j}}(q_j(s_j, S_{\text{-}j})v(s_j, S_{\text{-}j})|s_j). \tag{II.5}$$

In addition, (q, x) is *incentive compatible* (IC) if telling the truth is a

[6] Myerson and Satterthwaite (1983) analyze bilateral bargaining mechanisms in the context of trade with pure private values.

[7] I abuse notation again by denoting $F_{\text{-}i}(.|s_i)$ the distribution of all agents' signals except agent i's $(S_{\text{-}i})$ conditional on agent i's signal, $i = 1, ..., n+m$.

Bayesian Nash equilibrium:

$$\mathbb{E}_{s_{\text{-}i}}(x_i(s_i', S_{\text{-}i}) + q_i(s_i', S_{\text{-}i})v(s_i', S_{\text{-}i})|s_i') \leq$$
$$\mathbb{E}_{s_{\text{-}i}}(x_i(s_i, S_{\text{-}i}) + q_i(s_i, S_{\text{-}i})v(s_i', S_{\text{-}i})|s_i') \quad \text{(II.6)}$$

for all $s_i, s_i' \in \mathcal{S}_i$, $i = 1, ..., n$ and

$$\mathbb{E}_{s_{\text{-}j}}(x_j(s_j', S_{\text{-}j}) - q_j(s_j', S_{\text{-}j})v(s_j', S_{\text{-}j})|s_j') \leq$$
$$\mathbb{E}_{s_{\text{-}j}}(x_j(s_j, S_{\text{-}j}) - q_j(s_j, S_{\text{-}j})v(s_j', S_{\text{-}j})|s_j') \quad \text{(II.7)}$$

for all $s_j, s_j' \in \mathcal{S}_j$, $j = n + 1, ..., n + m$.

II.D Bilateral Case: A *No-Trade* Theorem

I analyze the possibility of trade with one buyer and one seller under two different scenarios: the continuous case ($F_{\text{-}i}(.|s_i)$ is absolutely continuous, $i = 1, 2$) and the finite case. I show that in the former there is no mechanism (q, x) involving positive probability of trade that is individually rational and incentive compatible. If there is no such mechanism, using the *revelation principle* we can assert that there is no equilibrium with positive probability in any trade environment (with voluntary participation) satisfying *Assumptions* 1-5 and the absolute continuity condition.

However, when the signal space is finite, equilibria with trade exist, although trade is restricted to take place only when both the seller receives her lowest possible signal ($\underline{s}_1$) and the buyer gets her highest signal ($\overline{s}_2$).

I simplify notation by denoting $x(s_1, s_2)$ the buyer's payment to the seller, $x(s_1, s_2) := -x_1(s_1, s_2) = x_2(s_1, s_2)$, and $q(s_1, s_2)$ the buyer's probability of getting the object, $q(s_1, s_2) := q_1(s_1, s_2) = q_2(s_1, s_2)$. The proof is in the *Appendix*.

Theorem 1 *(a) (**Continuous Case**) If $F_1(.|s_2)$ and $F_2(.|s_1)$ are absolutely continuous with densities $f_{\text{-}i}(s_{\text{-}i}|s_i) > \eta > 0$ $(i = 1, 2)$ for all $s \in \mathcal{S}$, there is no IR* *and IC bargaining mechanism involving positive probability of trade in any bilateral trade environment satisfying Assumptions 1-5.*

*(b) (**Finite Case**) Assume $\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2$ is finite. The only IR and IC bargaining mechanisms satisfying Assumptions 1-5 that exist involve no trade for all $s \in \mathcal{S} \backslash \{(\underline{s}_1, \overline{s}_2)\}$.*

The proof of *Theorem 1* consists of two parts. First, I show that the zero sum game nature of the trading environment implies that IR constraints hold with equality almost surely. Given this, if there is trade with positive probability for some seller's signal $s_1 > \underline{s}_1$ and for some buyer's signal $s_2 < \overline{s}_2$, in order to satisfy seller's IC constraints we need to violate buyer's IC constraints. This result is driven by the strict monotonicity of $v(s_1, s_2)$. To see how, assume that the seller gets $\underline{s}_1$ and that IR holds with equality for $\underline{s}_1$. For the seller's IC constraint under $\underline{s}_1$ to be satisfied, the expected payment under $s_1 > \underline{s}_1$ has to be strictly lower than the expected net value of the object $(q(s_1, s_2)v(s_1, s_2))$, when expectations (taken over buyer's signals) are conditional on $\underline{s}_1$. Otherwise, by strict monotonicity, it will be profitable for the seller to lie and report $s_1 > \underline{s}_1$.[8] This implies that q and x, on average, favor the buyer when low signals are more likely. But given this, a buyer with a very high signal has an incentive to report a low signal, provided that, by strict monotonicity, the object is more valuable to him than to a buyer receiving the lower (reported) signal. Hence, the scenario in which positive probability of trading does not violate IR and IC constraints involves $q(s) > 0$ only for $s = (\underline{s}_1, \overline{s}_2)$, which is a non-null event in the finite case by the full support assumption.

In sum, the strict monotonicity of $v(s_1, s_2)$, coupled with IR constraints holding with equality, forces net values to be bigger than payments for low signal profiles in order to satisfy low-signal seller's IC constraints, but this provides incentives for high-signal buyers to lie. Thus, the only way all constraints are satisfied

[8] Strict monotonicity implies that $\mathbb{E}_{s_2}(q(s_1, s_2)v(\underline{s}_1, s_2)|\underline{s}_1) < \mathbb{E}_{s_2}(q(s_1, s_2)v(s_1, s_2)|\underline{s}_1)$. Hence, if the last term is not strictly bigger than $\mathbb{E}_{s_2}(x(s_1, s_2)|\underline{s}_1)$, a seller receiving $\underline{s}_1$ will get a strictly positive payoff from reporting s_1.

is when net values and payments are zero (except maybe for $(\underline{s}_1, \overline{s}_2)$).

II.E The Multilateral Case

A no-trade result like *Theorem 1* does not exist for the case with three or more traders, except when only one trader receives more than one signal.[9] In fact, there are mechanisms (q, x) leading to trade with probability one. However, except in boundary cases, to elicit trade the probability that payments equal values needs to be strictly less than one (*Theorem 2*). As an example, consider the following trade environment.

