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Geometry of Calabi-Yau Moduli

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Abstract of the Dissertation

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Professor Kefeng Liu, Chair

In this thesis, we study the geometry of the moduli space and the Teichmüller space of Calabi-Yau manifolds, which mainly involves the following two aspects: the (locally, globally) Hermitian symmetric property of the Teichmüller space and the first Chern form of the moduli space with the Weil-Petersson and Hodge metrics.

In the first part, we define the notation of quantum correction for the Teichmüller space $\mathcal{T}$ of Calabi-Yau manifolds. Under the assumption of vanishing of weak quantum correction, we prove that the Teichmüller space $\mathcal{T}$, with the Weil-Petersson metric, is a locally symmetric space. For Calabi-Yau threefolds, we show that the vanishing of strong quantum correction is equivalent to that the image $\Phi(\mathcal{T})$ of the Teichmüller space $\mathcal{T}$ under the period map Φ is an open submanifold of a globally Hermitian symmetric space W of the same dimension as $\mathcal{T}$. Finally, for Hyperkähler manifolds of dimension $2n \geq 4$, we find globally defined families of $(2,0)$ and $(2n,0)$-classes over the Teichmüller space of polarized Hyperkähler manifolds.

In the second part, we prove that the first Chern form of the moduli space of polarized Calabi-Yau manifolds, with the Hodge metric or the Weil-Petersson metric, represents the first Chern class of the canonical extensions of the tangent bundle to the compactification of the moduli space with normal crossing divisors.

The dissertation of Changyong Yin is approved.

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To my family

for their endless love, support and encouragement

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Vita

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CHAPTER 1

Introduction

Moduli spaces of general polarized algebraic varieties are studied exrensively by algebraic geometers. However, there are two classes of moduli spaces where the methods of differential geometry are equally powerful. These are the moduli spaces of curves and the moduli space of polarized Calabi-Yau manifolds. The Weil-Petersson metric and Hodge metric are the main tools to investigate the geometry of such moduli spaces, under which these moduli spaces are Kählerian.

In this thesis, we study the Teichmüller space of polarized and marked Calabi-Yau manifolds and the moduli spaces of polarized Calabi-Yau manifolfds with level m structure with $m \geq 3$. Recall that a compact projective manifold X of complex dimension n with $n \geq 3$ is called a Calabi-Yau manifold, if it has a trivial canonical bundle and satisfies $H^i(X, \mathcal{O}_X) = 0$ for $0 < i < n$. A polarized and marked Calabi-Yau manifold is a triple (X, L, γ) of a Calabi-Yau manifold X, an ample line bundle L over X and a basis γ of the integral middle homology group modulo torsion, $H_n(X, \mathbb{Z})/\mathrm{Tor}$. And, a level m structure of a Calabi-Yau manifold X with $m \geq 3$ is a basis of the quotient space $(H_n(X, \mathbb{M})/\mathrm{Tor})/m(H_n(X, \mathbb{Z})/\mathrm{Tor})$.

For the moduli space of polarized Calabi-Yau manifolds with level m structure, we have the following theorem, which is a reformulation of [Szedröi99, Theorem 2.2]. One can also refer to [Popp77] and [Viehweg95] for more details about the construction of the moduli space of Calabi-Yau manifolds.

Theorem 1.0.1. *Let (X, L) be a polarized Calabi-Yau manifold with level m structure with $m \geq 3$, then there exists a quasi-projective complex manifold $\mathcal{M}_m$ with a versal family of*

Calabi-Yau maniflods,

$$\mathcal{X}_{\mathcal{M}_m} \longrightarrow \mathcal{M}_m, \tag{1.1}$$

which contains X as a fiber, and polarized by an ample line bundle $\mathcal{L}_{\mathcal{M}_m}$ on the versal family $\mathcal{X}_{\mathcal{M}_m}$.

The Teichmüller space $\mathcal{T}$ of polarized and marked Calabi-Yau manifolds is actually the universal cover of the smooth moduli space $\mathcal{M}_m$ of polarized Calabi-Yau manifolds with level m structure with $m \geq 3$. We denote by $\mathcal{U} \to \mathcal{T}$ the pull-back of the family 1.1 via the covering map $\pi_m : \mathcal{T} \to \mathcal{M}_m$, then we have

Proposition 1.0.2. *The Teichmüller space $\mathcal{T}$ is a simply connected smooth complex manifold, and the family*

$$\mathcal{U} \longrightarrow \mathcal{T} \tag{1.2}$$

containing X as a fiber, is local Kuranishi at each point of the Teichmüller space $\mathcal{T}$.

Our result mainly involves the following two aspects: the (locally, globally) Hermitian symmetric property of the Teichmüller space $\mathcal{T}$ of polarized and marked Calabi-Yau manifolds and the first Chern forms of the moduli space $\mathcal{M}_m$ of polarized Calabi-Yau manifolds with level m structure with $m \geq 3$.

In the first part, we define the notation of quantum correction for the Teichmüller space $\mathcal{T}$ of polarized and marked Calabi-Yau manifolds. Under the assumption of vanishing of weak quantum correction, we prove that the Teichmüller space $\mathcal{T}$, with the Weil-Petersson metric, is a locally Hermitian symmetric space. For Calabi-Yau threefolds, we show that the vanishing of strong quantum correction is equivalent to that the image $\Phi(\mathcal{T})$ of the Teichmüller space $\mathcal{T}$ under the period map Φ is an open submanifold of a globally Hermitian symmetric space W of the same dimension as $\mathcal{T}$. Finally, for Hyperkähler manifold of dimension $2n \geq 4$, we find globally defined families of $(2,0)$ and $(2n,0)$-classes over the Teichmüller space of polarized Hyperkähler manifolds.

Fix $p \in \mathcal{T}$, let X be the corresponding Calabi-Yau manifold in the versal family $\mathcal{U} \to \mathcal{T}$ and $\{\varphi_1, \cdots, \varphi_N\} \in \mathbb{H}^{0,1}(X, T^{1,0}X)$ be an orthonormal basis with respect to the Calabi-Yau metric over X. Then we can construct the following smooth family of Betrami differentials

$$\Phi(t) = \sum_{|I|\geq 1} t^I \varphi_I, \qquad \varphi_I \in A^{0,1}(X, T^{1,0}X),$$

which describes the deformation of complex structures in a neighbourhood of $p \in \mathcal{T}$.

Our essential idea is to consider the strong quantum correction at any point $p \in \mathcal{T}$, which comes from the quantum correction of Yukawa coupling in the Kodaira-Spencer theory developed in [Bershadshy-Cecotti-Ooguri-Vafa94, Chapter 5]. It can be simply described as the following identity of cohomology classes,

$$[\Xi(t)] = [\Omega^c(t)] - \left[\exp(\sum_{i=1}^N t_i\varphi_i) \lrcorner \Omega\right],$$

where Ω is a holomorphic $(n,0)$-form over X and $\Omega^c(t)$ is the canonical family of holomorphic $(n,0)$-forms $e^{\Phi(t)} \lrcorner \Omega$ in a neighborhood of $p \in \mathcal{T}$. And the weak quantum correction at $p \in \mathcal{T}$ is defined as the lowest order expansion of the strong quantum correction $[\Xi(t)]$ with respect to t, i.e.,

$$[\Xi(t)]_1 = \sum_{i,j,k=1}^N t_i t_j t_k [\varphi_i \lrcorner \varphi_{jk} \lrcorner \Omega].$$

When $n = 3$, i.e., for Calabi-Yau threefolds, vanishing of strong quantum correction implies vanishing of quantum correction of the Yukawa coupling in physics literatures, the reader can refer to [Bershadshy-Cecotti-Ooguri-Vafa94] for details about the quantum correction of the Yukawa coupling in the Kodaira-Spencer theory.

We first have the following result which characterizes the Teichmüller space $\mathcal{T}$ of Calabi-Yau manifolds when the weak quantum correction vanishies at any point $p \in \mathcal{T}$.

Theorem 1.0.3. *Let $\mathcal{T}$ be the Teichmüller space of polarized and marked Calabi-Yau manifolds. Vanishing of weak quantum correction at any point $p \in \mathcal{T}$, i.e., $[\Xi(t)]_1 = 0$, implies that $\mathcal{T}$, with the Weil-Petersson metric, is a locally Hermitian symmetric space.*

Here, a locally Hermitian symmetric space is a smooth manifold satisfying $\nabla R = 0$, i.e., its curvature tensor is parallel, which is not necessarily complete. Moreover, for polarized and marked Calabi-Yau threefolds, we found the following equivalent condition for the vanishing of strong quantum correction at any point $p \in \mathcal{T}$.

Theorem 1.0.4. *Let $\mathcal{T}$ be the Teichmüller space of polarized and marked Calabi-Yau threefolds and $\Phi : \mathcal{T} \to D$ be the period map. Then the following are equivalent:*

1. *The strong quantum correction vanishes at any point $p \in \mathcal{T}$;*

2. *With respect to the Hodge metric, the image $\Phi(\mathcal{T})$ is an open submanifold of a globally Hermitian symmetric space W of the same dimension as $\mathcal{T}$, which is also a totally geodesic submanifold of the period domain D.*

In Section 3.4, we study Hyperkähler manifolds. Let $\mathcal{T}$ be the Teichmüller space of polarized Hyperkähler manifolds. A (irreducible) Hyperkähler manifold is a compact and simply-connected Kähler manifold of complex dimension $2n \geq 4$ such that there exists a non-zero holomorphic non-degenerate $(2,0)$-form $\Omega^{2,0}$ on X.

Fix $p \in \mathcal{T}$, let X be the corresponding Hyperkähler manifold in the versal family $\mathcal{U} \to \mathcal{T}$, with $\dim_{\mathbb{C}} X = 2n$, and $\Omega^{2,0}$ be a nowhere vanishing $(2,0)$-form over X. By explicitly computing the Taylor expansions of the canonical families $[\mathbb{H}(e^{\Phi(t)} \lrcorner \Omega^{2,0})]$ and $[e^{\Phi(t)} \lrcorner \wedge^n \Omega^{2,0}]$, we show that the strong quantum correction vanishes at any point $p \in \mathcal{T}$. Therefore the Teichmüller spaces of polarized Hyperkähler manifolds are locally Hermitian symmetric with the Weil-Petersson metric, which is affirmed without using the Torelli theorem for Hyperkähler manifolds. Then, we show that these local expansions are actually global defined on the Teichmüller spaces.

Theorem 1.0.5. *Fix $p \in \mathcal{T}$, let X be the corresponding Hyperkähler manifold in the versal family $\mathcal{U} \to \mathcal{T}$ and $\Omega^{2,0}$ be a nowhere vanishing $(2,0)$-form over X, then, in a neighborhood U of p, there exist local families of $(2,0)$ and $(2n,0)$-classes defined by the canonical families $[\mathbb{H}(e^{\Phi(t)} \lrcorner \Omega^{2,0})]$ and $[e^{\Phi(t)} \lrcorner \wedge^n \Omega^{2,0}]$. Furthermore their expansions are actually globally defined* *over the Teichmüller space $\mathcal{T}$, i.e.,*

$$[\Omega^{2,0}] + \sum_{i=1}^{N}[\varphi_i \lrcorner \Omega^{2,0}]t_i + \frac{1}{2}\sum_{i=1}^{N}[\varphi_i \lrcorner \varphi_j \lrcorner \Omega^{2,0}]t_i t_j \in H^{2,0}(X_t),$$

$$\left[\wedge^n \Omega^{2,0}\right] + \sum_{i=1}^{N}\left[\varphi_i \lrcorner \wedge^n \Omega^{2,0}\right] t_i$$
$$+\frac{1}{k!}\sum_{k=1}^{2n}\left(\sum_{1\leq i_1\leq \ldots \leq i_k \leq N}[\varphi_{i_1}\lrcorner \ldots \lrcorner \varphi_{i_k}\lrcorner \wedge^n \Omega^{2,0}]t_{i_1}t_{i_2}\cdots t_{i_k}\right) \in H^{2n,0}(X_t)$$

are globally defined over $\mathcal{T}$.

In the second part, we study the moduli space $\mathcal{M}_m$ of polarized Calabi-Yau manifolds with level m structure with $m \geq 3$, which is called the Calabi-Yau moduli in this paper for simplicity. We prove that the first Chern form of the Calabi-Yau moduli, with the Hodge metric or the Weil-Petersson metric, represents the first Chern class of the canonical extensions of the tangent bundle to the compactification of the Calabi-Yau moduli with normal crossing divisors.

Over the Calabi-Yau moduli $\mathcal{M}_m$, we construct various Hodge bundles. The holomorphic bundle H^n over $\mathcal{M}_m$, whose fiber is the primitive cohomology group $H^n_{pr}(X_p, \mathbb{C})$ at each point $p \in \mathcal{M}_m$, carries a polarized Hodge structure of weight n. Then the holomorphic bundle $End(H^n) \to \mathcal{M}_m$ defines a variation of polarized Hodge structure over $\mathcal{M}_m$, which is defined over $\mathbb{Z}$. Then, with the Hodge metric, we have the following useful observation,

Theorem 1.0.6. *Let $\mathcal{M}_m$ be the moduli space of polarized Calabi-Yau manifolds with level m structure with $m \geq 3$. Then the holomorphic vector bundle $End(H^n)$ defines a variation of polarized Hodge structure over $\mathcal{M}_m$, which is defined over $\mathbb{Z}$. Moreover, with the natural Hodge metric over the Calabi-Yau moduli $\mathcal{M}_m$, the tangent bundle*

$$T\mathcal{M}_m \hookrightarrow End(H^n), \tag{1.3}$$

is a holomorphic subbundle of $End(H^n) \to \mathcal{M}_m$ with the induced Hodge metric.

Then, by the important results for the integrability of Chern forms of subbundles and quotient bundles of a variation of polarized Hodge structure over a quasi-projective manifold as given by [Cattani-Kaplan-Schimid86] and [Kollár85], see Theorem 4.1.2 and Theorem 4.1.3, we can get that the first Chern form of the Calabi-Yau moduli are integrable with the induced Hodge metric. More precisely, as a subbundle of the Hodge bundle $End(H^n) \to \mathcal{M}_m$, the tangent bundle $T\mathcal{M}_m$ can be canonically extended, denoted by $\widetilde{T\mathcal{M}_m} \longrightarrow \overline{\mathcal{M}}_m$. And, the same canonical extension was used by [Lu-Sun06], the reader can refer to [Lu-Sun06, Remark 4.3].

Here, we want to emphasize that the canonical extension $\widetilde{T\mathcal{M}_m} \to \overline{\mathcal{M}}_m$ of the tangent bundle is not the tangent bundle of the compactification of the moduli space, $T\overline{\mathcal{M}}_m \to \overline{\mathcal{M}}_m$. By using the canonical extension $\widetilde{T\mathcal{M}_m} \to \overline{\mathcal{M}}_m$, we have,

Theorem 1.0.7. *The first Chern form of the Calabi-Yau moduli $\mathcal{M}_m$ with the induced Hodge metric define currents over the compactification $\overline{\mathcal{M}}_m$ with normal crossing boundary divisors. Moreover, let R_H be the curvature form of $T\mathcal{M}_m$ with the induced Hodge metric, then we have*

$$\left(\frac{-1}{2\pi i}\right)^N \int_{T\mathcal{M}_m} (trR_H)^N = c_1(\widetilde{T\mathcal{M}_m})^N$$

where $N = \dim_{\mathbb{C}} \mathcal{M}_m$ and $\widetilde{T\mathcal{M}_m}$ the canonical extension of the tangent bundle $T\mathcal{M}_m$.

Another direct and easy consequence is that the other Chern forms of the Hodge bundles on the Calabi-Yau moduli with the induced Hodge metrics are all integrable.

In this thesis, we focus on Calabi-Yau manifolds. Actually our method only needs the fact that the moduli space of the manifolds with certain structures are smooth and quasi-projective and the period map is locally injective (the local Torelli theorem). So our results can be easily extended to more general projective manifolds, including Calabi-Yau manifolds, Hyperkähler manifolds, many hypersurfaces and complete intersections in projective spaces. Here, we only summarize the results into the following theorem:

Theorem 1.0.8. *Let $\mathcal{M}$ be the moduli space of polarized projective manifolds with certain structure. Assume that $\mathcal{M}$ is smooth and quasi-projective. If the period map from $\mathcal{M}$ to the period domain is locally injective, then the first Chern form of the moduli space $\mathcal{M}$ with* *the induced Hodge metric defines currents over the compactification $\overline{\mathcal{M}}$ with normal crossing boundary divisors. Moreover, the first Chern form represents the first Chern class of the corresponding canonical extension $\widetilde{T\mathcal{M}} \longrightarrow \overline{\mathcal{M}}$ of the tangent bundle.*

By a similar argument, one can show that the Chern forms of the moduli space $\mathcal{M}$ with the Weil-Petersson metric define currents over the compactification $\overline{\mathcal{M}}$ of the moduli space $\mathcal{M}$, and the first Chern form also represents the first Chern class of the corresponding canonical extension of the tangent bundle.

This thesis is organized as follows. In Chapter 2, we review some necessary concepts and results for the study of the moduli space and Teichmüller space of Calabi-Yau manifodls. In Section 2.1, we briefly review the construction of the moduli space and the Teichmüller space of Calabi-Yau manifolds, the local deformation theory of Calabi-Yau manifolds and the construction of the canonical family of $(n, 0)$-forms. In Section 2.2, we recall the formal variation of Hodge structure and local period map for a variation of Hodge structure over a connected and simply connected parametrized space. In Section 2.3, we summarize the Weil-Petersson metric and Hodge metric over the moduli space and Teichmüller space, which will be used to investigate the differential geometry of the Calabi-Yau moduli.

In Chapter 3, we study the (locally, globally) Hermitian symmetric property of the Teichmüller space of polarized and marked Calabi-Yau manifolds. In Section 3.1, the definition and criteria of Hermitian symmetric space are introduced. Also, we define the quantum correction of the Teichmüller space $\mathcal{T}$, which originally comes from physics literatures. In Section 3.2, we review the Weil-Petersson geometry of the Teichmüller space $\mathcal{T}$, and derive a local formula for the covariant derivatives of the curvature tensor ∇R in terms of the flat affine coordinate t. Under the assumption of no weak quantum correction at any point $p \in \mathcal{T}$, we prove that the Teichmüller space $\mathcal{T}$ is a locally Hermitian symmetric space with the Weil-Petersson metric. We remark that the results in Sections 2.1.2 to 3.2 actually all hold for both Calabi-Yau and Hyperkähler manifolds. In Section 3.3, for Calabi-Yau threefolds, we show that vanishing of the strong quantum correction is equivalent to, with the Hodge metric, that the image $\Phi(\mathcal{T})$ of the Teichmüller space $\mathcal{T}$ under the period map Φ is an open submanifold of a globally Hermitian symmetric space W with the same dimension as $\mathcal{T}$. In Section 3.4, we construct a globally defined families of $(2,0)$ and $(2n,0)$-classes over the Teichmüller space $\mathcal{T}$ of polarized Hyperkähler manifolds with $\dim_{\mathbb{C}} X = 2n$.

In Chapter 4, we study the first Chern form of the Calabi-Yau moduli, with the Weil-Petersson metric and Hodge metric. In Section 4.1, we review the essential estimates for the degeneration of the Hodge metric of a variation of polarized Hodge structure near a normal crossing divisor , which was used to derive the integrability of the Chern forms of subbundles and quotient bundles of the variation of polarized Hodge structure over a quasi-projective manifold. In Section 4.2, we construct various Hodge bundles over the Calabi-Yau moduli $\mathcal{M}_m$. Then, by a key observation that the tangent bundle of Calabi-Yau moduli is a subbundle of the variation of polarized Hodge structure $End(H^n) \to \mathcal{M}_m$, we prove that the first Chern form of the Calabi-Yau moduli $\mathcal{M}_m$ are integrable, with the Hodge metric, which represents the first Chern class of the canonical extension $\widetilde{T\mathcal{M}_m} \to \overline{\mathcal{M}}_m$ of the tangent bundle. In Section 4.3, By the isomorphism $T\mathcal{M}_m \cong (F^n)^* \otimes F^{n-1}/F^n$ with the Weil-Petersson metric over $\mathcal{M}_m$, we show that the Chern forms of the Calabi-Yau moduli $\mathcal{M}_m$ are integrable, equipped with the Weil-Petersson metric. Moreover, the first Chern form represent the first Chern class of the quotient bundle $(F^n\widetilde{)^* \otimes F}^{n-1}/(F\widetilde{^n)^* \otimes} F^n \to \overline{\mathcal{M}}_m$, where $(F\widetilde{^n)^* \otimes F}^{n-1} \to \overline{\mathcal{M}}_m$ and $(F\widetilde{^n)^* \otimes} F^n \to \overline{\mathcal{M}}_m$ are the canonical extensions of the Hodge bundles $(F^n)^* \otimes F^{n-1} \to \mathcal{M}_m$ and $(F^n)^* \otimes F^n \to \mathcal{M}_m$ respectively.

CHAPTER 2

Calabi-Yau Moduli and Hodge Structure

In this chapter, we review some necessary concepts and results for the study of the moduli space and Teichmüller space of Calabi-Yau manifodls, including the construction of moduli spaces, the variation of Hodge structure, the period domain, the period map, the Weil-Petersson metric and the Hodge metric.

2.1 Locally Geometric Structure of the Moduli Space

In Section 2.1.1, we review the construction of the moduli space and the Teichmülller space of Calabi–Yau manifolds based on the works of Popp [Popp77], Viehweg [Viehweg95] and Szendröi [Szedröi99]. In Section 2.1.2 and Section 2.1.3, the smooth family of Betrami differentials $\Phi(t)$ and the canonical family of $(n,0)$-forms $e^{\Phi(t)}\lrcorner\Omega$ over the deformation space of Calabi-Yau manifolds are introduced. The results in Section 2.1.2 and Section 2.1.3 also hold for polarized Hyperkähler manifolds.

