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Analytic torsion of noncompact manifolds

A dissertation submitted in partial satisfaction
of the requirements for the degree

Doctor of Philosophy
in
Mathematics

by

Debin Liu

Committee in charge:

Professor Xianzhe Dai, Chair
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Professor Rugang Ye

June 2026

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May 2026

Analytic torsion of noncompact manifolds

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by

Debin Liu

To my family.

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- [2] X. Dai and D. Liu, "Adiabatic Limit and Analytic Torsion of Vector Bundles", 54 pages, arXiv preprint arXiv:2512.18602, 2025
- [3] D. Liu and P. Zhang, "Hermitian-Einstein Metrics for Higgs Bundles Over Complete Hermitian Manifolds", 15 pages, *Acta Math Sci*, 40, 211–225 (2020) doi:10.1007/s10473-020-0114-z

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- [4] *The 30th Southern California Geometric Analysis Seminar*, UC Irvine
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- [7] *Workshop on Minimal Surfaces and Mean Curvature Flow 2024*, Peking University
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- [9] *Frontiers of Geometric Analysis*, UC Santa Cruz June 2024
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- [11] *The 29th Southern California Geometric Analysis Seminar*, UC San Diego Apr. 2024
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Jan. 2026

Abstract

Analytic torsion of noncompact manifolds

by

Debin Liu

Motivated by the desingularization of certain orbifolds and the construction of geometric quantization, we consider a vector bundle $E^{n+k} \rightarrow M^n$ over a closed manifold M , and introduce a regularization of analytic torsion on E by replacing the Hodge Laplacian in the torsion zeta function with a Witten Laplacian. We also investigate the relationship of the primary and secondary invariants between the total space and the base manifold. By passing to the adiabatic limit, we show that the index of E coincides with the index of M . For analytic torsion, with the additional assumption that k is even and n is odd, we show that the Quillen metric associated with the analytic torsion of E is independent of the metric chosen, and equals the usual Quillen metric of M .

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Chapter 1

Introduction

A fundamental problem in geometry and topology is the classification of manifolds in terms of their homeomorphism types. Unfortunately, this problem is difficult even for 3-manifolds, as two lens spaces may share exactly the same homotopic and homological information, but are still not homotopy equivalent. In 1935, in order to classify 3-dimensional lens spaces, K. Reidemeister [1] introduced a topological invariant, called R-torsion, associated with the simplicial chain complex of universal covers. It is the first topological invariant that can distinguish closed manifolds that are homotopically equivalent but not homeomorphic. In fact, 3-dimensional lens spaces can be completely classified using their R-torsions and the fundamental groups.

For a closed manifold M , its R-torsion $\tau(M)$ is always a positive number and can be expressed in terms of the determinant of the combinatorial Laplacian. In the 1970's, D. B. Ray and I. M. Singer [2] [3] introduced analytic torsion as an analytic analogue of R-torsion. It is defined by replacing the combinatorial Laplacian with the Hodge Laplacian. Besides the similarity in the definitions, they also observed that R-torsion and analytic torsion enjoy the same properties and indeed agree on lens spaces. This led them to conjecture that on a closed manifold satisfying a certain acyclic property,

R-torsion coincides with the analytic torsion. This conjecture was verified by J. Cheeger [4] [5] and W. Müller [6] independently, and this result is now known as Cheeger-Müller theorem.

We will discuss analytic torsion in more detail in Section 2.6. For clarity, here we briefly introduce its construction. As the manifold M is closed, the heat operator is of trace class. Using supertrace, one can define the torsion zeta function to be

$$\zeta_T(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \operatorname{Tr}_s (N e^{-t\Delta}) dt, \quad (1.1)$$

where $\Gamma(s)$ is the Gamma function, N is the number operator, and Δ is the Hodge Laplacian with its null space removed. Intuitively, ζ_T defines a holomorphic function only when $\operatorname{Re}(s)$ is sufficiently large. But it can be proved, by using the asymptotic expansion of the heat kernel, that $\zeta_T(s)$ has an analytic continuation as a meromorphic function that is regular near $s = 0$. The Ray-Singer analytic torsion is then defined as follows.

Definition 1.0.1 *The analytic torsion $T(M)$ of (M, g) is defined by*

$$\log T(M) = -\frac{1}{2} \zeta'_T(0). \quad (1.2)$$

Analytic torsion has remarkable applications in seemingly unrelated fields and problems. For instance, in number theory it has been used by N. Bergeron and A. Venkatesh [7] to quantify the exponential growth of torsion in the homology of arithmetic groups (see also the work of F. Calegari and A. Venkatesh [8], where analytic torsion is used to study the arithmetic Langlands program). On the other hand, D. Fried [9] introduced a Ruelle-type zeta function $R(s)$ for a closed oriented hyperbolic manifold M , which is defined in terms of the lengths of prime closed geodesics. Notably, he showed that its

value at zero is related to the corresponding analytic torsion as $R(0) = T(M)^2$. This suggests that analytic torsion can be used to count the number of closed periodic orbits of a hyperbolic geodesic flow. Analytic torsion can also be used to study the volume of hyperbolic manifolds. In [10], W. Müller considered the analytic torsions $T(M, \rho_m)$ of a closed hyperbolic 3-manifold M associated with a family of standard representations $\{\rho_m\}$ of $\mathrm{SL}(2; \mathbb{C})$. He then proved that the volume $\mathrm{vol}(M)$ can be completely determined by this family of analytic torsions $T(M, \rho_m)$ in the sense that when m tends to infinity, the leading coefficient of the asymptotic formula of $T(M, \rho_m)$ is given by $\mathrm{vol}(M)$.

Despite the well developed theories and the far-reaching applications of analytic torsions of closed manifolds, very little is known for noncompact manifolds. The main reason behind this is that the Hodge Laplacian in general may have a nonempty continuous spectrum, which prevents the heat operator from being trace-class. To make sense of the torsion zeta function in this case, one must introduce a regularization to recover the trace-class property. In fact, all previous work along this line relies on a local regularization of the heat trace. This is usually achieved by taking a compact exhaustion $\{M_Y\}$ and defining the regularized heat trace to be the constant term in the asymptotic formula of the heat trace over M_Y as sending $Y \rightarrow \infty$.

In this dissertation, following [11] and [12], we will introduce a new global regularization of the heat operator using Witten deformation. This is based on a joint work with Xianzhe Dai [13]. Instead of a general complete noncompact manifold, we will consider a vector bundle E over a closed manifold M . The motivation for this consideration is twofold. First, the distance function along the fibers provides us with a natural Morse-Bott function, which will be used in the construction of the Witten deformation. Second, this setting arises naturally as a local model in many contexts.

For example, in the work of X. Dai and K.-I. Yoshikawa [14], the authors introduced a holomorphic torsion invariant for a type of compact complex surface Y with

certain orbifold singularities. Their idea was to desingularize the double covering X of Y to obtain a $K3$ surface $\tilde{X}$. In this situation, the singular set of Y must be discrete: $\text{Sing}(Y) = \{p_1, \dots, p_k\}$, and there is a blowing-down of a disjoint union of $\mathbb{C}P^1$'s:

$$\pi : (\tilde{X}, \tilde{X}^\theta) \rightarrow (X, \text{Sing } X), \text{ where } \tilde{X}^\theta = E_1 \sqcup \dots \sqcup E_k, \text{ and each } E_i \cong \mathbb{C}P^1. \quad (1.3)$$

Moreover, they constructed a two-parameter family of Kähler metrics on $\tilde{X}$, which converges to an orbifold Kähler metric on X , by gluing the Eguchi-Hanson instanton at each p_i and a Kähler metric on X . By taking a tubular neighborhood near the blowing-up of each singular point, the local picture is modeled on an open neighborhood of the zero section of $T^*\mathbb{C}P^1 \rightarrow \mathbb{C}P^1$, where the cotangent bundle $T^*\mathbb{C}P^1$ is equipped with a family of asymptotically locally Euclidean (ALE) Kähler metrics. Therefore, to compute the torsion invariant using this local data, one is forced to study the analytic torsion of the cotangent bundle.

Another example comes from geometric quantization. Let (X, ω) be a symplectic manifold that admits a Hamiltonian action of a compact connected Lie group G with Lie algebra $\mathfrak{g}$. This induces a smooth Morse-Bott function $\mu : X \rightarrow \mathfrak{g}^*$ which is usually called the moment map. If $0 \in \mathfrak{g}^*$ is a regular value of μ and G acts on $X_0 := \mu^{-1}(0)$ freely, then X_0 is a G -invariant co-isotropic submanifold of X , and the quotient $X_G := X_0/G$ is a smooth manifold with a unique symplectic form ω_G induced by ω . In this way, one obtains the Marsden-Weinstein symplectic reduction (X_G, ω_G) . X is called pre-quantizable if the symplectic form is integrable: $[\omega] \in H^2(X; \mathbb{Z})$. In this situation, there is an induced line bundle $L \rightarrow X$, called the pre-quantum line bundle of X , and a finite dimensional Hilbert space X_{quantum} . It turns out that X_{quantum} comes with a natural G action, and the Guillemin-Sternberg conjecture [15] predicts that the geometric quantization should commute with the symplectic reduction, namely $(X_{\text{quantum}})_G = (X_G)_{\text{quantum}}$. Inspired by

this conjecture, it is natural to wonder if analytic torsion "commutes" with the symplectic reduction. To attack this problem, it is crucial to understand how the analytic torsion behaves near X_0 , for which the local picture is the normal bundle $NX_0 \rightarrow X_0$.

The dissertation is organized as follows. In Chapter 2 we collect some necessary preliminaries in geometry, algebra, and analysis. In Chapter 3, we introduce a Witten Laplacian with a particular Morse-Bott function as the potential function and calculate the asymptotic expansion of the supertrace. Then we show the equivalence between two indexes. In Chapter 4, we introduce a regularization of analytic torsion for a vector bundle and show that the corresponding Quillen metric is independent of the metric chosen. In Chapter 5, we prove that the Quillen metric of the total space of a vector bundle coincides with the Quillen metric of the base manifold.

Chapter 2

Preliminary

In this chapter, we provide a brief review of the geometry of fibration, Clifford multiplication, Dirac operator and Witten deformation, functional calculus for closed operators, and the construction of some spectral topological invariants on closed manifolds.

2.1 Elementary geometry of fiber bundles

Let $Y^k \hookrightarrow E^{k+n} \xrightarrow{\pi} M^n$ be a smooth fibration with base manifold M and fiber Y . Then there is a unique subbundle $VE \subset TE$, called the vertical subbundle, which is defined as the kernel of the projection:

$$VE = \ker \pi_* = \{X \in \Gamma(TE) \mid \pi_*(X) = 0\}. \quad (2.1)$$

For any $x \in M$, we denote by $Y_x = \pi^{-1}(x)$ the fiber of E at x . Then for any $p \in Y_x$, $V_p E$ is exactly the tangent space of the fiber Y_x at p , namely $V_p E \cong T_p Y_x$.

Now we fix an arbitrary Ehresmann connection on the fiber bundle $E \rightarrow M$. This

amounts to a splitting of the tangent bundle:

$$TE = HE \oplus VE, \quad (2.2)$$

where HE is called the horizontal subbundle associated with this connection. One may observe that as $(\pi_*)_p : T_p E \rightarrow T_{\pi(p)} M$ is a linear map with kernel $\ker(\pi_*)_p = V_p E$, there is a natural identification: $H_p E \cong T_p E / V_p E \cong T_{\pi(p)} M$.

Let us introduce some notations. Throughout this paper, we will use P^H and P^V to denote the projections of TE onto HE and VE respectively. Let $X \in \Gamma(TE)$ denote an arbitrary vector field on E , with $X^H \in \Gamma(HE)$ and $X^V \in \Gamma(VE)$ being its horizontal and vertical components. As a consequence, we have $X = X^H + X^V$. We denote by $U \in \Gamma(TM)$ a tangent vector on the base manifold M . It is well-known that U admits a unique horizontal lift $\tilde{U} \in \Gamma(HE)$ satisfying $\pi_*(\tilde{U}) = U$. If no confusion arises, we may also write $\tilde{U}$ as U . Besides, we let $W \in \Gamma(VE)$ denote a vertical vector field.

