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The Tightness of the Kesten-Stigum Reconstruction Bound

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The Tightness of the Kesten-Stigum Reconstruction Bound

A dissertation submitted in partial satisfaction
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Doctor of Philosophy in Mathematics

by

Wenjian Liu

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Abstract of the Dissertation

The Tightness of the Kesten-Stigum Reconstruction Bound

by

Wenjian Liu

Doctor of Philosophy in Mathematics

University of California, Los Angeles, 2013

Professor Kefeng Liu, Chair

It is well known that reconstruction problems, as the cross-disciplinary subject, have been studied in numerous contexts including statistical physics, information theory and computational biology. My major contributions to the this field are to figure out the tightness of the Kesten-Stigum reconstruction bound for both the $2q$-state symmetric model with triple mutation probabilities and the asymmetric binary channel on trees. Furthermore, we determine asymptotics for the reconstruction thresholds on regular trees of large degree.

The dissertation of Wenjian Liu is approved.

David Gieseker

Gang Liu

Hongquan Xu

Kefeng Liu, Committee Chair

University of California, Los Angeles

2013

To my family,

for their constant source of

love, concern, support and strength all these years.

TABLE OF CONTENTS

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VITA

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CHAPTER 1

INTRODUCTION

1.1 Preliminaries.

We start with the following broadcasting process that stands as a discrete, irreducible, aperiodic, and reversible Markov chain. Let $\mathbb{T} = (\mathbb{V}, \mathbb{E}, \rho)$ be a tree with nodes $\mathbb{V}$, edges $\mathbb{E}$ and root $\rho \in \mathbb{V}$. Each edge of the tree acts as a channel on a finite characters set $\mathcal{C}$, whose elements are configurations on $\mathbb{T}$, denoted by σ. We set a probability transition matrix $\mathbf{M} = (M_{ij})$ as the noisy communication channel on each edge. The state of the root ρ, denoted by σ_ρ, is chosen according to an initial distribution π on $\mathcal{C}$. This symbol is then propagated in the tree as follows. For each vertex v having as a parent u, the spin at v is defined according to the probabilities

$$\mathbf{P}(\sigma_v = j \mid \sigma_u = i) = M_{ij}$$

with $i, j \in \mathcal{C}$. Roughly speaking, the problem of reconstruction is the following: consider all the symbols received at the vertices of the nth generation. Does this configuration contain a non-vanishing information on the letter transmitted by the root, as n goes to ∞?

In this paper, we will restrict our attention to regular d-ary trees, that is the infinite rooted tree where every vertex has exactly d offspring(every vertex has degree $d+1$ except the root which has degree d). Let $\sigma(n)$ denote the spins at distance n from the root and let $\sigma^i(n)$ denote $\sigma(n)$ conditioned on $\sigma_\rho = i$. Two important channels will be mainly investigated in this thesis. First consider a characters set $\mathcal{C} = \mathcal{C}_1 \bigcup \mathcal{C}_2$, consisting of two categories $\mathcal{C}_1 = \{1, \ldots, q\}$ and $\mathcal{C}_2 = \{q+1, \ldots, 2q\}$ with $q \geq 2$, and the state of the root ρ is chosen according to the uniform distribution on $\mathcal{C}$. Moreover, a $2q \times 2q$ probability transition matrix $\mathbf{M} = (M_{ij})_{2q\times 2q}$ is defined as following:

$$M_{ij} = \begin{cases} p_0 & \text{if} \quad i = j, \\ \\ p_1 & \text{if } i \neq j \text{ and } i, j \text{ are in the same category,} \\ \\ p_2 & \text{if } i \neq j \text{ and } i, j \text{ are in different categories,} \end{cases}$$

where p_0, p_1 and p_2 are all nonnegative, as well as $p_0 + (q-1)p_1 + qp_2 = 1$.

The second model taken into account in Chapter 5 is the asymmetric binary channel with the configuration set $\mathcal{C} = \{1, 2\}$, whose transition matrix is of the form

$$\mathbf{M} = \frac{1}{2}\left[\begin{pmatrix} 1+\theta & 1-\theta \\ 1-\theta & 1+\theta \end{pmatrix} + \Delta \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\right].$$

First of all, we would like to give the formal definition of the reconstruction.

Definition 1.1.1. The reconstruction problem for the infinite tree $\mathbb{T}$ is *solvable* if for some $i, j \in \mathcal{C}$,

$$\limsup_{n\to\infty} d_{TV}(\sigma^i(n), \sigma^j(n)) > 0$$

where d_{TV} is the total variation distance. When the $\limsup$ is 0 we will say the model has *non-reconstruction* on $\mathbb{T}$.

For any channel $\mathbf{M}$, it is well known that the reconstruction problem is connected tightly to λ, the second largest eigenvalue by absolute value of $\mathbf{M}$. An important general bound was obtained by Kesten and Stigum (KS)[1][2]: the reconstruction problem is solvable if $d|\lambda|^2 > 1$(λ may be a complex number), which is known as the Kesten-Stigum bound. On the other side, for larger noise ($d|\lambda|^2 < 1$) one may wonder whether reconstruction is possible exploiting the whole set of symbols received at the nth generation, through a clever use of the correlations between the symbols received on the leaves. The answer depends on the channel. A particularly important example is provided by q-state symmetric channels, that is, Potts channels in the statistical mechanics terminology, with the transition matrix

$$\mathbf{M} = \begin{pmatrix} p_0 & p_1 & \cdots & p_1 \\ p_1 & p_0 & \cdots & p_1 \\ \vdots & \vdots & \ddots & \vdots \\ p_1 & p_1 & \cdots & p_0 \end{pmatrix}$$

whose eigenvalues are $\lambda = p_0 - p_1$ and 1. In [3], this model has been completely investigated by means of the recursive structure of the tree, and more importantly, the author successfully accomplished that non-reconstruction is equivalent to $\lim_{n\to\infty} x_n = 0$, where $x_n = \mathbf{EP}(\sigma_\rho = 1 \mid \sigma^1(n)) - \frac{1}{q}$. Therefore the key method is to analyze the recursion relation between x_n and x_{n+1}. As a result, most parts of the conjecture in [4] have been solved as follows.

Theorem. *When $q \geq 5$, for every d the Kesten-Stigum bound is not sharp. Moreover for fixed q, a precise asymptotic result for the threshold Λ of reconstruction as d goes to infinity:*

$$\lim_{d\to\infty} d\Lambda^2 = C_q$$

where $0 < C_q < 1$ is a constant depending on q. When $q = 3$ there exists a $d_{\min}$ such that for $d \geq d_{\min}$ the Kesten-Stigum bound is sharp for both the ferromagnetic and antiferrmagnetic channels.

In contrast to the symmetric model in [3], our $2q$ model turns out to be much more challenging, because of an additional class of mutation that would complicate the second largest eigenvalue λ. Despite of these troubles, inspired by the techniques in [3], we could still refer to the recursion approach, however, it would need much improvements, such as introducing additional auxiliary parameters y_n and z_n after x_n. More specifically, the blueprint of my proof is to research a second order recursion relation, which can be counted as a nonlinear dynamical system.

There is no difficulty of calculating the eigenvalues of $2q$ symmetric channel $\mathbf{M}$: first set

$$\mathbf{A} = \begin{pmatrix} p_0-\lambda & p_1 & \cdots & p_1 \\ p_1 & p_0-\lambda & \cdots & p_1 \\ \vdots & \vdots & \ddots & \vdots \\ p_1 & p_1 & \cdots & p_0-\lambda \end{pmatrix}_{q\times q} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} p_2 & p_2 & \cdots & p_2 \\ p_2 & p_2 & \cdots & p_2 \\ \vdots & \vdots & \ddots & \vdots \\ p_2 & p_2 & \cdots & p_2 \end{pmatrix}_{q\times q}.$$

and then

$$
\begin{aligned}
& \det(\mathbf{M}-\lambda\mathbf{I}) \\
= \ & \det\begin{pmatrix}
p_0-\lambda & p_1 & \cdots & p_1 & p_2 & p_2 & \cdots & p_2 \\
p_1 & p_0-\lambda & \cdots & p_1 & p_2 & p_2 & \cdots & p_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
p_1 & p_1 & \cdots & p_0-\lambda & p_2 & p_2 & \cdots & p_2 \\
p_2 & p_2 & \cdots & p_2 & p_0-\lambda & p_1 & \cdots & p_1 \\
p_2 & p_2 & \cdots & p_2 & p_1 & p_0-\lambda & \cdots & p_1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
p_2 & p_2 & \cdots & p_2 & p_1 & p_1 & \cdots & p_0-\lambda
\end{pmatrix} \\
= \ & \det\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{A} \end{pmatrix} \\
= \ & \det(\mathbf{A}+\mathbf{B})\det(\mathbf{A}-\mathbf{B}) \\
= \ & [(p_0+p_2-\lambda)+(q-1)(p_1+p_2)][(p_0-p_2-\lambda)+(q-1)(p_1-p_2)](p_0-p_1-\lambda)^{2(q-1)}.
\end{aligned}
$$

Therefore the eigenvalues are $\lambda_1 = p_0 - p_1$, $\lambda_2 = p_0 + (q-1)p_1 - qp_2$, and $\lambda_3 = p_0 + (q-1)p_1 + qp_2 = 1$, where we have two candidates λ_1 and λ_2 for λ, the second largest eigenvalue by absolute value, by which a big trouble is caused. Because non-reconstruction happens at most $d|\lambda|^2 = 1$, in order to investigate the tightness of the Kesten-Stigum bound, without loss of generality, assume $\frac{1}{2} \le d|\lambda|^2 \le 1$ in the following context.

1.2 Background.

Beyond fundamental interest of determining the reconstruction threshold of a Markov random field in probability, this problem is relevant to statistical physics, biology, and information theory(the problem being equivalent to computing the information capacity of the tree network), and so on.

The reconstruction problem is intimately related to statistical physics. Consider a model of Potts spins $\mathcal{C} = \{1, \ldots, q\}$ on a finite rooted tree with n generations, denoted by $\mathbb{T}_n$ Suppose that the energy of a configuration $\underline{x} = \{x_i \in \mathcal{C} : i \in \mathbb{T}_n\}$ is given by:

$$E(\underline{x}) = -J \sum_{(i,j)\in\mathbb{T}_n} \delta_{x_i x_j}$$

where (i, j) denotes pairs of spins connected by an edge of the tree. Let $\underline{X}$ be the random configuration produced by the broadcast process with the symmetric Potts channel up to generation n, when the transmitted symbol is uniformly random in $\mathcal{C}$. Then

$$\mathbf{P}(\underline{X} = \underline{x}) = \frac{1}{Z} e^{-\beta E(\underline{x})}$$

where the normalizing constant $Z = Z(\beta)$ is the partition function and the free parameter β is the inverse temperature [5]. In other words, the broadcast process allows to construct one particular Gibbs measure (state) associated to the energy function: in the statistical physics terminology this is the free-boundary measure. On an infinite tree the Gibbs measure is not the unique for this energy function. Even if more than one Gibbs state exists, the free-boundary state can be extremal. It turns out that the reconstruction problem is solvable if and only if the Gibbs state with free boundary conditions is not extremal.

Given the strong connection between extremality of Gibbs states and spatial decay of correlations [6], there is the relation of the reconstructibility with the existence of a dynamical glass phase. In recent years, an ongoing effort has been devoted to the study of glassy models on sparse random graphs. These are graphs which contain cycles but locally look like a tree (e.g. uniformly random graphs with given degree). One of the most widespread features of these models, is the occurrence of glass phases in which the Boltzmann measure gets split into an exponential number of lumps (also referred to as clusters or pure states). This phenomenon is usually studied by solving some one-step replica symmetry breaking (1RSB) distributional equations. It was shown that these equations, as well as the criterion used to detect glass phases, do indeed coincide with the solvability of an appropriate reconstruction problem.

Reconstruction thresholds on trees are also believed to determine the dynamic phase transitions in many constraint satisfaction problems including random K-SAT and random colorings on random graphs[7][8]. The reconstruction threshold is also believed to play an important role in the efficiency of the Glauber dynamics on trees and random graphs. In [9] it was shown that the mixing time for the the Glauber dynamics on trees is $n^{1+\Omega(1)}$ when the model has reconstruction and slower than at higher temperature when the mixing time is $O(n \log n)$. In the case of the Ising model this is tight, the mixing time is $O(n \log n)$ when $d\lambda^2 < 1$.

Phylogenetic reconstruction is a major task of systematic biology [10]. It was shown in [11] that for binary symmetric channels, also called CFN models in evolutionary biology, the sampling efficiency of phylogenetic reconstruction is determined by the reconstruction threshold. Thus, if for all edges of the tree, it holds that $2\theta^2 > 1$ the tree can be recovered efficiently from $O(\log n)$ samples. If $2\theta^2 < 1$, then [12] implies that $n\Omega(1)$ samples are needed. In fact, the proof of the lower bound in [12] implies the lower bound $n\Omega(1)$ whenever the reconstruction problem is "exponentially" unsolvable. The results in [13] imply $n\Omega(1)$ lower bounds for phylogenetic reconstruction for asymmetric channels such that $2\theta^2 < 1$ and δ sufficiently small. The details are omitted from this extended abstract. It is natural to conjecture that this is tight and that if $2\theta^2 > 1$ then phylogenetic reconstruction may be achieved with $O(\log n)$ sequences. Indeed my results in this thesis could improve the preceding topic further.

1.3 Main Results.

From Chapter 2 to Chapter 4, we will specify the $2q$ symmetric model first.

Theorem 1.3.1. *Assume $|\lambda_2| \leq |\lambda_1|$. When $q \geq 4$, for every d the Kesten-Stigum bound is not sharp, that is, the reconstruction is solvable for some $\lambda = \lambda_1$ even if $d\lambda_1^2 < 1$.*

The case of $q = 3$ is kind of the threshold with respect to q, so we need more delicate analysis to figure out the reconstruction. Therefore we defer this discussion to a subsequent paper. On the other side, if q is small, say, $q = 2$, there would be no longer the reconstruction for large degree.

Theorem 1.3.2. *When $q = 2$, there exists a $D > 0$ such that for $d \geq D$ the Kesten-Stigum bound is sharp, that is, there is non-reconstruction whenever $|\lambda| < d^{-\frac{1}{2}}$.*

In light of the techniques in symmetric models, it is feasible to deal with the asymmetric binary channel with the initial distribution $\pi = (\pi_1, \pi_2)$ analogously. It is easy to see that the second largest eigenvalue by absolute value is θ. Proposition 12 of [14] implies that for any asymmetric channel, given d and π, reconstruction is monotone in $|\theta|$, say, there exist the thresholds $\theta^- < 0 < \theta^+$ such that there is non-reconstruction when $\theta \in (\theta^-, \theta^+)$ while it is reconstructible given $\theta < \theta^-$ or $\theta > \theta^+$.

Theorem 1.3.3. *When $\Delta^2 > \frac{1}{3}(1-\theta)^2$, for every d the Kesten-Stigum bound is not sharp.*

Theorem 1.3.4. *When $\Delta^2 < \frac{1}{3}(1-\theta)^2$, there exists a $D = D(\pi) > 0$ such that for $d > D$ the Kesten-Stigum bound is tight.*

Furthermore with the assistance of the central limit theorem and gaussian approximation, we could figured out the precise asymptotic result for the threshold $\Theta = \theta^+$ or θ^- for fixed π as d goes to infinity.

Theorem 1.3.5. *When $\Delta^2 > \frac{1}{3}(1-\theta)^2$, $\lim_{d\to\infty} d\Theta^2 = C_\pi$, where $0 < C_\pi < 1$ is a constant depending only on π.*

CHAPTER 2

SECOND ORDER RECURSION RELATION

2.1 Notations.

Before further reading, let's introduce the notation in our proofs. Let $u_1, \ldots, u_d$ be the children of ρ and $\mathbb{T}_v$ be the subtree of descendants of $v \in \mathbb{T}$. Furthermore, if we set $d(\cdot, \cdot)$ as the graph-metric distance on $\mathbb{T}$, denote the nth level of the tree by $L_n = \{v \in \mathbb{V} : d(\rho, v) = n\}$. With the notation above, let $\sigma(n)$ and $\sigma_j(n)$ denote the spins on L_n and $L_n \cap \mathbb{T}_{u_j}$ respectively. For a configuration A on L_n define the posterior function

$$f_n(i, A) = \mathbf{P}(\sigma_\rho = i \mid \sigma(n) = A).$$

By the recursive nature of the tree for a configuration A on $L(n+1) \cap \mathbb{T}_{u_j}$ we can further give the equivalent form

$$f_n(i, A) = \mathbf{P}(\sigma_{u_j} = i \mid \sigma_j(n+1) = A).$$

Now for any $1 \leq i \leq 2q$, define a collection of random variables

$$X_i(n) = f_n(i, \sigma(n))$$

to describe the posterior probability of state i at the root given the random configuration $\sigma(n)$ of the leaves, and analogously,

$$X^{(1)}(n) = f_n(1, \sigma^1(n)),$$

$$X^{(2)}(n) = f_n(2, \sigma^1(n)),$$

and

$$X^{(3)}(n) = f_n(q+1, \sigma^1(n)).$$

By symmetry the collections $\{f_n(i,\sigma^1(n)) : 2 \le i \le q\}$ and $\{f_n(i,\sigma^1(n)) : q+1 \le i \le 2q\}$ are exchangeable respectively, in addition, $f_n(j,\sigma^i(n))$ is distributed as

$$f_n(j,\sigma^i(n)) \sim \begin{cases} X^{(1)}(n) & \text{if } i = j, \\ \\ X^{(2)}(n) & \text{if } i \neq j \text{ are in the same category,} \\ \\ X^{(3)}(n) & \text{if } i \neq j \text{ are in different categories.} \end{cases}$$

Lase denote the corresponding moments of former random variables, which would be the main objectives in this paper,

$$x_n = \mathbf{E}\left(X^{(1)}(n) - \frac{1}{2q}\right), \quad y_n = \mathbf{E}\left(X^{(2)}(n) - \frac{1}{2q}\right), \quad z_n = \mathbf{E}\left(X^{(3)}(n) - \frac{1}{2q}\right),$$

and

$$u_n = \mathbf{E}\left(X^{(1)}(n) - \frac{1}{2q}\right)^2, \quad v_n = \mathbf{E}\left(X^{(2)}(n) - \frac{1}{2q}\right)^2, \quad w_n = \mathbf{E}\left(X^{(3)}(n) - \frac{1}{2q}\right)^2.$$

2.2 Basic Facts.

First from the symmetric property of the tree and the fact $\sum_{i=1}^{2q} X_i(n) \equiv 1$, it follows that for any $1 \le i \le 2q$ and nonnegative $n \in \mathbb{Z}$, $\mathbf{E}X_i(n) = \frac{1}{2q}$ is always true and

$$\left(x_n + \frac{1}{2q}\right) + (q-1)\left(y_n + \frac{1}{2q}\right) + q\left(z_n + \frac{1}{2q}\right) = 1,$$

which implies

$$x_n + (q-1)y_n + qz_n = 0.$$

Besides, there are also some nontrivial results concerning these quantities.

Lemma 2.2.1. *For any $n \in \mathbb{N}\bigcup\{0\}$, we have*

$$x_n = \mathbf{E}\sum_{i=1}^{2q}\left(X_i(n) - \frac{1}{2q}\right)^2 \ge 0,$$

meanwhile,

$$z_n \le 0; \quad x_n + z_n \ge 0.$$

Proof. Starting with the definition and applying the Bayes' rule yield

$$\begin{aligned}
x_n + \frac{1}{2q} &= \mathbf{E}X^{(1)}(n) \\
&= \mathbf{E}\mathbf{P}(\sigma_\rho = 1 \mid \sigma^1(n)) \\
&= \sum_A f_n(1,A)\mathbf{P}(\sigma(n) = A \mid \sigma_\rho = 1) \\
&= \sum_A \frac{\mathbf{P}(\sigma_\rho = 1 \bigcap \sigma(n) = A)}{\mathbf{P}(\sigma_\rho = 1)} f_n(1,A) \\
&= 2q \sum_A \mathbf{P}(\sigma(n) = A) f_n^2(1,A) \\
&= 2q\mathbf{E}X_1^2(n) \\
&= \sum_{i=1}^{2q} \mathbf{E}X_i^2(n),
\end{aligned}$$

and thus

$$0 \le \mathbf{E}\sum_{i=1}^{2q}\left(X_i(n) - \frac{1}{2q}\right)^2 = \sum_{i=1}^{2q}\mathbf{E}X_i^2(n) - \frac{2}{2q}\sum_{i=1}^{2q}\mathbf{E}X_i(n) + \frac{1}{2q} = x_n. \tag{2.2.1}$$

Next let's consider the covariance matrix of random variables $\left\{X_i(n) - \frac{1}{2q}\right\}_1^{2q}$. First point taken into account is to express covariances in terms of x_n, y_n and z_n. The same trick above is on again,

$$\begin{aligned}
y_n + \frac{1}{2q} &= \mathbf{E}X^{(2)}(n) \\
&= \sum_A f_n(2,A)\mathbf{P}(\sigma(n) = A \mid \sigma_\rho = 1) \\
&= 2q\sum_A \mathbf{P}(\sigma(n) = A) f_n(1,A) f_n(2,A) \\
&= 2q\mathbf{E}X_1(n)X_2(n),
\end{aligned}$$

so for any $i_1 < i_2$ in the same category, it is concluded that, from the symmetric property of the tree,

$$\mathbf{E}\left(X_{i_1}(n) - \frac{1}{2q}\right)\left(X_{i_2}(n) - \frac{1}{2q}\right) = \mathbf{E}\left(X_1 - \frac{1}{2q}\right)\left(X_2 - \frac{1}{2q}\right) = \mathbf{E}X_1X_2 - \frac{1}{4q^2} = \frac{y_n}{2q}.$$

Similarly as above,

$$z_n = 2q\mathbf{E}X_1(n)X_{q+1}(n)$$

and thus

$$\mathbf{E}\left(X_{i_1}(n)-\frac{1}{2q}\right)\left(X_{i_2}(n)-\frac{1}{2q}\right)=\frac{z_n}{2q}.$$

Therefore the covariance matrix is

$$\Sigma_X(n)=\begin{pmatrix}
\frac{x_n}{2q} & \frac{y_n}{2q} & \cdots & \frac{y_n}{2q} & \frac{z_n}{2q} & \frac{z_n}{2q} & \cdots & \frac{z_n}{2q}\\
\frac{y_n}{2q} & \frac{x_n}{2q} & \cdots & \frac{y_n}{2q} & \frac{z_n}{2q} & \frac{z_n}{2q} & \cdots & \frac{z_n}{2q}\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\
\frac{y_n}{2q} & \frac{y_n}{2q} & \cdots & \frac{x_n}{2q} & \frac{z_n}{2q} & \frac{z_n}{2q} & \cdots & \frac{z_n}{2q}\\
\frac{z_n}{2q} & \frac{z_n}{2q} & \cdots & \frac{z_n}{2q} & \frac{x_n}{2q} & \frac{y_n}{2q} & \cdots & \frac{y_n}{2q}\\
\frac{z_n}{2q} & \frac{z_n}{2q} & \cdots & \frac{z_n}{2q} & \frac{y_n}{2q} & \frac{x_n}{2q} & \cdots & \frac{y_n}{2q}\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\
\frac{z_n}{2q} & \frac{z_n}{2q} & \cdots & \frac{z_n}{2q} & \frac{y_n}{2q} & \frac{y_n}{2q} & \cdots & \frac{x_n}{2q}
\end{pmatrix}_{2q\times 2q}$$

whose eigenvalues are 0, $\frac{x_n+(q-1)y_n-qz_n}{2q}$, and $\frac{x_n-y_n}{2q}$ respectively. It is well known that the covariance matrix of a multivariate probability distribution is always positive semi-definite, which implies that all eigenvalues are nonnegative, i.e. $x_n+(q-1)y_n-qz_n\geq 0$ and $x_n-y_n\geq 0$. Combining these results and the fact $x_n+(q-1)y_n+qz_n=0$ completes the proof. □

Next it would be of interest to reveal the relation identities between means and covariances.

Lemma 2.2.2. *For any $n\in\mathbb{N}\bigcup\{0\}$, we have*

(i) $x_n=u_n+(q-1)v_n+qw_n$;

(ii) $\mathbf{E}\left(X^{(1)}(n)-\frac{1}{2q}\right)\left(X^{(2)}(n)-\frac{1}{2q}\right)=v_n+\frac{y_n-x_n}{2q}$;

(iii) $\mathbf{E}\left(X^{(1)}(n)-\frac{1}{2q}\right)\left(X^{(3)}(n)-\frac{1}{2q}\right)=w_n+\frac{z_n-x_n}{2q}$;

(iv) $\mathbf{E}\left(X^{(2)}(n)-\frac{1}{2q}\right)\left(X^{(3)}(n)-\frac{1}{2q}\right)=-\frac{w_n}{q-1}-\frac{z_n}{2q(q-1)}-\frac{y_n}{2q}$;

(v) $\mathbf{E}\left(f_n(q+1,\sigma^1(n))-\frac{1}{2q}\right)\left(f_n(2q,\sigma^1(n))-\frac{1}{2q}\right)=-\frac{w_n}{q-1}-\frac{z_n}{2(q-1)}$;

(vi) $\mathbf{E}\left(f_n(2,\sigma^1(n))-\frac{1}{2q}\right)\left(f_n(q,\sigma^1(n))-\frac{1}{2q}\right)=-\frac{2v_n}{q-2}-\frac{z_n}{2(q-1)}+\frac{qw_n}{(q-1)(q-2)}$.

Proof. Let's display our discussion in virtue of the total probability formula, starting with the identity (2.2.1),

$$\begin{aligned}
x_n &= \mathbf{E}\sum_{i=1}^{2q}\left(X_i(n)-\frac{1}{2q}\right)^2 \\
&= \sum_{j=1}^{2q}\mathbf{E}\left(\sum_{i=1}^{2q}\left(X_i(n)-\frac{1}{2q}\right)^2 \mid \sigma_\rho = j\right)\mathbf{P}(\sigma_\rho = j) \\
&= \sum_{j=1}^{2q}\frac{1}{2q}\left[\mathbf{E}\left(X^{(1)}(n)-\frac{1}{2q}\right)^2+(q-1)\mathbf{E}\left(X^{(2)}(n)-\frac{1}{2q}\right)^2+q\mathbf{E}\left(X^{(3)}(n)-\frac{1}{2q}\right)^2\right] \\
&= \mathbf{E}\left(X^{(1)}(n)-\frac{1}{2q}\right)^2+(q-1)\mathbf{E}\left(X^{(2)}(n)-\frac{1}{2q}\right)^2+q\mathbf{E}\left(X^{(3)}(n)-\frac{1}{2q}\right)^2 \\
&= u_n+(q-1)v_n+qw_n.
\end{aligned}$$

Applying the analogous trick in Lemma 2.2.1 gives

$$\begin{aligned}
& \mathbf{E}X^{(1)}(n)X^{(2)}(n) \\
= & \sum_A \mathbf{P}(\sigma_\rho = 1 \mid \sigma(n)=A)\mathbf{P}(\sigma_\rho = 2 \mid \sigma(n)=A)P(\sigma(n)=A \mid \sigma_\rho = 1) \\
= & \sum_A [\mathbf{P}(\sigma_\rho = 1 \mid \sigma(n)=A)]^2\mathbf{P}(\sigma(n)=A \mid \sigma_\rho = 2) \\
= & \mathbf{E}\left(X^{(2)}(n)\right)^2
\end{aligned}$$

and hence (ii) follows immediately as

$$\mathbf{E}\left(X^{(1)}(n)-\frac{1}{2q}\right)\left(X^{(2)}(n)-\frac{1}{2q}\right) = \mathbf{E}\left(X^{(2)}-\frac{1}{2q}\right)^2+\frac{y_n-x_n}{2q} = v_n+\frac{y_n-x_n}{2q}.$$

Similarly, (iii) turns out to be true, by means of $\mathbf{E}X^{(1)}X^{(3)} = \mathbf{E}\left(X^{(3)}\right)^2$. Finally (iv), (v) and (vi) can be handled as before, plus considering the symmetry,

$$\begin{aligned}
& \mathbf{E}X^{(2)}X^{(3)} \\
= & \sum_A \mathbf{P}(\sigma_\rho = 2 \mid \sigma(n)=A)\mathbf{P}(\sigma_\rho = q+1 \mid \sigma(n)=A)\mathbf{P}(\sigma(n)=A \mid \sigma_\rho = 1) \\
= & \sum_A \mathbf{P}(\sigma_\rho = 1 \mid \sigma(n)=A)\mathbf{P}(\sigma_\rho = 2 \mid \sigma(n)=A)\mathbf{P}(\sigma(n)=A \mid \sigma_\rho = q+1) \\
= & \sum_A \mathbf{P}(\sigma_\rho = q+1 \mid \sigma(n)=A)\mathbf{P}(\sigma_\rho = 2q \mid \sigma(n)=A)\mathbf{P}(\sigma(n)=A \mid \sigma_\rho = 1) \\
= & \mathbf{E}f_n(q+1,\sigma^1(n))f_n(2q,\sigma^1(n)).
\end{aligned}$$

To obtain the expression of $\mathbf{E}X^{(2)}X^{(3)}$, first recall

$$\begin{aligned} z_n + \frac{1}{2q} &= \mathbf{E}X^{(3)}(n) \\ &= \mathbf{E}f_n(q+1,\sigma^1(n))\sum_{i=1}^{2q} f(i,\sigma^1(n)) \\ &= \mathbf{E}X^{(1)}(n)X^{(3)}(n) + (q-1)\mathbf{E}X^{(2)}(n)X^{(3)}(n) + \mathbf{E}(X^{(3)})^2 \\ &\quad +(q-1)\mathbf{E}f_n(q+1,\sigma^1(n))f_n(2q,\sigma^1(n)) \\ &= \mathbf{E}X^{(1)}(n)X^{(3)}(n) + 2(q-1)\mathbf{E}X^{(2)}(n)X^{(3)}(n) + \mathbf{E}(X^{(3)})^2, \end{aligned}$$

and it implies that $\mathbf{E}X^{(2)}X^{(3)} = \frac{1}{2(q-1)}\left(z_n + \frac{1}{2q} - 2\mathbf{E}(X^{(3)})^2\right)$, further leading to

$$\begin{aligned} &\mathbf{E}\left(X^{(2)} - \frac{1}{2q}\right)\left(X^{(3)} - \frac{1}{2q}\right) \\ = \;& \frac{1}{2(q-1)}\left(z_n + \frac{1}{2q} - 2\mathbf{E}\left(X^{(3)} - \frac{1}{2q}\right)^2 - \frac{2z_n}{q} - \frac{1}{2q^2}\right) - \frac{1}{2q}\left(y_n + \frac{1}{2q} + z_n + \frac{1}{2q}\right) + \frac{1}{4q^2} \\ = \;& -\frac{w_n}{q-1} - \frac{z_n}{2q(q-1)} - \frac{y_n}{2q} \end{aligned}$$

and thus

$$\begin{aligned} &\mathbf{E}\left(f_n(q+1,\sigma^1(n)) - \frac{1}{2q}\right)\left(f_n(2q,\sigma^1(n)) - \frac{1}{2q}\right) \\ =\;& \mathbf{E}f_n(q+1,\sigma^1(n))f_n(2q,\sigma^1(n)) - \frac{2}{2q}\left(z_n + \frac{1}{2q}\right) + \frac{1}{4q^2} \\ =\;& \mathbf{E}X^{(2)}X^{(3)} - \frac{z_n}{q} - \frac{1}{4q^2} \\ =\;& \mathbf{E}\left(X^{(2)} - \frac{1}{2q}\right)\left(X^{(3)} - \frac{1}{2q}\right) + \frac{1}{2q}\mathbf{E}(X^{(2)} + X^{(3)}) - \frac{1}{4q^2} - \frac{z_n}{q} - \frac{1}{4q^2} \\ =\;& -\frac{w_n}{q-1} - \frac{z_n}{2q(q-1)} - \frac{y_n}{2q} + \frac{y_n}{2q} - \frac{z_n}{2q} \\ =\;& -\frac{w_n}{q-1} - \frac{z_n}{2(q-1)}. \end{aligned}$$

As in the preceding discussion, starting with

$$\begin{aligned} y_n + \frac{1}{2q} &= \mathbf{E}X^{(2)}(n) \\ &= \mathbf{E}f_n(2,\sigma^1(n))\sum_{i=1}^{2q} f(i,\sigma^1(n)) \\ &= \mathbf{E}X^{(1)}(n)X^{(2)}(n) + \mathbf{E}(X^{(2)})^2 + (q-2)\mathbf{E}f_n(2,\sigma^1(n))f_n(q,\sigma^1(n)) + q\mathbf{E}X^{(2)}X^{(3)} \\ &= 2\mathbf{E}(X^{(2)})^2 + (q-2)\mathbf{E}f_n(2,\sigma^1(n))f_n(q,\sigma^1(n)) + q\mathbf{E}X^{(2)}X^{(3)}, \end{aligned}$$

we conclude

$$\mathbf{E}f_n(2,\sigma^1(n))f_n(q,\sigma^1(n)) = \frac{1}{q-2}\left[y_n + \frac{1}{2q} - 2\mathbf{E}(X^{(2)})^2 - q\mathbf{E}X^{(2)}X^{(3)}\right],$$

as well as,

$$\begin{aligned}
& \mathbf{E}\left(f_n(2,\sigma^1(n)) - \frac{1}{2q}\right)\left(f_n(q,\sigma^1(n)) - \frac{1}{2q}\right) \\
= & \frac{1}{q-2}\left(y_n + \frac{1}{2q} - 2\mathbf{E}(X^{(2)})^2 - q\mathbf{E}X^{(2)}X^{(3)}\right) - \frac{2}{2q}\left(y_n + \frac{1}{2q}\right) + \frac{1}{4q^2} \\
= & \frac{1}{q-2}\left[y_n + \frac{1}{2q} - 2\left(v_n + \frac{y_n}{q} + \frac{1}{4q^2}\right) - \frac{q}{2(q-1)}\left(z_n + \frac{1}{2q} - 2w_n - \frac{2z_n}{q} - \frac{1}{2q^2}\right)\right] \\
& -\frac{y_n}{q} - \frac{1}{4q^2} \\
= & -\frac{2v_n}{q-2} - \frac{z_n}{2(q-1)} + \frac{qw_n}{(q-1)(q-2)}.
\end{aligned}$$

□

2.3 Means and Covariances of Y_{ij}.

Define

$$Y_{ij}(n) = f_n\left(i, \sigma_j^1(n+1)\right),$$

and taking advantage of the symmetries of the model, it is apparent that for $j = 1,\ldots,d$, the random vectors $(Y_{ij})_{i=1}^{2q}$ are independent.

