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Author
Hauge, P.S.

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P. S. Hauge, R. H. Muller, and C. G. Smith

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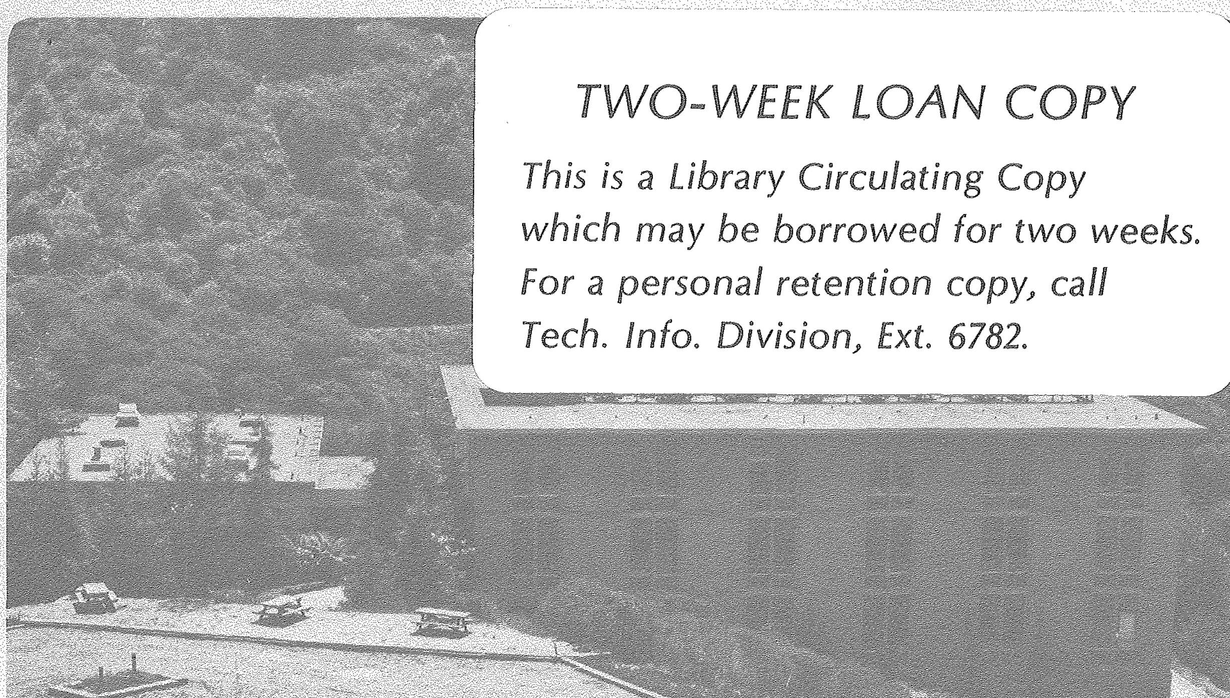

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DISCLAIMER

CONVENTIONS AND FORMULAS FOR USING THE MUELLER-STOKES CALCULUS IN ELLIPSOMETRY

P. S. HAUGE

IBM Thomas J. Watson Research Center, Yorktown Heights, New York 10598, USA

R. H. MULLER

Inorganic Materials Research Division, Lawrence Berkeley Laboratory and Department of Chemical Engineering; University of California Berkeley, California 94720, USA

and

C. G. SMITH

Bell Laboratories, Murray Hill, New Jersey 07974, USA

ABSTRACT

An analysis of the compatability of ellipsometry conventions with the Mueller-Stokes calculus has shown that for the two approaches to be consistent, modifications to the 1968 Nebraska ellipsometry conventions are necessary. A sign convention is proposed which specifies the ellipticity angle and the fourth Stokes parameter to be positive when the tip of the instantaneous electric-field vector describes a right-handed helix in space. Polarization of this type will be designated as right-handed and represented by a point on the northern half of the Poincaré sphere. The consequences of using this convention, along with the previously adopted $e^{i(\omega t+\delta)}$ convention for the electric field, are seen in the relationships among several representations of polarization state. These include the magnitudes and phases of the electric-field components, the Jones vector, coherency matrix, Stokes parameters, Poincaré sphere, polarization ratio, ellipsometric parameters ψ and Δ, and the azimuth and ellipticity of the polarization ellipse. The formalism for converting the Jones matrix into the matrix that transforms the coherency vector or the Stokes vector (Mueller matrix) is described. Mueller matrices for a linear retarder and an isotropic reflecting surface and the description of a compensator-analyzer polarimeter are given.

1. Introduction

A convention for absolute phase of the electric-field vector wave has been established by adopting the $e^{i(\omega t+\delta)}$ time /phase dependence [1,2] . However, unambiguous and comprehensive sign conventions have not been adopted for parameters affected by the phase *difference* of orthogonal components of the electric field. In ellipsometry, the ellipticity angle and the fourth Stokes parameter carry the sign of the phase difference. The growing use of the Mueller-Stokes calculus with photometric ellipsometry makes it increasingly desirable to adopt an unambiguous designation for phase differences.

Born and Wolf [3] note that it is traditional to call polarization right- (left-) handed when an observer looking into the source would see the polarization ellipse described in a clockwise (counterclockwise) sense. As pointed out by Clarke [4], this is equivalent to calling the polarization right- (left-) handed when the helix described by the instantaneous locus of the tip of the electric-field vector in space is right- (left-) handed.

Opposite names are obtained, however, from the convention in which the fingers of the appropriate hand curl in the direction of the trace of the ellipse while the thumb points in the direction of propagation. Thus, when applied to the handedness of polarization state, the terms "right" or "left" can have an uncertain interpretation.

Since the ellipticity angle and the fourth Stokes parameter change their sign with changes in handedness, one must also associate the terms "positive" and "negative" with the two types of polarization. Unfortunately, the literature in the fields of optics, astronomy and radio polarimetry contains examples of all four combinations of terms for each type of polarization [4].

A further point of disagreement among authors is the relation of handedness to the Poincaré sphere. The north pole of the sphere variously represents right [3] or left [5,6] circular polarization, and the positive direction for the fourth Stokes parameter is variously to the north [3,7,8] or south [5]. Finally, the direction of increasing azimuth on the sphere is variously to the west [5,9] or east [3,6,7].

We propose in this paper a convention that specifies the signs of the ellipticity angle and the fourth Stokes parameter to be positive whenever the instantaneous electric-field vector describes a right-handed helix in space. The ellipse of this type of polarization is described in a clockwise sense when viewed by a observer looking into the source. We further propose that polarization of this type be designated as right-handed, and represented by a point in the northern half of the Poincaré sphere.

The adoption of a sign convention affects the relationships among the various representations of polarization state: the magnitudes and phases of orthogonal electric-field components, the Jones vector, coherency matrix, Stokes parameters, Poincaré sphere, polarization ratio, ellipsometric parameters and the azimuth and ellipticity of the polarization ellipse. The relationships appropriate for the proposed sign convention are all given in appendices at the end of the paper.

Also given are the transformation matrices for the coherency vector and Stokes vector (Mueller matrix) derived from the corresponding Jones matrix, using the proposed convention. Examples of Mueller matrices for a linear retarder and an isotropic reflecting surface as well as the Mueller-Stokes description of a compensator-analyzer polarimeter are included.

These conventions differ from previous conventions principally in the ellipticity angle ϵ and in the Poincaré sphere. Ellipticity differs by defining ϵ to be positive for right-handed polarization, while the north pole on the Poincaré sphere represents right-handed, rather than left-handed polarization, to be consistent with the fourth Stokes parameter. These differences, discussed in §4, are required to establish consistency with conventions used for the Mueller-Stokes calculus, without requiring major changes to the calculus.

2. Established conventions

Previously adopted definitions for the electric field, Jones vector, coherency matrix, polarization ratio, and ellipsometric parameters ψ and Δ are given for reference in this section.

2.1 Electric field

According to the Muller-Nebraska convention [1,2] the Cartesian components of the electric field are written using the $e^{i(\omega t+\delta)}$ time/phase dependence as

$$E_x(z,t) = \mathrm{Re}\{E_{x0}\, e^{i(\omega t-2\pi z/\lambda+\delta_x)}\}, \tag{2.1}$$

$$E_y(z,t) = \mathrm{Re}\{E_{y0}\, e^{i(\omega t-2\pi z/\lambda+\delta_y)}\}. \tag{2.2}$$

2.2 Jones vector

The Jones vector is a concise representation of a totally polarized (quasi-)monochromatic wave. It is defined as the vector of complex electric-field components with the explicit time and space dependence removed. The positive phase dependence is retained.

$$\mathbb{E} \equiv \begin{bmatrix} E_x \\ E_y \end{bmatrix} = \begin{bmatrix} E_{x0}\, e^{i\delta_x} \\ E_{y0}\, e^{i\delta_y} \end{bmatrix}. \tag{2.3}$$

E_{x0}, E_{y0}, δ_x and δ_y may have a stochastic time dependence about an average value, as in the case of quasi-monochromatic or partially polarized light. Time-averaged values, denoted by $\langle\ \rangle$, are then used.