2.1.1 The Construction of the Teichmüller Space

In this section, we briefly review the construction of the moduli space and Teichmüller space of polarized Calabi-Yau manifolds. A pair (X, L) consisting of a Calabi-Yau manifold X of complex dimension n with $n \geq 3$ and an ample line bundle $L \to X$ is called a polarized Calabi-Yau manifold. By abuse of notation, the Chern class of L will be also denoted by L and thus $L \in H^2(X, \mathbb{Z})$. Let $\{\gamma_1, \cdots, \gamma_N\}$ be a basis of the integral homology group modulo torsion, $H_n(X, \mathbb{Z})/Tor$, with $\dim(H_n(X, \mathbb{Z})/Tor) = N$.

Definition 2.1.1. *Let the pair (X, L) be a polarized Calabi-Yau manifold, we call the triple $(X, L, \{\gamma_1, \cdots, \gamma_N\})$ a polarized and marked Calabi-Yau manifold.*

We first recall the concept of Kuranishi family of compact complex manifolds, we refer to [Shimizu-Ueno02, Pages 8-10], [Popp77, Page 94] or [Viehweg95, Page 19] for equivalent definitions and more details. A family of compact manifolds $\pi : \mathcal{W} \to B$ is versal at a point $p \in B$ if it is satisfies the following conditions:

1. Given a complex analytic family $\iota : \mathcal{V} \to S$ of compact complex manifolds with a point $s \in S$ and a biholomorphic map $f_0 : V_s = \iota^{-1}(s) \to U_p = \pi^{-1}(p)$, then there exists a holomorphic map g from a neighbourhood $\mathcal{N} \subset S$ of the point $s \in S$ and a holomorphic map $f : \iota^{-1}(N) \to \mathcal{W}$ with $\iota^{-1}(N) \subset \mathcal{V}$ such that they satisfy that $g(s) = p$ and $f|_{\iota^{-1}(s)} = f_0$ with the following commutative diagram

$$\begin{array}{ccc} \iota^{-1}(N) & \xrightarrow{f} & \mathcal{W} \\ \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \pi} \\ N & \xrightarrow{g} & B. \end{array}$$

2. For all g satisfying the above condition, the tangent map $(dg)_s$ is uniquely determined.

If a family $\pi : \mathcal{W} \to B$ is versal at every point $p \in B$, then it is a versal family on B. If a complex analytic family satisfies the above condition 1, then the family is call complete at $p \in B$. If a complex analytic family $\pi : \mathcal{X} \to S$ of compact complex manifolds is complete at each point of S and versal at the point $0 \in S$, then the family $\pi : \mathcal{X} \to S$ is called the Kuranishi family of the complex maniflod $X = \pi^{-1}(0)$. The base space S is called the Kuranishi space. If the family is complete at each point of a neighbourhood of $0 \in S$ and versal at 0, then this family is called a local Kuranishi family at $0 \in S$. In particular, by definition, if the family is versal at each point of S, then it is local Kuranishi at each point of S.

Let (X, L) be a polarized Calabi-Yau manifold. A level m structure with $m \geq 3$ of a Calabi-Yau manifold X is a basis of the quotient space $(H_n(X, \mathbb{Z})/\mathrm{Tor})/m(H_n(X, \mathbb{Z})/\mathrm{Tor})$. For deformation of polarized Calabi-Yau manifolds with level m structure, we have the following theorem, which is a reformulation of [Szedröi99, Theorem 2.2]. One can also refer [Popp77] and [Viehweg95] for more details about the construction of the moduli space of Calabi-Yau manifolds.

Theorem 2.1.2. *Let (X, L) be a polarized Calabi-Yau manifold with level m structure with $m \geq 3$, then there exists a quasi-projective complex manifold $\mathcal{M}_m$ with a versal family of Calabi-Yau maniflods,*

$$\mathcal{X}_{\mathcal{M}_m} \longrightarrow \mathcal{M}_m, \tag{2.1}$$

which contains X as a fiber, and polarized by an ample line bundle $\mathcal{L}_{\mathcal{M}_m}$ on the versal family $\mathcal{X}_{\mathcal{M}_m}$.

The Teichmülller space is the moduli space of equivalent classes of polarized and marked Calabi-Yau manifolds. More precisely, a polarized and marked Calabi-Yau manifold is a triple (X, L, γ), where X is a Calabi-Yau manifold, L is a polarization on X and γ is a basis of $H_n(X, \mathbb{Z})/Tor$. Two triples (X, L, γ) and (X', L', γ') are equivalent if there exists a bihomorphic map $f : X \to X'$ with

$$\begin{aligned} f^*L' &= L, \\ f_*\gamma &= \gamma', \end{aligned}$$

then $[X, L, \gamma] = [X', L', \gamma']$ is an element of the Teichmüller space $\mathcal{T}$. Because a basis γ of $H_n(X, \mathbb{Z})/Tor$ naturally induces a basis of $(H_n(X, \mathbb{Z})/Tor)/m(H_n(X, \mathbb{Z})/Tor)$, we have a natural map $\pi_m : \mathcal{T} \to \mathcal{M}_m$, where $\mathcal{T}$ is the Teichmüller space of polarized and marked Calabi-Yau manifolds.

By the definition, it is not hard to show that the Teichmüller space is precisely the universal cover of $\mathcal{M}_m$ with the covering map $\pi_m : \mathcal{T} \to \mathcal{M}_m$. In fact, as the same construction in [Szedröi99, Section 2], we can also realize that the Teichmüller space $\mathcal{T}$ as a quotient space of the universal cover of the Hilbert scheme of Calabi-Yau manifolds by special linear group $SL(N + 1, \mathbb{C})$. Here the dimension is given by $N + 1 = p(k)$ where p is the Hilbert polynomial of each fiber (X, L) and k satisfies that for any polarized algebraic variety $(\widetilde{X}, \widetilde{L})$ with Hilbert polynomial p, the line bundle $\widetilde{L}^{\otimes k}$ is very ample. Under this construction, the Teichmüller space is automoatically simply connected, and there is a natural covering map $\pi_m : \mathcal{T} \to \mathcal{M}_m$. On the other hand, in [Chen-Guan-Liu12, Theorem 2.5 and Corollary 2.8], the authors also proved that $\pi_m : \mathcal{T} \to \mathcal{M}_m$ is a universal covering map, and consequently that $\mathcal{T}$ is the universal cover space of $\mathcal{M}_m$ for some m. Thus, by the uniqueness of universal covering space, these two constructions of $\mathcal{T}$ are equivalent to each other.

We denote by $\mathcal{U} \to \mathcal{T}$ the pull-back of the family 2.1 via the covering map π_m, then we have

Proposition 2.1.3. *The Teichmüller space $\mathcal{T}$ is a simply connected smooth complex manifold, and the family*

$$\mathcal{U} \longrightarrow \mathcal{T} \tag{2.2}$$

containing X as a fiber, is local Kuranishi at each point of the Teichmüller space $\mathcal{T}$.

Proof. For the first half, because $\mathcal{M}_m$ is a connected and smooth complex manifold, its universal cover $\mathcal{T}$ is thus a connected and simply connected smooth complex manifold. For the second half, we know that the family 2.1 is a versal family at each point of $\mathcal{M}_m$ and that π_m is locally biholomorphic , therefore the pull-back family via π_m is also versal at each point of $\mathcal{T}$. Then by the definition of local Kuranishi family, we get that $\mathcal{U} \to \mathcal{T}$ is local Kuranishi at each point of $\mathcal{T}$. $\square$

Actually, the family $\mathcal{U} \to \mathcal{T}$ is local Kuranishi at each point is essentially due to the local Torelli theorem for Calabi-Yau manifolds. In fact, the Kodaira-Spencer map of the family $\mathcal{U} \to \mathcal{T}$

$$\kappa : T_p^{1,0}\mathcal{T} \to H^{0,1}(X_p, T^{1,0}X_p),$$

is an isomorphism for each point $p \in \mathcal{T}$. Then by theorem in [Shimizu-Ueno02, Page 9], we conclude that $\mathcal{U} \to \mathcal{T}$ is versal at each point $p \in \mathcal{T}$. Moreover, the well Bogomolov-Tian-Todorov Lemma ([Bogomolov78],[Tian87] and [Todorov89]) implies that $\dim_{\mathbb{C}} \mathcal{T} =$

$N = h^{n-1,1}$. We refer the reader to [Kodaira-Morrow06, Chapter 4] for more details about deformation of complex structures and the Kodaira-Spencer map.

Note that the Teichmüller space $\mathcal{T}$ does not depend on the choice of level m. In fact, let m_1, m_2 be two different positive integers, $\mathcal{U}_1 \to \mathcal{T}_1$ and $\mathcal{U}_2 \to \mathcal{T}_2$ are two versal families constructed via level m_1 and level m_2 respectively as above, both of which contain X as a fiber. By using the fact that $\mathcal{T}_1$ and $\mathcal{T}_2$ are simply connected and the definition of versal families, we have a biholomorphic map $f : \mathcal{T}_1 \to \mathcal{T}_2$, such that the versal family $\mathcal{U}_1 \to \mathcal{T}_1$ is the pull- back of the versal family $\mathcal{U}_2 \to \mathcal{T}_2$ by the map f. Thus these two families are isomorphic to each other.

In this thesis, we call $\mathcal{M}_m$ with $m \geq 3$ the Calabi-Yau moduli for simplicity and $\mathcal{T}$ the Teichmüller space of Calabi-Yau manifolds.

2.1.2 Local Deformation of Calabi-Yau Manifolds

Fix $p \in \mathcal{T}$, let (X, L) be the corresponding polarized Calabi-Yau manifold in the versal family $\mathcal{U} \to \mathcal{T}$. Yau's solution of Calabi conjecture assigns a unique Calabi-Yau metric g on X, whose imaginary part $\omega = \operatorname{Im} g \in L$ is the corresponding Kähler form. Under the Calabi-Yau metric g, we have the following lemma which follows from the Calabi-Yau theorem directly,

Lemma 2.1.4. *Let Ω be a nowhere vanishing holomorphic $(n,0)$-form on X such that*

$$(\frac{\sqrt{-1}}{2})^n(-1)^{\frac{n(n-1)}{2}}\Omega \wedge \overline{\Omega} = \omega^n.$$

Then the map $\iota : A^{0,1}(X, T^{1,0}X) \to A^{n-1,1}(X)$ given by $\iota(\varphi) = \varphi \lrcorner \Omega$ is an isometry with respect to the natural Hermitian inner product on both spaces induced by the Calabi-Yau metric g. Furthermore, the map ι preserves the Hodge decomposition.

With the Calabi-Yau metric g, we have a precise description of the local deformation of the polarized Calabi-Yau manifolds. By the Hodge theory, we have the following identification

$$T^{1,0}_p\mathcal{T} \cong \mathbb{H}^{0,1}(X, T^{1,0}X),$$

where X is the corresponding fiber over $p \in \mathcal{T}$ in the versal family $\mathcal{U} \to \mathcal{T}$. By the Kodaira-Spencer-Kuranishi theory, we have the following convergent power series expansion of the Betrami differentials, which is now well-known as the Bogomolov-Tian-Todorov Theorem [Bogomolov78, Tian87, Todorov89].

Theorem 2.1.5. *Let X be a Calabi-Yau manifold and $\{\varphi_1, \cdots, \varphi_N\} \in \mathbb{H}^{0,1}(X, T^{1,0}X)$ be a harmonic basis. Then for any nontrivial holomorphic $(n,0)$-form Ω on X, we can construct a smooth power series of Betrami differentials as follows*

$$\Phi(t) = \sum_{|I| \geq 1} t^I \varphi_I = \sum_{\substack{\nu_1 + \cdots + \nu_N \geq 1, \\ each\ \nu_i \geq 0, i = 1, 2, \cdots, N}} \varphi_{\nu_1 \cdots \nu_N} t_1^{\nu_1} \cdots t_N^{\nu_N} \in A^{0,1}(X, T_X^{1,0}), \tag{2.3}$$

where $\varphi_{0 \cdots \nu_i \cdots 0} = \varphi_i$. This power series has the following properties:

1) $\overline{\partial}\Phi(t) = \frac{1}{2}[\Phi(t), \Phi(t)]$, *the integrability condition;*
2) $\overline{\partial}^* \varphi_I = 0$ *for each multi-index I with $|I| \geq 1$;*
3) $\varphi_I \lrcorner \Omega$ *is ∂-exact for each I with $|I| \geq 2$.*
4) *it converges when $|t| < \epsilon$.*

For more about the convergent radius, the reader can refer to [Liu-Rao-Yang14, Theorem 4.4]. This theorem will be used to define the flat affine coordinates $\{t_1, \cdots, t_N\}$ around any point $p \in \mathcal{T}$, for a given orthonormal basis $\{\varphi_1, \cdots, \varphi_N\}$ of $\mathbb{H}^{0,1}(X, T^{1,0}X)$ with respect to the Calabi-Yau metric over X.

2.1.3 Canonical Family of $(n,0)$-Classes

Based on the construction of the smooth family $\Phi(t)$ of Beltrami differentials in Theorem 2.1.5, we can construct a canonical family of holomorphic $(n,0)$-forms on the deformation spaces of Calabi-Yau manifolds. Here we just list the results we need, the reader can refer [Liu-Rao-Yang14, Section 5.1] for details.

Let X be an n-dimensional Calabi-Yau manifold and $\{\varphi_1, \cdots, \varphi_N\} \in \mathbb{H}^{0,1}(X, T^{1,0}X)$ a harmonic basis where $N = \dim \mathbb{H}^{0,1}(X, T^{1,0}X)$. As constructed in Theorem 2.1.5, there exists a smooth family of Beltrami differentials in the following form

$$\Phi(t) = \sum_{i=1}^{N} \varphi_i t_i + \sum_{|I|\geq 2} \varphi_I t^I = \sum_{\nu_1+\cdots+\nu_N\geq 1} \varphi_{\nu_1\cdots\nu_N} t_1^{\nu_1}\cdots t_N^{\nu_N} \in A^{0,1}(X, T_X^{1,0})$$

for $t \in \mathbb{C}^N$ with $|t| < \epsilon$. It is easy to check that the map

$$e^{\Phi(t)} \lrcorner : A^0(X, K_X) \to A^0(X_t, K_{X_t}) \tag{2.4}$$

is a well-defined linear isomorphism. We can write down this map explicitly in the local coordinate systems. Let $\{z^\alpha\}_{\alpha=1}^n$ and $\{w^\alpha\}_{\alpha=1}^n$ be local holomorphic coordinate systems on X and X_t respectively. For the Beltrami differential $\Phi(t)$, we write it locally as

$$\Phi(t) = (\Phi(t))^\alpha_{\overline{\beta}} d\overline{z}^\beta \otimes \frac{\partial}{\partial z^\alpha} = \Phi^\alpha_{\overline{\beta}} d\overline{z}^\beta \otimes \frac{\partial}{\partial z^\alpha}.$$

If $\sigma = f(z)dz^1 \wedge \cdots \wedge dz^n \in A^0(X, K_X)$, the isomorphism is defined as

$$e^{\Phi(t)} \lrcorner \sigma = f(z)(dz^1 + \Phi(dz^1)) \wedge \cdots \wedge (dz^n + \Phi(dz^n)). \tag{2.5}$$

This map is well-defined, in fact, we need to verify that (I) this map does not depend on the coordinate system $\{z^\alpha\}_{\alpha=1}^n$, and also (II) $e^{\Phi(t)} \lrcorner \sigma$ is a section of $A^{n,0}(X_t, L_t)$. (I) is obvious. For (II), since

$$dw^\alpha = \frac{\partial w^\alpha}{\partial z^\beta} dz^\beta + \frac{\partial w^\alpha}{\partial \overline{z}^\gamma} d\overline{z}^\gamma = \frac{\partial w^\alpha}{\partial z^\beta}\left(dz^\beta + \Phi(dz^\beta)\right),$$

we can write $e^{\Phi(t)} \lrcorner \sigma$ as

$$e^{\Phi(t)} \lrcorner \sigma = \frac{f(z)}{\det\left(\frac{\partial w^\alpha}{\partial z^\beta}\right)} dw^1 \wedge \cdots \wedge dw^n. \tag{2.6}$$

By the natural diffeomorphism $G_t = (w^1, \cdots, w^n) : X_0 \to X_t$, we can regard f as a function $(G_t^{-1})^*(f)$ on X_t. Now it is obvious that $e^{\Phi(t)} \lrcorner \sigma$ is a well-defined section of $A^{n,0}(X_t, L_t)$.

Proposition 2.1.6. *For any smooth $(n, 0)$-form $\Omega \in A^{n,0}(X)$, the section $e^{\Phi(t)} \lrcorner \Omega \in A^{n,0}(X_t)$ is holomorphic with respect to the complex structure $J_{\Phi(t)}$ induced by $\Phi(t)$ on X_t if and only if*

$$\overline{\partial}\Omega + \partial(\Phi(t) \lrcorner \Omega) = 0. \tag{2.7}$$

Proof. This is a direct consequence of the following formula, which is [Liu-Rao-Yang14, Corollary 3.5],

$$e^{-\Phi(t)} \lrcorner d\ (e^{\Phi(t)} \lrcorner \Omega) = \overline{\partial}\Omega + \partial(\Phi(t) \lrcorner \Omega).$$

In fact, the operator d can be decomposed as $d = \overline{\partial}_t + \partial_t$, where $\overline{\partial}_t$ and ∂_t denote the $(0,1)$-part and $(1,0)$-part of d, with respect to the complex structure $J_{\Phi(t)}$ induced by $\Phi(t)$ on X_t. Note that $e^{\Phi(t)} \lrcorner \Omega \in A^{n,0}(X_t)$ and so

$$\partial_t (e^{\Phi(t)} \lrcorner \Omega) = 0.$$

Hence,

$$e^{-\Phi(t)} \lrcorner \overline{\partial}_t\ (e^{\Phi(t)} \lrcorner \Omega) = \overline{\partial}\Omega + \partial(\Phi(t) \lrcorner \Omega),$$

which implies the assertion. □

Theorem 2.1.7. *Let Ω be a nontrivial holomorphic $(n,0)$-form on the Calabi-Yau manifold X and $X_t = (X_t, J_{\Phi(t)})$ be the deformation of X induced by the smooth family $\Phi(t)$ of Beltrami differentials on X as constructed in Theorem 2.1.5. Then, for $|t| < \epsilon$,*

$$\Omega^c(t) := e^{\Phi(t)} \lrcorner \Omega \tag{2.8}$$

defines a canonical family of holomorphic $(n,0)$-forms on X_t which depends on t holomorphically.

Proof. Since Ω is holomorphic, and $\Phi(t)$ is smooth, by Proposition 2.1.6, we only need to show that

$$\partial(\Phi(t) \lrcorner \Omega) = 0$$

in the distribution sense. In fact, for any test form η on X,

$$(\Phi(t) \lrcorner \Omega, \partial^* \eta) = \lim_{k \to \infty} \left(\left(\sum_{|I| \le k} \varphi_I t^I \right) \lrcorner \Omega, \partial^* \eta \right) = \lim_{k \to \infty} \left(\sum_{i=1}^{N} t_i \varphi_i \lrcorner \Omega + \sum_{2 \le |I| \le k} t^I \partial \psi_I, \eta \right) = 0,$$

as $\varphi_i \lrcorner \Omega$, $1 \le i \le N$ are harmonic and $\varphi_I \lrcorner \Omega = \partial \psi_I$ are ∂-exact for $|I| \ge 2$ by Theorem 2.1.5. □

Corollary 2.1.8. *Let $\Omega^c(t) := e^{\Phi(t)} \lrcorner \Omega$ be the canonical family of holomorphic $(n,0)$-forms as constructed in Theorem 2.1.7. Then for $|t| < \epsilon$, there holds the following expansion of $[\Omega^c(t)]$ in cohomology classes,*

$$[\Omega^c(t)] = [\Omega] + \sum_{i=1}^{N} [\varphi_i \lrcorner \Omega] t_i + O(|t|^2), \tag{2.9}$$

where $O(|t|^2)$ denotes the terms in $\bigoplus_{j=2}^{n} H^{n-j,j}(X)$ of order at least 2 in t.

2.2 Variation of Hodge Structure and Period Map

In this section, we review the definition of formal variation of Hodge structure and the period domain. Then we proceed to the study of local period map $\Phi : S \to D$, for a variation of Hodge structure parametrised by a simply connected base S. This map associates to $s \in S$ the Hodge filtration over $s \in S$. For more details, the reader can refer to [Voisin02].

2.2.1 Definition of Variation of Hodge Structure

Let $H_{\mathbb{R}}$ be a real vector space with a $\mathbb{Z}$-structure defined by a lattice $H_{\mathbb{Z}} \subset H_{\mathbb{R}}$, and let $H_{\mathbb{C}}$ be the complexification of $H_{\mathbb{R}}$. A Hodge structure of weight n on $H_{\mathbb{C}}$ is a decomposition

$$H_{\mathbb{C}} = \bigoplus_{k=0}^{n} H^{k,n-k}, \quad \text{with} \quad H^{n-k,k} = \overline{H^{k,n-k}}.$$

The integers $h^{k,n-k} = \dim_{\mathbb{C}} H^{k,n-k}$ are called the Hodge numbers. To each Hodge structure of weight n on $H_{\mathbb{C}}$, one assigns the Hodge filtration:

$$H_{\mathbb{C}} = F^0 \supset \cdots \supset F^n, \tag{2.10}$$

with $F^k = H^{n,0} \oplus \cdots \oplus H^{k,n-k}$ and $f^k = \dim_{\mathbb{C}} F^k = \sum_{i=k}^{n} h^{i,n-i}$. This filtration satisfies that

$$H_{\mathbb{C}} = F^k \oplus \overline{F^{n-k+1}}, \quad \text{for } 0 \leq k \leq n. \tag{2.11}$$

Conversely, every decreasing filtration (2.10), with the property (2.11) and fixed dimensions $\dim_{\mathbb{C}} F^k = f^k$, determines a Hodge structure $\{H^{k,n-k}\}_{k=0}^{n}$ with

$$H^{k,n-k} = F^k \cap \overline{F^{n-k}}.$$

A polarization for a Hodge structure of weight n consists of the data of a Hodge-Riemann bilinear form Q over $\mathbb{Z}$, which is symmetric for even n, skew symetric for odd n, such that

$$Q(H^{k,n-k}, H^{r,n-r}) \;\; = 0 \;\text{ unless }\; k = n-r, \tag{2.12}$$

$$i^{2k-n}Q(v,\overline{v}) \;\; > 0 \;\text{ if } v \in H^{k,n-k}, v \neq 0. \tag{2.13}$$

In terms of the Hodge filtration $H_{\mathbb{C}} = F^0 \supseteq F^1 \supseteq \cdots \supseteq F^n$, the relations (2.12) and (2.13) can be written as

$$Q\left(F^k, F^{n-k+1}\right) = 0, \tag{2.14}$$

$$Q\left(Cv, \overline{v}\right) > 0 \;\text{ if }\; v \neq 0, \tag{2.15}$$

where C is the Weil operator given by $Cv = i^{2k-n}v$ when $v \in H^{k,n-k} = F^k \cap \overline{F^{n-k}}$.