Since the moments of Y_{ij} would play a key role in further calculation, it is necessary to figure out them in the first place.

(i) for each $1 \le j \le d$,

$$\begin{aligned}
& \mathbf{E}\left(Y_{1j}(n) - \frac{1}{2q}\right) \\
= & p_0\mathbf{E}\left(X^{(1)}(n) - \frac{1}{2q}\right) + (q-1)p_1\mathbf{E}\left(X^{(2)}(n) - \frac{1}{2q}\right) + qp_2\mathbf{E}\left(X^{(3)}(n) - \frac{1}{2q}\right) \\
= & p_0x_n + (q-1)p_1y_n + qp_2z_n \\
= & \lambda_1 x_n + (\lambda_1 - \lambda_2)z_n;
\end{aligned}$$

(ii) for $2 \le i \le q$,

$$\begin{aligned}
& \mathbf{E}\left(Y_{ij}(n) - \frac{1}{2q}\right) \\
= \;& p_1 \mathbf{E}\left(X^{(1)}(n) - \frac{1}{2q}\right) + [p_0 + (q-2)p_1]\mathbf{E}\left(X^{(2)}(n) - \frac{1}{2q}\right) + qp_2\mathbf{E}\left(X^{(3)}(n) - \frac{1}{2q}\right) \\
= \;& p_1 x_n + [p_0 + (q-2)p_1] y_n + qp_2 z_n \\
= \;& -\frac{\lambda_1}{q-1} x_n - \frac{\lambda_1 + (q-1)\lambda_2}{q-1} z_n;
\end{aligned}$$

(iii) for $q+1 \le i \le 2q$, by means of the identity $\sum_{i=1}^{2q} Y_{ij}(n) \equiv 1$, it follows immediately

$$\mathbf{E}\left(Y_{ij}(n) - \frac{1}{2q}\right) = -\frac{1}{q}\sum_{i=1}^{q} \mathbf{E}\left(Y_{ij}(n) - \frac{1}{2q}\right) = \lambda_2 z_n;$$

(iv) when $i = 1$,

$$\begin{aligned}
& \mathbf{E}\left(Y_{1j}(n) - \frac{1}{2q}\right)^2 \\
= \;& p_0 \mathbf{E}\left(X^{(1)}(n) - \frac{1}{2q}\right)^2 + (q-1)p_1\mathbf{E}\left(X^{(2)}(n) - \frac{1}{2q}\right)^2 + qp_2\mathbf{E}\left(X^{(3)}(n) - \frac{1}{2q}\right)^2 \\
= \;& p_0 u_n + (q-1)p_1 v_n + qp_2 w_n \\
= \;& \frac{1 + \lambda_2 - 2\lambda_1}{2q} x_n + \lambda_1 u_n + (\lambda_1 - \lambda_2) w_n;
\end{aligned}$$

(v) for $2 \le i \le q$,

$$\begin{aligned}
& \mathbf{E}\left(Y_{ij}(n) - \frac{1}{2q}\right)^2 \\
= \;& p_1 \mathbf{E}\left(X^{(1)}(n) - \frac{1}{2q}\right)^2 + [p_0 + (q-2)p_1]\mathbf{E}\left(X^{(2)}(n) - \frac{1}{2q}\right)^2 + qp_2\mathbf{E}\left(X^{(3)}(n) - \frac{1}{2q}\right)^2 \\
= \;& p_1 u_n + [p_0 + (q-2)p_1] v_n + qp_2 w_n \\
= \;& \left(\frac{1}{2q} + \frac{\lambda_2}{2q} + \frac{\lambda_1}{q(q-1)}\right) x_n - \frac{\lambda_1}{q-1} u_n - \frac{\lambda_1 + (q-1)\lambda_2}{q-1} w_n;
\end{aligned}$$

(vi) for $q+1 \le i \le 2q$,

$$\begin{aligned}
& \mathbf{E}\left(Y_{ij}(n) - \frac{1}{2q}\right)^2 \\
= \;& p_2 \mathbf{E}\left(X^{(1)}(n) - \frac{1}{2q}\right)^2 + (q-1)p_2\mathbf{E}\left(X^{(2)}(n) - \frac{1}{2q}\right)^2 + [p_0 + (q-1)p_1]\mathbf{E}\left(X^{(3)}(n) - \frac{1}{2q}\right)^2 \\
= \;& p_2 u_n + (q-1)p_2 v_n + [p_0 + (q-1)p_1] w_n \\
= \;& \frac{1 - \lambda_2}{2q} x_n + \lambda_2 w_n;
\end{aligned}$$

(vii) for $2 \le i \le q$,

$$\begin{aligned}
& \mathbf{E}\left(Y_{1j}(n) - \frac{1}{2q}\right)\left(Y_{ij}(n) - \frac{1}{2q}\right) \\
= \;& (p_0 + p_1)\mathbf{E}\left(X^{(1)}(n) - \frac{1}{2q}\right)\left(X^{(2)}(n) - \frac{1}{2q}\right) \\
& +(q-2)p_1\mathbf{E}\left(f_n(2, \sigma^1(n)) - \frac{1}{2q}\right)\left(f_n(q, \sigma^1(n)) - \frac{1}{2q}\right) \\
& +qp_2\mathbf{E}\left(f_n(q+1, \sigma^1(n)) - \frac{1}{2q}\right)\left(f_n(2q, \sigma^1(n)) - \frac{1}{2q}\right) \\
= \;& (p_0 - p_1)v_n + \frac{q(p_1 - p_2)}{q-1}w_n - \frac{(q-2)p_1 + qp_2}{2(q-1)}z_n + \frac{p_0 + p_1}{2q}(y_n - x_n) \\
= \;& \frac{(q+2)\lambda_1 - \lambda_2 - 1}{2q(q-1)}x_n - \frac{z_n}{2(q-1)} - \frac{\lambda_1}{q-1}u_n - \frac{(q+1)\lambda_1 - \lambda_2}{q-1}w_n;
\end{aligned}$$

(viii) for $q+1 \le i \le 2q$,

$$\begin{aligned}
& \mathbf{E}\left(Y_{1j}(n) - \frac{1}{2q}\right)\left(Y_{ij}(n) - \frac{1}{2q}\right) \\
= \;& (p_0 + p_2)\mathbf{E}\left(X^{(1)}(n) - \frac{1}{2q}\right)\left(X^{(3)}(n) - \frac{1}{2q}\right) \\
& +[(q-1)p_1 + (q-1)p_2]\mathbf{E}\left(X^{(2)}(n) - \frac{1}{2q}\right)\left(X^{(3)}(n) - \frac{1}{2q}\right) \\
= \;& (p_0 + p_2)\left[w_n + \frac{1}{2q}(z_n - x_n)\right] + (q-1)(p_1 + p_2)\left[-\frac{w_n}{q-1} - \frac{z_n}{2q(q-1)} - \frac{y_n}{2q}\right] \\
= \;& -\frac{\lambda_1}{2q}x_n + \frac{z_n}{2q} + \lambda_1 w_n;
\end{aligned}$$

(ix) for $1 < i_1 < i_2 \le q$,

$$\begin{aligned}
& \mathbf{E}\left(Y_{i_1 j}(n) - \frac{1}{2q}\right)\left(Y_{i_2 j}(n) - \frac{1}{2q}\right) \\
= \;& [p_0 + (q-3)p_1]\mathbf{E}\left(f_n(2, \sigma^1(n)) - \frac{1}{2q}\right)\left(f_n(q, \sigma^1(n)) - \frac{1}{2q}\right) \\
& +2p_1\mathbf{E}\left(X^{(1)}(n) - \frac{1}{2q}\right)\left(X^{(2)}(n) - \frac{1}{2q}\right) \\
& +qp_2\mathbf{E}\left(f_n(q+1, \sigma^1(n)) - \frac{1}{2q}\right)\left(f_n(2q, \sigma^1(n)) - \frac{1}{2q}\right) \\
= \;& \left[-\frac{2(q+2)\lambda_1 + (q-2)\lambda_2}{2q(q-1)(q-2)} - \frac{1}{2q(q-1)}\right]x_n - \frac{z_n}{2(q-1)} \\
& +\frac{2\lambda_1}{(q-1)(q-2)}u_n + \frac{2(q+1)\lambda_1 + (q-2)\lambda_2}{(q-1)(q-2)}w_n;
\end{aligned}$$

(x) for $1 < i_1 \le q < i_2 \le 2q$,

$$
\begin{aligned}
& \mathbf{E}\left(Y_{i_1 j}(n) - \frac{1}{2q}\right)\left(Y_{i_2 j}(n) - \frac{1}{2q}\right) \\
= \; & [p_0 + (q-2)p_1]\mathbf{E}\left(X^{(2)}(n) - \frac{1}{2q}\right)\left(X^{(3)}(n) - \frac{1}{2q}\right) \\
& + p_1 \mathbf{E}\left(X^{(1)}(n) - \frac{1}{2q}\right)\left(X^{(3)}(n) - \frac{1}{2q}\right) + p_2 \mathbf{E}\left(X^{(1)}(n) - \frac{1}{2q}\right)\left(X^{(3)}(n) - \frac{1}{2q}\right) \\
& + (q-1)p_2 \mathbf{E}\left(X^{(2)}(n) - \frac{1}{2q}\right)\left(X^{(3)}(n) - \frac{1}{2q}\right) \\
= \; & [p_0 + (q-2)p_1 + (q-1)p_2]\mathbf{E}\left(X^{(2)}(n) - \frac{1}{2q}\right)\left(X^{(3)}(n) - \frac{1}{2q}\right) \\
& + (p_1 + p_2)\mathbf{E}\left(X^{(1)}(n) - \frac{1}{2q}\right)\left(X^{(3)}(n) - \frac{1}{2q}\right) \\
= \; & [p_0 + (q-2)p_1 + (q-1)p_2]\left[-\frac{w_n}{q-1} - \frac{z_n}{2q(q-1)} - \frac{y_n}{2q}\right] \\
& + (p_1 + p_2)\left[w_n + \frac{1}{2q}(z_n - x_n)\right] \\
= \; & -\frac{p_0 - p_1}{q-1} w_n - \frac{p_0 - p_1}{2q(q-1)} z_n - \frac{p_0 + (q-2)p_1 + (q-1)p_2}{2q} y_n - \frac{p_1 + p_2}{2q} x_n \\
= \; & \frac{\lambda_1}{2q(q-1)} x_n + \frac{z_n}{2q} - \frac{\lambda_1}{q-1} w_n;
\end{aligned}
$$

(xi) for $q + 1 \le i_1 < i_2 \le 2q$,

$$
\begin{aligned}
& \mathbf{E}\left(Y_{i_1 j}(n) - \frac{1}{2q}\right)\left(Y_{i_2 j}(n) - \frac{1}{2q}\right) \\
= \; & [p_0 + (q-1)p_1]\mathbf{E}\left(f_n(q+1, \sigma^1(n)) - \frac{1}{2q}\right)\left(f_n(2q, \sigma^1(n)) - \frac{1}{2q}\right) \\
& + (q-2)p_2 \mathbf{E}\left(f_n(2, \sigma^1(n)) - \frac{1}{2q}\right)\left(f_n(q, \sigma^1(n)) - \frac{1}{2q}\right) \\
& + 2p_2 \mathbf{E}\left(X^{(1)}(n) - \frac{1}{2q}\right)\left(X^{(2)}(n) - \frac{1}{2q}\right) \\
= \; & [p_0 + (q-1)p_1]\left(-\frac{w_n}{q-1} - \frac{z_n}{2(q-1)}\right) \\
& + (q-2)p_2\left[-\frac{2v_n}{q-2} - \frac{z_n}{2(q-1)} + \frac{q w_n}{(q-1)(q-2)}\right] + 2p_2\left[v_n + \frac{1}{2q}(y_n - x_n)\right] \\
= \; & -\frac{p_0 + (q-1)p_1 - q p_2}{q-1} w_n - \frac{p_0 + (q-1)p_1 + (q-2)p_2}{2(q-1)} z_n + \frac{p_2}{q}(y_n - x_n) \\
= \; & \frac{\lambda_2 - 1}{2q(q-1)} x_n - \frac{z_n}{2(q-1)} - \frac{\lambda_2}{q-1} w_n.
\end{aligned}
$$

2.4 Distributional Recursion.

The following standard relation follows from the Markov random field property:

$$
\begin{aligned}
& f_{n+1}(1, A) \\
= \ & \mathbf{P}(\sigma_\rho = 1 \mid \sigma_{n+1} = A) \\
= \ & \frac{\mathbf{P}(\sigma_\rho = 1 \bigcap \sigma_{n+1} = A)}{\mathbf{P}(\sigma_{n+1} = A)} \\
= \ & \frac{\mathbf{P}(\sigma_\rho = 1 \bigcap \sigma_{n+1} = A)}{\sum_{i=1}^{2q} \mathbf{P}(\sigma_\rho = i \bigcap \sigma_{n+1} = A)} \\
= \ & \frac{\prod_{j=1}^{d} \left[p_0 f_n(1, A_j) + p_1 \sum_{i=2}^{q} f_n(i, A_j) + p_2 \sum_{i=q+1}^{2q} f_n(i, A_j)\right]}{\sum_{i=1}^{2q} \prod_{j=1}^{d} \sum_{\ell=1}^{2q} M_{i\ell} f(\ell, A_j)} \\
= \ & \frac{\prod_{j=1}^{d} \left[1 + 2q(p_0 - p_2)\left(f_n(1, A_j) - \frac{1}{2q}\right) + 2q(p_1 - p_2) \sum_{i=2}^{q} \left(f_n(i, A_j) - \frac{1}{2q}\right)\right]}{\sum_{i=1}^{2q} \prod_{j=1}^{d} \sum_{\ell=1}^{2q} M_{i\ell} f(\ell, A_j)} \\
= \ & \left\{\prod_{j=1}^{d} \left[1 + 2q(p_0 - p_2)\left(f_n(1, A_j) - \frac{1}{2q}\right) + 2q(p_1 - p_2) \sum_{i=2}^{q} \left(f_n(i, A_j) - \frac{1}{2q}\right)\right]\right\} \times \\
& \left\{\sum_{i=1}^{q} \prod_{j=1}^{d} \left[p_0 f_n(i, A_j) + p_1 \sum_{1 \le \ell \ne i \le q} f_n(\ell, A_j) + p_2 \sum_{\ell=q+1}^{2q} f_n(\ell, A_j)\right]\right. \\
& \left. + \sum_{i=q+1}^{2q} \prod_{j=1}^{d} \left[p_0 f_n(i, A_j) + p_1 \sum_{q+1 \le \ell \ne i \le 2q} f_n(\ell, A_j) + p_2 \sum_{\ell=1}^{q} f_n(\ell, A_j)\right]\right\}^{-1} \\
= \ & \left\{\prod_{j=1}^{d} \left[1 + 2q(p_0 - p_2)\left(f_n(1, A_j) - \frac{1}{2q}\right) + 2q(p_1 - p_2) \sum_{i=2}^{q} \left(f_n(i, A_j) - \frac{1}{2q}\right)\right]\right\} \times \\
& \left\{\sum_{i=1}^{q} \prod_{j=1}^{d} \left[1 + 2q(p_0 - p_2)\left(f_n(i, A_j) - \frac{1}{2q}\right) + 2q(p_1 - p_2) \sum_{1 \le \ell \ne i \le q} \left(f_n(\ell, A_j) - \frac{1}{2q}\right)\right]\right. \\
& \left. + \sum_{i=q+1}^{2q} \prod_{j=1}^{d} \left[1 + 2q(p_0 - p_2)\left(f_n(i, A_j) - \frac{1}{2q}\right) + 2q(p_1 - p_2) \sum_{q+1 \le \ell \ne i \le 2q} \left(f_n(\ell, A_j) - \frac{1}{2q}\right)\right]\right\}^{-1}
\end{aligned}
$$

and similarly,

$$
\begin{aligned}
& f_{n+1}(q+1, A) \\
= \; & \mathbf{P}(\sigma_\rho = q+1 \mid \sigma_{n+1} = A) \\
= \; & \left\{ \prod_{j=1}^{d} \left[1 + 2q(p_0 - p_2)\left(f_n(2, A_j) - \frac{1}{2q} \right) + 2q(p_1 - p_2) \sum_{1 \le i \ne 2 \le q} \left(f_n(i, A_j) - \frac{1}{2q} \right) \right] \right\} \times \\
& \left\{ \sum_{i=1}^{q} \prod_{j=1}^{d} \left[1 + 2q(p_0 - p_2)\left(f_n(i, A_j) - \frac{1}{2q} \right) + 2q(p_1 - p_2) \sum_{1 \le \ell \ne i \le q} \left(f_n(\ell, A_j) - \frac{1}{2q} \right) \right] \right. \\
& \left. + \sum_{i=q+1}^{2q} \prod_{j=1}^{d} \left[1 + 2q(p_0 - p_2)\left(f_n(i, A_j) - \frac{1}{2q} \right) + 2q(p_1 - p_2) \sum_{q+1 \le \ell \ne i \le 2q} \left(f_n(\ell, A_j) - \frac{1}{2q} \right) \right] \right\}^{-1}.
\end{aligned}
$$

Next consider the preceding recursive formula by taking $A = \sigma^1(n+1)$, then there are two random variables

$$
X^{(1)}(n+1) = f_{n+1}(1, \sigma^1(n+1)) = \frac{Z_1}{\sum_{i=1}^{2q} Z_i}
$$

and

$$
X^{(3)}(n+1) = f_{n+1}(q+1, \sigma^1(n+1)) = \frac{Z_{q+1}}{\sum_{i=1}^{2q} Z_i},
$$

where

(a) if $1 \le i \le q$

$$
\begin{aligned}
Z_i = Z_i(n) &= \prod_{j=1}^{d} \left[1 + 2q(p_0 - p_2)\left(Y_{ij} - \frac{1}{2q} \right) + 2q(p_1 - p_2) \sum_{1 \le \ell \ne i \le q} \left(Y_{\ell j} - \frac{1}{2q} \right) \right] \\
&= \prod_{j=1}^{d} \left[1 + 2q(p_0 - p_1)\left(Y_{ij} - \frac{1}{2q} \right) - 2q(p_1 - p_2) \sum_{q+1 \le \ell \le 2q} \left(Y_{\ell j} - \frac{1}{2q} \right) \right] \\
&= \prod_{j=1}^{d} \left[1 + 2q\lambda_1 \left(Y_{ij} - \frac{1}{2q} \right) + 2(\lambda_1 - \lambda_2) \sum_{q+1 \le \ell \le 2q} \left(Y_{\ell j} - \frac{1}{2q} \right) \right]
\end{aligned}
$$

(b) if $q+1 \le i \le 2q$

$$
\begin{aligned}
Z_i = Z_i(n) &= \prod_{j=1}^{d} \left[1 + 2q(p_0 - p_2)\left(Y_{ij} - \frac{1}{2q} \right) + 2q(p_1 - p_2) \sum_{q+1 \le \ell \ne i \le 2q} \left(Y_{\ell j} - \frac{1}{2q} \right) \right] \\
&= \prod_{j=1}^{d} \left[1 + 2q(p_0 - p_1)\left(Y_{ij} - \frac{1}{2q} \right) - 2q(p_1 - p_2) \sum_{1 \le \ell \le q} \left(Y_{\ell j} - \frac{1}{2q} \right) \right] \\
&= \prod_{j=1}^{d} \left[1 + 2q\lambda_1 \left(Y_{ij} - \frac{1}{2q} \right) + 2(\lambda_1 - \lambda_2) \sum_{1 \le \ell \le q} \left(Y_{\ell j} - \frac{1}{2q} \right) \right].
\end{aligned}
$$

To carry out the further proof, it is convenient to derive some identities of $Z_i(n)$'s.

Lemma 2.4.1. *For any nonnegative* $n \in \mathbb{Z}$ *and* $1 \le i \le 2q$*, we have*

$$\mathbf{E}Z_1(n)Z_i(n) = \mathbf{E}Z_i(n)^2$$

and given any $2 \le i_1 \le q < q+1 \le i_2 \le 2q$*, then*

$$\mathbf{E}Z_{i_1}(n)Z_{i_2}(n) = \mathbf{E}Z_{q+1}(n)Z_{2q}(n).$$

Proof. When $i = 1$, the result is trivial. If $2 \le i \le 2q$, for any configurations $A = (A_1, \ldots, A_d)$ on the $(n+1)$th level, where A_j denote the spins on $L_{n+1} \cap \mathbb{T}_{u_j}$,we have

$$Z_i(A) = 2q\frac{\mathbf{P}(\sigma(n+1) = A)}{\prod_{j=1}^{d}\mathbf{P}(\sigma_j(n+1) = A_j)}\mathbf{P}(\sigma_\rho = i \mid \sigma(n+1) = A)$$

By the symmetry of the tree,

$$\begin{aligned}
& \mathbf{E}Z_1Z_i \\
= \;& (2q)^2\sum_A\left(\frac{\mathbf{P}(\sigma(n+1) = A)}{\prod_{j=1}^{d}\mathbf{P}(\sigma_j(n+1) = A_j)}\right)^2\mathbf{P}(\sigma_\rho = 1 \mid \sigma(n+1) = A)\mathbf{P}(\sigma_\rho = i \mid \sigma(n+1) = A) \\
& \cdot\mathbf{P}(\sigma(n+1) = A \mid \sigma_\rho = 1) \\
= \;& (2q)^2\sum_A\left(\frac{\mathbf{P}(\sigma(n+1) = A)}{\prod_{j=1}^{d}\mathbf{P}(\sigma_j(n+1) = A_j)}\right)^2\mathbf{P}^2(\sigma_\rho = 1 \mid \sigma(n+1) = A)\mathbf{P}(\sigma(n+1) = A \mid \sigma_\rho = i) \\
= \;& (2q)^2\sum_A\left(\frac{\mathbf{P}(\sigma(n+1) = A)}{\prod_{j=1}^{d}\mathbf{P}(\sigma_j(n+1) = A_j)}\right)^2\mathbf{P}^2(\sigma_\rho = i \mid \sigma(n+1) = A)\mathbf{P}(\sigma(n+1) = A \mid \sigma_\rho = 1) \\
= \;& \mathbf{E}Z_i^2.
\end{aligned}$$

Similarly, given arbitrary $2 \le i_1 \le q < i_2 \le 2q$,

$$\begin{aligned}
& \mathbf{E}Z_{i_1}Z_{i_2} \\
= \;& (2q)^2\sum_A\left(\frac{\mathbf{P}(\sigma(n+1) = A)}{\prod_{j=1}^{d}\mathbf{P}(\sigma_j(n+1) = A_j)}\right)^2\mathbf{P}(\sigma_\rho = 1 \mid \sigma(n+1) = A)\mathbf{P}(\sigma_\rho = i_1 \mid \sigma(n+1) = A) \\
& \cdot\mathbf{P}(\sigma(n+1) = A \mid \sigma_\rho = i_2) \\
= \;& (2q)^2\sum_A\left(\frac{\mathbf{P}(\sigma(n+1) = A)}{\prod_{j=1}^{d}\mathbf{P}(\sigma_j(n+1) = A_j)}\right)^2\mathbf{P}(\sigma_\rho = q+1 \mid \sigma(n+1) = A)\mathbf{P}(\sigma_\rho = 2q \mid \sigma(n+1) = A) \\
& \cdot\mathbf{P}(\sigma(n+1) = A \mid \sigma_\rho = 1) \\
= \;& \mathbf{E}Z_{q+1}Z_{2q}.
\end{aligned}$$

□

2.5 Main Expansion of x_{n+1} and z_{n+1}.

In this section, we are about to figure out the second order recursion relation of x_{n+1} and z_{n+1}, specifically speaking, their major expansions, which would play a crucial rule in our further discussion. First let's take care of approximating means and variances of Z_is by extending the results in [3, Lemma 2.6.] and Taylor series approximations.

Lemma 2.5.1. *Recall that $|\lambda| = \max\{|\lambda_1|, |\lambda_2|\}$. For $1 \le j \le d$ and $1 \le i \le i' \le 2q$, there exists a constant $C = C(q)$ such that*

$$\mathbf{E}\left[2q\lambda_1\left(Y_{ij} - \frac{1}{2q}\right) + 2(\lambda_1 - \lambda_2)\sum_{\substack{\forall \ell \in \mathcal{C}_i \\ i \notin \mathcal{C}_i}}\left(Y_{\ell j} - \frac{1}{2q}\right)\right] \le C\lambda^2 x_n$$

and

$$\Bigg|\mathbf{E}\left[2q\lambda_1\left(Y_{i1} - \frac{1}{2q}\right) + 2(\lambda_1 - \lambda_2)\sum_{\substack{\forall \ell \in \mathcal{C}_i \\ i \notin \mathcal{C}_i}}\left(Y_{\ell 1} - \frac{1}{2q}\right)\right]$$
$$\left[2q\lambda_1\left(Y_{i'1} - \frac{1}{2q}\right) + 2(\lambda_1 - \lambda_2)\sum_{\substack{\forall \ell \in \mathcal{C}_{i'} \\ i' \notin \mathcal{C}_{i'}}}\left(Y_{\ell 1} - \frac{1}{2q}\right)\right]\Bigg| \le C\lambda^2 x_n.$$

Proof. The key of the proof is to estimate means and covariances of the previous random variables, denoted by Λ_is and ①s respectively:

$$\begin{aligned}
\Lambda_1 &= \mathbf{E}\left[2q\lambda_1\left(Y_{1j} - \frac{1}{2q}\right) + 2(\lambda_1 - \lambda_2)\sum_{q+1 \le \ell \le 2q}\left(Y_{\ell j} - \frac{1}{2q}\right)\right] \\
&= 2q\lambda_1\left[\lambda_1 x_n + (\lambda_1 - \lambda_2) z_n\right] + 2q(\lambda_1 - \lambda_2)\lambda_2 z_n \\
&= 2q\lambda_1^2 x_n + 2q(\lambda_1^2 - \lambda_2^2) z_n;
\end{aligned}$$

for any $2 \le i \le q$,

$$\begin{aligned}
\Lambda_2 &= \mathbf{E}\left[2q\lambda_1\left(Y_{ij} - \frac{1}{2q}\right) + 2(\lambda_1 - \lambda_2)\sum_{q+1 \le \ell \le 2q}\left(Y_{\ell j} - \frac{1}{2q}\right)\right] \\
&= 2q\lambda_1\left[-\frac{\lambda_1}{q-1} x_n - \frac{\lambda_1 + (q-1)\lambda_2}{q-1} z_n\right] + 2q(\lambda_1 - \lambda_2)\lambda_2 z_n \\
&= -\frac{2q\lambda_1^2}{q-1} x_n - \left(\frac{2q\lambda_1^2}{q-1} + 2q\lambda_2^2\right) z_n;
\end{aligned}$$

for any $q+1 \le i \le 2q$,

$$\begin{aligned}\Lambda_3 &= \mathbf{E}\left[2q\lambda_1\left(Y_{(q+1)j}-\frac{1}{2q}\right)+2(\lambda_2-\lambda_1)\sum_{q+1\le\ell\le 2q}\left(Y_{\ell j}-\frac{1}{2q}\right)\right]\\ &= [2q\lambda_1+2q(\lambda_2-\lambda_1)]\lambda_2 z_n\\ &= 2q\lambda_2^2 z_n.\end{aligned}$$