Example 1 *There are three traders, one seller ($n = 1$) and two buyers ($m = 2$). Each of the traders receives a private signal $s_i \in \{0, 1\}$, with $\mathbb{P}(s_i = 0) = \mathbb{P}(s_i = 1) = \frac{1}{2}$. The conditional distribution of s_{-i} given s_i is given by*

$$\mathbb{P}(s_{-i}|s_i) = \begin{cases} \frac{1}{2} & \textit{if } s_{-i} = (0,0),\ s_i = 0 \\ \frac{1}{8} & \textit{if } s_{-i} = (0,0),\ s_i = 1 \\ \frac{3}{16} & \textit{if } s_{-i} \in \{(0,1),(1,0)\} \\ \frac{1}{8} & \textit{if } s_{-i} = (1,1),\ s_i = 0 \\ \frac{1}{2} & \textit{if } s_{-i} = (1,1),\ s_i = 1, \end{cases} \tag{II.8}$$

and the expected value of the asset conditional on the vector of signals is

$$v(s) = \begin{cases} 0 & \textit{if } s = (0,0,0) \\ \frac{1}{4} & \textit{if } \sum s_i = 1 \\ \frac{3}{4} & \textit{if } \sum s_i = 2 \\ 1 & \textit{if } s = (1,1,1) \end{cases} \tag{II.9}$$

In this setting, it is possible to find a direct bargaining mechanism (x, q) such that the object is traded for all s. In particular, we can find a payment

[9] In such a case, it is easy to show that in the finite case trade can only happen when the trader with a signal space that is not a singleton either receives the lowest possible signal (seller) or the highest possible one (buyer).

function $x(s)$ such that the probability of trade for buyer i is

$$q_i(s) = \begin{cases} 0 & \text{if } s_i < s_j \\ \frac{1}{2} & \text{if } s_i = s_j \\ 1 & \text{if } s_i > s_j, \end{cases} \tag{II.10}$$

where j denotes the other buyer. The above probabilities mean that the buyer with the highest signal receives the object for sure except when both buyers have the same signal, in which case each buyer receives the object with probability one half.[10]

The IR constraints (II.4)-(II.5) in this zero-sum environment hold with equality.[11] Thus, the IR constraints for buyer $i \in \{2, 3\}$ given (II.8)-(II.10) are, respectively,[12]

$$\frac{1}{2}x_i(0,(0,0)) + \frac{3}{16}[x_i(0,(1,0)) + x_i(0,(0,1))] + \frac{1}{8}x_i(0,(1,1)) = \frac{3}{128}$$

and

$$\frac{1}{8}x_i(1,(0,0)) + \frac{3}{16}[x_i(1,(1,0)) + x_i(1,(0,1))] + \frac{1}{2}x_i(1,(1,1)) = \frac{63}{128}.$$

Let $\Sigma x_i(s_i, s_{-i}) = x_2(s_2,(s_1,s_3)) + x_3(s_3,(s_1,s_2))$. The IR constraints for the seller are given by

$$\frac{1}{2}\Sigma x_i(0,(0,0)) + \frac{3}{16}[\Sigma x_i(1,(0,0)) + \Sigma x_i(0,(0,1))] + \frac{1}{8}\Sigma x_i(1,(0,1)) = \frac{3}{16}$$

and

$$\frac{1}{8}\Sigma x_i(0,(1,0)) + \frac{3}{16}[\Sigma x_i(1,(1,0)) + \Sigma x_i(0,(1,1))] + \frac{1}{2}\Sigma x_i(1,(1,1)) = \frac{13}{16}.$$

Given that the IR constraints hold with equality, the IC constraints (II.6) for buyer i reduce to the following inequalities:

$$\frac{1}{2}x_i(1,(0,0)) + \frac{3}{16}[x_i(1,(1,0)) + x_i(1,(0,1))] + \frac{1}{8}x_i(1,(1,1)) \geq \frac{15}{128}$$

[10] These trade probabilities are similar to what happens in symmetric equilibria of common value auctions.

[11] This is shown in the proof of *Theorem 2*.

[12] Recall that the notation followed is $x_i(s_i, s_{-i})$. For instance, $x_i(0,(1,0))$ denotes the payment from buyer i to the seller when the signals received are zero for both buyers one for the seller.

and

$$\frac{1}{8}x_i(0,(0,0)) + \frac{3}{16}[x_i(0,(1,0)) + x_i(0,(0,1))] + \frac{1}{2}x_i(0,(1,1)) \geq \frac{11}{128}.$$

Similarly, the IC constraints for the seller are

$$\frac{1}{2}\Sigma x_i(0,(1,0)) + \frac{3}{16}[\Sigma x_i(1,(1,0)) + \Sigma x_i(0,(1,1))] + \frac{1}{8}\Sigma x_i(1,(1,1)) \leq \frac{3}{16}$$

and

$$\frac{1}{8}\Sigma x_i(0,(0,0)) + \frac{3}{16}[\Sigma x_i(1,(0,0)) + \Sigma x_i(0,(0,1))] + \frac{1}{2}\Sigma x_i(1,(0,1)) \leq \frac{13}{16}.$$

If we further require that a buyer does not pay when he does not receive the object with positive probability ($x_i(0,(.,1)) = 0$ for $i = 2, 3$),[13] then

(i) the payments when the object is (in expectation) *least* valuable are strictly less than the value: $x_i(0,(0,0)) < v(0,0,0) = 0$ for $i = 2, 3$,[14]

(ii) for some buyer i, the payment when the object is *most* valuable is larger than the value: $x_i(1,(1,1)) > v(1,1,1) = 1$,[15] and

(iii) for some buyer i, $x_i(1,(1,0)) < v(0,0,0)$.

An example of such a mechanism (x, q) is given by (II.10) and the following symmetric payment function:

$$x_i(s_i, s_{-i}) = \begin{cases} -\frac{3}{16} & \text{if } s_i = 0,\ s_{-i} = (0,0) \\ \frac{5}{8} & \text{if } s_i = 0,\ s_{-i} = (1,0) \\ 0 & \text{if } s_i = 0,\ s_{-i} = (s_1,1) \\ \frac{3}{4} & \text{if } s_i = 1,\ s_{-i} = (0,0) \\ -\frac{5}{2} & \text{if } s_i = 1,\ s_{-i} = (1,0) \\ \frac{3}{8} & \text{if } s_i = 1,\ s_{-i} = (0,1) \\ \frac{51}{32} & \text{if } s_i = 1,\ s_{-i} = (1,1). \end{cases} \tag{II.11}$$

[13] This is a common feature in many trade environments, such as double auctions.

[14] This is easy to check by subtracting the buyers' first IR constraint from the second IC constraint and setting $x_i(0,(1,1)) = 0$.

[15] This is due to the restrictions that (i) and the seller's second IR constraint and first IC constraint impose on $\Sigma x_i(1,(1,1))$.

As illustrated by this example, incentive compatibility prevents payments to be equal to values almost everywhere, except if buyers (sellers) only trade when they receive their highest (lowest) signal. This is formally stated in the following theorem. Let $q_i(s_i) = \int_{s_{\text{-}i} \in \mathcal{S}_{\text{-}i}} q_i(s_i, s_{\text{-}i}) dF_{\text{-}i}(s_{\text{-}i}|s_i)$.

Theorem 2 *If* (x, q) *satisfies Assumptions 1-5, then* $\mathbb{P}(-x_i(s_i, S_{-i}) = v(s_i, S_{-i})|q_i(s_i) > 0, s_i > \underline{s}_i) < 1$ *for all* $i \in \{1, ..., n\}$ *and* $\mathbb{P}(x_j(s_j, S_{-j}) = v(x_j(s_j, S_{-j})|q_j(s_j) > 0, s_j < \overline{s}_j) < 1$ *for all* $j \in \{n+1, ..., n+m\}$.