2.3 Coherency matrix

The coherency matrix gives the correlation of the field components with themselves and each other. It is capable of representing partially polarized waves. It is defined [3] as the direct product of the Jones vector with its Hermetian adjoint.

$$\mathbf{J} \equiv \langle \mathbb{E} \times \mathbb{E}^\dagger \rangle = \begin{bmatrix} \langle E_x E_x^* \rangle & \langle E_x E_y^* \rangle \\ \langle E_y E_x^* \rangle & \langle E_y E_y^* \rangle \end{bmatrix} = \begin{bmatrix} J_{xx} & J_{xy} \\ J_{yx} & J_{yy} \end{bmatrix}. \tag{2.4}$$

Using the phase convention of eq. (2.3), the elements of this matrix are given by

$$J_{xx} = \langle E_{x0}^2 \rangle, \tag{2.5}$$

$$J_{xy} = \langle E_{x0} E_{y0} \, e^{-i(\delta_y - \delta_x)} \rangle, \tag{2.6}$$

$$J_{yx} = \langle E_{x0} E_{y0} \, e^{i(\delta_y - \delta_x)} \rangle, \tag{2.7}$$

$$J_{yy} = \langle E_{y0}^2 \rangle. \tag{2.8}$$

It is easily seen that when the alternative $e^{-i(\omega t+\delta)}$ convention is used, the coherency matrix is transposed: If

$$E'_{x,y} = \mathrm{Re}\{ E_{x0,y0} \, e^{-i(\omega t - 2\pi z/\lambda + \delta_{x,y})} \}, \tag{2.9}$$

then

$$\mathbf{E}' = \begin{bmatrix} E_x{}' \\ E_y{}' \end{bmatrix} = \begin{bmatrix} E_{x0} \, e^{-i\delta_x} \\ E_{y0} \, e^{-i\delta_y} \end{bmatrix} \tag{2.10}$$

and

$$J'_{xy} = \langle E'_x E'^*_y \rangle = \langle E_{x0} E_{y0} \, e^{i(\delta_y - \delta_x)} \rangle = J_{yx} \, . \tag{2.11}$$

Equations (2.10) and (2.11) are also obtained with the $e^{i(\omega t - \delta)}$ convention. It is necessary that the convention of eq. (2.3) be consistently followed to avoid conflicts in relating the coherency matrix to other polarization parameters. Because of the close relationship of the elements of the coherency matrix to the Stokes parameters, we make special note of this point.

2.4 Complex polarization ratio

The complex ratio of the elements of the Jones vector suppresses the absolute amplitude and phase information of the wave, leaving only a description of polarization state. Although both the x-to-y [10,11] and y-to-x [3,9] ratios are found in the literature, the latter appears more frequently and is adopted here. Following the notation of Azzam and Bashara [9], we write

$$\chi \equiv E_y/E_x = (E_{y0}/E_{x0}) \, e^{i(\delta_y - \delta_x)}. \tag{2.12}$$

It is sometimes convenient to express this ratio in trigonometric form by using the angles α and δ in the relation

$$\chi = \tan\alpha\, e^{i\delta}, \tag{2.13}$$

where $\tan\alpha = E_{y0}/E_{x0}$ and $\delta = \delta_y - \delta_x$.

2.5 Ellipsometric parameters

The parameters ψ and Δ describe the interaction of polarized light with an optical system. Explicitly, we have [1]

$$\rho \equiv R_p/R_s = \tan\psi\, e^{i\Delta} = \chi_{\mathrm{inc}}/\chi_{\mathrm{ref}} \tag{2.14}$$

for the case of oblique reflection from an isotropic surface having the reflection coefficients R_{p} and R_{s} for the p and s components. If we postulate linearly polarized incident light having an azimuth of 45°, then $\chi_{\mathrm{inc}} = 1$, and the polarization state of the reflected light is given by

$$\chi_{\mathrm{ref}} = \cot\psi\, e^{-i\Delta}. \tag{2.15}$$

Thus, with the assumption that $\chi_{\mathrm{inc}} = 1$, ψ and Δ provide an unambiguous description of the reflected polarization state.

3. Proposed conventions

In this section, we define handedness independently of mathematical convention, according to the instantaneous helix of the electric field and the rotation of the field vector in a transverse plane. From the established time and phase convention, the condition for right-handedness is stated in terms of the component phase difference. The same condition is then expressed in terms of the coherency matrix, polarization ratio and ellipsometric parameter Δ. Finally we propose choices for the signs of ellipticity angle and the fourth Stokes parameter as well as the hemisphere of the Poincaré sphere which is to represent right-handed polarization. Some of the more important relationships are then given; a complete listing is in the appendices.

3.1 Definition of right-handed polarization

We follow the traditional (helix) convention for naming handedness: when the instantaneous electric field forms a right-handed helix in space, so that its point of intersection with a transverse plane rotates in a clockwise sense to an observer facing the source, we call the polarization right-handed [3]. This convention, almost universally followed in the field of optics, is illustrated in fig. 1.

If we consider the transverse plane at $z = 0$, eqs. (2.1) and (2.2) describe the electric field components as

$$E_x(t) = E_{x0} \cos(\omega t + \delta_x), \tag{3.1}$$

$$E_y(t) = E_{y0} \cos(\omega t + \delta_y). \tag{3.2}$$

The condition for clockwise rotation, and thus for right-handed polarization, is [3]

$$\sin(\delta_y - \delta_x) > 0. \tag{3.3}$$

From eq. (2.7), a state of polarization represented by the coherency matrix J is right-handed when

$$\mathrm{Im}\,(J_{yx}) = E_{x0}E_{y0} \sin(\delta_y - \delta_x) > 0. \tag{3.4}$$

In terms of the polarization ratio χ, eqs. (2.12) and (2.13) give the condition for right-handedness as

$$\mathrm{Im}\,(\chi) = \tan\alpha \sin(\delta_y - \delta_x) > 0. \tag{3.5}$$

Thus, right-handed polarization states lie in the upper half of the complex polarization plane. Finally, eqs. (2.15) and (3.5) combine to give the condition for right-handedness in terms of the ellipsometric parameter Δ:

$$\sin \Delta < 0. \tag{3.6}$$

Equation (3.6) states that when light, linearly polarized at $45°$ (or any azimuth θ such that $\tan \theta > 0$), is obliquely incident on the bare surface of an absorbing material ($0° < \Delta < 180°$), the reflected light has *left-handed* polarization.

3.2 Choice of sign for ellipticity

The azimuth of a polarized light wave is simply the angular position of the ellipse's major axis measured from a reference orientation in a counterclockwise direction for an observer looking into the source. The reference orientation is taken as the plane of incidence or scattering, the horizontal position, or the x axis. We use the symbol θ to denote azimuth. Ellipticity is defined as the angle ϵ whose tangent has magnitude equal to the ratio of the minor to major axis (axial ratio) of the polarization ellipse, and whose sign indicates the handedness. See fig. 2.

We propose the sign convention which makes the ellipticity positive when the polarization is right-handed. That is, for right-handed polarization,

$$0 < \epsilon \leq 45°. \tag{3.7}$$

Using this convention, the Jones vector of the wave is given by [9]

$$\mathbb{E} = A\, e^{i\phi}\, \mathbb{R}(-\theta) \begin{bmatrix} \cos \epsilon \\ i \sin \epsilon \end{bmatrix} = A\, e^{i\phi} \begin{bmatrix} \cos \theta \cos \epsilon - i \sin \theta \sin \epsilon \\ \sin \theta \cos \epsilon + i \cos \theta \sin \epsilon \end{bmatrix}, \tag{3.8}$$

where $\mathbb{R}$ is the Cartesian coordinate rotation matrix, A is the overall amplitude and ϕ is the overall phase of the wave.

3.3 Choice of sign for fourth Stokes parameter

The four real quantities known as the Stokes parameters provide a useful description of polarization state in terms of measurable intensities. They may be defined phenomenologically in terms of intensities measured with the use of ideal polarization filters which transmit linear polarizations with azimuths of $0°$, $45°$, $90°$, and $135°$, as well as right- and left-handed circular polarizations [3,5]. To be consistent with our choice of sign for ellipticity, we choose the convention which makes

$$S_3 > 0 \tag{3.9}$$

for right-handed polarization. The defining equations for the Stokes parameters thus take the form [5,7-9,12-16]

$$S_0 \equiv \langle E_x E_x^* \rangle + \langle E_y E_y^* \rangle = \langle E_{x0}^2 \rangle + \langle E_{y0}^2 \rangle, \tag{3.10}$$

$$S_1 \equiv \langle E_x E_x^* \rangle - \langle E_y E_y^* \rangle = \langle E_{x0}^2 \rangle - \langle E_{y0}^2 \rangle, \tag{3.11}$$

$$S_2 \equiv 2\mathrm{Re}\ \langle E_x^* E_y \rangle = 2\langle E_{x0} E_{y0} \cos\ (\delta_y - \delta_x) \rangle, \tag{3.12}$$

$$S_3 \equiv 2\mathrm{Im}\ \langle E_x^* E_y \rangle = 2\langle E_{x0} E_{y0} \sin\ (\delta_y - \delta_x) \rangle. \tag{3.13}$$

Combining eqs. (2.3), (3.8) and (3.10)-(3.13) gives the relation of the Stokes parameters of totally polarized light to the azimuth and ellipticity:

$$S_1 = S_0 \cos 2\epsilon \ \cos 2\theta, \tag{3.14}$$

$$S_2 = S_0 \cos 2\epsilon \ \sin 2\theta, \tag{3.15}$$

$$S_3 = S_0 \sin 2\epsilon . \tag{3.16}$$

From eqs. (2.4)-(2.8) we obtain the relationship between the Stokes parameters as we have defined them and the coherency matrix:

$$S_0 = J_{xx} + J_{yy}, \tag{3.17}$$

$$S_1 = J_{xx} - J_{yy}, \tag{3.18}$$

$$S_2 = 2\mathrm{Re}\ (J_{yx}) = J_{xy} + J_{yx}, \tag{3.19}$$

$$S_3 = 2\mathrm{Im}\ (J_{yx}) = i(J_{xy} - J_{yx}). \tag{3.20}$$

The use of different time and phase conventions for the electric field or inconsistent use of the present convention leads to the opposite sign in eq. (3.20), even when positive values of S_3 indicate right-handedness [3,9,15,17-21]. Consistent use of the present convention leads to eq. (3.20), however [12,14,22-24]. The relationship between the Stokes parameters and the coherency matrix appears in matrix form in appendices A4 and A5.

3.4 *Choice of hemisphere on Poincaré sphere*

A state of polarization described by the azimuth and ellipticity angles (θ, ϵ) is represented on the Poincaré sphere by a point having longitude equal to 2θ and latitude equal to 2ϵ. The Cartesian coordinates of a point on the sphere are simply the Stokes parameters of the corresponding polarization state. We make the natural choice of having north latitudes specify positive values of S_3 and ϵ, and therefore represent right-handed polarization states. According to eqs. (3.14) and (3.15), longitude (azimuth) is taken to be zero for the meridian defined by $S_2 = 0$ and $S_1 > 0$. In order for S_1, S_2, and S_3 to form a right-handed system of coordinates, the positive direction for S_2 is taken to be east of S_1. These conventions are illustrated in fig. 3.