Next consider covariances:

$$\begin{aligned}① &= \mathbf{E}\left[\lambda_1\left(Y_{1j}-\frac{1}{2q}\right)+\frac{\lambda_1-\lambda_2}{q}\sum_{q+1\le\ell\le 2q}\left(Y_{\ell j}-\frac{1}{2q}\right)\right]^2\\ &= \lambda_1^2\mathbf{E}\left(Y_{11}-\frac{1}{2q}\right)^2+2\lambda_1\frac{\lambda_1-\lambda_2}{q}\mathbf{E}\left(Y_{11}-\frac{1}{2q}\right)\sum_{q+1\le\ell\le 2q}\left(Y_{\ell 1}-\frac{1}{2q}\right)\\ &\quad+\left(\frac{\lambda_1-\lambda_2}{q}\right)^2\mathbf{E}\left[\sum_{q+1\le\ell\le 2q}\left(Y_{\ell 1}-\frac{1}{2q}\right)\right]^2\\ &= \lambda_1^2\mathbf{E}\left(Y_{11}-\frac{1}{2q}\right)^2+2q\lambda_1\frac{\lambda_1-\lambda_2}{q}\mathbf{E}\left(Y_{11}-\frac{1}{2q}\right)\left(Y_{(q+1)1}-\frac{1}{2q}\right)\\ &\quad+q\left(\frac{\lambda_1-\lambda_2}{q}\right)^2\mathbf{E}\left(Y_{(q+1)1}-\frac{1}{2q}\right)^2+q(q-1)\left(\frac{\lambda_1-\lambda_2}{q}\right)^2\mathbf{E}\left(Y_{(q+1)1}-\frac{1}{2q}\right)\left(Y_{(2q)1}-\frac{1}{2q}\right)\\ &= \frac{1-4\lambda_1+3\lambda_2}{2q}\lambda_1^2 x_n+\frac{\lambda_1^2-\lambda_2^2}{2q}z_n+\lambda_1^3 u_n+3(\lambda_1-\lambda_2)\lambda_1^2 w_n;\end{aligned}$$

for any $2 \le i \le q$,

$$\begin{aligned}② &= \mathbf{E}\left[\lambda_1\left(Y_{ij}-\frac{1}{2q}\right)+\frac{\lambda_1-\lambda_2}{q}\sum_{q+1\le\ell\le 2q}\left(Y_{\ell j}-\frac{1}{2q}\right)\right]^2\\ &= \lambda_1^2\mathbf{E}\left(Y_{i1}-\frac{1}{2q}\right)^2+2\lambda_1(\lambda_1-\lambda_2)\mathbf{E}\left(Y_{i1}-\frac{1}{2q}\right)\left(Y_{(q+1)1}-\frac{1}{2q}\right)\\ &\quad+q\left(\frac{\lambda_1-\lambda_2}{q}\right)^2\mathbf{E}\left(Y_{(q+1)1}-\frac{1}{2q}\right)^2+q(q-1)\left(\frac{\lambda_1-\lambda_2}{q}\right)^2\mathbf{E}\left(Y_{(q+1)1}-\frac{1}{2q}\right)\left(Y_{(2q)1}-\frac{1}{2q}\right)\\ &= \lambda_1^2\left[\left(\frac{1}{2q}+\frac{\lambda_2}{2q}+\frac{\lambda_1}{q(q-1)}\right)x_n-\frac{\lambda_1}{q-1}u_n-\frac{\lambda_1+(q-1)\lambda_2}{q-1}w_n\right]\\ &\quad+2\lambda_1(\lambda_1-\lambda_2)\left(\frac{\lambda_1}{2q(q-1)}x_n+\frac{z_n}{2q}-\frac{\lambda_1}{q-1}w_n\right)\\ &\quad+q\left(\frac{\lambda_1-\lambda_2}{q}\right)^2\left(\frac{1-\lambda_2}{2q}x_n+\lambda_2 w_n\right)+q(q-1)\left(\frac{\lambda_1-\lambda_2}{q}\right)^2\left(\frac{(\lambda_2-1)x_n-qz_n}{2q(q-1)}-\frac{\lambda_2}{q-1}w_n\right)\\ &= \left(\frac{1}{2q}+\frac{4\lambda_1+(q-3)\lambda_2}{2q(q-1)}\right)\lambda_1^2 x_n+\frac{\lambda_1^2-\lambda_2^2}{2q}z_n-\frac{\lambda_1^3}{q-1}u_n-\frac{3\lambda_1+(q-3)\lambda_2}{q-1}\lambda_1^2 w_n;\end{aligned}$$

for any $q+1 \le i \le 2q$,

$$
\begin{aligned}
③ &= \mathbf{E}\left[\lambda_1\left(Y_{ij}-\frac{1}{2q}\right)+\frac{\lambda_2-\lambda_1}{q}\sum_{q+1\le\ell\le 2q}\left(Y_{\ell j}-\frac{1}{2q}\right)\right]^2 \\
&= \left\{\left[(q-1)\lambda_1+\frac{\lambda_2}{q}\right]^2+(q-1)\left(\frac{\lambda_2-\lambda_1}{q}\right)^2\right\}\mathbf{E}\left(Y_{(q+1)j}-\frac{1}{2q}\right)^2+\left\{\frac{q-1}{q^2}(\lambda_2-\lambda_1)[(2q-2)\lambda_1\right. \\
&\quad \left. +2\lambda_2]+\frac{(q-1)(q-2)}{q^2}(\lambda_2-\lambda_1)^2\right\}\mathbf{E}\left(Y_{(q+1)j}-\frac{1}{2q}\right)\left(Y_{(2q)j}-\frac{1}{2q}\right) \\
&= \frac{(q-1)\lambda_1^2+\lambda_2^2}{q}\left(\frac{1-\lambda_2}{2q}x_n+\lambda_2 w_n\right) \\
&\quad +\frac{(q-1)(\lambda_2^2-\lambda_1^2)}{q}\left(\frac{\lambda_2-1}{2q(q-1)}x_n-\frac{z_n}{2(q-1)}-\frac{\lambda_2}{q-1}w_n\right) \\
&= \frac{\lambda_1^2}{2q}x_n+\frac{\lambda_1^2-\lambda_2^2}{2q}z_n+\lambda_1^2\lambda_2\left(w_n-\frac{x_n}{2q}\right);
\end{aligned}
$$

for $2 \le i_1 < i_2 \le q$,

$$
\begin{aligned}
④ &= \mathbf{E}\left[\lambda_1\left(Y_{i_1 j}-\frac{1}{2q}\right)+\frac{\lambda_1-\lambda_2}{q}\sum_{q+1\le\ell\le 2q}\left(Y_{\ell j}-\frac{1}{2q}\right)\right] \\
&\quad \left[\lambda_1\left(Y_{i_2 j}-\frac{1}{2q}\right)+\frac{\lambda_1-\lambda_2}{q}\sum_{q+1\le\ell\le 2q}\left(Y_{\ell j}-\frac{1}{2q}\right)\right] \\
&= \lambda_1^2\mathbf{E}\left(Y_{21}-\frac{1}{2q}\right)\left(Y_{q1}-\frac{1}{2q}\right)+2\lambda_1(\lambda_1-\lambda_2)\mathbf{E}\left(Y_{21}-\frac{1}{2q}\right)\left(Y_{(q+1)1}-\frac{1}{2q}\right) \\
&\quad +\frac{(\lambda_1-\lambda_2)^2}{q^2}\left[q\mathbf{E}\left(Y_{(q+1)1}-\frac{1}{2q}\right)^2+q(q-1)\mathbf{E}\left(Y_{(q+1)1}-\frac{1}{2q}\right)\left(Y_{(2q)1}-\frac{1}{2q}\right)\right] \\
&= \lambda_1^2\left\{\left[-\frac{2(q+2)\lambda_1+(q-2)\lambda_2}{2q(q-1)(q-2)}-\frac{1}{2q(q-1)}\right]x_n-\frac{z_n}{2(q-1)}+\frac{2\lambda_1}{(q-1)(q-2)}u_n\right. \\
&\quad \left.+\frac{2(q+1)\lambda_1+(q-2)\lambda_2}{(q-1)(q-2)}w_n\right\}+2\lambda_1(\lambda_1-\lambda_2)\left\{\frac{\lambda_1}{2q(q-1)}x_n+\frac{z_n}{2q}-\frac{\lambda_1}{q-1}w_n\right\} \\
&\quad +\frac{(\lambda_1-\lambda_2)^2}{q}\left\{\left[\frac{1-\lambda_2}{2q}x_n+\lambda_2 w_n\right]+(q-1)\left[\frac{\lambda_2-1}{2q(q-1)}x_n-\frac{z_n}{2(q-1)}-\frac{\lambda_2}{q-1}w_n\right]\right\} \\
&= -\frac{\lambda_1^2}{2q(q-1)}x_n-\frac{\lambda_1^2+(q-1)\lambda_2^2}{2q(q-1)}z_n+\frac{2\lambda_1^3}{(q-1)(q-2)}\left(u_n-\frac{x_n}{2q}\right) \\
&\quad +\frac{6\lambda_1+(3q-6)\lambda_2}{(q-1)(q-2)}\lambda_1^2\left(w_n-\frac{x_n}{2q}\right);
\end{aligned}
$$

for $q+1 \leq i \leq 2q$,

$$\begin{aligned}
⑤ &= \mathbf{E}\left[\lambda_1\left(Y_{2j}-\frac{1}{2q}\right)-\frac{\lambda_1-\lambda_2}{q}\sum_{1\leq \ell\leq q}\left(Y_{\ell j}-\frac{1}{2q}\right)\right] \\
&\quad \left[\lambda_1\left(Y_{ij}-\frac{1}{2q}\right)-\frac{\lambda_1-\lambda_2}{q}\sum_{q+1\leq \ell\leq 2q}\left(Y_{\ell j}-\frac{1}{2q}\right)\right] \\
&= \frac{\lambda_2-\lambda_1}{q}\left(\lambda_1-\frac{\lambda_1-\lambda_2}{q}q\right)\mathbf{E}\left(Y_{11}-\frac{1}{2q}\right)\left(Y_{(q+1)1}-\frac{1}{2q}\right) \\
&\quad +\left(\lambda_1-(q-1)\frac{\lambda_1-\lambda_2}{q}\right)\left(\lambda_1-\frac{\lambda_1-\lambda_2}{q}q\right)\mathbf{E}\left(Y_{21}-\frac{1}{2q}\right)\left(Y_{(q+1)1}-\frac{1}{2q}\right) \\
&= \frac{\lambda_2^2-\lambda_1\lambda_2}{q}\left(-\frac{\lambda_1}{2q}x_n+\frac{z_n}{2q}+\lambda_1 w_n\right)+\frac{\lambda_1\lambda_2+(q-1)\lambda_2^2}{q}\left(\frac{\lambda_1}{2q(q-1)}x_n+\frac{z_n}{2q}-\frac{\lambda_1}{q-1}w_n\right) \\
&= \frac{\lambda_2^2}{2q}z_n-\frac{\lambda_1^2\lambda_2}{q-1}\left(w_n-\frac{x_n}{2q}\right);
\end{aligned}$$

for any $2 \leq i \leq q$,

$$\begin{aligned}
⑥ &= \mathbf{E}\left[\lambda_1\left(Y_{1j}-\frac{1}{2q}\right)+\frac{\lambda_1-\lambda_2}{q}\sum_{q+1\leq \ell\leq 2q}\left(Y_{\ell j}-\frac{1}{2q}\right)\right] \\
&\quad \left[\lambda_1\left(Y_{ij}-\frac{1}{2q}\right)+\frac{\lambda_1-\lambda_2}{q}\sum_{q+1\leq \ell\leq 2q}\left(Y_{\ell j}-\frac{1}{2q}\right)\right] \\
&= -\frac{\lambda_1^2}{2q(q-1)}(x_n+z_n)-\frac{\lambda_2^2}{2q}z_n-\frac{\lambda_1^3}{q-1}\left(u_n-\frac{x_n}{2q}\right)-\frac{3\lambda_1+(q-3)\lambda_2}{q-1}\lambda_1^2\left(w_n-\frac{x_n}{2q}\right);
\end{aligned}$$

for any $q+1 \leq i \leq 2q$,

$$\begin{aligned}
⑦ &= \mathbf{E}\left[\lambda_1\left(Y_{1j}-\frac{1}{2q}\right)+\frac{\lambda_1-\lambda_2}{q}\sum_{q+1\leq \ell\leq 2q}\left(Y_{\ell j}-\frac{1}{2q}\right)\right] \\
&\quad \left[\lambda_1\left(Y_{ij}-\frac{1}{2q}\right)+\frac{\lambda_1-\lambda_2}{q}\sum_{1\leq \ell\leq q}\left(Y_{\ell j}-\frac{1}{2q}\right)\right] \\
&= \frac{(q-1)\lambda_1\lambda_2+\lambda_2^2}{q}\mathbf{E}\left(Y_{11}-\frac{1}{2q}\right)\left(Y_{(q+1)1}-\frac{1}{2q}\right) \\
&\quad +\frac{q-1}{q}(\lambda_2^2-\lambda_1\lambda_2)\mathbf{E}\left(Y_{21}-\frac{1}{2q}\right)\left(Y_{(q+1)1}-\frac{1}{2q}\right) \\
&= \frac{(q-1)\lambda_1\lambda_2+\lambda_2^2}{q}\left(-\frac{\lambda_1}{2q}x_n+\frac{z_n}{2q}+\lambda_1 w_n\right)+\frac{q-1}{q}(\lambda_2^2-\lambda_1\lambda_2)\left(\frac{\lambda_1}{2q(q-1)}x_n+\frac{z_n}{2q}-\frac{\lambda_1}{q-1}w_n\right) \\
&= \frac{\lambda_2^2}{2q}z_n+\lambda_1^2\lambda_2\left(w_n-\frac{x_n}{2q}\right).
\end{aligned}$$

As a result of Lemma 2.2.1 and Lemma 2.2.2, it is easy to get that $x_n \geq |z_n|$ and $0 \leq u_n, w_n \leq x_n$, which, together with the fact of $|\lambda| \leq d^{-\frac{1}{2}} < 1$ completes the proof. □

Lemma 2.5.2. *For each positive integer k, there exists a $C = C(k, q)$ only depending on k and q such that for each $0 \le k_1, \dots, k_{2q} \le k$ and $1 \le j \le d$,*

$$\mathbf{E} \prod_{i=1}^{2q} Z_i^{k_i} \le C,$$

$$\left| \mathbf{E} \prod_{i=1}^{2q} Z_i^{k_i} - 1 - d \left\{ \mathbf{E} \prod_{i=1}^{2q} \left[1 + 2q\lambda_1 \left(Y_{ij} - \frac{1}{2q} \right) + 2(\lambda_1 - \lambda_2) \sum_{\substack{\forall \ell \in \mathcal{C}_i \\ i \notin \mathcal{C}_i}} \left(Y_{\ell j} - \frac{1}{2q} \right) \right]^{k_i} - 1 \right\} \right| \le C x_n^2$$

and

$$\left| \mathbf{E} \prod_{i=1}^{2q} Z_i^{k_i} - 1 - d \left\{ \mathbf{E} \prod_{i=1}^{2q} \left[1 + 2q\lambda_1 \left(Y_{ij} - \frac{1}{2q} \right) + 2(\lambda_1 - \lambda_2) \sum_{\substack{\forall \ell \in \mathcal{C}_i \\ i \notin \mathcal{C}_i}} \left(Y_{\ell j} - \frac{1}{2q} \right) \right]^{k_i} - 1 \right\} \right.$$
$$\left. - \frac{d(d-1)}{2} \left\{ \mathbf{E} \prod_{i=1}^{2q} \left[1 + 2q\lambda_1 \left(Y_{ij} - \frac{1}{2q} \right) + 2(\lambda_1 - \lambda_2) \sum_{\substack{\forall \ell \in \mathcal{C}_i \\ i \notin \mathcal{C}_i}} \left(Y_{\ell j} - \frac{1}{2q} \right) \right]^{k_i} - 1 \right\}^2 \right| \le C x_n^3.$$

Proof. In the first place, we would like to discuss nonnegative integral exponents $s_1, \dots, s_q$ with $s_i \le k_i$ for all i. The following C_is denote the constants depending only on q and k.

(a) If $s_h \ge 2$ for some h, it follows from the fact $0 \le Y_{ij} \le 1$ and Lemma 2.5.1 that

$$\begin{aligned}
& \left| \mathbf{E} \prod_{i=1}^{2q} \left[2q\lambda_1 \left(Y_{i1} - \frac{1}{2q} \right) + 2(\lambda_1 - \lambda_2) \sum_{\substack{\forall \ell \in \mathcal{C}_i \\ i \notin \mathcal{C}_i}} \left(Y_{\ell 1} - \frac{1}{2q} \right) \right]^{s_i} \right| \\
\le \;& \mathbf{E} \left[2q\lambda_1 \left(Y_{h1} - \frac{1}{2q} \right) + 2(\lambda_1 - \lambda_2) \sum_{\substack{\forall \ell \in \mathcal{C}_h \\ h \notin \mathcal{C}_h}} \left(Y_{\ell 1} - \frac{1}{2q} \right) \right]^2 (6q|\lambda|)^{s_h - 2} \prod_{1 \le i \ne h \le 2q} (6q|\lambda|)^{s_i} \\
\le \;& C_1 \lambda^2 x_n,
\end{aligned}$$

where the last inequality comes from $|\lambda| \le d^{-\frac{1}{2}} < 1$.

(b) If $s_h = s_{h'} = 1$ holds for some $h \neq h'$, then Lemma 2.5.1 also yields

$$\begin{aligned} & \left| \mathbf{E} \prod_{i=1}^{2q} \left[2q\lambda_1 \left(Y_{i1} - \frac{1}{2q} \right) + 2(\lambda_1 - \lambda_2) \sum_{\substack{\forall \ell \in \mathcal{C}_i \\ i \notin \mathcal{C}_i}} \left(Y_{\ell 1} - \frac{1}{2q} \right) \right]^{s_i} \right| \\ \leq & \left| \mathbf{E} \left[2q\lambda_1 \left(Y_{h1} - \frac{1}{2q} \right) + 2(\lambda_1 - \lambda_2) \sum_{\substack{\forall \ell \in \mathcal{C}_h \\ h \notin \mathcal{C}_h}} \left(Y_{\ell 1} - \frac{1}{2q} \right) \right] \right. \\ & \left. \left[2q\lambda_1 \left(Y_{h'1} \frac{1}{2q} \right) + 2(\lambda_1 - \lambda_2) \sum_{\substack{\forall \ell \in \mathcal{C}_{h'} \\ h' \notin \mathcal{C}_{h'}}} \left(Y_{\ell 1} - \frac{1}{2q} \right) \right] \right| \prod_{\substack{1 \leq i \leq 2q \\ i \neq h \text{ or } h'}} (6q|\lambda|)^{s_i} \\ \leq & \ C_2 \lambda^2 x_n. \end{aligned}$$

(c) If $s_h = 1$ and $s_i = 0$ for all $i \neq h$,

$$\begin{aligned} & \left| \mathbf{E} \prod_{i=1}^{2q} \left[2q\lambda_1 \left(Y_{i1} - \frac{1}{2q} \right) + 2(\lambda_1 - \lambda_2) \sum_{\substack{\forall \ell \in \mathcal{C}_i \\ i \notin \mathcal{C}_i}} \left(Y_{\ell 1} - \frac{1}{2q} \right) \right]^{s_i} \right| \\ = & \left| \mathbf{E} \left[2q\lambda_1 \left(Y_{h1} - \frac{1}{2q} \right) + 2(\lambda_1 - \lambda_2) \sum_{\substack{\forall \ell \in \mathcal{C}_h \\ h \notin \mathcal{C}_h}} \left(Y_{\ell 1} - \frac{1}{2q} \right) \right] \right| \leq C_3 \lambda^2 x_n. \end{aligned}$$

Therefore, combining the preceding results together, we conclude

$$\begin{aligned} & \left| \mathbf{E} \prod_{i=1}^{2q} \left[1 + 2q\lambda_1 \left(Y_{i1} - \frac{1}{2q} \right) + 2(\lambda_1 - \lambda_2) \sum_{\substack{\forall \ell \in \mathcal{C}_i \\ i \notin \mathcal{C}_i}} \left(Y_{\ell j} - \frac{1}{2q} \right) \right]^{k_i} - 1 \right| \\ = & \left| \sum_{(h_1, \ldots, h_{2q})} \mathbf{E} \prod_{i=1}^{2q} \binom{k_i}{h_i} \left[2q\lambda_1 \left(Y_{i1} - \frac{1}{2q} \right) + 2(\lambda_1 - \lambda_2) \sum_{\substack{\forall \ell \in \mathcal{C}_i \\ i \notin \mathcal{C}_i}} \left(Y_{\ell j} - \frac{1}{2q} \right) \right]^{h_i} - 1 \right| \\ = & \left| \mathbf{E} \sum_{i=1}^{2q} k_i \left[2q\lambda_1 \left(Y_{i1} - \frac{1}{2q} \right) + 2(\lambda_1 - \lambda_2) \sum_{\substack{\forall \ell \in \mathcal{C}_i \\ i \notin \mathcal{C}_i}} \left(Y_{\ell j} - \frac{1}{2q} \right) \right] \right. \\ & \left. + \sum_{\substack{(h_1, \ldots, h_{2q}) \\ \sum_{i=1}^{2q} h_i \geq 2}} \mathbf{E} \prod_{i=1}^{2q} \binom{k_i}{h_i} \left[2q\lambda_1 \left(Y_{i1} - \frac{1}{2q} \right) + 2(\lambda_1 - \lambda_2) \sum_{\substack{\forall \ell \in \mathcal{C}_i \\ i \notin \mathcal{C}_i}} \left(Y_{\ell j} - \frac{1}{2q} \right) \right]^{h_i} \right| \\ \leq & \ C_4 \lambda^2 x_n. \end{aligned}$$

Next write

$$x = \mathbf{E}\prod_{i=1}^{2q}\left[1 + 2q\lambda_1\left(Y_{i1} - \frac{1}{2q}\right) + 2(\lambda_1 - \lambda_2)\sum_{\substack{\forall \ell \in \mathcal{C}_i \\ i \notin \mathcal{C}_i}}\left(Y_{\ell j} - \frac{1}{2q}\right)\right]^{k_i} - 1$$

and then

$$\mathbf{E}\prod_{i=1}^{2q} Z_i^{k_i} \quad = \quad (1+x)^d$$

since for any $1 \leq i \leq 2q$, the terms in the product of Z_i are independent and identically distributed. Referring to the binomial expansion, in tandem with the Remainder Theorem, there is no difficulty in concluding that if $d|x| \leq C_5$ for some constant $C_5 > 0$, then

$$\left|(1+x)^d - \sum_{m=0}^{n}\binom{d}{m}x^m\right| \leq \sum_{m=n+1}^{\infty}\frac{d^m}{m!}|x|^m \leq e^{C_5}|dx|^{n+1}.$$

Finally applying the facts of $0 \leq x_n \leq 1$ and $d\lambda^2 \leq 1$ completes the proof. □

As regards x_{n+1} and z_{n+1}, we could expand them out in virtue of the identity

$$\frac{a}{s+r} = \frac{a}{s} - \frac{ar}{s^2} + \frac{r^2}{s^2}\frac{a}{s+r}. \tag{2.5.1}$$

Specifically, taking $a = Z_1$, $s = 2q$ and $r = \sum_{i=1}^{2q} Z_i - 2q$ yields

$$\begin{aligned} x_{n+1} + \frac{1}{2q} &= \mathbf{E}\frac{Z_1}{\sum_{i=1}^{2q} Z_i} \\ &= \mathbf{E}\frac{Z_1}{2q} - \mathbf{E}\frac{Z_1(\sum_{i=1}^{2q} Z_i - 2q)}{(2q)^2} + \mathbf{E}\frac{Z_1}{\sum_{i=1}^{2q} Z_i}\frac{(\sum_{i=1}^{2q} Z_i - 2q)^2}{(2q)^2}, \end{aligned}$$

as well as

$$\begin{aligned} z_{n+1} + \frac{1}{2q} &= \mathbf{E}\frac{Z_{q+1}}{\sum_{i=1}^{2q} Z_i} \\ &= \mathbf{E}\frac{Z_{q+1}}{2q} - \mathbf{E}\frac{Z_{q+1}(\sum_{i=1}^{2q} Z_i - 2q)}{(2q)^2} + \mathbf{E}\frac{Z_{q+1}}{\sum_{i=1}^{2q} Z_i}\frac{(\sum_{i=1}^{2q} Z_i - 2q)^2}{(2q)^2}. \end{aligned}$$

Then we have to distract our attention to taking care of terms in expansions above respectively. The following R_is denote the remainder terms bounded by $O_q(x_n^3)$, where O_q-constants depend only on q.

(i) When $i = 1$,

$$\mathbf{E}Z_1 = \prod_{j=1}^{d} \mathbf{E}\left[1 + 2q\lambda_1\left(Y_{1j} - \frac{1}{2q}\right) + 2(\lambda_1 - \lambda_2)\sum_{q+1\le\ell\le 2q}\left(Y_{\ell j} - \frac{1}{2q}\right)\right]$$
$$= 1 + d\Lambda_1 + \frac{d(d-1)}{2}\Lambda_1^2 + R_1;$$

(ii) For $2 \le i \le q$,

$$\mathbf{E}Z_i = \mathbf{E}Z_2 = \prod_{j=1}^{d} \mathbf{E}\left[1 + 2q\lambda_1\left(Y_{2j} - \frac{1}{2q}\right) + 2(\lambda_1 - \lambda_2)\sum_{q+1\le\ell\le 2q}\left(Y_{\ell j} - \frac{1}{2q}\right)\right]$$
$$= 1 + d\Lambda_2 + \frac{d(d-1)}{2}\Lambda_2^2 + R_2;$$

(iii) For $q + 1 \le i \le 2q$,

$$\mathbf{E}Z_i = \mathbf{E}Z_{q+1} = \prod_{j=1}^{d} \mathbf{E}\left[1 + 2q\lambda_1\left(Y_{(q+1)j} - \frac{1}{2q}\right) + 2(\lambda_1 - \lambda_2)\sum_{1\le\ell\le q}\left(Y_{\ell j} - \frac{1}{2q}\right)\right]$$
$$= \prod_{j=1}^{d} \mathbf{E}\left[1 + 2q\lambda_1\left(Y_{(q+1)j} - \frac{1}{2q}\right) + 2(\lambda_2 - \lambda_1)\sum_{q+1\le\ell\le 2q}\left(Y_{\ell j} - \frac{1}{2q}\right)\right]$$
$$= 1 + d\Lambda_3 + \frac{d(d-1)}{2}\Lambda_3^2 + R_3.$$

Next consider covariances of Z_is:

(a) beginning with $\mathbf{E}Z_1^2$,

$$\mathbf{E}Z_1^2 = \mathbf{E}\prod_{j=1}^{d}\left[1 + 2q\lambda_1\left(Y_{1j} - \frac{1}{2q}\right) + 2(\lambda_1 - \lambda_2)\sum_{q+1\le\ell\le 2q}\left(Y_{\ell j} - \frac{1}{2q}\right)\right]^2$$
$$= 1 + d\Pi_1 + \frac{d(d-1)}{2}\Pi_1^2 + R_4$$

where

$$\Pi_1 = \mathbf{E}\left[1 + 2q\lambda_1\left(Y_{1j} - \frac{1}{2q}\right) + 2(\lambda_1 - \lambda_2)\sum_{q+1\le\ell\le 2q}\left(Y_{\ell j} - \frac{1}{2q}\right)\right]^2 - 1$$
$$= 2\Lambda_1 + 4q^2 \text{①}$$
$$= 4q\lambda_1^2 x_n + 4q(\lambda_1^2 - \lambda_2^2)z_n + 4q^2 \text{①}$$
$$= 4q\lambda_1^2 x_n + 4q(\lambda_1^2 - \lambda_2^2)z_n + 4q^2\left[\frac{1 - 4\lambda_1 + 3\lambda_2}{2q}\lambda_1^2 x_n + \frac{\lambda_1^2 - \lambda_2^2}{2q}z_n + \lambda_1^3 u_n + 3(\lambda_1 - \lambda_2)\lambda_1^2 w_n\right]$$
$$= 6q\lambda_1^2 x_n + 6q(\lambda_1^2 - \lambda_2^2)z_n + 4q^2\lambda_1^3\left(u_n - \frac{x_n}{2q}\right) + 12q^2\lambda_1^2(\lambda_1 - \lambda_2)\left(w_n - \frac{x_n}{2q}\right);$$

(b) For $2 \le i \le q$,

$$\begin{aligned} \mathbf{E} Z_i^2 &= \mathbf{E} \prod_{j=1}^{d} \left[1 + 2q\lambda_1 \left(Y_{ij} - \frac{1}{2q} \right) + 2(\lambda_1 - \lambda_2) \sum_{q+1 \le \ell \le 2q} \left(Y_{\ell j} - \frac{1}{2q} \right) \right]^2 \\ &= 1 + d\Pi_2 + \frac{d(d-1)}{2} \Pi_2^2 + R_5 \end{aligned}$$

where

$$\begin{aligned} \Pi_2 &= \mathbf{E} \left[1 + 2q\lambda_1 \left(Y_{ij} - \frac{1}{2q} \right) + 2(\lambda_1 - \lambda_2) \sum_{q+1 \le \ell \le 2q} \left(Y_{\ell j} - \frac{1}{2q} \right) \right]^2 - 1 \\ &= 2\Lambda_2 + 4q^2 \textcircled{2} \\ &= -\frac{4q\lambda_1^2}{q-1} x_n - \left(\frac{4q\lambda_1^2}{q-1} + 4q\lambda_2^2 \right) z_n \\ &\quad + 4q^2 \left[\left(\frac{1}{2q} + \frac{4\lambda_1 + (q-3)\lambda_2}{2q(q-1)} \right) \lambda_1^2 x_n + \frac{\lambda_1^2 - \lambda_2^2}{2q} z_n - \frac{\lambda_1^3}{q-1} u_n - \frac{3\lambda_1 + (q-3)\lambda_2}{q-1} \lambda_1^2 w_n \right] \\ &= \frac{2q(q-3)}{q-1} \lambda_1^2 x_n + \left(\frac{2q(q-3)}{q-1} \lambda_1^2 - 6q\lambda_2^2 \right) z_n - \frac{4q^2}{q-1} \lambda_1^3 \left(u_n - \frac{x_n}{2q} \right) \\ &\quad - 4q^2 \frac{3\lambda_1 + (q-3)\lambda_2}{q-1} \lambda_1^2 w_n; \end{aligned}$$

(c) When $q + 1 \le i \le 2q$,

$$\begin{aligned} \mathbf{E} Z_i^2 &= \mathbf{E} \prod_{j=1}^{d} \left[1 + 2q\lambda_1 \left(Y_{ij} - \frac{1}{2q} \right) + 2(\lambda_1 - \lambda_2) \sum_{1 \le \ell \le q} \left(Y_{\ell j} - \frac{1}{2q} \right) \right]^2 \\ &= 1 + d\Pi_3 + \frac{d(d-1)}{2} \Pi_3^2 + R_6 \end{aligned}$$

where

$$\begin{aligned} \Pi_3 &= \mathbf{E} \left[1 + 2q\lambda_1 \left(Y_{ij} - \frac{1}{2q} \right) + 2(\lambda_2 - \lambda_1) \sum_{q+1 \le \ell \le 2q} \left(Y_{\ell j} - \frac{1}{2q} \right) \right]^2 - 1 \\ &= 2\Lambda_3 + \textcircled{3} \\ &= 4q\lambda_2^2 z_n + 2q\lambda_1^2 x_n + 2q(\lambda_1^2 - \lambda_2^2) z_n + 4q^2 \lambda_1^2 \lambda_2 \left(u_n - \frac{x_n}{2q} \right) \\ &= 2q\lambda_1^2 x_n + 2q(\lambda_1^2 + \lambda_2^2) z_n + 4q^2 \lambda_1^2 \lambda_2 \left(w_n - \frac{x_n}{2q} \right). \end{aligned}$$

By Lemma 2.4.1, it is known that $\mathbf{E} Z_1 Z_i = \mathbf{E} Z_i^2$, and hence we can skip the calculations of these covariances.