Proof. First, it is straightforward to show that IR constraints hold with equality almost surely (henceforth *a.s.*). Notice that pure common values plus the requirement that $\sum_i x_i(s) = 0$ and $\sum_{i=1}^{n} q_i(s) = \sum_{i=n+1}^{n+m} q_i(s)$ for all $s \in \mathcal{S}$ imply that

$$\mathbb{E}\left(-\sum_{i=1}^{n+m} x_i(S) - v(S)\left[\sum_{i=1}^{n} q_i(S) - \sum_{i=n+1}^{n+m} q_i(S)\right]\right) = \sum_{i=1}^{n+m} \mathbb{E}(\pi_i(S)) = 0.$$

By *Assumption 3* (full support) this is only true if $\pi_i(s_i) = 0$ *a.s.*, $i = 1, ..., n+m$, provided that $\pi_i(s_i)$ is nonnegative by IR.

Now assume that $\mathbb{P}(-x_i(s_i, S_{-i}) = v(S)|q_i(s_i) > 0) = 1$ for some seller i and signal $s_i > \underline{s}_i$. By strict monotonicity $v(s_i', s_{-i}) < v(s_i, s_{-i})$ for all $s_i' < s_i$ and all s_{-i}. But then, if $q_i(s_i) > 0$ she can earn a strictly positive payoff by reporting s_i when her true signal is s_i', thus violating incentive compatibility. A symmetric argument holds for buyers. ■

II.F Appendix

Proof of *Theorem 1*. Denote $A_1 \subseteq \mathcal{S}_1$ and $A_2 \subseteq \mathcal{S}_2$ the sets for which (II.4) and (II.5) hold with equality, respectively. Accordingly, IC constraints simplify to

$$\mathbb{E}_{s_2}(x(s_1, S_2)|s_1') \leq \mathbb{E}_{s_2}(q(s_1, S_2)v(s_1', S_2)|s_1') \tag{II.12}$$

for all $s_1' \in A_1$ and all $s_1 \in \mathcal{S}_1$, and

$$\mathbb{E}_{s_1}(x(S_1, s_2)|s_2') \geq \mathbb{E}_{s_1}(q(S_1, s_2)v(S_1, s_2')|s_2') \tag{II.13}$$

for all $s_2' \in A_2$ and all $s_2 \in \mathcal{S}_2$.

PART (a): the continuous case

Assume trade occurs with positive probability, i.e. the sets $\mathcal{S}_i^* = \{s_i \in \mathcal{S}_i\text{: } \mathbb{E}_{s_{\text{-}i}}(q(s_i, S_{\text{-}i})|s_i) > 0\}$ are non-null, $i = 1, 2$. By (II.12) and the strict monotonicity of $v(\,.\,,.)$ we have that, for all $s_1 > s_1'$ with $s_1 \in \mathcal{S}_1^*$ and $s_1' \in A_1$,

$$\begin{aligned} \mathbb{E}_{s_2}(x(s_1, S_2)|s_1') &\leq \mathbb{E}_{s_2}(q(s_1, S_2)v(s_1', S_2)|s_1') \\ &< \mathbb{E}_{s_2}(q(s_1, S_2)v(s_1, S_2)|s_1'). \end{aligned} \tag{II.14}$$

These inequalities lead to the following result, whose validity we assume for the moment.

Claim 1 *If trade occurs with positive probability there exists a small enough signal $s_1' \in A_1$ such that, for all $s_2' \in A_2$,*

$$\mathbb{E}_{s_1}[\mathbb{E}_{s_2}(x(S_1, S_2)|s_1')|s_2'] < \mathbb{E}_{s_1}[\mathbb{E}_{s_2}(q(S_1, S_2)v(S_1, S_2)|s_1')|s_2']. \tag{II.15}$$

Note that $F_1(.|s_2') \times F_2(.|s_1')$ induces a well defined product measure on $\sigma(\mathcal{S}_1 \times \mathcal{S}_2)$, the σ-field generated by $\mathcal{S}_1 \times \mathcal{S}_2$. In addition, it is easy to check that both $x(S_1, S_2)$ and $q(S_1, S_2)v(S_1, S_2)$ are integrable with respect to this product measure.[16] Therefore, we can apply Fubini's theorem and switch the order of integration on both sides of (II.15):

$$\mathbb{E}_{s_2}[\mathbb{E}_{s_1}(x(S_1, S_2)|s_2')|s_1'] < \mathbb{E}_{s_2}[\mathbb{E}_{s_1}(q(S_1, S_2)v(S_1, S_2)|s_2')|s_1'] \tag{II.16}$$

[16] The latter is integrable since both $q(.,.)$ and $v(\,.\,,.)$ are bounded. Given this and that $\mathbb{E}_{s_2}(x(s_1, S_2)|\underline{s}_1) \leq \mathbb{E}_{s_2}(q(s_1, S_2)v(s_1, S_2)|\underline{s}_1)$ holds for all $s_1 \in \mathcal{S}_1$ the former is also integrable.

Given that $\{s_2 \in A_2\}$ is a probability one event and that $\mathcal{S}_2^*$ is non-null, this strict inequality implies that there exist $s_2 \in \mathcal{S}_2^*$ and a high enough signal $s_2' \in A_2$ satisfying $s_2 < s_2'$ such that

$$\mathbb{E}_{s_1}(x(S_1, s_2)|s_2') < \mathbb{E}_{s_1}(q(S_1, s_2)v(S_1, s_2)|s_2') \leq \mathbb{E}_{s_1}(q(S_1, s_2)v(S_1, s_2')|s_2')$$

where the last inequality is due to the strict monotonicity of $v(\,.\,,.)$.

But since (II.13) holds for all $s_2' \in A_2$, the buyer's IC constraint for s_2' is violated. Hence, the only mechanism that satisfies IR and IC constraints involves $q(s_1, s_2)$ equal to zero *a.s.*

PART (b): the finite case

By *Assumption 3* and the finiteness of $\mathcal{S}$, all $s \in \mathcal{S}$ occur with positive probability. Hence, IR constraints hold with equality and (II.12)-(II.13) are satisfied for all seller and buyer's signals.

First, I show that $q(s_1, s_2)$ can not be greater than zero for more than one $s \in \mathcal{S}$. Assume that there exist two seller's signals with positive probability of trade. In this case, (II.15) is satisfied for $s_1' = \underline{s}_1$. Applying Fubini's theorem for $s_2' = \overline{s}_2$ we have that

$$\mathbb{E}_{s_2}[\mathbb{E}_{s_1}(x(S_1, S_2)|\bar{s}_2)|\underline{s}_1] < \mathbb{E}_{s_2}[\mathbb{E}_{s_1}(q(S_1, S_2)v(S_1, S_2)|\bar{s}_2)|\underline{s}_1]$$

Note that for this inequality to hold, there needs to exist a signal $s_2 \in \mathcal{S}_2$ such that

$$\mathbb{E}_{s_1}(x(S_1, s_2)|\bar{s}_2) < \mathbb{E}_{s_1}(q(S_1, s_2)v(S_1, s_2)|\bar{s}_2) \leq \mathbb{E}_{s_1}(q(S_1, s_2)v(S_1, \bar{s}_2)|\bar{s}_2).$$

If this inequality holds for $s_2 \neq \bar{s}_2$ the buyer has an incentive to report untruthfully whenever he receives $\bar{s}_2$, thus violating (II.13). On the other hand, if the only signal for which it holds is $\bar{s}_2$, then it would imply a violation of the IR constraint, since this constraint holds with equality. A similar argument applies when there exist two buyer's signals with positive probability of trade.