Certain points on the Poincaré sphere are frequently labeled H and V (where the positive and negative S_1 axes intersect with the equator) and R and L (at the two poles) to denote respectively horizontal and vertical linear polarization and right- and left-handed circular polarization. From the choices made above, the labels R and L appear respectively at the north and south poles. H and V do not always convey useful ellipsometric information, since the reference axis is not always horizontal. Depending on the orientation of the system being measured, the plane of incidence or scattering (reference azimuth) may have arbitrary orientation. The traditional symbols (H, V) may thus be considered to be interchangeable with any of the symbol pairs (p, s), (x, y), $(\parallel, \perp)$, (TM, TE), (0, 90), (l, r) or (π, σ) to indicate transverse directions which are respectively parallel and perpendicular to the plane of incidence or scattering.

The ellipsometric parameters (ψ, Δ) as well as the auxiliary angles (α, δ) are related to the Poincaré sphere as shown in fig. 4. The inclination of the plane defined by the S_1 axis and the point P on the sphere relative to the equatorial plane is the component phase difference $\delta = \delta_y - \delta_x$. The ellipsometric phase angle Δ is simply $360° - \delta$ [see eqs. (2.13) and (2.15)] and gives the inclination measured in the opposite direction. The angles 2α and 2ψ (both positive) give the inclination of point P to the S_1 axis, measured from opposite sides of the origin.

Points on the unit-radius Poincaré sphere are uniquely related to values of χ in the complex plane by means of a stereographic projection [3,9]. The projection which relates the sphere and plane as we have defined them makes the center of the sphere coincide with the origin of the plane [24]. The radius of the sphere is normalized to the value of S_0. The positive real and imaginary axes coincide with the positive S_2 and S_3 axes respectively. The projection is made from the point of intersection of the negative S_1 axis with the equator. The respective projections of the real and imaginary axes of the plane onto the equator and lines of 0° and 180° longitude of the sphere are shown in fig. 5.

3.5 Summary

The results of this section are summarized in the following statements, which are all true for right-handed polarization.

- By definition:

  1) The helix is right-handed.

  2) The electric-field vector rotates clockwise.

- By the Muller-Nebraska time and phase convention [1,2]:

  3) $\sin(\delta_y - \delta_x) > 0$.

  4) $\mathrm{Im}(J_{yx}) > 0$.

  5) $\mathrm{Im}(\chi) > 0$.

  6) $\sin\Delta < 0$.

- By proposed convention:

  7) $\sin 2\epsilon > 0$.

  8) $S_3 > 0$.

  9) The point is in the northern hemisphere.

4. Alternate convention for ellipticity and fourth Stokes parameter

We consider here the alternate sign convention for ϵ and S_3, namely that they be taken to be positive for left-handed polarization. Equations (3.8), (3.13) and (3.20) would then read

$$\mathbb{E} = A\ e^{i\phi}\ \mathbb{R}(-\theta)\begin{bmatrix} \cos\epsilon \\ -i\sin\epsilon \end{bmatrix} = A\ e^{i\phi}\begin{bmatrix} \cos\theta\cos\epsilon + i\sin\theta\sin\epsilon \\ \sin\theta\cos\epsilon - i\cos\theta\sin\epsilon \end{bmatrix}, \tag{4.1}$$

$$S_3 \equiv 2\mathrm{Im}\ \langle E_x E_y^* \rangle = 2\langle E_{x0}E_{y0}\sin(\delta_x - \delta_y)\rangle, \tag{4.2}$$

and

$$S_3 = 2\mathrm{Im}\ (J_{xy}) = i(J_{yx} - J_{xy}), \tag{4.3}$$

while eq. (3.16) would remain unchanged:

$$S_3 = S_0 \sin 2\epsilon. \tag{4.4}$$

North latitude on the Poincaré sphere, while again corresponding to positive values of S_3 and ϵ, would now represent left-handed polarization states. This convention has been used by Muller in a discussion of coordinate system conventions [1] and in a review of photometric ellipsometers [6], and is illustrated in fig. 6.

This convention has the advantage of giving $\sin(\delta_x - \delta_y)$, $\sin 2\epsilon$, S_3 and $\sin\Delta$ all the same sign. Moreover, positive ellipticity would correspond to an increase with time of the azimuth of the electric-field vector, an aesthetically pleasing choice. Finally, the polarization ratio could be redefined as $\chi = E_x/E_y$, making the angles (α,δ) identical to (ψ,Δ) and relating the parameters χ and ρ in a more natural manner: $\chi_{\mathrm{ref}} = \rho\chi_{\mathrm{inc}}$.

The choice between the conventions is thus not a clear-cut decision. Ellipsometric usage tends to make $E_{y(s)}$ the reference component by defining $\rho = R_p/R_s$, while the common definition of the Stokes parameters [eqs. (3.10)-(3.13)] makes $E_{x(p)}$ the reference component. Thus, when the Mueller-Stokes calculus is used for the analysis of photometric ellipsometry (to which it is otherwise well suited), conflicts, taking the form of unwanted minus signs, invariably occur (see appendix A8.)

The preferred convention of section 3 has the following advantages: The x axis is universally used as the azimuth reference. By defining $\chi = E_y/E_x$, the result is the natural $\chi = \tan\theta$ when $\epsilon = 0$. The relationship of ellipticity to the Jones vector is commonly given by eq. (3.8) in ellipsometry literature. Several monographs on polarized light [5,7,9] define the Mueller matrix of a linear retarder in keeping with the proposed convention (see appendix A12). Finally, the relationship between the Jones and Mueller matrices (see appendix A10) is also consistent with the literature [9,12] .

5. Summary of mathematical conventions

We have proposed nomenclature and sign conventions for parameters related to phase differences in order to reduce ambiguity in mathematical treatments of photometric ellipsometry and polarimetry. Although an author is always free to set forth and use his own conventions, the aim of this proposal is to free him of that task if he so chooses. The mathematical conventions are summarized here:

- Time and phase dependence (previously adopted [1,2]):

$$E_x(z,t) = \mathrm{Re}\{E_{x0}\ \mathrm{e}^{i(\omega t - 2\pi z/\lambda + \delta_x)}\}, \tag{5.1}$$

$$E_y(z,t) = \mathrm{Re}\{E_{y0}\ \mathrm{e}^{i(\omega t - 2\pi z/\lambda + \delta_y)}\}. \tag{5.2}$$

These equations specify the form of the Jones vector

$$\mathbf{E} = \begin{bmatrix} E_{x0}\ \mathrm{e}^{i\delta_x} \\ E_{y0}\ \mathrm{e}^{i\delta_y} \end{bmatrix} \tag{5.3}$$

and the coherency matrix

$$\mathbf{J} = \begin{bmatrix} \langle E_{x0}^2 \rangle & \langle E_{x0}E_{y0}\ \mathrm{e}^{-i(\delta_y - \delta_x)} \rangle \\ \langle E_{y0}E_{x0}\ \mathrm{e}^{i(\delta_y - \delta_x)} \rangle & \langle E_{y0}^2 \rangle \end{bmatrix}. \tag{5.4}$$

- Complex polarization ratio:

$$\chi = (E_{y0}/E_{x0})\ \mathrm{e}^{i(\delta_y - \delta_x)}. \tag{5.5}$$

- Ellipticity angle:

$$\sin 2\epsilon = \frac{2E_{x0}E_{y0}}{E_{x0}^2 + E_{y0}^2} \sin(\delta_y - \delta_x)\ . \tag{5.6}$$

- Fourth Stokes parameter:

$$S_3 = 2\langle E_{x0}E_{y0} \sin(\delta_y - \delta_x) \rangle. \tag{5.7}$$

The conventions adhere to the traditional usage of calling a state of polarization right-handed when the polarization ellipse is described in a clockwise sense. Since this happens whenever $\sin(\delta_y - \delta_x) > 0$, the ellipticity and fourth Stokes parameter are positive for right-handed polarization. Finally, right-handed polarization states are represented by points on the northern half of the Poincaré sphere.

APPENDICES

The appendices give expressions for various descriptions of polarization state according to the proposed conventions, and illustrate the use of the Mueller-Stokes calculus in photometric ellipsometry. Definitions and relations among various descriptions of polarization state, intensity and degree of polarization are given in appendices A1-A9. (The use of ψ and Δ to describe the polarization state of light emerging from the system they characterize presumes 45^o linearly polarized incident light.) Appendix A10 takes the Jones matrix for a nondepolarizing system and derives from it the matrices which transform the coherency and Stokes vectors. Mueller matrices for an isotropic reflecting surface and a linear retarder are given in A11 and A12, and the Mueller-Stokes calculus is applied to the analysis of a compensator-analyzer polarimeter in A13.

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A1. Electric field - Cartesian representation

Components:

$$E_x(z,t) = \mathrm{Re}\{E_{x0}(t)\ \mathrm{e}^{i[\omega t - 2\pi z/\lambda + \delta_x(t)]}\}, \tag{A1.1}$$

$$E_y(z,t) = \mathrm{Re}\{E_{y0}(t)\ \mathrm{e}^{i[\omega t - 2\pi z/\lambda + \delta_y(t)]}\}. \tag{A1.2}$$

Vector:

$$\mathbb{E}(z,t) = \mathrm{Re}\left\{\begin{bmatrix} E_x \\ E_y \end{bmatrix} \mathrm{e}^{i(\omega t - 2\pi z/\lambda)}\right\}. \tag{A1.3}$$

E_{x0}, E_{y0}, δ_x, and δ_y may have a stochastic time dependence about an average value, as in the case of quasi-monochromatic waves or partial polarization.