Regarding the integrability of these subbundles, one has the following classic results:

Proposition 2.1.1 (1) VE is integrable, namely $[W_1, W_2] \in \Gamma(VE)$ for any vertical vector fields $W_1, W_2 \in \Gamma(VE)$.

(2) In general HE may not be integrable. In fact, for any $U_1, U_2 \in \Gamma(TM)$ with horizontal lifts $\tilde{U}_1, \tilde{U}_2 \in \Gamma(HE)$, one has $[\tilde{U}_1, \tilde{U}_2]^H = \widetilde{[U_1, U_2]}$.

(3) For any $U \in \Gamma(HE)$ and $W \in \Gamma(VE)$, $[U, W] \in \Gamma(VE)$ is always vertical.

Now we construct Riemannian structures on this fibration. Suppose g_M is a Riemannian metric on M . When there is no risk of confusion, we will also denote the pullback metric π^*g_M on TE by g_M . Let $g_Y(x)$ be a family of symmetric $(0, 2)$ -tensors that depends smoothly on $x \in M$ and induces a Riemannian metric on the each fiber Y_x . Using

the pointwise identification $H_p E \cong T_{\pi(p)} M$ and $V_p E \cong T_p Y_{\pi(p)}$, we can equip TE with a metric

$$g_E = g_M + g_Y \quad (2.3)$$

by requiring that g_E preserves the splitting. As a result, the splitting $TE = HE \oplus VE$ is orthogonal with respect to g_E . Since clearly $(\pi_*)_p : H_p E \rightarrow T_{\pi(p)} M$ is an isometry for each $p \in E$, the projection $\pi : (E, g_E) \rightarrow (M, g_M)$ is a Riemannian submersion.

Now we introduce a natural connection, first considered by J.-M. Bismut [16, section 1(b)] (see also [17, section 4(a)]), that faithfully reflects the splitting structure of the tangent bundle. Let ∇^{LC} denote the Levi-Civita connection of E with respect to g_E .

Definition 2.1.2 *The connection ∇^E on TE is defined by*

$$\begin{aligned} \nabla_{\tilde{U}_1}^E \tilde{U}_2 &= \nabla_{U_1}^M U_2, & \nabla_{\tilde{U}}^E W &= P^V(\nabla_{\tilde{U}}^{LC} W) \\ \nabla_W^E \tilde{U} &= 0, & \nabla_{W_1}^E W_2 &= P^V(\nabla_{W_1}^{LC} W_2), \end{aligned} \quad (2.4)$$

where $U_i, U \in \Gamma(TM)$, $W_i, W \in \Gamma(VE)$. We let S denote the difference tensor $S = \nabla^L - \nabla^E$.

Using the definition above, one can easily verify that this connection is compatible with the metric g_E . For later purpose, it is useful to compute all the components of S .

Proposition 2.1.3 *Let $U_i, U \in \Gamma(TM)$, $W_i, W \in \Gamma(VE)$. Then the only nonvanishing components of S are*

$$\langle S(W_1)W_2, U \rangle = -\langle S(W_1)U, W_2 \rangle = -\langle \nabla_{W_1}^{LC} U, W_2 \rangle \quad (2.5)$$

and

$$\langle S(U_1)W, U_2 \rangle = -\langle S(U_1)U_2, W \rangle = \langle S(W)U_1, U_2 \rangle = -\frac{1}{2}\langle [U_1, U_2], W \rangle. \quad (2.6)$$

Moreover, we have

$$(1) \ S(U_1)U_2 \in \Gamma(VE), \ S(\cdot)W \in \Gamma(HE).$$

$$(2) \ S(W)U \in \Gamma(HE) \oplus \Gamma(VE).$$

$$(3) \ \langle S(\cdot)\cdot, \cdot \rangle \text{ is independent of the choice of } g_M.$$

2.2 Clifford multiplication

We start with the algebraic preliminaries. Let V be an n -dimensional real vector space, equipped with an inner product g_V . Let $\{e_1, \dots, e_n\}$ be an orthonormal basis of V , and $\{e^1, \dots, e^n\}$ be its dual basis. Then the Clifford algebra $\text{Cl}(V^*)$ is the algebra spanned by V^* , subject to the relations

$$e^i \cdot e^j + e^j \cdot e^i = -2\delta_{ij}. \quad (2.7)$$

The canonical isomorphism (as vector spaces) between $\text{Cl}(V^*)$ and the exterior algebra ΛV^* is given by identifying $e^{i_1}e^{i_2}\dots e^{i_k}$ with $e^{i_1} \wedge e^{i_2} \wedge \dots \wedge e^{i_k}$, provided that i_j 's are mutually distinct. Under this isomorphism, the left Clifford multiplication by e^i becomes $c(e^i) = e^i \wedge -\iota_{e_i}$. In particular, this equips ΛV^* with a Clifford module structure $c : \text{Cl}(V^*) \rightarrow \text{End}(\Lambda V^*)$. When restricting to $\Lambda^p V^*$, the right Clifford multiplication by e^i is given by $(-1)^p \hat{c}(e^i) := (-1)^p (e^i \wedge + \iota_{e_i})$. Moreover, as an algebra, $\text{End}(\Lambda V^*)$ is generated by 1 and $c(e^i), \hat{c}(e^i)$'s. In the sequel, we will call $c(e^i)$ and $\hat{c}(e^i)$ the left and right Clifford

multiplications of $\text{Cl}(V^*)$ on ΛV^* . It is not hard to verify that for any $i, j \in \{1, 2, \dots, n\}$,

$$\begin{aligned} c(e^i)c(e^j) + c(e^j)c(e^i) &= -2\delta_{ij}, \\ \hat{c}(e^i)\hat{c}(e^j) + \hat{c}(e^j)\hat{c}(e^i) &= 2\delta_{ij}, \\ c(e^i)\hat{c}(e^j) + \hat{c}(e^j)c(e^i) &= 0. \end{aligned} \tag{2.8}$$

In particular, the last equality shows that c and $\hat{c}$ always anti-commute.

Remark 2.2.1 *Suppose we rescale the metric by a factor ε^{-2} , i.e. we consider the inner product $\varepsilon^{-2}g_V$. Using the isometry $(V, \varepsilon^{-2}g_V) \rightarrow (V, g_V)$ induced by $\varepsilon e_i := \tilde{e}_i \mapsto e_i$, we can identify $\text{Cl}_\varepsilon(V^*)$ with $\text{Cl}(V^*)$ as algebras. In this way, we see that the natural left and right Clifford multiplications of $\text{Cl}_\varepsilon(V^*)$ on ΛV^* are given by*

$$\begin{aligned} c(\tilde{e}^i) &= \tilde{e}^i \wedge -\iota_{\tilde{e}_i}, \\ \hat{c}(\tilde{e}^i) &= \tilde{e}^i \wedge +\iota_{\tilde{e}_i}, \end{aligned} \tag{2.9}$$

and they are independent of ε .

Now we turn to the geometric side. Consider a Riemannian manifold (E^n, g_E) . The space of differential forms comes with a natural $\mathbb{Z}_2$ -grading structure:

$$\Omega^*E = \Omega^+E \oplus \Omega^-E, \tag{2.10}$$

where Ω^+E and Ω^-E denote the space of even (respectively odd) forms. This leads to a $\mathbb{Z}_2$ -grading on the endomorphism bundle:

$$\Gamma(\text{End}(\Lambda^*E)) = \Gamma(\text{End}^+(\Lambda^*E)) \oplus \Gamma(\text{End}^-(\Lambda^*E)). \tag{2.11}$$

In particular, $A \in \Gamma(\text{End}^+(\Lambda^*E))$ if $A : \Omega^\pm E \rightarrow \Omega^\pm E$, and $A \in \Gamma(\text{End}^-(\Lambda^*E))$ if

$A : \Omega^\pm E \rightarrow \Omega^\mp E$. Observe that under the splitting (2.10), every $A \in \Gamma(\text{End}(\Lambda^* E))$ can be expressed as

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad (2.12)$$

where $A_{11}, A_{22} \in \Gamma(\text{End}^+(\Lambda^* E))$, and $A_{12}, A_{21} \in \Gamma(\text{End}^-(\Lambda^* E))$.

Definition 2.2.2 *The (global) supertrace $\text{Tr}_s : \Gamma(\text{End}(\Lambda^* E)) \rightarrow C^\infty(M)$ is defined by*

$$\text{Tr}_s A = \text{Tr} A_{11} - \text{Tr} A_{22}. \quad (2.13)$$

The supertrace can be elegantly formulated using Clifford multiplication. To see this, let $\{e_1, \dots, e_n\}$ be a local orthonormal frame. Then locally $\Omega^1(E)$ acts on $\Gamma(\text{End}(\Lambda^* E))$ via Clifford multiplications $c(e^i), \hat{c}(e^i)$ which again satisfy (2.8). Moreover, as a $C^\infty(E)$ -algebra, $\Gamma(\text{End}(\Lambda^* E))$ is generated by 1 and the $c(e^i), \hat{c}(e^i)$'s. For simplicity, for $I = \{i_1, \dots, i_p\} \subset \{1, 2, \dots, n\}$, we let $c(e^I)$ denote $c(e^{i_1})c(e^{i_2}) \cdots c(e^{i_p})$, and define $\hat{c}(e^I)$ similarly. The following "miraculous cancellation" formula is well-known:

Proposition 2.2.3 *For two sets of indices $I, I' \subset \{1, 2, \dots, n\}$,*

$$\text{Tr}_s c(e^I) \hat{c}(e^{I'}) = \begin{cases} (-1)^{\frac{n(n+1)}{2}} 2^n, & \text{if } I = I' = \{1, \dots, n\}; \\ 0, & \text{if otherwise.} \end{cases} \quad (2.14)$$

Sometimes it is useful to separate the contributions of left and right Clifford multiplications. Here we introduce a construction following the idea in [18, section 4(d)]. Consider a vector bundle $T^*(E) \oplus T^*(\hat{E})$ over E that consists of two copies of $T^*(E)$. Here, to avoid confusion, we use $T^*(\hat{E})$ to denote the second copy of $T^*(E)$.

We require that the two copies $T^*(E)$ and $T^*(\hat{E})$ act on $\Lambda^*(E)$ only via left (respectively right) Clifford multiplication. To be more precise, let $e^1, \dots, e^n$ denote a local

orthonormal basis of $T^*(E)$, and $\hat{e}^1, \dots, \hat{e}^n$ denote the corresponding orthonormal basis of $T^*(\hat{E})$. Set $c(e^i) = e^i \wedge -\iota_{e_i}$ and $\hat{c}(e^i) = \hat{e}^i + \iota_{\hat{e}_i}$. Then in view of $\Lambda^*(E \oplus \hat{E}) = \Lambda^*E \hat{\otimes} \Lambda^*\hat{E}$, $c(e^i)$ (respectively $\hat{c}(e^i)$) only acts on Λ^*E (respectively on $\Lambda^*\hat{E}$). Clearly Proposition 2.2.3 still holds with respect to the induced $\mathbb{Z}_2$ -grading on $\text{End}(\Lambda^*E \hat{\otimes} \Lambda^*\hat{E})$.

In general, for any $a = \sum a_{I,I'} c(e^I) \hat{c}(e^{I'}) \in \text{End}(\Lambda^*E \hat{\otimes} \Lambda^*\hat{E})$, let $a' := \sum a_{I,I'} e^I \wedge e^{I'}$ be the associated differential form obtained by replacing the Clifford multiplications with $e^I \wedge e^{I'}$. Moreover, we define the Berezin integral $\int^B a'$ to be the coefficient of the top form term. Then Proposition 2.2.3 implies

Proposition 2.2.4

$$\text{Tr}_s a = (-1)^{\frac{n(n+1)}{2}} 2^n \int^B a'. \quad (2.15)$$

2.3 Dirac operator and Witten deformation

In this section we review some basic facts about the Dirac operator and the corresponding Witten deformation. Let (E^n, g_E) be a (possibly noncompact) Riemannian manifold, and $\{e_1, \dots, e_n\}$ be a local orthonormal frame. Then the Levi-Civita connection induces a Clifford connection ∇ on Λ^*E , namely it satisfies $\nabla_X(c(v)\alpha) = c(\nabla_X v)\alpha + c(v)\nabla_X \alpha$ for any $X \in \Gamma(T E)$, $v \in \Omega^1(E)$, and $\alpha \in \Omega^*(E)$.