Second, I show that when the only $s \in \mathcal{S}$ for which $q(s_1, s_2)$ can be greater than zero is $(\underline{s}_1, \overline{s}_2)$. Assume there exists $s_1 > \underline{s}_1$ with $q(s_1, s_2) > 0$ for some $s_2 \in \mathcal{S}_2$. Then,

$$x(s_1, s_2) = q(s_1, s_2)v(s_1, s_2) > q(s_1, s_2)v(\underline{s}_1, s_2),$$

which violates seller's IC constraint for $\underline{s}_1$. A similar argument applies to any $s_2 < \overline{s}_1$ with $q(s_1, s_2) > 0$ for some $s_1 \in \mathcal{S}_1$.

Finally, it is straightforward to see that any mechanism such that $q(s) = a1_{\{s=(\underline{s}_1,\overline{s}_2)\}}$ and $x(s) = q(s)v(s)$, with $a \in (0, 1]$ satisfies IR constraints and (II.12)-(II.13).

Proof of *Claim 1.* First, note that $q(s_1, s_2) = 0$ *a.s.* for any $s_1 \in \mathcal{S}_1 \backslash \mathcal{S}_1^*$, provided $\mathbb{E}_{s_2}(q(s_1, S_2)|s_1) = 0$ and $q(s_1, s_2) \geq 0$. This, in conjunction with (II.12) and the absolute continuity of $F_2(.|s_1')$, implies that

$$\mathbb{E}_{s_2}(x(s_1, S_2)|s_1') \leq \mathbb{E}_{s_2}(q(s_1, S_2)v(s_1', S_2)|s_1') = 0$$

a.s. for all $s_1 \in \mathcal{S}_1 \backslash \mathcal{S}_1^*$. Hence, we have that the left hand side of (II.15) satisfies

$$\mathbb{E}_{s_1}[\mathbb{E}_{s_2}(x(S_1, S_2)|s_1')|s_2'] \leq$$
$$\mathbb{E}_{s_1}[\mathbb{E}_{s_2}(x(S_1, S_2)|s_1')[1_{\{\{S_1<s_1'\}\bigcap \mathcal{S}_1^*\}} + 1_{\{\{S_1>s_1'\}\bigcap \mathcal{S}_1^*\}}]|s_2']. \quad \text{(II.17)}$$

Likewise,

$$\mathbb{E}_{s_1}[\mathbb{E}_{s_2}(q(S_1, S_2)v(S_1, S_2)|s_1')|s_2'] =$$
$$\mathbb{E}_{s_1}[\mathbb{E}_{s_2}(q(S_1, S_2)v(S_1, S_2)|s_1')[1_{\{\{S_1<s_1'\}\bigcap \mathcal{S}_1^*\}} + 1_{\{\{S_1>s_1'\}\bigcap \mathcal{S}_1^*\}}]|s_2']. \quad \text{(II.18)}$$

We have to show that there exists a small enough $s_1' \in A_1$ for which the right hand side of (II.17) is strictly smaller than the right hand side of (II.18) or, alternatively, that

$$\overbrace{\mathbb{E}_{s_1}[\mathbb{E}_{s_2}([x(S_1, S_2) - q(S_1, S_2)v(S_1, S_2)]|s_1')1_{\{\{S_1<s_1'\}\bigcap \mathcal{S}_1^*\}}|s_2']}^{\varphi(s_1',s_2')} <$$
$$\underbrace{\mathbb{E}_{s_1}[\mathbb{E}_{s_2}([q(S_1, S_2)v(S_1, S_2) - x(S_1, S_2)]|s_1')1_{\{\{S_1>s_1'\}\bigcap \mathcal{S}_1^*\}}|s_2']}_{\psi(s_1',s_2')}. \quad \text{(II.19)}$$

First, notice that by (II.12),

$$\begin{aligned}\varphi(s_1', s_2') &\leq \mathbb{E}_{s_1}[\mathbb{E}_{s_2}(q(S_1,S_2)[v(s_1',S_2) - v(S_1,S_2)]|s_1')1_{\{\{S_1<s_1'\}\bigcap \mathcal{S}_1^*\}}|s_2'] \\ &\leq (\overline{v} - \underline{v})\mathbb{P}(\{S_1 < s_1'\} \cap \mathcal{S}_1^* | s_2') =: \underline{\varphi}(s_1', s_2'), \qquad \text{(II.20)}\end{aligned}$$

where $\overline{v}$ and $\underline{v}$ denote the maximum and minimum values in $\mathcal{V}$, respectively.

By *Assumption 1* $(\overline{v} - \underline{v})$ is bounded. In addition, the absolute continuity of $F_1(.|s_2')$ implies that $\mathbb{P}(\mathcal{S}_1^* \cap (\mathcal{S}_1 \backslash A_1)|s_2') = 0$. Hence, by arguing again the absolute continuity of $F_1(.|s_2')$, given any $\varepsilon > 0$ we can find a small enough $s_1' \in A_1$ such that $\mathbb{P}(\{S_1 < s_1'\} \cap \mathcal{S}_1^* | s_2') < \varepsilon$. On the other hand, we have that

$$\begin{aligned}\psi(s_1', s_2') &\geq \mathbb{E}_{s_1}[\mathbb{E}_{s_2}(q(S_1,S_2)[v(S_1,S_2) - v(s_1',S_2)]|s_1')1_{\{\{S_1>s_1'\}\bigcap \mathcal{S}_1^*\}}|s_2'] \\ &= \int\limits_{\substack{s_1 > s_1', \\ s_1 \in \mathcal{S}_1^*}} \Big[\int\limits_{s_2 \in \mathcal{S}_2} q(s_1,s_2)[v(s_1,s_2) - v(s_1',s_2)] f_2(s_2|s_1') ds_2 \Big] f_1(s_1|s_2') ds_1 \\ &\geq \int\limits_{\substack{s_1 > s_1', \\ s_1 \in \mathcal{S}_1^*}} \Big[\int\limits_{s_2 \in \mathcal{S}_2} q(s_1,s_2)[v(s_1,s_2) - v(s_1',s_2)] \eta^2 ds_2 \Big] ds_1 =: \underline{\psi}(s_1', s_2').\end{aligned}$$

The inner integral is strictly positive for any $s_1 \in \mathcal{S}_1^*$ such that $s_1 > s_1'$ by strict monotonicity of $v(\,.\,,.)$. In addition, since $\mathbb{P}(\mathcal{S}_1^* \cap A_1 | s_2') = \mathbb{P}(\mathcal{S}_1^* | s_2') > 0$, by the absolute continuity of $F_1(.|s_2')$ there is a small enough $s_1' \in A_1$ such that the set of s_1 over which we integrate $(\{S_1 > s_1'\} \cap \mathcal{S}_1^*)$ has positive Lebesgue measure, implying that $\underline{\psi}(s_1', s_2') > 0$. Moreover, $\underline{\psi}(s_1', s_2')$ gets larger as s_1' gets smaller given that both the inner integral and $\{S_1 > s_1'\} \cap \mathcal{S}_1^*$ are bigger for smaller s_1'. Therefore, we can find a small enough $s_1' \in A_1$ such that

$$\varphi(s_1', s_2') \leq \underline{\varphi}(s_1', s_2') < \underline{\psi}(s_1', s_2') \leq \psi(s_1', s_2'). \ \blacksquare$$

This completes the proof of *Theorem 1*. ■

III

Long-Run Implementation in Repeated Public Good Games with Incomplete Information

Abstract

Despite predictions of complete free riding in one shot public good games, repeated interaction allows for any level of public provision, so long as it is feasible and Pareto superior to no provision at all. We investigate the long run effects of weakening the information players have about each others' preferences. We find that when agents are patient enough any provision level can be attained in the long run.