A2. Intensity and degree of polarization

Two parameters that are of interest in photometric ellipsometry and polarimetry, but which are independent of handedness conventions are the intensity I and degree of polarization p of a beam of light. These may be expressed in terms of the field component magnitudes, the Stokes parameters and the coherency matrix as follows:

Defining relationships:

$$I \equiv \langle E_{x0}^2 \rangle + \langle E_{y0}^2 \rangle, \tag{A2.1}$$

$$p \equiv \text{fraction of energy in polarized component of beam.} \tag{A2.2}$$

Other formulas:

$$I = S_0, \tag{A2.3}$$

$$I = J_{xx} + J_{yy} = \mathrm{Tr}\ (\mathbf{J}), \tag{A2.4}$$

$$p = (S_1^2 + S_2^2 + S_3^2)^{\frac{1}{2}} / S_0, \tag{A2.5}$$

$$p = \left[1 - \frac{4\ \det(\mathbf{J})}{\mathrm{Tr}^2(\mathbf{J})} \right]^{\frac{1}{2}}, \tag{A2.6}$$

where $\det(\mathbf{J})$ and $\mathrm{Tr}(\mathbf{J})$ are respectively the determinant and trace of the coherency matrix $\mathbf{J}$ of the wave.

A3. Jones vector (totally polarized wave)

Defining relationship:

$$\mathbb{E} \equiv \begin{bmatrix} E_x \\ E_y \end{bmatrix} = \begin{bmatrix} E_{x0}\ e^{i\delta_x} \\ E_{y0}\ e^{i\delta_y} \end{bmatrix}. \tag{A3.1}$$

Other formulas:

$$\mathbb{E} = \frac{e^{i\delta_x}}{E_{x0}} \begin{bmatrix} J_{xx} \\ J_{yx} \end{bmatrix} = \frac{e^{i\delta_y}}{E_{y0}} \begin{bmatrix} J_{xy} \\ J_{yy} \end{bmatrix}, \tag{A3.2}$$

$$\mathbb{E} = \frac{e^{i\delta_x}}{2E_{x0}} \begin{bmatrix} S_0 + S_1 \\ S_2 + iS_3 \end{bmatrix} = \frac{e^{i\delta_y}}{2E_{y0}} \begin{bmatrix} S_2 - iS_3 \\ S_0 - S_1 \end{bmatrix}, \tag{A3.3}$$

$$\mathbb{E} = E_{x0}\ e^{i\delta_x} \begin{bmatrix} 1 \\ \chi \end{bmatrix} = \frac{E_{x0}}{\cos\alpha} \begin{bmatrix} \cos\alpha\ e^{i\delta_x} \\ \sin\alpha\ e^{i\delta_y} \end{bmatrix}, \tag{A3.4}$$

$$\mathbb{E} = E_{y0}\ e^{i\delta_y} \begin{bmatrix} \rho \\ 1 \end{bmatrix} = \frac{E_{y0}}{\cos\psi} \begin{bmatrix} \sin\psi\ e^{i\delta_x} \\ \cos\psi\ e^{i\delta_y} \end{bmatrix}, \tag{A3.5}$$

$$\mathbb{E} = \frac{E_M\ e^{i(\delta_x - \delta_a)}}{\cos\epsilon}\ \mathbb{R}(-\theta) \begin{bmatrix} \cos\epsilon \\ i\sin\epsilon \end{bmatrix} = \frac{E_M\ e^{i(\delta_x - \delta_a)}}{\cos\epsilon} \begin{bmatrix} \cos\theta\cos\epsilon - i\sin\theta\sin\epsilon \\ \sin\theta\cos\epsilon + i\cos\theta\sin\epsilon \end{bmatrix}, \tag{A3.6}$$

where E_M is the ellipse major axis, $\mathbb{R}$ is the Cartesian coordinate rotation matrix given by eq. (A13.2) and

$$\tan\delta_a = \tan\theta\tan\epsilon. \tag{A3.7}$$

A4. Coherency matrix (partially polarized wave)

Defining relationship:

$$\mathbf{J} \equiv \langle \mathbb{E} \times \mathbb{E}^{\dagger} \rangle = \begin{bmatrix} \langle E_x E_x^* \rangle & \langle E_x E_y^* \rangle \\ \langle E_y E_x^* \rangle & \langle E_y E_y^* \rangle \end{bmatrix} = \begin{bmatrix} J_{xx} & J_{xy} \\ J_{yx} & J_{yy} \end{bmatrix}. \qquad \text{(A4.1)}$$

Other formulas:

$$\mathbf{J} = \begin{bmatrix} \langle E_{x0}^2 \rangle & \langle E_{x0} E_{y0}\, \mathrm{e}^{-i(\delta_y - \delta_x)} \rangle \\ \langle E_{x0} E_{y0}\, \mathrm{e}^{i(\delta_y - \delta_x)} \rangle & \langle E_{y0}^2 \rangle \end{bmatrix}, \qquad \text{(A4.2)}$$

$$\mathbf{J} = \frac{1}{2} \left\{ S_0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + S_1 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + S_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + S_3 \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \right\}, \qquad \text{(A4.3)}$$

$$\mathbf{J} = \frac{1}{2} \begin{bmatrix} S_0 + S_1 & S_2 - iS_3 \\ S_2 + iS_3 & S_0 - S_1 \end{bmatrix}, \qquad \text{(A4.4)}$$

$$\begin{bmatrix} J_{xx} \\ J_{xy} \\ J_{yx} \\ J_{yy} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -i \\ 0 & 0 & 1 & i \\ 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{bmatrix}, \qquad \text{(A4.5)}$$

$$\mathbf{J} = \frac{I}{2} \begin{bmatrix} 1 + p \cos 2\alpha & p \sin 2\alpha\, \mathrm{e}^{-i\delta} \\ p \sin 2\alpha\, \mathrm{e}^{i\delta} & 1 - p \cos 2\alpha \end{bmatrix}, \qquad \text{(A4.6)}$$

$$\mathbf{J} = \frac{I}{2} \begin{bmatrix} 1 - p \cos 2\psi & p \sin 2\psi\, \mathrm{e}^{i\Delta} \\ p \sin 2\psi\, \mathrm{e}^{-i\Delta} & 1 + p \cos 2\psi \end{bmatrix}, \qquad \text{(A4.7)}$$

$$\mathbf{J} = \frac{I}{2} \begin{bmatrix} 1 + p \cos 2\theta \cos 2\epsilon & p(\sin 2\theta \cos 2\epsilon - i \sin 2\epsilon) \\ p(\sin 2\theta \cos 2\epsilon + i \sin 2\epsilon) & 1 - p \cos 2\theta \cos 2\epsilon \end{bmatrix}. \qquad \text{(A4.8)}$$

A5. Stokes parameters (partially polarized wave)

Defining relationship:

$$\mathrm{S} \equiv \left\{\langle E_{x0}^2\rangle + \langle E_{y0}^2\rangle,\ \langle E_{x0}^2\rangle - \langle E_{y0}^2\rangle,\ 2\langle E_{x0}E_{y0}\cos(\delta_y-\delta_x)\rangle,\ 2\langle E_{x0}E_{y0}\sin(\delta_y-\delta_x)\rangle\right\}. \quad (\mathrm{A5.1})$$

Other formulas:

$$\mathrm{S} = \left\{\langle E_xE_x^*\rangle + \langle E_yE_y^*\rangle,\ \langle E_xE_x^*\rangle - \langle E_yE_y^*\rangle,\ \langle E_xE_y^*\rangle + \langle E_yE_x^*\rangle,\ i[\langle E_xE_y^*\rangle - \langle E_yE_x^*\rangle]\right\}, \quad (\mathrm{A5.2})$$

$$\mathrm{S} = \left\{\langle E_xE_x^*\rangle + \langle E_yE_y^*\rangle,\ \langle E_xE_x^*\rangle - \langle E_yE_y^*\rangle,\ 2\mathrm{Re}\,\langle E_yE_x^*\rangle,\ 2\mathrm{Im}\,\langle E_yE_x^*\rangle\right\}, \quad (\mathrm{A5.3})$$

$$\mathrm{S} = \left\{\mathrm{Tr}\left(\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\mathrm{J}\right),\ \mathrm{Tr}\left(\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\mathrm{J}\right),\ \mathrm{Tr}\left(\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\mathrm{J}\right),\ \mathrm{Tr}\left(\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}\mathrm{J}\right)\right\}, \quad (\mathrm{A5.4})$$

$$\mathrm{S} = \{(J_{xx} + J_{yy}),\ (J_{xx} - J_{yy}),\ (J_{xy} + J_{yx}),\ i(J_{xy} - J_{yx})\}, \quad (\mathrm{A5.5})$$

$$\mathrm{S} = \{(J_{xx} + J_{yy}),\ (J_{xx} - J_{yy}),\ 2\mathrm{Re}(J_{yx}),\ 2\mathrm{Im}(J_{yx})\}, \quad (\mathrm{A5.6})$$

$$\begin{bmatrix}S_0\\ S_1\\ S_2\\ S_3\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 1\\ 1 & 0 & 0 & -1\\ 0 & 1 & 1 & 0\\ 0 & i & -i & 0\end{bmatrix}\begin{bmatrix}J_{xx}\\ J_{xy}\\ J_{yx}\\ J_{yy}\end{bmatrix}, \quad (\mathrm{A5.7})$$

$$\mathrm{S} = I\ \{1,\ p\cos 2\alpha,\ p\sin 2\alpha\cos\delta,\ p\sin 2\alpha\sin\delta\}, \quad (\mathrm{A5.8})$$

$$\mathrm{S} = I\ \{1,\ -p\cos 2\psi,\ p\sin 2\psi\cos\Delta,\ -p\sin 2\psi\sin\Delta\}, \quad (\mathrm{A5.9})$$

$$\mathrm{S} = I\ \{1,\ p\cos 2\epsilon\cos 2\theta,\ p\cos 2\epsilon\sin 2\theta,\ p\sin 2\epsilon\}. \quad (\mathrm{A5.10})$$

A6. Polarization ratio (totally polarized wave)