Definition 2.2.1. *Let S be a connected complex manifold, a variation of polarized Hodge structure of weight n over S consists of a polarized local system $H_{\mathbb{Z}}$ of $\mathbb{Z}$-modules and a filtration of the associated holomorphic vector bundle H:*

$$\cdots \supseteq F^{k-1} \supseteq F^k \supseteq \cdots \tag{2.16}$$

by holomorphic subbundles F^k which satisfy:

1. $H^{k,n-k} = F^k \oplus \overline{F^{n-k+1}}$ *as C^∞ bundles, where the conjugation is taking relative to the local system of real vectorspace $H_{\mathbb{R}} := H_{\mathbb{Z}} \otimes \mathbb{R}$.*

2. $\nabla(F^k) \subseteq \Omega^1_S \otimes F^{k-1}$, *where ∇ denotes the flat connection on H.*

We refer to the holomorphic subbundles F^k as the Hodge bundles of the variation of polarized Hodge structure. And for each $s \in S$, we have the Hodge decomposition:

$$H_s = \bigoplus_{k=0}^{n} H_s^{k,n-k}; \qquad H_s^{k,n-k} = \overline{H_s^{n-k,k}} \tag{2.17}$$

where $H^{k,n-k}$ is the C^∞ subbundle of H defined by:

$$H^{k,n-k} = F^k \cap \overline{F^{n-k}}.$$

2.2.2 Definition of Period Map

In this section, we defines the local period map for the variation of Hodge structure. Starting from a variation of polarized Hodge structure of weight n parametrised by a simply connected complex manifold S, we can construct the period domain D and its dual $\check{D}$ by fixing a point $s \in S$ as reference point.

The classifying space or the period domain D for polarized Hodge structures with Hodge numbers $\{h^{k,n-k}\}_{k=0}^{n}$ is the space of all such Hodge filtrations

$$D = \left\{ F^n \subset \cdots \subset F^0 = H_{\mathbb{C}} \mid \dim F^k = f^k, (3.15) \text{ and } (3.16) \text{ hold} \right\}.$$

The compact dual $\check{D}$ of D is

$$\check{D} = \left\{ F^n \subset \cdots \subset F^0 = H_{\mathbb{C}} \mid \dim F^k = f^k \text{ and } (3.15) \text{ hold} \right\}.$$

It's easy to see that the classifying space or the period domain $D \subset \check{D}$ is an open subset of the compact dual $\check{D}$. Now, we can define the local period map $\varphi : S \to D$ for the variation of Hodge structure.

Definition 2.2.2. *The period map*

$$\varphi : S \to D$$

is the map which to $s \in S$ associates the subspace

$$\cdots \supseteq F_s^{k-1} \supseteq F_s^k \supseteq \cdots,$$

i.e., the filtration of fibers of the Hodge bundles over $s \in S$.

2.3 Weil-Petersson and Hodge Metric over Calabi-Yau Moduli

There are various metrics over the moduli space and Teichmüller space of Calabi-Yau manifolds, among which the Weil-Petersson metric and Hodge metric are the main tools to investingate the geometry of such moduli spaces. In this section, we review the definitions of Weil-Petersson metric and Hodge metric over the moduli space and Teichmüller space.

2.3.1 The Weil-Petersson Geometry

The local Kuranishi family of polarized Calabi-Yau manifolds $\pi : \mathcal{X} \to S$ is smooth by the Bogomolov-Tian-Todorov theorem [Tian87, Todorov89]. One can assign the unique Ricci-flat or Calabi-Yau metric $g(s)$ on the fiber $X := X_s$ in the polarization Kähler class [Yau78]. Then, on the fiber X, the Kodaira-Spencer theory gives rise to an injective map $\rho : T_sS \longrightarrow H^1(X, T^{1,0}X) \cong \mathbb{H}^{0,1}(X, T^{1,0}X)$, the space of harmonic representatives. The metric $g(s)$ induces a metric on $A^{0,1}(X, T^{1,0}X)$. The reader may also refer to [Wang03] for the discussion. For $v, w \in T_sS$, one then defines the Weil-Petersson metric on S by

$$g_{WP}(v,w) := \int_X \langle \rho(v), \rho(w) \rangle_{g(s)} dvol_g(s). \tag{2.18}$$

Let $\dim X = n$, by the fact that the global holomorphic $(n,0)$-form $\Omega := \Omega(s)$ is flat with respect to $g(s)$, it can be shown [Tian87] that

$$g_{WP}(v,w) = -\frac{\widetilde{Q}(i(v)\Omega, \overline{i(w)\Omega})}{\widetilde{Q}(\Omega, \overline{\Omega})}. \tag{2.19}$$

Here, for convenience, we write $\widetilde{Q}(\cdot,\cdot) = (\sqrt{-1})^n Q(\cdot,\cdot)$, where Q is the intersection product. Therefore, $\widetilde{Q}$ has alternating signs in the successive primitive cohomology groups $H^{p,q}_{pr} \subset H^{p,q}, \quad p+q=n$.

The formula (4.5) implies that the natural map $H^1(X, T^{1,0}X) \longrightarrow \mathrm{Hom}(H^{n,0}, H^{n-1,1})$ via the interior product $v \mapsto v \lrcorner \Omega$ is an isometry from the tangent space T_sS to $(H^{n,0})^* \otimes H^{n-1,1}$. So the Weil-Petersson metric is precisely the metric induced from the first piece of the Hodge metric on the horizontal tangent bundle over the period domain. Let F^n denote the Hodge bundle induced by $H^{n,0}$. A simple calculation in formal Hodge theory shows that

$$\omega_{WP} = Ric(F^n) = -\partial\overline{\partial} \log \widetilde{Q}(\Omega, \overline{\Omega}) = -\frac{\widetilde{Q}(\partial_i\Omega, \overline{\partial_j\Omega})}{\widetilde{Q}(\Omega, \overline{\Omega})} + \frac{\widetilde{Q}(\partial_i\Omega, \overline{\Omega})\ \widetilde{Q}(\Omega, \overline{\partial_j\Omega})}{\widetilde{Q}(\Omega, \overline{\Omega})^2}, \tag{2.20}$$

where ω_{WP} is the 2-form associated to g_{WP}. In particular, g_{WP} is Kähler and is independent of the choice of Ω. In fact, g_{WP} is also independent of the choice of the polarization. Next, we define

$$K_i = -\partial_i \log \widetilde{Q}(\Omega, \overline{\Omega}) = -\frac{\widetilde{Q}(\partial_i\Omega, \overline{\Omega})}{\widetilde{Q}(\Omega, \overline{\Omega})}$$

and

$$D_i\Omega = \partial_i\Omega + K_i\Omega$$

for $1 \leq i \leq N$. And it is easy to check that $D_i\Omega$ is the projection of $\partial_i\Omega$ into $H^{n-1,1}$ with respect to the quadratic form $\widetilde{Q}(\cdot,\cdot)$. And if we denote the Christoffel symbol of the Weil-Petersson metric by Γ^k_{ij}, it is easy to check that

$$D_jD_i\Omega = \partial_jD_i\Omega - \Gamma^k_{ij}D_k\Omega + K_jD_i\Omega,$$

is the projection of $\partial_jD_j\Omega$ into $H^{n-2,2}$. The reader can refer to [Lu-Sun04] for details of these notations.

2.3.2 Period Map and the Hodge Metric on Calabi-Yau Moduli

For any point $p \in \mathcal{T}$, let (X_p, L_p) be the corresponding fiber in the versal family $\mathcal{U} \to \mathcal{T}$, which is a polarized and marked Calabi–Yau manifold. Since the Teichmüller space is simply connected and we have fixed the basis of the middle homology group modulo torsions, we identify the basis of $H_n(X,\mathbb{Z})/Tor$ to a lattice $\wedge$ as in [Szedröi99]. This gives us a canonical identification of the middle dimensional cohomology of X_p to that of the background manifold M, that is, $H^n(M) \simeq H^n(X_p)$. Therefore, we can use this to identify $H^n(X_p)$ for all fibers over $\mathcal{T}$. Thus we get a canonically trivial bundle $H^n(M) \times \mathcal{T}$.

The period map from $\mathcal{T}$ to D is defined by assigning to each point $p \in \mathcal{T}$ the Hodge structure on X_p, that is

$$\Phi : \mathcal{T} \to D, \qquad p \mapsto \Phi(p) = \{F^n(X_p) \subset \cdots \subset F^0(X_p)\}$$

The period map has several good properties, and one may refer to [Voisin02, Chapeter 10] for details. Among them, one of the most important is the following Griffiths transversality: the period map Φ is a holomorphic map and its tangent map satisfies that

$$\Phi_*(v) \in \bigoplus_{k=1}^{n} \mathrm{Hom}(F^k_p/F^{k+1}_p, F^{p-1}_p/F^k_p) \quad \text{for any} \quad p \in \mathcal{T} \ \text{ and } \ v \in T^{1,0}_p\mathcal{T},$$

with $F^{n+1} = 0$, or equivalently, $\Phi_*(v) \in \bigoplus_{k=1}^{n} \mathrm{Hom}(F_p^k, F_p^{k-1})$.

For the Calabi-Yau moduli $\mathcal{M}_m$, we have the following period map:

$$\Phi_m : \mathcal{M}_m \longrightarrow D/\Gamma, \tag{2.21}$$

where Γ denotes the global monodromy group which acts properly and discontinuously on the period domain D. By going to finite covers of $\mathcal{M}_m$ and D/Γ, we may also assume D/Γ is smooth without loss of generality.

In [Griffiths-Schmid69], Griffiths and Schmid studied the so-called Hodge metric on the period domain D which is the natural homogeneous metric on D. We denote it by h. In particular, this Hodge metric is a complete homogeneous metric. By local Torelli theorem for Calabi–Yau manifolds, we know that $\Phi_{\mathcal{T}}, \Phi$ are both locally injective. Thus the pull-backs of h by $\Phi_{\mathcal{T}}$ and Φ on $\mathcal{T}$ and $\mathcal{M}_m$ respectively are both well-defined Kähler metrics by [Griffiths-Schmid69] and [Lu99]. By abuse of notation, we still call these pull-back metrics the Hodge metrics. For explicit formula of the Hodge metric over moduli space of polarized Calabi-Yau manifolds, especially for threefolds, the reader can refer to [Lu99], [Lu01-1] and [Lu01-2] for details.

CHAPTER 3

Quantum Correction and Moduli Space of Calabi-Yau manifolds

In this chapter, we define quantum correction for the Teichmüller space $\mathcal{T}$ of Calabi-Yau manifolds. Under the assumption of vanishing of weak quantum correction, we prove that the Teichmüller space $\mathcal{T}$, with the Weil-Petersson metric, is a locally Hermitian symmetric space. For Calabi-Yau threefolds, we show that the vanishing of strong quantum correction is equivalent to that the image $\Phi(\mathcal{T})$ of the Teichmüller space $\mathcal{T}$ under the period map Φ is an open submanifold of a globally Hermitian symmetric space W of the same dimension as $\mathcal{T}$. Finally, for Hyperkähler manifold of dimension $2n \geq 4$, we find globally defined families of $(2,0)$ and $(2n,0)$-classes over the Teichmüller space of polarized Hyperkähler manifolds.

3.1 Hermitian Symmetric Space and Quantum Correction

In this section, we review the concepts of (globally, locally) Hermitian Symmetric spaces and define the notation of quantum correction for the Teichmüller spaces. In Section 3.1.1, we review the definitions of locally Hermitian symmetric spaces and globally Hermitian symmetric spaces. In Section 3.1.2, we define the notation of quantum correction for the Teichmüller space of polarized and marked Calabi-Yau manifolds, which originally comes from the quantum correction of Yukawa coupling in the Kodaira-Spencer theory developed in [Bershadshy-Cecotti-Ooguri-Vafa94]. The definition of quantum correction also applies to the Teichmüller space of polarized Hyperkähler manifolds.

3.1.1 Hermitian Symmetric Space

First let us review some basic definitions of symmetric spaces, the reader can refer to [Kobayashi-Nomizu69, Chapter 11] or [Zheng00, Chapter 3] for details. Let N be a Riemannian manifold, $p \in N$, and $r_p > 0$ the injective radius at the point p. Consider the diffeomorphism s_p from the geodesic ball $B_{r_p}(p)$ onto N defined by

$$s_p(\exp_p(X)) = \exp_p(-X), \quad \forall X \in B_{r_p}(0) \subset T_pN. \tag{3.1}$$

The map s_p is called the geodesic symmetry at p. It has p as an isolated fixed point, and $(s_p)_{*p} = -id$. In general, it is not an isometry.

Definition 3.1.1. *A Riemannian manifold N is called a locally Riemannian symmetric space, if for any point $p \in N$, the geodesic symmetry s_p is an isometry on $B_{r_p}(p)$. N is called a globally Riemannian symmetric space if, for any point $p \in N$, there exists an isometry in its isometry group $I(N)$ whose restriction on $B_{r_p}(p)$ is s_p.*

Clearly, globally Riemannian symmetric spaces are locally Riemannian symmetric spaces. Applying the theorem of Cartan-Ambrose-Hicks [Cartan46, Ambrose56, Hicks59, Hicks66] to the map s_p and the isometry $I = -id$ at T_pN, we immediately get the following lemma

Lemma 3.1.2. *A Riemannian manifold N is a locally Riemannian symmetric space if and only if $\bigtriangledown R = 0$, i.e., the curvature tensor is parallel. Also, if a locally Riemannian symmetric space is complete and simply-connected, then it is a globally Riemannian symmetric space. Two locally Riemannian symmetric spaces are locally isometric if they have the same curvature at one point.*

Now, let us consider the complex case,

Definition 3.1.3. *A Hermitian manifold N is a locally Hermitian symmetric space if, for any point $p \in N$, $s_p : \exp_p(X) \to \exp_p(-X), \forall X \in T_pN$ is a local automorphism around p of N, i.e., s_p leaves the Levi-Civita connection ∇ and complex structure J invariant. It is called a globally Hermitian symmetric space if it is connected and for any point $p \in N$ there exists an involutive automorphism s_p of N with p as an isolated fixed point.*

Similarly, in terms of the curvature tensor, we have the following characterization of locally Hermitian symmetric spaces.

Theorem 3.1.4. *A Hermitian manifold is a locally Hermitian symmetric space if and only if*

$$\nabla R = 0 = \nabla J. \tag{3.2}$$

where ∇ is the Levi-Civita connection associated to the underlying Riemannian metric.

Corollary 3.1.5. *Let N be a Kähler manifold , if N is a locally Riemannian symmetric space, then N is a locally Hermitian symmetric space.*

For the Riemannican curvature tensor, Nomizu and Ozeki [Nomizu-Ozeki62] and later Nomizu, without assuming completeness, proved the following proposition.

Proposition 3.1.6. *(Nomizu and Ozeki [Nomizu-Ozeki62], Nomizu) For a Riemannian manifold (N, g), if $\nabla^k R = 0$ for some $k \geq 1$, then $\nabla R = 0$.*

3.1.2 Quantum Correction

In this section, we will define the notation of quantum correction for the Teichmüller space of polarized and marked Calabi-Yau manifolds. Our motivation for quantum correction comes from the Kodaira-Spencer theory developed in [Bershadshy-Cecotti-Ooguri-Vafa94, Chapter 5].

The physical fields of the Kodaira-Spencer theory are differential forms of type $(0,1)$ on X with coefficients $(1,0)$-vectors, i.e., sections $\psi \in C^\infty(X, T^{*0,1}X \otimes T^{1,0}X)$ with

$$\partial(\psi \lrcorner \Omega) = 0,$$

where Ω is a nowhere vanishing $(n,0)$-form normalized as in Lemma 2.1.4. Then the Kodaira-Spencer action is given as follows

$$\lambda^2 S(\psi, \varphi | p) = \frac{1}{2}\int_X \psi \lrcorner \Omega \wedge \frac{1}{\partial}\overline{\partial}(\psi \lrcorner \Omega) + \frac{1}{6}\int_X ((\psi + \varphi) \wedge (\psi + \varphi)) \lrcorner \Omega \wedge (\psi + \varphi) \lrcorner \Omega,$$

where λ is the coupling constant. The Euler-Lagrange equation of this action is

$$\overline{\partial}(\varphi \lrcorner \Omega) + \frac{1}{2}\partial((\psi + \varphi) \wedge (\psi + \varphi)) \lrcorner \Omega = 0.$$

They also concluded that the Kodaira-Spencer action is a closed string theory action at least up to cubic order. In a properly regularized Kodaira-Spencer theory, the partition function should satisfy

$$e^{W(\lambda,\varphi|t,\bar{t})} = \int D\psi e^{S(\lambda,\varphi|t,\bar{t})}. \tag{3.3}$$

The effective action $W(\lambda, \varphi|t, \bar{t})$ is physically computed in [Bershadshy-Cecotti-Ooguri-Vafa94] in the flat affine coordinate $t = (t_1, t_2, \cdots, t_N)$. The term $W_0(\lambda, \varphi|t, \bar{t})$ in front of λ^{-2} satisfies

$$W_0(\lambda, \varphi|t, \bar{t}) = \lambda^2 S_0(\varphi, \psi|t, \bar{t}), \tag{3.4}$$

where $\psi(t)$ and $W_0(\lambda, \varphi|t, \bar{t})$ satisfy

$$\frac{\partial \psi(t)}{\partial t_i}|_{t=0} = \varphi_i, \quad \overline{\partial}(\psi \lrcorner \Omega) + \frac{1}{2}((\varphi + \psi(t)) \wedge (\varphi + \psi(t))) \lrcorner \Omega = 0 \tag{3.5}$$

and

$$\frac{\partial^3 W_0(\varphi|t, \bar{t})}{\partial t_i \partial t_j \partial t_k} = C_{ijk}(t_1, t_2, \cdots, t_N). \tag{3.6}$$

Here $W_0(\varphi|t, \bar{t})$ may be viewed as the effective action for the massless modes from which the massive modes will be integrated out. It is quite amazing that integrating the massive modes has only the effectivity of taking derivatives of the Yukawa coupling. For example the four point function give rise to $\nabla_l C_{ijk}$, the five point function to $\nabla_s \nabla_l C_{ijk}$ and the six point function to $\nabla_r \nabla_s \nabla_l C_{ijk}$. Thus all the discussion suggests us to define the quantum correction of the Yukawa coupling as

$$\sum_{s_1+\cdots+s_N=1}^{\infty} C_{ijk,s_1,\cdots,s_N} t_1^{s_1} \cdots t_N^{s_N}. \tag{3.7}$$

On the other hand, besides the canonical family of holomorphic $(n, 0)$-forms, we can define the classic canonical family as

$$\Omega^{cc}(t) = \exp(\sum_{i=1}^{N} t_i \varphi_i) \lrcorner \Omega \in A^n(X). \tag{3.8}$$

Proposition 3.1.7. *Let $\mathcal{T}$ be the Teichmüller space of polarized and marked Calabi-Yau threefolds. Fix $p \in \mathcal{T}$, let X be the corresponding fiber in the versal family $\mathcal{U} \to \mathcal{T}$, Ω be a nontrivial holomorphic $(n,0)$-form over X and $\{\varphi_i\}_{i=1}^N$ be an orthonormal basis of $\mathbb{H}^{0,1}(X, T^{1,0}X)$ with respect to the Calabi-Yau metric. If the cohomology class*

$$[\Xi(t)] = [\Omega^c(t)] - [\Omega^{cc}(t)] = [\Omega^c(t)] - [\exp(\sum_{i=1}^N t_i\varphi_i)\lrcorner\Omega] = 0,$$

then the quantum correction of the Yukawa coupling vanishes, i.e.,

$$\sum_{s_1+\cdots+s_N=1}^{\infty} C_{ijk,s_1,\cdots,s_N} t_1^{s_1}\cdots t_N^{s_N} = 0.$$

Moreover, $\sum\limits_{i,j,k=1}^{N} t_i t_j t_k [\varphi_i \lrcorner \varphi_{jk} \lrcorner \Omega] = 0$ if and only if the first order quantum correction of the Yukawa coupling vanishes, i.e.,

$$\sum_{s_1+\cdots+s_N=1} C_{ijk,s_1,\cdots,s_N} t_1^{s_1}\cdots t_N^{s_N} = 0.$$

Proof. From the definition of Yukawa coupling, with the flat affine coordinate $t = (t_1, \cdots, t_N)$, we have

$$\begin{aligned} C_{ijk}(t_1, t_2, \cdots, t_N) &= \int_X \Omega^c(t) \wedge \frac{\partial^3 \Omega^c(t)}{\partial t_i \partial t_j \partial t_k} \\ &= \int_X \Omega^c(t) \wedge \frac{\partial^3}{\partial t_i \partial t_j \partial t_k} \left(\exp(\sum_{i=1}^N t_i\varphi_i)\lrcorner\Omega + \Xi(t) \right), \end{aligned}$$

where $\Xi(t) = \Omega^c(t) - \exp(\sum\limits_{i=1}^N t_i\varphi_i)\lrcorner\Omega$. Direct computation shows that

$$\frac{\partial^3}{\partial t_i \partial t_j \partial t_k} \left(\exp(\sum_{i=1}^N t_i\varphi_i)\lrcorner\Omega \right) = \varphi_i \lrcorner \varphi_j \lrcorner \varphi_k \lrcorner \Omega,$$

thus the term

$$\int_X \Omega^c(t) \wedge \frac{\partial^3}{\partial t_i \partial t_j \partial t_k} \left(\exp(\sum_{i=1}^N t_i\varphi_i)\lrcorner\Omega \right)$$

has order zero with respect to t. Thus the quantum correction of Yukawa coupling satisfies

$$\sum_{s_1+\cdots+s_N=1}^{\infty} C_{ijk,s_1,\cdots,s_N} t_1^{s_1}\cdots t_N^{s_N} = \int_X \Omega^c(t) \wedge \Xi(t).$$