(d) For $2 \le i_1 < i_2 \le q$,

$$
\begin{aligned}
\mathbf{E} Z_{i_1} Z_{i_2} &= \mathbf{E} Z_2 Z_q \\
&= \mathbf{E} \prod_{j=1}^{d} \left[1 + 2q\lambda_1 \left(Y_{2j} - \frac{1}{2q} \right) + 2(\lambda_1 - \lambda_2) \sum_{q+1 \le \ell \le 2q} \left(Y_{\ell j} - \frac{1}{2q} \right) \right] \\
&\quad \left[1 + 2q\lambda_1 \left(Y_{qj} - \frac{1}{2q} \right) + 2(\lambda_1 - \lambda_2) \sum_{q+1 \le \ell \le 2q} \left(Y_{\ell j} - \frac{1}{2q} \right) \right] \\
&= 1 + d\Pi_4 + \frac{d(d-1)}{2} \Pi_4^2 + R_7
\end{aligned}
$$

where

$$
\begin{aligned}
\Pi_4 &= \mathbf{E} \left[1 + 2q\lambda_1 \left(Y_{2j} - \frac{1}{2q} \right) + 2(\lambda_1 - \lambda_2) \sum_{q+1 \le \ell \le 2q} \left(Y_{\ell j} - \frac{1}{2q} \right) \right] \\
&\quad \left[1 + 2q\lambda_1 \left(Y_{qj} - \frac{1}{2q} \right) + 2(\lambda_1 - \lambda_2) \sum_{q+1 \le \ell \le 2q} \left(Y_{\ell j} - \frac{1}{2q} \right) \right] - 1 \\
&= 2\Lambda_2 + 4q^2 \text{④} \\
&= -\frac{4q\lambda_1^2}{q-1} x_n - \left(\frac{4q\lambda_1^2}{q-1} + 4q\lambda_2^2 \right) z_n + 4q^2 \left[-\frac{\lambda_1^2}{2q(q-1)} x_n - \frac{\lambda_1^2 + (q-1)\lambda_2^2}{2q(q-1)} z_n \right. \\
&\quad \left. + \frac{2\lambda_1^3}{(q-1)(q-2)} \left(u_n - \frac{x_n}{2q} \right) + \frac{6\lambda_1 + (3q-6)\lambda_2}{(q-1)(q-2)} \lambda_1^2 \left(w_n - \frac{x_n}{2q} \right) \right] \\
&= -\frac{6q\lambda_1^2}{q-1} x_n - \left(\frac{6q\lambda_1^2}{q-1} + 6q\lambda_2^2 \right) z_n + \frac{8q^2\lambda_1^3}{(q-1)(q-2)} \left(u_n - \frac{x_n}{2q} \right) \\
&\quad + 4q^2 \frac{6\lambda_1 + (3q-6)\lambda_2}{(q-1)(q-2)} \lambda_1^2 \left(w_n - \frac{x_n}{2q} \right);
\end{aligned}
$$

(e) For $q + 1 \le i \le 2q$,

$$
\begin{aligned}
\mathbf{E} Z_2 Z_i &= \mathbf{E} Z_2 Z_{q+1} \\
&= \mathbf{E} \prod_{j=1}^{d} \left[1 + 2q\lambda_1 \left(Y_{2j} - \frac{1}{2q} \right) + 2(\lambda_1 - \lambda_2) \sum_{q+1 \le \ell \le 2q} \left(Y_{\ell j} - \frac{1}{2q} \right) \right] \\
&\quad \left[1 + 2q\lambda_1 \left(Y_{(q+1)j} - \frac{1}{2q} \right) + 2(\lambda_1 - \lambda_2) \sum_{1 \le \ell \le q} \left(Y_{\ell j} - \frac{1}{2q} \right) \right] \\
&= 1 + d\Pi_5 + \frac{d(d-1)}{2} \Pi_5^2 + R_8
\end{aligned}
$$

where

$$\begin{aligned}
\Pi_5 &= \mathbf{E}\prod_{j=1}^{d}\left[1+2q\lambda_1\left(Y_{2j}-\frac{1}{2q}\right)+2(\lambda_1-\lambda_2)\sum_{q+1\le\ell\le 2q}\left(Y_{\ell j}-\frac{1}{2q}\right)\right] \\
&\quad \left[1+2q\lambda_1\left(Y_{(q+1)j}-\frac{1}{2q}\right)+2(\lambda_1-\lambda_2)\sum_{1\le\ell\le q}\left(Y_{\ell j}-\frac{1}{2q}\right)\right]-1 \\
&= \Lambda_2+\Lambda_3+4q^2 \text{⑤} \\
&= -\frac{2q\lambda_1^2}{q-1}x_n-\frac{2q\lambda_1^2}{q-1}z_n+4q^2\left[\frac{\lambda_2^2}{2q}z_n-\frac{\lambda_1^2\lambda_2}{q-1}\left(w_n-\frac{x_n}{2q}\right)\right] \\
&= -\frac{2q\lambda_1^2}{q-1}x_n+\left(-\frac{2q\lambda_1^2}{q-1}+2q\lambda_2^2\right)z_n-\frac{4q^2}{q-1}\lambda_1^2\lambda_2\left(w_n-\frac{x_n}{2q}\right).
\end{aligned}$$

Finally taken together, all the calculation above yield

$$\mathbf{E}Z_1\left(\sum_{i=1}^{2q}Z_i-2q\right) = \frac{d(d-1)}{2}[\Pi_1^2+(q-1)\Pi_2^2+q\Pi_3^2-2q\Lambda_1^2]+R_9$$

and

$$\begin{aligned}
&\mathbf{E}\left(\sum_{i=1}^{2q}Z_i-2q\right)^2 \\
&= \sum_{i=1}^{2q}\mathbf{E}Z_i^2+2\sum_{i=2}^{2q}\mathbf{E}Z_1Z_i+2\sum_{i_1=2}^{2q}\sum_{i_2=i_1+1}^{2q}\mathbf{E}Z_{i_1}Z_{i_2}-4q\sum_{i=1}^{2q}\mathbf{E}Z_i+4q^2 \\
&= \frac{d(d-1)}{2}\{[\Pi_1^2+(q-1)\Pi_2^2+q\Pi_3^2]+2[(q-1)\Pi_2^2+q\Pi_3^2] \\
&\quad +[(q-1)(q-2)\Pi_4^2+2q(q-1)\Pi_5^2+q(q-1)\Pi_5^2]-4q\Lambda_1^2-4q(q-1)\Lambda_2^2-4q^2\Lambda_3^2\}+R_{10} \\
&= \frac{d(d-1)}{2}[\Pi_1^2+3(q-1)\Pi_2^2+3q\Pi_3^2+(q-1)(q-2)\Pi_4^2+3q(q-1)\Pi_5^2 \\
&\quad -4q\Lambda_1^2-4q(q-1)\Lambda_2^2-4q^2\Lambda_3^2]+R_{10}.
\end{aligned}$$

Finally we obtain the main expansion of

$$\begin{aligned}
x_{n+1} &= \mathbf{E}\frac{Z_1}{2q}-\mathbf{E}\frac{Z_1(\sum_{i=1}^{2q}Z_i-2q)}{(2q)^2}+\mathbf{E}\frac{Z_1}{\sum_{i=1}^{2q}Z_i}\frac{(\sum_{i=1}^{2q}Z_i-2q)^2}{(2q)^2}-\frac{1}{2q} \\
&= \mathbf{E}\frac{Z_1}{2q}-\mathbf{E}\frac{Z_1(\sum_{i=1}^{2q}Z_i-2q)}{(2q)^2}+\mathbf{E}\frac{(\sum_{i=1}^{2q}Z_i-2q)^2}{(2q)^3}-\frac{1}{2q} \\
&\quad +\mathbf{E}\left(\frac{Z_1}{\sum_{i=1}^{2q}Z_i}-\frac{1}{2q}\right)\frac{(\sum_{i=1}^{2q}Z_i-2q)^2}{(2q)^2} \\
&= d\frac{\Lambda_1}{2q}+\frac{d(d-1)}{2}\mathcal{X}_2+R \\
&= d\lambda_1^2x_n+(d\lambda_1^2-d\lambda_2^2)z_n+\frac{d(d-1)}{2}\mathcal{X}_2+R_x
\end{aligned}$$

where $R_x = \mathbf{E}\left(\frac{Z_1}{\sum_{i=1}^{2q} Z_i} - \frac{1}{2q}\right)\frac{(\sum_{i=1}^{2q} Z_i - 2q)^2}{(2q)^2} + O_q(x_n^3)$ and

$$\begin{aligned}
\mathcal{X}_2 &= \frac{1}{2q}\Lambda_1^2 - \frac{1}{(2q)^2}[\Pi_1^2 + (q-1)\Pi_2^2 + q\Pi_3^2 - 2q\Lambda_1^2] \\
&\quad + \frac{1}{(2q)^3}[\Pi_1^2 + 3(q-1)\Pi_2^2 + 3q\Pi_3^2 + (q-1)(q-2)\Pi_4^2 + 3q(q-1)\Pi_5^2 \\
&\quad -4q\Lambda_1^2 - 4q(q-1)\Lambda_2^2 - 4q^2\Lambda_3^2] \\
&= \frac{2q-1}{2q^2}\Lambda_1^2 - \frac{q-1}{2q^2}\Lambda_2^2 - \frac{1}{2q}\Lambda_3^2 - \frac{2q-1}{(2q)^3}\Pi_1^2 - \frac{(q-1)(2q-3)}{(2q)^3}\Pi_2^2 \\
&\quad - \frac{2q-3}{8q^2}\Pi_3^2 + \frac{(q-1)(q-2)}{(2q)^3}\Pi_4^2 + \frac{3(q-1)}{8q^2}\Pi_5^2 \\
&= \frac{2q-1}{2q^2}\{2q\lambda_1^2 x_n + 2q(\lambda_1^2 - \lambda_2^2)z_n\}^2 - \frac{q-1}{2q^2}\left\{-\frac{2q\lambda_1^2}{q-1}x_n - \left(\frac{2q\lambda_1^2}{q-1} + 2q\lambda_2^2\right)z_n\right\}^2 - \frac{1}{2q}\{2q\lambda_2^2 z_n\}^2 \\
&\quad - \frac{2q-1}{(2q)^3}\left\{6q\lambda_1^2 x_n + 6q(\lambda_1^2 - \lambda_2^2)z_n\right\}^2 - \frac{(q-1)(2q-3)}{(2q)^3}\left\{\frac{2q(q-3)}{q-1}\lambda_1^2(x_n + z_n) - 6q\lambda_2^2 z_n\right\}^2 \\
&\quad - \frac{2q-3}{8q^2}\left\{2q\lambda_1^2 x_n + 2q(\lambda_1^2 + \lambda_2^2)z_n\right\}^2 + \frac{(q-1)(q-2)}{(2q)^3}\left\{-\frac{6q\lambda_1^2}{q-1}x_n - \left(\frac{6q\lambda_1^2}{q-1} + 6q\lambda_2^2\right)z_n\right\}^2 \\
&\quad + \frac{3(q-1)}{8q^2}\left\{-\frac{2q\lambda_1^2}{q-1}x_n + \left(-\frac{2q\lambda_1^2}{q-1} + 2q\lambda_2^2\right)z_n\right\}^2 + O_q\left(|\lambda_1|^5 x_n^2\left(\left|\frac{u_n}{x_n} - \frac{1}{2q}\right| + \left|\frac{w_n}{x_n} - \frac{1}{2q}\right|\right)\right) \\
&= \frac{q(2q-5)}{q-1}\lambda_1^4(x_n + z_n)^2 - 4q\lambda_1^2\lambda_2^2(x_n + z_n)z_n - 4q\lambda_2^4 z_n^2 \\
&\quad + O_q\left(|\lambda_1|^5 x_n^2\left(\left|\frac{u_n}{x_n} - \frac{1}{2q}\right| + \left|\frac{w_n}{x_n} - \frac{1}{2q}\right|\right)\right).
\end{aligned}$$

Hence we come up with

$$\begin{aligned}
x_{n+1} &= d\lambda_1^2 x_n + (d\lambda_1^2 - d\lambda_2^2)z_n + \frac{d(d-1)}{2}\left(\frac{q(2q-5)}{q-1}\lambda_1^4(x_n + z_n)^2 - 4q\lambda_1^2\lambda_2^2(x_n + z_n)z_n - 4q\lambda_2^4 z_n^2\right. \\
&\quad \left. + O_q\left(|\lambda_1|^5 x_n^2\left(\left|\frac{u_n}{x_n} - \frac{1}{2q}\right| + \left|\frac{w_n}{x_n} - \frac{1}{2q}\right|\right)\right)\right) + R_x \\
&= d\lambda_1^2 x_n + (d\lambda_1^2 - d\lambda_2^2)z_n + \frac{d(d-1)}{2}\left(\frac{q(2q-5)}{q-1}\lambda_1^4(x_n + z_n)^2 - 4q\lambda_1^2\lambda_2^2(x_n + z_n)z_n - 4q\lambda_2^4 z_n^2\right) \\
&\quad + O_q\left(|\lambda_1| x_n^2\left(\left|\frac{u_n}{x_n} - \frac{1}{2q}\right| + \left|\frac{w_n}{x_n} - \frac{1}{2q}\right|\right)\right) + R_x.
\end{aligned}$$

The argument of z_{n+1} is substantially the same as x_{n+1}, by noting that, instead of Z_1, there is an analogous discussion such as

$$\begin{aligned}
\mathbf{E}Z_{q+1}\left(\sum_{i=1}^{2q} Z_i - 2q\right) &= \mathbf{E}Z_{q+1}^2 + \mathbf{E}Z_1 Z_{q+1} + \sum_{i=2}^{q}\mathbf{E}Z_{q+1}Z_i + \sum_{i=q+2}^{2q}\mathbf{E}Z_{q+1}Z_i - 2q\mathbf{E}Z_{q+1} \\
&= \frac{d(d-1)}{2}[2\Pi_3^2 + (2q-2)\Pi_5^2 - 2q\Lambda_3^2] + R_{11},
\end{aligned}$$

and therefore

$$\begin{aligned}
z_{n+1} &= \mathbf{E}\frac{Z_{q+1}}{2q} - \mathbf{E}\frac{Z_{q+1}(\sum_{i=1}^{2q} Z_i - 2q)}{(2q)^2} + \mathbf{E}\frac{Z_{q+1}}{\sum_{i=1}^{2q} Z_i}\frac{(\sum_{i=1}^{2q} Z_i - 2q)^2}{(2q)^2} - \frac{1}{2q} \\
&= \mathbf{E}\frac{Z_{q+1}}{2q} - \mathbf{E}\frac{Z_{q+1}(\sum_{i=1}^{2q} Z_i - 2q)}{(2q)^2} + \mathbf{E}\frac{(\sum_{i=1}^{2q} Z_i - 2q)^2}{(2q)^3} - \frac{1}{2q} \\
&\quad + \mathbf{E}\left(\frac{Z_{q+1}}{\sum_{i=1}^{2q} Z_i} - \frac{1}{2q}\right)\frac{(\sum_{i=1}^{2q} Z_i - 2q)^2}{(2q)^2} \\
&= d\frac{\Lambda_3}{2q} + \frac{d(d-1)}{2}\mathcal{Z}_2 + R_z,
\end{aligned}$$

where $R_z = \mathbf{E}\left(\frac{Z_{q+1}}{\sum_{i=1}^{2q} Z_i} - \frac{1}{2q}\right)\frac{(\sum_{i=1}^{2q} Z_i - 2q)^2}{(2q)^2} + O_q(x_n^3)$ and

$$\begin{aligned}
\mathcal{Z}_2 &= \frac{1}{2q}\Lambda_3^2 - \frac{1}{(2q)^2}[2\Pi_3^2 + (2q-2)\Pi_5^2 - 2q\Lambda_3^2] \\
&\quad + \frac{1}{(2q)^3}[\Pi_1^2 + 3(q-1)\Pi_2^2 + 3q\Pi_3^2 + (q-1)(q-2)\Pi_4^2 + 3q(q-1)\Pi_5^2 \\
&\quad - 4q\Lambda_1^2 - 4q(q-1)\Lambda_2^2 - 4q^2\Lambda_3^2] \\
&= \left(\frac{q}{q-1}\lambda_1^4(x_n+z_n)^2 - 4q\lambda_2^4 z_n^2\right) + O_q\left(|\lambda_1|^5 x_n^2\left(\left|\frac{u_n}{x_n} - \frac{1}{2q}\right| + \left|\frac{w_n}{x_n} - \frac{1}{2q}\right|\right)\right).
\end{aligned}$$

Last plugging in gives

$$\begin{aligned}
z_{n+1} &= d\lambda_2^2 z_n - \frac{d(d-1)}{2}\left(\frac{q}{q-1}\lambda_1^4(x_n+z_n)^2 - 4q\lambda_2^4 z_n^2\right) \\
&\quad + O_q\left(|\lambda_1| x_n^2\left(\left|\frac{u_n}{x_n} - \frac{1}{2q}\right| + \left|\frac{w_n}{x_n} - \frac{1}{2q}\right|\right)\right) + R_z.
\end{aligned}$$

To the end, taking substitutions of $X_n = x_n + z_n$ and $Z_n = -z_n$, this two dimensional recursive formula becomes the canonical form of the linear diagonal part,

$$\begin{aligned}
X_{n+1} &= d\lambda_1^2 X_n + \frac{d(d-1)}{2}\left(\frac{2q(q-3)}{q-1}\lambda_1^4 X_n^2 + 4q\lambda_1^2\lambda_2^2 X_n Z_n\right) \\
&\quad + O_q\left(|\lambda_1| x_n^2\left(\left|\frac{u_n}{x_n} - \frac{1}{2q}\right| + \left|\frac{w_n}{x_n} - \frac{1}{2q}\right|\right)\right) + R_x + R_z \qquad (2.5.2)
\end{aligned}$$

and

$$\begin{aligned}
Z_{n+1} &= d\lambda_2^2 Z_n + \frac{d(d-1)}{2}\left(\frac{q}{q-1}\lambda_1^4 X_n^2 - 4q\lambda_2^4 Z_n^2\right) \\
&\quad + O_q\left(|\lambda_1| x_n^2\left(\left|\frac{u_n}{x_n} - \frac{1}{2q}\right| + \left|\frac{w_n}{x_n} - \frac{1}{2q}\right|\right)\right) - R_z. \qquad (2.5.3)
\end{aligned}$$

CHAPTER 3

RECONSTRUCTION FOR $q \geq 4$

3.1 Equivalent Condition for Non-reconstruction.

If the reconstruction problem is solvable, then $\sigma(n)$ contains significant information on the root variable. This may be expressed in several equivalent ways [14, Proposition 14].

Lemma 3.1.1. *The non-reconstruction is equivalent to*

$$\lim_{n\to\infty} x_n = 0.$$

Proof. The maximum-likelihood algorithm, which is the optimal reconstruction algorithm of σ_ρ given $\sigma(n)$, is successful with probability

$$\Delta(n) = \mathbf{E} \max_{1\leq i\leq 2q} X_i(n).$$

The inequality $x_n + \frac{1}{2q} \leq \Delta_n$ was shown in [4] by noting that the algorithm that choose σ_ρ randomly according to probabilities X_i is correct with probability $x_n + \frac{1}{2q}$.

On the other side, the identity (2.2.1) indicates that $x_n = \mathbf{E}\sum_{i=1}^{2q}\left(X_i(n) - \frac{1}{2q}\right)^2$ and then by the Cauchy-Schwartz inequality, we have

$$\begin{aligned}
\Delta_n &\leq \frac{1}{2q} + \mathbf{E}\max_i \left|X_i(n) - \frac{1}{2q}\right| \\
&\leq \frac{1}{2q} + \left(\mathbf{E}\max_i \left(X_i(n) - \frac{1}{2q}\right)^2\right)^{\frac{1}{2}} \\
&\leq \frac{1}{2q} + \left(\mathbf{E}\sum_{i=1}^{2q} \left(X_i(n) - \frac{1}{2q}\right)^2\right)^{\frac{1}{2}} \\
&= \frac{1}{2q} + x_n^{\frac{1}{2}}.
\end{aligned}$$

To sum up, we come up with the inequalities

$$x_n \leq \Delta_n - \frac{1}{2q} \leq x_n^{\frac{1}{2}},$$

following that $\lim_{n\to\infty} x_n = 0$ is equivalent to $\lim_{n\to\infty} \Delta_n = \frac{1}{2q}$, which is in turn equivalent to non-reconstruction [14]. $\square$

3.2 Concentration Analysis.

Before investigating the concentration, we would like to introduce a significant lemma showing that x_n does not drop from a very large value to a very small one.

Lemma 3.2.1. *For any $\varrho > 0$ there exists a constant $\gamma = \gamma(q, d, \varrho) > 0$ such that for all n when* $\min\{|\lambda_1|, |\lambda_2|\} > \varrho$,

$$x_{n+1} \geq \gamma x_n.$$

Proof. For a configuration A on $\mathbb{T}_{u_1} \cap L(n+1)$ define

$$\begin{aligned}
f^*_{n+1}(1, A) &= \mathbf{P}(\sigma_\rho = 1 \mid \sigma_1(n+1) = A) \\
&= \frac{1}{2q} \frac{\mathbf{P}(\sigma_1(n+1) = A \mid \sigma_\rho = 1)}{\mathbf{P}(\sigma_1(n+1) = A)} \\
&= \frac{1}{2q} (\mathbf{P}(\sigma_1(n+1) = A))^{-1} [p_0 \mathbf{P}(\sigma_1(n+1) = A \mid \sigma_{u_1} = 1) \\
&\quad + p_1 \sum_{i=2}^{q} \mathbf{P}(\sigma_1(n+1) = A \mid \sigma_{u_1} = i) + p_2 \sum_{i=q+1}^{2q} \mathbf{P}(\sigma_1(n+1) = A \mid \sigma_{u_1} = i)] \\
&= \frac{1}{2q} + p_0 \left(f_n(1, A) - \frac{1}{2q} \right) + p_1 \sum_{i=2}^{q} \left(f_n(i, A) - \frac{1}{2q} \right) + p_2 \sum_{i=q+1}^{2q} \left(f_n(i, A) - \frac{1}{2q} \right) \\
&= \frac{1}{2q} + \lambda_1 \left(f_n(1, A) - \frac{1}{2q} \right) + \frac{\lambda_1 - \lambda_2}{q} \sum_{i=q+1}^{2q} \left(f_n(i, A) - \frac{1}{2q} \right)
\end{aligned}$$

and then

$$\begin{aligned}
\mathbf{E} f^*_{n+1}(1, \sigma_1^1(n+1)) &= \frac{1}{2q} + \lambda_1 \mathbf{E} \left(Y_{11}(n) - \frac{1}{2q} \right) + \frac{\lambda_1 - \lambda_2}{q} q \mathbf{E} \left(Y_{q+11}(n) - \frac{1}{2q} \right) \\
&= \frac{1}{2q} + \frac{\lambda_2^2 + (q-1)\lambda_1^2}{q} x_n + \frac{(q-1)(\lambda_2^2 - \lambda_1^2)}{q} y_n \\
&= \frac{1}{2q} + \lambda_1^2 x_n + (\lambda_1^2 - \lambda_2^2) z_n.
\end{aligned}$$

The estimator that chooses a state with probability $f_{n+1}(i, \sigma_1(n+1))$ correctly reconstructs the root with probability $\frac{1}{2q} + \lambda_1^2 x_n + (\lambda_1^2 - \lambda_2^2) z_n$. Since this probability must be less than the maximum-likelihood estimation, it follows that

$$\frac{1}{2q} + \lambda_1^2 x_n + (\lambda_1^2 - \lambda_2^2) z_n \leq \Delta_{n+1} \leq \frac{1}{2q} + x_{n+1}^{\frac{1}{2}}.$$

If $\lambda_1^2 \geq \lambda_2^2$, by noting that $x_n + z_n \geq 0$ in Lemma 2.2.1, it follows

$$\begin{aligned} \lambda_2^2 x_n &\leq \lambda_2^2 x_n + (\lambda_1^2 - \lambda_2^2)(x_n + z_n) \\ &= \lambda_1^2 x_n + (\lambda_1^2 - \lambda_2^2) z_n \\ &\leq x_{n+1}^{\frac{1}{2}}. \end{aligned}$$

On the other side, if $\lambda_1^2 \leq \lambda_2^2$, then $\lambda_1^2 x_n \leq x_{n+1}^{\frac{1}{2}}$ follows from $z_n \leq 0$ immediately. To sum up, we always have

$$\min\{\lambda_1^2, \lambda_2^2\} x_n \leq x_{n+1}^{\frac{1}{2}}.$$

Next choose $\varepsilon = \varrho^2$, by the expansion of x_{n+1}, it is known that there exists a $\delta = \delta(\varepsilon, q) > 0$ when $x_n < \delta$,

$$x_{n+1} \geq (d \min\{\lambda_1^2, \lambda_2^2\} - \varepsilon) x_n \geq (d-1) \varrho^2 x_n.$$

On the other hand, if $x_n \geq \delta$, then $x_{n+1} \geq (\min\{\lambda_1^2, \lambda_2^2\} x_n)^2 \varrho^4 \delta x_n$. Finally making $\gamma = \min\{(d-1)\varrho^2, \varrho^4 \delta\}$ completes the proof. $\square$

Next we come up with a crucial lemma, indicating that the fixed finite vertices long way from the root have only a small effect on the root, which would play a crucial role in concentration analysis.

Lemma 3.2.2. *For any $\varepsilon > 0$ and positive integer k there exists $M = M(q, \varepsilon, k)$ such that for any collection of vertices $v_1, \ldots, v_k \in L(M)$,*

$$\sup_{i, i_1, \ldots, i_k} \left| \mathbf{P}(\sigma_\rho = i \mid \sigma_{v_j} = i_j, 1 \leq j \leq k) - \frac{1}{2q} \right| \leq \varepsilon$$

Proof. Denote the transition matrices at distance s by

$$U_s = M_{1,1}^s, \quad V_s = M_{1,2}^s, \quad \text{and} \quad W_s = M_{1,q+1}^s,$$

and then it is convenient to figure out the iterative formulae of them such as

$$\begin{cases} U_s = p_0 U_{s-1} + (q-1)p_1 V_{s-1} + qp_2 W_{s-1} \\ V_s = p_1 U_{s-1} + [p_0 + (q-2)p_1] V_{s-1} + qp_2 W_{s-1} \\ W_s = p_2 U_{s-1} + (q-1)p_2 V_{s-1} + [p_0 + (q-1)p_1] W_{s-1}. \end{cases}$$

To evaluate this three order recursive system, let's start with the difference of the first two equation

$$U_s - V_s = \lambda_1 (U_{s-1} - V_{s-1}),$$

and then considering the initial conditions of $U_0 = 1$ and $V_0 = W_0 = 0$, it follows that

$$U_s - V_s = \lambda_1^s. \tag{3.2.1}$$

Taking all the formulae above in tandem with the stationary property of

$$U_s + (q-1)V_s + qW_s = 1$$

we conclude that

$$\begin{aligned} U_s &= p_0 U_{s-1} + (q-1)p_1 V_{s-1} + qp_2 W_{s-1} \\ &= (p_0 - p_2) U_{s-1} + (q-1)(p_1 - p_2)(U_s - \lambda_1^s) + p_2 \\ &= \lambda_2 U_{s-1} + \frac{1-\lambda_2}{2q} - \frac{q-1}{q}(\lambda_2 - \lambda_1)\lambda_1^s. \end{aligned}$$

Therefore the general solutions are

$$\begin{cases} U_s = \frac{1}{2q} + \left(1 - \frac{1}{2q}\right)\lambda_2^s - \left(1 - \frac{1}{q}\right)\lambda_1(\lambda_2^s - \lambda_1^s) \\ V_s = \frac{1}{2q} - \frac{1}{2q}\lambda_2^s + \left(1 - \frac{q-1}{q}\lambda_1\right)(\lambda_2^s - \lambda_1^s) \\ W_s = \frac{1}{2q} - \frac{1}{2q}\lambda_2^s - \left(1 - \frac{1}{q}\right)(1-\lambda_1)(\lambda_2^s - \lambda_1^s) \end{cases}$$

and thus under the conditions of $d\lambda^2 \leq 1$, it is concluded that U_s, V_s and W_s all range in the same small interval such as

$$\frac{1}{2q} - 3d^{-\frac{s}{2}} \leq U_s, V_s, W_s \leq \frac{1}{2q} + 3d^{-\frac{s}{2}};$$

For fixed q, d and k, define

$$B(s) = \frac{\frac{1}{2q} + 3d^{-\frac{s}{2}}}{\frac{1}{2q} - 3d^{-\frac{s}{2}}}$$

and let γ be a sufficiently large integer, depending only on q, such that $B^k(\gamma) \le 1+\varepsilon$, since $d^{-s/2} \le 2^{-s/2} \to 0$ as $s \to \infty$ implies $B(s) \to 1$.

Fix an integer M such that $M > k\gamma$. Now choose any k vertices on the Mth level $v_1, \ldots, v_k \in L(M)$. For $0 \le \ell \le M$ define

$$n_\ell = \#\{v \in L(\ell) : |\mathbb{T}_v \cap \{v_1, \ldots, v_k\}| > 0\},$$

the number of vertices with distance ℓ from the root that have at least one decedent in the set $\{v_1, \ldots, v_k\}$. Then from the definition, it is trivial to come up with that n_ℓ is an increasing integer valued function with respect to ℓ from $n_0 = 1$ to $n_M = k$, which, by the pigeonhole principle, implies that there must exist some ℓ such that $n_\ell = n_{\ell+\gamma}$.

Next denote $\{w_1, \ldots, w_{n_\ell}\}$ and $\{\overline{w}_1, \ldots, \overline{w}_{n_\ell}\}$ by the vertices in the sets $\{v \in L(\ell+\gamma) : |\mathbb{T}_v \cap \{v_1, \ldots, v_k\}| > 0\}$ and $\{v \in L(\ell) : |\mathbb{T}_v \cap \{v_1, \ldots, v_k\}| > 0\}$ respectively, such that w_j is the descendent of $\overline{w}_j$, and then

$$\mathbf{P}(\sigma_{w_j} = i_2 \mid \sigma_{\overline{w}_j} = i_1) = M^\gamma_{i_1 i_2}.$$

Then for any $i, i', i_1, \ldots, i_{n_\ell} \in \mathcal{C}$, the following could be derived from Bayes' Rule,

$$\begin{aligned}
& \frac{\mathbf{P}(\sigma_\rho = i \mid \sigma_{w_j} = i_j, 1 \le j \le n_\ell)}{\mathbf{P}(\sigma_\rho = i' \mid \sigma_{w_j} = i_j, 1 \le j \le n_\ell)} \\
= & \frac{\mathbf{P}(\sigma_{w_j} = i_j, 1 \le j \le n_\ell \mid \sigma_\rho = i)}{\mathbf{P}(\sigma_{w_j} = i_j, 1 \le j \le n_\ell \mid \sigma_\rho = i')} \\
= & \frac{\sum_{h_1,\ldots,h_{n_\ell} \in \mathcal{C}} \mathbf{P}(\forall j\ \sigma_{w_j} = i_j \mid \forall j\ \sigma_{\overline{w}_j} = h_j) \mathbf{P}(\forall j\ \sigma_{\overline{w}_j} = h_j \mid \sigma_\rho = i)}{\sum_{h_1,\ldots,h_{n_\ell} \in \mathcal{C}} \mathbf{P}(\forall j\ \sigma_{w_j} = i_j \mid \forall j\ \sigma_{\overline{w}_j} = h_j) \mathbf{P}(\forall j\ \sigma_{\overline{w}_j} = h_j \mid \sigma_\rho = i')} \\
= & \frac{\sum_{h_1,\ldots,h_{n_\ell} \in \mathcal{C}} \mathbf{P}(\forall j\ \sigma_{\overline{w}_j} = h_j \mid \sigma_\rho = i) \prod_{j=1}^{n_\ell} M^\gamma_{h_j i_j}}{\sum_{h_1,\ldots,h_{n_\ell} \in \mathcal{C}} \mathbf{P}(\forall j\ \sigma_{\overline{w}_j} = h_j \mid \sigma_\rho = i') \prod_{j=1}^{n_\ell} M^\gamma_{h_j i_j}} \\
\le & B^{n_\ell}(\gamma) \frac{\sum_{h_1,\ldots,h_{n_\ell} \in \mathcal{C}} \mathbf{P}(\forall j\ \sigma_{\overline{w}_j} = h_j \mid \sigma_\rho = i)}{\sum_{h_1,\ldots,h_{n_\ell} \in \mathcal{C}} \mathbf{P}(\forall j\ \sigma_{\overline{w}_j} = h_j \mid \sigma_\rho = i')} \\
\le & B^k(\gamma) \\
\le & (1+\varepsilon),
\end{aligned}$$

where the third equality comes from the Markov random field property, implying that the σ_{w_j} are conditionally independent given the $\sigma_{\overline{w_j}}$. The preceding formula yields that

$$\frac{1-\varepsilon}{2q} \le \mathbf{P}(\sigma_\rho = i \mid \sigma_{w_j} = i_j, 1 \le j \le n_\ell) \le \frac{1+\varepsilon}{2q}.$$

Finally the proof is completed as following

$$\begin{aligned} & \sup_{i,i_1,\dots,\, i_k} \left| \mathbf{P}(\sigma_\rho = i \mid \sigma_{v_j} = i_j, 1 \le j \le k) - \frac{1}{2q} \right| \\ \le & \sup_{i,i_1,\dots,\, i_{n_\ell}} \left| \mathbf{P}(\sigma_\rho = i \mid \sigma_{w_j} = i_j, 1 \le j \le n_\ell) - \frac{1}{2q} \right| \\ \le & \ \varepsilon, \end{aligned}$$

since σ_ρ is conditionally independent of the collection $\sigma_{v_1}, \dots, \sigma_{v_k}$ given $\sigma_{w_1}, \dots, \sigma_{w_{n_\ell}}$. □

With the assistance of (3.2.1) we are able to set up the following lemma immediately.

Lemma 3.2.3. *Assume $\lambda_1 \neq 0$. For any nonnegative $n \in \mathbb{Z}$, we always have*

$$x_n + z_n > 0.$$

Proof. In Lemma 2.2.1 we have proved that $x_n + z_n \ge 0$, so here it suffices to exclude the equality. Now we refer to the contradiction and assume $x_n + z_n = 0$ for some $n \in \mathbb{N}$. It follows that if $i \neq j$ are in the same configuration set, then

$$\begin{aligned} \mathbf{E}(X_i(n) - X_j(n))^2 &= 2EX_i^2(n) - 2EX_i(n)X_j(n) \\ &= 2\frac{x_n + \frac{1}{2q}}{2q} - 2\frac{y_n + \frac{1}{2q}}{2q} \\ &= \frac{x_n - y_n}{q} \\ &= \frac{x_n + z_n}{q-1} \\ &= 0. \end{aligned}$$

Therefore $X_1(n) = X_2(n) = \cdots X_q(n)$ and $X_{q+1}(n) = X_{q+2}(n) = \cdots X_{2q}(n)$ a.s., that is, for any configuration combination A on the nth level, we always have

$$\mathbf{P}(\sigma_\rho = 1 \mid \sigma(n) = A) = \mathbf{P}(\sigma_\rho = 2 \mid \sigma(n) = A).$$

Denote the leftmost vertex on the nth level by $v_n(1)$, and it follows

$$\mathbf{P}(\sigma_\rho = 1 \mid \sigma_{v_n(1)} = 1) = \mathbf{P}(\sigma_\rho = 2 \mid \sigma_{v_n(1)} = 1).$$

Last from the reversible property of the channel, we can conclude that

$$\lambda_1^n = U_n - V_n = \mathbf{P}(\sigma_\rho = 1 \mid \sigma_{v_n(1)} = 1) - \mathbf{P}(\sigma_\rho = 2 \mid \sigma_{v_n(1)} = 1) = 0,$$

i.e., $\lambda_1 = 0$, a contradiction to the assumption of $\lambda_1 \neq 0$. □

The next lemma establishes concentration of the posterior distributions when x_n is small.