Defining relationship:

$$\chi \equiv E_y/E_x. \tag{A6.1}$$

Other formulas:

$$\chi = (E_{y0}/E_{x0})\ e^{i(\delta_y-\delta_x)}, \tag{A6.2}$$

$$\chi = J_{yx}/J_{xx} = J_{yy}/J_{xy}, \tag{A6.3}$$

$$\chi = \frac{S_2+iS_3}{S_0+S_1} = \frac{S_0-S_1}{S_2-iS_3}, \tag{A6.4}$$

$$\chi = \tan\alpha\ e^{i\delta}, \tag{A6.5}$$

$$\chi = \cot\psi\ e^{-i\Delta}, \tag{A6.6}$$

$$\chi = \frac{\tan\theta + i\tan\epsilon}{1-i\tan\theta\tan\epsilon} = \frac{\sin\theta\cos\epsilon + i\sin\epsilon\cos\theta}{\cos\theta\sin\epsilon - i\sin\epsilon\sin\theta}. \tag{A6.7}$$

A7. Auxiliary angles (α, δ) (totally polarized wave)

Defining relationship:

$$\tan\alpha\, e^{i\delta} = E_y/E_x = \chi. \tag{A7.1}$$

Other formulas:

$$\tan\alpha = E_{y0}/E_{x0}, \tag{A7.2}$$

$$\delta = \delta_y - \delta_x, \tag{A7.3}$$

$$\cos 2\alpha = \frac{J_{xx} - J_{yy}}{J_{xx} + J_{yy}}, \tag{A7.4}$$

$$\delta = \arg(J_{yx}), \tag{A7.5}$$

$$\cos 2\alpha = S_1/S_0, \tag{A7.6}$$

$$\tan\delta = S_3/S_2, \tag{A7.7}$$

$$\alpha = \tan^{-1}|\chi|, \tag{A7.8}$$

$$\delta = \arg(\chi), \tag{A7.9}$$

$$\alpha = 90^\circ - \psi, \tag{A7.10}$$

$$\delta = 360^\circ - \Delta, \tag{A7.11}$$

$$\cos 2\alpha = \cos 2\epsilon \cos 2\theta, \tag{A7.12}$$

$$\tan\delta = \tan 2\epsilon / \sin 2\theta. \tag{A7.13}$$

A8. Ellipsometric parameters (describing χ_{ref} when $\chi_{\rm inc} = 1$; totally polarized wave)

Defining relationship:

$$\tan\psi\, e^{i\Delta} = \rho = E_x/E_y = 1/\chi. \tag{A8.1}$$

Other formulas:

$$\tan\psi = E_{x0}/E_{y0}, \tag{A8.2}$$

$$\Delta = \delta_x - \delta_y, \tag{A8.3}$$

$$\cos 2\psi = -\frac{J_{xx}-J_{yy}}{J_{xx}+J_{yy}}, \tag{A8.4}$$

$$\Delta = -\arg(J_{yx}), \tag{A8.5}$$

$$\cos 2\psi = -S_1/S_0, \tag{A8.6}$$

$$\tan\Delta = -S_3/S_2, \tag{A8.7}$$

$$\psi = \tan^{-1}\frac{1}{|\chi|}, \tag{A8.8}$$

$$\Delta = -\arg(\chi), \tag{A8.9}$$

$$\psi = 90^\circ - \alpha, \tag{A8.10}$$

$$\Delta = 360^\circ - \delta, \tag{A8.11}$$

$$\cos 2\psi = -\cos 2\epsilon\, \cos 2\theta, \tag{A8.12}$$

$$\tan\Delta = -\tan 2\epsilon / \sin 2\theta. \tag{A8.13}$$

A9. Azimuth and ellipticity angles

Defining relationships:

$$\theta = \text{azimuth of major axis}, \tag{A9.1}$$

$$|\tan \epsilon| = \text{axial ratio}, \tag{A9.2}$$

$$S_3 \tan \epsilon > 0. \tag{A9.3}$$

Other formulas:

$$\tan 2\theta = \frac{2E_{x0}E_{y0}}{E_{x0}^2 - E_{y0}^2} \cos(\delta_y - \delta_x), \tag{A9.4}$$

$$\sin 2\epsilon = \frac{2E_{x0}E_{y0}}{E_{x0}^2 + E_{y0}^2} \sin(\delta_y - \delta_x), \tag{A9.5}$$

$$\tan 2\theta = \frac{J_{xy} + J_{yx}}{J_{xx} - J_{yy}} = \frac{2\mathrm{Re}(J_{yx})}{J_{xx} - J_{yy}}, \tag{A9.6}$$

$$\sin 2\epsilon = \frac{i(J_{xy} - J_{yx})}{p(J_{xx} + J_{yy})} = \frac{2\mathrm{Im}(J_{yx})}{p(J_{xx} + J_{yy})}, \tag{A9.7}$$

$$\tan 2\theta = S_2/S_1, \tag{A9.8}$$

$$\sin 2\epsilon = S_3/pS_0, \tag{A9.9}$$

$$\tan 2\theta = \frac{2\mathrm{Re}(\chi)}{1 - |\chi|^2}, \tag{A9.10}$$

$$\sin 2\epsilon = \frac{2\mathrm{Im}(\chi)}{1 + |\chi|^2}, \tag{A9.11}$$

$$\tan 2\theta = \tan 2\alpha \cos \delta, \tag{A9.12}$$

$$\sin 2\epsilon = \sin 2\alpha \sin \delta, \tag{A9.13}$$

$$\tan 2\theta = -\tan 2\psi \cos \Delta, \tag{A9.14}$$

$$\sin 2\epsilon = -\sin 2\psi \sin \Delta. \tag{A9.15}$$

A10. Transformation matrices for coherency and Stokes vectors

Analogous to the relationship of the Stokes parameters to the Jones vector of totally polarized light is the relationship of the Mueller matrix to the Jones matrix of a nondepolarizing optical system. We begin with the transformation matrix for the Jones vector (Jones matrix):

$$\mathbb{E}' = \begin{bmatrix} E_x' \\ E_y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} E_x \\ E_y \end{bmatrix} = \begin{bmatrix} aE_x + bE_y \\ cE_x + dE_y \end{bmatrix}. \tag{A10.1}$$

Forming the expression $\mathbb{J}' = \langle \mathbb{E}' \times \mathbb{E}'^{\dagger} \rangle$ element by element, the transformation matrix for the coherency vector is obtained:

$$\begin{bmatrix} J_{xx}' \\ J_{xy}' \\ J_{yx}' \\ J_{yy}' \end{bmatrix} = \begin{bmatrix} aa^* & ab^* & ba^* & bb^* \\ ac^* & ad^* & bc^* & bd^* \\ ca^* & cb^* & da^* & db^* \\ cc^* & cd^* & dc^* & dd^* \end{bmatrix} \begin{bmatrix} J_{xx} \\ J_{xy} \\ J_{yx} \\ J_{yy} \end{bmatrix}. \tag{A10.2}$$

From the relationship between the coherency and Stokes vectors [see eqs. (3.17)-(3.20)]:

$$\begin{bmatrix} S_0' \\ S_1' \\ S_2' \\ S_3' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & i & -i & 0 \end{bmatrix} \begin{bmatrix} J_{xx}' \\ J_{xy}' \\ J_{yx}' \\ J_{yy}' \end{bmatrix}, \tag{A10.3}$$

$$\begin{bmatrix} J_{xx} \\ J_{xy} \\ J_{yx} \\ J_{yy} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -i \\ 0 & 0 & 1 & i \\ 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{bmatrix}, \tag{A10.4}$$

we obtain

$$\mathbf{S}' = \mathbf{M}\mathbf{S}, \tag{A10.5}$$

where $\mathbf{M}$ is the real 4×4 Mueller matrix corresponding to the Jones matrix in eq. (A10.1) and having elements given by

$$M_{00} = (aa^* + bb^* + cc^* + dd^*)/2 \quad = (A + B + C + D)/2, \tag{A10.6}$$

$$M_{01} = (aa^* - bb^* + cc^* - dd^*)/2 \quad = (A - B + C - D)/2, \tag{A10.7}$$

$$M_{02} = (ab^* + ba^* + cd^* + dc^*)/2 \quad = \mathrm{Re}[ab^* + cd^*], \tag{A10.8}$$

$$M_{03} = -i(ab^* - ba^* + cd^* - dc^*)/2 \quad = \mathrm{Im}[ab^* + cd^*], \tag{A10.9}$$

$$M_{10} = (aa^* + bb^* - cc^* - dd^*)/2 \quad = (A + B - C - D)/2, \tag{A10.10}$$

$$M_{11} = (aa^* - bb^* - cc^* + dd^*)/2 \quad = (A - B - C + D)/2, \tag{A10.11}$$

$$M_{12} = (ab^* + ba^* - cd^* + dc^*)/2 \quad = \mathrm{Re}[ab^* - cd^*], \tag{A10.12}$$

$$M_{13} = -i(ab^* - ba^* - cd^* + dc^*)/2 \quad = \mathrm{Im}[ab^* - cd^*], \tag{A10.13}$$

$$M_{20} = (ac^* + bd^* + ca^* + db^*)/2 \quad = \mathrm{Re}[ac^* + bd^*], \tag{A10.14}$$

$$M_{21} = (ac^* - bd^* - ca^* + db^*)/2 \quad = \mathrm{Re}[ac^* - bd^*], \tag{A10.15}$$

$$M_{22} = (ad^* + bc^* + cb^* + da^*)/2 \quad = \mathrm{Re}[ad^* + bc^*], \tag{A10.16}$$

$$M_{23} = -i(ad^* - bc^* + cb^* - da^*)/2 \quad = \mathrm{Im}[ad^* - bc^*], \tag{A10.17}$$

$$M_{30} = i(ac^* + bd^* - ca^* - db^*)/2 \quad = -\mathrm{Im}[ac^* + bd^*], \tag{A10.18}$$

$$M_{31} = i(ac^* - bd^* - ca^* - db^*)/2 \quad = -\mathrm{Im}[ac^* - bd^*], \tag{A10.19}$$

$$M_{32} = i(ad^* + bc^* - cb^* - da^*)/2 \quad = -\mathrm{Im}[ad^* + bc^*], \tag{A10.20}$$

$$M_{33} = (ad^* - bc^* - cb^* + da^*)/2 \quad = \mathrm{Re}[ad^* - bc^*], \tag{A10.21}$$

where

$$A = aa^*,\ B = bb^*,\ C = cc^*,\ D = dd^*. \tag{A10.22}$$

This result is consistent with that given by van de Hulst [12]. An important special case of interest in ellipsometry is the diagonal Jones matrix, for which $b = c = 0$. **M** then simplifies as follows:

$$\mathbf{M} = \begin{bmatrix} (A+D)/2 & (A-D)/2 & 0 & 0 \\ (A-D)/2 & (A+D)/2 & 0 & 0 \\ 0 & 0 & \mathrm{Re}(ad^*) & \mathrm{Im}(ad^*) \\ 0 & 0 & -\mathrm{Im}(ad^*) & \mathrm{Re}(ad^*) \end{bmatrix}. \tag{A10.23}$$