Definition 2.3.1 *The Dirac operator of Λ^*E is a map $D : \Omega^*(E) \rightarrow \Omega^*(E)$ defined by*

$$D\alpha = c(e^i) \nabla_{e_i} \alpha. \quad (2.16)$$

It is clear that (2.16) yields a global operator, and there holds $D = d + d^*$, where d^* is the formal adjoint of d with respect to the metric g_E . Consequently, the square of the Dirac operator gives the Hodge Laplacian: $D^2 = \Delta$.

Using the Bochner-Lichnerowicz-Weitzenbock formula, one can expression the Hodge Laplacian in terms of the connection Laplacian (see the definition below) and Riemannian curvature.

Definition 2.3.2 *The connection Laplacian $\nabla^*\nabla : \Omega^*(E) \rightarrow \Omega^*(E)$ is defined by*

$$\nabla^*\nabla\alpha = - \sum_i (\nabla_{e_i} \nabla_{e_i} \alpha - \nabla_{\nabla_{e_i} e_i} \alpha).$$

Proposition 2.3.3 (Bochner-Lichnerowicz-Weitzenbock formula) *Let R^E denote the scalar curvature of the metric g_E , and R_{ijkl} denote the components of the Riamnnian curvature. Then*

$$D^2 = \nabla^*\nabla + \frac{R^E}{4} + \frac{1}{8} R_{ijkl} c(e^i) c(e^j) \hat{c}(e^k) \hat{c}(e^l).$$

To compute the connection Laplacian, we pick a local coordinate $\{x_1, \dots, x_n\}$ near some $p_0 \in E$ so that the metric takes the form

$$g_{ij} = \delta_{ij} + O(|x|^2). \quad (2.17)$$

Let $\{e_1, \dots, e_n\}$ be the local orthonormal frame obtained by parallel transporting the coordinate vector fields at p_0 along the radial geodesics. Using (2.17) and the fact that locally the curvature 1-form acts on $\Gamma(TE)$ anti-symmetrically, we find that the action of $\nabla^*\nabla$ on $\Omega^*(E)$ is given by

$$\nabla^*\nabla = - \sum_i \left[e_i + \frac{1}{8} \sum_{j,k,l} (R_{ijkl}^L(0)x^l + O(|x|^2)) (c(e^j)c(e^k)) - \hat{c}(e^j)\hat{c}(e^k) \right]^2 \quad (2.18)$$

Therefore Proposition 2.3.3 yields the following:

Corollary 2.3.4

$$\begin{aligned}
D^2 = & - \sum_i \left[e_i + \frac{1}{8} \sum_{j,k,l} (R_{ijkl}^L(0)x^l + O(|x|^2)) (c(e^j)c(e^k)) - \hat{c}(e^j)\hat{c}(e^k) \right]^2 \\
& + \frac{R^E}{4} + \frac{1}{8} R_{ijkl} c(e^i) c(e^j) \hat{c}(e^k) \hat{c}(e^l).
\end{aligned}$$

Now we introduce the Witten deformation. Let $h \in C^\infty(E)$ be a smooth function and $\tau \in \mathbb{R}_+$ be a positive parameter.

Definition 2.3.5 *The Witten deformations by h of the exterior derivative d and its adjoint d^* are given by*

$$\begin{aligned}
d_{\tau h} &= e^{-\tau h} \circ d \circ e^{\tau h}, \\
d_{\tau h}^* &= e^{\tau h} \circ d^* \circ e^{-\tau h}.
\end{aligned}$$

The corresponding Dirac operator $D_{\tau h} := d_{\tau h} + d_{\tau h}^$ is called the Witten-Dirac operator, and the Witten Laplacian is $D_{\tau h}^2 = d_{\tau h} d_{\tau h}^* + d_{\tau h}^* d_{\tau h}$.*

It is easy to see from the definition that

$$\begin{aligned}
d_{\tau h} &= d + \tau dh \wedge, \\
d_{\tau h}^* &= d^* + \tau \iota_{\nabla h}.
\end{aligned}$$

Therefore as an analogue of Corollary 2.3.4, one has the following formula for Witten Laplacian.

Corollary 2.3.6

$$\begin{aligned}
D_{\tau h}^2 = & - \sum_i \left[e_i + \frac{1}{8} \sum_{j,k,l} (R_{ijkl}^L(0)x^l + O(|x|^2)) (c(e^j)c(e^k)) - \hat{c}(e^j)\hat{c}(e^k) \right]^2 \\
& + \frac{R^E}{4} + \frac{1}{8} R_{ijkl} c(e^i)c(e^j)\hat{c}(e^k)\hat{c}(e^l) + \tau \text{Hess } h(e_i, e_j) c(e^i)\hat{c}(e^j) + \tau^2 |\nabla h|^2.
\end{aligned}$$

Witten deformation was introduced by the theoretical physicist Edward Witten in his seminal paper [19]. Witten's goal was to provide a new, purely analytic proof of the classical Morse inequalities. To do this, he borrowed ideas from supersymmetric quantum mechanics and applied them to differential geometry. The core idea of Witten deformation is to modify the standard differential operators of de Rham cohomology using a real parameter τ and a Morse function h , and then study the asymptotic behavior of the spectrum as that parameter goes to infinity. As seen in Corollary 2.3.6, when τ approaches infinity, the potential energy $\tau^2 |\nabla h|^2$ dominates the remaining terms everywhere except at the critical points of h . This forces the values of eigenfunctions corresponding to small eigenvalues to concentrate near the critical points.

2.4 Riesz-Dunford functional calculus

In this section, we review the so-called Riesz-Dunford functional calculus for closed operators. This serves as a powerful tool for analyzing the properties of operators constructed from (local-)holomorphic functions. For the purpose of this paper, we will present only a baby version of it. For more details, we refer to [20, Chapter VII].

To begin with, let H be a Hilbert space, $T : H \rightarrow H$ be a closed linear operator (possibly unbounded). Let $\text{spec}(T)$ denote the spectrum of T . Suppose f is a complex function over $\mathbb{C}$, which is analytic near $\text{spec}(T) \cup \{\infty\}$.

Theorem 2.4.1 *Suppose*

- (1) $V \subset \mathbb{C}$ is an open subset containing $\text{spec}(T)$;
- (2) The boundary Γ of V consists of a finite number of Jordan curves, equipped with positive orientation with respect to V ;
- (3) f is holomorphic on $V \cup \Gamma$.

Then $f(T)$ is a well-defined linear operator over H , and we have the following Cauchy-type integral formula

$$f(T) = \frac{1}{2\pi i} \int_{\Gamma} f(\lambda)(\lambda - T)^{-1} d\lambda, \quad (2.19)$$

provided that the integral on the right-hand side is absolutely convergent. Moreover, $\text{spec}(f(T)) = f(\text{spec}(T) \cup \{\infty\})$.

As an example, suppose $D : H \rightarrow H$ is an unbounded self-adjoint operator. Then from basic functional analysis theory, we know that $\text{spec}(D) \subset \mathbb{R}$. Consider the function $f(z) = e^{-tz^2}$, where $t > 0$ is any fixed number. If we choose Γ properly such that $(\lambda - D)^{-1}$ is uniformly bounded, Theorem 2.4.1 implies that

$$e^{-tD^2} = f(D) = \frac{1}{2\pi i} \int_{\Gamma} e^{-t\lambda^2}(\lambda - D)^{-1} d\lambda. \quad (2.20)$$

Moreover, if $\{D_n\}$ is a sequence of such operators that converges to D in the norm resolvent sense (i.e. $(\lambda - D_n)^{-1}$ converges to $(\lambda - D)^{-1}$ under operator norm topology), then it is easy to see that $e^{-tD_n^2}$ and e^{-tD^2} are all bounded operators, and

$$\lim_{n \rightarrow \infty} e^{-tD_n^2} = e^{-tD^2}$$

in operator norm sense.

2.5 Index of closed manifolds

In this section and the next section, we review the definitions of index and analytic torsion of a closed manifold. For our purpose later, we shall restrict our discussion to the exterior algebra bundle without any twisting (equivalently, twisted with a trivial bundle). The discussion for the general case with a non-trivial twisting is in fact completely parallel to ours.

We start by discussing the spectral properties of the Hodge Laplacian. Let (M, g) be an n -dimensional closed Riemannian manifold. It is well-known that the de Rham complex

$$(\Omega^*(M), d) : 0 \rightarrow \Omega^0(M) \xrightarrow{d_0} \Omega^1(M) \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} \Omega^n(M) \xrightarrow{d_n} 0 \quad (2.21)$$

gives the de Rham cohomology:

$$H_{dR}^k(M; \mathbb{R}) = \ker d_k / \operatorname{im} d_{k-1}. \quad (2.22)$$

Moreover, regarding the kernel of the Hodge Laplacian, one has the following isomorphism due to Hodge theory.

Theorem 2.5.1 (Hodge) *Let Δ_k denote the restriction of Δ on $\Omega^k(M)$. Then for each $0 \leq k \leq n$, $\ker \Delta_k$ is a finite dimensional vector space, and*

$$\ker \Delta_k \cong H_{dR}^k(M).$$

As M is compact, one has the following result describing the entire spectrum of the Hodge Laplacian.

Theorem 2.5.2 *The spectrum of Δ is discrete and diverges to infinity:*

$$0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots .$$

Moreover, each eigenspace is finite dimensional.

A lot of spectral invariants can be constructed using the (super)trace of heat type operators. For this reason, we now digress to review several facts about trace class operators and Fredholm operators. Suppose $\mathcal{H}$ is a Hilbert space with an orthonormal basis $\{e_i\}$. Let $T : \mathcal{H} \rightarrow \mathcal{H}$ be a compact operator. It follows from classical functional analysis theory that T^*T is a compact self-adjoint operator, and hence $\text{spec}(T^*T) = \{0\} \cup \{s_i\}_{i=1}^\infty$ consists of eigenvalues of finite multiplicity except 0.

Definition 2.5.3 (1) *The p -trace norm of T is defined to be*

$$\|T\|_p = \left[\sum_{i=1}^{\infty} s_i^{p/2} \right]^{1/p} .$$

In particular, $\|T\|_1 = \sum_{i=1}^{\infty} s_i^{1/2}$ is called the trace norm of T .

(2) *T is said to be of trace class if the trace norm is finite: $\|T\|_1 < \infty$. In this case, the trace of T is defined to be*

$$\text{Tr}(T) = \sum_{i=1}^{\infty} \langle T e_i, e_i \rangle .$$

Clearly, the trace norm and the trace are related by $\|T\|_p = \left[\text{Tr}(T^*T)^{p/2} \right]^{1/p}$. In particular, $\|T\|_1 = \text{Tr}(T^*T)^{1/2}$.

Remark 2.5.4 *If T is self-adjoint, its spectrum $\sigma(T) = \{\lambda_i\}$ is purely discrete. In this*

case, the trace of T can be written as the sum of all eigenvalues:

$$\mathrm{Tr}(T) = \|T\|_1 = \sum_i \lambda_i.$$

Next, we discuss the index of an operator, which can be defined as the difference between the dimensions of the kernel and that of the cokernel. It is a finite number only when both of the dimensions are finite. That is:

Definition 2.5.5 *Let $\mathcal{H}_0, \mathcal{H}_1$ be two Hilbert spaces, and $A : \mathcal{H}_0 \rightarrow \mathcal{H}_1$ be a closed operator. Then A is called a Fredholm operator, if it has a dense range, and both $\ker P$ and $\mathrm{coker} P$ are finite dimensional. In this situation, the index of A is defined to be*

$$\mathrm{ind} A = \dim \ker A - \dim \mathrm{coker} A.$$

In many cases, one can express the index in terms of the traces of heat type operators.