Lemma 3.2.4. *Assume* $\min\{|\lambda_1|, |\lambda_2|\} > \varrho$ *for some* $\varrho > 0$. *Given arbitrary* $\varepsilon, \alpha > 0$ *there exist constants* $C = C(q, d, \varepsilon, \alpha, \varrho) > 0$ *and* $N = N(q, \varepsilon, \alpha, \varrho)$ *such that whenever* $n > N$,

$$\mathbf{P}\left(\left|\frac{Z_1}{\sum_{i=1}^{2q} Z_i} - \frac{1}{2q}\right| > \varepsilon\right) \le C x_n^\alpha.$$

Proof. Suppose that k is an integer with $k > \alpha$. In Lemma 3.2.3 choose M to hold with bound $\frac{\varepsilon}{2}$. Then let $v_1, \ldots, v_{|L(M)|}$ denote the vertices on $L(M)$ and define

$$F(i, v) = f_{n+1-M}(i, \sigma_v^1(n+1))$$

where $\sigma_v^1(n+1)$ denotes the spins of vertices in $\mathbb{T}_v \cap L(n+1)$. The distribution of $F(i, v)$ would be discussed as follows,

$$F(i,v) \sim \begin{cases} X^{(1)}(n+1-M) & \text{if } \sigma_v^1 = i, \\ \\ X^{(2)}(n+1-M) & \text{if } \sigma_v^1 = j:\ i, j \text{ come from the same category}, \\ \\ X^{(3)}(n+1-M) & \text{if } \sigma_v^1 = j:\ i, j \text{ are in the different categories}. \end{cases}$$

By the Markov random field property, the vectors $(F(1, v), \ldots, F(2q, v))$ are conditionally independent on $\sigma(M)$ for different $v \in L(M)$. Repeating the technique of recursion relation in Section 2.4, a posterior probability of a vertex can be written as a function of the posterior probabilities of its children such as

$$\frac{Z_1}{\sum_{i=1}^{2q} Z_i} = f_{n+1}(1, \sigma^1(n+1)) = H(\mathcal{F})$$

for some function $H(\mathcal{F})$ with the vector variable

$$\mathcal{F} := (F(1, v_1), \ldots, F(1, v_{|L(M)|}), \ldots, (F(2q, v_1), \ldots, F(2q, v_{|L(M)|})).$$

There is no difficulty in finding that $H(\mathcal{F}) = \frac{1}{2q}$ when all the entries in $\mathcal{F}$ are identically $\frac{1}{2q}$ and $H(\mathcal{F})$ is a continuous function of the vector $\mathcal{F}$. Therefore by Lemma 3.2.3, if there are at most k vertices $v \in L(M)$ such that for some $1 \leq i \leq 2q$, $F(i,v) \neq \frac{1}{2q}$ then

$$\left| H(\mathcal{F}) - \frac{1}{2q} \right| < \frac{\varepsilon}{2}.$$

As a result, there exists a $\delta = \delta(\varepsilon) > 0$ such that if $\mathcal{F}$ satisfies

$$\# \left\{ v \in L(M) : \max_{1 \leq i \leq 2q} \left| F(i,v) - \frac{1}{2q} \right| > \delta \right\} \leq k$$

then

$$\left| H(\mathcal{F}) - \frac{1}{2q} \right| < \varepsilon.$$

On the other side, from Chebyshev's inequality and Lemma 2.2.2, we could estimate

$$\begin{aligned}
& \mathbf{P}\left(\max_{1 \leq i \leq 2q} \left| F(i,v) - \frac{1}{2q} \right| > \delta \right) \\
\leq\ & \mathbf{P}\left(\left| X^{(1)}(n+1-M) - \frac{1}{2q} \right| > \delta \right) + (q-1)\mathbf{P}\left(\left| X^{(2)}(n+1-M) - \frac{1}{2q} \right| > \delta \right) \\
& + q\mathbf{P}\left(\left| X^{(3)}(n+1-M) - \frac{1}{2q} \right| > \delta \right) \\
\leq\ & \delta^{-2} \left[\mathbf{E}\left(X^{(1)}(n+1-M) - \frac{1}{2q} \right)^2 + (q-1)\mathbf{E}\left(X^{(2)}(n+1-M) - \frac{1}{2q} \right)^2 \right. \\
& \left. + q\mathbf{E}\left(X^{(3)}(n+1-M) - \frac{1}{2q} \right)^2 \right] \\
=\ & \frac{x_{n+1-M}}{\delta^2}.
\end{aligned}$$

Finally based on the conditional independence given $\sigma(M)$, the random variables $\max_{1 \leq i \leq 2q} \left| F(i,v) - \frac{1}{2q} \right|$ further turn out to be independent because of the symmetry of the model implying that they do not in fact depend on the spins in $L(M)$. To the end, under the assumption of $\min\{|\lambda_1|, |\lambda_2|\} > \varrho$, there exist constants $C = C(q,d,\varepsilon,\alpha,\varrho)$ and $N = N(q,\varepsilon,\alpha,\varrho)$ such that whenever $n > N$,

$$\begin{aligned}
\mathbf{P}\left(\left| \frac{Z_1}{\sum_{i=1}^{2q} Z_i} - \frac{1}{2q} \right| > \varepsilon \right) & \leq \mathbf{P}\left(\# \left\{ v \in L(M) : \max_{1 \leq i \leq 2q} \left| F(i,v) - \frac{1}{2q} \right| > \delta \right\} > k \right) \\
& \leq \mathbf{P}\left(\mathbf{B}\left(|L(M)|, \frac{x_{n+1-M}}{\delta^2} \right) > k \right) \\
& \leq C' x_{n+1-M}^{\alpha} \\
& \leq C x_n^{\alpha},
\end{aligned}$$

where $\mathbf{B}(\cdot,\cdot)$ denotes the binomial distribution, and we use the fact of $k > \alpha$ and Lemma 3.2.1.

□

With the preceding concentration result, we are able to bound R_x and R_z in (2.5.2) and (2.5.3).

Proposition 3.2.5. *Assume* $\min\{|\lambda_1|, |\lambda_2|\} > \varrho$ *for some* $\varrho > 0$. *For any* $\varepsilon > 0$, *there exist* $N = N(q, \varepsilon, \varrho)$ *and* $\delta = \delta(q, d, \varepsilon, \varrho) > 0$ *such that if* $n > N$ *and* $x_n \leq \delta$ *then* $|R_x| \leq \varepsilon x_n^2$ *and* $|R_x| \leq \varepsilon x_n^2$.

Proof. For any $\eta > 0$ and $1 \leq i \leq 2q$, applying Cauchy-Schwartz inequality gives

$$
\begin{aligned}
& \left| \mathbf{E} \frac{Z_1}{\sum_{i=1}^{2q} Z_i} \frac{(\sum_{i=1}^{2q} Z_i - 2q)^2}{(2q)^2} - \mathbf{E} \frac{1}{2q} \frac{(\sum_{i=1}^{2q} Z_i - 2q)^2}{(2q)^2} \right| \\
\leq\ & \mathbf{E} \frac{(\sum_{i=1}^{2q} Z_i - 2q)^2}{(2q)^2} \left| \frac{Z_1}{\sum_{i=1}^{2q} Z_i} - \frac{1}{2q} \right| \\
\leq\ & \eta \mathbf{E} \left(\frac{(\sum_{i=1}^{2q} Z_i - 2q)^2}{(2q)^2}; \left| \frac{Z_1}{\sum_{i=1}^{2q} Z_i} - \frac{1}{2q} \right| \leq \eta \right) + \mathbf{E} \left(\frac{(\sum_{i=1}^{2q} Z_i - 2q)^2}{(2q)^2} \mathbf{I} \left(\left| \frac{Z_1}{\sum_{i=1}^{2q} Z_i} - \frac{1}{2q} \right| > \eta \right) \right) \\
\leq\ & \eta \mathbf{E} \left(\frac{(\sum_{i=1}^{2q} Z_i - 2q)^2}{(2q)^2} \right) + \left(\mathbf{E} \frac{(\sum_{i=1}^{2q} Z_i - 2q)^4}{(2q)^4} \right)^{\frac{1}{2}} \left(\mathbf{P} \left(\left| \frac{Z_1}{\sum_{i=1}^{2q} Z_i} - \frac{1}{2q} \right| > \eta \right) \right)^{\frac{1}{2}}.
\end{aligned}
$$

From the calculation for distributional recursion and Lemma 2.5.2, $\mathbf{E} \left(\frac{(\sum_{i=1}^{2q} Z_i - 2q)^2}{(2q)^2} \right) \leq C_1(q) x_n^2$ and $\mathbf{E} \frac{(\sum_{i=1}^{2q} Z_i - 2q)^4}{(2q)^4} \leq C_2(q)$. Besides, by taking $\alpha = 6$ in Lemma 3.2.4, there exist $C_3 = C_3(q, d, \eta, \varrho)$ and $N = N(q, \eta, \varrho)$ such that whenever $n > N$,

$$
\mathbf{P} \left(\left| \frac{Z_1}{\sum_{i=1}^{2q} Z_i} - \frac{1}{2q} \right| > \eta \right) \leq C_3 x_n^6.
$$

and thus there is a $C_4 = C_4(q, d, \eta)$ such that

$$
\begin{aligned}
|R_x| &= \left| \mathbf{E} \left(\frac{Z_1}{\sum_{i=1}^{2q} Z_i} - \frac{1}{2q} \right) \frac{(\sum_{i=1}^{2q} Z_i - 2q)^2}{(2q)^2} + O_q(x_n^3) \right| \\
&\leq \eta C_1 x_n^2 + C_4 x_n^3.
\end{aligned}
$$

Finally it suffices to take $C_1 \eta = \frac{\varepsilon}{2}$ and $\delta = \frac{\varepsilon}{2C_4}$, so if $x_n \leq \delta$, then $R_x \leq \varepsilon x_n^2$. Repeating the similar discussion yields $R_z \leq \varepsilon x_n^2$. □

In the following we would like to estimate the terms $u_n - \frac{x_n}{2q}$ and $w_n - \frac{x_n}{2q}$ when x_n is small.

Lemma 3.2.6. *Assume $|\lambda_1| = |\lambda_2|$ or $\frac{|\lambda_1|}{|\lambda_2|} \geq \kappa$ for some $\kappa > 1$, where $\lambda_2 = 0$ is allowed since in that case we could choose any $\kappa > 1$. For any $\varepsilon > 0$, there exist $N = N(q, \kappa, \varepsilon)$ and $\delta = \delta(q, \kappa, \varepsilon) > 0$ such that if $n \geq N$ and $x_n \leq \delta$ then*

$$\left|\frac{u_n}{x_n} - \frac{1}{2q}\right| < \varepsilon \quad \text{and} \quad \left|\frac{w_n}{x_n} - \frac{1}{2q}\right| < \varepsilon.$$

Proof. Applying the identity (2.5.1) again, we have

$$\begin{aligned}
u_{n+1} &= \mathbf{E}\frac{\left(Z_1 - \frac{1}{2q}\sum_{i=1}^{2q} Z_i\right)^2}{\left(\sum_{i=1}^{2q} Z_i\right)^2} \\
&= \frac{1}{4q^2}\mathbf{E}\left(Z_1 - \frac{1}{2q}\sum_{i=1}^{2q} Z_i\right)^2 - \frac{1}{16q^4}\mathbf{E}\left(Z_1 - \frac{1}{2q}\sum_{i=1}^{2q} Z_i\right)^2\left(\left(\sum_{i=1}^{2q} Z_i\right)^2 - 4q^2\right) \\
&\quad + \frac{1}{16q^4}\mathbf{E}\frac{\left(Z_1 - \frac{1}{2q}\sum_{i=1}^{2q} Z_i\right)^2}{\left(\sum_{i=1}^{2q} Z_i\right)^2}\left(\left(\sum_{i=1}^{2q} Z_i\right)^2 - 4q^2\right).
\end{aligned}$$

Next let's estimate these expectations term by term and we remark that all the O_q constants in the following context only depend on q:

$$\begin{aligned}
&\mathbf{E}\left(Z_1 - \frac{1}{2q}\sum_{i=1}^{2q} Z_i\right)^2 \\
= \;& \mathbf{E}\left((Z_1 - 1) - \frac{1}{2q}\left(\sum_{i=1}^{2q} Z_i - 2q\right)\right)^2 \\
= \;& \mathbf{E}(Z_1 - 1)^2 - \frac{2}{2q}\mathbf{E}(Z_1 - 1)\left(\sum_{i=1}^{2q} Z_i - 2q\right) + \frac{1}{4q^2}E\left(\sum_{i=1}^{2q} Z_i - 2q\right)^2 \\
= \;& \mathbf{E}Z_1^2 - 2\mathbf{E}Z_1 + 1 + +O_q(x_n^2) \\
= \;& (1 + d\Pi_1) - 2(1 + d\Lambda_1) + 1 + O_q(x_n^2) \\
= \;& 2dq\lambda_1^2 x_n + 2dq(\lambda_1^2 - \lambda_2^2)z_n + 4dq^2\lambda_1^3\left(u_n - \frac{x_n}{2q}\right) + 12dq^2\lambda_1^2(\lambda_1 - \lambda_2)\left(w_n - \frac{x_n}{2q}\right) + O_q(x_n^2)
\end{aligned}$$

and similarly,

$$\mathbf{E}\left(Z_1 - \frac{1}{2q}\sum_{i=1}^{2q} Z_i\right)^2\left(\left(\sum_{i=1}^{2q} Z_i\right)^2 - 4q^2\right) = O_q(x_n^2),$$

as well as

$$\mathbf{E}\left(\left(\sum_{i=1}^{2q} Z_i\right)^2 - 4q^2\right)^2 = O_q(x_n^2).$$

Substituting these bounds into the expansion of u_{n+1}, plus noting that

$$\frac{\left(Z_1 - \frac{1}{2q}\sum_{i=1}^{2q} Z_i\right)^2}{\left(\sum_{i=1}^{2q} Z_i\right)^2} \leq 1$$

it comes up with the following expansion

$$\begin{aligned} u_{n+1} &= \frac{d\lambda_1^2 x_n + d(\lambda_1^2 - \lambda_2^2) z_n}{2q} + d\lambda_1^3 \left(u_n - \frac{x_n}{2q}\right) + 3d\lambda_1^2(\lambda_1 - \lambda_2)\left(w_n - \frac{x_n}{2q}\right) + O_q(x_n^2) \\ &= \frac{x_{n+1}}{2q} + d\lambda_1^3 \left(u_n - \frac{x_n}{2q}\right) + 3d\lambda_1^2(\lambda_1 - \lambda_2)\left(w_n - \frac{x_n}{2q}\right) + O_q(x_n^2) \end{aligned}$$

by means of $x_{n+1} = d\lambda_1^2 x_n + d(\lambda_1^2 - \lambda_2^2) z_n + O_q(x_n^2)$. Moreover, the similar expansion of w_{n+1} would be

$$\begin{aligned} w_{n+1} &= \mathbf{E}\frac{\left(Z_{q+1} - \frac{1}{2q}\sum_{i=1}^{2q} Z_i\right)^2}{\left(\sum_{i=1}^{2q} Z_i\right)^2} \\ &= \frac{1}{4q^2}\mathbf{E}\left(Z_{q+1} - \frac{1}{2q}\sum_{i=1}^{2q} Z_i\right)^2 - \frac{1}{16q^4}\mathbf{E}\left(Z_{q+1} - \frac{1}{2q}\sum_{i=1}^{2q} Z_i\right)^2 \left(\left(\sum_{i=1}^{2q} Z_i\right)^2 - 4q^2\right) \\ &\quad + \frac{1}{16q^4}\mathbf{E}\frac{\left(Z_{q+1} - \frac{1}{2q}\sum_{i=1}^{2q} Z_i\right)^2}{\left(\sum_{i=1}^{2q} Z_i\right)^2}\left(\left(\sum_{i=1}^{2q} Z_i\right)^2 - 4q^2\right) \\ &= \frac{1}{4q^2}\mathbf{E}(Z_{q+1} - 1)^2 + O_q(x_n^2) \\ &= \frac{1}{4q^2} d(\Pi_3 - 2\Lambda_3) + O_q(x_n^2) \\ &= \frac{d\lambda_1^2 x_n + d(\lambda_1^2 - \lambda_2^2) z_n}{2q} + d\lambda_1^2\lambda_2\left(w_n - \frac{x_n}{2q}\right) + O_q(x_n^2) \\ &= \frac{x_{n+1}}{2q} + d\lambda_1^2\lambda_2\left(w_n - \frac{x_n}{2q}\right) + O_q(x_n^2) \end{aligned}$$

and thus

$$\frac{w_{n+1}}{x_{n+1}} - \frac{1}{2q} = d\lambda_1^2\lambda_2 \frac{x_n}{x_{n+1}}\left(\frac{w_n}{x_n} - \frac{1}{2q}\right) + O_q\left(\frac{x_n^2}{x_{n+1}}\right). \qquad (3.2.2)$$

Next let's display our discussion in the XOZ plane. First consider the case of $\kappa > 1$. Then in the small neighborhood of $(0,0)$, because of $d\lambda_2^2 < \kappa^2 d|\lambda_2^2| \leq d\lambda_1^2 < 1$ and $X_n > 0$, the discrete trajectories approach to the origin point "tangential" to the X-axis if x_n is small enough for some n [15]. Besides, the conclusion of Lemma 3.2.3 excludes the trajectory along Z-axis. Then for some $M > 1$, there exist absolute constants $N_1 = N_1(q, \kappa, M)$ and $\delta_1 = \delta_1(q, \kappa, M)$ such that if $n \geq N_1$ and $x_n \leq \delta_1$, we have $X_n \geq M Z_n$ and $\frac{1}{M(M+1)} d\lambda_1^2 x_n + O_q(x_n^2) > 0$ simultaneously, where the remainder term $O_q(x_n^2)$ comes from the expansion of x_{n+1}. Consequently, we have $x_n + z_n = X_n \geq \frac{M}{M+1} x_n$, which yields, in connection with the result of $z_n \leq 0$ in Lemma 2.2.1,

$$
\begin{aligned}
\frac{x_n}{x_{n+1}} &= \frac{x_n}{d\lambda_1^2 x_n + d(\lambda_1^2 - \lambda_2^2) z_n + O_q(x_n^2)} \\
&\leq \frac{x_n}{\frac{M}{M+1} d\lambda_1^2 x_n + O_q(x_n^2)} \\
&\leq \frac{x_n}{\left(1 - \frac{1}{M}\right) d\lambda_1^2 x_n} \\
&= \frac{M}{M-1} \frac{1}{d\lambda_1^2}. \qquad (3.2.3)
\end{aligned}
$$

The second case taken into account is $|\lambda_1| = |\lambda_2|$. In view of $\frac{1}{2} \leq d\lambda^2 = d\lambda_1^2 \leq 1$, there also exist absolute constants $N_2 = N_2(q, M)$ and $\delta_2 = \delta_2(q, M)$ such that if $n \geq N_2$ and $x_n \leq \delta_2$ then

$$
\frac{x_n}{x_{n+1}} = \frac{x_n}{d\lambda_1^2 x_n + O_q(x_n^2)} \leq \frac{x_n}{\left(1 - \frac{1}{M}\right) d\lambda_1^2 x_n} = \frac{M}{M-1} \frac{1}{d\lambda_1^2}.
$$

For fixed k, it is known by (2.5.2) and (2.5.3) that

$$
|x_{n+1} - (d\lambda_1^2 X_n + d\lambda_2^2 Z_n)| \leq C(q) x_n^2,
$$

and then there exists a $\delta_3 = \delta_3(q, \kappa, M, k) < \delta_1, \delta_2$ such that if $x_n < \delta_3$ then $x_{n+\ell} < 2\delta_3$ for any $1 \leq \ell \leq k$. Therefore for any positive integer k, iterating (3.2.2) k times yields

$$
\begin{aligned}
\frac{w_{n+k}}{x_{n+k}} - \frac{1}{2q} &= d\lambda_1^2 \lambda_2 \frac{x_{n+k-1}}{x_{n+k}} \left(\frac{w_{n+k-1}}{x_{n+k-1}} - \frac{1}{2q} \right) + O_q\left(x_{n+k-1} \frac{x_{n+k-1}}{x_{n+k}} \right) \\
&= (d\lambda_1^2 \lambda_2)^k \left(\prod_{\ell=1}^{k} \frac{x_{n+\ell-1}}{x_{n+\ell}} \right) \left(\frac{w_n}{x_n} - \frac{1}{2q} \right) + R,
\end{aligned}
$$

where

$$
(d\lambda_1^2 \lambda_2)^k \left(\prod_{\ell=1}^{k} \frac{x_{n+\ell-1}}{x_{n+\ell}} \right) \leq (d\lambda_1^2 |\lambda_2|)^k \left(\frac{M}{M-1} \frac{1}{d\lambda_1^2} \right)^k = \left(\frac{M}{M-1} |\lambda_2| \right)^k
$$

and the remainder term

$$
|R| \leq 2C\delta_3 \left(\sum_{i=1}^{k} \left(\frac{M}{M-1} \frac{1}{d\lambda_1^2} \right)^i (d\lambda_1^2 |\lambda_2|)^{i-1} \right) \leq 2C\delta_3 \frac{1 - \left(\frac{M}{M-1} |\lambda_2| \right)^k}{1 - \left(\frac{M}{M-1} |\lambda_2| \right)} \frac{M}{M-1} \frac{1}{d\lambda_1^2}
$$

with C denoting the O_q constant in (3.2.2). From the identity $x_n = u_n + (q-1)v_n + qw_n$, it is easy to obtain $0 \le \frac{w_n}{x_n} \le \frac{1}{q}$, which implies $\left|\frac{w_n}{x_n} - \frac{1}{2q}\right| \le \frac{1}{2q}$. Next from $|\lambda_2| \le |\lambda_1| \le d^{-\frac{1}{2}} \le \frac{1}{\sqrt{2}}$, it can be guaranteed that $\frac{M}{M-1}|\lambda_2| < 1$ by choosing arbitrary $M \ge 4$, say, $M = 4$. Therefore it is feasible to take $k = k(\varepsilon)$ sufficiently large and $\delta = \delta(q, \kappa, k, \varepsilon) = \delta(q, \kappa, \varepsilon) < \delta_3$ sufficiently small to make sure that $\frac{w_{n+k}}{x_{n+k}} - \frac{1}{2q}$ is arbitrarily small. Finally choosing $N = N(q, \kappa, \varepsilon, k) = N(q, \kappa, \varepsilon) > N_1 + k, N_2 + k$, plus noting that if $x_n \le \delta$ then $x_{n-k} \le (\frac{M}{M-1}\frac{1}{d\lambda_1^2})^k x_n \le (\frac{8}{3})^k \delta$, the previous result completes the proof of

$$\left|\frac{w_n}{x_n} - \frac{1}{2q}\right| < \varepsilon.$$

Finally the case of $\frac{u_n}{x_n} - \frac{1}{2q}$ follows by plugging in the first result and processing similarly as above. □

3.3 Proof of Theorem 1.3.1.

Proof. First for any fixed $\varrho > 0$, consider $\varrho < |\lambda_2| < |\lambda_1|$. By Lemma 3.1.1 it suffices to establish that when $d\lambda_1^2$ is close enough to 1, X_n does not converge to 0, which implies x_n does not converge to 0 either, in virtue of $0 \le X_n = x_n + z_n \le x_n$. Therefore it is convenient to see $\lambda_2 > \varrho$ fixed and just λ_1 varying, and then without loss of generality, assume $d\lambda_1^2 > \frac{1+d\lambda_2^2}{2}$. Consequently choose $\kappa = \kappa(d, \lambda_2) = \sqrt{\frac{1+d\lambda_2^2}{2d\lambda_2^2}} > 1$ and thus $\frac{|\lambda_1|}{|\lambda_2|} \ge \kappa$.

As the discussion in Lemma 3.2.6, display our proof in the XOZ plane. With the condition of $q \ge 4$ and (2.5.2), it is apparent that

$$\begin{aligned} X_{n+1} &= d\lambda_1^2 X_n + \frac{d(d-1)}{2}\left(\frac{2q(q-3)}{q-1}\lambda_1^4 X_n^2 + 4q\lambda_1^2\lambda_2^2 X_n Z_n\right) \\ &\quad + O_q\left(|\lambda_1| x_n^2 \left(\left|\frac{u_n}{x_n} - \frac{1}{2q}\right| + \left|\frac{w_n}{x_n} - \frac{1}{2q}\right|\right)\right) + R_x + R_z \\ &\ge d\lambda_1^2 X_n + \frac{d(d-1)}{2}\frac{2q(q-3)}{q-1}\lambda_1^4 X_n^2 \\ &\quad - C X_n^2 \left(\left|\frac{u_n}{x_n} - \frac{1}{2q}\right| + \left|\frac{w_n}{x_n} - \frac{1}{2q}\right|\right) - |R_x| - |R_z|, \end{aligned}$$

where $C = C(q) > 0$ is a constant and the last inequality comes from $|\lambda_1| \le d^{-\frac{1}{2}} < 1$. Then by Lemma 3.2.6 and Proposition 3.2.5, there exist $N = N(q, \kappa, \varrho)$ and $\delta = \delta(q, d, \kappa, \varrho) > 0$ such that if $n \ge N$ and $x_n \le \delta$, then in the small neighborhood of the origin point $(0, 0)$, we have $X_n > Z_n$

and thus $X_n > \frac{x_n}{2}$, which enables the positive quadratic term of X_n^2, as the major one, to control remainder terms, meanwhile, the following estimates hold simultaneously:

$$\left|\frac{u_n}{x_n} - \frac{1}{2q}\right| \le \frac{1}{8C}\frac{d(d-1)}{2}\frac{2q(q-3)}{q-1}\lambda_1^4;$$
$$\left|\frac{w_n}{x_n} - \frac{1}{2q}\right| \le \frac{1}{8C}\frac{d(d-1)}{2}\frac{2q(q-3)}{q-1}\lambda_1^4,$$

and

$$|R_x| \le \frac{1}{32}\frac{d(d-1)}{2}\frac{2q(q-3)}{q-1}\lambda_1^4 x_n^2 \le \frac{1}{8}\frac{d(d-1)}{2}\frac{2q(q-3)}{q-1}\lambda_1^4 X_n^2;$$
$$|R_z| \le \frac{1}{32}\frac{d(d-1)}{2}\frac{2q(q-3)}{q-1}\lambda_1^4 x_n^2 \le \frac{1}{8}\frac{d(d-1)}{2}\frac{2q(q-3)}{q-1}\lambda_1^4 X_n^2.$$

Therefore taken together, the preceding results give the following crucial estimate

$$\begin{aligned} X_{n+1} &\ge d\lambda_1^2 X_n + \frac{1}{2}\frac{d(d-1)}{2}\frac{2q(q-3)}{q-1}\lambda_1^4 X_n^2 \\ &= X_n\left[d\lambda_1^2 + \frac{1}{2}\frac{d(d-1)}{2}\frac{2q(q-3)}{q-1}\lambda_1^4 X_n\right]. \end{aligned} \tag{3.3.1}$$

Take $\varepsilon = \min\{\frac{1}{4}\gamma^N, \gamma\delta\} > 0$, where $\gamma = \gamma(q, d, \varrho) > 0$ is the constant in Lemma 3.2.1. Because $q \ge 4$ and ε is independent of λ_1, we can choose $\lambda_1 < d^{-\frac{1}{2}}$ to make

$$d\lambda_1^2 + \frac{1}{2}\frac{d(d-1)}{2}\frac{2q(q-3)}{q-1}\lambda_1^4 \varepsilon > 1.$$

Since $x_0 = 1 - \frac{1}{2q} > \frac{1}{2}$, it is concluded that $x_n \ge \frac{1}{2}\gamma^n \ge 2\varepsilon$ when $n \le N$, in addition, $X_N \ge \frac{X_N + Z_N}{2} = \frac{x_N}{2} \ge \varepsilon$. Now suppose $X_n \ge \varepsilon$ for some $n \ge N$. Then display our discussion of X_n as follows:

(1) If $X_n \ge 2\gamma^{-1}\varepsilon$, then

$$X_{n+1} \ge \frac{x_{n+1}}{2} \ge \frac{\gamma x_n}{2} \ge \frac{\gamma X_n}{2} \ge \varepsilon;$$

(2) If $\varepsilon \le X_n \le 2\gamma^{-1}\varepsilon$, then $x_n \le \frac{X_n}{2} \le \gamma^{-1}\varepsilon \le \delta$, and thus it follows by (3.3.1) that

$$\begin{aligned} x_{n+1} &\ge X_{n+1} \\ &\ge X_n\left[d\lambda_1^2 + \frac{1}{2}\frac{d(d-1)}{2}\frac{2q(q-3)}{q-1}\lambda_1^4 X_n\right] \\ &\ge X_n\left[d\lambda_1^2 + \frac{1}{2}\frac{d(d-1)}{2}\frac{2q(q-3)}{q-1}\lambda_1^4 \varepsilon\right] \\ &\ge X_n \\ &\ge \varepsilon. \end{aligned}$$

Finally show by induction that for all n that $x_n \geq X_n \geq \varepsilon$. Consequently it is established that the Kesten-Stigum bound is not tight.

The second case taken into account is $|\lambda_1| = |\lambda_2|$, under which there are two equal multipliers in this nonlinear second order point mapping, the origin point must be a star node. Although the principal axis is undetermined, just by the comparison of the quadratic terms and $q \geq 4$, it is concluded that

$$\begin{aligned} & \frac{d(d-1)}{2}\left(\frac{2q(q-3)}{q-1}\lambda_1^4 X_n^2 + 4q\lambda_1^4 X_n Z_n\right) - \frac{d(d-1)}{2}\left(\frac{q}{q-1}\lambda_1^4 X_n^2 - 4q\lambda_1^4 Z_n^2\right) \\ = & \frac{d(d-1)}{2}\left(\frac{2q^2-7q}{q-1}\lambda_1^4 X_n^2 + 4q\lambda_1^4 X_n Z_n + 4q\lambda_1^4 Z_n^2\right) \\ \geq & \frac{d(d-1)}{2}\lambda_1^4 x_n^2, \end{aligned}$$

and thus the decay rate of X_n is much slower than Z_n when n is sufficiently large. Therefore in light of the preceding discussion, there still exist $N = N(q)$ and $\delta = \delta(q)$ such that if $n \geq N$ and $x_n \leq \delta$, we have $X_n \geq Z_n$ and thus $x_n = X_n + Z_n \leq 2X_n$. Then the rest would be the same as the first part.

Last consider the case of $0 = |\lambda_2| < |\lambda_1|$. Due to $|\lambda_2| = 0$, Lemma 3.2.1 loses its effectiveness, however, in connection with the previous discussion, it suffices to make a subtle adjustment.

Lemma 3.3.1. *Assume* $d\lambda_1^2 \geq \frac{1}{2}$, $\lambda_2 = 0$ *and* $\lim_{n\to\infty} x_n = 0$. *Then there still exists a constant* $0 < \gamma = \gamma(d,q) < 1$ *such that for all* $n \geq 0$

$$x_{n+1} \geq \gamma x_n.$$

Proof. Under the assumptions of $d\lambda_1^2 \geq \frac{1}{2}$ and the fact that the less $d\lambda_1^2$ is, the faster x_n approaches to 0, it is concluded that there exists a large $N = N(q) > 0$ such that when $n > N$, we always have $Z_n < X_n$ that implies $X_n > \frac{x_n}{2}$, and $\frac{1}{4}d\lambda_1^2 x_n + O_q(x_n^2) > 0$, where $O_q(x_n^2)$ is the error of the linear approximation of x_{n+1}. Consequently,

$$\frac{x_n}{x_{n+1}} = \frac{x_n}{d\lambda_1^2 X_n + O_q(x_n^2)} \leq \frac{x_n}{\frac{1}{4}d\lambda_1^2 x_n} = \frac{4}{d\lambda_1^2} \leq 8. \qquad (3.3.2)$$

Since here $\lambda_1 \neq 0$ implies $x_n \geq X_n > 0$ from Lemma 3.2.3, it is feasible to define a positive function

$$\Gamma(\lambda_1, d, q) = \min\left\{\frac{1}{8}, \frac{x_{m+1}}{x_m} : 0 \le m \le N\right\}.$$

Now fix d and q, from the construction of x_n, it is natural that x_n is continuous for the parameter λ_1, so is Γ. Based on the assumption, we have $\sqrt{\frac{1}{2d}} \le |\lambda_1| \le \sqrt{\frac{1}{d}}$, that is, λ_1 ranges in a compact set, and then there exists a $\widetilde{\lambda_1} \in \left[-\sqrt{\frac{1}{d}}, -\sqrt{\frac{1}{2d}}\right] \bigcup \left[\sqrt{\frac{1}{2d}}, \sqrt{\frac{1}{d}}\right]$ such that $\Gamma_{\min} = \Gamma(\widetilde{\lambda_1}, d, q) > 0$. Finally choosing $\gamma(d, q) = \Gamma(\widetilde{\lambda_1}, d, q) > 0$ completes the proof. □

Finally apply contradiction by assuming that the Kesten-Stigum bound is tight, say, $\lim_{n\to\infty} x_n = 0$ for any λ_1 with $(2d)^{-\frac{1}{2}} \le |\lambda_1| < d^{-\frac{1}{2}}$, which implies $X_n \to 0$ and $Z_n \to 0$ simultaneously as n tends to ∞. With the assistance of Lemma 3.3.1, it is pleasant to see that Lemma 3.2.1 recovers from the trouble caused by $\lambda_2 = 0$, accordingly results of Lemma 3.2.4 and Proposition 3.2.5 still work out for this model. Thus there exist $N = N(q)$ and $\delta = \delta(q, d)$ such that if $n \ge N$ and $x_n \le \delta$ then (3.3.1) holds. The rest would be exactly the same as the discussion in the first part of this section to present that there exists a $\varepsilon > 0$ such that $x_n \ge \varepsilon$ holds for all n, a contradiction to the assumption. Thus we establish the non-tightness of the Kesten-Stigum bound. □

CHAPTER 4

NON-RECONSTRUCTION FOR $q = 2$.

In this chapter we present the asymptotic behavior of the symmetric model, say, what happens as d grows, under which the interactions become weaker. Moreover we could utilize this technique to solve the non-reconstruction when $q = 2$. It worth mentioning that this asympotics approach is of more power than just solving the case of $q = 2$, but also could be engaged in the investigation of the reconstruction corresponding to $q = 3$ and the asymptotic reconstruction threshold concerning with λ_1 and λ_2. However, in light of the more delicate analysis in those topics, we defer these discussions to a subsequent paper.