A11. Mueller matrix for oblique reflection from an isotropic surface

Jones matrix T: ($x = p$; $y = s$)

$$\mathrm{T}_{\mathrm{ref}} = \begin{bmatrix} r_p\,\mathrm{e}^{i\Delta_p} & 0 \\ 0 & r_s\,\mathrm{e}^{i\Delta_s} \end{bmatrix}. \tag{A11.1}$$

Thus, in eq. (A10.1), $a = r_p\,\mathrm{e}^{i\Delta_p}$, $b = c = 0$, $d = r_s\,\mathrm{e}^{i\Delta_s}$. Using the definition

$$\rho \equiv \tan\psi\,\mathrm{e}^{i\Delta} = (r_p/r_s)\,\mathrm{e}^{i(\Delta_p - \Delta_s)}, \tag{A11.2}$$

T may be written

$$\mathrm{T}_{\mathrm{ref}} = r_s\,\mathrm{e}^{i\Delta_s}\begin{bmatrix} \rho & 0 \\ 0 & 1 \end{bmatrix}, \tag{A11.3}$$

and by eq. (A10.23) the Mueller matrix for the reflecting surface is given by

$$\mathrm{M}_{\mathrm{ref}} = \frac{r_p^2 + r_s^2}{2}\begin{bmatrix} 1 & -\cos 2\psi & 0 & 0 \\ -\cos 2\psi & 1 & 0 & 0 \\ 0 & 0 & \sin 2\psi\cos\Delta & \sin 2\psi\sin\Delta \\ 0 & 0 & -\sin 2\psi\sin\Delta & \sin 2\psi\cos\Delta \end{bmatrix}. \tag{A11.4}$$

A12. Mueller matrix for linear retarder

Jones matrix T: ($x = f$, fast axis; $y = s$, slow axis; δ_f, δ_s represent retardations associated with the $e^{-i(2\pi z/\lambda)}$ term in the electric field)

$$\mathrm{T}_{\mathrm{ret}} = \begin{bmatrix} T_f\, e^{-i\delta_f} & 0 \\ 0 & T_s\, e^{-i\delta_s} \end{bmatrix}. \tag{A12.1}$$

Thus, in eq. (A10.1) $a = T_f\, e^{-i\delta_f}$, $b = c = 0$, $d = T_s\, e^{-i\delta_s}$. Using the definition

$$\rho_C \equiv \tan\psi_C\, e^{-i\delta_C} = (T_s/T_f)\, e^{-i(\delta_s - \delta_f)}, \tag{A12.2}$$

T may be written

$$\mathrm{T}_{\mathrm{ret}} = T_f\, e^{-i\delta_f} \begin{bmatrix} 1 & 0 \\ 0 & \rho_C \end{bmatrix}, \tag{A12.3}$$

and by eq. (A10.23) the Mueller matrix for the retarder is given by

$$\mathrm{M}_{\mathrm{ret}} = \frac{T_f^2 + T_s^2}{2} \begin{bmatrix} 1 & \cos 2\psi_C & 0 & 0 \\ \cos 2\psi_C & 1 & 0 & 0 \\ 0 & 0 & \sin 2\psi_C \cos\delta_C & \sin 2\psi_C \sin\delta_C \\ 0 & 0 & -\sin 2\psi_C \sin\delta_C & \sin 2\psi_C \cos\delta_C \end{bmatrix}. \tag{A12.4}$$

For an ideal quarter wave retarder, $\delta_C = 90°$ and $\psi_C = 45°$ ($T_s = T_f = T$). Equation (A12.4) then becomes

$$\mathrm{M}_{\mathrm{ret}} = T^2 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}. \tag{A12.5}$$

When the fast axis of the retarder has an azimuth $C \neq 0$, the coordinate system rotation matrices [eq. (A13.2)] are used to transform either of the above two expressions. The appropriate Mueller matrix is then given by $\mathbb{R}(-C)\, \mathrm{M}_{\mathrm{ret}}\, \mathbb{R}(C)$.

A13. Mueller-Stokes analysis of a compensator-analyzer polarimeter

As an illustration of the use of the proposed sign conventions, we derive the expression for the intensity transmitted by a linear retarder C and a linear analyzer A in terms of the Stokes parameters of the incident radiation. Various conditions under which the Stokes parameters may be determined are then described. The Stokes vector $\mathbf{S}'$ of the transmitted light in the analyzer reference frame is given by

$$\mathbf{S}' = \mathbf{M}_{\text{anal}}\, \mathbb{R}(A - C)\, \mathbf{M}_{\text{ret}}\, \mathbb{R}(C)\, \mathbf{S}, \tag{A13.1}$$

where C and A are respectively the azimuths of the retarder and analyzer, $\mathbb{R}$ is the coordinate system rotation matrix

$$\mathbb{R}(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\theta & \sin 2\theta & 0 \\ 0 & -\sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{A13.2}$$

and $\mathbf{M}_{\text{anal}}$ is the Mueller matrix of a linear analyzer:

$$\mathbf{M}_{\text{anal}} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{A13.3}$$

By combining eq. (A12.4) with eqs. (A13.1)-(A13.3), the detected intensity S_0' is given in terms of the input Stokes parameters by [25]

$$\begin{aligned} 4S_0'/(T_f^2 + T_s^2) = {} & S_0[s \cos(2C - 2A) + 1] \\ & + S_1[f \cos(4C - 2A) + s \cos 2C + (1 - f) \cos 2A] \\ & + S_2[f \sin(4C - 2A) + s \sin 2C + (1 - f) \sin 2A] \\ & + S_3[-r \sin(2C - 2A)], \end{aligned} \tag{A13.4}$$

where

$$s = \cos 2\psi_C \cong 0, \tag{A13.5}$$

$$p = \sin 2\psi_C \cos \delta_C \cong 0, \tag{A13.6}$$

$$r = \sin 2\psi_C \sin \delta_C \cong 1, \tag{A13.7}$$

$$f = (1 - p)/2 \cong 0.5. \tag{A13.8}$$

The approximations above are equalities for an ideal quarter-wave plate. In that case, using eq. (A12.5), and setting $T = 1$ and $A = 0$, eq. (A13.4) becomes

$$2S_0' = [S_0 + S_1/2] + [S_1/2] \cos 4C + [S_2/2] \sin 4C \; - \; S_3 \sin 2C \, . \tag{A13.9}$$

This equation shows that all four Stokes parameters may be determined from Fourier analysis of the intensity of light transmitted by a rotating compensator followed by a fixed linear analyzer [10b,25].

If A and δ_C are variable, $\psi_C = 45°$ and we fix $C = 0$, eq. (A13.4) becomes

$$2S_0' = S_0 + S_1 \cos 2A + [S_2 \cos \delta_C + S_3 \sin \delta_C] \sin 2A. \tag{A13.10}$$

This equation shows that one may use a fixed compensator with a rotating analyzer to determine two of the Stokes parameters directly plus a linear combination of the remaining two parameters with a single measurement [10].

If C is fixed with respect to A so that $(A - C) = 45°$, $\psi_C = 45°$ and δ_C is modulated, eq. (A13.4) becomes

$$2S_0' = S_0 + [S_1 \cos 2A + S_2 \sin 2A] \cos \delta_C + S_3 \sin \delta_C. \tag{A13.11}$$

From this we see that by using phase modulation, obtained for example by piezo-birefringence, [26] one may again determine three of the four input Stokes parameters with a single measurement. In either of the last two cases, a second measurement can be made to determine the remaining Stokes parameter.

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Figure captions

Fig. 1. Right-handed elliptical polarization. The instantaneous locus of the tip of the electric field vector forms a right-handed helix in space. As the helix propagates in the positive z direction, the point of intersection of the helix with a transverse plane (e.g. $z = 0$) traces an ellipse in a clockwise sense as the viewer faces the source.