Therefore, $[\Xi(t)]=0$ implies that the quantum correction of the Yukawa coupling vanishes. Moreover, we have

$$
\begin{aligned}
& \sum_{s_1+\cdots+s_N=1} C_{ijk,s_1,\cdots,s_N} t_1^{s_1}\cdots t_N^{s_N}=0; \\
\Longleftrightarrow\ & \int_X \Omega\wedge\varphi_i\lrcorner\varphi_j\lrcorner\varphi_I\lrcorner\Omega+\int_X \varphi_i\lrcorner\Omega\wedge\varphi_j\lrcorner\varphi_I\lrcorner\Omega=0 \quad \text{for any} \quad 1\le i,j\le N \quad \text{and} \quad |I|=2; \\
\Longleftrightarrow\ & \int_X \varphi_i\lrcorner\Omega\wedge\varphi_j\lrcorner\varphi_I\lrcorner\Omega=0, \quad \text{for any} \quad 1\le i,j\le N \quad \text{and} \quad |I|=2; \\
& (\text{as } \varphi_i\lrcorner\Omega\wedge\varphi_j\lrcorner\varphi_I\lrcorner\Omega=\Omega\wedge\varphi_i\lrcorner\varphi_j\lrcorner\varphi_I\lrcorner\Omega); \\
\Longleftrightarrow\ & \mathbb{H}(\varphi_j\lrcorner\varphi_I\lrcorner\Omega)=0, \quad \text{for any} \quad 1\le j\le N \quad \text{and} \quad |I|=2; \\
& \big(\text{as} \quad \{[\varphi_i\lrcorner\Omega]\}_{i=1}^N \quad \text{is a basis of} \quad \mathbb{H}^{2,1}(X)\big); \\
\Longleftrightarrow\ & \sum_{i,j,k=1}^N t_it_jt_k[\varphi_i\lrcorner\varphi_{jk}\lrcorner\Omega]=0.
\end{aligned}
$$

□

Lemma 3.1.8. *Under the conditions as Proposition 3.1.7, the form* $\Xi(t)$ *are identically zero, i.e.,* $\Xi(t)=0$ *if and only if* $[\varphi_i,\varphi_j]=0$ *for all* $1\le i,j\le N$*. And, for* $|t|<\epsilon$*, there holds the following expansion of* $[\Xi(t)]$ *in cohomology classes,*

$$
[\Xi(t)]=\sum_{i,j,k=1}^N t_it_jt_k[\varphi_i\lrcorner\varphi_{jk}\lrcorner\Omega]+O(|t|^4). \tag{3.9}
$$

where $O(|t|^4)$ *denotes the terms of order at least* 4 *in* t*.*

Proof. From the construction of the smooth family 2.3 of Beltrami differentials, see [Morrow-Kodaira71, page 162] or [Todorov89], we have

$$
\varphi_K=-\frac{1}{2}\overline{\partial}^* G\Big(\sum_{I+J=K}[\varphi_I,\varphi_J]\Big) \quad \text{for} \quad |K|\ge 2. \tag{3.10}
$$

Thus we have

$$\begin{aligned}
\Xi(t)=0 &\iff \left(\exp\left(\sum_{|I|\geq 1}\varphi_I t^I\right)-\exp\left(\sum_{i=1}^N \varphi_i t_i\right)\right)\lrcorner\Omega=0\\
&\iff \sum_{|I|\geq 1}\varphi_I t^I=\sum_{i=1}^N\varphi_i t_i\\
&\iff \varphi_I=0 \quad \text{for} \quad |I|\geq 2\\
&\iff [\varphi_i,\varphi_j]=0 \quad \text{for} \quad 1\leq i,j\leq N \quad \text{by Formula (3.10).}
\end{aligned}$$

Moreover, by the property that $\varphi_I\lrcorner\Omega=\partial\psi_I$ for $|I|\geq 2$ by Theorem 2.1.5, the cohomology class of the quantum correction satisfies

$$\begin{aligned}
[\Xi(t)] &= \left[\left(\exp\left(\sum_{|I|\geq 1}\varphi_I t^I\right)-\exp\left(\sum_{i=1}^N\varphi_i t_i\right)\right)\lrcorner\Omega\right]\\
&= \left[\sum_{i,j=1}^N(\varphi_{ij}\lrcorner\Omega)t_it_j+\sum_{i,j,k=1}^N(\varphi_i\lrcorner\varphi_{jk}\lrcorner\Omega+\varphi_{ijk}\lrcorner\Omega)t_it_jt_k+O(|t|^4)\right]\\
&= \sum_{i,j,k=1}^N t_it_jt_k[\varphi_i\lrcorner\varphi_{jk}\lrcorner\Omega]+O(|t|^4).
\end{aligned}$$

□

Thus the lowest order quantum correction has the form $\sum_{i,j,k=1}^N t_it_jt_k[\varphi_i\lrcorner\varphi_{jk}\lrcorner\Omega]$, so we have the following definitions,

Definition 3.1.9. *We define the cohomology class* $[\Xi(t)]$ *to be the strong quantum correction for the Teichmüller space* $\mathcal{T}$ *at* $p\in\mathcal{T}$ *and the cohomology class*

$$[\Xi(t)]_1=\sum_{i,j,k=1}^N t_it_jt_k[\varphi_i\lrcorner\varphi_{jk}\lrcorner\Omega],$$

to be the weak quantum correction for the Teichmüller space $\mathcal{T}$ *at* $p\in\mathcal{T}$.

Remark 3.1.10. *For Calabi-Yau threefolds, by Proposition 3.1.7, vanishing of strong quantum correction at any point* $p\in\mathcal{T}$ *implies that the quantum correction of Yukawa coupling vanishes at* $p\in\mathcal{T}$. *Moreover, vanishing of weak quantum correction at* $p\in\mathcal{T}$ *is equivalent to that the first order quantum correction of Yukawa coupling vanishes at* $p\in\mathcal{T}$.

3.2 Quantum Correction and the Weil-Petersson Metric

In Section 3.2.1, we derive a local formula for ∇R in the flat affine coordinate of the Teichmüller space with the Weil-Petersson metric. In Section 3.2.2, under the assumption of vanishing of weak quantum correction at any point $p \in \mathcal{T}$, we prove that $\mathcal{T}$ is a locally Hermitian symmetric space with the Weil-Petersson metric by using the formula of ∇R. The results in this section also hold for the Teichmüller space of polarized Hyperkähler manifolds.

3.2.1 Property of the Curvature Tensor

To simplify the notation, we abstract the discussion by considering a variations of polarized Hodge structure $H \to S$ of weight n with $h^{n,0} = 1$ and a smooth base S. Also, we always assume that it is effectively parametrized in the sense that the infinitesimal period map

$$\Phi_{*,s} : T_sS \longrightarrow \mathrm{Hom}(H^{n,0}, H^{n-1,1}) \oplus \mathrm{Hom}(H^{n-1,1}, H^{n-2,2}) \oplus \cdots \tag{3.11}$$

is injective in the first piece. Then the Weil-Petersson metric g_{WP} on S is defined by formula (4.5). In our abstract setting, instead of using $H^{p,q}_{pr}$ in the geometric case, we will write $H^{p,q}$ directly for simplicity.

Theorem 3.2.1. *For a given effectively parametrized ploarized variation of Hodge structure $H \to S$ of weight n with $h^{n,0} = 1$ and smooth S, the Riemannian curvature of the Weil-Petersson metric g_{WP} on S satisfies:*

1. *Its Riemannian curvature tensor is*

$$R_{i\bar{j}k\bar{l}} = g_{i\bar{j}}g_{k\bar{l}} + g_{i\bar{l}}g_{k\bar{j}} - \frac{\widetilde{Q}(D_kD_i\Omega^c(t), \overline{D_lD_j\Omega^c(t)})}{\widetilde{Q}(\Omega^c(t), \overline{\Omega^c(t)})}.$$

2. *The covariant derivative of the Riemannian curvature tensor is*

$$\nabla_r R_{i\bar{j}k\bar{l}} = \frac{\widetilde{Q}(\nabla_r\nabla_k\nabla_i\Omega^c(t), \overline{D_lD_j\Omega^c(t)})}{\widetilde{Q}(\Omega^c(t), \overline{\Omega^c(t)})};$$

$$\nabla_{\bar{r}} R_{i\bar{j}k\bar{l}} = \frac{\widetilde{Q}(D_kD_i\Omega^c(t), \overline{\nabla_r\nabla_l\nabla_j\Omega^c(t)})}{\widetilde{Q}(\Omega^c(t), \overline{\Omega^c(t)})}.$$

The main idea of the proof is that, when we use the canonical family $[\Omega^c(t)]$ of $(n,0)$-classes constructed in Corollary 2.1.8 to express the Weil-Petersson metric, the flat affine coordinate $t=(t_1,\cdots,t_N)$ is normal at the point $t=0$. Since the problem is local, we may assume that S is a disk in $\mathbb{C}^N$, where $N=\dim H^{n-1,1}$, around $t=0$. The first part of the above theorem for the curvature formula of the Weil-Petersson metric is due to Strominger. See [Wang03] and [Lu-Sun04].

Proof. Let $\Omega^c(t)$ be the canonical family of holomorphic $(n,0)$-forms constructed in Theorem 2.1.7, so we have

$$
\begin{aligned}
\Omega^c(t) &= \Omega + \sum_{i=1}^N t_i\varphi_i\lrcorner\Omega + \sum_{|I|\geq 2} t^I\ \varphi_I\lrcorner\Omega + \sum_{k\geq 2}\wedge^k\Phi(t)\lrcorner\Omega \\
&= \Omega + \sum_{i=1}^N t_i\varphi_i\lrcorner\Omega + \frac{1}{2!}\sum_{i,j=1}^N t_it_j(\varphi_i\lrcorner\varphi_j\lrcorner\Omega + \varphi_{ij}\lrcorner\Omega) \\
&\quad + \frac{1}{3!}\sum_{i,j<k} t_it_jt_k(\varphi_i\lrcorner\varphi_j\lrcorner\varphi_k\lrcorner\Omega + \varphi_i\lrcorner\varphi_{jk}\lrcorner\Omega + \varphi_{ijk}\lrcorner\Omega) + O(|\ t\ |^4) \\
&= a_0 + \sum_{i=1}^N a_it_i + \cdots + \sum_{|I|=k} a_It^I + \cdots.
\end{aligned}
$$

And the coefficients satisfy $\widetilde{Q}(a_0,a_0)=1, \widetilde{Q}(a_i,a_j)=-\delta_{ij}$ and $\widetilde{Q}(a_0,a_i)=\widetilde{Q}(a_0,a_I)=\widetilde{Q}(a_i,a_I)=0$ for $|I|\geq 2$. For multi-indices I and J, we set $q_{I,\overline{J}} := \widetilde{Q}(a_I,\overline{a}_J)$. Then we have

$$
\begin{aligned}
q(t): &= \widetilde{Q}(\Omega^c(t),\overline{\Omega^c(t)}) \\
&= 1 - \sum_i t_i\overline{t_i} + \sum_{i,j,k,l}\frac{1}{2!2!}q_{ik,\overline{jl}}t_it_k\overline{t_jt_l} \\
&\quad + \sum_{i,j,k,l,r}\frac{1}{2!3!}q_{ik,\overline{jlr}}t_it_k\overline{t_jt_lt_r} + \sum_{i,j,k,l,r}\frac{1}{2!3!}q_{ikr,\overline{jl}}t_it_kt_r\overline{t_jt_k} + O(t^6),
\end{aligned}
$$

where

$$
\begin{aligned}
q_{ik,\overline{jl}} &= \widetilde{Q}(\varphi_i\lrcorner\varphi_k\lrcorner\Omega,\ \ \overline{\varphi_j\lrcorner\varphi_l\lrcorner\Omega}), \\
q_{ik,\overline{jlr}} &= \frac{1}{3}\widetilde{Q}(\varphi_i\lrcorner\varphi_k\lrcorner\Omega,\ \ \overline{\varphi_j\lrcorner\varphi_{lr}\lrcorner\Omega + \varphi_l\lrcorner\varphi_{jr}\lrcorner\Omega + \varphi_r\lrcorner\varphi_{jl}\lrcorner\Omega}), \\
q_{ikr,\overline{jl}} &= \frac{1}{3}\widetilde{Q}(\varphi_i\lrcorner\varphi_{kr}\lrcorner\Omega + \varphi_k\lrcorner\varphi_{ir}\lrcorner\Omega + \varphi_r\lrcorner\varphi_{ik}\lrcorner\Omega,\ \ \overline{\varphi_j\lrcorner\varphi_l\lrcorner\Omega}).
\end{aligned}
$$

Thus, the Weil-Petersson metric can be expressed as

$$\begin{aligned} g_{k\bar{l}} &= -\partial_k \partial_{\bar{l}} \ \log \ q = q^{-2}(\partial_k q \partial_{\bar{l}} q - q \partial_k \partial_{\bar{l}} q) \\ &= (1 + 2\sum_i t_i \overline{t_i} + \cdots)[t_l t_{\bar{k}} - (1 - \sum_i t_i \overline{t_i})(-\delta_{kl} + \sum_{i,j} q_{ik,jl} t_i \overline{t_j} \\ &\quad + \sum_{i,j,r} q_{ik,\overline{jlr}} t_i \overline{t_j t_r} + \sum_{i,j,r} q_{ikr,\overline{jl}} t_i t_r \overline{t_l} + \cdots)] \\ &= \delta_{kl} + \delta_{kl} \sum_i t_i \overline{t_i} + t_l \overline{t_k} - \sum_{i,j} q_{ik,jl} t_i \overline{t_j} + \sum_{i,j,r} q_{ik,\overline{jlr}} t_i \overline{t_j t_r} + \sum_{i,j,r} q_{ikr,\overline{jl}} t_i t_r \overline{t_l} + \cdots . \end{aligned}$$

As a result, the Weil-Petersson metric g is already in its geodesic normal form at $t = 0$, so the Christoffel symbols at the point $t = 0$ are zero, i.e. $\Gamma_{ij}^k(0) = 0$ for any $1 \leq i, j, k \leq N$. So the full curvature tensor at $t = 0$ is given by

$$R_{i\bar{j}k\bar{l}}(0) = \frac{\partial^2 g_{k\bar{l}}}{\partial t_i \partial \overline{t_j}}(0) = \delta_{ij}\delta_{kl} + \delta_{il}\delta_{kj} + q_{ik,\overline{jl}}.$$

Rewrite this in its tensor form then gives the formula in the theorem.

By using the well-known formula:

$$\bigtriangledown_r R_{i\bar{j}k\bar{l}} = \frac{\partial}{\partial t_r} R_{i\bar{j}k\bar{l}} - \Gamma_{ri}^q R_{q\bar{j}k\bar{l}} - \Gamma_{rk}^q R_{i\bar{j}q\bar{l}},$$

$$\bigtriangledown_{\bar{r}} R_{i\bar{j}k\bar{l}} = \frac{\partial}{\partial \overline{t_r}} R_{i\bar{j}k\bar{l}} - \overline{\Gamma_{rj}^q} R_{i\bar{q}k\bar{l}} - \overline{\Gamma_{rl}^q} R_{i\bar{j}k\bar{q}},$$

at the point $t = 0$, we have

$$\bigtriangledown_r R_{i\bar{j}k\bar{l}}(0) = \frac{\partial}{\partial t_r} R_{i\bar{j}k\bar{l}}(0); \qquad \bigtriangledown_{\bar{r}} \ R_{i\bar{j}k\bar{l}}(0) = \frac{\partial}{\partial \overline{t_r}} R_{i\bar{j}k\bar{l}}(0).$$

as $\Gamma_{ij}^k(0) = 0$ for any $1 \leq i, j, k \leq N$. And, from the formula of Riemannian curvature for Kähler manifold

$$R_{i\bar{j}k\bar{l}} = \frac{\partial^2 g_{i\bar{j}}}{\partial t_k \partial \overline{t_l}} - g^{p\bar{q}} \frac{\partial g_{i\bar{q}}}{\partial t_k} \frac{\partial g_{j\bar{k}}}{\partial \overline{t_l}}, \tag{3.12}$$

each term of $\nabla_r R_{i\bar{j}k\bar{l}}$ or $\nabla_{\bar{r}} R_{i\bar{j}k\bar{l}}$ includes first derivative of $g_{k\bar{l}}$ as a factor, except $\frac{\partial^3 g_{k\bar{l}}}{\partial t_i \partial \overline{t_j} \partial t_r}$, $\frac{\partial^3 g_{k\bar{l}}}{\partial t_i \partial \overline{t_j} \partial \overline{t_r}}$, thus it is zero at $t = 0$ from the expression of $g_{k\bar{l}}$. So we have

$$\nabla_r R_{i\bar{j}k\bar{l}}(0) = \frac{\partial^3 g_{k\bar{l}}}{\partial t_i \partial \overline{t_j} \partial t_r}(0) = q_{ikr,\overline{jl}},$$

$$\nabla_{\bar{r}} R_{i\bar{j}k\bar{l}}(0) = \frac{\partial^3 g_{k\bar{l}}}{\partial t_i \partial \overline{t_j} \partial \overline{t_r}}(0) = q_{ik,\overline{jlr}}.$$

Rewrite this in its tensor form, we get the formula. □

3.2.2 Quantum Correction and the Weil-Petersson Metric

In this section, we consider the locally Hermitian symmetric property of the Teichmüller space $\mathcal{T}$ of polarized and marked Calabi-Yau manifolds.

Theorem 3.2.2. *Let $\mathcal{T}$ be the Teichmüller space of polarized and marked Calabi-Yau manifolds. Vanishing of weak quantum correction at any point $p \in \mathcal{T}$, i.e., $[\Xi(t)]_1 = 0$, implies that $\mathcal{T}$, with the Weil-Petersson metric, is a locally Hermitian symmetric space.*

Proof. Fix $p \in \mathcal{T}$, let X be the corresponding fiber in the versal family $\mathcal{U} \to \mathcal{T}$. If the weak quantum correction vanishes at point $p \in \mathcal{T}$, i.e.,

$$[\Xi(t)]_1 = \sum_{i,j,k=1}^{N} t_i t_j t_k [\varphi_i \lrcorner \varphi_{jk} \lrcorner \Omega] = 0,$$

then $[\varphi_i \lrcorner \varphi_{jk} \lrcorner \Omega] + [\varphi_j \lrcorner \varphi_{ik} \lrcorner \Omega] + [\varphi_k \lrcorner \varphi_{ij} \lrcorner \Omega] = 0$ for any $1 \leq i, j, k \leq N$. So, from Theorem 3.2.1, we have

$$\nabla_r R_{i\bar{j}k\bar{l}}(p) = \frac{\partial^3 g_{k\bar{l}}}{\partial t_i \partial \overline{t_j} \partial t_r}(p) = q_{ikr,\overline{jl}} = \frac{1}{3}\widetilde{Q}(\varphi_i \lrcorner \varphi_{kr} \lrcorner \Omega + \varphi_k \lrcorner \varphi_{ir} \lrcorner \Omega + \varphi_r \lrcorner \varphi_{ik} \lrcorner \Omega, \ \ \overline{\varphi_j \lrcorner \varphi_l \lrcorner \Omega}) = 0,$$

$$\nabla_{\bar{r}} R_{i\bar{j}k\bar{l}}(p) = \frac{\partial^3 g_{k\bar{l}}}{\partial t_i \partial \overline{t_j} \partial \overline{t_r}}(p) = q_{ik,\overline{jlr}} = \frac{1}{3}\widetilde{Q}(\varphi_i \lrcorner \varphi_k \lrcorner \Omega, \ \ \overline{\varphi_j \lrcorner \varphi_{lr} \lrcorner \Omega + \varphi_l \lrcorner \varphi_{jr} \lrcorner \Omega + \varphi_r \lrcorner \varphi_{jl} \lrcorner \Omega}) = 0,$$

i.e., $\nabla R = 0$. So $\mathcal{T}$ is a locally Hermitian symmetric space by Lemma 3.1.2. □

On the other hand, by the definition of locally Hermitian symmetric spaces, the following condition can also guarantee the locally Hermitian symmetric property for the Teichmüller space $\mathcal{T}$.

Theorem 3.2.3. *Let $\mathcal{T}$ be the Teichmüller space of polarized and marked Calabi-Yau manifolds and $\Omega^c(t)$ be the canonical family of holomorphic $(n,0)$-forms constructed in Theorem 2.1.7. If the Weil-Petersson potential $\widetilde{Q}(\Omega^c_t, \overline{\Omega^c_t})$ only has finite terms, i.e., a polynormial in terms of the flat affine coordinate t, then $\mathcal{T}$, with the Weil-Petersson metric, is a locally Hermitian symmetric space. Furthermore, if $\mathcal{T}$ is complete, then $\mathcal{T}$, with the Weil-Petersson metric, is a globally Hermitian symmetric space.*

Proof. Because of Proposition 3.1.6, to prove a Kähler manifold is a locally Hermitian symmetric space, we only need to show its curvature tensor satisfies $\nabla^m R = 0$ for some positive integer m. If the Weil-Petersson potential $\widetilde{Q}(\Omega^c(t), \overline{\Omega^c}(t))$ only has finite terms, i.e., it is a polynomial of the flat affine coordinate $t = (t_1, t_2, \cdots, t_N)$. Then, in the flat affine coordinate t, the coefficients of the Weil-Petersson metric and its curvature tensor

$$\begin{aligned} g_{k\bar{l}} &= -\partial_k \partial_{\bar{l}} \log \widetilde{Q}(\Omega^c(t), \overline{\Omega^c(t)}) \\ R_{i\bar{j}k\bar{l}} &= \frac{\partial^2 g_{i\bar{j}}}{\partial t_k \partial \overline{t_l}} - g^{p\bar{q}} \frac{\partial g_{i\bar{q}}}{\partial t_k} \frac{\partial g_{p\bar{j}}}{\partial \overline{t_l}}. \end{aligned}$$

is a polynomial of variable $(t_1, t_2, \cdots, t_N, \overline{t_1}, \overline{t_2}, \cdots, \overline{t_N})$.