Theorem 2.5.6 *Suppose that*

- (1) $A : \mathcal{H}_0 \rightarrow \mathcal{H}_1$ is a Fredholm operator;
- (2) A^*A and AA^* have discrete spectra, where $A^* : \mathcal{H}_1 \rightarrow \mathcal{H}_0$ denotes the adjoint of A ;
- (3) for every $t > 0$, e^{-tA^*A} and e^{-tAA^*} are trace-class operators.

Then there holds

$$\mathrm{ind} A = \mathrm{Tr} \left(e^{-tA^*A} \right) - \mathrm{Tr} \left(e^{-tAA^*} \right),$$

for any $t > 0$. In particular, the difference on the right hand side is independent of t .

Now we return to the geometric side. As is easily verified, the Dirac operator D is self-adjoint and Fredholm, and the operators D^+D^- and D^-D^+ satisfy the assumptions in

Theorem 2.5.6. As a consequence, we see that $\text{ind } D = 0$. Taking this into consideration, to construct a nontrivial invariant, one may instead consider the index of a "component" of D . To be more precise, we recall that the endomorphism bundle has a natural $\mathbb{Z}_2$ -grading given by

$$\text{End}(\Lambda^* M) = \text{End}^+(\Lambda^* M) \oplus \text{End}^-(\Lambda^* M).$$

Under this splitting, D has the following 2×2 matrix expression:

$$D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}, \quad (2.23)$$

where $D^- : \Omega^-(M) \rightarrow \Omega^+(M)$ and $D^+ : \Omega^+(M) \rightarrow \Omega^-(M)$.

Definition 2.5.7 *The index of M , denoted by $\text{ind}(M)$, is defined to be the index of D^+ .*

By Definition 2.2.2 and Theorem 2.5.6, we find that

$$\text{ind}(M) = \text{Tr}_s e^{-t\Delta}. \quad (2.24)$$

A priori, $\text{ind}(M)$ may still depend on the Hodge Laplacian Δ , and hence the metric chosen. However, as M is closed, it follows from Hodge theory that $\text{ind}(M)$ is independent of g_M and is in fact the same as the Euler characteristic $\chi(M)$. The index of M is a primary invariant in the sense that it is constructed using cohomology groups directly and only depends on the zero spectrum of Hodge Laplacian. In particular, it is a homotopy invariant.

2.6 Analytic torsion of closed manifolds

In contrast to index, analytic torsion is a secondary invariant that lies in the determinant line associated with the de Rham cohomology groups. It can be defined using the non-zero spectrum of Hodge Laplacian. If the primary invariant $\text{ind}(M)$ vanishes, then analytic torsion can be canonically identified with a positive real number, and it is a homeomorphism invariant but not a homotopy invariant.

We start by reviewing the definition of Reidemeister torsion (or R-torsion for short). Here we adopt the spectral interpretation first introduced by Ray-Singer [2]. As in the last section, let (M, g) be an n -dimensional closed Riemannian manifold. We assume that M is endowed with a smooth triangulation, which induces a simplicial chain complex of real vector spaces:

$$\mathcal{C} : C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0. \quad (2.25)$$

We equip complex $\mathcal{C}$ with a metric $g_{\mathcal{C}} = \{g_i\}$ by choosing an inner product g_i on each C_i . Let $\partial_p^* : C_{p-1} \rightarrow C_p$ be the adjoint of ∂_p with respect to g_i . Then the combinatorial Laplacian

$$\Delta_p^{(c)} = \partial_{p+1} \partial_{p+1}^* + \partial_p^* \partial_p : C_p \rightarrow C_p \quad (2.26)$$

is a self-adjoint linear map. As a consequence, $\Delta_p^{(c)}$ only has nonnegative eigenvalues. The R-torsion will be defined in terms of the determinants of the combinatorial Laplacians. This suggests that one shall renormalize the determinant by excluding the zero eigenvalues. For this reason, we let $\underline{\Delta}_p^{(c)}$ denote $\Delta_p^{(c)}$ with the null space removed. Equivalently, $\underline{\Delta}_p^{(c)}$ is the restriction of $\Delta_p^{(c)}$ onto the eigenspaces corresponding to nonzero eigenvalues.

Definition 2.6.1 *The R-torsion $\tau(M)$ of M is defined by*

$$\log \tau(M) = \frac{1}{2} \sum_{p=0}^n (-1)^p p \log \det \underline{\Delta}_p^{(\mathcal{C})}. \quad (2.27)$$

In general, $\tau(M)$ may depend on the metric $g_{\mathcal{C}}$, and hence is not a topological invariant. However, one can always couple it with an induced metric to eliminate the dependence on $g_{\mathcal{C}}$ and obtain a topological invariant. To be more precise, we consider the trivial chain complex:

$$H(\mathcal{C}) : H^n(\mathcal{C}) \xrightarrow{0} H^{n-1}(\mathcal{C}) \xrightarrow{0} \cdots \xrightarrow{0} H^1(\mathcal{C}) \xrightarrow{0} H^0(\mathcal{C}). \quad (2.28)$$

The determinant line $\lambda_{\mathcal{C}}$ associated with $H(\mathcal{C})$ is defined by

$$\lambda_{\mathcal{C}} = \bigotimes_{i=0}^n (\det H^i(\mathcal{C}))^{(-1)^i}, \quad (2.29)$$

where $\det H^i(\mathcal{C}) = \Lambda^{\dim \det H^i(\mathcal{C})} \det H^i(\mathcal{C})$ is the usual determinant line. Following the idea of [21], one can make the following definition:

Definition 2.6.2 *The Quellin metric associated with $(\mathcal{C}, g_{\mathcal{C}})$ is defined by*

$$\|\cdot\|_{\lambda_{\mathcal{C}}} = \tau(M) |\cdot|_{\lambda_{\mathcal{C}}}, \quad (2.30)$$

where $|\cdot|_{\lambda_{\mathcal{C}}}$ denotes the metric on $\lambda_{\mathcal{C}}$ induced by $g_{\mathcal{C}}$. In particular, $\|\cdot\|_{\lambda_{\mathcal{C}}}$ defines a metric on $\lambda_{\mathcal{C}}$.

Proposition 2.6.3 *$\|\cdot\|_{\lambda_{\mathcal{C}}}$ does not depend on the choice of triangulation and the metric $g_{\mathcal{C}}$. In fact, it is invariant under homeomorphism, but not under homotopy equivalence.*

In their celebrated paper [2], Ray and Singer introduced analytic torsion as an analytic

analogue of R-torsion. Their idea was to replace the combinatorial Laplacian $\Delta_p^{(c)}$ with the Hodge Laplacian Δ_p , and thus they were facing the problem of making sense of $\det \Delta_p$. To resolve this issue, they made the following observation: Suppose $\lambda > 0$ is a positive eigenvalue of $\Delta_p^{(c)}$. Then the Mellin transformation yields

$$\lambda^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-t\lambda} dt. \quad (2.31)$$

On the other hand, by differentiating at $s = 0$, one finds that

$$\left. \frac{d}{ds} \right|_{s=0} \lambda^{-s} = -\log \lambda. \quad (2.32)$$

From the computation above, we deduce that

$$\log \det \underline{\Delta}_p^{(c)} = \sum_{\text{eigenvalue } \lambda_i \neq 0} \log \lambda_i = - \left. \frac{d}{ds} \right|_{s=0} \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{Tr } e^{-t\underline{\Delta}_p^{(c)}} dt. \quad (2.33)$$

Inspired by this, one may use the derivative on the right-hand-side of (2.33) to define $\det \Delta_p$. To be more precise, we recall that as M is compact, the heat operator $e^{-t\Delta}$ is of trace class for any $t > 0$. We still use $\underline{\Delta}$ to denote the operator obtained by removing the null space from the Hodge Laplacian Δ . Let $N : \Omega^*(M) \rightarrow \Omega^*(M)$ denote the number operator given by $N|_{\Omega^p(M)} = p$.

Definition 2.6.4 *The torsion zeta function is defined as*

$$\zeta_T(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{Tr}_s (N e^{-t\underline{\Delta}}) dt. \quad (2.34)$$

When this function is differentiable at $s = 0$, the analytic torsion $T(M)$ is defined by

$$\log T(M) = -\frac{1}{2} \zeta'_T(0). \quad (2.35)$$

Using Poincare duality, one can check that analytic torsion is trivial for manifolds of even dimensions.

Proposition 2.6.5 *Suppose M is a closed, even-dimensional Riemannian manifold. Then $T(M) \equiv 1$.*

Following [2], we now discuss the metric dependence of analytic torsion. Roughly speaking, $T(M)$ can be viewed as an element of the determinant line

$$\lambda_M := \bigotimes_{i=0}^n (\det H_{dR}^i(M; \mathbb{R}))^{(-1)^i}.$$

As each identification between λ_M and $\mathbb{R}^1$ amounts to choosing a metric on M , analytic torsion $T(M)$ by itself is not a topological invariant. As in the case of R-torsion, to recover the topological invariance, one shall consider the Quillen metric defined by $\|\cdot\|_{\lambda_M} := T(M)|\cdot|_{\lambda_M}$, where $|\cdot|_{\lambda_M}$ is the metric induced by the L^2 -metric on $H^i(M; \mathbb{R})$.

In light of Proposition 2.6.5, in the following we always assume that $\dim M$ is odd. Suppose $g(\sigma)$ is a smooth family of Riemannian metrics on M parameterized by $\sigma \in [0, 1]$. One has the following variation formula for the supertrace:

Lemma 2.6.6 *Let $\alpha = *^{-1}\dot{*}$. Then*

$$\frac{\partial}{\partial \sigma} \text{Tr}_s (N e^{-t\Delta}) = t \frac{\partial}{\partial t} \text{Tr}_s (\alpha e^{-t\Delta})$$

Since the torsion zeta function is defined as an integral of a supertrace, the lemma above leads to

Theorem 2.6.7 *Let $H : L^2(\Lambda^* M) \rightarrow \ker \Delta$ denote the orthogonal projection onto the*

space of harmonic forms. Then

$$\frac{\partial}{\partial \sigma} \log T(E, g(\sigma)) = -\frac{1}{2} \text{Tr}_s(\alpha H).$$

For each σ , let $\|\cdot\|_{\lambda_M, \sigma}$ denote the Quillen metric corresponding to $g(\sigma)$. Then

$$\frac{\partial}{\partial \sigma} \|\cdot\|_{\lambda, \tau h, g(\sigma)} = 0.$$

In particular, the Quillen metric $\|\cdot\|_{\lambda_M, \sigma}$ doesn't depend on the choice of $g(\sigma)$.

Chapter 3

Adiabatic Limit and Index of Vector Bundles

This chapter is devoted to the study of the adiabatic limit of the heat operator when the time is large. As a result, we will prove that the index of the total space E coincides with the index of M . Passing to the adiabatic limit refers to the geometric degeneration in which the metric is blown up along certain directions. This blow-up process usually reveals important information about the metric that cannot be seen using regular methods. In particular, it is useful in the study of geometric invariants. Historically, the study of adiabatic limit was initiated by E. Witten in [22], where he related the adiabatic limit of the η -invariant to the "global anomaly".

We will start with the case when the fiber is compact and provide a slightly different proof of some results in [23]. By introducing a Morse-Bott function and considering the Witten Laplacian, we show that these results remain true when the fiber is complete and noncompact. This provides us with a machinery that leads to the convergence of the index under the adiabatic limit.

In what follows, we consider a fiber bundle $Y \hookrightarrow E \xrightarrow{\pi} M$ over a closed manifold M .

Let $\|\cdot\|_0 = \|\cdot\|_{L^2(\Lambda^*E)}$ denote the standard L^2 -norm of sections. If $A : L^2(\Lambda^*E) \rightarrow L^2(\Lambda^*E)$ is a bounded linear operator, we use $\|\cdot\|_{0,0}$ to denote the L^2 -operator norm. Besides, there is no parity restriction on the dimensions of E and M in this chapter.