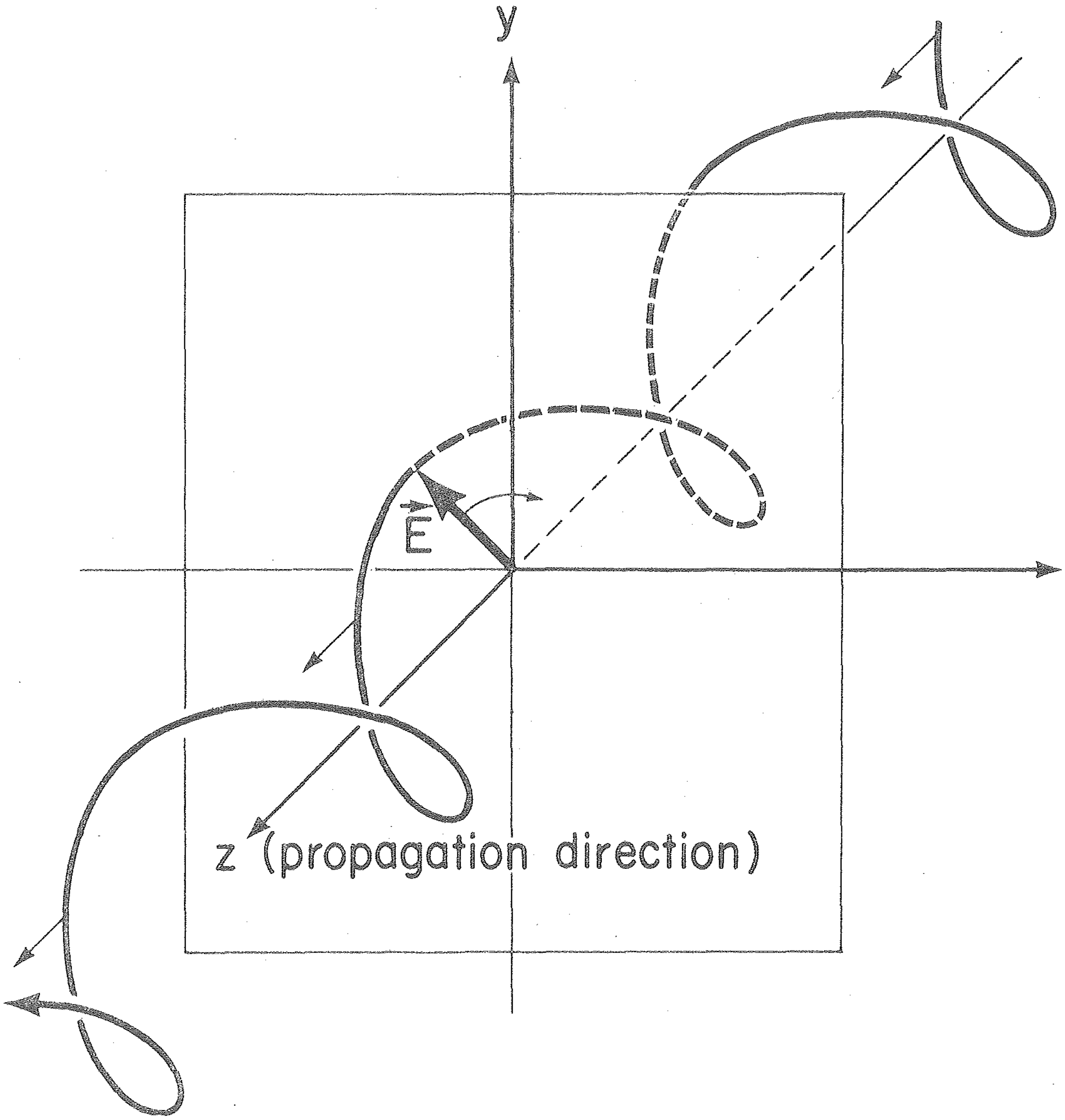
Fig. 2. Definition of azimuth θ and ellipticity ϵ. For right-handed polarization (as shown), both ϵ and S_3 are positive.

Fig. 3. Relationship of the Poincaré sphere to the Stokes parameters, azimuth θ and ellipticity ϵ of a polarization state represented by the point P. Azimuth increases to the east from a zero value at the meridian defined by $S_2 = 0$ and $S_1 > 0$. Ellipticity increases to the north from a zero value at the equator. The north pole represents right-handed circular polarization. The S_1, S_2 and S_3 axes form a right-handed Cartesian coordinate system.

Fig. 4. Relationship of the Poincaré sphere to the ellipsometric parameters ψ, Δ and the auxiliary angles α, δ (see text) of a polarization state represented by the point P. The angles 2α and 2ψ are measured to the line connecting P and the origin respectively from the positive and negative S_1 axes and are complementary. The angles Δ and δ give the inclination of the plane of the point P and the S_1 axis to the positive S_2 half of the equatorial plane. They are measured clockwise and counterclockwise respectively when viewed as shown. Their sum is 360°.

Fig. 5. The relationship of the unit-radius Poincaré sphere to the Cartesian complex polarization plane (χ plane.) The origin of the plane and the center of the sphere coincide, and the projection is made from the intersection of the negative S_1 axis and the equator. The projections of the real and imaginary axes of the plane respectively onto the equator and the 0° and 180° meridians of the sphere are shown.

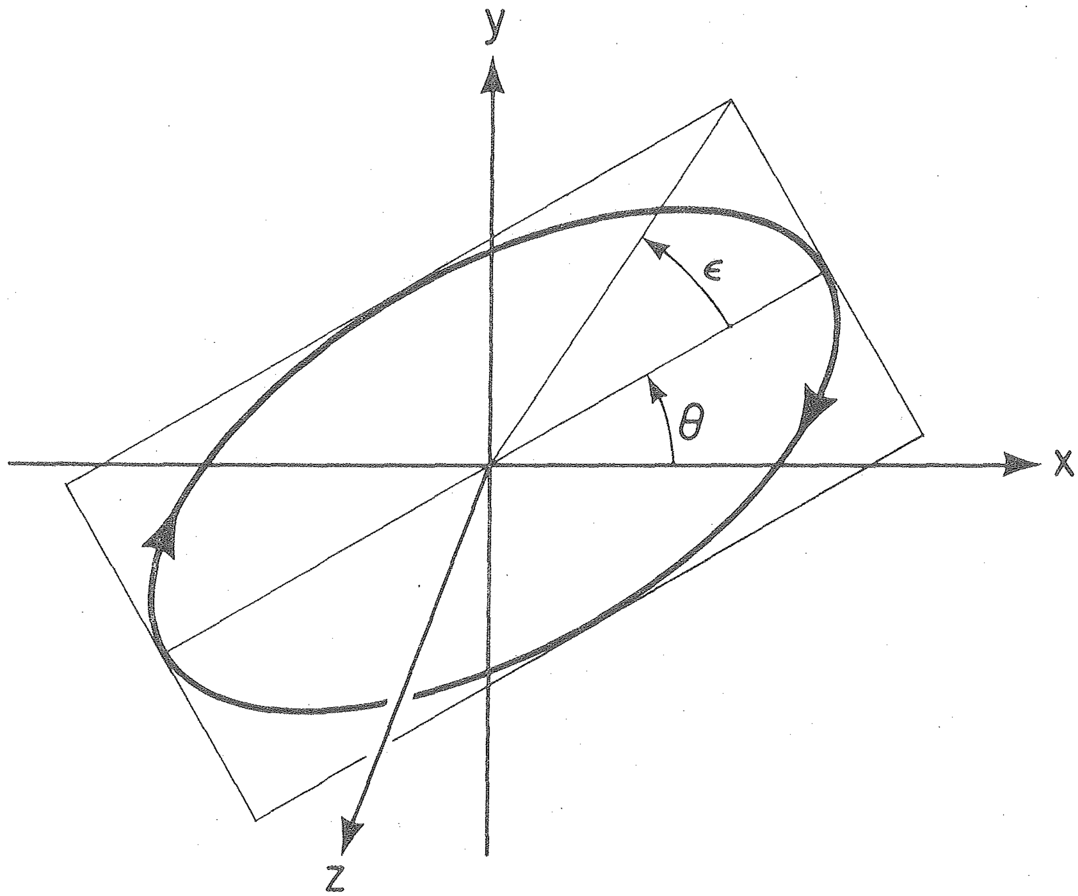
Fig. 6. The Poincaré sphere with the previously used conventions for azimuth θ and ellipticity angle γ. The north pole represents left-handed circular polarization. θ increases to the east, and γ (positive for left-handed polarizaion) increases to the north.