JEL Classification: *C73, D82, H41.*

Keywords: *Public good games, repeated games, incomplete information, belief-invariant Nash equilibrium.*

III.A Introduction

The Folk Theorem asserts that cooperation can be sustained in non-zero sum repeated games through punishment threats. This result is based on two informational assumptions. First, payoff functions are observable. Thus, players are aware of what payoffs can be implemented in equilibrium and how to punish a deviation. Second, players have enough information about others' past actions to detect with high probability deviations from the equilibrium path.[1] While imperfect monitoring of players' actions has received much attention, little is known about what happens when there is incomplete information about players' payoff functions, which depend on privately observed types. In this context, players may not be able to induce adherence to some strategy profiles satisfying feasibility and individual rationality in the complete information game. Ignorance of a player's payoff function may preclude other players punishing a deviation. Moreover, detecting a deviation may not be possible in strategies that require the revelation of private information. We investigate the latter case in a general class of infinitely repeated public good games with discounting. These games have the property that the existence of private information hinders players' ability to detect a deviation but it does not affect their ability to find suitable punishments. Hence, we are able to isolate the effect of the detection problem on the set of equilibrium payoffs from enforcement issues.

We study long run implementation in a class of dynamic public good games defined as follows. A finite number of players play repeatedly a stage game in which they choose how much to contribute to a public good. Individual contributions are observed at the end of each stage (perfect monitoring). A player's payoff in the stage game is a function of both the vector of contributions and the player's type, which she privately observes before the start of the game and re-

[1]For the Folk Theorem in games of complete information, see Aumann and Shapley (1976) and Rubinstein (1977, 1979) for games with no discounting and Fudenberg and Maskin (1986) for the discounted case. Fudenberg, Levine, and Maskin (1994) study games with imperfect monitoring.

mains fixed throughout.[2] Given any combination of types, free riding (i.e. zero contribution) is a strictly dominant strategy in the stage game. Payoffs of the overall game are represented by the discounted sum of expected stage game payoffs. These so called infinitely repeated games with incomplete information are classical Bayesian games: Nature picks a type profile according to a commonly known prior probability, and players play repeatedly the stage game selected by Nature.

We find that long run implementation is possible as long as there is little but strictly positive discounting: After a *screening* phase in which contributions may not be close to the desired level, any feasible and individually rational provision can be sustained. This asymptotic result holds even if priors over type profiles are private knowledge, provided the strategies proposed are best responses regardless of the beliefs players hold about other players' types. That is, they represent a *belief-invariant Nash equilibrium.*

With the exception of Chakrabarti (2003), who studies a Cournot-type oligopoly game with privately observed costs, the existing literature on repeated games with incomplete information has focused on two-player games. Aumann, Maschler, and Stearns (1968) introduce the analysis of games with no discounting. In their model one player has complete information while the other neither knows her type nor observes her payoffs after each stage. In this case, equilibrium may even fail to exist.[3] As for the discounted case, Cripps and Thomas (2003) study the "known own payoffs" case (i.e. the uninformed player's payoffs are invariant to Nature's choice of types). They find that when players are patient enough and the probability of one of the types goes to one, the Folk Theorem holds.[4]

[2]In particular, for any given type, individual payoffs depend positively on the overall sum of contributions and negatively on own contributions. This implies that players can punish a player by contributing zero regardless of her type. Thus, detected deviations are always punishable.

[3]See Hart (1985) for a characterization of the equilibrium payoff set. On the other hand, Neyman and Sorin (1998) find that the equilibrium set is non-empty when both players remain uninformed.

[4]This results does not hold with no discounting. Shalev (1994) shows that the set of equilibrium payoffs is strictly smaller than the set of feasible and individually rational payoffs.

III.B The Model

Consider a game $\Gamma(P,\delta)$ with a finite number of players $I \geq 2$, defined as follows. According to the nondegenerate probability distribution P, Nature picks a type profile $\theta = (\theta_1, \theta_2, ..., \theta_I)$ from the finite support of P, denoted by Θ^I, where Θ is the set of individual types. Any given player i only observes her own type $\theta_i \in \Theta$. After Nature's move, players enter the infinitely repeated game $\Gamma(\theta,\delta)$, in which payoffs of the stage game $G(\theta)$ are determined by the *realized* type profile θ. Players' actions are observed at the end of each stage.

$G(\theta) = \left\{A, \{U(\theta_i)\}_{i=1}^I\right\}$ is a public good game, in which $A = [0,c] \subset \mathbb{R}_+$ is the set of possible individual contributions, and $U(\theta_i) : A^I \longrightarrow \mathbb{R}$ is the payoff function, which is a function of both a player's own contribution (a_i), and the level of provision of the public good $(\sum\limits_{j=1}^{I} a_j)$:

$$U(a|\theta_i) = U(a_i, \sum_{j=1}^{I} a_j|\theta_i).$$

Assumption 1 *For any $\theta_i \in \Theta$, $U(.,.|\theta_i)$ is continuous, strictly decreasing in its first argument and strictly increasing in its second argument.*

Assumption 2 *(Incentive to free ride) For any $\theta_i \in \Theta$, $U(r, g+r|\theta_i) > U(r', g+r'|\theta_i)$ for any $g \geq 0$, and any $r' > r \geq 0$.*

The last assumption implies that, for any profile θ, contributing zero would be a strictly dominant strategy in the stage game $G(\theta)$.

Let $\underline{u}(\theta_i) = \min_{a_{-i}\in A^{I-1}} \max_{a_i \in A} U(a_i, a_{-i}|\theta_i)$ be player i's minmax payoff when her type is θ_i. We have that $\underline{u}(\theta_i) = U(0,0|\theta_i)$ for any $\theta_i \in \Theta$, by *Assumptions 1* and *2*. Denote opponents' minmaxing strategy by $\underline{a}_{-i}(\theta_i)$. Note that $\underline{a}_{-i}(\theta_i) = (0,...,0)$ for all $\theta_i \in \Theta$, so $\Gamma(P,\delta)$ is a game with type-independent minmaxing. This property plays an important role in our construction. It allows players to punish a detected deviation regardless of the information they have about the deviator's type.

A payoff vector $v = (v_1, ..., v_I)$ is individually rational (IR) under θ if $v_i \geq \underline{u}(\theta_i)\ \forall i$. We will also say that a contribution vector is IR, meaning that its associated payoff vector is IR.