3.1 Compact fiber case

Suppose the fiber Y is a compact manifold. We equip TE with a metric g_E as in (2.3), and a connection ∇^E defined by (2.4). We further consider a family of metrics with an adiabatic parameter along the horizontal direction:

$$g_\varepsilon = \frac{1}{\varepsilon^2}g_M + g_Y. \quad (3.1)$$

For any $p \in M$, let $x^1, \dots, x^n$ be a normal coordinate centered at p . Let $\{e_1, \dots, e_n\}$ be a locally orthonormal frame around $p \in M$ of TM such that $e_i(p) = \frac{\partial}{\partial x^i}(p)$ and $\nabla_{e_i}^L e_j(p) = 0$. Let $\{f_1, \dots, f_k\}$ be an orthonormal frame for TY . The connection ∇^E respects the horizontal-vertical decomposition, but may not be orthogonal in general. To fix this, consider the following connection

$$\nabla^u = \nabla^E - \frac{1}{2} \sum_{i=1}^k \langle S(f_i) f_i, \cdot \rangle.$$

We denote by $d^{*,\varepsilon}$ the formal dual of the exterior differential operator d with respect to the metric g_ε . Let $D_\varepsilon = d + d^{*,\varepsilon}$ be the corresponding Dirac operator. Then

$$\begin{aligned} D_\varepsilon &= \varepsilon c(\tilde{e}^i) \nabla_{e_i}^u + c(f^j) \nabla_{f_j}^E - \frac{\varepsilon^2}{4} c(\tilde{e}^\alpha) c(\tilde{e}^\beta) c(T)(e_\alpha, e_\beta) \\ &:= \varepsilon \tilde{D}_M + D_Y - \frac{\varepsilon^2}{4} T, \end{aligned} \quad (3.2)$$

where $c(T)$ is the Clifford multiplication of the torsion tensor. We refer to [17, P. 56] for

the proof of (3.2).

Although of primary interest to us is the heat operator corresponding to D_ε , a detailed analysis of the resolvent of $\frac{1}{\varepsilon}D_\varepsilon$ is necessary to understand the spectrum of D_ε . Formally, since $\frac{1}{\varepsilon}D_\varepsilon$ is self-adjoint, we have the following integration formula for the heat operator

$$e^{-\frac{t}{\varepsilon^2}D_\varepsilon^2} = \frac{1}{2\pi i} \int_{\Gamma} e^{-t\lambda^2} \left(\lambda - \frac{1}{\varepsilon}D_\varepsilon \right)^{-1} d\lambda, \quad (3.3)$$

provided that the integral on the right-hand side is uniformly convergent. This can be done by choosing the contour Γ properly. Given that $\frac{1}{\varepsilon}D_\varepsilon$ is unbounded, Γ should be taken such that Γ , together with the infinity, surrounds its spectrum. We postpone the definition of Γ to (3.10).

Now, as in [23, Section 2.2], by assuming that $\ker D_Y$ is a vector bundle, we can consider the following decomposition:

$$L^2(\Lambda^*E) = L^2(\ker D_Y)^\perp \oplus L^2(\ker D_Y). \quad (3.4)$$

Let p and $p^\perp$ be the projections onto $L^2(\ker D_Y)$ and $L^2(\ker D_Y)^\perp$, respectively. Using (3.4), we can write the rescaled Dirac operator as $\frac{1}{\varepsilon}D_\varepsilon = \begin{pmatrix} A_1 & A_2 \\ A_2^* & A_3 \end{pmatrix}$, where

$$\begin{aligned} A_1 &= p^\perp \tilde{D}_M p^\perp + \frac{1}{\varepsilon}D_Y - \varepsilon p^\perp \frac{T}{4} p^\perp, \\ A_2 &= p^\perp \tilde{D}_M p - \varepsilon p^\perp \frac{T}{4} p, \\ A_3 &= D_0 - \varepsilon p \frac{T}{4} p, \end{aligned} \quad (3.5)$$

and A_2^* is the dual operator of A_2 . Here we set $D_0 := p \tilde{D}_M p$ for simplicity.

The following lemma plays a crucial role in proving the convergence of the heat

operator. The proof essentially follows the argument in [17, Proposition 4.41], [23, Lemma 2.6], and is omitted here.

Lemma 3.1.1

(1) *There exists a positive constant λ_0 such that $\text{spec}(|A_1|) \subset [\frac{\lambda_0}{\varepsilon}, +\infty)$.*

(2) *The operators A_2 and A_2^* are bounded.*

For any $\lambda \notin \text{spec}(D_0)$, we consider the operator

$$S_\varepsilon = \lambda - \frac{1}{\varepsilon} D_\varepsilon = \begin{pmatrix} \lambda - A_1 & -A_2 \\ -A_2^* & \lambda - A_3 \end{pmatrix}. \quad (3.6)$$

It is obvious that S_ε is invertible, and its inverse is given by

$$S_\varepsilon^{-1} = \begin{pmatrix} T_1 & T_2 \\ T_2^* & T_3 \end{pmatrix}, \quad (3.7)$$

where

$$\begin{aligned} T_1 &= (\lambda - A_1 - A_2(\lambda - A_3)^{-1}A_2^*)^{-1} \\ &= (\lambda - A_1)^{-1} (I - (\lambda - A_1)^{-1}A_2(\lambda - A_3)^{-1}A_2^*)^{-1}, \\ T_2 &= (\lambda - A_1)^{-1}A_2 (\lambda - A_3 - A_2^*(\lambda - A_1)^{-1}A_2)^{-1} \\ &= (\lambda - A_1)^{-1}A_2(\lambda - A_3)^{-1} (I - (\lambda - A_3)^{-1}A_2^*(\lambda - A_1)^{-1}A_2)^{-1}, \\ T_3 &= (\lambda - A_3 - A_2^*(\lambda - A_1)^{-1}A_2)^{-1} \\ &= (\lambda - A_3)^{-1} (I - (\lambda - A_3)^{-1}A_2^*(\lambda - A_1)^{-1}A_2)^{-1}. \end{aligned}$$

We introduce the following notations:

$$\begin{aligned} B_1 &= (\lambda - A_1)^{-1} A_2 (\lambda - A_3)^{-1} A_2^*, \\ B_2 &= (\lambda - A_3)^{-1} A_2^* (\lambda - A_1)^{-1} A_2. \end{aligned} \tag{3.8}$$

Therefore

$$\begin{aligned} T_1 &= (\lambda - A_1)^{-1} (I - B_1)^{-1}, \\ T_2 &= (\lambda - A_1)^{-1} A_2 (\lambda - A_3)^{-1} (I - B_2)^{-1}, \\ T_3 &= (\lambda - A_3)^{-1} (I - B_2)^{-1}. \end{aligned} \tag{3.9}$$

Since $\frac{1}{\varepsilon}D_\varepsilon$ and D_0 are self-adjoint, both of their spectra are real. As the fiber is compact, it follows from [23, Theorem 1.5] that an eigenvalue of $\frac{1}{\varepsilon}D_\varepsilon$ either is bounded from below by a uniform constant $\lambda_0 > 0$, or decays at least linearly as $\varepsilon \rightarrow 0$. Consequently, we can pick a constant $b > 0$ such that $20b < \min\{\lambda_0, \lambda_1(D_0)\}$, where $\lambda_1(D_0)$ is the smallest nonzero eigenvalue of D_0 . Thus by choosing ε small enough, one has $\text{spec}(\frac{1}{\varepsilon}D_\varepsilon) \subset [0, b] \cup [20b, \infty)$.

To define the integral, consider the contour $\Gamma = \Gamma' \cup \Gamma'' \subset \mathbb{C}$, where

$$\begin{aligned} \Gamma' &= \partial B_b(0), \\ \Gamma'' &= \{|\text{Re } z| \geq 4b, |\text{Im } z| = 2b\} \cup \{|\text{Re } z| = 4b, |\text{Im } z| \leq 2b\}. \end{aligned} \tag{3.10}$$

Here Γ' and Γ'' take the counterclockwise orientations. We first estimate the components of $(\lambda - \frac{1}{\varepsilon}D_\varepsilon)^{-1}$ along the contour.

Lemma 3.1.2 *For any $\lambda \in \Gamma$, we have*

$$(1) \lim_{\varepsilon \rightarrow 0} \|T_1\|_{0,0} = \lim_{\varepsilon \rightarrow 0} \|T_2\|_{0,0} = 0.$$

$$(2) \lim_{\varepsilon \rightarrow 0} \|T_3 - (\lambda - D_0)^{-1}\|_{0,0} = 0.$$

Proof: Let us begin by estimating the operators that appear in the definitions of T_1 , T_2 and T_3 . For any $\lambda \in \Gamma$ fixed, we choose $\varepsilon > 0$ sufficiently small such that $|\lambda| \leq \frac{\lambda_0}{2\varepsilon}$. It follows from lemma 3.1.1 that

$$\|(A_1 - \lambda)u\|_0 \geq \|A_1 u\|_0 - |\lambda| \|u\|_0 \geq \left(\frac{\lambda_0}{\varepsilon} - |\lambda| \right) \|u\|_0,$$

which implies

$$\|(A_1 - \lambda)^{-1}\|_{0,0} \leq \frac{1}{\frac{\lambda_0}{\varepsilon} - |\lambda|} \leq \frac{2\varepsilon}{\lambda_0}. \quad (3.11)$$

Similarly, since D_0 is self-adjoint, we have

$$\begin{aligned} \|(A_3 - \lambda)u\|_0 &\geq \|(D_0 - \lambda)u\|_0 - \left\| \varepsilon p \frac{T}{4} p u \right\|_0 \\ &\geq b \|u\|_0 - \varepsilon C_1 \|u\|_0 = (b - \varepsilon C_1) \|u\|_0. \end{aligned}$$

Thus

$$\|(A_3 - \lambda)^{-1}\|_{0,0} \leq \frac{1}{b - \varepsilon C_1}. \quad (3.12)$$

By combining (3.11), (3.12), and the fact that $\|A_2\|_{0,0} \leq \sqrt{C_2}$ is bounded, we obtain

$$\|B_1\|_{0,0} = \|B_2\|_{0,0} \leq \frac{2C_2}{\lambda_0} \frac{\varepsilon}{b - \varepsilon C_1} < 1, \quad (3.13)$$

provided that ε is small enough. In fact, $\lim_{\varepsilon \rightarrow 0} \|B_i\| = 0$ for $i = 1, 2$. As a result,

$$\|(I - B_i)^{-1}\|_{0,0} \leq \frac{1}{1 - \|B_i\|_{0,0}} \leq \frac{1}{1 - \frac{2C_2}{\lambda_0} \frac{\varepsilon}{b - \varepsilon C_1}} \rightarrow 1, \quad \text{as } \varepsilon \rightarrow 0. \quad (3.14)$$

Using (3.11), (3.12) and (3.13), we obtain the following estimates for T_i 's.

$$\|T_1\|_{0,0} = \|(\lambda - A_1)^{-1}(I - B_1)^{-1}\|_{0,0} \leq \frac{2\varepsilon}{\lambda_0} \frac{1}{1 - \frac{2C_2}{\lambda_0} \frac{\varepsilon}{b - \varepsilon C_1}}, \quad (3.15)$$

$$\begin{aligned} \|T_2\|_{0,0} &= \|(\lambda - A_1)^{-1}A_2(\lambda - A_3)^{-1}(I - B_2)^{-1}\|_{0,0} \\ &\leq \frac{2\varepsilon}{\lambda_0} \frac{C_2}{b - \varepsilon C_1} \frac{1}{1 - \frac{2C_2}{\lambda_0} \frac{\varepsilon}{b - \varepsilon C_1}}. \end{aligned} \quad (3.16)$$

In particular, $\lim_{\varepsilon \rightarrow 0} \|T_1\|_{0,0} = \lim_{\varepsilon \rightarrow 0} \|T_2\|_{0,0} = 0$.