ELLIPTIC POLARIZATION
Right-handed helix

XBL 798-2629

FIGURE 1

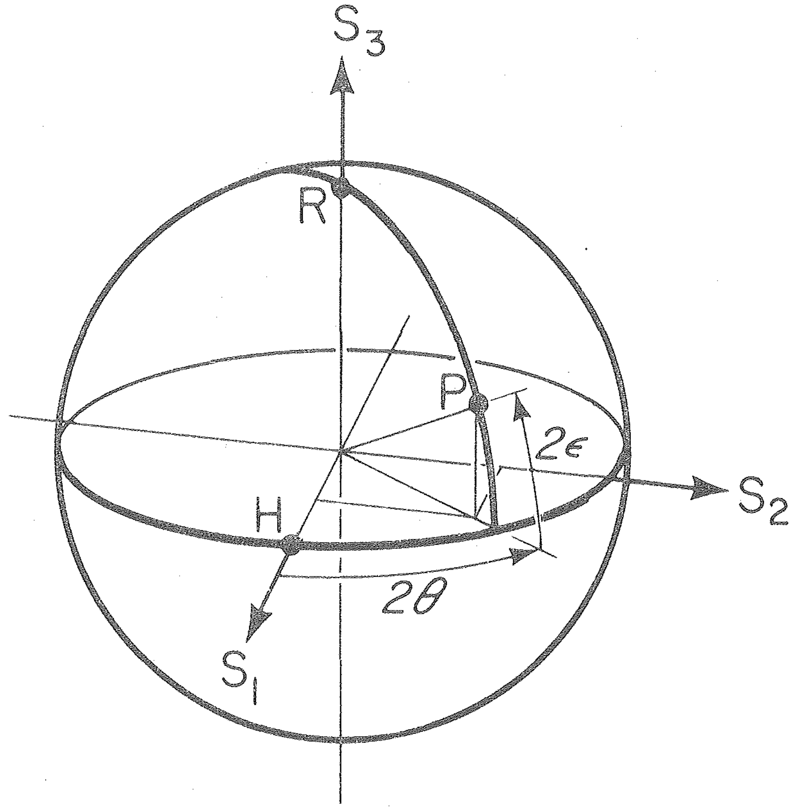


RIGHT-HANDED ELLIPSE

Azimuth θ and Ellipticity angle ϵ

XBL 798-2631

FIGURE 2

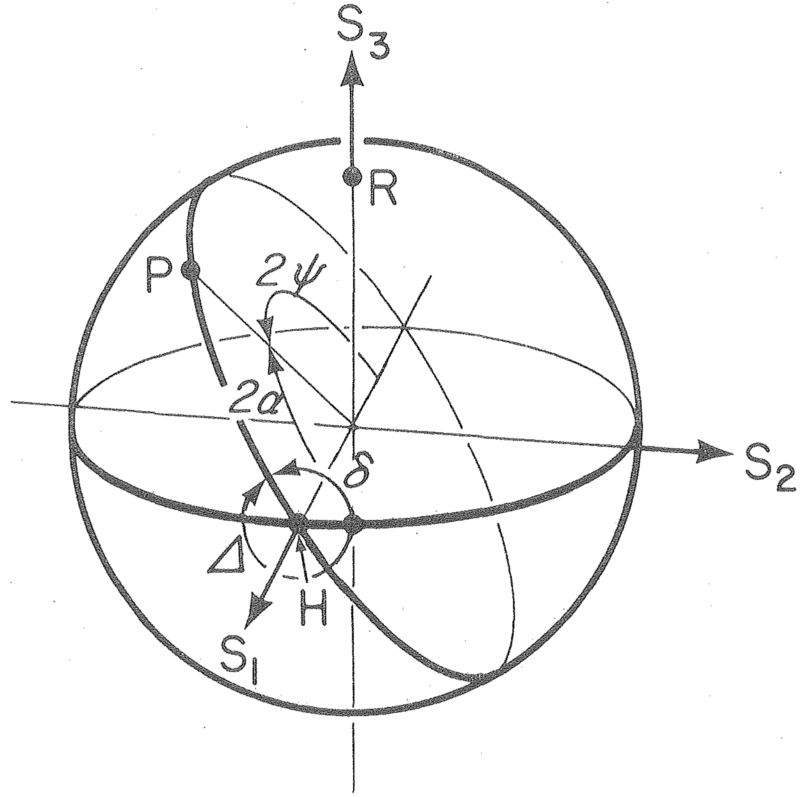


POINCARÉ SPHERE
State of polarization P

Stokes parameters, Azimuth θ and
Ellipticity angle ε

XBL 798-2633

FIGURE 3

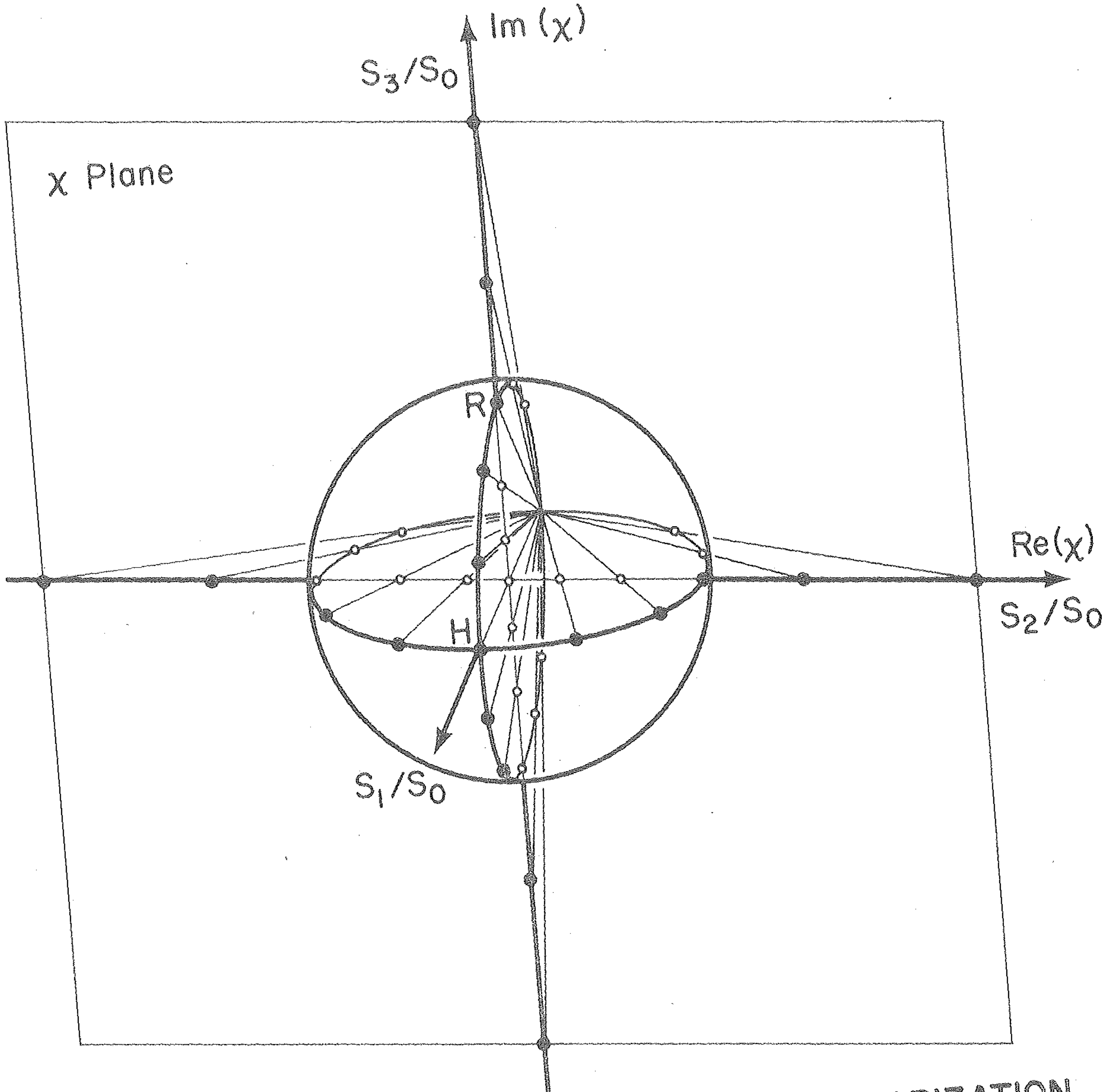


POINCARÉ SPHERE

Ellipsometer parameters ψ, Δ and auxiliary angles α, δ

XBL 798-2632

FIGURE 4



POINCARÉ SPHERE AND COMPLEX POLARIZATION PLANE

XBL 798-2630

FIGURE 5

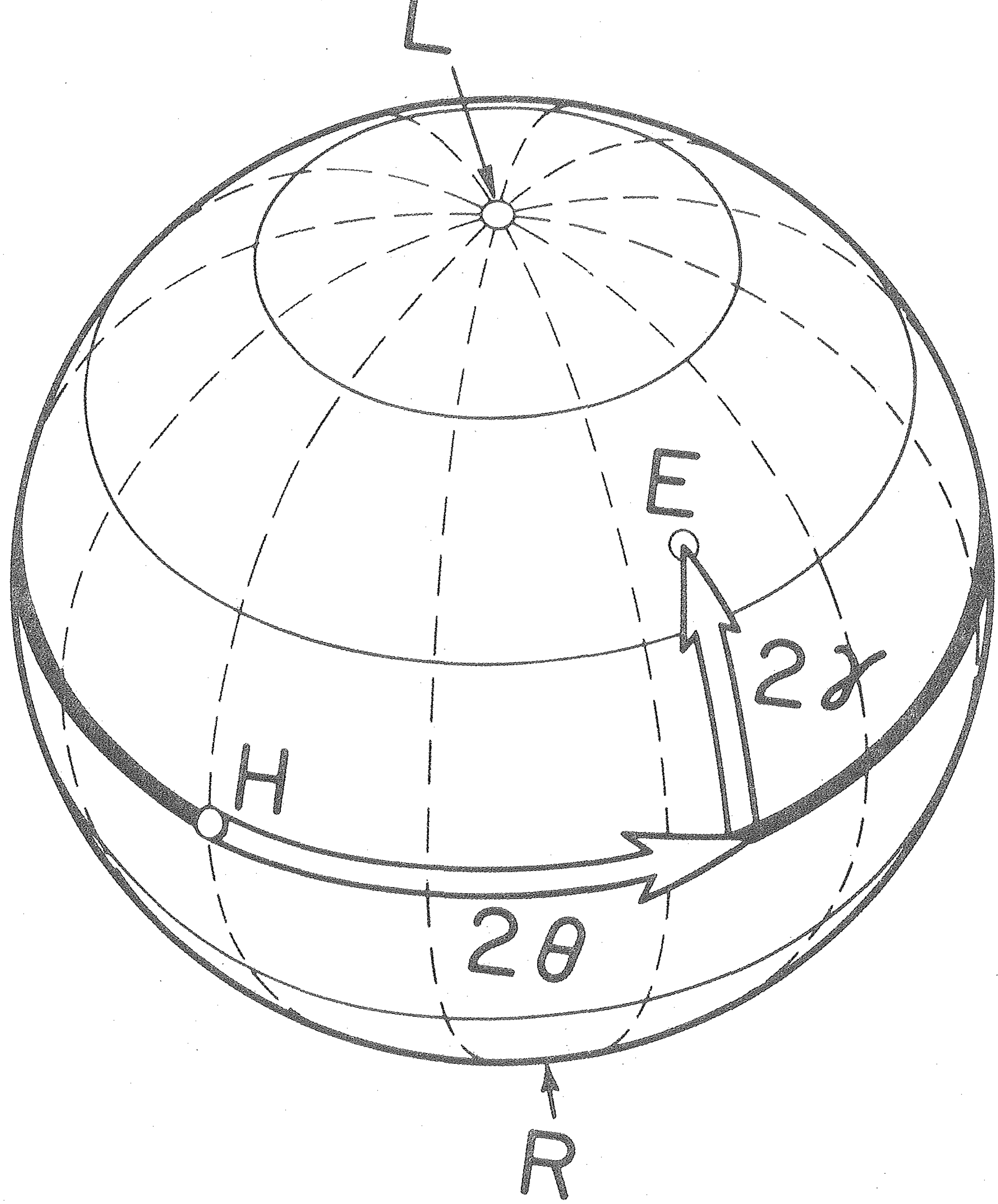


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FIGURE 6