4.1 Gauss Approximation.

Define

(A) when $1 \leq i \leq q$

$$U_{ij} = \log \left[1 + 2q\lambda_1 \left(Y_{ij} - \frac{1}{2q} \right) + 2(\lambda_1 - \lambda_2) \sum_{q+1 \leq \ell \leq 2q} \left(Y_{\ell j} - \frac{1}{2q} \right) \right];$$

(B) when $q + 1 \leq i \leq 2q$

$$U_{ij} = \log \left[1 + 2q\lambda_1 \left(Y_{ij} - \frac{1}{2q} \right) + 2(\lambda_1 - \lambda_2) \sum_{1 \leq \ell \leq q} \left(Y_{\ell j} - \frac{1}{2q} \right) \right],$$

and $U_j = (U_{1j}, \ldots, U_{(2q)j}) \in R^q$.

In order to set up a distribution approximation, it is desired to estimate the means and covariances of the U_{ij}.

Lemma 4.1.1. *There exist constants C and D depending only on q such that when $d > D$,*

$$\left| d\mathbf{E}U_{1j} - d\{q\lambda_1^2 x_n + q(\lambda_1^2 - \lambda_2^2)z_n\} \right| \leq Cd^{-\frac{1}{2}};$$

when $2 \leq i \leq q$,

$$\left| d\mathbf{E}U_{ij} + d\left\{ \frac{q(q+1)}{q-1}\lambda_1^2 x_n + \left(\frac{q(q+1)}{q-1}\lambda_1^2 + q\lambda_2^2 \right) z_n \right\} \right| \leq Cd^{-\frac{1}{2}};$$

for $q+1 \leq i \leq 2q$,

$$\left| d\mathbf{E}U_{ij} + d\left\{ q\lambda_1^2 x_n + q(\lambda_1^2 - 3\lambda_2^2)z_n \right\} \right| \leq Cd^{-\frac{1}{2}}.$$

Meanwhile, for any $1 \leq i \leq 2q$,

$$\left| d\mathbf{Var}(U_{ij}) - d\{2q\lambda_1^2 x_n + 2q(\lambda_1^2 - \lambda_2^2)z_n\} \right| \leq Cd^{-\frac{1}{2}};$$

for any $1 \leq i_1 < i_2 \leq 2q$ in the same category,

$$\left| d\mathbf{Cov}(U_{i_1 j}, U_{i_2 j}) + d\left\{ \frac{2q}{q-1}\lambda_1^2 x_n + \frac{2q}{q-1}(\lambda_1^2 + (q-1)\lambda_2^2)z_n \right\} \right| \leq Cd^{-\frac{1}{2}};$$

for any $1 \leq i_1 \leq q < i_2 \leq 2q$,

$$\left| d\mathbf{Cov}(U_{i_1 j}, U_{i_2 j}) - d2q\lambda_2^2 z_n \right| \leq Cd^{-\frac{1}{2}}.$$

Proof. For any $1 \leq j \leq d$, write

$$a_j = 2q\lambda_1\left(Y_{1j} - \frac{1}{2q}\right) - 2(\lambda_2 - \lambda_1)\sum_{q+1\leq \ell \leq 2q}\left(Y_{\ell j} - \frac{1}{2q}\right),$$

$$b_j = 2q\lambda_1\left(Y_{2j} - \frac{1}{2q}\right) + 2(\lambda_1 - \lambda_2)\sum_{q+1\leq \ell \leq 2q}\left(Y_{\ell j} - \frac{1}{2q}\right),$$

and

$$c_j = 2q\lambda_1\left(Y_{(q+1)j} - \frac{1}{2q}\right) + 2(\lambda_1 - \lambda_2)\sum_{1\leq \ell \leq q}\left(Y_{\ell j} - \frac{1}{2q}\right).$$

Using the Taylor series expansion of $\log(1+w)$, there exists a constant $W > 0$ such that when $|w| < W$,

$$\left| \log(1+w) - w + \frac{w^2}{2} \right| \leq |w|^3 \tag{4.1.1}$$

is always true. If we take D sufficiently large such that when $d > D$, $|\lambda| \le d^{-\frac{1}{2}}$ is small enough to make

$$|a_j| = \left| 2q\lambda_1 \left(Y_{1j} - \frac{1}{2q} \right) + 2(\lambda_1 - \lambda_2) \sum_{q+1 \le \ell \le 2q} \left(Y_{\ell j} - \frac{1}{2q} \right) \right| \le 6q|\lambda| \le W.$$

Therefore

$$\left| \mathbf{E} U_{1j} - \mathbf{E} \left(a_j - \frac{a_j^2}{2} \right) \right| \quad \le \quad \mathbf{E}|a_j^3| \le 216q^3|\lambda|^3 \le 216q^3 d^{-\frac{3}{2}}.$$

On the other hand, we have the explicit formula

$$\begin{aligned}
& \mathbf{E} \left(a_j - \frac{a_j^2}{2} \right) \\
= \; & \Lambda_1 - 2q^2 \text{①} \\
= \; & 2\Lambda_1 - \frac{\Pi_1}{2} \\
= \; & [q\lambda_1^2 x_n + q(\lambda_1^2 - \lambda_2^2) z_n] - 2q^2\lambda_1^3 \left(u_n - \frac{x_n}{2q} \right) - 6q^2\lambda_1^2(\lambda_1 - \lambda_2) \left(w_n - \frac{x_n}{2q} \right).
\end{aligned}$$

Taken together, two previous results establish the first estimate

$$\begin{aligned}
& \left| \mathbf{E}(U_{1j}) - [q\lambda_1^2 x_n + q(\lambda_1^2 - \lambda_2^2) z_n] \right| \\
\le \; & 216q^3 d^{-\frac{3}{2}} + 2q^2|\lambda_1|^3 \left| u_n - \frac{x_n}{2q} \right| + 6q^2\lambda_1^2(|\lambda_1| + |\lambda_2|) \left| w_n - \frac{x_n}{2q} \right| \\
\le \; & 240q^3 d^{-\frac{3}{2}}
\end{aligned}$$

in view of $\left| u_n - \frac{x_n}{2q} \right| < 1$ and $\left| w_n - \frac{x_n}{2q} \right| < 1$. Similarly the second and third estimates of this lemma turn out to be true by noticing

$$\begin{aligned}
& \mathbf{E} \left(b_j - \frac{b_j^2}{2} \right) \\
= \; & \Lambda_2 - 2q^2 \text{②} \\
= \; & 2\Lambda_2 - \frac{\Pi_2}{2} \\
= \; & -\frac{q(q+1)}{q-1}\lambda_1^2 x_n - \left(\frac{q(q+1)}{q-1}\lambda_1^2 + q\lambda_2^2 \right) z_n \\
& + \frac{2q^2}{q-1}\lambda_1^3 \left(u_n - \frac{x_n}{2q} \right) + \frac{2q^2}{q-1}\lambda_1^2[3\lambda_1 + (q-3)\lambda_2] \left(w_n - \frac{x_n}{2q} \right)
\end{aligned}$$

and

$$\begin{aligned}
& \mathbf{E}\left(c_j - \frac{c_j^2}{2}\right) \\
= \; & 2\Lambda_3 - \frac{\Pi_3}{2} \\
= \; & -q\lambda_1^2 x_n - q(\lambda_1^2 - 3\lambda_2^2) z_n - 2q^2\lambda_1^2\lambda_2 \left(w_n - \frac{x_n}{2q}\right).
\end{aligned}$$

In order to complete this proof, it suffices to consider covariances by means of the analogous analysis:

(1) when $i = 1$

$$\begin{aligned}
\mathbf{Var}(U_{1j}) &= \mathbf{E}U_{1j}^2 - (\mathbf{E}U_{1j})^2 \\
&= \mathbf{E}a_j^2 + O(|\lambda_1|^3) \\
&= \Pi_1 - 2\Lambda_1 + O(|\lambda_1|^3) \\
&= 2q\lambda_1^2 x_n + 2q(\lambda_1^2 - \lambda_2^2) z_n + O(d^{-\frac{3}{2}});
\end{aligned}$$

(2) when $2 \le i \le q$

$$\begin{aligned}
\mathbf{Var}(U_{ij}) &= \mathbf{E}b_j^2 + O(|\lambda_1|^3) \\
&= \Pi_2 - 2\Lambda_2 + O(|\lambda_1|^3) \\
&= 2q\lambda_1^2 x_n + 2q(\lambda_1^2 - \lambda_2^2) z_n + O(d^{-\frac{3}{2}});
\end{aligned}$$

(3) when $q + 1 \le i \le 2q$

$$\begin{aligned}
\mathbf{Var}(U_{ij}) &= \mathbf{E}c_j^2 + O(|\lambda_1|^3) \\
&= \Pi_3 - 2\Lambda_3 + O(|\lambda_1|^3) \\
&= 2q\lambda_1^2 x_n + 2q(\lambda_1^2 - \lambda_2^2) z_n + O(d^{-\frac{3}{2}});
\end{aligned}$$

(4) when $2 \le i \le q$, by noting that $\mathbf{E}Z_1 Z_i = \mathbf{E}Z_i^2$,

$$\begin{aligned}
\mathbf{Cov}(U_{1j}, U_{ij}) &= \mathbf{E}U_{1j}U_{ij} - (\mathbf{E}U_{1j})(\mathbf{E}U_{ij}) \\
&= \mathbf{E}a_j b_j + O(|\lambda_1|^3) \\
&= \Pi_2 - \Lambda_1 - \Lambda_2 + O(|\lambda_1|^3) \\
&= -\frac{2q}{q-1}\lambda_1^2 x_n - \frac{2q}{q-1}(\lambda_1^2 + (q-1)\lambda_2^2) z_n + O(d^{-\frac{3}{2}});
\end{aligned}$$

(5) when $q+1 \le i \le 2q$, by noting that $\mathbf{E}Z_1Z_i = \mathbf{E}Z_i^2$,

$$\begin{aligned}\mathbf{Cov}(U_{1j}, U_{ij}) &= \mathbf{E}a_jc_j + O(|\lambda_1|^3) \\ &= \Pi_3 - \Lambda_1 - \Lambda_3 + O(|\lambda_1|^3) \\ &= 2q\lambda_2^2 z_n + O(d^{-\frac{3}{2}});\end{aligned}$$

(6) when $2 \le i_1 \le q < i_2 \le 2q$

$$\begin{aligned}\mathbf{Cov}(U_{i_1j}, U_{i_2j}) &= \mathbf{E}b_jc_j + O(|\lambda_1|^3) \\ &= \Pi_6 - \Lambda_2 - \Lambda_3 + O(|\lambda_1|^3) \\ &= 2q\lambda_2^2 z_n + O(d^{-\frac{3}{2}})\end{aligned}$$

(7) when $2 \le i_1 < i_2 \le q$

$$\begin{aligned}\mathbf{Cov}(U_{i_1j}, U_{i_2j}) &= \Pi_5 - 2\Lambda_2 + O(|\lambda_1|^3) \\ &= -\frac{2q}{q-1}\lambda_1^2 x_n - \frac{2q}{q-1}(\lambda_1^2 + (q-1)\lambda_2^2) z_n + O(d^{-\frac{3}{2}});\end{aligned}$$

(8) when $q+1 \le i_1 < i_2 \le 2q$

$$\begin{aligned}\mathbf{Cov}(U_{i_1j}, U_{i_2j}) &= \Pi_7 - 2\Lambda_3 + O(|\lambda_1|^3) \\ &= -\frac{2q}{q-1}\lambda_1^2 x_n - \frac{2q}{q-1}(\lambda_1^2 + (q-1)\lambda_2^2) z_n + O(d^{-\frac{3}{2}}).\end{aligned}$$

□

Define two $2q$ dimensional vectors $\mu = (\mu_i)_1^{2q}$ and $\nu = (\nu_i)_1^{2q}$ by

$$\mu_i = \begin{cases} q & \text{if } i = 1, \\ -\frac{q(q+1)}{q-1} & \text{if } 2 \le i \le q, \\ -q & \text{if } q+1 \le i \le 2q \end{cases} \quad \text{and} \quad \nu_i = \begin{cases} q & \text{if } i = 1, \\ q & \text{if } 2 \le i \le q, \\ -3q & \text{if } q+1 \le i \le 2q. \end{cases}$$

In addition, $\Sigma(U) = (\Sigma_{ij}(U))_{2q\times 2q}$ and $\Sigma(V) = (\Sigma_{ij}(V))_{2q\times 2q}$ denote $2q \times 2q$ symmetric positive semi-definite matrices with

$$\Sigma_{ij}(U) = \begin{cases} 2q & \text{if } i = j, \\ \\ -\frac{2q}{q-1} & \text{if } i \neq j \text{ are in the same category,} \\ \\ 0 & \text{if } i \text{ and } j \text{ come from different categories} \end{cases}$$

and

$$\Sigma_{ij}(V) = \begin{cases} 2q & \text{if } i = j, \\ \\ 2q & \text{if } i \neq j \text{ are in the same class,} \\ \\ -2q & \text{if } i \text{ and } j \text{ are from different classes.} \end{cases}$$

Let $s = d\lambda_1^2(x_n + z_n)$ and $t = -d\lambda_2^2 z_n$. Under results of the preceding estimates, we construct a multivariate Gaussian distribution $(W_i)_1^{2q}$ given by $W_i = s\mu_i + \sqrt{s}U_i + t\nu_i + \sqrt{t}V_i$, where $\{U_i\}_1^{2q}$ and $\{V_i\}_1^{2q}$ are two sequences of independent normal random variables with expectations 0 and covariance matrices $\Sigma(U)$ and $\Sigma(V)$ respectively. Next define a $2q$-variable function

$$\psi(w_1, \ldots, w_{2q}) = \frac{e^{w_1}}{\sum_{i=1}^{2q} e^{w_i}},$$

which is positive, analytic and bounded by 1 and

$$f(s,t) = \mathbf{E}\psi(W_1, \ldots, W_{2q}) - \frac{1}{2q} = \mathbf{E}\frac{e^{W_1}}{\sum_{i=1}^{2q} e^{W_i}} - \frac{1}{2q}.$$

Then the proposition 4.2 in [3] established by using Central Limit Theorem, Gaussian approximation and Portmanteau Theorem, leads to the following lemma.

Lemma 4.1.2. *For each $\varepsilon > 0$ there exists a $D = D(\varepsilon, q)$ such that for all n when $d > D$,*

$$\left|X_{n+1} - g(d\lambda_1^2 X_n, d\lambda_2^2 Z_n)\right| \leq \varepsilon,$$

$$\left|Z_{n+1} - h(d\lambda_1^2 X_n, d\lambda_2^2 Z_n)\right| \leq \varepsilon,$$

and

$$\left|x_{n+1} - f(d\lambda_1^2 X_n, d\lambda_2^2 Z_n)\right| \leq \varepsilon.$$

Since $0 < X_n \leq 1 - \frac{1}{q}$ and $0 \leq Z_n \leq \frac{1}{2q}$, we will focus on the behavior of $f(s,t)$ in the area $\left[0, 1 - \frac{1}{q}\right] \times \left[0, \frac{1}{2q}\right]$.

Lemma 4.1.3. *The functions $f(s,t)$, $g(s,t)$ and $h(s,t)$ have the continuous partial derivatives with respect to $s \in \left(0, 1 - \frac{1}{q}\right]$ and $t \in \left(0, \frac{1}{2q}\right]$ respectively, in addition, $f(s,t)$ is increasing of s and t separately.*

Proof. In view of the similar definition of f, g and h, it suffices to verify the differentiability of f with respect to s and the rests would follow similarly. When $s > 0$,

$$\begin{aligned}
\mathbf{E}\left|\frac{\partial}{\partial s}\frac{e^{W_1}}{\sum_{i=1}^{2q} e^{W_i}}\right| &= \mathbf{E}\left|\frac{\sum_{i=1}^{2q} e^{W_i - W_1}\left(\frac{\partial W_i}{\partial s} - \frac{\partial W_1}{\partial s}\right)}{(\sum_{i=1}^{2q} e^{W_i - W_1})^2}\right| \\
&\leq \mathbf{E}\sum_{i=1}^{2q}\left|\frac{\partial W_i}{\partial s} - \frac{\partial W_1}{\partial s}\right| \\
&= \mathbf{E}\sum_{i=1}^{2q}\left|\mu_i - \mu_1 + \frac{U_i - U_1}{2\sqrt{s}}\right| \\
&< \infty
\end{aligned}$$

Therefore f is differentiable with respect to s, as desired.

Next it is in turn to show the monotonicity of $f(s,t)$. First let $(U_1', U_2', \ldots, U_{2q}')$ and $(V_1', V_2', \ldots, V_{2q}')$ be an independent copy of $(U_1, U_2, \ldots, U_{2q})$ and $(V_1, V_2, \ldots, V_{2q})$ respectively. Then when $0 \leq s' < s$ and $0 \leq t' < t$, we can construct the equivalent distributions such as

$$\sqrt{s}(U_1, U_2, \ldots, U_{2q}) \sim \sqrt{s'}(U_1, U_2, \ldots, U_{2q}) + \sqrt{s - s'}(U_1', U_2', \ldots, U_{2q}');$$

$$\sqrt{t}(V_1, V_2, \ldots, V_{2q}) \sim \sqrt{t'}(V_1, V_2, \ldots, V_{2q}) + \sqrt{t - t'}(V_1', V_2', \ldots, V_{2q}').$$

Next we will focus on the discussion of the monotonicity with respect to s, since the case of t could be handled similarly. It is obviously that $(U_1' - U_1', U_2' - U_1', \ldots, U_{2q}' - U_1')$ is also a multivariate gaussian distribution with mean $\mu' = (0, \ldots, 0)$ and variances such as

$$\mathbf{Var}(U_i' - U_1') = \begin{cases} 0 & \text{if } i = 1, \\ \frac{4q^2}{q-1} & \text{if } 2 \leq i \leq q, \\ 4q & \text{if } q + 1 \leq i \leq 2q. \end{cases}$$

It is well known that if W is distributed as $\mathbf{N}(\mu, \sigma^2)$, the expectation of the exponential random variable could be estimated as

$$\mathbf{E}e^{W} = e^{\mu+\frac{\sigma^2}{2}}, \tag{4.1.2}$$

based on which, we are allowed to evaluate the conditional expectation given $\{U_j\}_1^{2q}$ and $\{V_j\}_1^{2q}$:

$$\begin{aligned}
&\mathbf{E}\left[\exp(\sqrt{s'}(U_i-U_1)+\sqrt{s-s'}(U_i'-U_1')+t(\nu_i-\nu_1)+\sqrt{t}(V_i-V_1)) \mid \{U_j\}_1^{2q},\{V_j\}_1^{2q}\right]\\
&= \begin{cases} 1 & \text{if } i=1,\\ \exp(\sqrt{s'}(U_i-U_1)+\frac{2q^2}{q-1}(s-s')+t(\nu_i-\nu_1)+\sqrt{t}(V_i-V_1)) & \text{if } 2\le i\le q,\\ \exp(\sqrt{s'}(U_i-U_1)+2q(s-s')+t(\nu_i-\nu_1)+\sqrt{t}(V_i-V_1)) & \text{if } q+1\le i\le 2q. \end{cases}
\end{aligned}$$

Then we can apply Jensen's inequality, plus noting that the function $\frac{1}{1+x}$ is convex, to derive

$$\begin{aligned}
f(s,t) &= \mathbf{E}\frac{\exp(s\mu_1+\sqrt{s}U_1+t\nu_1+\sqrt{t}V_1)}{\sum_{i=1}^{2q}\exp(s\mu_i+\sqrt{s}U_i+t\nu_i+\sqrt{t}V_i)}-\frac{1}{2q}\\
&= \mathbf{E}\Bigg[1+\sum_{i=2}^{2q}\exp\Big(s(\mu_i-\mu_1)+\sqrt{s'}(U_i-U_1)+\sqrt{s-s'}(U_i'-U_1')\\
&\quad +t(\nu_i-\nu_1)+\sqrt{t}(V_i-V_1)\Big)\Bigg]^{-1}-\frac{1}{2q}\\
&\ge \mathbf{E}\Bigg\{1+\mathbf{E}\Bigg[\sum_{i=2}^{2q}\exp\Big(s(\mu_i-\mu_1)+\sqrt{s'}(U_i-U_1)+\sqrt{s-s'}(U_i'-U_1')\\
&\quad +t(\nu_i-\nu_1)+\sqrt{t}(V_i-V_1)\Big) \mid \{U_j\}_1^{2q},\{V_j\}_1^{2q}\Bigg]\Bigg\}^{-1}-\frac{1}{2q}\\
&= \mathbf{E}\Bigg[1+\sum_{i=2}^{q}\exp\left(-\frac{2q^2s}{q-1}+\sqrt{s'}(U_i-U_1)+\frac{2q^2}{q-1}(s-s')+t(\nu_i-\nu_1)+\sqrt{t}(V_i-V_1)\right)\\
&\quad +\sum_{i=q+1}^{2q}\exp\Big(-2qs+\sqrt{s'}(U_i-U_1)+2q(s-s')+t(\nu_i-\nu_1)+\sqrt{t}(V_i-V_1)\Big)\Bigg]^{-1}-\frac{1}{2q}\\
&= \mathbf{E}\Bigg[1+\sum_{i=2}^{q}\exp\left(-\frac{2q^2s'}{q-1}+\sqrt{s'}(U_i-U_1)+t(\nu_i-\nu_1)+\sqrt{t}(V_i-V_1)\right)\\
&\quad +\sum_{i=q+1}^{2q}\exp\Big(-2qs'+\sqrt{s'}(U_i-U_1)+t(\nu_i-\nu_1)+\sqrt{t}(V_i-V_1)\Big)\Bigg]^{-1}-\frac{1}{2q}\\
&= f(s',t),
\end{aligned}$$

as desired. □

4.2 Taylor expansions of $f(s,t)$, $g(s,t)$ and $h(s,t)$.

The Taylor expansions of $g(s,t)$ and $h(s,t)$ in the small neighborhood of the origin $(0,0)$ would be of interest to us. By taking $a = e^{W_1}$, $y = 2q$ and $x = \sum_{i=1}^{2q} e^{W_i} - 2q$ in the identity

$$\frac{a}{x+y} = \left(\sum_{i=1}^{4}(-1)^{i-1}\frac{ax^{i-1}}{y^i}\right) + (-1)^4\frac{x^4}{y^4}\frac{a}{x+y},$$

we have

$$\begin{aligned} f(s,t) &= \mathbf{E}\left(\sum_{i=1}^{4}(-1)^{i-1}\frac{e^{W_1}(\sum_{i=1}^{2q} e^{W_i} - 2q)^{i-1}}{(2q)^i}\right) + \mathbf{E}\left(\frac{(\sum_{i=1}^{2q} e^{W_i} - 2q)^4}{(2q)^4}\frac{e^{W_1}}{\sum_{i=1}^{2q} e^{W_i}}\right) - \frac{1}{2q} \\ &= s + t + \frac{q(2q-5)}{2(q-1)}s^2 + 2qst - 2qt^2 + \frac{2q^2(q^2-10q+17)}{3(q-1)^2}s^3 + \frac{2(q-6)q^2}{q-1}s^2t - 4q^2st^2 \\ &\quad + \frac{20q^2}{3}t^3 + O(s^4+t^4), \end{aligned}$$

where we use the formula (4.1.2) and facts of

$$\mathbf{E}\left(\frac{\sum_{i=1}^{2q} e^{W_i} - 2q}{2q}\right)^4 = O(s^4+t^4)$$

and

$$0 < \frac{e^{W_1}}{\sum_{i=1}^{2q} e^{W_i}} < 1.$$

Similarly,

$$g(s,t) = s + \frac{q(q-3)}{q-1}s^2 + 2qst + \frac{q^2(2q^2-21q+39)}{3(q-1)^2}s^3 + \frac{2q^2(q-5)}{q-1}s^2t - 4q^2st^2 + O(s^4+t^4)$$

and

$$h(s,t) = t + \frac{q}{2(q-1)}s^2 - 2qt^2 + \frac{q^2(q-5)}{3(q-1)^2}s^3 - \frac{2q^2}{q-1}s^2t + \frac{20q^2}{3}t^3 + O(s^4+t^4).$$

4.3 Proof of Theorem 1.3.2.

Lemma 4.3.1. *When $q = 2$, for all $(s,t) \in \left[0, 1-\frac{1}{q}\right] \times \left[0, \frac{1}{2q}\right] \setminus (0,0)$, we have*

$$f(s,t) < s + t.$$

Proof. When $q = 2$,

$$\mu_i = \begin{cases} 2 & \text{if } i = 1, \\ -6 & \text{if } i = 2, \\ -2 & \text{if } i = 3, 4, \end{cases} \qquad \nu_i = \begin{cases} 2 & \text{if } i = 1, \\ 2 & \text{if } i = 2, \\ -6 & \text{if } i = 3, 4, \end{cases}$$

$$\Sigma_{ij}(U) = \begin{cases} 4 & \text{if } i = j, \\ -4 & \text{if } i \neq j \text{ are in the same category}, \\ 0 & \text{if } i \text{ and } j \text{ are in different categories}, \end{cases}$$

and

$$\Sigma_{ij}(V) = \begin{cases} 4 & \text{if } i = j, \\ 4 & \text{if } i \neq j \text{ are in the same category}, \\ -4 & \text{if } i \text{ and } j \text{ are in different categories}. \end{cases}$$

By examining the preceding results, it is obtained that for $i \neq j$ in the same category,

$$\mathbf{E}(U_i + U_j)^2 = 2\mathbf{E}U_i^2 + 2\mathbf{E}U_iU_j = 0,$$

which implies that $U_i = -U_j \ a.s.$, and similarly we can conclude that $V_i = V_j \ a.s.$ from

$$\mathbf{E}(V_i - V_j)^2 = \mathbf{E}V_i^2 + \mathbf{E}V_j^2 - 2\mathbf{E}V_i\mathbf{E}V_j = 2q + 2q - 2 \times 2q = 0,$$

whereas if i and j belong to different classes, we have

$$\mathbf{E}(V_i + V_j)^2 = \mathbf{E}V_i^2 + \mathbf{E}V_j^2 + 2\mathbf{E}V_i\mathbf{E}V_j = 2q + 2q - 2 \times 2q = 0,$$

which gives $V_i = -V_j \ a.s.$

Therefore it is convenient to reduce the random variables in the expectation of f as

$$\begin{aligned} f(s,t) &= \mathbf{E}\frac{\exp(s\mu_1 + \sqrt{s}U_1 + t\nu_1 + \sqrt{t}V_1)}{\sum_{i=1}^4 \exp(s\mu_i + \sqrt{s}U_i + t\nu_i + \sqrt{t}V_i)} - \frac{1}{2q} \\ &= \mathbf{E}\frac{1}{1 + e^{-8s+\sqrt{s}(-2U_1)} + e^{-4s+\sqrt{s}(U_3-U_1)-8t+\sqrt{t}(-2V_1)} + e^{-4s+\sqrt{s}(-U_3-U_1)-8t+\sqrt{t}(-2V_1)}} - \frac{1}{4}, \end{aligned}$$

where U_1, U_3 and V_1 are independent and identical normal random variables with mean 0 and variance 4. For the rest part of the proof, we could refer to Mathematica. $\square$

Proof of **Theorem 1.3.2.** Suppose $d\lambda^2 = \max\{d\lambda_1^2, d\lambda_2^2\} < 1$. By means of (2.5.2) and (2.5.3), taking $\epsilon = \frac{1}{2}(1 - d\lambda^2) > 0$ there exists a constant $\delta = \delta(q, \epsilon)$ such that suppose $x_n < \delta$ then

$$\begin{aligned} x_{n+1} &= X_{n+1} + Z_{n+1} \\ &\leq (d\lambda_1^2 + \epsilon)X_n + (d\lambda_2^2 + \epsilon)Z_n \\ &\leq (d\lambda^2 + \epsilon)(X_n + Z_n) \\ &= (1 - \epsilon)x_n < \delta, \end{aligned}$$

in addition, induction in tandem with $0 < 1 - \epsilon < 1$ implies $\lim_{n\to\infty} x_n = 0$ and thus there is non-reconstruction. So here it suffices to find some m such that $x_m < \delta$. Otherwise assume all $x_n \geq \delta$. Next define $\varepsilon = \frac{1}{2}\min_{s+t\geq\delta}(s + t - f(s,t))$. Since $s + t - f(s,t)$ is continuous and positive in the compact set $\left[0, 1 - \frac{1}{q}\right] \times \left[0, \frac{1}{2q}\right] \bigcap \{s + t \geq \delta\}$, which is bounded away from the origin, we can conclude $\varepsilon > 0$ by Lemma 4.3.1.

Then by Lemma 4.1.2 there exists a $D = D(\varepsilon, q) > 0$ such that when $d > D$,

$$\begin{aligned} x_{n+1} &< f(d\lambda_1^2 X_n, d\lambda_2^2 Z_n) + \varepsilon \\ &\leq f(X_n, Z_n) + \varepsilon \\ &\leq X_n + Z_n - 2\varepsilon + \varepsilon \\ &= x_n - \varepsilon, \end{aligned}$$

where the second inequality is from Lemma 4.1.3, say, $f(s,t)$ is increasing with respect to s and t simultaneously. So here if we choose N large enough to make $N > \varepsilon^{-1}$, $x_N - N\varepsilon < 0$, a contradiction to $x_n \geq 0$ for all n. $\square$

In Theorem 1.3.2, by taking $\lambda_1 = \lambda_2$, i.e. $p_1 = p_2$, our model becomes the 4-state symmetric Potts model, which was left as an unsolved problem in [3].

Proposition 4.3.2. *For the symmetric channel corresponds to the* 4*-state Potts model on the tree, there exists a* D *such that for* $d \geq D$ *the Kesten-Stigum bound is sharp for both the ferromagnetic and antiferrmagnetic channels.*

CHAPTER 5

ASYMMETRIC BINARY CHANNELS

In this chapter, we will focus on the asymmetric binary channel

$$\mathbf{M} = \frac{1}{2}\left[\begin{pmatrix} 1+\theta & 1-\theta \\ 1-\theta & 1+\theta \end{pmatrix} + \Delta \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\right].$$

where the parameter Δ describes the deviation from the symmetric model. Obviously it should have some restriction, such as

$$|\theta| + |\Delta| \leq 1$$

to make this matrix well-defined. Furthermore it is clear that the second eigenvalue of the channel $\mathbf{M}$ is θ. Recall that $|\Theta| \leq d^{-\frac{1}{2}}$ is the exact threshold for non-solvability, which had not been known until [13], though.

Theorem. *For all $d \geq 2$, there exists a $\Delta_0 > 0$ such that for all $|\Delta| \leq \Delta_0$, the reconstruction problem for M on the d-ary tree $\mathbb{T}$ is not solvable if $d\theta^2 \leq 1$, that is, $d\Theta^2 = 1$.*

But this theorem has just established the existence of Δ_0 without estimating the range to keep Kesten-Stigum bound tight. In this chapter we will establish the critical condition of Δ to keep $d\Theta^2 = 1$. In light of the specified approaches engaged in previous chapters, we will only give the brief discussion for most similar results in the following context. Since $d\theta^2 > 1$ always guarantee the reconstruction, it suffices to consider $\frac{1}{2} \leq d\theta^2 \leq 1$ in this chapter.

5.1 Preliminary Works.

First notice that the stationary distribution $\pi = (\pi_1, \pi_2)$ of $\mathbf{M}$ is given by

$$\pi_1 = \frac{1}{2} - \frac{\Delta}{2(1-\theta)}, \quad \text{and} \quad \pi_2 = \frac{1}{2} + \frac{\Delta}{2(1-\theta)}.$$

Next according as the posterior functions in Chapter 1, define

$$X^+ = X^+(n) = f_n(1, \sigma^1(n));$$

$$X^- = X^-(n) = f_n(2, \sigma^2(n)),$$

and for $1 \le j \le d$,

$$Y_j = Y_j(n) = f_n(1, \sigma_j^1(n+1)),$$

where the random variables $\{Y_j\}_{1\le j\le d}$ are independent and identical in distribution. Furthermore, it is apparent that $\mathbf{E}(X_1) = \pi_1$ and $\mathbf{E}(X_2) = \pi_2$. Last denote similarly the objective quantities

$$x_n = \mathbf{E}(X^+(n) - \pi_1) = \mathbf{E}f_n(1, \sigma^1(n)) - \pi_1$$

and

$$z_n = \mathbf{E}(X^+(n) - \pi_1)^2 = \mathbf{E}(f_n(1, \sigma^1(n)) - \pi_1)^2.$$

Next let's introduce some analogous identities as developed in Section 2.2 and 2.3.