On the other hand, by the second resolvent identity, one has

$$(\lambda - A_3)^{-1} - (\lambda - D_0)^{-1} = -(\lambda - A_3)^{-1} \left(\varepsilon p \frac{T}{4} p \right) (\lambda - D_0)^{-1},$$

which implies

$$\|(\lambda - A_3)^{-1} - (\lambda - D_0)^{-1}\|_{0,0} \leq \frac{C_3}{b^2 - \varepsilon b C_1} \varepsilon.$$

Therefore a direct computation shows

$$\begin{aligned} &\|T_3 - (\lambda - D_0)^{-1}\|_{0,0} \\ &\leq \|T_3 - (\lambda - D_0)^{-1}(I - B_2)^{-1}\|_{0,0} + \|(\lambda - D_0)^{-1}(I - B_2)^{-1} - (\lambda - D_0)^{-1}\|_{0,0} \\ &\leq \|(\lambda - A_3)^{-1} - (\lambda - D_0)^{-1}\|_{0,0} \cdot \|(I - B_2)^{-1}\|_{0,0} \\ &\quad + \|(\lambda - D_0)^{-1}\|_{0,0} \cdot \|(I - B_2)^{-1} - I\|_{0,0} \\ &\leq o(\varepsilon) + C\|(I - B_2)^{-1}B_2\|_{0,0} = o(\varepsilon). \end{aligned} \quad (3.17)$$

From this we see that as $\varepsilon \rightarrow 0$, T_3 converges to $(\lambda - D_0)^{-1}$ in the operator norm sense. ■

In the end, we prove the following main result of this section.

Theorem 3.1.3 *Suppose $\ker D_Y$ is a vector bundle over M . Let $\delta > 0$ and $t > 0$ be*

fixed. Then

$$\lim_{\varepsilon \rightarrow 0} \left\| e^{-\frac{t}{\varepsilon^2} D_\varepsilon^2} - \begin{pmatrix} 0 & 0 \\ 0 & e^{-t D_0^2} \end{pmatrix} \right\|_{0,0} = 0.$$

Proof: We first show that the heat operator admits an integral expression in terms of the resolvent. Following the proof of lemma 3.1.2, we have

$$\begin{aligned} \left\| \int_{\Gamma} e^{-t\lambda^2} \left(\lambda - \frac{1}{\varepsilon} D_\varepsilon \right)^{-1} d\lambda \right\|_{0,0} &\leq \int_{\Gamma} |e^{-t\lambda^2}| \left\| \left(\lambda - \frac{1}{\varepsilon} D_\varepsilon \right)^{-1} \right\|_{0,0} d\lambda \\ &\leq 2 \int_{\mathbb{R}} e^{-t(x^2 - 4b^2)} \left\| \left(\lambda - \frac{1}{\varepsilon} D_\varepsilon \right)^{-1} \right\|_{0,0} dx \\ &\leq 2(C + o(\varepsilon)) \int_{\mathbb{R}} e^{-t(x^2 - 4b^2)} dx \\ &= (C' + o(\varepsilon)) e^{4tb^2}. \end{aligned}$$

In particular, the integral is bounded for fixed ε and t . Therefore, Theorem 2.4.1 implies

$$e^{-\frac{t}{\varepsilon^2} D_\varepsilon^2} = \int_{\Gamma} e^{-t\lambda^2} \left(\lambda - \frac{1}{\varepsilon} D_\varepsilon \right)^{-1} d\lambda \quad (3.18)$$

for any $\varepsilon > 0$. To distinguish the contributions of different λ 's to the integral, we split the region as $\Gamma = \{\lambda \in \Gamma : |\lambda| \leq \frac{\lambda_0}{2\varepsilon}\} \cup \{\lambda \in \Gamma : |\lambda| > \frac{\lambda_0}{2\varepsilon}\}$.

For the first part, we consider the substitution $\tau = \varepsilon\lambda$. Let $\Gamma_\varepsilon = \{z \in \mathbb{C} \mid \frac{z}{\varepsilon} \in \Gamma\}$. A direct computation shows that the following estimate is uniform in ε :

$$\left| \int_{|\lambda| \leq \frac{\lambda_0}{2\varepsilon}, \lambda \in \Gamma} e^{-t\lambda^2} \varepsilon d\lambda \right| = \left| \int_{|\tau| \leq \frac{\lambda_0}{2}, \tau \in \Gamma_\varepsilon} e^{-\frac{t}{\varepsilon^2} \tau^2} d\tau \right| \leq C < \infty \quad (3.19)$$

This, together with (3.15), (3.16) and (3.17), implies

$$\begin{aligned}
& \lim_{\varepsilon \rightarrow 0} \left\| \int_{|\lambda| \leq \frac{\lambda_0}{2\varepsilon}, \lambda \in \Gamma} e^{-t\lambda^2} \begin{pmatrix} T_1 & T_2 \\ T_2^* & T_3 - (\lambda - D_0)^{-1} \end{pmatrix} d\lambda \right\|_{0,0} \\
& \leq \lim_{\varepsilon \rightarrow 0} \int_{|\lambda| \leq \frac{\lambda_0}{2\varepsilon}, \lambda \in \Gamma} |e^{-t\lambda^2}| o(\varepsilon) dx \\
& \leq \lim_{\varepsilon \rightarrow 0} \int_{\Gamma} |e^{-t\lambda^2}| o(\varepsilon) dx \\
& = 0.
\end{aligned} \tag{3.20}$$

To estimate the second part, we observe that if $\lambda = x + iy \in \Gamma$ satisfies $|\lambda| > \frac{\lambda_0}{2\varepsilon}$ for some ε small, then $|y| = 2b$ and $x^2 > \frac{\lambda_0^2}{4\varepsilon^2} - 4b^2$. Thus, letting $X = \varepsilon x$,

$$\begin{aligned}
& \lim_{\varepsilon \rightarrow 0} \int_{\{(x, 2b) \in \mathbb{C} | x^2 > \lambda_0^2/4\varepsilon^2 - 4b^2\}} e^{-t(x^2 - 4b^2)} \left\| \left(\lambda - \frac{1}{\varepsilon} D_\varepsilon \right)^{-1} \right\|_{0,0} dx \\
& \leq \lim_{\varepsilon \rightarrow 0} \int_{\{(x, 2b) \in \mathbb{C} | x^2 > \lambda_0^2/4\varepsilon^2 - 4b^2\}} e^{-t(x^2 - 4b^2)} \frac{1}{2b} dx \\
& = \lim_{\varepsilon \rightarrow 0} \int_{\{(X, 2b) \in \mathbb{C} | X^2 > \lambda_0^2/4 - 4b^2\varepsilon^2\}} e^{4tb^2} e^{-t\frac{X^2}{\varepsilon^2}} \frac{1}{2b\varepsilon} dX \\
& = 0,
\end{aligned} \tag{3.21}$$

Combining (3.20) and (3.21), we complete the proof. ■

3.2 Noncompact fiber case

Now we discuss the case that the fiber is complete but not compact. Assume $Y = \mathbb{R}^k$ is the k -dimensional Euclidean space, i.e. E is a real vector bundle of rank k . Similar to the compact case, we equip E with a Riemannian metric g_E given in (2.3). Moreover, we assume that for each $x \in M$, the metric $g_Y(x)$ is a rescale of the Euclidean metric on $Y = \mathbb{R}^k$. In other words, locally $g_Y(x) = f(x)g_{\mathbb{R}^k}$ for some smooth function $f(x)$. As the

fiber is noncompact, we shall no longer expect the spectrum of $|A_1|$ to have a uniformly positive lower bound.

To begin with, consider a function h on E that is locally defined by

$$h(x, y) = \frac{1}{2}(y^1)^2 + \frac{1}{2}(y^2)^2 + \cdots + \frac{1}{2}(y^k)^2, \quad (3.22)$$

where $y^1, \dots, y^k$ are the coordinates of the fibers. We will always pick a local chart so that $|\nabla h|^2 = |y|^2$. By choosing the transition functions to be $O(k)$ -valued, h can be defined globally. It is easy to see that h is a Morse-Bott function on E , namely the set of critical points form a submanifold (in fact it is the image of the 0-section), and at any point of the critical submanifold, the Hessian $\text{Hess } h$ is nondegenerate along the normal direction.

For $\tau \in \mathbb{R}$ positive, let $D_{\tau h}^2$ be the Witten Laplacian defined in Definition 2.3.5. As an analogue of (3.2), we have

$$D_{\varepsilon, \tau h} = \varepsilon \tilde{D}_M + D_Y + \tau \hat{c}(dh) - \frac{\varepsilon^2}{4} T. \quad (3.23)$$

Let $\tilde{D}_Y := D_Y + \tau \hat{c}(dh)$. Suppose for now that $\ker \tilde{D}_Y$ is a vector bundle. Consider the following decomposition of the L^2 -space:

$$L^2(\Lambda^* E) = L^2(\ker \tilde{D}_Y)^\perp \oplus L^2(\ker \tilde{D}_Y). \quad (3.24)$$

Then we have $\frac{1}{\varepsilon}D_{\varepsilon,\tau h} = \begin{pmatrix} \tilde{A}_1 & \tilde{A}_2 \\ \tilde{A}_2^* & \tilde{A}_3 \end{pmatrix}$, where $\tilde{A}_2^*$ is the adjoint of $\tilde{A}_2$, and

$$\begin{aligned} \tilde{A}_1 &= \frac{1}{\varepsilon}D_Y + \frac{1}{\varepsilon}\tau\hat{c}(dh) + p^\perp\tilde{D}_Mp^\perp + \varepsilon p^\perp\frac{T}{4}p^\perp, \\ \tilde{A}_2 &= p^\perp\tilde{D}_Mp + \varepsilon p^\perp\frac{T}{4}p, \\ \tilde{A}_3 &= D_0 + \varepsilon p\frac{T}{4}p. \end{aligned}$$

An argument similar to the proof of Lemma 3.1.1 shows that $\text{spec}(|\tilde{A}_1|) \subset [\frac{\tilde{\lambda}_0(\tau)}{\varepsilon}, +\infty)$, and $\tilde{A}_2$ is a bounded operator.

To construct a contour as in the compact case, we need a uniform lower bound on the spectrum (except 0) of $\frac{1}{\varepsilon}D_{\varepsilon,\tau h}$.

Proposition 3.2.1 *There exists $\lambda'_0 > 0$ such that for any $\varepsilon > 0$ sufficiently small, we have the following uniform bound for non-zero eigenvalues λ'_ε of $\frac{1}{\varepsilon}D_{\varepsilon,\tau h}$:*

$$|\lambda'_\varepsilon| \geq \lambda'_0, \quad \forall \lambda'_\varepsilon \neq 0 \in \text{spec}\left(\frac{1}{\varepsilon}D_{\varepsilon,\tau h}\right).$$

Proof: We will follow [23, section 4.2] and derive the uniform bound by comparing two spectral sequences. For references of spectral sequence, we refer to [23, section 4.2], [24, section 1], and [25, chapter 3.5].

It is clear that every λ'_ε corresponds to a non-zero eigenvalue $\lambda_\varepsilon = \varepsilon\lambda'_\varepsilon$ of $D_{\varepsilon,\tau h}$. To obtain a uniform lower bound, it suffices to consider λ_ε which decays to 0 as $\varepsilon \rightarrow 0$. Let $G_{\Lambda_r^0}$ be the direct sum of eigenspaces of $D_{\varepsilon,\tau h}$, such that the corresponding eigenvalues decay at least like ε^r , i.e.

$$G_{\Lambda_r^0} = \bigoplus_{\lambda_\varepsilon = o(\varepsilon^r)} E(\lambda_\varepsilon).$$

Define $E_r := \lim_{\varepsilon \rightarrow 0} G_{\Lambda_r^0}$. Then it is easy to check that $(E_r, \varepsilon^{-r}d_{\tau h})$ forms a spectral

sequence. A direct observation is that the sequence terminates at E_k , i.e. $E_{k-1} \neq E_k = E_{k+1} = \cdots = E_\infty$, if and only if every $\lambda_\varepsilon \neq 0$ decays at most like ε^{k-1} .