On the other hand, from the proof of Theorem 3.2.1, the flat affine coordinate t is a normal coordinate at the point $t = 0$. So we know that the Christoffel symbols vanish at the point $t = 0$, i.e., $\Gamma^k_{ij}(0) = 0$ for any $1 \leq i, j, k \leq N$. Thus, at the point $t = 0$, the covariant derivative $\nabla_p T = \partial_p T$ for any $(0, m)$-tensor T. Therefore, for a large enough integer m, we have $\nabla^m R(0) = 0$. Thus the Teichmüller space $\mathcal{T}$ is a locally Hermitian symmetric space with the Weil-Petersson metric. □

In particular, we have the following corollary,

Corollary 3.2.4. *If the canonical family of $(n, 0)$-classes $[\Omega^c(t)]$ constructed in Corollary 2.1.8 has finite terms, i.e., a polynormial in terms of the flat affine coordinate t, then $\mathcal{T}$, with the Weil-Petersson metric, is a locally Hermitian symmetric space.*

Proof. The proof follows directly from Theorem 3.2.3. □

3.3 Quantum Correction and Calabi-Yau Threefolds

In Section 3.3.1, we review some basic properties of the period domain from Lie group and Lie algebra point of view. In Section 3.3.2, for Calabi-Yau threefolds, we show that vanishing of strong quantum correction is equivalent to that the image $\Phi(\mathcal{T})$ of the Teichmüller space

$\mathcal{T}$ under the period map Φ is an open submanifold of a globally Hermitian symmetric space W with the same dimension as $\mathcal{T}$.

3.3.1 Period Domain

Let us briefly recall some properties of the period domain from Lie group and Lie algebra point of view. All results in this section are well-known to the experts in the subject. One may either skip this section or refer to [Griffiths-Schmid69] and [Schmid73] for most of the details, here we use the summary in [Chen-Guan-Liu13, Section 3.1] to fix notations.

A pair (X, L) consisting of a Calabi–Yau manifold X of complex dimension n with $n \geq 3$ and an ample line bundle L over X is called a polarized Calabi–Yau manifold. By abuse of notation, the Chern class of L will also be denoted by L and thus $L \in H^2(X, \mathbb{Z})$. The Poincaré bilinear form Q on $H^n_{pr}(X, \mathbb{Q})$ is defined by

$$Q(u, v) = (-1)^{\frac{n(n-1)}{2}} \int_X u \wedge v$$

for any d-closed n-forms u, v on X. Furthermore, Q is nondegenerate and can be extended to $H^n_{pr}(X, \mathbb{C})$ bilinearly. Let $f^k = \sum_{i=k}^n h^{i,n-i}$ and $F^k = F^k(X) = H^{n,0}_{pr}(X) \oplus \cdots \oplus H^{k,n-k}_{pr}(X)$, from which we have the decreasing filtration $H^n_{pr}(X, \mathbb{C}) = F^0 \supset \cdots \supset F^n$. We know that

$$\dim_{\mathbb{C}} F^k = f^k, \tag{3.13}$$

$$H^n_{pr}(X, \mathbb{C}) \ = F^k \oplus \overline{F^{n-k+1}}, \quad \text{and} \quad H^{k,n-k}_{pr}(X) = F^k \cap \overline{F^{n-k}}. \tag{3.14}$$

In terms of the Hodge filtration, then the Hodge-Riemann relations are

$$Q\left(F^k, F^{n-k+1}\right) = 0, \quad \text{and} \tag{3.15}$$

$$Q\left(Cv, \overline{v}\right) > 0 \quad \text{if} \quad v \neq 0, \tag{3.16}$$

where C is the Weil operator given by $Cv = \left(\sqrt{-1}\right)^{2k-n} v$ for $v \in H^{k,n-k}_{pr}(X)$. The period domain D for polarized Hodge structures with data (3.13) is the space of all such Hodge filtrations

$$D = \left\{H^n_{pr}(X, \mathbb{C}) = F^0 \supseteq \cdots \supseteq F^n \mid (3.13), (3.15) \text{ and } (3.16) \text{ hold}\right\}.$$

The compact dual $\check{D}$ of D is

$$\check{D} = \left\{ H^n_{pr}(X, \mathbb{C}) = F^0 \supseteq \cdots \supseteq F^n \mid (3.13) \text{ and } (3.15) \text{ hold} \right\}.$$

The period domain $D \subseteq \check{D}$ is an open subset. Let us introduce the notion of an adapted basis for the given Hodge decomposition or Hodge filtration. For any $p \in \mathcal{T}$ and $f^k = \dim F^k_p$ for any $0 \leq k \leq n$, We call a basis

$$\zeta = \{\zeta_0, \zeta_1, \cdots, \zeta_N, \cdots, \zeta_{f^{k+1}}, \cdots, \zeta_{f^k-1}, \cdots, \zeta_{f^2}, \cdots, \zeta_{f^1-0}, \zeta_{f^0-1}\}$$

of $H^n(X)$ an adapted basis for the given filtration

$$F^0 \supseteq \cdots \supseteq F^{n-1} \supseteq F^n,$$

if it satisfies $F^k = \mathrm{Span}_{\mathbb{C}}\{\zeta_0, \cdots, \zeta_{f^k-1}\}$ with $\dim_{\mathbb{C}} F^k = f^k$.

The orthogonal group of the bilinear form Q in the definition of Hodge structure is a linear algebraic group, defined over $\mathbb{Q}$. Let us simply denote $H = H^n(X)$ and $H_{\mathbb{R}} = H^n(X, \mathbb{R})$. The group of the $\mathbb{C}$-rational points is

$$G_{\mathbb{C}} = \{g \in GL(H) | \ Q(gu, gv) = Q(u, v) \text{ for all } u, v \in H\},$$

which acts on $\check{D}$ transitively. The group of real points in $G_{\mathbb{C}}$ is

$$G_{\mathbb{R}} = \{g \in GL(H_{\mathbb{R}}) | \ Q(gu, gv) = Q(u, v) \text{ for all } u, v \in H_{\mathbb{R}}\},$$

which acts transitively on D as well.

Consider the period map $\Phi : \mathcal{T} \to D$. Fix $p \in \mathcal{T}$ and the image $O := \Phi(p) = \{F^n_p \subset \cdots \subset F^0_p\} \in D$ may be referred as the base points or the reference point. A linear transformation $g \in G_{\mathbb{C}}$ preserves the base point if and only if $gF^k_p = F^k_p$ for each k. Thus it gives the identification

$$\check{D} \simeq G_{\mathbb{C}}/B \quad \text{with} \quad B = \{g \in G_{\mathbb{C}} | \ gF^k_p = F^k_p, \text{ for any } k\}.$$

Similarly, one obtains an analogous identification

$$D \simeq G_{\mathbb{R}}/V \hookrightarrow \check{D} \quad \text{with} \quad V = G_{\mathbb{R}} \cap B,$$

where the embedding corresponds to the inclusion $G_{\mathbb{R}}/V = G_{\mathbb{R}}/G_{\mathbb{R}} \cap B \subseteq G_{\mathbb{C}}/B$. The Lie algebra $\mathfrak{g}$ of the complex Lie group $G_{\mathbb{C}}$ can be described as

$$\mathfrak{g} = \{X \in \mathrm{End}(H) |\ Q(Xu, v) + Q(u, Xv) = 0, \text{ for all } u, v \in H\}.$$

It is a simple complex Lie algebra, which contains $\mathfrak{g}_0 = \{X \in \mathfrak{g}|\ XH_{\mathbb{R}} \subseteq H_{\mathbb{R}}\}$ as a real form, i.e. $\mathfrak{g} = \mathfrak{g}_0 \oplus i\mathfrak{g}_0$. With the inclusion $G_{\mathbb{R}} \subseteq G_{\mathbb{C}}$, $\mathfrak{g}_0$ becomes Lie algebra of $G_{\mathbb{R}}$. One observes that the reference Hodge structure $\{H_p^{k,n-k}\}_{k=0}^n$ of $H^n(M)$ induces a Hodge structure of weight zero on $\mathrm{End}(H^n(M))$, namely,

$$\mathfrak{g} = \bigoplus_{k\in\mathbb{Z}} \mathfrak{g}^{k,-k} \quad \text{with} \quad \mathfrak{g}^{k,-k} = \{X \in \mathfrak{g}|XH_p^{r,n-r} \subseteq H_p^{r+k,n-r-k}\}.$$

Since the Lie algebra $\mathfrak{b}$ of B consists of those $X \in \mathfrak{g}$ that preserves the reference Hodge filtration $\{F_p^n \subset \cdots \subset F_p^0\}$, one thus has

$$\mathfrak{b} = \bigoplus_{k\geq 0} \mathfrak{g}^{k,-k}.$$

The Lie algebra $\mathfrak{v}_0$ of V is $\mathfrak{v}_0 = \mathfrak{g}_0 \cap \mathfrak{b} = \mathfrak{g}_0 \cap \mathfrak{b} \cap \overline{\mathfrak{b}} = \mathfrak{g}_0 \cap \mathfrak{g}^{0,0}$. With the above isomorphisms, the holomorphic tangent space of $\check{D}$ at the base point is naturally isomorphic to $\mathfrak{g}/\mathfrak{b}$.

Let us consider the nilpotent Lie subalgebra $\mathfrak{n}_+ := \oplus_{k\geq 1}\mathfrak{g}^{-k,k}$. Then one gets the holomorphic isomorphism $\mathfrak{g}/\mathfrak{b} \cong \mathfrak{n}_+$. Since D is an open set in $\check{D}$, we have the following relation:

$$T^{1,0}_{O,h}D = T^{1,0}_{O,h}\check{D} \cong \mathfrak{b} \oplus \mathfrak{g}^{-1,1}/\mathfrak{b} \hookrightarrow \mathfrak{g}/\mathfrak{b} \cong \mathfrak{n}_+. \tag{3.17}$$

We define the unipotent group $N_+ = \exp(\mathfrak{n}_+)$.

Remark 3.3.1. *With a fixed base point, we can identify N_+ with its unipotent orbit in $\check{D}$ by identifying an element $c \in N_+$ with $[c] = cB$ in $\check{D}$; that is, $N_+ = N_+($ base point $) \cong N_+B/B \subseteq \check{D}$. In particular, when the base point O is in D, we have $N_+ \cap D \subseteq D$. We can also identify a point $\Phi(p) = \{F_p^0 \supseteq F_p^1 \supseteq \cdots \supseteq F_p^n\} \in D$ with any fixed adapted basis of the corresponding Hodge filtration, we have matrix representations of elements in the above Lie groups and Lie algebras. For example, elements in N_+ can be realized as nonsingular block upper triangular matrices with identity blocks in the diagonal; elements in B can be realized as nonsingular block lower triangular matrices.*

3.3.2 Quantum Correction and Calabi-Yau Threefolds

For any $p \in \mathcal{T}$, let (X_p, L) be the corresponding fiber in the versal family $\mathcal{X} \to \mathcal{T}$, which is a polarized and marked Calabi–Yau manifold. The period map from $\mathcal{T}$ to D is defined by assigning each point $p \in \mathcal{T}$ the Hodge structure on X_p, that is

$$\Phi : \mathcal{T} \to D, \qquad p \mapsto \Phi(p) = \{F^0(X_p) \supseteq \cdots \supseteq F^n(X_p)\}.$$

In [Griffiths-Schmid69], Griffiths and Schmid studied the Hodge metric over the period domain D. In particular, this Hodge theory is a complete homogenous metric. Consider the period map on the Teichmüller space $\Phi : \mathcal{T} \to D$. By local Torelli theorem for Calabi-Yau manifolds, we know that the period map Φ is locally injective. Thus it follows from [Griffiths-Schmid69] that the pull-back of the Hodge metric over D by Φ on $\mathcal{T}$ is a well-defined Kähler metric. We will call the pull-back metric the Hodge metric over Teichmüller space $\mathcal{T}$, still denoted by h. For explicit formula of the Hodge metric over moduli space of polarized Calabi-Yau manifolds, especially for threefolds, the reader can refer to [Lu99], [Lu01-1] and [Lu01-2] for details.

Theorem 3.3.2. *Let $\mathcal{T}$ be the Teichmüller space of polarized and marked Calabi-Yau threefolds and $\Phi : \mathcal{T} \to D$ be the period map. Then the following conditions are equivalent:*

1. *The strong quantum correction vanishes at any point $p \in \mathcal{T}$.*
2. *With respect to the Hodge metric, the image $\Phi(\mathcal{T})$ is an open submanifold of a globally Hermitian symmetric space W of the same dimension as $\mathcal{T}$, which is also a totally geodesic submanifold of the period domain D.*

This theorem also implies that, under the assumption of vanishing of strong quantum correction at any point $p \in \mathcal{T}$, the Teichmüller space $\mathcal{T}$ is a locally Hermitian symmetric space and its image $\Phi(\mathcal{T})$ under the period map Φ is a totally geodesic submanifold of the period domain D, both with the natural Hodge metrics. Moreover, assuming the global Torelli theorem in [Chen-Guan-Liu13] for Calabi-Yau manifolds, another consequence of this theorem is that the period map Φ embed the Teichmüller space $\mathcal{T}$ as a Zariski-open subset in the totally geodesic submanifold W.

Proof. Fix $p \in \mathcal{T}$, let X be the corresponding Calabi-Yau threefold in the versal family $\mathcal{U} \to \mathcal{T}$ and $\varphi_1, \cdots, \varphi_N \in \mathbb{H}^{0,1}(X, T^{1,0}X)$ be an orthonormal basis with respect to the Calabi-Yau metric. The orthonormal basis $\varphi_1, \cdots, \varphi_N$ will be used to define the flat affine coordinate t around $p \in \mathcal{T}$ by Theorem 2.1.5. Also, fix a nowhere vanishing holomorphic $(3,0)$-form Ω over X and $\eta_i = \varphi_i \lrcorner \Omega \in \mathbb{H}^{2,1}(X)$, then $[\Omega], [\eta_1], \cdots, [\eta_N], [\overline{\eta}_1], \cdots, [\overline{\eta}_N], [\overline{\Omega}]$ is a basis of $H^3(X)$ adapted the Hodge filtration of X.

Step 1: we give the expansion for the classic canonical family $[\Omega^{cc}(t)]$. Let's assume

$$\begin{bmatrix} [\varphi_i \lrcorner \eta_1] \\ \vdots \\ [\varphi_i \lrcorner \eta_N] \end{bmatrix} = A_i \begin{bmatrix} [\overline{\eta}_1] \\ \vdots \\ [\overline{\eta}_N] \end{bmatrix},$$

for some $N \times N$-matrix A_i, then, from the identity $[\varphi_i \lrcorner \overline{\eta}_j] = \delta_{ij}[\overline{\Omega}]$, we have

$$\begin{bmatrix} [\varphi_i \lrcorner \Omega] \\ [\varphi_i \lrcorner \eta] \\ [\varphi_i \lrcorner \overline{\eta}] \\ [\varphi_i \lrcorner \overline{\Omega}] \end{bmatrix} = \begin{bmatrix} 0 & e_i & & \\ & 0_{N\times N} & A_i & \\ & & 0_{N\times N} & e_i^T \\ & & & 0 \end{bmatrix} \begin{bmatrix} [\Omega] \\ [\eta] \\ [\overline{\eta}] \\ [\overline{\Omega}] \end{bmatrix} = E_i \begin{bmatrix} [\Omega] \\ [\eta] \\ [\overline{\eta}] \\ [\Omega] \end{bmatrix}, \tag{3.18}$$

where $e_i = (0, \cdots, 1, \cdots, 0)$, $[\eta] = \Big[[\eta_1], \cdots, [\eta_N]\Big]^T$, $[\overline{\eta}] = \Big[[\overline{\eta}_1], \cdots, [\overline{\eta}_N]\Big]^T$, $[\varphi_i \lrcorner \eta] = \Big[[\varphi_i \lrcorner \eta_1], \cdots, [\varphi_i \lrcorner \eta_N]\Big]^T$ and $[\varphi_i \lrcorner \overline{\eta}] = \Big[[\varphi_i \lrcorner \overline{\eta}_1], \cdots, [\varphi_i \lrcorner \overline{\eta}_N]\Big]^T$.

Moreover, if we define $A(t) = \sum_{i=1}^N t_i A_i$, then it is easy to check that

$$\sum_{i,j=1}^N t_i t_j [\varphi_i \lrcorner \varphi_j \lrcorner \Omega] = \sum_{i=1}^N t_i (t_1, \cdots, t_N) \begin{bmatrix} [\varphi_i \lrcorner \eta_1] \\ \vdots \\ [\varphi_i \lrcorner \eta_N] \end{bmatrix} = (t_1, \cdots, t_N) A(t) \begin{bmatrix} [\overline{\eta}_1] \\ \vdots \\ [\overline{\eta}_N] \end{bmatrix}.$$

And, from the identity $[\varphi_i \lrcorner \overline{\eta}_j] = \delta_{ij}[\overline{\Omega}]$, we know that

$$\begin{aligned}
\frac{1}{3!}\sum_{i,j,k=1}^{N} t_i t_j t_k [\varphi_i \lrcorner \varphi_j \lrcorner \varphi_k \lrcorner \Omega] &= \frac{1}{3!}\left(\sum_{i=1}^{N} t_i \varphi_i\right) \lrcorner \left(\sum_{j,k=1}^{N} t_j t_k [\varphi_j \lrcorner \varphi_k \lrcorner \Omega]\right) \\
&= \frac{1}{3!}(t_1, \cdots, t_N) A(t) \left(\sum_{i=1}^{N} t_i \varphi_i\right) \lrcorner \begin{bmatrix} [\overline{\eta}_1] \\ \vdots \\ [\overline{\eta}_N] \end{bmatrix} \\
&= \frac{1}{3!}(t_1, \cdots, t_N) A(t) \begin{bmatrix} t_1 \\ \vdots \\ t_N \end{bmatrix}.
\end{aligned}$$

Therefore, the classic canonical family

$$\Omega^{cc}(t) = \exp(\sum_{i=1}^{N} t_i \varphi_i) \lrcorner \Omega = \Omega + \sum_{i=1}^{N} t_i \varphi_i \lrcorner \Omega + \frac{1}{2!}\sum_{i,j=1}^{N} t_i t_j \varphi_i \lrcorner \varphi_j \lrcorner \Omega + \frac{1}{3!}\sum_{i,j,k=1}^{N} t_i t_j t_k \varphi_i \lrcorner \varphi_j \lrcorner \varphi_k \lrcorner \Omega,$$

has the following expansion

$$[\Omega^{cc}(t)] = \left[1, (t_1, \cdots, t_N), \tfrac{1}{2!}(t_1, \cdots, t_N) A(t), \tfrac{1}{3!}(t_1, \cdots, t_N) A(t) (t_1, \cdots, t_N)^T\right] \begin{bmatrix} [\Omega] \\ [\eta] \\ [\overline{\eta}] \\ [\overline{\Omega}] \end{bmatrix}. \quad (3.19)$$

Step 2: Let's construct a globally Hermitian symmetric space $W \subset D$ with the same dimension as the Teichmüller space $\mathcal{T}$. Define $O := \Phi(p)$ to be the base point or reference point. If we define $E_i := \Phi_*(\frac{\partial}{\partial t_i}) \in \mathfrak{g}^{-1,1}$, then $\mathfrak{a} = \mathrm{span}_{\mathbb{C}}\{E_1, E_2, \cdots, E_N\}$ is an abelian Lie subalgebra, see [Carlso-Müller-Stach-Peters03, Lemma 5.5.1, Page 173]. So we can define

$$W := \exp(\sum_{i=1}^{N} \tau_i E_i) \cap D,$$

which is a totally geodesic submanifold of D and a globally Herimitian symmetric space with respect to the natural Hodge metric. Let's define a map $\rho : \mathbb{C}^N \to \check{D}$ given by

$$\rho(\tau) = \exp(\sum_{i=1}^{N} \tau_i E_i)$$

where $\tau = (\tau_1, \cdots, \tau_N)$ is the standard coordinate of $\mathbb{C}^N$. Then the Hodge-Riemann bilinear relations define a bounded domain $\daleth \subset \mathbb{C}^N$, which is biholomorphic to W via the map ρ. As the $E_i's$ commute with each other, thus, for any $z, w \in \daleth$, we have

$$\begin{aligned}
&\rho(z)(\rho(w))^{-1} = \exp(\sum_{i=1}^N z_i E_i)\exp(-\sum_{i=1}^N w_i E_i) = \exp(\sum_{i=1}^N (z_i - w_i)E_i) = I \\
&\Longleftrightarrow z_i = w_i \quad \text{for} \quad 1 \le i \le N,
\end{aligned}$$

i.e., ρ is one-to-one, which means that it defines a global coordinate τ over W.

Step 3: $1 \Rightarrow 2$. First we show that there exists a local coordinate chart U_p around $p \in \mathcal{T}$ such that $\Phi(U_p) \subset W$. For any point $\widetilde{p} \in U_p$ with the flat affine coordinate t, there is a unique upper-triangle matrix $\sigma(t) \in N_+$, i.e., an nonsingular upper triangular block matrices with identity blocks in the diagonal, such that

$$\sigma(t)\left[[\Omega], [\eta], [\overline{\eta}], [\overline{\Omega}]\right]^T \tag{3.20}$$

is a basis of $H^3(X_{\widetilde{p}})$ adapted to the Hodge filtration at $X_{\widetilde{p}}$.