A pure strategy for player i in $\Gamma(\theta, \delta)$ is a sequence of maps $s_i = \{s_i^t\}_{t=1}^{\infty}$, $s_i^t : H^{t-1} \rightarrow A$, where H^t denotes the set of all possible histories of play $h_i^t = (\theta_i, a^1, a^2, ..., a^t)$.[5] Players discount future payoffs according to discount factors $\{\delta_i\}_{i=1}^I$, with $0 < \delta \leq \delta_i < 1$ for any i. $\{\delta_i\}_{i=1}^I$ are common knowledge. The payoff for player i when players follow strategy profile $s = (s_1, ..., s_I)$ and the realized type vector is θ is given by

$$v_i(s|\theta) = (1 - \delta_i)\sum_{t=1}^{\infty}\delta_i^{t-1}U(a^t|\theta_i).$$

III.C Belief-Invariant Nash Equilibrium

Our characterization of $\Gamma(P, \delta)$'s payoff set applies to both PBE and Nash equilibrium (NE). In addition, we introduce the concept of *belief-invariant* Nash equilibrium (BINE), which we use in the long run implementation result stated in the next section.

A strategy profile s is a NE of $\Gamma(P, \delta)$ if

$$E_P\left[v_i(s|\theta)|\theta_i\right] \geq E_P\left[v_i(s_i', s_{-i}|\theta)|\theta_i\right]\ \forall s_i' \text{ and all } \theta_i \in \Theta,\ i = 1, 2, ..., I. \quad \text{(III.1)}$$

Since P is common knowledge, a NE strategy profile only needs to satisfy (1) for the prior probability distribution P. However, strategies satisfying (1) for any well-defined probability distribution constitute a NE even if players have private and differing beliefs about the likelihood of each $\theta \in \Theta^I$. Thus, NE implemented with these *belief-invariant* strategies are robust to weakening standard Bayesian assumptions about players' knowledge of P.

[5] Since the set of contribution vectors is convex, we do not need to take into account mixed or correlated strategies of the stage game, given that for any such strategy profile we can always find a contribution vector that yields the same expected payoffs.

Definition 1 *A strategy s_i is a belief-invariant best response to s_{-i} for type θ_i player i if $E_P\left[v_i(s_i, s_{-i}|\theta)|\theta_i\right] \geq E_P\left[v_i(s_i', s_{-i}|\theta)|\theta_i\right]$ for all s_i' and all $P \in \Delta(\Theta^I)$, where $\Delta(\Theta^I)$ is the space of probability distributions over type profiles.*[6]

Definition 2 *A strategy profile s is a belief-invariant Nash equilibrium (BINE) of $\Gamma(P, \delta)$ if $E_{P_i}\left[v_i(s|\theta)|\theta_i\right] \geq E_{P_i}\left[v_i(s_i', s_{-i}|\theta)|\theta_i\right]$ for all s_i', all $P_i \in \Delta(\Theta^I)$ and all $\theta_i \in \Theta$, $i = 1, 2, ..., I$.*

In a BINE, players on the equilibrium path are best responding to opponents' strategies given any well-defined probability distribution over opponents' types. In other words, BINE strategies are invariant to individual beliefs about players' types.

III.D Long Run Implementation

Let $\mathcal{U}_\theta$ be the convex hull of the set of feasible payoff vectors of $G(\theta)$ that satisfy strict IR for all players. According to the Folk Theorem, any payoff vector in $\mathcal{U}_\theta$ can be supported as the average payoff vector in a subgame perfect equilibrium of $\Gamma(\theta, \delta)$, provided δ is high enough (Fudenberg and Maskin (1986)). However, in $\Gamma(P, \delta)$, the set of feasible and strictly IR payoffs depends on which type profile is selected by Nature. Accordingly, we define a mapping (*payoff target*), which uniquely assigns a payoff vector to any possible type profile, as the object of our implementation results.

Definition 3 *A payoff target for $\Gamma(P, \delta)$ is a mapping $w : \Theta^I \to \mathbb{R}^I$ such that $w(\theta) \in \mathcal{U}_\theta$ for any $\theta \in \Theta^I$.*

[6] Belief-invariant best responses are logically related to the "belief free strategies" defined in Ely, Hörner, and Olszewski (2005). The latter refer to best responses that are independent of the opponent's private histories in two-player games with imperfect private monitoring. In such games, each player observes her own action and a private signal about the other player's action. Notice that in $\Gamma(P, \delta)$ actions are observable. Thus, the only private information is the player's type. Since belief-invariant best responses are independent of the other players' types, they are trivially independent of the "private histories" of the other players and could therefore be considered belief free. Both equilibrium concepts rule out sequential equilibria in which players best responses depend on the particular beliefs about opponents private histories (either types in this setting or signals in the imperfect monitoring case).

We show in *Proposition 1* that any payoff target can be supported in the long run by a BINE, provided $\Gamma(P, \delta)$ satisfies two mild assumptions. *Assumption 3* is a weak separability condition, which is implied in games where the standard single crossing property holds. It ensures that no two distinct types equally rank all contribution vectors belonging to any I-dimensional convex subset of A^I. Before stating this assumption, we define *utility equivalent* types in a set X of contribution vectors as types that share the same ranking over all contribution vectors belonging to X.[7] For any $\varepsilon > 0$, let $B(a; \varepsilon)$ be the ε -neighborhood of contribution vector a.

Definition 4 *If for a given set* $X \subseteq A^I$, $U(a|\theta_i) = b + dU(a|\theta_i')$ *with* $d > 0$ *for all* $a \in X$, *then types* θ_i',θ_i *are utility equivalent in* X.

Assumption 3 *Given any contribution vector* $a \in interior(A^I)$ *and any scalar* $\varepsilon > 0$, *there are no two distinct types in* Θ *that are utility equivalent in* $B(a; \varepsilon)$.

That is, for any pair of types and a contribution vector a, we can always find two contribution vectors close to a such that the first type strictly prefers one while the second type prefers the other.

The following assumption implies the existence of a positive contribution vector that is strictly Pareto superior to zero provision under any possible type profile. This assumption is satisfied, for instance, if there is a strictly IR lump-sum contribution vector when nature selects all players to have the lowest WTC among all types in Θ.

Assumption 4 *There exists a contribution vector* $a^* >> 0$ *that is strictly IR for all players under any* $\theta \in \Theta^I$.

Proposition 1 states that, if the game satisfies *Assumptions 1-4* and δ is close to one, any payoff target w can be the continuation payoff in a BINE for any period after $T(\delta)$. That is, the continuation payoff vector at any period $t > T$ is $w(\theta)$, where θ is the realized type profile. That is, any feasible provision scheme can be sustained in the long run provided it is Pareto superior to zero provision.

[7] This is a local version of the definition provided by Abreu, Dutta, and Smith (1994).

Proposition 1 *Assume* $\Gamma(P, \delta)$ *satisfies Assumptions 1-4. Then, for any payoff target* w*, there is a discount factor* $\underline{\delta} < 1$ *and a finite period* $T(\delta)$ *such that, for all* $\delta \in (\underline{\delta}, 1)$*, there is a BINE of* $\Gamma(P, \delta)$ *with continuation payoffs at any* $t > T$ *equal to* $w(\theta)$ *for any* $\theta \in \Theta$*.*[8]

This result is based both on the existence of type-independent minmaxing (*Assumptions 1* and *2*), and on the introduction of rewards for truthful reporting at the beginning of the game (*Assumptions 3* and *4*). While the former enables players to punish observable deviations (contributions different from those on the equilibrium path), the latter prevents undetected deviations (i.e. untruthful reporting).