Lemma 5.1.1. *We have that*

$$x_n + \pi_1 = \mathbf{E}X^+ = \frac{1}{\pi_1}\mathbf{E}(X_1^2)$$

and

$$x_n = \frac{1}{\pi_1}\mathbf{E}(X_1 - \pi_1)^2 = \mathbf{E}(X^+(n) - \pi_1)^2 + \frac{\pi_2}{\pi_1}\mathbf{E}(X^-(n) - \pi_2)^2 \ge z_n \ge 0.$$

Proof. By Bayes' rule, we have

$$\begin{aligned}
\mathbf{E}X^+ &= \mathbf{E}f_n(1, \sigma^1(n)) \\
&= \sum_A f_n(1, A)\mathbf{P}(\sigma(n) = A \mid \sigma_\rho = 1) \\
&= \frac{1}{\pi_1}\sum_A \mathbf{P}(\sigma_\rho = 1 \mid \sigma(n) = A)\mathbf{P}(\sigma(n) = A) f_n(1, A) \\
&= \frac{1}{\pi_1}\sum_A f_n(1, A)^2\mathbf{P}(\sigma(n) = A) \\
&= \frac{1}{\pi_1}\mathbf{E}(X_1^2)
\end{aligned}$$

and similarly,

$$\mathbf{E}X^- = \mathbf{E}f_n\left(2, \sigma^2(n)\right) = \frac{1}{\pi_2}\mathbf{E}(X_2^2).$$

Then it follows

$$x_n = \frac{1}{\pi_1}\left(\mathbf{E}(X_1^2) - \pi_1^2\right) = \frac{1}{\pi_1}\mathbf{E}(X_1 - \pi_1)^2$$

from the fact of $\mathbf{E}(X_1) = \pi_1$. Last we illustrate the relation between x_n and z_n by Total Probability Formula,

$$\begin{aligned}
x_n &= \frac{1}{\pi_1}\mathbf{E}(X_1 - \pi_1)^2 \\
&= \frac{1}{\pi_1}\left[\mathbf{P}(\sigma_\rho = 1)\mathbf{E}\big((X_1 - \pi_1)^2 \mid \sigma_\rho = 1\big) + \mathbf{P}(\sigma_\rho = 2)\mathbf{E}\big((X_2 - \pi_2)^2 \mid \sigma_\rho = 2\big)\right] \\
&= \frac{1}{\pi_1}\left[\pi_1\mathbf{E}(X^+(n) - \pi_1)^2 + \pi_2\mathbf{E}(X^-(n) - \pi_2)^2\right] \\
&= \mathbf{E}(X^+(n) - \pi_1)^2 + \frac{\pi_2}{\pi_1}\mathbf{E}(X^-(n) - \pi_2)^2 \\
&\geq z_n,
\end{aligned}$$

as desired. □

Proposition 5.1.2. *For each $1 \leq j \leq d$, we have*

$$\mathbf{E}(Y_j - \pi_1) = \theta x_n$$

and

$$\mathbf{E}(Y_j - \pi_1)^2 = \theta z_n + \pi_1(1-\theta)x_n.$$

Proof. If $\sigma^1_{u_j} = 1$, Y_j is distributed according to $X^+(n)$, while to $1 - X^-(n)$ given $\sigma^1_{u_j} = 2$. Therefore Lemma 5.1.1 yields

$$\begin{aligned}
\mathbf{E}(Y_j - \pi_1) &= \mathbf{P}(\sigma^1_{u_j} = 1)\mathbf{E}(X^+(n) - \pi_1) + \mathbf{P}(\sigma^1_{u_j} = 2)\mathbf{E}(1 - X^-(n) - \pi_1) \\
&= M_{11}x_n - M_{12}\frac{\pi_1}{\pi_2}x_n \\
&= \theta x_n,
\end{aligned}$$

as well as,

$$\begin{aligned}
\mathbf{E}(Y_j - \pi_1)^2 &= \mathbf{P}(\sigma^1_{u_j} = 1)\mathbf{E}(X^+(n) - \pi_1)^2 + \mathbf{P}(\sigma^1_{u_j} = 2)\mathbf{E}(1 - X^-(n) - \pi_1)^2 \\
&= M_{11}\mathbf{E}(X^+(n) - \pi_1)^2 + M_{12}\mathbf{E}(X^-(n) - \pi_2)^2 \\
&= M_{11}z_n + M_{12}\frac{\pi_1}{\pi_2}(x_n - z_n) \\
&= \theta z_n + \pi_1(1-\theta)x_n.
\end{aligned}$$

□

5.2 Recursive Formula.

To determine the reconstruction, it suffices to research the asymptotic behavior of x_n when n is large enough, because the non-reconstruction is equivalent to $\lim_{n\to\infty} x_n = 0$ as showed in Lemma 3.1.1. Then from the Markov random field property, we have

$$\begin{aligned} f_{n+1}(1,A) &= \frac{\pi_1 \prod_{j=1}^d \left[\frac{M_{11}}{\pi_1} f_n(1,A_j) + \frac{M_{12}}{\pi_2} f_n(2,A_j)\right]}{\pi_1 \prod_{j=1}^d \left[\frac{M_{11}}{\pi_1} f_n(1,A_j) + \frac{M_{12}}{\pi_2} f_n(2,A_j)\right] + \pi_2 \prod_{j=1}^d \left[\frac{M_{21}}{\pi_1} f_n(1,A_j) + \frac{M_{22}}{\pi_2} f_n(2,A_j)\right]} \\ &= \frac{\pi_1 \prod_{j=1}^d \left[1 + \frac{\theta}{\pi_1}(f_n(1,A_j) - \pi_1)\right]}{\pi_1 \prod_{j=1}^d \left[1 + \frac{\theta}{\pi_1}(f_n(1,A_j) - \pi_1)\right] + \pi_2 \prod_{j=1}^d \left[1 - \frac{\theta}{\pi_2}(f_n(1,A_j) - \pi_1)\right]}. \qquad (5.2.1) \end{aligned}$$

Next conditioning the root to be 1 and setting $A = \sigma^1(n+1)$, it can be expressed as

$$X^+(n+1) = \frac{\pi_1 Z_1}{\pi_1 Z_1 + \pi_2 Z_2},$$

where

$$Z_1 = \prod_{j=1}^d \left[1 + \frac{\theta}{\pi_1}(f_n(1,A_j) - \pi_1)\right] = \prod_{j=1}^d \left[1 + \frac{\theta}{\pi_1}(Y_j(n) - \pi_1)\right]$$

and

$$Z_2 = \prod_{j=1}^d \left[1 - \frac{\theta}{\pi_2}(f_n(1,A_j) - \pi_1)\right] = \prod_{j=1}^d \left[1 - \frac{\theta}{\pi_2}(Y_j(n) - \pi_1)\right].$$

By means of the identity (2.5.1), we still focus on the main expansion of x_{n+1} as before

$$\begin{aligned} x_{n+1} &= \mathbf{E}X^+(n+1) - \pi_1 \\ &= \mathbf{E}(\pi_1 Z_1) - \mathbf{E}[\pi_1 Z_1(\pi_1 Z_1 + \pi_2 Z_2 - 1)] + \mathbf{E}\left[(\pi_1 Z_1 + \pi_2 Z_2 - 1)^2 \frac{\pi_1 Z_1}{\pi_1 Z_1 + \pi_2 Z_2}\right] - \pi_1. \end{aligned}$$

Next estimate preceding terms one by one as showed in Lemma 2.5.2 and it is remarked that the following R_is denote the remainder terms bounded by $O_\pi(x_n^3)$:

$$\begin{aligned} \mathbf{E}Z_1 &= \mathbf{E}\prod_{j=1}^d \left[1 + \frac{\theta}{\pi_1}(Y_j - \pi_1)\right] \\ &= 1 + \frac{d\theta}{\pi_1}\mathbf{E}(Y_1 - \pi_1) + \frac{d(d-1)}{2}\left[\frac{\theta}{\pi_1}\mathbf{E}(Y_1 - \pi_1)\right]^2 + R_1 \\ &= 1 + \frac{d\theta^2}{\pi_1}x_n + \frac{d(d-1)}{2}\frac{\theta^4}{\pi_1^2}x_n^2 + R_1; \end{aligned}$$

$$
\begin{aligned}
\mathbf{E}Z_1^2 &= \mathbf{E}\prod_{j=1}^{d}\left[1+\frac{\theta}{\pi_1}(Y_j-\pi_1)\right]^2 \\
&= 1+d\left\{\mathbf{E}\left[1+\frac{\theta}{\pi_1}(Y_1-\pi_1)\right]^2-1\right\}+\frac{d(d-1)}{2}\left\{\mathbf{E}\left[1+\frac{\theta}{\pi_1}(Y_1-\pi_1)\right]^2-1\right\}^2+R_2 \\
&= 1+d\left[\frac{\theta^2}{\pi_1}(3-\theta)x_n+\frac{\theta^3}{\pi_1^2}z_n\right]+\frac{d(d-1)}{2}\left[\frac{\theta^2}{\pi_1}(3-\theta)x_n+\frac{\theta^3}{\pi_1^2}z_n\right]^2+R_2;
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{E}Z_1Z_2 &= \mathbf{E}\prod_{j=1}^{d}\left[1+\frac{\theta}{\pi_1}(Y_j-\pi_1)\right]\left[1-\frac{\theta}{\pi_2}(Y_j-\pi_1)\right] \\
&= 1+d\left[\theta^2\left(\frac{1}{\pi_1}+\frac{\theta-2}{\pi_2}\right)x_n-\frac{\theta^3}{\pi_1\pi_2}z_n\right]+\frac{d(d-1)}{2}\left[\theta^2\left(\frac{1}{\pi_1}+\frac{\theta-2}{\pi_2}\right)x_n-\frac{\theta^3}{\pi_1\pi_2}z_n\right]^2+R_3.
\end{aligned}
$$

Consequently taken together, the preceding results yield

$$
\begin{aligned}
& \mathbf{E}\pi_1Z_1(\pi_1Z_1+\pi_2Z_2-1) \\
=\ & \pi_1^2\mathbf{E}Z_1^2+\pi_1\pi_2\mathbf{E}Z_1Z_2-\pi_1\mathbf{E}Z_1 \\
=\ & \pi_1^2\frac{d(d-1)}{2}\left[\frac{\theta^2}{\pi_1}(3-\theta)x_n+\frac{\theta^3}{\pi_1^2}z_n\right]^2+\pi_1\pi_2\frac{d(d-1)}{2}\left[\theta^2\left(\frac{1}{\pi_1}+\frac{\theta-2}{\pi_2}\right)x_n-\frac{\theta^3}{\pi_1\pi_2}z_n\right]^2 \\
& -\frac{d(d-1)}{2}\frac{\theta^4}{\pi_1}x_n^2+R_4.
\end{aligned}
$$

Now treating Z_2 in the similar way as before yields

$$
\begin{aligned}
\mathbf{E}Z_2 &= \mathbf{E}\prod_{j=1}^{d}\left[1-\frac{\theta}{\pi_2}(Y_j-\pi_1)\right] \\
&= 1-\frac{d\theta}{\pi_2}\mathbf{E}(Y_1-\pi_1)+\frac{d(d-1)}{2}\left[\frac{\theta}{\pi_2}\mathbf{E}(Y_1-\pi_1)\right]^2+R_5 \\
&= 1-\frac{d\theta^2}{\pi_2}x_n+\frac{d(d-1)}{2}\frac{\theta^4}{\pi_2^2}x_n^2+R_5
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbf{E}Z_2^2 &= \mathbf{E}\prod_{j=1}^{d}\left[1-\frac{\theta}{\pi_2}(Y_j-\pi_1)\right]^2 \\
&= 1+d\left\{\mathbf{E}\left[1-\frac{\theta}{\pi_2}(Y_1-\pi_1)\right]^2-1\right\}+\frac{d(d-1)}{2}\left\{\mathbf{E}\left[1-\frac{\theta}{\pi_2}(Y_1-\pi_1)\right]^2-1\right\}^2+R_6 \\
&= 1+d\left\{\frac{\theta^2}{\pi_2}\left[\frac{\pi_1}{\pi_2}(1-\theta)-2\right]x_n+\frac{\theta^3}{\pi_2^2}z_n\right\}+\frac{d(d-1)}{2}\left\{\frac{\theta^2}{\pi_2}\left[\frac{\pi_1}{\pi_2}(1-\theta)-2\right]x_n+\frac{\theta^3}{\pi_2^2}z_n\right\}^2+R_6.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
& \mathbf{E}(\pi_1 Z_1 + \pi_2 Z_2 - 1)^2 \\
= \; & \pi_1^2 \mathbf{E}(Z_1^2) + \pi_2^2 \mathbf{E}(Z_2^2) + 2\pi_1\pi_2 \mathbf{E}(Z_1 Z_2) - 2\pi_1 \mathbf{E}(Z_1) - 2\pi_2 \mathbf{E}(Z_2) + 1 \\
= \; & \pi_1^2 \frac{d(d-1)}{2} \left[\frac{\theta^2}{\pi_1}(3-\theta)x_n + \frac{\theta^3}{\pi_1^2} z_n \right]^2 + \pi_2^2 \frac{d(d-1)}{2} \left\{ \frac{\theta^2}{\pi_2} \left[\frac{\pi_1}{\pi_2}(1-\theta) - 2 \right] x_n + \frac{\theta^3}{\pi_2^2} z_n \right\}^2 \\
& + \pi_1\pi_2 d(d-1) \left[\theta^2 \left(\frac{1}{\pi_1} + \frac{\theta - 2}{\pi_2} \right) x_n - \frac{\theta^3}{\pi_1 \pi_2} z_n \right]^2 - d(d-1)\frac{\theta^4}{\pi_1} x_n^2 - d(d-1)\frac{\theta^4}{\pi_2} x_n^2 + R_7.
\end{aligned}
$$

The purpose of the following lemma is to describe how close the linear term in the recursive expansion approaches to x_{n+1}.

Lemma 5.2.1. *For any $\varepsilon > 0$, there exists a constant $\delta = \delta(\pi, \varepsilon)$ such that for all n, if $x_n < \delta$ then*

$$|x_{n+1} - d\theta^2 x_n| \leq \varepsilon x_n.$$

Proof. It is natural that $Z_1, Z_2 \geq 0$, and thus $0 \leq \frac{\pi_1 Z_1}{\pi_1 Z_1 + \pi_2 Z_2} \leq 1$. Next substitute the preceding results into the expression of x_{n+1} and then get

$$
\begin{aligned}
& |x_{n+1} - d\theta^2 x_n| \\
= \; & \left| \mathbf{E}(\pi_1 Z_1) - \mathbf{E}\pi_1 Z_1(\pi_1 Z_1 + \pi_2 Z_2 - 1) + \mathbf{E}(\pi_1 Z_1 + \pi_2 Z_2 - 1)^2 \frac{\pi_1 Z_1}{\pi_1 Z_1 + \pi_2 Z_2} - \pi_1 - d\theta^2 x_n \right| \\
\leq \; & \left| \mathbf{E}(\pi_1 Z_1) - \mathbf{E}\pi_1 Z_1(\pi_1 Z_1 + \pi_2 Z_2 - 1) - \pi_1 - d\theta^2 x_n \right| + \mathbf{E}(\pi_1 Z_1 + \pi_2 Z_2 - 1)^2 \\
\leq \; & C_1 x_n^2 + C_2 x_n^3 \\
\leq \; & \varepsilon x_n,
\end{aligned}
$$

where $C_1 = C_1(\pi)$ and $C_2 = C_2(\pi)$ depend only on π, the second last inequality follows from the fact $0 \leq z_n \leq x_n$, and the last holds if $x_n < \delta$ for $\delta = \delta(\pi, \varepsilon)$ small enough. □

Referring to Lemma 3.2.2, it is known that fixed finite different vertices far away from the root can effect the root little, based on which, it is possible to exploit the concentration analysis. Although the blueprint of the proof is similar to Lemma 3.2.2, it is still worth illustrating afresh due to some qualitative changes caused by differences of models.

Lemma 5.2.2. *For any $\varepsilon > 0$ and positive integer k there exists $M = M(\pi, \varepsilon, k)$ such that for any collection of vertices $v_1, \dots, v_k \in L(M)$,*

$$\sup_{i_1,\dots,i_k} \left| \mathbf{P}(\sigma_\rho = 1 \mid \sigma_{v_j} = i_j, 1 \le j \le k) - \pi_1 \right| \le \varepsilon$$

Proof. Denote the transition matrices at distance s by

$$U_s = M^s_{1,1}, \quad V_s = M^s_{2,2}$$

and it is natural to see that $M^s_{1,2} = 1 - U_s$ and $M^s_{2,1} = 1 - V_s$. As a result, it follows that

$$\begin{cases} U_s = M_{1,1}U_{s-1} + M_{1,2}(1 - V_{s-1}) \\ V_s = M_{2,1}(1 - U_{s-1}) + M_{2,2}V_{s-1} \end{cases}$$

which yields a second order recursive formula

$$U_s - (1+\theta)U_{s-1} + \theta U_{s-2} = 0$$

with the initial conditions $U_0 = 1$ and $U_1 = M_{1,1} = \pi_1 + \pi_2\theta$. Then the general solutions are

$$U_s = \pi_1 + \pi_2\theta^s \quad \text{and} \quad V_s = \pi_2 + \pi_1\theta^s. \tag{5.2.2}$$

Consequently under the condition of $d\theta^2 \le 1$ we have

$$\pi_1 - \pi_2 d^{-s/2} \le M^s_{1,1} \le \pi_1 + \pi_2 d^{-s/2};$$

$$\pi_2 - \pi_1 d^{-s/2} \le M^s_{2,2} \le \pi_2 + \pi_1 d^{-s/2};$$

$$\pi_2 - \pi_2 d^{-s/2} \le M^s_{1,2} \le \pi_2 + \pi_2 d^{-s/2};$$

$$\pi_1 - \pi_1 d^{-s/2} \le M^s_{2,1} \le \pi_1 + \pi_1 d^{-s/2}.$$

For fixed π, d and k, define

$$B(s) = \max\left\{ \frac{\pi_1 + \pi_2 d^{-s/2}}{\pi_1 - \pi_2 d^{-s/2}}, \frac{\pi_2 + \pi_1 d^{-s/2}}{\pi_2 - \pi_1 d^{-s/2}}, \frac{1 + d^{-s/2}}{1 - d^{-s/2}} \right\}$$

and let γ be a sufficiently large integer such that

$$B^k(\gamma) \le 1 + \varepsilon$$

since $d^{-s/2} \to 0$ implies $B(s) \to 1$ as $s \to \infty$.

According as the procedure of Lemma 3.2.2, it is convenient to skip the rest analogous steps. By Bayes' Rule and the Markov random field property, for any $i_1, \ldots, i_{n_\ell} \in \mathcal{C}$, we have

$$\begin{aligned}
& \frac{\mathbf{P}(\sigma_\rho = 1 \mid \sigma_{w_j} = i_j, 1 \le j \le n_\ell)}{\mathbf{P}(\sigma_\rho = 2 \mid \sigma_{w_j} = i_j, 1 \le j \le n_\ell)} \\
= \; & \frac{\pi_1}{\pi_2} \frac{\mathbf{P}(\sigma_{w_j} = i_j, 1 \le j \le n_\ell \mid \sigma_\rho = 1)}{\mathbf{P}(\sigma_{w_j} = i_j, 1 \le j \le n_\ell \mid \sigma_\rho = 2)} \\
= \; & \frac{\pi_1}{\pi_2} \frac{\sum_{h_1, \ldots, h_{n_\ell} \in \mathcal{C}} \mathbf{P}(\forall j \; \sigma_{w_j} = i_j \mid \forall j \; \sigma_{\overline{w}_j} = h_j) \mathbf{P}(\forall j \; \sigma_{\overline{w}_j} = h_j \mid \sigma_\rho = 1)}{\sum_{h_1, \ldots, h_{n_\ell} \in \mathcal{C}} \mathbf{P}(\forall j \; \sigma_{w_j} = i_j \mid \forall j \; \sigma_{\overline{w}_j} = h_j) \mathbf{P}(\forall j \; \sigma_{\overline{w}_j} = h_j \mid \sigma_\rho = 2)} \\
= \; & \frac{\pi_1}{\pi_2} \frac{\sum_{h_1, \ldots, h_{n_\ell} \in \mathcal{C}} \mathbf{P}(\forall j \; \sigma_{\overline{w}_j} = h_j \mid \sigma_\rho = 1) \prod_{j=1}^{n_\ell} M^\gamma_{h_j i_j}}{\sum_{h_1, \ldots, h_{n_\ell} \in \mathcal{C}} \mathbf{P}(\forall j \; \sigma_{\overline{w}_j} = h_j \mid \sigma_\rho = 2) \prod_{j=1}^{n_\ell} M^\gamma_{h_j i_j}} \\
\le \; & \frac{\pi_1}{\pi_2} B^{n_\ell}(\gamma) \frac{\sum_{h_1, \ldots, h_{n_\ell} \in \mathcal{C}} \mathbf{P}(\forall j \; \sigma_{\overline{w}_j} = h_j \mid \sigma_\rho = 1)}{\sum_{h_1, \ldots, h_{n_\ell} \in \mathcal{C}} \mathbf{P}(\forall j \; \sigma_{\overline{w}_j} = h_j \mid \sigma_\rho = 2)} \\
\le \; & \frac{\pi_1}{\pi_2} B^k(\gamma) \\
\le \; & \frac{\pi_1}{\pi_2} (1 + \varepsilon),
\end{aligned}$$

which implies that

$$\pi_1 - \varepsilon \le \mathbf{P}(\sigma_\rho = 1 \mid \sigma_{w_j} = i_j, 1 \le j \le n_\ell) \le \pi_1 + \varepsilon.$$

To the end, since σ_ρ is conditionally independent of the collection $\sigma_{v_1}, \ldots, \sigma_{v_k}$ given $\sigma_{w_1}, \ldots, \sigma_{w_{n_\ell}}$, it is concluded that

$$\begin{aligned}
& \sup_{i_1, \ldots, i_k} \left| \mathbf{P}(\sigma_\rho = 1 \mid \sigma_{v_j} = i_j, 1 \le j \le k) - \pi_1 \right| \\
\le \; & \sup_{i_1, \ldots, i_{n_\ell}} \left| \mathbf{P}(\sigma_\rho = 1 \mid \sigma_{w_j} = i_j, 1 \le j \le n_\ell) - \pi_1 \right| \\
\le \; & \varepsilon.
\end{aligned}$$

□

Before the concentration lemmas, it is necessary to verify that x_n does not drop from a very large value to a very small one as discussed in Lemma 3.2.1, but with the distinct approach.

Lemma 5.2.3. *If $\theta \neq 0$, then $x_n > 0$ holds for all $n \ge 0$. Consequently, under the assumption of $\lim_{n \to \infty} x_n = 0$ and $d\theta^2 \ge \frac{1}{2}$, there exists a constant $0 < \gamma = \gamma(d, \pi) < 1$ such that for all $n \ge 0$*

$$x_{n+1} \ge \gamma x_n.$$

Proof. In Lemma 5.1.1 we have proved that $x_n \geq 0$, so here it suffices to exclude the equality. Applying the contradiction, assume $x_n = 0$ for some $n \geq 1$. By Lemma 5.1.1, it follows that

$$\frac{1}{\pi_1}\mathbf{E}(X_1 - \pi_1)^2 = x_n = 0$$

Therefore $X_1(n) = \pi_1$ and thus $X_2(n) = \pi_2$ a.s., and thus by Bayes' rule for any configuration combination A on the nth level, we always have

$$\begin{aligned}
\mathbf{P}(\sigma(n) = A \mid \sigma_\rho = 1) &= \frac{1}{\pi_1}\mathbf{P}(\sigma_\rho = 1 \mid \sigma(n) = A)\mathbf{P}(\sigma(n) = A) \\
&= \mathbf{P}(\sigma(n) = A) \\
&= \frac{1}{\pi_2}\mathbf{P}(\sigma_\rho = 2 \mid \sigma(n) = A)\mathbf{P}(\sigma(n) = A) \\
&= \mathbf{P}(\sigma(n) = A \mid \sigma_\rho = 2).
\end{aligned}$$

Denote the leftmost vertex on the nth level by $v_n(1)$, and the preceding formula implies

$$\mathbf{P}(\sigma_{v_n(1)} = 1 \mid \sigma_\rho = 1) = \mathbf{P}(\sigma_{v_n(1)} = 1 \mid \sigma_\rho = 2).$$

Therefore from the formula (5.2.2), we have $X_n = 1 - Y_n$, which further implies $\theta = 0$, a contradiction.

Under the assumption of $d\theta^2 \geq \frac{1}{2}$ and $\lim_{n\to\infty} x_n = 0$, it is concluded that the less $d\theta^2$ is, the faster x_n approaches to 0, and thus there exists a large $N = N(\pi) > 0$ such that when $n > N$,

$$\frac{x_n}{x_{n+1}} = \frac{x_n}{d\theta^2 x_n + O_\pi(x_n^2)} \leq \frac{x_n}{\frac{9}{10}d\theta^2 x_n} = \frac{10}{9}\frac{1}{d\theta^2} \leq \frac{20}{9}. \tag{5.2.3}$$

Since here $\theta \neq 0$ implies $x_n > 0$, it is feasible to define a positive function

$$\Gamma(\theta, d, \pi) = \min\left\{\frac{9}{20}, \frac{x_{m+1}}{x_m} : 0 \leq m \leq N\right\}.$$

Now fix d and π, from the construction of x_n, it is natural that x_n is continuous for the parameter θ, so is Γ. Based on the assumption, we have $\sqrt{\frac{1}{2}\frac{1}{d}} \leq |\theta| \leq \sqrt{\frac{1}{d}}$, that is, θ ranges in a compact set, and then there exists a $\widetilde{\theta} \in [-\sqrt{\frac{1}{d}}, -\sqrt{\frac{1}{2d}}] \cup [\sqrt{\frac{1}{2d}}, \sqrt{\frac{1}{d}}]$ such that $\Gamma_{\min} = \Gamma(\widetilde{\theta}, d, \pi)$, where $\Gamma(\widetilde{\theta}, d, \pi) > 0$ follows from the first part of this lemma. Finally choosing $\gamma(d, \pi) = \Gamma(\widetilde{\theta}, d, \pi) > 0$ completes the proof. □

With the previous Lemmas, we are able to obtain the concentration estimates of $\frac{\pi_1 Z_1}{\pi_1 Z_1+\pi_2 Z_2}$ and $\frac{z_n}{x_n}$ after Lemma 3.2.4 and Lemma 3.2.6 respectively, when x_n is small.

Lemma 5.2.4. *Assume* $\lim_{n\to\infty} x_n = 0$. *Then for any* $\varepsilon, \alpha > 0$ *there exist constants* $C = C(\pi, \varepsilon, \alpha)$ *and* $N = N(\pi, \varepsilon, \alpha)$ *such that when* $n > N$,

$$\mathbf{P}\left(\left|\frac{\pi_1 Z_1}{\pi_1 Z_1 + \pi_2 Z_2} - \pi_1\right| > \varepsilon\right) \leq C x_n^\alpha$$

Proof. Next fix k an integer with $k > \alpha$. In Lemma 5.2.2 choose M to hold with bound $\varepsilon/2$. Then let $v_1, \ldots, v_{|L(M)|}$ denote the vertices in $L(M)$ and define

$$W(v) = f_{n+1-M}(1, \sigma_v^1(n+1))$$

where $\sigma_v^1(n+1)$ denotes the spins of vertices in $\mathbb{T}_v \cap L(n+1)$. Then $W(v)$ would be distributed as

$$W(v) \sim \begin{cases} X^+(n+1-M) & \text{if } \sigma_v^1 = 1, \\ \\ 1 - X^-(n+1-M) & \text{if } \sigma_v^1 = 2. \end{cases} \tag{5.2.4}$$

Repeating recursion formula (5.2.1) yields a function

$$H(W_1, \ldots, W_{|L(M)|}) = f_n(1, \sigma^1(n+1)) = \frac{\pi_1 Z_1}{\pi_1 Z_1 + \pi_2 Z_2},$$

where $W_i = W(v_i)$ for $1 \leq i \leq |L(M)|$. Based on the discussion in Lemma 3.2.4, it is known that if x_n is sufficiently small, $W(v)$ should be close enough to π_1, which implies that $H(W_1, \ldots, W_{|L(M)|})$ is sufficiently around π_1. Therefore by Lemma 5.2.2 if there are at most k vertices in $L(M)$ such that $W(v) \neq \pi_1$ then

$$|H(W_1, \ldots, W_{|L(M)|}) - \pi_1| < \varepsilon/2.$$

Since H is a continuous function for all W_i, there exists some $\delta = \delta(\varepsilon) > 0$ such that if

$$\#\left\{v \in L(M) : |W(v) - \pi_1| > \delta\right\} \leq k$$

then

$$|H(W_1, \ldots, W_{|L(M)|}) - \pi_1| < \varepsilon.$$

Next by Chebyshev's inequality and relying on (5.2.4), the following holds,

$$\begin{aligned}\mathbf{P}(|W(v)-\pi_1|>\delta) &\le \delta^{-2}[\mathbf{E}(X^+(n+1-M)-\pi_1)^2+\mathbf{E}(X^-(n+1-M)-\pi_2)^2] \\ &\le \frac{\delta^{-2}}{\min\left\{1,\frac{\pi_2}{\pi_1}\right\}}x_{n+1-M}.\end{aligned}$$

As random variables, $|W(v)-\pi_1|$ for distinct v are conditionally independent given $\sigma(M)$, and therefore there exist suitable constants $C(\pi,\varepsilon,\alpha)$ and $N(\pi,\varepsilon,\alpha)$ such that whenever $n>N$,

$$\begin{aligned}&\mathbf{P}\left(\left|\frac{\pi_1 Z_1}{\pi_1 Z_1+\pi_2 Z_2}-\pi_1\right|>\varepsilon\right)\\ \le\ &\mathbf{P}(\#\{v\in L(M):|W(v)-\pi_1|>\delta\}>k)\\ =\ &\sum_A \mathbf{P}(\#\{v\in L(M):|W(v)-\pi_1|>\delta\}>k\mid\sigma(M)=A)\mathbf{P}(\sigma(M)=A)\\ \le\ &\sum_A \mathbf{P}\left[\mathbf{B}\left(|L(M)|,\frac{\delta^{-2}}{\min\left\{1,\frac{\pi_2}{\pi_1}\right\}}x_{n+1-M}\right)>k\right]\mathbf{P}(\sigma(M)=A)\\ \le\ &C'x^{\alpha}_{n+1-M}\\ \le\ &Cx^{\alpha}_n\end{aligned}$$

where we use the fact $k>\alpha$ and the last inequality comes from (5.2.3). □

Lemma 5.2.5. *Assume* $\lim_{n\to\infty}x_n=0$. *For any* $\varepsilon>0$, *there exist* $N=N(\pi,\varepsilon)$ *and* $\delta=\delta(\pi,\varepsilon)$ *such that if* $n>N$ *and* $x_n<\delta$,

$$\left|\frac{z_n}{x_n}-\pi_1\right|\le\varepsilon.$$

Proof. Taking $a=(Z_1-Z_2)^2$, $r=(\pi_1Z_1+\pi_2Z_2)^2-1$ and $s=1$ in the identity (2.5.1) yields

$$\begin{aligned}z_{n+1} &= \mathbf{E}\left(\frac{\pi_1 Z_1}{\pi_1 Z_1+\pi_2 Z_2}-\pi_1\right)^2\\ &= \pi_1^2\pi_2^2\mathbf{E}\frac{(Z_1-Z_2)^2}{1+(\pi_1Z_1+\pi_2Z_2)^2-1}\\ &= \pi_1^2\pi_2^2\left\{\mathbf{E}(Z_1-Z_2)^2-\mathbf{E}(Z_1-Z_2)^2[(\pi_1Z_1+\pi_2Z_2)^2-1]\right.\\ &\quad\left.+\mathbf{E}[(\pi_1Z_1+\pi_2Z_2)^2-1]^2\frac{(Z_1-Z_2)^2}{(\pi_1Z_1+\pi_2Z_2)^2}\right\}.\end{aligned}$$

Next let's estimate these expectations term by term and we remark that C_is and O_π-constants in the following context only depend on π. Then

$$\begin{aligned}
\mathbf{E}(Z_1-Z_2)^2 &= \mathbf{E}(Z_1^2+Z_2^2-2Z_1Z_2) \\
&= 1+d\left[\frac{\theta^2}{\pi_1}(3-\theta)x_n+\frac{\theta^3}{\pi_1^2}z_n\right]+1+d\left\{\frac{\theta^2}{\pi_2}\left[\frac{\pi_1}{\pi_2}(1-\theta)-2\right]x_n+\frac{\theta^3}{\pi_2^2}z_n\right\} \\
&\quad -2\left\{1+d\left[\theta^2\left(\frac{1}{\pi_1}+\frac{\theta-2}{\pi_2}\right)x_n-\frac{\theta^3}{\pi_1\pi_2}z_n\right]\right\}+O_\pi(x_n^2) \\
&= d\theta^2\left(\frac{1-\theta}{\pi_1\pi_2^2}x_n+\frac{\theta}{\pi_1^2\pi_2^2}z_n\right)+O_\pi(x_n^2),
\end{aligned}$$

and

$$\mathbf{E}(Z_1-Z_2)^2[(\pi_1Z_1+\pi_2Z_2)^2-1=O_\pi(x_n^2),$$

in addition, relying on $\pi_1^2\pi_2^2\frac{(Z_1-Z_2)^2}{(\pi_1Z_1+\pi_2Z_2)^2}\le 1$, we could come up with

$$\pi_1^2\pi_2^2\mathbf{E}[(\pi_1Z_1+\pi_2Z_2)^2-1]^2\frac{(Z_1-Z_2)^2}{(\pi_1Z_1+\pi_2Z_2)^2} \le \mathbf{E}[(\pi_1Z_1+\pi_2Z_2)^2-1]^2=O_\pi(x_n^2).$$