Also, we know that $[\Omega^c(t)] = [e^{\Phi(t)} \lrcorner \Omega]$ is a basis of $F^3(X_{\widetilde{p}})$, so $[\frac{\partial}{\partial t_i}\Omega^c(t)] \in F^2(X_{\widetilde{p}}), 1 \le i \le N$ by Griffiths' transversality. By the assumption of the vanishing of strong quantum correction at $p \in \mathcal{T}$ and Formula 3.19, we have

$$[\Omega^c(t)] \;=\; [\Omega^{cc}(t)] = \left[1, (t_1, \cdots, t_N), \tfrac{1}{2!}(t_1, \cdots, t_N)A(t), \tfrac{1}{3!}(t_1, \cdots, t_N)A(t)(t_1, \cdots, t_N)^T\right]\begin{bmatrix}[\Omega] \\ [\eta] \\ [\overline{\eta}] \\ [\overline{\Omega}]\end{bmatrix}.$$

And, for any $1 \le i \le N$, we have

$$\begin{aligned}
[\frac{\partial \Omega^c(t)}{\partial t_i}] \;&=\; [\varphi_i \lrcorner \Omega] + \sum_{j=1}^N t_j [\varphi_i \lrcorner \varphi_j \lrcorner \Omega] + \frac{1}{2!}\sum_{j,k=1}^N t_j t_k [\varphi_i \lrcorner \varphi_j \lrcorner \varphi_k \lrcorner \Omega] \\
&=\; [\eta_i] + (t_1, \cdots, t_N)\begin{bmatrix}\varphi_i \lrcorner \eta_1 \\ \vdots \\ \varphi_i \lrcorner \eta_N\end{bmatrix} + \frac{1}{2}(t_1, \cdots, t_N)A(t)\begin{bmatrix}[\varphi_i \lrcorner \overline{\eta}_1] \\ \vdots \\ [\varphi_i \lrcorner \overline{\eta}_N]\end{bmatrix},
\end{aligned}$$

which implies that

$$\begin{bmatrix}[\frac{\partial\Omega^c(t)}{\partial t_1}]\\ \vdots \\ [\frac{\partial\Omega^c(t)}{\partial t_N}]\end{bmatrix} = [\eta]+A(t)[\overline{\eta}]+\frac{1}{2!}(t_1,\cdots,t_N)A(t)\ [\overline{\Omega}]$$

$$= \begin{bmatrix}0_{n\times 1}, I_{n\times n}, A(t), \frac{1}{2}A^T(t)(t_1,\cdots,t_N)^T\end{bmatrix}\begin{bmatrix}[\Omega]\\ [\eta]\\ [\overline{\eta}]\\ [\overline{\Omega}]\end{bmatrix}.$$

Thus $\{[\Omega^c(t)],[\frac{\partial\Omega^c(t)}{\partial t_1}]\cdots[\frac{\partial\Omega^c(t)}{\partial t_N}]\}$ is a basis of $F^2(X_{\widetilde{p}})$ as they are linearly independent, which implies that the unique matrix $\sigma(t)\in N_+$ has the form

$$\sigma(t)=\begin{bmatrix} I & (t_1,\cdots,t_N) & \frac{1}{2}(t_1,\cdots,t_N)A(t) & \frac{1}{3!}(t_1,\cdots,t_N)A(t)(t_1,\cdots,t_N)^T\\ & I & A(t) & \frac{1}{2}A^T(t)(t_1,\cdots,t_N)^T\\ & & I & *\\ & & & I\end{bmatrix},$$

where $*$ represents an unknown column vector. Meanwhile, we know $\sigma(t)\in G_{\mathbb{C}}$, i.e.,

$$\sigma(T)^T\begin{bmatrix} & & & 1\\ & & -I_{N\times N} & \\ & I_{N\times N} & & \\ -1 & & & \end{bmatrix}\sigma(t)=\begin{bmatrix} & & & 1\\ & & -I_{N\times N} & \\ & I_{N\times N} & & \\ -1 & & & \end{bmatrix},$$

which implies that

$$\sigma(t)=\begin{bmatrix} I & (t_1,\cdots,t_N) & \frac{1}{2}(t_1,\cdots,t_N)A(t) & \frac{1}{3!}(t_1,\cdots,t_N)A(t)(t_1,\cdots,t_N)^T\\ & I & A(t) & \frac{1}{2}A^T(t)(t_1,\cdots,t_N)^T\\ & & I & (t_1,\cdots,t_N)^T\\ & & & I\end{bmatrix}.$$

Direct computation shows that

$$\sigma(t) = \exp\left(\begin{bmatrix} 0 & (t_1, \cdots, t_N) & & \\ & 0_{N\times N} & A & \\ & & 0_{N\times N} & (t_1, \cdots, t_N)^T \\ & & & 0 \end{bmatrix}\right), \tag{3.21}$$

i.e., $\Phi(\widetilde{p}) \in W$ for any point $\widetilde{p} \in U_p$, which implies that $\Phi(U_p) \subset W$.

Now we prove that $\Phi(\mathcal{T}) \subset W$ by an open-closed argument. For any point $q \in \mathcal{T}$ satisfying $\Phi(q) \in \Phi(\mathcal{T}) \cap W$, by the same argument for the point $q \in \mathcal{T}$, there exists a neighborhood U_q of q such that $\Phi(U_q) \subset W$, i.e., $\Phi(U_q) \subset \Phi(\mathcal{T}) \cap W$. By the local Torelli theorem for Calabi-Yau manifolds, $\Phi(U_q) \subset \Phi(\mathcal{T}) \cap W$ is an open neighborhood of $\Phi(q)$ as $\mathcal{T}$ and W having the same dimension, thus $\Phi(\mathcal{T}) \cap W \subset \Phi(\mathcal{T})$ is an open subset of $\Phi(\mathcal{T})$. On the other hand, $W = \exp(\sum_{i=1}^{N} \tau_i E_i) \cap D \subset D$ is a closed subset of the period domain D. The closedness of $W \subset D$ implies that $\Phi(\mathcal{T}) \cap W \subset \Phi(\mathcal{T})$ is also a closed subset. Therefore, $\Phi(\mathcal{T}) \cap W = \Phi(\mathcal{T})$, i.e., $\Phi(\mathcal{T}) \subset W$ as $\Phi(\mathcal{T})$ is connected and $\Phi(\mathcal{T}) \cap W$ is not empty.

Step 4: $2 \Rightarrow 1$. We only need to show that the strong quantum correction vanishes at point $p \in \mathcal{T}$. Assume the flat affine coordinate of $\widetilde{p} \in U_p$ is t where U_p is a local coordinate chart around $p \in \mathcal{T}$, then we know the canonical family of $(3,0)$-classes is given by

$$[\Omega^c(t)] = [e^{\Phi(t)} \lrcorner \Omega] \in F^3(X_{\widetilde{p}}).$$

Moreover, from the fact that $\Phi(\widetilde{p}) \in W$, there exists $\tau = (\tau_1, \cdots, \tau_N) \in \daleth$ such that

$$\begin{aligned} \Phi(\widetilde{p}) &= \exp(\sum_{i=1}^{N} \tau_i E_i) \begin{bmatrix} [\Omega] \\ [\eta] \\ [\overline{\eta}] \\ [\overline{\Omega}] \end{bmatrix} \\ &= \begin{bmatrix} 1 & (\tau_1, \cdots, \tau_N) & \frac{1}{2}(\tau_1, \cdots, \tau_N) A(\tau) & \frac{1}{3!}(\tau_1, \cdots, \tau_N) A(\tau) (\tau_1, \cdots, \tau_N)^T \\ & I_{N\times N} & A(\tau) & \frac{1}{2} A^T(\tau) (\tau_1, \cdots, \tau_N)^T \\ & & I_{N\times N} & (\tau_1, \cdots, \tau_N)^T \\ & & & 1 \end{bmatrix} \begin{bmatrix} [\Omega] \\ [\eta] \\ [\overline{\eta}] \\ [\overline{\Omega}] \end{bmatrix}, \end{aligned}$$

which is a basis adapted to the Hodge filtration over $X_{\widetilde{p}}$. In particular, we know the first element in the basis

$$\begin{aligned} A_{\widetilde{p}}(t) &= (1,(\tau_1,\cdots,\tau_N),\frac{1}{2!}(\tau_1,\cdots,\tau_N)A(\tau),\frac{1}{3!}(\tau_1,\cdots,\tau_N)A(\tau)(\tau_1,\cdots,\tau_N)^T)\begin{bmatrix}[\Omega]\\ [\eta]\\ [\overline{\eta}]\\ [\overline{\Omega}]\end{bmatrix} \\ &\in F^3(X_{\widetilde{p}})=H^{3,0}(X_{\widetilde{p}}).\end{aligned}$$

Thus, by the fact that $H^{3,0}(X_{\widetilde{p}})\cong\mathbb{C}$ and $[\Omega^c(t)]\in H^{3,0}(X_{\widetilde{p}})$, there exists a constant $\lambda\in\mathbb{C}$ such that

$$A_{\widetilde{p}}(t)=\lambda[\Omega^c(t))].$$

Also, we have

$$\mathrm{Pr}_{H^{3,0}(X)}(A_{\widetilde{p}}(t))=\mathrm{Pr}_{H^{3,0}(X)}([\Omega^c(t)])=[\Omega],$$

so $A_{\widetilde{p}}(t)=[\Omega^c(t))]$. Thus

$$\mathrm{Pr}_{H^{2,1}(X)}(\Omega^c(t))=\mathrm{Pr}_{H^{2,1}(X)}(A_{\widetilde{p}}(t))=\sum_{i=1}^N\tau_i[\eta_i], \tag{3.22}$$

and, from Corollary 2.1.8,

$$[\Omega^c(t)]=[\Omega]+\sum_{i=1}^N t_i[\eta_i]+A(t), \tag{3.23}$$

where $A(t)\in H^{1,2}(X)\oplus H^{0,3}(X)$. Then, project to $H^{2,1}(X)$, we have

$$\sum_{i=1}^N t_i[\eta_i]=\sum_{i=1}^N\tau_i[\eta_i],$$

which implies that $\tau_i=t_i,\ 1\le i\le N$ as $\{[\eta_i]\}_{i=1}^N$ is a basis of $H^{2,1}(X)$. Therefore,

$$[\Omega^c(t)]=(1,(t_1,\cdots,t_N),\frac{1}{2!}(t_1,\cdots,t_N)A(t),\frac{1}{3!}(t_1,\cdots,t_N)A(t)(t_1,\cdots,t_N)^T)\begin{bmatrix}[\Omega]\\ [\eta]\\ [\overline{\eta}]\\ [\overline{\Omega}]\end{bmatrix}=[\Omega^{cc}(t)],$$

by Formula 3.19, i.e., the strong quantum correction vanishes at $p\in\mathcal{T}$. □

3.4 Example: Compact Hyperkähler Manifolds

In Section 3.4.1, we review some preliminary results about Hyperkähler manifolds and the period domain of weight 2. In Section 3.4.2, we construct the canonical families of $(2,0)$ and $(2n,0)$-classes by using the canonical families of smooth forms $e^{\Phi(t)} \lrcorner \Omega^{2,0}$ and holomorphic forms $e^{\Phi(t)} \lrcorner \wedge^n \Omega^{2,0}$. In Section 3.4.3, we prove the expansions of the canonical families of $(2,0)$ and $(2n,0)$-classes are actually globally defined over the Teichmüller space $\mathcal{T}$.

3.4.1 Preliminary Results

In this section, we will review some preliminary results about Hyperkähler manifolds and the period domain. We define Hyperkähler manifolds as follows,

Definition 3.4.1. *Let X be a compact and simply-connected Kähler manifold of complex dimension $2n \geq 4$ such that there exists a non-zero holomorphic non-degenerate $(2,0)$-form $\Omega^{2,0}$ on X, unique up to a constant such that $\det(\Omega^{2,0}) \neq 0$ at each point $x \in X$ and $H^1(X, \mathcal{O}_X) = 0$. Then X is called a Hyperkähler manifold.*

The conditions on the holomorphic $(2,0)$-form $\Omega^{2,0}$ imply that $\dim_{\mathbb{C}} H^2(X, \mathcal{O}_{\mathrm{X}}) = 1$. A pair (X, L) consisting of a Hyperkähler manifold X of complex dimension $2n$ with $2n \geq 4$ and an ample line bundle L over X is called a polarized Hyperkähler manifold. By abuse of notation, the Chern class of L will also be denoted by L and thus $L \in H^2(X, \mathbb{Z})$. Let $\omega = \omega_g$ correspond to the Calabi-Yau metric in the class L, then

$$\mathbb{H}^{0,1}_L(X, T^{1,0}X) = \{\varphi \in \mathbb{H}^{0,1}(X, T^{1,0}X) | [\varphi \lrcorner \omega] = 0\}$$

And we know that if $\varphi \in \mathbb{H}^{0,1}_L(X, T^{1,0}X)$ is harmonic, then $\varphi \lrcorner \omega$ is a harmonic $(0,2)$-form. Thus we have the identification

$$\mathbb{H}^{0,1}_L(X, T^{1,0}X) = \{\varphi \in \mathbb{H}^{0,1}(X, T^{1,0}X) | \varphi \lrcorner \omega = 0\}$$

Furthermore, the primitive cohomology groups satisfy:

$$\begin{aligned} H_{pr}^{1,1}(X) &\cong \mathbb{H}_{pr}^{1,1}(X) = \{\eta \in H^{1,1}(X) | \eta \wedge \omega^{2n-1} = 0\} \\ H_{pr}^{2}(X) &= H^{2,0}(X) \oplus H_{pr}^{1,1}(X) \oplus H^{0,2}(X) \end{aligned}$$

The primitive cohomology group $H_{pr}^2(X)$ carry a nondegenerate bilinear form, the so-called Hodge bilinear form

$$Q(\alpha, \beta) = -\int_X \omega^{2n-2} \wedge \alpha \wedge \beta, \qquad \alpha, \beta \in H_{pr}^2(X), \tag{3.24}$$

which is evidently defined over $\mathbb{Q}$.

We consider the decreasing Hodge filtration $H_{pr}^2(M, \mathbb{C}) = F^0 \supset F^1 \supset F^2$ with condition

$$\dim_C F^2 = 1, \quad \dim_{\mathbb{C}} F^1 = b_2 - 2, \qquad \dim_{\mathbb{C}} F^0 = b_2 - 1. \tag{3.25}$$

Then the Hodge-Riemann relations are

$$Q(F^k, F^{3-k}) = 0, \tag{3.26}$$

$$Q(Cv, \overline{v}) > 0 \quad \text{if} \quad v \neq 0, \tag{3.27}$$

where C is the Weil operator given by $Cv = (\sqrt{-1})^{2k-2} v$ for $v \in H_{pr}^{k,2-k}(M) = F^k \cap \overline{F^{2-k}}$. The period domain D for polarized Hodge structures with data 3.25 is the space of all such Hodge filtrations

$$D = \{H_{pr}^2(X, \mathbb{C}) = F^0 \supseteq F^1 \supseteq F^2 | (3.25), \ \ (3.26) \ \text{ and } \ (3.27) \ \text{ hold}\}.$$

The compact dual $\check{D}$ of D is

$$\check{D} = \{H_{pr}^2(X, \mathbb{C}) = F^0 \supseteq F^1 \supseteq F^2 | (3.25) \ \text{ and } \ (3.26) \ \text{ hold}\}.$$

The period domain $D \subseteq \check{D}$ is an open subset. We may identify the period space D with the Grassmannian of positive 2-plans in $L^\perp$, and this gives us

$$D \cong SO(b_2 - 3, 2)/SO(2) \times SO(b_2 - 3),$$

which implies that the period domain D is a global Hermitian symmetric space.

Let $\mathcal{T} = \mathcal{T}_L$ be the Teichmüller space of the polarized (irreducible) Hyperkähler manifold (X, L), which is a smooth complex manifold. The reader can refer to [Verbitsky09] or [Huybrechts04] for the construction of Teichmüller space and moduli space of polarized Hyperkähler manifolds. The following result follows from [Verbitsky09] or [Chen-Guan-Liu13, Chen-Guan-Liu14],

Theorem 3.4.2. *The period map*

$$\Phi : \mathcal{T} \to D$$

is injective.

3.4.2 Local Family of $(2,0)$- and $(2n,0)$-Classes

In this section, we derive the expansions of the canonical families $[\mathbb{H}(e^{\Phi(t)} \lrcorner \Omega^{2,0})]$ and $[e^{\Phi(t)} \lrcorner \wedge^n \Omega^{2,0}]$, where $\Omega^{2,0}$ is a nowhere vanishing holomorphic $(2,0)$-form over the Hyperkäher manifold X. First we have the following Bochner's principle for compact Ricci-flat manifolds:

Proposition 3.4.3. *(Bochner's principle) On a compact Kähler Ricci-flat manifold, any holomorphic tensor field (covariant or contravariant) is parallel.*

The proof rests on the following formula, which follows from a tedious but straightforward computation [Bochner-Yano53, Page 142]: if τ is any tensor field,

$$\Delta(\|\tau\|^2) = \|\nabla\tau\|^2.$$

Therefore $\Delta(\|\tau\|^2)$ is nonnegative, hence 0 since its mean value over X is 0 by the Stokes' formula. It follows that τ is parallel.

Then we consider the canonical family of smooth $(2,0)$-forms $\Omega^{c;2,0}(t) = e^{\Phi(t)} \lrcorner \Omega^{2,0}$ whose harmonic projection has the following expansion,

Theorem 3.4.4. *Fix $p \in \mathcal{T}$, let (X, L) be the corresponding polarized Hyperkähler manifold, $\Omega^{2,0}$ be a nonzero holomorphic nondegenerate $(2,0)$-form over X and $\{\varphi_i\}_{i=1}^N$ be an orthonormal basis of $\mathbb{H}^{0,1}_L(X, T^{1,0}X)$ with respect to the Calabi-Yau metric. Then, in a neighborhood U of p, there exists a canonical family of smooth $(2,0)$-forms,*

$$\Omega^{c;2,0}(t) = e^{\Phi(t)} \lrcorner \Omega^{2,0},$$

which defines a canonical family of $(2,0)$-classes

$$[\mathbb{H}(\Omega^{c;2,0}(t))] = [\Omega^{2,0}] + \sum_{i=1}^{N} [\varphi_i \lrcorner \Omega^{2,0}] t_i + \frac{1}{2} \sum_{i=1}^{N} \left[\varphi_i \lrcorner \varphi_j \lrcorner \Omega^{2,0}\right] t_i t_j. \tag{3.28}$$

Proof. Let us consider the canonical family of smooth $(2,0)$-forms

$$\begin{aligned} e^{\Phi(t)} \lrcorner \Omega^{2,0} = & \sum_{i=1}^{N} \varphi_i \lrcorner \Omega^{2,0} t^i + \tfrac{1}{2} \sum_{i,j=1}^{N} \left(\varphi_i \lrcorner \varphi_j \lrcorner \Omega^{2,0} + \varphi_{ij} \lrcorner \Omega^{2,0}\right) t_i t_j \\ & + \sum_{|I| \geq 3} \left(\varphi_I \lrcorner \Omega^{2,0} + \sum_{J+K=I} \varphi_J \lrcorner \varphi_K \lrcorner \Omega^{2,0} \right). \end{aligned} \tag{3.29}$$

We claim that

Claim 3.4.5. *Suppose, for any multi-index J, K with $|J| \geq 2$, the harmonic projection $\mathbb{H}\left(\varphi_J \lrcorner \varphi_K \lrcorner \Omega^{2,0}\right) = c \cdot \overline{\Omega^{2,0}}$ for some constant c, then we have*

$$\mathbb{H}\left(\varphi_J \lrcorner \varphi_K \lrcorner \Omega^{2,0} \wedge \wedge^n \Omega^{2,0} \wedge \wedge^{n-1} \overline{\Omega^{2,0}}\right) = c \cdot \wedge^n \Omega^{2,0} \wedge \wedge^n \overline{\Omega^{2,0}}. \tag{3.30}$$

Proof. By the Hodge decomposition, we have

$$\varphi_J \lrcorner \varphi_K \lrcorner \Omega^{2,0} = c \cdot \overline{\Omega^{2,0}} + d\alpha_1 + d^* \alpha_2.$$

The proof bases on the following facts:

$$(\overline{\partial} \varphi)_{A_p, \overline{\alpha B_q}} = \sum_{\alpha} \nabla_{\overline{\alpha}} \varphi_{A_p, \overline{B}_q}, \qquad (\overline{\partial^*} \varphi)_{A_p, \overline{B}_q} = (-1)^{p+1} \sum_{\alpha, \beta} g^{\overline{\beta}\alpha} \nabla_\alpha \varphi_{A_p, \overline{\beta B_q}}, \tag{3.31}$$

and their conjugate, which can be found in [Morrow-Kodaira71]. From the formula

$$\nabla(\alpha \wedge \beta) = \nabla \alpha \wedge \beta + \alpha \wedge \nabla \beta, \tag{3.32}$$

and $\nabla\Omega^{2,0}=0$, which comes from the Bochner principal 3.4.3, we have

$$\nabla(\alpha\wedge\wedge^n\Omega^{2,0}\wedge\wedge^{n-1}\overline{\Omega^{2,0}})=\nabla\alpha\wedge\wedge^n\Omega^{2,0}\wedge\wedge^{n-1}\overline{\Omega^{2,0}}. \tag{3.33}$$

From the formulas 3.31 and their conjugate, we have

$$d(\alpha_1\wedge\wedge^n\Omega^{2,0}\wedge\wedge^{n-1}\overline{\Omega^{2,0}})=d\alpha_1\wedge\wedge^n\Omega^{2,0}\wedge\wedge^{n-1}\overline{\Omega^{2,0}},$$
$$d^*(\alpha_2\wedge\wedge^n\Omega^{2,0}\wedge\wedge^{n-1}\overline{\Omega^{2,0}})=d^*\alpha_2\wedge\wedge^n\Omega^{2,0}\wedge\wedge^{n-1}\overline{\Omega^{2,0}}.$$

Thus we have

$$\begin{aligned}&\varphi_J\lrcorner\varphi_K\lrcorner\Omega^{2,0}\wedge\wedge^n\Omega^{2,0}\wedge\wedge^{n-1}\overline{\Omega^{2,0}}\\ =\;& c\cdot\wedge^n\Omega^{2,0}\wedge\wedge^n\overline{\Omega^{2,0}}+d(\alpha_1\wedge\wedge^n\Omega^{2,0}\wedge\wedge^{n-1}\overline{\Omega^{2,0}})+d^*(\alpha_2\wedge\wedge^n\Omega^{2,0}\wedge\wedge^{n-1}\overline{\Omega^{2,0}}),\end{aligned}$$

which implies our claim. □

Direct computations show that

$$\left(\varphi_J\lrcorner\varphi_K\lrcorner\Omega^{2,0}\right)\wedge\wedge^n\Omega^{2,0}=\left(\varphi_J\lrcorner\Omega^{2,0}\right)\wedge\left(\varphi_K\lrcorner\Omega^{2,0}\right)\wedge\wedge^{n-1}\Omega^{2,0} \tag{3.34}$$

as a smooth $(2n,2)$-form. Therefore, for any multi-index J,K with $|J|\geq 2$, we have

$$\begin{aligned}&\int_X(\varphi_J\lrcorner\varphi_K\lrcorner\Omega^{2,0})\wedge\Omega^{2,0}\wedge\wedge^{n-1}\Omega^{2,0}\wedge\wedge^{n-1}\overline{\Omega^{2,0}}\\ =\;&\int_X(\varphi_J\lrcorner\Omega^{2,0})\wedge(\varphi_K\lrcorner\Omega^{2,0})\wedge\wedge^{n-1}\Omega^{2,0}\wedge\wedge^{n-1}\overline{\Omega^{2,0}}\\ =\;&\frac{1}{n}\int_X(\varphi_J\lrcorner\wedge^n\Omega^{2,0})\wedge(\varphi_K\lrcorner\Omega^{2,0})\wedge\wedge^{n-1}\overline{\Omega^{2,0}}\\ =\;&\frac{1}{n}\int_X\partial\psi_J\wedge(\varphi_K\lrcorner\Omega^{2,0})\wedge\wedge^{n-1}\overline{\Omega^{2,0}}\\ =\;&\frac{1}{n}\int_X\partial(\psi_J\wedge(\varphi_K\lrcorner\Omega^{2,0})\wedge\wedge^{n-1}\overline{\Omega^{2,0}})\qquad(\text{as }\ \varphi_K\lrcorner\Omega^{2,0}\wedge\wedge^{n-1}\overline{\Omega^{2,0}}\ \text{ is d-closed}).\\ =\;&0.\end{aligned}$$

On the other hand, by Claim 3.4.5 and the Stokes' formula, we have

$$0=\int_X(\varphi_J\lrcorner\varphi_K\lrcorner\Omega^{2,0})\wedge\Omega^{2,0}\wedge\wedge^{n-1}\Omega^{2,0}\wedge\wedge^{n-1}\overline{\Omega^{2,0}}=c\cdot\int_X\wedge^n\Omega^{2,0}\wedge\wedge^n\overline{\Omega^{2,0}}$$

So we have $c=0$, i.e., $\mathbb{H}\left((\varphi_J\lrcorner\varphi_K\lrcorner\Omega^{2,0})\right)=0$ for any multiple-index J,K with $|J|\geq 2$.