The strategies used to prove *Proposition 1* consist of three different phases of play on the equilibrium path, coupled with off-equilibrium minmax threats. In the first period, each player reveals her type (θ_i) through her initial contribution. This *communication* phase is followed by a *screening* phase in which a rewarding cycle is played repeatedly. Each cycle consists of I periods, one per player. In each period $i = 1, .., I$ of the cycle, players choose a strictly IR contribution vector $a(\theta_i; i)$, which depends solely on player i's report. This vector has the property that player i's payoff will be higher given player i's truthful revelation in the communication phase ($u(a(\theta_i; i)|\theta_i) > u(a(\theta_i'; i)|\theta_i)$ for all $\theta_i' \neq \theta_i$). By *Assumptions 3* and *4*, $a(\theta_i; i)$ exist for all $\theta_i \in \Theta$. Once this phase is over, players play any feasible and strictly IR contribution scheme (*implementation* phase). In the presence of positive discounting, players can deter untruthful reporting by choosing the appropriate length of the screening phase. If the screening phase is played long enough, the discounted value of rewards for telling the truth will outweigh any discounted potential gains from lying to be obtained during the implementation phase. In addition, by strict IR of $\{a(\theta_i; i)\}$ any detected deviation can be deterred with minmax threats for patient enough players (δ close to one).

[8] The proof is in the Appendix.

The belief-invariant feature of these strategies comes from the fact that a player's incentives for truthful reporting are independent of other players' types (i.e. $a(\theta_i; i)$ does not depend on θ_j, for any $j \neq i$). Therefore, no matter what a player believes about Nature's choice of types, it is always a best response to report her true type. Once all players have reported their types there is no uncertainty about the sequence of contributions to be played.

Remark 1 *(Consistency off the equilibrium path) By Assumption 2 punishment threats are credible: not deviating when minmaxing a player is a belief-invariant best response. Therefore, the BINE strategies proposed show a higher consistency than that required in PBE.*[9]

Remark 2 *Proposition 1 does not hold in the undiscounted case as payoffs in the implementation phase are the only relevant factor for players with $\delta_i = 1$. Thus, truthful reporting may not be incentive compatible (IC) for certain payoff targets.*

We conclude by pointing out two important limitations of *Proposition 1*. First, average payoffs may not be close to the payoff target. Second, the length of the screening phase depends on the discount factor. Hence, it is natural to ask whether an approximate Folk Theorem is possible. The answer to this question is not straightforward: It depends on the particular parameters of the game. For instance, Cripps and Thomas (2003) show that when there are two players, one of them with only one possible type while the other has two types, an approximate folk theorem holds if the probability of one of second player's types is sufficiently close to zero. However, a proof of a folk theorem type of result is not available for the general case.

III.E Appendix

Proof of *Proposition 1*. Consider the following strategy profile s:

[9] The notion of PBE requires that equilibrium strategies are best responses under beliefs consistent with Bayesian updating. This means that initial beliefs about types have to be derived from P. In this case, beliefs are irrelevant, so any belief system will support on- and off-equilibrium play.

On the equilibrium path:

A) *Communication period*: In the first period each player i chooses a contribution $m(\theta_i)$ so as to signal her type θ_i.[10] The vector of contributions in the first period is denoted by $m(\theta)$.

B) *Screening phase*: Players play cyclically (N times, where N is a function of δ) the sequence of I contribution vectors $(a(\theta;1), a(\theta;2), ..., a(\theta;I))$ where $\theta = (\theta_1, ..., \theta_I)$ are the reported types in the first period. Each vector $a(\theta;i)$ satisfies the following conditions:[11]

(i) *Individual Rationality (IR)*: $U(a(\theta;i)|\theta_j) \geq \underline{u}(\theta_j)$ $\forall j$, with strict inequality when $i = j$.

(ii) *Incentive Compatibility (IC)*: $U(a(\theta;i)|\theta_i) > U(a(\theta_i', \theta_{-i};i)|\theta_i)$ for any $\theta_i' \neq \theta_i$, where $a(\theta_i', \theta_{-i};i)$ would be the contribution vector to be played if the reported types were $(\theta_1, ..., \theta_{i-1}, \theta_i', \theta_{i+1}, ..., \theta_I)$.

(iii) θ_{-i}-*Independence*: $a(\theta;i)$ is independent of the report of any player $j \neq i$. That is, $a(\theta;i) = a(\theta_i;i)$ for all $\theta_i \in \Theta$.

C) *Implementation phase*: If no deviation has occurred, a contribution vector yielding $w(\theta)$ is played.

Off the equilibrium path:

A) If a player i does not send a meaningful signal (i.e. with positive probability on the equilibrium path), players do not contribute from $t = 2$ on.

B) If a player deviates at the screening or the implementation phase, players do not contribute from the period after the deviation on.

By *Assumption* 4, we know that there exists a vector a^* strictly IR for any report θ. Since $U(.,.|.)$ is continuous, there exists an $\varepsilon > 0$ such that any vector $a \in B(\varepsilon, a^*)$ is also strictly IR.

[10] Since there are infinitely many actions in A and finite types, it is possible to uniquely assign a different contribution level $m(\theta_i)$ to each type θ_i.

[11] Hence, the length of the screening phase is $N \times I$.

By *Assumption 3*, for any pair of distinct types $\theta_i, \theta_i' \in \Theta$ we can find two contribution vectors $\mathsf{a}(\theta_i, \theta_i'), \mathsf{a}(\theta_i', \theta_i) \in B(\varepsilon, a^*)$ such that $U(\mathsf{a}(\theta_i, \theta_i')|\theta_i) > U(\mathsf{a}(\theta_i', \theta_i)|\theta_i)$ and $U(\mathsf{a}(\theta_i', \theta_i)|\theta_i') > U(\mathsf{a}(\theta_i, \theta_i')|\theta_i')$.[12]

For each type θ_i we select a set $A^*(\theta_i) = \{\mathsf{a}(\theta_i, \theta_i')\}_{\theta_i' \neq \theta_i}$ of $\|\Theta\| - 1$ contribution vectors satisfying these inequalities, one for each element of $\Theta \setminus \{\theta_i\}$. We then order the $\|\Theta\|\,(\|\Theta\| - 1)$ vectors in $A_i^* = \cup_{\theta_i \in \Theta} A^*(\theta_i)$ in ascending order with respect to type-θ_i utility, breaking ties arbitrarily, and assign them strictly increasing weights adding up to one. Finally, we define $a(\theta; i)$ as the resulting convex combination of the vectors in A_i^*.[13] By defining $a(\theta; i)$ in this way we ensure that it satisfies (i)-(iii). Note that, by (iii), player i's report cannot affect the selection of $a(\theta; j)$ for $j \neq i$.