Therefore the recursion of z_{n+1} would be expressed as

$$z_{n+1}=d\theta^2[\pi_1(1-\theta)x_n+\theta z_n]+O_\pi(x_n^2).$$

Lemmas 5.2.1 implies $\left|\frac{d\theta^2x_n}{x_{n+1}}-1\right|\le C_1\frac{x_n^2}{x_{n+1}}$, and thus if let $N_1=N_1(\pi)$ be the constant in the proof of Lemma 5.2.3 to guarantee (5.2.3), when $n>N_1$,

$$\begin{aligned}
&\left|\frac{z_{n+1}}{x_{n+1}}-\left[\pi_1(1-\theta)+\theta\frac{z_n}{x_n}\right]\right| \\
=& \left|\frac{z_{n+1}}{x_{n+1}}-\frac{d\theta^2x_n}{x_{n+1}}\left[\pi_1(1-\theta)+\theta\frac{z_n}{x_n}\right]\right|+\left|\left(\frac{d\theta^2x_n}{x_{n+1}}-1\right)\left[\pi_1(1-\theta)+\theta\frac{z_n}{x_n}\right]\right| \\
\le& C_2\frac{x_n^2}{x_{n+1}}+C_3\frac{x_n^2}{x_{n+1}} \\
\le& C_4x_{n+1},
\end{aligned}$$

where the second term of the first inequality comes from the fact of $0\le\frac{z_n}{x_n}\le 1$. For any $k\in\mathbb{N}$, iterating the preceding inequality k times yields

$$\begin{aligned}
\left|\frac{z_{n+k}}{x_{n+k}}-\left[\pi_1(1-\theta^k)+\theta^k\frac{z_n}{x_n}\right]\right| &\le \sum_{\ell=1}^{k}\left|\pi_1(1-\theta^{k-\ell})+\theta^{k-\ell}\frac{z_{n+\ell}}{x_{n+\ell}}-\pi_1(1-\theta^{k-\ell+1})-\theta^{k-\ell+1}\frac{z_{n+\ell-1}}{x_{n+\ell-1}}\right| \\
&\le \sum_{\ell=1}^{k}|\theta|^{k-\ell}\left|\frac{z_{n+\ell}}{x_{n+\ell}}-\left[\pi_1(1-\theta)+\theta\frac{z_{n+\ell-1}}{x_{n+\ell-1}}\right]\right| \\
&\le C_4\sum_{\ell=1}^{k}|\theta|^{k-\ell}x_{n+\ell-1}.
\end{aligned}$$

For fixed k, by Lemma 5.2.1 there exists a $\delta = \delta(\pi, k)$ such that if $x_n < \delta$, $x_{n+\ell} < 2\delta$ for any $1 \leq \ell \leq k$. Therefore based on the fact of $|\theta| \leq d^{-\frac{1}{2}} \leq \frac{1}{\sqrt{2}}$, taking $k = k(\varepsilon)$ large enough and $\delta = \delta(\pi, \varepsilon, k) = \delta(\pi, \varepsilon)$ sufficiently small, we obtain

$$\left|\frac{z_{n+k}}{x_{n+k}} - \pi_1\right| \leq |\theta|^k + 2\delta C_4 \sum_{\ell=1}^{k} |\theta|^{k-\ell} = |\theta|^k + 2\delta C_4 \frac{1-|\theta|^k}{1-|\theta|} \leq \varepsilon,$$

where the first inequality relies on $\left|\frac{z_n}{x_n} - \pi_1\right| < 1$. Finally, by choosing $N = N(\pi, \varepsilon) > N_1 + k$, plus noting that if $x_n < \delta$ then $x_{n-k} \leq (\frac{20}{9})^k x_n < (\frac{20}{9})^k \delta$ from (5.2.3), the previous result completes the proof. □

5.3 Condition for Reconstruction.

Theorem 5.3.1. *Assume* $\lim_{n\to\infty} x_n = 0$. *When* $\Delta^2 > \frac{1}{3}(1-\theta)^2$, *there exist* $N = N(\pi)$ *and* $\delta = \delta(\pi)$ *such that if* $n \geq N$ *and* $x_n \leq \delta$,

$$x_{n+1} \geq d\theta^2 x_n + \frac{1}{2}\frac{(1-6\pi_1\pi_2)}{\pi_1\pi_2^2}\frac{d(d-1)}{2}\theta^4 x_n^2.$$

Proof. First let's review the recursion formula

$$x_{n+1} = \mathbf{E}(\pi_1 Z_1) - \mathbf{E}\pi_1 Z_1(\pi_1 Z_1 + \pi_2 Z_2 - 1) + \mathbf{E}(\pi_1 Z_1 + \pi_2 Z_2 - 1)^2 \frac{\pi_1 Z_1}{\pi_1 Z_1 + \pi_2 Z_2} - \pi_1.$$

For any $\varepsilon > 0$, suppose that $N' = N'(\pi, \varepsilon)$ and $C = C(\pi, \varepsilon)$ are constants from Lemma 5.2.4, when $n > N'$, combining Cauchy-Schwartz inequality and Lemma 5.2.4 gives

$$\begin{aligned}
&\left|\mathbf{E}(\pi_1 Z_1 + \pi_2 Z_2 - 1)^2 \frac{\pi_1 Z_1}{\pi_1 Z_1 + \pi_2 Z_2} - \pi_1 \mathbf{E}(\pi_1 Z_1 + \pi_2 Z_2 - 1)^2\right| \\
\leq\ & \mathbf{E}(\pi_1 Z_1 + \pi_2 Z_2 - 1)^2 \left|\frac{\pi_1 Z_1}{\pi_1 Z_1 + \pi_2 Z_2} - \pi_1\right| \\
\leq\ & \varepsilon \mathbf{E}(\pi_1 Z_1 + \pi_2 Z_2 - 1)^2 + \mathbf{E}(\pi_1 Z_1 + \pi_2 Z_2 - 1)^2 \mathbf{I}\left(\left|\frac{\pi_1 Z_1}{\pi_1 Z_1 + \pi_2 Z_2} - \pi_1\right| > \varepsilon\right) \\
\leq\ & \varepsilon \mathbf{E}(\pi_1 Z_1 + \pi_2 Z_2 - 1)^2 + \mathbf{P}\left(\left|\frac{\pi_1 Z_1}{\pi_1 Z_1 + \pi_2 Z_2} - \pi_1\right| > \varepsilon\right)^{\frac{1}{2}} \left[\mathbf{E}(\pi_1 Z_1 + \pi_2 Z_2 - 1)^4\right]^{\frac{1}{2}} \\
\leq\ & \varepsilon \mathbf{E}(\pi_1 Z_1 + \pi_2 Z_2 - 1)^2 + C x_n^3.
\end{aligned}$$

Now it is in turn to estimate

$$
\begin{aligned}
& \mathbf{E}(\pi_1 Z_1) - \mathbf{E}\pi_1 Z_1(\pi_1 Z_1 + \pi_2 Z_2 - 1) + \pi_1 \mathbf{E}(\pi_1 Z_1 + \pi_2 Z_2 - 1)^2 \\
= \;& \pi_1 + d\theta^2 x_n + \frac{d(d-1)}{2}\theta^4 x_n^2 \left\{ \left(\frac{2}{\pi_1} - \frac{2}{\pi_2}\right) - \pi_2 \left[3 + \frac{\theta}{\pi_1}\left(\frac{z_n}{x_n} - \pi_1\right)\right]^2 \right. \\
& + \pi_1\pi_2(\pi_1 - \pi_2)\left[\left(\frac{1}{\pi_1} - \frac{2}{\pi_2}\right) - \frac{\theta}{\pi_1\pi_2}\left(\frac{z_n}{x_n} - \pi_1\right)\right]^2 \\
& \left. + \pi_1 \left[\frac{\pi_1}{\pi_2} - 2 + \frac{\theta}{\pi_2}\left(\frac{z_n}{x_n} - \pi_1\right)\right]^2 \right\} + R \\
= \;& \pi_1 + d\theta^2 x_n + \frac{1 - 6\pi_1\pi_2}{\pi_1\pi_2^2}\frac{d(d-1)}{2}\theta^4 x_n^2 + S + R
\end{aligned}
$$

where $R = O_\pi(x_n^3)$ and $S = O_\pi\left(\left|\frac{z_n}{x_n} - \pi_1\right| x_n^2\right)$. By means of Lemma 5.2.5 and the assumption, plus choosing sufficiently small $\varepsilon = \varepsilon(\pi)$, there exists a constant $N = N(\pi) > N'$ and $\delta = \delta(\pi)$ such that if $n \geq N$ and $x_n \leq \delta$, then

$$
\varepsilon \mathbf{E}(\pi_1 Z_1 + \pi_2 Z_2 - 1)^2 + C x_n^3 + |S| + |R| \leq \frac{1}{2}\frac{1 - 6\pi_1\pi_2}{\pi_1\pi_2^2}\frac{d(d-1)}{2}\theta^4 x_n^2.
$$

Finally, combining all the results above together gives

$$
x_{n+1} \geq d\theta^2 x_n + \frac{1}{2}\frac{(1 - 6\pi_1\pi_2)}{\pi_1\pi_2^2}\frac{d(d-1)}{2}\theta^4 x_n^2,
$$

as desired. □

Proof of **Theorem 1.3.3.** To accomplish the proof, it suffices to show that when $d\theta^2$ is close enough to 1, x_n does not converge to 0. Referring to contradiction, assume $\lim_{n\to\infty} x_n = 0$ for any $\frac{1}{2} \leq d\theta^2 < 1$, under which Theorem 5.3.1 and Lemma 5.2.3 are activated simultaneously. Consequently there is a $\gamma = \gamma(d, \pi) > 0$ such that

$$
x_{n+1} \geq \gamma x_n
$$

for all n. Furthermore, since the initial point is $x_0 = 1 - \pi_1 = \pi_2$, then

$$
x_n \geq \pi_2 \gamma^n.
$$

Next take N and δ in Theorem 5.3.1 such that if $n \geq N$ and $x_n < \delta$, then

$$
x_{n+1} \geq d\theta^2 x_n + \frac{1}{2}\frac{(1 - 6\pi_1\pi_2)}{\pi_1\pi_2^2}\frac{d(d-1)}{2}\theta^4 x_n^2. \tag{5.3.1}
$$

Thus define $\varepsilon = \min\{\pi_2\gamma^N, \delta\gamma\} > 0$, and then by Lemma 5.2.3 and $x_0 = \pi_2$, it follows that $x_n \geq \varepsilon$ when $n \leq N$. Because of $\Delta^2 > \frac{1}{3}(1-\theta)^2$, namely, $1 - 6\pi_1\pi_2 > 0$, by choosing suitable $|\theta| < d^{-\frac{1}{2}}$, it is feasible to make

$$d\theta^2 + \frac{1}{2}\frac{(1-6\pi_1\pi_2)}{\pi_1\pi_2^2}\frac{d(d-1)}{2}\theta^4\varepsilon \geq 1, \tag{5.3.2}$$

since ε is independent of θ. Therefore, suppose $x_n \geq \varepsilon$ for some $n \geq N$. If $x_n \geq \varepsilon\gamma$, then by Lemma 5.2.3, $x_{n+1} \geq \gamma x_n \geq \varepsilon$. If $\varepsilon \leq x_n \leq \gamma^{-1}\varepsilon \leq \delta$, by (5.3.1) and (5.3.2), then

$$\begin{aligned} x_{n+1} &\geq d\theta^2 x_n + \frac{1}{2}\frac{(1-6\pi_1\pi_2)}{\pi_1\pi_2^2}\frac{d(d-1)}{2}\theta^4 x_n^2 \\ &\geq x_n\left[d\theta^2 + \frac{1}{2}\frac{(1-6\pi_1\pi_2)}{\pi_1\pi_2^2}\frac{d(d-1)}{2}\theta^4\varepsilon\right] \\ &\geq x_n \\ &\geq \varepsilon. \end{aligned}$$

Finally show by induction that $x_n \geq \varepsilon$ for all n, a contradiction to our initial assumption. Therefore $\lim_{n\to\infty} x_n \neq 0$ holds for some $d\theta^2 < 1$, that is, the Kesten-Stigum bound is not tight. □

5.4 Large Degree Asymptotics.

Imitating the way of handling the large degree asymptotics in Chapter 4, define

$$U_j = \log\left[1 + \frac{\theta}{\pi_1}(Y_j - \pi_1)\right] \quad \text{and} \quad V_j = \log\left[1 - \frac{\theta}{\pi_2}(Y_j - \pi_1)\right]$$

with $1 \leq j \leq d$.

Lemma 5.4.1. *There exist constants $C = C(\pi)$ and $D = D(\pi)$ such that whenever $d > D$*

$$\left|d\mathbf{E}U_j - \frac{d\theta^2}{2\pi_1}x_n\right| \leq Cd^{-\frac{1}{2}};$$

$$\left|d\mathbf{E}V_j + \frac{1+\pi_2}{2\pi_2^2}d\theta^2 x_n\right| \leq Cd^{-\frac{1}{2}};$$

$$\left|d\mathbf{Var}(U_j) - \frac{d\theta^2}{\pi_1}x_n\right| \leq Cd^{-\frac{1}{2}};$$

$$\left|d\mathbf{Var}(V_j) - \frac{\pi_1}{\pi_2^2}d\theta^2 x_n\right| \leq Cd^{-\frac{1}{2}};$$

$$\left| d\mathbf{Cov}(U_j, V_j) + \frac{d\theta^2}{\pi_2} x_n \right| \le C d^{-\frac{1}{2}}.$$

Proof. Since $|Y_j - \pi_1| \le 1$ and $d\theta^2 \le 1$, if we take d large enough, i.e., find a large D such that whenever $d > D$, $\frac{\theta}{\pi_1}(Y_j - \pi_1)$ is small enough to guarantee (4.1.1). Then

$$\begin{aligned}
\left| \mathbf{E} U_j - \frac{\theta^2}{2\pi_1} x_n \right| &\le \mathbf{E} \left| U_j - \frac{\theta}{\pi_1}(Y_j - \pi_1) + \frac{1}{2}\left[\frac{\theta}{\pi_1}(Y_j - \pi_1)\right]^2 \right| \\
&\quad + \left| \mathbf{E}\frac{\theta}{\pi_1}(Y_j - \pi_1) - \mathbf{E}\frac{1}{2}\left[\frac{\theta}{\pi_1}(Y_j - \pi_1)\right]^2 - \frac{\theta^2}{2\pi_1} x_n \right| \\
&\le \mathbf{E}\frac{\theta^3}{\pi_1^3}|Y_j - \pi_1|^3 + \frac{\theta^3}{2\pi_1^2}|z_n - \pi_1 x_n| \\
&\le \frac{\theta^3}{\pi_1^3} + \frac{\theta^3}{2\pi_1^2} \\
&\le C(\pi) d^{-\frac{3}{2}},
\end{aligned}$$

for some constant $C = C(\pi)$, where the third inequality follows from $0 \le z_n \le x_n \le 1$. The rest inequalities follow similarly. $\square$

Under results of the preceding estimates, define a 2-dimensional vector $\mu = (\mu_1, \mu_2)$ with

$$\begin{cases} \mu_1 = \frac{1}{2\pi_1} \\ \\ \mu_2 = -\frac{1+\pi_2}{2\pi_2^2} \end{cases}$$

and 2×2-covariance matrix Σ with

$$\begin{pmatrix} \frac{1}{\pi_1} & -\frac{1}{\pi_2} \\ -\frac{1}{\pi_2} & \frac{\pi_1}{\pi_2^2} \end{pmatrix}$$

which is a positive semi-definite symmetric 2×2-matrix. Next Let (G_1, G_2) have a Gaussian distribution $\mathbf{N}(\mu, \Sigma)$.

Next define

$$\psi(w_1, w_2) = \frac{\pi_1 e^{w_1}}{\pi_1 e^{w_1} + \pi_2 e^{w_2}},$$

and then we have the expression in terms of ψ as

$$\begin{aligned} x_{n+1} &= \mathbf{E}\frac{\pi_1 Z_1}{\pi_1 Z_1 + \pi_2 Z_2} - \pi_1 \\ &= \mathbf{E}\frac{\pi_1 \exp\left(\sum_{j=1}^d U_j\right)}{\pi_1 \exp\left(\sum_{j=1}^d U_j\right) + \pi_2 \exp\left(\sum_{j=1}^d V_j\right)} - \pi_1 \\ &= \mathbf{E}\psi\left(\sum_{j=1}^d U_j, \sum_{j=1}^d V_j\right) - \pi_1. \end{aligned}$$

Now if (W_1, W_2) has a Gaussian distribution $\mathbf{N}(0, \Sigma)$, then $(s\mu_1 + \sqrt{s}W_1, s\mu_2 + \sqrt{s}W_2)$ is distributed according to $\mathbf{N}(s\mu, s\Sigma)$. Last denote

$$\begin{aligned} g(s) &= \mathbf{E}\psi(s\mu_1 + \sqrt{s}W_1, s\mu_2 + \sqrt{s}W_2) - \pi_1 \\ &= \mathbf{E}\frac{\pi_1 e^{s\mu_1+\sqrt{s}W_1}}{\pi_1 e^{s\mu_1+\sqrt{s}W_1} + \pi_2 e^{s\mu_2+\sqrt{s}W_2}} - \pi_1. \end{aligned}$$

Again by Proposition 4.2 in [3] the following lemma comes immediately.

Lemma 5.4.2. *For arbitrary $\varepsilon > 0$ there exists a $D = D(\varepsilon, \pi)$ such that whenever $d > D$,*

$$\left|x_{n+1} - g(d\theta^2 x_n)\right| \leq \varepsilon.$$

In order to estimate x_{n+1}, it suffices to research the property of $g(s)$ in the interval $[0, \pi_2]$ in virtue of $0 \leq x_n \leq \pi_2$ and $d\theta^2 \leq 1$.

Lemma 5.4.3. *The function $g(s)$ is continuously differentiable and increasing on the interval $(0, \pi_2]$.*

Proof. When $s > 0$,

$$\begin{aligned} \mathbf{E}\left|\frac{\partial}{\partial s}\frac{\pi_1 e^{s\mu_1+\sqrt{s}W_1}}{\pi_1 e^{s\mu_1+\sqrt{s}W_1} + \pi_2 e^{s\mu_2+\sqrt{s}W_2}}\right| &= \mathbf{E}\left|\frac{\partial}{\partial s}\frac{1}{1 + \frac{\pi_2}{\pi_1}e^{s(\mu_2-\mu_1)+\sqrt{s}(W_2-W_1)}}\right| \\ &= \mathbf{E}\left|\frac{\frac{\pi_2}{\pi_1}e^{s(\mu_2-\mu_1)+\sqrt{s}(W_2-W_1)}}{\left(1 + \frac{\pi_2}{\pi_1}e^{s(\mu_2-\mu_1)+\sqrt{s}(W_2-W_1)}\right)^2}\left(\mu_2 - \mu_1 + \frac{W_2 - W_1}{2\sqrt{s}}\right)\right| \\ &\leq \frac{1}{4}\mathbf{E}\left|\mu_2 - \mu_1 + \frac{W_2 - W_1}{2\sqrt{s}}\right| < \infty, \end{aligned}$$

where we use the fact that $\left|\frac{\frac{\pi_2}{\pi_1}e^t}{\left(1+\frac{\pi_2}{\pi_1}e^t\right)^2}\right| \leq \frac{1}{4}$ holds for any $t \in \mathbb{R}$. Then we establish the differentiability with respect to s.

Now let (W_1', W_2') be an independent copy of (W_1, W_2). Thus if $0 \leq s' < s$, it is feasible to construct the equivalent distributions such as

$$\sqrt{s}(W_1, W_2) \sim \sqrt{s'}(W_1, W_2) + \sqrt{s-s'}(W_1', W_2').$$

Because of $(W_1, W_2) \sim \mathbf{N}(0, \Sigma)$, we get $\mathbf{E}(W_2 - W_1) = 0$ and

$$\mathbf{Var}(W_2 - W_1)^2 \;=\; \mathbf{E}W_2^2 + \mathbf{E}W_1^2 - 2\mathbf{E}W_1W_2 = \frac{1}{\pi_1} + \frac{\pi_1}{\pi_2^2} - 2\left(-\frac{1}{\pi_2}\right) = \frac{1}{\pi_1\pi_2^2},$$

which implies that $W_2 - W_1$ and $W_2' - W_1'$ are both distributed as $\mathbf{N}(0, a)$ with $a = \frac{1}{\pi_1\pi_2^2}$.

Next by (4.1.2), we are allowed to estimate the conditional expectation given W_1 and W_2 as

$$\mathbf{E}\left[\exp(\sqrt{s'}(W_2 - W_1) + \sqrt{s-s'}(W_2' - W_1')) \mid \{W_1, W_2\}\right] = \exp\left[\sqrt{s'}(W_i - W_1) + \frac{a}{2}(s - s')\right].$$

Then we can apply Jensen's inequality, plus noting that the function $\frac{1}{1+x}$ is convex and $\mu_2 - \mu_1 = -\frac{1+\pi_2}{2\pi_2^2} - \frac{1}{2\pi_1} = -\frac{1}{2\pi_1\pi_2^2} = -\frac{a}{2}$, to get

$$\begin{aligned}
g(s) &= \mathbf{E}\frac{1}{1 + \frac{\pi_2}{\pi_1}e^{s(\mu_2-\mu_1)+\sqrt{s}(W_2-W_1)}} - \pi_1 \\
&= \mathbf{E}\frac{1}{1 + \frac{\pi_2}{\pi_1}e^{\frac{-as}{2}+\sqrt{s'}(W_2-W_1)+\sqrt{s-s'}(W_2'-W_1')}} - \pi_1 \\
&\geq \mathbf{E}\frac{1}{1 + \frac{\pi_2}{\pi_1}e^{-\frac{as}{2}}\mathbf{E}\left[e^{\sqrt{s'}(W_2-W_1)+\sqrt{s-s'}(W_2'-W_1')} \mid \{W_1, W_2\}\right]} - \pi_1 \\
&= \mathbf{E}\frac{1}{1 + \frac{\pi_2}{\pi_1}e^{-\frac{as'}{2}}\mathbf{E}e^{\sqrt{s'}(W_2-W_1)}} - \pi_1 \\
&= \mathbf{E}\frac{1}{1 + \frac{\pi_2}{\pi_1}e^{s'(\mu_2-\mu_1)+\sqrt{s'}(W_2-W_1)}} - \pi_1 \\
&= g(s'),
\end{aligned}$$

as desired. □

Next let's turn to the Taylor expansions of $g(s)$ in the small neighborhood of $s = 0$.

Lemma 5.4.4. *For small $s > 0$, we have*

$$g(s) = s + \frac{1 - 6\pi_1\pi_2}{2\pi_1\pi_2^2}s^2 + \frac{1 - 24\pi_1\pi_2 + 90\pi_1^2\pi_2^2}{6\pi_1^2\pi_2^4}s^3 + O(s^4).$$

Proof. Now define $W = s(\mu_2 - \mu_1) + \sqrt{s}(W_2 - W_1)$, and based on the calculations in Lemma 5.4.3, it is apparent that $W \sim \mathbf{N}\left(-\frac{as}{2}, as\right)$. Then there is no difficulty in evaluating the following moments:

$$\begin{aligned}
&\mathbf{E}(e^W - 1) = e^{\frac{-as}{2} + \frac{as}{2}} - 1 = e^0 - 1 = 0;\\
&\mathbf{E}(e^W - 1)^2 = e^{as} - 1 = as + \frac{a^2s^2}{2} + \frac{a^3s^3}{6} + O(s^4);\\
&\mathbf{E}(e^W - 1)^3 = e^{3as} - 3e^{as} + 2 = 3a^2s^2 + 4a^3s^3 + O(s^4);\\
&\mathbf{E}(e^W - 1)^4 = e^{6as} - 4e^{3as} + 6e^{as} - 3 = 3a^2s^2 + 19a^3s^3 + O(s^4);\\
&\mathbf{E}(e^W - 1)^5 = e^{10as} - 5e^{6as} + 10e^{3as} - 10e^{as} + 4 = 30a^3s^3 + O(s^4);\\
&\mathbf{E}(e^W - 1)^6 = e^{15as} - 6e^{10as} + 15e^{6as} - 20e^{3as} + 15e^{as} - 5 = 15a^3s^3 + O(s^4);\\
&\mathbf{E}(e^W - 1)^7 = e^{21as} - 7e^{15as} + 21e^{10as} - 35e^{6as} + 35e^{3as} - 21e^{as} + 6 = O(s^4).
\end{aligned}$$

Last in view of the identity

$$\frac{1}{1 + \pi_2(e^W - 1)} = \sum_{n=1}^{6}(-1)^{n-1}(e^W - 1)^n - \pi_2^7(e^W - 1)^7\frac{1}{1 + \pi_2(e^W - 1)},$$

plugging in previous results yields

$$\begin{aligned}
\frac{g(s) + \pi_1}{\pi_1} &= \frac{1}{\pi_1}\mathbf{E}\frac{\pi_1 \exp(s\mu_1 + \sqrt{s}W_1)}{\pi_1 \exp(s\mu_1 + \sqrt{s}W_1) + \pi_2 \exp(s\mu_2 + \sqrt{s}W_2)}\\
&= \mathbf{E}\frac{1}{1 + \pi_2(e^W - 1)}\\
&= \mathbf{E}\left[\sum_{n=0}^{6}(-1)^n\pi_2^n(e^W - 1)^n - \pi_2^7(e^W - 1)^7\frac{1}{1 + \pi_2(e^W - 1)}\right]\\
&= 1 + \frac{1}{\pi_1}\left(s + \frac{1 - 6\pi_1\pi_2}{2\pi_1\pi_2^2}s^2 + \frac{1 - 24\pi_1\pi_2 + 90\pi_1^2\pi_2^2}{6\pi_1^2\pi_2^4}s^3 + O(s^4)\right),
\end{aligned}$$

that is,

$$g(s) = s + \frac{1 - 6\pi_1\pi_2}{2\pi_1\pi_2^2}s^2 + \frac{1 - 24\pi_1\pi_2 + 90\pi_1^2\pi_2^2}{6\pi_1^2\pi_2^4}s^3 + O(s^4)$$

as desired. □

5.5 Asymptotic Estimation of the Reconstruction Threshold.

Following is the proof of Theorem 1.3.5 and indicates the approach to the reconstruction threshold.

Theorem 5.5.1. *When $\Delta^2 > \frac{1}{3}(1-\theta)^2$, define*

$$\omega^* = \inf\{\omega : there\ exists\ a\ 0 < s^* < \pi_2\ such\ that\ g(\omega s^*) = s^*\}.$$

Then $0 < \omega^* < 1$ and for any $\delta > 0$ there exists a $D = D(\pi, \delta)$ such that if $d > D$ then the model has reconstruction when $d\theta^2 \geq \omega^* + \delta$ but does not have reconstruction when $d\theta^2 \leq \omega^* - \delta$. In other words,

$$\lim_{d\to\infty} d\Theta^2 = \omega^*.$$

Proof. By Lemma 5.4.4,

$$g(s) = s + \frac{1 - 6\pi_1\pi_2}{2\pi_1\pi_2^2}s^2 + \frac{1 - 24\pi_1\pi_2 + 90\pi_1^2\pi_2^2}{6\pi_1^2\pi_2^4}s^3 + O(s^4),$$

so if $s > 0$ is small enough then $g(s) > s$. Thus for any $0 < \omega < 1$, the set $\{0 < s < \pi_2 : g(\omega s) \geq s\}$ is a compact set bounded away from 0. From the continuity of $g(s)$, builded in Lemma 5.4.3,

$$\{0 < s < \pi_2 : g(\omega^* s) = s\} = \bigcap_{\omega* < \omega < 1} \{0 < s < \pi_2 : g(\omega s) \geq s\}$$

is still nonempty compact. Next take $s^* \in \{0 < s < \pi_2 : g(\omega^* s) = s\}$ and $d\theta^2 = \omega^* + \delta$, then

$$g\left[(\omega^* + \delta)\left(s^* \frac{\omega^*}{\omega^* + \delta}\right)\right] = g(s^*\omega^*) = s^* > s^* \frac{\omega^*}{\omega^* + \delta}.$$

Take d large enough to make Lemma 5.4.2 hold with $0 < \varepsilon < s^* - s^* \frac{\omega^*}{\omega^*+\delta}$. When $x_n > s^* \frac{\omega^*}{\omega^*+\delta}$,

$$\begin{aligned} x_{n+1} &\geq g[(\omega^* + \delta)x_n] - \varepsilon \\ &> g\left[(\omega^* + \delta)\left(s^* \frac{\omega^*}{\omega^* + \delta}\right)\right] - \left(s^* - s^* \frac{\omega^*}{\omega^* + \delta}\right) \\ &= s^* \frac{\omega^*}{\omega^* + \delta} \end{aligned}$$

and hence $\inf_n x_n \geq s^* \frac{\omega^*}{\omega^*+\delta}$ establishes reconstruction.

Last when $d\theta^2 < \omega^*$, we have $g(d\theta^2 s) \leq \frac{d\theta^2}{\omega^*}s$. Take $\varepsilon = \frac{1}{2}(1 - \omega^*) > 0$ from Lemma 5.2.1 there exists a constant $\delta = \delta(\pi, \varepsilon)$ such that for all n, if $x_n < \delta$ then

$$x_{n+1} \leq d\theta^2 x_n + \varepsilon x_n \leq \frac{1}{2}(1 + \omega^*)x_n,$$

where $\frac{1}{2}(1+\omega^*) < 1$ implies $\lim_{n\to\infty} x_n = 0$ and thus there is non-reconstruction. So here it suffices to find some m such that $x_m < \delta$. Otherwise assume all $x_n \geq \delta$. Then using Lemma 5.4.2 again with d sufficiently large gives that for $\varepsilon > 0$ small enough such that $\varepsilon \leq \frac{1}{2}\left(1 - \frac{d\theta^2}{\omega^*}\right)\delta$, we have

$$x_{n+1} \leq g(d\theta^2 x_n) + \varepsilon \leq \frac{d\theta^2}{\omega^*} x_n + \varepsilon \leq \frac{1}{2}\left(1 + \frac{d\theta^2}{\omega^*}\right) x_n$$

where $\frac{1}{2}\left(1 + \frac{d\theta^2}{\omega^*}\right) < 1$ that implies $x_n \to 0$ as $n \to \infty$, a contradiction to $x_n \geq \delta$ for all n. □

Referring to Mathematica, we are able to establish the following result.

Lemma 5.5.2. *When $\Delta^2 < \frac{1}{3}(1-\theta)^2$, for all $0 < s \leq \pi_2$, we have*

$$g(s) < s.$$

Proof of **Theorem 1.3.4.** For $d\theta^2 < 1$, taking $\eta = \frac{1}{2}(1 - d\theta^2) > 0$ in Lemma 5.2.1, there exists a constant $\delta = \delta(\pi, \eta)$ such that if $x_n < \delta$ for some n, then

$$x_{n+1} \leq (d\theta^2 + \eta)x_n = (1-\eta)x_n < \delta.$$

Then it is possible to apply induction, in tandem with $0 < 1 - \eta < 1$, to conclude $\lim_{n\to\infty} x_n = 0$ and thus non-reconstruction.

So here it suffices to find some m such that $x_m < \delta$. Otherwise assume $x_n \geq \delta$ for any positive integer n. Next define $\varepsilon = \frac{1}{2}\min_{s\geq\delta}(s - g(s))$. Since $s - g(s)$ is continuous and positive in $[\delta, \pi_2]$, it follows $\varepsilon > 0$ by Lemma 5.5.2. Then by Lemma 5.4.2, there exists a $D = D(\varepsilon, \pi) > 0$ such that when $d > D$,

$$|x_{n+1} - g(d\theta^2 x_n)| < \varepsilon,$$

and then

$$\begin{aligned} x_{n+1} &< g(d\theta^2 x_n) + \varepsilon \\ &\leq g(x_n) + \varepsilon \\ &\leq x_n - 2\varepsilon + \varepsilon \leq x_n - \varepsilon, \end{aligned}$$

where the second inequality is from Lemma 5.4.3, say, $g(s)$ is increasing in $[0, \pi_2]$. Thus if choose N large enough to make $N > \varepsilon^{-1}$, $x_N - N\varepsilon < 0$ would be true, a contradiction to $x_n \geq 0$ for all n. □

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