Next, for the $(1,1)$-form $\varphi_I\lrcorner\Omega^{2,0}$, we claim that

Claim 3.4.6. *1.* $\varphi_i \lrcorner \Omega^{2,0}$ *is harmonic for* $1 \leq i \leq N$.

2. For any multi-index I *with* $|I| \geq 2$, $\varphi_I \lrcorner \Omega^{2,0}$ *is* ∂*-exact, which implies that*

$$\mathbb{H}(\varphi_I \lrcorner \Omega^{2,0}) = 0.$$

Proof. 1. As φ_i is harmonic and $\Omega^{2,0}$ is parallel with respect to Levi-Civita connection. So, from the formula 3.31 and

$$\nabla(\varphi_i \lrcorner \Omega^{2,0}) = \nabla\varphi_i \lrcorner \Omega^{2,0} + \varphi_i \lrcorner \nabla\Omega^{2,0},$$

we have

$$d(\varphi_i \lrcorner \Omega^{2,0}) = d\varphi_i \lrcorner \Omega^{2,0} = 0,$$
$$d^*(\varphi_i \lrcorner \Omega^{2,0}) = d^*\varphi_i \lrcorner \Omega^{2,0} = 0,$$

i.e., $\varphi_i \lrcorner \Omega^{2,0}$ is harmonic for $1 \leq i \leq N$.

2. As $\Omega^{2,0}$ is a nowhere vanishing holomorphic $(2,0)$-form, so we can define $\Omega^{*2,0} \in A^0(X, \wedge^2 T^{1,0}X)$ by requiring $\langle \Omega^{2,0}, \Omega^{*2,0} \rangle = 1$ pointwise on X. Actually, in a local coordinate chart $\{z_1, z_2, \cdots, z_{2n}\}$, we can assume

$$\begin{aligned} \Omega^{2,0} &= \sum_{i,j=1}^{2n} a_{ij} dz_i \wedge dz_j \quad \text{with} \quad a_{ij} = -a_{ji} \\ \Omega^{*2,0} &= \sum_{i,j=1}^{2n} b_{ij} \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j} \quad \text{with} \quad b_{ij} = -b_{ji}. \end{aligned}$$

Then, if we define matrices $A = (a_{ij})$ and $B = (b_{ij})$, then $\det(A) \neq 0$ and

$$\langle \Omega^{2,0}, \Omega^{*2,0} \rangle = \sum_{i,j=1}^{2n} a_{ij} b_{ij} = tr(AB^T) = 1,$$

so, locally, the matrix B can be defined by

$$B = \frac{1}{2n}(A^{-1})^T.$$

And it is easy to check that this definition is independent of the local coordinates and $\nabla\Omega^{*2,0} = 0$ by the Bochner's principle 3.4.3. Then, by Theorem 2.1.5, we have

$$\varphi_I \lrcorner \wedge^n \Omega^{2,0} = \partial_X \psi_I, \qquad |I| \geq 2,$$

which implies that

$$\varphi_I \lrcorner \Omega^{2,0} = \wedge^{n-1}\Omega^{*2,0} \lrcorner (\varphi_I \lrcorner \wedge^n \Omega^{2,0}) = \wedge^{n-1}\Omega^{*2,0} \lrcorner \partial \psi_I = \partial(\wedge^{n-1}\Omega^{*2,0} \lrcorner \psi_I),$$

by formulas 3.31 and 3.32. □

Thus, the harmonic projection of the family of $(2,0)$-forms $e^{\Phi(t)} \lrcorner \Omega^{2,0}$ is given by

$$\mathbb{H}\left(e^{\Phi(t)} \lrcorner \Omega^{2,0}\right) = \Omega^{2,0} + \sum_{i=1}^{N} \varphi_i \lrcorner \Omega^{2,0} t_i + \frac{1}{2}\sum_{i=1}^{N} \mathbb{H}\left(\varphi_i \lrcorner \varphi_j \lrcorner \Omega^{2,0}\right) t_i t_j.$$

Theorem 3.4.4 is proved. □

Corollary 3.4.7. *Fix $p \in \mathcal{T}$, let (X, L) be the corresponding polarized Hyperkähler manifold, $\Omega^{2,0}$ be a non-zero holomorphic non-degenerate $(2,0)$-form over X and $\{\varphi_i\}_{i=1}^N$ be an orthonormal basis of $\mathbb{H}_L^{0,1}(X, T^{1,0}X)$ with respect to the Calabi-Yau metric. Then, in a neighborhood U of p, there exists a canonical family of holomorphic $(2n,0)$-forms,*

$$\Omega^c(t) = e^{\Phi(t)} \lrcorner \wedge^n \Omega^{2,0}$$

which defines a canonical family of $(2n,0)$-classes

$$[\Omega^c(t)] = \left[\wedge^n \Omega^{2,0}\right] + \sum_{i=1}^{N} \left[\varphi_i \lrcorner \wedge^n \Omega^{2,0}\right] t_i + \frac{1}{k!}\sum_{k=1}^{2n}\left(\sum_{1\le i_1 \le \ldots \le i_k \le N} \left[\varphi_{i_1} \lrcorner \ldots \lrcorner \varphi_{i_k} \lrcorner \wedge^n \Omega^{2,0}\right] t_{i_1} t_{i_2} \cdots t_{i_k}\right). \tag{3.35}$$

In particular, the expansion implies that the Teichmüller space $\mathcal{T}$ is a locally Hermitian symmetric space.

Proof. From the Proposition 2.1.7, we have the harmonic projection

$$\mathbb{H}(e^{\Phi(t)} \lrcorner \wedge^n \Omega^{2,0}) \in \mathbb{H}^{2n,0}(X_t).$$

And from Theorem 3.4.4, we know that

$$\mathbb{H}(e^{\Phi(t)} \lrcorner \Omega^{2,0}) \in \mathbb{H}^{2,0}(X_t),$$

therefore, we have

$$\mathbb{H}[\wedge^n \mathbb{H}(e^{\Phi(t)} \lrcorner \Omega^{2,0})] \in \mathbb{H}^{2n,0}(X_t).$$

Because of $\dim_{\mathbb{C}} \mathbb{H}^{2n,0}(X_t) = 1$, there exists $\lambda \in \mathbb{C}$ such that

$$\mathbb{H}(e^{\Phi(t)} \lrcorner \wedge^n \Omega^{2,0}) = \lambda \mathbb{H}[\wedge^n \mathbb{H}(e^{\Phi(t)} \lrcorner \Omega^{2,0})]. \tag{3.36}$$

On the other hand, we have

$$\mathrm{Pr}_{\mathbb{H}^{2n,0}(X)} \left(\mathbb{H}(e^{\Phi(t)} \lrcorner \wedge^n \Omega^{2,0}) \right) = \mathrm{Pr}_{\mathbb{H}^{2n,0}(X)} \left(\mathbb{H}[\wedge^n \mathbb{H}(e^{\Phi(t)} \lrcorner \Omega^{2,0})] \right) = \wedge^n \Omega^{2,0}. \tag{3.37}$$

Thus $\lambda = 1$, i.e.,

$$\begin{aligned} [\Omega^c(t)] = [e^{\Phi(t)} \lrcorner \wedge^n \Omega^{2,0}] &= \left[\wedge^n \left(\Omega^{2,0} + \sum_{i=1}^N \varphi_i \lrcorner \Omega^{2,0} t_i + \frac{1}{2} \sum_{i=1}^N \left(\varphi_i \lrcorner \varphi_j \lrcorner \Omega^{2,0} \right) t_i t_j \right) \right] \\ = \quad & \left[\wedge^n \Omega^{2,0} \right] + \sum_{i=1}^N \left[\varphi_i \lrcorner \wedge^n \Omega^{2,0} \right] t_i + \frac{1}{k!} \sum_{k=1}^{2n} \left(\sum_{0 \le i_1 \le \ldots \le i_k} \left[\varphi_{i_1} \lrcorner \ldots \lrcorner \varphi_{i_k} \lrcorner \wedge^n \Omega^{2,0} \right] t_{i_1} t_{i_2} \cdots t_{i_k} \right), \end{aligned}$$

by the formula 3.34. Thus, $[\Omega^c(t)]$ is a polynomial in terms of the flat affine coordinate $t = (t_1, \cdots, t_N)$. So, by the Corollary 3.2.4, the Teichmüller space $\mathcal{T}$ of polarized Hyperkähler manifolds is a locally Hermitian symmetric space. □

3.4.3 Global Family of (2,0)- and (2n,0)-Classes

In this section, we will show that the flat affine coordinate t is globally defined over the Teichmüller space $\mathcal{T}$, so do the expansions of the canonical families of $(2,0)$ and $(2n,0)$-classes.

Fix $p \in \mathcal{T}$, let (X, L) be the corresponding polarized Hyperkäher manifold. Fix a nowhere vanishing holomorphic $(2,0)$-form $\Omega^{2,0}$ over X and a basis of harmonic form $\eta_1, \cdots, \eta_N \in \mathbb{H}^{1,1}_{pr}(X)$. Then $[\Omega^{2,0}], [\eta_1], \cdots, [\eta_N], [\overline{\Omega^{2,0}}]$ is a basis of $H^2_{pr}(X, \mathbb{C})$. We normalize this basis such that $Q([\Omega^{2,0}], [\overline{\Omega^{2,0}}]) = -1$ and $Q([\eta_i], [\eta_j]) = \delta_{ij}$. And there exists an orthonormal basis $\{\varphi_i\}_{i=1}^N \subset \mathbb{H}^{0,1}(X, T^{1,0}X)$ such that $\varphi_i \lrcorner \Omega^{2,0} = \eta_i$ for $1 \le i \le N$, which will be used to defined the flat affine coordinate t around $p \in \mathcal{T}$ by Theorem 2.1.5.

Let $O = \Phi(p)$ be the base point or reference pint, we can parameterize the period domain D. Let $e_i = (0, \cdots, 1, \cdots, 0)$ with $1 \leq i \leq N$ be the standard basis of $\mathbb{C}^N$. Here we view e_i as a row vector, then we can define

$$E_i = \begin{bmatrix} 0 & e_i & 0 \\ 0 & 0 & e_i^T \\ 0 & 0 & 0 \end{bmatrix} \in \mathfrak{g}^{-1,1}.$$

It follows that $E_i E_j = 0$ if $i \neq j$ and

$$E_i^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

which implies that

$$\exp(\sum_{i=1}^N t_i E_i) = \begin{bmatrix} 1 & (\tau_1, \cdots, \tau_N) & \frac{1}{2}\sum_{i=1}^N \tau_i^2 \\ 0 & I_{N\times N} & (\tau_1, \cdots, \tau_N)^T \\ 0 & 0 & 1 \end{bmatrix}.$$

Let $\daleth \subset \mathbb{C}^N$ be the domain enclosed by the real hypersurface

$$1 - \sum_{i=1}^N |\tau_i|^2 + \frac{1}{4}|\sum_{i=1}^N \tau_i^2| = 0.$$

Let $\tau = (\tau_1, \cdots, \tau_N)$ be the standard coordinates on $\mathbb{C}^N$, then the map $\rho : \daleth \to D$ given by $\rho(t) = \exp(\sum_{i=1}^N \tau_i E_i)$ is a biholormorphic map. This is the Harish-Chandra realization [Harish-Chandra56] of the period domain D. Moreover, from the the global Torelli theorem (cf. [Verbitsky09] and [Chen-Guan-Liu13, Chen-Guan-Liu14]) for Hyperkähler manifolds, we know that the map

$$\rho^{-1} \circ \Phi : \mathcal{T} \to \daleth, \tag{3.38}$$

is an injective map. So the coordinate $\tau = (\tau_1, \cdots, \tau_N)$ are global coordinates on the Teichmüller space $\mathcal{T}$. We call it the Harish-Chandra coordinate. We know that, in a neighborhood of $p \in \mathcal{T}$, there is another flat affine coordinate t. Actually, these two coordinates coincide, and we have the following theorem:

Theorem 3.4.8. *Fix $p \in \mathcal{T}$, in a neighborhood U of p, the global Harish-Chandra coordinate τ coincide with the flat affine coordinate t. So the flat affine coordinate t is globally defined and the expansions of the canonical families of cohomology classes 3.28 and 3.35, i.e.,*

$$[\Omega^{2,0}] + \sum_{i=1}^{N} [\varphi_i \lrcorner \Omega^{2,0}] t_i + \frac{1}{2} \sum_{i=1}^{N} [\varphi_i \lrcorner \varphi_j \lrcorner \Omega^{2,0}] t_i t_j \in H^{2,0}(X_t),$$

$$\left[\wedge^n \Omega^{2,0}\right] + \sum_{i=1}^{N} \left[\varphi_i \lrcorner \wedge^n \Omega^{2,0}\right] t_i$$

$$+ \frac{1}{k!} \sum_{k=1}^{2n} \left(\sum_{1 \le i_1 \le \ldots \le i_k \le N} [\varphi_{i_1} \lrcorner \ldots \lrcorner \varphi_{i_k} \lrcorner \wedge^n \Omega^{2,0}] t_{i_1} t_{i_2} \cdots t_{i_k} \right) \in H^{2n,0}(X_t).$$

are globally defined over the Teichmüller space $\mathcal{T}$.

Proof. Let t be the flat affine coordinate of $q \in U_p$ where U_p is a local coordinate chart and τ be the Harish-Chandra coordinate of $q \in U_p$. We only need to show that $t = \tau$. From the definition of τ, we know that

$$\exp\left(\sum_{i=1}^{N} \tau_i E_i\right) \begin{bmatrix} [\Omega^{2,0}] \\ [\eta_1] \\ \vdots \\ [\eta_N] \\ [\overline{\Omega^{2,0}}] \end{bmatrix} = \begin{bmatrix} 1 & (\tau_1, \cdots, \tau_N) & \frac{1}{2}\sum_{i=1}^{N} \tau_i^2 \\ 0 & I_{N\times N} & (\tau_1, \cdots, \tau_N)^T \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} [\Omega^{2,0}] \\ [\eta_1] \\ \vdots \\ [\eta_N] \\ [\overline{\Omega^{2,0}}] \end{bmatrix},$$

is a basis of $H^2(X_q)$ adapted to the Hodge filtration of X_q. Consider the first element in the basis, we have

$$B_q(t) = [\Omega^{2,0}] + \sum_{i=1}^{N} \tau_i [\eta_i] + \frac{1}{2} \sum_{i=1}^{N} \tau_i^2 [\overline{\Omega^{2,0}}] \in H^{2,0}(X_q).$$

On the other hand, from Theorem 3.4.4, we know that

$$\begin{aligned} [\mathbb{H}(\Omega^{c;2,0}(t))] &= [\Omega^{2,0}] + \sum_{i=1}^{N} [\varphi_i \lrcorner \Omega^{2,0}] t_i + \frac{1}{2} \sum_{i=1}^{N} \left[\varphi_i \lrcorner \varphi_j \lrcorner \Omega^{2,0}\right] t_i t_j \\ &= [\Omega^{2,0}] + \sum_{i=1}^{N} [\eta_i] t_i + \frac{1}{2} \sum_{i=1}^{N} t_i^2 [\overline{\Omega^{2,0}}] \in H^{2,0}(X_q), \end{aligned}$$

in the flat affine coordinate t. Thus, by the fact that $H^{2,0}(X_q) \cong \mathbb{C}$, there exists a constant $\lambda \in \mathbb{C}$ such that

$$B_q(t) = \lambda[\mathbb{H}(\Omega^{c;2,0}(t))].$$

Moreover, we have

$$[\Omega^{2,0}] = \mathrm{Pr}_{H^{2,0}(X)}(B_q(t)) = \lambda \mathrm{Pr}_{H^{2,0}(X)}([\mathbb{H}(\Omega^{c;2,0}(t)0]) = \lambda[\Omega^{2,0}],$$

so $\lambda = 1$, i.e., $B_q(t) = [\mathbb{H}(\Omega^{c;2,0}(t))]$. Thus we know $t_i = \tau_i,\ 1 \le i \le N$, as $[\Omega^{2,0}], [\eta_1], \cdots, [\eta_N]$, $[\overline{\Omega^{2,0}}]$ is a basis of $H^2(X)$. $\square$

CHAPTER 4

Remarks on Chern Classes of Calabi-Yau Moduli

In this chapter, we prove that the first Chern form of the Calabi-Yau moduli $\mathcal{M}_m$, with the Hodge metric or the Weil-Petersson metric, represents the first Chern class of the canonical extensions of the tangent bundle to the compactification of the moduli space with normal crossing divisors.

4.1 Chern Forms of the Hodge Bundles

In Section 4.1.1, some essential estimates for the degeneration of Hodge metric of a variation of polarized Hodge structure near a normal crossing divisor was reviewed, which was used to derive the integrability of the Chern forms of subbundles and quotient bundles of a variation of polarized Hodge structure over a quasi-projective manifold in Section 4.1.2.

4.1.1 Degeneration of Hodge Structures

In this section, we consider a variation of polarized Hodge structure over S, where S is a quasi-projective manifold with $\dim_{\mathbb{C}} S = k$. For the definition of variation of Hodge structure, the reader can refer to Section 2.2. Let $\overline{S}$ be its compactification such that $\overline{S} - S$ is a divisor of normal crossings.