Next, we consider Serre's filtration

$$\begin{aligned} F^i &:= \{a(z, y)dx^\alpha \wedge dy^\beta \mid |\alpha| \geq i\} \\ &= \bigoplus_{p \geq i} \Lambda^p M \otimes \Lambda^* Y. \end{aligned}$$

This induces the Leray spectral sequence $(\bar{E}_r, \bar{d}_{r, \tau h})$. It is well known that

$$\begin{aligned} \bar{E}_1 &\cong \Lambda^* M \otimes \ker(\tilde{D}_Y|_{\Lambda^* Y}) \cong \Lambda^* M \otimes \text{span}_{\mathbb{R}} \left\{ e^{-\tau \frac{|y|^2}{2}} \right\} \\ \bar{E}_2 &\cong H^*(\bar{E}_1) \cong H_{dR}^* M \otimes \text{span}_{\mathbb{R}} \left\{ e^{-\tau \frac{|y|^2}{2}} \right\} \\ \bar{E}_\infty &\cong \ker(D_{\varepsilon, \tau h}). \end{aligned} \tag{3.25}$$

By Theorem 1.3 in [12], for $\varepsilon > 0$ small enough, we have

$$\ker(D_{\varepsilon, \tau h}) \cong H_{dR}^*(E, U_c) = H_{dR}^*(E) \cong H_{dR}^*(M), \tag{3.26}$$

where the second equality comes from the fact that $U_c := \{p \in E \mid h(p) < -c < 0\}$ is an empty set, and the last equality holds true because de Rham cohomology groups are homotopy invariant. As a consequence, we see that $\bar{E}_2 \cong \bar{E}_\infty$, and therefore the Leray spectral sequence terminates at $\bar{E}_2$.

Now notice that the proof of Theorem 0.2 in [23] extends to our situation trivially. In particular, we have $E_r \cong \bar{E}_r$. As a result, every λ_ε has exactly linear decay. Thus we can find a constant $\lambda'_0 > 0$ such that

$$|\lambda'_\varepsilon| = \frac{1}{\varepsilon} |\lambda_\varepsilon| \geq \lambda'_0.$$

■

Now if we take $b > 0$ so that $20b < \min\{\lambda'_0, \lambda_1(D_0)\}$, the rest of the argument in section 3.1 can be extended to the noncompact case without any change.

It turns out that, in the noncompact setting, the assumption that $\ker \tilde{D}_Y$ is a vector bundle is redundant. To see this, we observe that since $T^*E = \pi^*T^*M \oplus V^*E$, we have

$$\Lambda^*E = \pi^*(\Lambda^*M) \otimes \Lambda^*(V^*E). \quad (3.27)$$

For simplicity, in the following we will still denote $\pi^*(\Lambda^*M)$ by Λ^*M .

Since $\tilde{D}_Y$ only acts on the $\Lambda^*(V^*E)$ part of the exterior algebra bundle, it is easy to see

$$\ker \tilde{D}_Y = \Lambda^*M \otimes \ker(\tilde{D}_Y|_{\Lambda^*(V^*E)}). \quad (3.28)$$

A direct computation in local coordinates shows

$$(\tilde{D}_Y|_{\Lambda^*(V^*E)})^2 = - \sum_{j=1}^k \frac{\partial^2}{\partial(y^j)^2} - k\tau + \tau|y|^2 + \tau \cdot 2 \sum_{j=1}^k f^j \wedge \iota_{f_j}.$$

It is a standard fact (see, for example, [26, Section 5.4]) that locally

$$\ker \tilde{D}_Y|_{\Lambda^*(V^*E)} = \ker(\tilde{D}_Y|_{\Lambda^*(V^*E)})^2 = \text{span}_{\mathbb{R}}\{e^{-\frac{\tau|y|^2}{2}}\}. \quad (3.29)$$

As we have seen, $e^{-\frac{\tau|y|^2}{2}}$ can be extended to a global function on E . As a result, $\ker \tilde{D}_Y|_{\Lambda^*(V^*E)} \cong E \times \mathbb{R}^1$ is a trivial line bundle over E . Therefore we end up with the following noncompact version of theorem 3.1.3.

Theorem 3.2.2 *Let $E \rightarrow M$ be a vector bundle over a compact manifold M . For any*

$t > 0$ fixed, we have

$$\lim_{\varepsilon \rightarrow 0} \left\| e^{-\frac{t}{\varepsilon^2} D_{\varepsilon, \tau h}^2} - \begin{pmatrix} 0 & 0 \\ 0 & e^{-t D_0^2} \end{pmatrix} \right\|_{0,0} = 0. \quad (3.30)$$

Remark 3.2.3 If we decompose $(\lambda - \frac{1}{\varepsilon} D_{\varepsilon, \tau h})^{-1}$ with respect to (3.24) as

$$\left(\lambda - \frac{1}{\varepsilon} D_{\varepsilon, \tau h} \right)^{-1} = \begin{pmatrix} \tilde{T}_1 & \tilde{T}_2 \\ \tilde{T}_2^* & \tilde{T}_3 \end{pmatrix}, \quad (3.31)$$

then from the proof of Lemma 3.1.2 one can actually deduce that for $|\lambda| \leq \frac{\lambda'_0}{2\varepsilon}$,

$$\|\tilde{T}_1\|_{0,0}, \|\tilde{T}_2\|_{0,0}, \|\tilde{T}_3 - (\lambda - D_0)^{-1}\|_{0,0} \leq C\varepsilon, \quad (3.32)$$

3.3 Index of vector bundle

In this section we show that the index of the total space E of a vector bundle coincides with that of the base manifold M . We begin by reviewing the construction of the (L^2 -) index of Witten Laplacian on the total space E (see [11], [12] for more details). Let $L^2 \Lambda^*(E)$ denote the space of L^2 differential forms on E . Set $\Omega_{(2)}^*(E) = L^2 \Lambda^*(E) \cap \Omega^*(E)$ to be the space of smooth L^2 differential forms. As can be seen easily, $\Omega_{(2)}^*(E)$ together with $d_{\tau h}$ forms a cochain complex:

$$\cdots \xrightarrow{d_{\tau h}} \Omega_{(2)}^{i-1}(E) \xrightarrow{d_{\tau h}} \Omega_{(2)}^i(E) \xrightarrow{d_{\tau h}} \Omega_{(2)}^{i+1}(E) \xrightarrow{d_{\tau h}} \cdots. \quad (3.33)$$

Denote by $H_{(2)}^*(E, d_{\tau h})$ the corresponding cohomology. The index of the Witten Laplacian is then defined as

$$\chi(E, d_{\tau h}) = \sum_{i=0}^{\dim E} (-1)^i \dim H_{(2)}^*(E, d_{\tau h}). \quad (3.34)$$

As the index of a closed manifold, $\chi(E, d_{\tau h})$ enjoys a heat operator expression:

$$\chi(E, d_{\tau h}) = \text{Tr}_s e^{-tD_{\tau h}^2}. \quad (3.35)$$

Notably, it was shown in [12] that $H_{(2)}^*(E, d_{\tau h})$ is independent of $\tau > 0$. For this reason, in the following we will simply denote $\chi(E, d_{\tau h})$ by $\chi(E)$.

As shown in (3.35), to understand the index, it is crucial to study the heat operator $e^{-tD_{\tau h}^2}$. In fact, we have the following convergence result for the heat trace, which plays a crucial role in relating two indexes.

Theorem 3.3.1 *For any $t > 0$,*

$$\lim_{\varepsilon \rightarrow 0} \text{Tr}_s \left(e^{-\frac{t}{\varepsilon^2} D_{\varepsilon, \tau h}^2} \right) = \text{Tr}_s \left(e^{-tD_0^2} \right). \quad (3.36)$$

Proof: Using Remark 3.2.3, the assertion follows from an argument as in [24, Section 5(c)-5(e)]. ■

Now we are ready to state and prove the main theorem of this chapter.

Theorem 3.3.2 *Suppose E is a vector bundle over a compact manifold M . Then*

$$H_{(2)}^*(E, d_{\tau h}) \simeq H_{dR}^*(M). \quad (3.37)$$

In particular,

$$\text{ind}(E) = \text{ind}(M). \quad (3.38)$$

Proof: Using [12], more specifically arguing as in the proof of [12, Theorem 4.1], one can show that the Witten deformed L^2 -cohomology of E is isomorphic to the (absolute)

de Rham cohomology:

$$H_{(2)}^*(E, d_{\tau h}) \cong H_{dR}^*(E) \cong H_{dR}^*(M). \quad (3.39)$$

Thus (3.38) follows directly. Here we also provide an alternative proof. Recall that $D_0 = p\tilde{D}_M p$ maps $\ker \tilde{D}_Y$ to itself. Using the splittings (3.27) and (3.28), we can write D_0 as a twisted Dirac operator

$$D_0 = D_M \otimes 1 + 1 \otimes D^u : \Lambda^* M \otimes \ker(\tilde{D}_Y|_{\Lambda^*(V^*E)}) \rightarrow \Lambda^* M \otimes \ker(\tilde{D}_Y|_{\Lambda^*(V^*E)}),$$

where D^u is the induced Dirac operator on $\ker(\tilde{D}_Y|_{\Lambda^*(V^*E)})$. With respect to the natural $\mathbb{Z}_2$ grading

$$\begin{aligned} \Lambda^+ E &= \left(\Lambda^+ M \otimes \Lambda^+(V^*E) \right) \oplus \left(\Lambda^- M \otimes \Lambda^-(V^*E) \right), \\ \Lambda^- E &= \left(\Lambda^- M \otimes \Lambda^+(V^*E) \right) \oplus \left(\Lambda^+ M \otimes \Lambda^-(V^*E) \right), \end{aligned}$$

the "positive" part of D_0 is $D_0^+ = D_M^+ \otimes 1 + 1 \otimes D^u$. One should notice that $D_M^+ \otimes 1$ and $1 \otimes D^u$ map $\Lambda_M^+ \otimes \ker(\tilde{D}_Y|_{\Lambda^*(V^*E)})$ to $\Lambda^- M \otimes \Lambda^+(V^*E)$ and $\Lambda^+ M \otimes \Lambda^-(V^*E)$ separately. When coupled with the projection to the kernel of $\tilde{D}_Y|_{\Lambda^*(V^*E)}$, $1 \otimes D^u$ is the zero map. Therefore the kernel and cokernel of D_0^+ split as

$$\begin{aligned} \ker D_0^+ &= \ker D_M^+ \otimes \ker(\tilde{D}_Y|_{\Lambda^*(V^*E)}), \\ \text{coker } D_0^+ &= \text{coker } D_M^+ \otimes \ker(\tilde{D}_Y|_{\Lambda^*(V^*E)}). \end{aligned} \quad (3.40)$$

Since $\ker(\tilde{D}_Y|_{\Lambda^*(V^*E)})$ is 1-dimensional, we obtain

$$\begin{aligned}
 \operatorname{ind} D_0^+ &= \dim \ker D_0^+ - \dim \operatorname{coker} D_0^+ \\
 &= \dim \ker D_M^+ - \dim \operatorname{coker} D_M^+ \\
 &= \operatorname{ind}(M).
 \end{aligned} \tag{3.41}$$

On the other hand, Theorem 3.3.1 and the heat equation formulation for index imply

$$\operatorname{ind} D_0^+ = \operatorname{Tr}_s e^{-tD_0^2} = \operatorname{Tr}_s \begin{pmatrix} 0 & 0 \\ 0 & e^{-tD_0^2} \end{pmatrix} = \lim_{\varepsilon \rightarrow 0} \operatorname{Tr}_s e^{-\frac{t}{\varepsilon^2} D_{\varepsilon, \tau h}^2} = \operatorname{ind}(E), \tag{3.42}$$

which proves the theorem. ■

Chapter 4

Analytic Torsion of Vector Bundles

4.1 Regularized torsion zeta function and analytic torsion

Let (M, g_M) be a closed n -dimensional Riemannian manifold, and $E \rightarrow M$ be a vector bundle over M with rank k . We impose the same assumption on the metric g_E as in Section 3.2. Locally we can always write the metric as

$$g_E = g_M + f(x)g_{\mathbb{R}^k}. \quad (4.1)$$

In general, the spectrum of the Hodge Laplacian Δ over E may contain a nonempty continuous spectrum. As a result, the heat operator $e^{-t\Delta}$ is not of trace class, and hence the torsion zeta function $\zeta(s)$ given in 2.34 is not well-defined even when $\operatorname{Re} s$ is large. In this section, to extend the theory discussed in Section 2.6 to the noncompact setting, we introduce a Witten deformation of the Hodge Laplacian as a regularization, and define an analytic torsion on E accordingly. Moreover, we show that the definition is valid by applying results in [11] and [12] and calculating the asymptotic expansion of

the supertrace of the heat operator.