Thus, strategy profile s exists. Now we need to check that it is a BINE. After observing θ_i, player i's payoff at $t = 1$ on the equilibrium path is given by $E_P[v_i(s|\theta)|\theta_i]$, with

$$v_i(s|\theta) = (1 - \delta_i)U(m(\theta)|\theta_i) + \delta_i \pi_i(\theta) + \delta_i^T w_i(\theta).$$

In this expression, $T = N \times I + 1$ and $\pi_i(\theta)$ represents the payoff obtained during the screening phase:

$$\begin{aligned} \pi_i(\theta) &= (1 - \delta_i) \sum_{j=1}^{I} \sum_{s=0}^{N} \delta_i^{(sI+j-1)} U(a(\theta; j)|\theta_i) \\ &= (1 - \delta_i) \frac{1 - \delta_i^{IN}}{1 - \delta_i^{I}} \sum_{j=1}^{I} \delta_i^{(j-1)} U(a(\theta; j)|\theta_i). \end{aligned} \tag{III.2}$$

It suffices to show that, for all i, , all $\theta \in \Theta$, and all strategies s_i' we have that $v_i(s(\theta)|\theta) \geq v_i(s_i', s_{-i}|\theta)$. In such a case, $E_P[v_i(s|\theta)|\theta_i] \geq E_p[v_i(s_i', s_{-i}|\theta_i)|\theta_i]$ for all $P \in \Delta(\Theta^I)$, which implies that s is a BINE.

We have two possible scenarios in which a player might deviate:

[12] This property is similar to the *payoff asymmetry* property stated in Abreu, Dutta, and Smith (1994). The main difference is that we deal with a continuous action space, whereas in their model the action space is discrete.

[13] This construction is similar to the one used by Abreu, Dutta, and Smith (1994) in their proof of *Lemma 2*.

1) *Undetected deviation*: Player i reports a type θ_i' different from the true one. In this scenario, her payoff satisfies the following inequality for any realized type profile $\theta \in \Theta$:

$$v_i(s_i^1, s_{-i}|\theta) \leq (1-\delta_i)\max_{a \in A^I} U(a|\theta_i) + \delta_i \pi_i'(\theta) + \delta_i^T \max_{a \in A^I} U(a|\theta_i),$$

where $\pi_i'(\theta)$ corresponds to the same expression as (4), except in that it uses $a(\theta_i', \theta_{-i}; i)$ instead of $a(\theta; i)$. By *IC* and θ_{-i}-*Independence* of vectors $\{a(\theta; i)\}$, we have that $\pi_i'(\theta) < \pi_i(\theta)$.

2) *Detected deviation*: Player i deviates in any other circumstance. If she deviates in period $t \geq 1$, her continuation payoffs will be:

$$v_{i,t}(s_i^2, s_{-i}|\theta) \leq (1-\delta_i)\max_{a \in A^I} U(a|\theta_i) + \delta_i \underline{u}(\theta_i).$$

The upper bound on payoffs is higher for undetected deviations as long as $\pi_i'(\theta) \geq \underline{u}(\theta_i)$. Assume this is true without loss of generality, so that we need only prove that reporting a false type is not a best response.

Assume that there is a type $\theta_i' \neq \theta_i$ for player i such that, by not reporting truthfully, she gets

$$v_i(s_i^1, s_{-i}|\theta) = (1-\delta_i)\max_{a \in A^I} U(a|\theta_i) + \delta_i \max_{a(\theta_i', \theta_{-i}; i)} \pi_i'(\theta) + \delta_i^T \max_{a \in A^I} U(a|\theta_i). \quad \text{(III.3)}$$

If we show that there is a discount factor $\underline{\delta}_i < 1$ such that, for any $\delta_i > \underline{\delta}_i$, there is a finite $T(\delta_i) = N(\delta_i) \times I + 1$ such that $v_i(s|\theta) \geq v_i(s_i^1, s_{-i}|\theta)$, then it would not be a best response for player i to report a false type.

First, note that $v_i(s|\theta) > (1-\delta_i)\min_{a \in A^I} U(a|\theta_i) + \delta_i \pi_i(\theta) + \delta_i^T \underline{u}(\theta_i)$.[14] Subtracting (5) from this inequality we get

$$v_i(s|\theta) - v_i(s_i^1, s_{-i}|\theta) > (1-\delta_i)\left[\min_{a \in A^I} U(a|\theta_i) - \max_{a \in A^I} U(a|\theta_i)\right] +$$
$$+\delta_i \left[\pi_i(\theta) - \max_{a(\theta_i', \theta_{-i}; i)} \pi_i'(\theta)\right] + \delta_i^T \left[\underline{u}(\theta_i) - \max_{a \in A^I} U(a|\theta_i)\right]. \quad \text{(III.4)}$$

[14]The inequality is strict since $w_i(\theta) > \underline{u}(\theta_i)$.

Further,

$$\pi_i(\theta) - \max \pi_i'(\theta) \geq (1-\delta_i)\frac{1-\delta_i^{IN}}{1-\delta_i^I}\delta_i^{(I-1)}\left[U(a(\theta;i)|\theta_i) - \max_{\theta_i'\neq\theta_i} U(a(\theta_i', \theta_{-i};i)|\theta_i)\right].$$

Since $U(a(\theta;i)|\theta_i) > U(a(\theta_i', \theta_{-i};i)|\theta_i)\ \forall\theta_i' \neq \theta_i$ by *IC*, the right-hand side of the above inequality is strictly greater than zero for $\delta_i < 1$ and N finite. Thus, the first and the third terms on the right hand side of (6) are strictly negative, whereas the second is strictly positive.

As $\delta_i \uparrow 1$, we have that: $(1-\delta_i) \rightarrow 0$; $(1-\delta_i^{IN}) \rightarrow 0$ at a slower rate than $(1-\delta_i)$ if $I \times N > 1$; $\frac{(1-\delta_i)}{1-\delta_i^I}\delta_i^I \rightarrow 1/I$ and; $\delta_i^T \rightarrow 1$.

In addition, for $\delta_i < 1$, as $N \rightarrow \infty$ the following is true: $(1-\delta_i^{IN}) \rightarrow 1$ and; $\delta_i^T = \delta_i^{NI+1} \rightarrow 0$.

Therefore, for a big enough discount factor δ_i (strictly smaller than one) there is a finite $N(\delta_i)$ such that the right hand side of (6) is nonnegative. This implies $v_i(s|\theta) - v_i(s_i^1, s_{-i}|\theta) > 0$.

In order to determine $\underline{\delta}$, pick the lowest discount factor so that

$$\frac{(1-\delta_i)\delta_i^I}{1-\delta_i^I}\left[U(a(\theta;i)|\theta_i) - \max_{\theta_i'\neq\theta_i} U(a^i(\theta_i', \theta_{-i})|\theta_i)\right] >$$
$$> (1-\delta_i)\left[\max_{a\in A^I} U(a|\theta_i) - \min_{a\in A^I} U(a|\theta_i)\right]$$

is satisfied for any type profile θ, $i = 1, ..., I$. Next, for any game $\Gamma(P, \delta)$ with discount factors $(\delta_1, ..., \delta_I)$ such that $\underline{\delta} < \delta_i < 1\ \forall i$, choose $N(\delta_i)$ so that (6) is satisfied $\forall\theta$. Finally, pick the highest $N(\delta_i)$ of all players.[15] ■

[15] If inequality (6) holds for $N(\delta_i)$, it also holds for any $N > N(\delta_i)$.

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