Let $(\mathcal{U}, s) \subset \overline{S}$ be a special coordinate neighborhood, i.e., a coordinate neighborhood isomophic to the polycylinder Δ^k such that

$$S \cap \mathcal{U} \cong \{s = (s_1, \cdots, s_l, \cdots, s_k) \in \Delta^k \mid \prod_{i=1}^{l} s_i \neq 0\} = (\Delta^*)^l \times \Delta^{k-l}\}.$$

where $\Delta,\ \Delta^*$ are the unit disk and the punctured unit disk in the complex plane, respectively. Consider the period map

$$\Phi : (\Delta^*)^l \times \Delta^{k-l} \longrightarrow \Gamma \backslash D,$$

where Γ is the monodromy group. Let U be the upper half plane of $\mathbb{C}$. Than $U^l \times \Delta^{k-l}$ is the universal covering space of $(\Delta^*)^l \times \Delta^{k-l}$, and we can lift Φ to a mapping

$$\widetilde{\Phi} : U^l \times \Delta^{k-l} \longrightarrow D.$$

Let $(z_1, \cdots, z_l, s_{l+1}, \cdots, s_k)$ be the coordinates of $U^l \times \Delta^{k-l}$ such that $s_i = e^{2\pi i z_i}$ for $1 \leq i \leq l$. Corresponding to each of the first l variables, we choose a monodromy transformation $\gamma_i \in \Gamma$, such that

$$\widetilde{\Phi}(z_1, \cdots, z_i + 1, \cdots z_l, s_{l+1}, \cdots, s_k) = \gamma_i(\widetilde{\Phi}(z_1, \cdots, z_i, \cdots z_l, s_{l+1}, \cdots, s_k))$$

holds identically for all variables. And the monodromy transformations γ_i's commute with each other. By a theorem of Borel(see [Schmid77], Lemma 4.5 on p. 230), after passing to a finite cover if necessary, the monodromy transformation γ_i around the punctures $s_i = 0$ is unipotent, i.e.,

$$\begin{cases} (\gamma_i - I)^{m-1} = 0 \\ [\gamma_i, \gamma_j] = 0, \end{cases}$$

for some positive integer m. Therefore, we can define the monodromy logarithm $N_i = \log \gamma_i$ by the Taylor's expansion

$$N_i = log\ \gamma_i := \sum_{j \geq 1} (-1)^{j+1} \frac{(\gamma_i - 1)^j}{j}, \quad \forall 1 \leq i \leq l,$$

then $N_i, 1 \leq i \leq l$ are nilpotent. Let $(v_.)$ be a flat multivalued basis of H over $\mathcal{U} \cap S$. The formula

$$(\widetilde{v_.})(s) := \exp\ (\frac{-1}{2\pi\sqrt{-1}} \sum_{i=1}^{l} \log\ s_i N_i)(v_.)(s)$$

give us a single-valued basis of H. Deligne's canonical extension $\widetilde{H}$ of H over $\mathcal{U}$ is generated by this basis $(\widetilde{v}_{\cdot})$(cf. [Schmid77]). And we have

Proposition 4.1.1. *If the local monodromy is unipotent, then the cononical extension is a vector bundle, otherwise it is a coherent sheaf.*

The construction of $\widetilde{H}$ is independent of the choice of the local coordinates $s_i' s$ and the flat multivalued basis $(v_{\cdot})$. For any holomorphic subbundle A of H, Deligne's canonical extension of A is defined to be $\widetilde{A} := \widetilde{H} \cap j_* A$ where $j : S \to \overline{S}$ is the inclusion map. Then we have the canonical extension of the Hodge filtration:

$$\widetilde{H} = \widetilde{F}^0 \supset \widetilde{F}^1 \supset \cdots \supset \widetilde{F}^n \supset 0,$$

which is also a filtration of locally free sheaves.

Let N be a linear combination of $N_i, 1 \leq i \leq l$, then N defines a weight flat filtration $W_{\bullet}(N)$ of H (cf. [Deligne71], [Schmid77]) by

$$0 \subset \cdots W_{i-1}(N) \subset W_i(N) \subset W_{i+1}(N) \subset \cdots \subset H.$$

Denote by $W_{\bullet}^j := W_{\bullet}(\sum_{\alpha=1}^{j} N_\alpha)$ for $j = 1, \cdots, l$, we can choose a flat multigrading

$$H = \sum_{\beta_1, \cdots, \beta_l} H_{\beta_1, \cdots, \beta_l},$$

such that

$$\bigcap_{j=1}^{l} W_{\beta_j}^j = \sum_{k_j \leq \beta_j} H_{k_1, \cdots, k_l}.$$

Let h be the Hodge metric on the variation of polarized Hodge structure H. In the special neighborhood $\mathcal{U}$, let v be a nonzero local multivalued flat section of a multigrading component $H_{k_1, \cdots, k_l}$, then $(\widetilde{v})(s) := \exp(\frac{-\sum_{i=1}^{l} \log s_i N_i}{2\pi\sqrt{-1}})v(s)$ is a local single-valued section of $\widetilde{H}$. And, there holds a norm estimate (Theorem 5.21 in [Cattani-Kaplan-Schimid86])

$$\| \widetilde{v}(s) \|_h \leq C_1 (\frac{-\log | s_1 |}{-\log | s_2 |})^{k_1/2} (\frac{-\log | s_2 |}{-\log | s_3 |})^{k_2/2} \cdots (-\log | s_l |)^{k_l/2}, \tag{4.1}$$

on the region

$$\Xi(N_1, \cdots, N_l) := \{(s_1, \cdots, s_l, \cdots, s_k) \in (\Delta^*)^l \times \Delta^{k-l} \mid\mid s_1 \mid\leq \cdots \mid s_k \mid\leq \epsilon\}$$

for some small $\epsilon > 0$, where C_1 is a positive constant dependent on the ordering of $\{N_1, N_2, \cdots, N_l\}$ and ϵ. Since the number of ordering of $\{N_1, N_2, \cdots, N_l\}$ is finite, for any flat multivalued local section v of H, there exist positive constants C_2 and M_2 such that

$$\parallel \widetilde{v}(s) \parallel_h \leq C_2(\prod_{i=1}^{l} -\log \mid s_i \mid)^{M_2}, \tag{4.2}$$

in the domain $\{(s_1, \cdots, s_l, \cdots, s_k) \mid 0 <\mid s_i \mid< \epsilon \ (i = 1, \cdots, l), \mid s_j \mid< \epsilon \ (j = l+1, \cdots, k)\}$. Moreover, since the dual H^* is also a variation of polarized Hodge structure, we then know that, for any flat multivalued local section v of H, there holds

$$C'(\prod_{i=1}^{l} -\log \mid s_i \mid)^{-M} \leq\parallel \widetilde{v}(s) \parallel_h \leq C''(\prod_{i=1}^{l} -\log \mid s_i \mid)^{M}, \tag{4.3}$$

where C' and C'' both only depend on ϵ.

4.1.2 Chern forms of the Hodge bundles

By the norm estimate in Section 4.1.1, E. Cattani, A. Kaplan and W. Schmid get the following result for the Chern forms of Hodge bundles over the quasi-projective manifold S, which is Corollary 5.23 in [Cattani-Kaplan-Schimid86].

Theorem 4.1.2. *Let S be a smooth variety, $\overline{S} \supset S$ be a smooth compactifiction such that $\overline{S} - S = D$ is a normal crossing divisor. If H is a variation of polarized Hodge structure over S with unipotent monodromies around D, then the Chern forms of Hodge metric on various Hodge bundles F^p/F^q define currents on the compactification $\overline{S}$. Moreover, the first Chern form represents the first Chern class of the canonical extension $\widetilde{F^p/F^q} \longrightarrow \overline{S}$.*

Base on this result, the proof of [Kollár85, Theorem 5.1] gives us the following result for any subbundle of the variation of Hodge structure H.

Theorem 4.1.3. *Let S be a smooth variety, $\overline{S} \supset S$ be a smooth compactifiction such that $\overline{S} - S = D$ is a normal crossing divisor. Let H be a variation of polarized Hodge structure over S with unipotent monodromies around D and A be a vector subbundle of H. Then the first Chern form of A with respect to the induced Hodge mtric is integrable. Moreover, let R_H be the curvature form with the induced Hodge metric over A, then we have*

$$\left(\frac{-1}{2\pi i}\right)^n \int_S (trR_H)^n = c_1(\widetilde{A})^n,$$

where $n = \dim_{\mathbb{C}} S$.

4.2 Chern Forms of the Calabi-Yau Moduli with the Hodge Metric

In Section 4.2.1, we construct various Hodge bundles over the Calabi-Yau moduli $\mathcal{M}_m$. In Section 4.2.2, by a key observation that the tangent bundle of the Calabi-Yau moduli is a subbundle of the variation of polarized Hodge structure $End(H^n) \to \mathcal{M}_m$, we get that the first Chern form of the Calabi-Yau moduli $\mathcal{M}_m$ are integrable with the Hodge metric.

4.2.1 Calabi-Yau Moduli and Hodge Bundles

For a polarized Calabi-Yau manifold X, the polarization L, which is an integer class, defines a map

$$L : H^n(X, \mathbb{Q}) \to H^{n+2}(X, \mathbb{Q}), \quad A \longmapsto L \wedge A.$$

We denote by $H^n_{pr}(X) = Ker(L)$ the primitive cohomology groups, where the coefficient ring can be $\mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$. We define $H^{k,n-k}_{pr}(X) = H^{k,n-k} \cap H^n_{pr}(X)$ and denote its dimension by $h^{k,n-k}$. Then we have the Hodge decomposition

$$H^n_{pr}(X) = H^{n,0}_{pr}(X) \oplus \cdots \oplus H^{0,n}_{pr}(X).$$

It is easy to see that for a polarized Calabi-Yau manifold, since $H^2(X, \mathcal{O}_X) = 0$, we have

$$H^{n,0}_{pr}(X) = H^{n,0}(X), \quad H^{n-1,1}_{pr}(X) = H^{n-1,1}(X).$$

The Poincaré bilinear form Q on $H^n_{pr}(X,\mathbb{Q})$ is defined by

$$Q(u,v) = (-1)^{\frac{n(n-1)}{2}} \int_X u \wedge v$$

for any d-closed n-forms u, v on X. Furthermore, Q is nondegenerate and can be extended to $H^n_{pr}(X,\mathbb{C})$ bilinearly. Moreover, it also satisfies the Hodge-Riemann relations 2.12 and 2.13. Therefore, the primitive cohomology groups of the fibers of the versal family $\mathcal{X}_{\mathcal{M}_m} \to \mathcal{M}_m$ of Calabi-Yau manifofds defines a polarized variation of Hodge structure over the moduli space $\mathcal{M}_m$, which is denoted by H^n in this thesis.

Actually, the flat bundle H^n contains a flat real subbundle $H^n_{\mathbb{R}}$, whose fiber corresponds to the subspaces $H^n_{pr}(X_p,\mathbb{R}) \subset H^n_{pr}(X_p)$; and $H^n_{\mathbb{R}}$, in turn, contains a flat lattice bundle $H^n_{\mathbb{Z}}$, whose fibers are the images of $H^n_{pr}(X_p,\mathbb{Z})$ in $H^n_{pr}(X_p,\mathbb{R})$. Moreover, there exist C^∞-subbundles $H^{p,q} \subset H^n$ with $p+q=n$, whose fibers over $p \in \mathcal{M}_m$ are $H^{p,q}_{pr}(X_p)$. For $0 \le k \le n$, $F^k = \oplus_{i \ge k} H^{i,n-i}$ are then holomorphic subbundles of H^n.

As the holomorphic bundle H^n defines a variation of polarized Hodge structure over $\mathcal{M}_m$, which is defined over $\mathbb{Z}$. Thus, by the functorial construction of variation of polarized Hodge structure, the holomorphic bundle $End(H^n) \to \mathcal{M}_m$ defines a variation of polarized Hodge structure over $\mathcal{M}_m$, which is defined over $\mathbb{Z}$. And then, we have the following main theorem in this section.

Proposition 4.2.1. *Let $\mathcal{M}_m$ be the Calabi-Yau moduli, then $End(H^n)$ defines a variation of polarized Hodge structure over $\mathcal{M}_m$, which is defined over $\mathbb{Z}$. Moreover, with the induced Hodge metric over the Calabi-Yau moduli $\mathcal{M}_m$, the tangent bundle*

$$T\mathcal{M}_m \hookrightarrow End(H^n), \tag{4.4}$$

is a holomorphic subbundle of $End(H^n) \to \mathcal{M}_m$ with the induced Hodge metric.

Proof. By Griffiths' transversality, we have the tangent map of the period map $\Phi_m : \mathcal{M}_m \to D/\Gamma$ satisfies that

$$(\Phi_m)_*(v) \in \bigoplus_{k=1}^{n} \mathrm{Hom}\left(F_p^k/F_p^{k+1}, F_p^{k-1}/F_p^k\right) \quad \text{for any} \quad p \in \mathcal{M}_m \ \text{ and } \ v \in T_p^{1,0}\mathcal{M}_m$$

with $F^{n+1} = 0$, or equivalently, $(\Phi_m)_*(v) \in \bigoplus_{k=0}^{n} \mathrm{Hom}(F_p^k, F_p^{k-1}) \subset End(H_p^n)$. Therefore, the image of the tangent bundle $\mathrm{Im}((\Phi_m)_*)$ is a subbundle of the holomorphic bundle $End(H^n) \to \mathcal{M}_m$.

Moreover, by the local Toreli theorem for Calabi-Yau manifolds, the map $(\Phi_m)_*$ is injective. So we can view the tangent bunlde $T\mathcal{M}_m$ as a subbundle of the holomorphic bundle $End(H^n) \to \mathcal{M}_m$ via the tangent map $(\Phi_m)_*$.

□

4.2.2 Chern Forms of Calabi-Yau Moduli with the Hodge Metric

As the Calabi-Yau moduli $\mathcal{M}_m$ is quasi-projective, see Theorem 2.1.2, we know that there is a compact projective manifold $\overline{\mathcal{M}}_m$ such that $\overline{\mathcal{M}}_m - \mathcal{M}_m = D$ is a normal crossing divisor. Also, the local mondromy of the variation of polarized Hodge structure around the divisor is at least quasi-unipotent. Thus after passing to a finite ramified cover if necessary, the local monodromy becomes unipotent. Therefore, without loss of generality, we can assume the Hodge bundles have canonical extensions, which are vector bundles over the compactification $\overline{\mathcal{M}}_m$ of the Calabi-Yau moduli $\mathcal{M}_m$, due to Proposition 4.1.1.

By Proposition 4.2.1, the tangent bundle of the moduli space $T\mathcal{M}_m$ is a holomorphic subbundle of the variation of Hodge structure $End(H^n) \to \mathcal{M}_m$. So we can make the canonical extensions to both $T\mathcal{M}_m \to \mathcal{M}_m$ and $End(H^n) \to \mathcal{M}_m$ to get

$$\begin{array}{ccc} \widetilde{T\mathcal{M}_m} & \hookrightarrow & \widetilde{End(H^n)} \\ \downarrow & & \downarrow \\ \overline{\mathcal{M}}_m & \longrightarrow & \overline{\mathcal{M}}_m. \end{array}$$

This is the same canonical extension of the tangent bundle used by the paper [Lu-Sun06], the reader can refer to [Lu-Sun06, Remark 4.3].

And we know the canonical extension $\widetilde{End(H^n)} \to \overline{\mathcal{M}}_m$ is generated by the basis

$$(\widetilde{v_\cdot})(s) := \exp\ (\frac{-1}{2\pi\sqrt{-1}} \sum_{i=1}^{l} \log\ s_i N_i)(v_\cdot)(s)$$

where $(v.)(s)$ is a flat multivaued basis of $End(H^n)$ locally. And, by the Estimate 4.2, we know that the Hodge norm of the basis $(\widetilde{v_\cdot})(s)$ have at most logarithmic singularities over the divisor D. Moreover, by the claim on [Zuo00, Page 297], the subbundle $T\overline{\mathcal{M}}_m(-\log D) \to \widetilde{End(H^n)}$ is generated by those local sections, whose Hodge norms have at most logarithmic singularities over the divisor D. Thus, the canonical extension $\widetilde{T\mathcal{M}_m} \to \overline{\mathcal{M}}_m$ and the log tangent bundle $T\overline{\mathcal{M}}_m(-\log D)$ have the same local generating sections, which implies that the canonical extension $\widetilde{T\mathcal{M}_m} \to \overline{\mathcal{M}}_m$ of the tangent bundle $T\mathcal{M}_m \to \mathcal{M}_m$ is the same as the log tangent bundle of the compactifiction, i.e., $T\overline{\mathcal{M}}_m(-\log D) \to \overline{\mathcal{M}}_m$.

In this paper, we will continue to use the canonical extension $\widetilde{T\mathcal{M}_m} \to \overline{\mathcal{M}}_m$ of the tangent bundle which is not the tangent bundle $T\overline{\mathcal{M}}_m$ of the compactification of the moduli space. And, by Theorem 4.1.2 and Theorem 4.1.3, we have

Theorem 4.2.2. *The first Chern form of the Calabi-Yau moduli $\mathcal{M}_m$ with the induced Hodge metric define currents over the compactification $\overline{\mathcal{M}}_m$ with normal crossing boundary divisors. Moreover, let R_H be the curvature form of $T\mathcal{M}_m$ with the induced Hodge metric, then we have*

$$\left(\frac{-1}{2\pi i}\right)^N \int_{T\mathcal{M}_m} (trR_H)^N = c_1(\widetilde{T\mathcal{M}_m})^N$$

where $N = \dim_{\mathbb{C}} \mathcal{M}_m$.

Proof. By Proposition 4.2.1, with the Hodge metric, the tangent bundle $T\mathcal{M}_m$ of the Calabi-Yau moduli $\mathcal{M}_m$ is a holomorphic subbundle of the variation of polarized Hodge structure $End(H^n) \to \mathcal{M}_m$. So $T\mathcal{M}_m$ has the canonical extension, which give us a holomorphic vector bundle $\widetilde{T\mathcal{M}_m} \subset \widetilde{End(H^n)}$ over $\overline{\mathcal{M}}_m$. Therefore, by Theorem 4.1.3, the first Chern form of $T\mathcal{M}_m \to \mathcal{M}_m$ define currents over the compactification $\overline{\mathcal{M}}_m$ of $\mathcal{M}_m$, which represent the first Chern class of the vector bundle $\widetilde{T\mathcal{M}_m} \to \overline{\mathcal{M}}_m$ with the induced Hodge metric. □

4.3 Chern Forms of the Calabi-Yau Moduli with the Weil-Petersson Metric

In this section, by the standard isomorphism $T\mathcal{M}_m \cong (F^n)^* \otimes F^{n-1}/F^n$ under the Weil-Petersson geometry of the Calabi-Yau moduli, we get that the Chern forms of Calabi-Yau moduli $\mathcal{M}_m$ are integrable with the Weil-Petersson metric.

For each fiber $X = X_s$, we assign the Calabi-Yau metric $g(s)$ in the polarization Kähler class. Using the fact that the global holomorphic n-form $\Omega = \Omega(s)$ is flat with respect to $g(s)$, it can be shown that the Weil-Petersson metric has the following expression

$$g_{WP}(v, w) = -\frac{\widetilde{Q}(i_v\Omega, \overline{i_w\Omega})}{\widetilde{Q}(\Omega, \overline{\Omega})}. \tag{4.5}$$

Here, for convenience, we write $\widetilde{Q}(\cdot,\cdot) = (\sqrt{-1})^n Q(\cdot,\cdot)$, where Q is the intersection product. The reader can refer to Section 2.3.1 and [Lu-Sun06] for details of the definition.

Formula (4.5) of the Weil-Pertersson metric implies that the natural map $H^1(X, T_X) \longrightarrow \mathrm{Hom}(F^n, F^{n-1}/F^n)$ via the interior product $v \longmapsto i_v\Omega$ is an isometry from the tangent bundle $T\mathcal{M}_m$ with the Weil-Petersson metric to the Hodge bundle $(F^n)^* \otimes F^{n-1}/F^n$ with the induced Hodge metric. So the Weil-Petersson metric is precisely the metric induced from the first piece of the Hodge metric on the horizontal tangent bundle over the period domain.

More precisely, for the Calabi-Yau moduli $\mathcal{M}_m$, we have the following period map from the moduli space to the period domain of Hodge structures:

$$\Phi_m : \mathcal{M}_m \longrightarrow D/\Gamma, \tag{4.6}$$

where Γ denotes the global monodromy group which acts properly and discontinuously on the period domain D. By going to finite covers of $\mathcal{M}_m$ and D/Γ, we may also assume D/Γ is smooth without loss of generality.

Thus, the differential of the period map gives us the infinitesimal period map at $p \in \mathcal{M}_m$:

$$(\Phi_m)_* : T_p\mathcal{M}_m \longrightarrow \mathrm{Hom}(F^n, F^{n-1}/F^n) \oplus \mathrm{Hom}(F^{n-1}/F^n, F^{n-2}/F^{n-1}) \oplus \cdots .$$

is an isomorphism in the first piece. By using this isomorphism and Theorem 4.1.3, we have the following result, which is [Lu-Douglas13, Theorem 6.3]. Our proof is different and much simpler.

Theorem 4.3.1. *The Chern forms of the Calabi-Yau moduli $\mathcal{M}_m$ with the Weil-Petersson metric define currents over the compactification $\overline{\mathcal{M}}_m$ of $\mathcal{M}_m$. Moreover, the first Chern form represent the first Chern class of the quotient bundle $\widetilde{(F^n)^* \otimes F^{n-1}}/\widetilde{(F^n)^* \otimes F^n} \to \overline{\mathcal{M}}_m$.*

Proof. Equipped with the Weil-Petersson metric, the tangent bundle $T\mathcal{M}_m$ of the Calabi-Yau moduli $\mathcal{M}_m$ is isomorphic to

$$(F^n)^* \otimes F^{n-1}/F^n \cong (F^n)^* \otimes F^{n-1}/(F^n)^* \otimes F^n,$$

which is a quotient of subbundles of the variation of polarized Hodge structure $End(H^n) \to \mathcal{M}_m$. Here the Hodge bundles F^k's are all equipped with their natural Hodge metrics. So, by Theorem 4.1.2, the Chern forms of $T\mathcal{M}_m$ define currents over the compactification $\overline{\mathcal{M}}_m$ of $\mathcal{M}_m$. Moreover, the first Chern form of the tangent bundle $T\mathcal{M}_m$ with the Weil-Petersson metric represent the first Chern class of the canonical extension

$$\widetilde{T\mathcal{M}}_m \cong \widetilde{(F^n)^* \otimes F^{n-1}}/\widetilde{(F^n)^* \otimes F^n}.$$

□

As a corollary, we have the following result on the Chern numbers,

Corollary 4.3.2. *Let f be an invariant polynomial on $Hom(T\mathcal{M}_m, T\mathcal{M}_m)$ and R_{WP} represent the curvature form of the Weil-Petersson metric on the Calabi-Yau moduli $\mathcal{M}_m$. Then we have*

$$\int_{\mathcal{M}_m} tr(f(R_{WP})) < \infty. \tag{4.7}$$

Proof. The proof follows directly from Theorem 4.3.1. □

As pointed out in the introduction, it follows from Theorem 4.1.2 easily that the first Chern form of all of the Hodge bundles with Hodge metrics also represent the Chern classes of their canonical extensions. Finally note that the Kähler form of the Weil-Petersson metric is equal to the first Chern form of the Hodge bundle F^n with its Hodge metric,

$$\omega_{WP} = c_1(F^n)_H,$$

so we easily deduce that the Weil-Petersson volume is finite and is a rational number, as proved in [Lu-Sun06] and [Todorov89] by computations.

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