Let h and $\Delta_{\tau h} = D_{\tau h}^2$ be the Morse-Bott function and the corresponding Witten Laplacian introduced in Section 3.2. It is easy to verify that the triple (E, g_E, h) is strongly tame (we refer to [27] and [12] for more details about strong tameness), namely:

- (1) (E, g_E) has bounded geometry,
- (2) $\limsup_{p \rightarrow \infty} \frac{|\nabla^2 h|}{|\nabla h|^2}(p) = 0$,
- (3) $\lim_{p \rightarrow \infty} |\nabla h| = \infty$.

Thus it follows from [12, Theorem 2.2] that the spectrum of $D_{\tau h}^2$ is discrete.

Let $K_{\tau h}$ be the heat kernel of $e^{-tD_{\tau h}^2}$. Then clearly $NK_{\tau h}$ is the heat kernel of $Ne^{-tD_{\tau h}^2}$. We first show that the heat operator is of trace class, and the McKean-Singer formula still holds true in the noncompact situation.

Lemma 4.1.1 *$Ne^{-tD_{\tau h}^2}$ is a trace class operator. Moreover, there holds*

$$\mathrm{Tr}_s \left(Ne^{-tD_{\tau h}^2} \right) = \int_E \mathrm{tr}_s \left(NK_{\tau h}(t, p, p) \right) d\mathrm{vol}_{g_E}, \quad (4.2)$$

where tr_s denotes the pointwise supertrace.

Proof: Let $V(y) = |\nabla h|^2$. One can check directly that it satisfies $tV(y) = V(t^{1/2}y)$. So by a similar argument as in [11, Section 6], we conclude that $Ne^{-tD_{\tau h}^2}$ is of trace class. Equation (4.2) follows from this directly. ■

Using this lemma, we introduce the following definition.

Definition 4.1.2 *We define the (Witten-)torsion zeta function to be*

$$\zeta_{\tau h}(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \mathrm{Tr}_s \left(Ne^{-tD_{\tau h}^2} \right) dt. \quad (4.3)$$

The regularized analytic torsion $T_{\tau h}(E)$ of E is defined by

$$\log T_{\tau h}(E) = -\frac{1}{2}\zeta'_{\tau h}(0), \quad (4.4)$$

provided that the function $\zeta_{\tau h}(s)$ is differentiable at $s = 0$.

We will devote the rest of this section to proving the differentiability of $\zeta_{\tau h}(s)$ at the origin. First, it follows from [11, Theorem 1.1] that, for $t \in (0, 1]$ and $\tau \in (0, \sqrt{t}]$, we have the following asymptotic expansion for $K_{\tau h}$ along the diagonal:

$$K_{\tau h}(t, p, p) \sim \frac{1}{(4\pi t)^{\frac{n+k}{2}}} e^{-t\tau^2|y|^2} \sum_{j=0}^{\infty} t^j A_{\tau,j}(p), \quad (4.5)$$

where $A_{\tau,j}$'s are smooth sections of $\text{End}(\Lambda^* E)$. Also, for r large enough we have the following remainder estimate:

$$\left| K_{\tau h} - \frac{1}{(4\pi t)^{\frac{n+k}{2}}} e^{-t\tau^2|y|^2} \sum_{j=0}^r t^j A_{\tau,j} \right| \leq C t^{\frac{r+1}{3} - \frac{n+k}{2}} \tau^{\frac{-2r+4}{3}} e^{-a\tilde{d}_{\tau}(t,y)}, \quad (4.6)$$

where $\tilde{d}_{\tau}$ is a parabolic distance satisfying $\tilde{d}_{\tau}(t, y) \geq \min\{c_1\tau|y|, c_2t\tau^2|y|^2\}$ for some positive constants c_1 and c_2 .

From section 2.2, we see that each $A_{\tau,j}$ can be written as

$$A_{\tau,j} = \sum_{\substack{I, I' \subset \{1, 2, \dots, n\} \\ J, J' \subset \{1, 2, \dots, k\}}} A_{\tau,j}^{I, J, I', J'} c(e^I f^J) \hat{c}(e^{I'} f^{J'}), \quad (4.7)$$

and $A_{\tau,j}^{I, J, I', J'}$'s are algebraic combinations of derivatives of the curvature of g_E and the

function h . In particular, we have

$$A_{\tau,j}^{I,J,I',J'} = \sum_{0 \leq l \leq j} \sum_{\sum_{i=1}^l |\alpha_i| \leq 2j} c_{\alpha_1, \dots, \alpha_l}^{I,J,I',J'}(\tau) \partial^{\alpha_1} h \dots \partial^{\alpha_l} h.$$

Joining this with the asymptotic expansion of the heat kernel $K_{\tau h}$ and Lemma 4.1.1, there holds

$$\begin{aligned} \text{Tr}_s \left(N e^{-t D_{\tau h}^2} \right) &\sim \frac{1}{(4\pi t)^{\frac{n+k}{2}}} \text{Tr}_s \left(N c(e^I f^J) c(e^{I'} f^{J'}) \right) \\ &\quad \sum_{j=0}^{\infty} \sum_{I,J,I',J'} t^j \left(\int_E e^{-t\tau^2 |y|^2} \sum_{0 \leq l \leq j} \sum_{\sum_{i=1}^l |\alpha_i| \leq 2j} c_{\alpha_1, \dots, \alpha_l}^{I,J,I',J'}(\tau) \partial^{\alpha_1} h \dots \partial^{\alpha_l} h d\text{vol}_{g_E} \right). \end{aligned}$$

The supertrace terms on the right are always real numbers, so it remains to evaluate the integrals above. Note that the derivative factors satisfy

$$(\partial^{\alpha_i} h)(y) = t^{\frac{|\alpha_i|}{2}-1} (\partial^{\alpha_i} h)(t^{1/2} y).$$

Then using the substitution $\tilde{y} = t^{1/2} y$, we find that

$$\begin{aligned} &\int_{\mathbb{R}^k} e^{-t\tau^2 |y|^2} \partial^{\alpha_1} h \dots \partial^{\alpha_l} h dy \\ &= \int_{\mathbb{R}^k} e^{-t\tau^2 |y|^2} t^{\frac{|\alpha_1|+\dots+|\alpha_l|}{2}-l} \partial^{\alpha_1} h(t^{1/2} y) \dots \partial^{\alpha_l} h(t^{1/2} y) dy \\ &= \int_{\mathbb{R}^k} e^{-\tau^2 |\tilde{y}|^2} t^{\frac{|\alpha_1|+\dots+|\alpha_l|}{2}-l} \partial^{\alpha_1} h(\tilde{y}) \dots \partial^{\alpha_l} h(\tilde{y}) t^{-k/2} d\tilde{y} \\ &= \tilde{C}_{\alpha_1, \dots, \alpha_l}(\tau) t^{\frac{|\alpha_1|+\dots+|\alpha_l|}{2}-l-k/2}. \end{aligned}$$

Therefore, by Fubini's theorem and the assumption that M is compact, we obtain

$$\begin{aligned}
& \int_E e^{-t\tau^2|y|^2} A_{\tau,j}^{I,J,I',J'}(p) d\text{vol}_{g_E} \\
&= \int_M \left(\int_{E_x} e^{-t\tau^2|y|^2} A_{\tau,j}^{I,J,I',J'}(x,y) dy \right) f(x) d\text{vol}_x \\
&= \sum_{0 \leq l \leq j} \sum_{\sum |\alpha_i| \leq 2j} \int_M c_{\alpha_1, \dots, \alpha_l}^{I,J,I',J'}(\tau) \left(\int_{\mathbb{R}^k} e^{-t\tau^2|y|^2} \partial^{\alpha_1} h \dots \partial^{\alpha_l} h dy \right) f(x) d\text{vol}_x \\
&= C_j^{I,J,I',J'}(\tau) \sum_{0 \leq l \leq j} \sum_{\sum |\alpha_i| \leq 2j} t^{\frac{|\alpha_1| + \dots + |\alpha_l|}{2} - l - \frac{k}{2}}.
\end{aligned}$$

This leads to the following asymptotic expansion for the supertrace:

Theorem 4.1.3 *For any $\tau \in (0, \sqrt{t}]$, we have*

$$\text{Tr}_s \left(N e^{-tD_{\tau h}^2} \right) \sim \frac{1}{(4\pi t)^{\frac{n+k}{2}}} \sum_{j=0}^{\infty} \sum_{l \leq j} \sum_{\sum |\alpha_i| \leq 2j} C_{\alpha_1, \dots, \alpha_l}(\tau) t^{j + \frac{\sum |\alpha_i|}{2} - l - \frac{k}{2}}$$

as $t \rightarrow 0$. Moreover, for r large enough, we have the corresponding remainder estimate:

$$\begin{aligned}
& \left| \text{Tr}_s \left(N e^{-tD_{\tau h}^2} \right) - \frac{1}{(4\pi t)^{\frac{n+k}{2}}} \sum_{j=0}^r \sum_{l \leq j} \sum_{\sum |\alpha_i| \leq 2j} C_{\alpha_1, \dots, \alpha_l}(\tau) t^{j + \frac{\sum |\alpha_i|}{2} - l - \frac{k}{2}} \right| \\
& \leq C(\tau, r) t^{\frac{r+3}{1} - 3n - 3k}.
\end{aligned}$$

Remark 4.1.4 *Obviously $j + \frac{\sum |\alpha_i|}{2} - l \in [0, 3j]$. The asymptotic expansion above can be expressed as*

$$\text{Tr}_s \left(N e^{-tD_{\tau h}^2} \right) \sim \frac{1}{(4\pi t)^{\frac{n+k}{2}}} \sum_{p=-k}^{\infty} a_p t^{\frac{p}{2}}. \quad (4.8)$$

In particular, the leading term is $a_{-k} t^{-\frac{n+2k}{2}}$. Moreover, for r sufficiently large, the re-

mainder estimate implies

$$\left| \text{Tr}_s \left(N e^{-t D_{\tau h}^2} \right) - \frac{1}{(4\pi t)^{\frac{n+k}{2}}} \sum_{p=-k}^r a_p t^{\frac{p}{2}} \right| \leq Ct.$$

As a consequence, we have the following main theorem of this section.

Theorem 4.1.5 *For any $\tau \in (0, 1)$, $\zeta_{\tau h}(s)$ admits an analytic continuation as a meromorphic function on $\mathbb{C}$, and it is analytic near 0. Consequently, $\log T_{\tau h}(E)$ is well-defined.*

Proof: Let D_q^2 denote the restriction of $D_{\tau h}^2$ on $\Lambda^q E$. Let $\Sigma_0 = \frac{1}{(4\pi t)^{\frac{n+k}{2}}} \sum_{p=-k}^r a_p t^{\frac{p}{2}}$ be a partial sum of the series in the asymptotic expansion, with r large enough such that Remark 4.1.4 applies.

We consider the two intervals $[0, 1]$ and $[1, +\infty]$ separately. Let us first deal with the second interval. Let λ_0 be the smallest nonzero eigenvalue of $N e^{-t D_{\tau h}^2}$. Then it is easy to see

$$\left| \text{Tr}_s (N e^{-t D_{\tau h}^2}) \right| \leq C e^{-\frac{\lambda_0}{2} t}$$

for some constant $C > 0$. Therefore integrating over the second interval yields a holomorphic function on $\mathbb{C}$.

For the first interval, it follows from a straightforward computation that

$$\begin{aligned} & \int_0^1 t^{s-1} \text{Tr}_s (N e^{-t D_{\tau h}^2}) dt \\ &= \int_0^1 t^{s-1} \left(\text{Tr}_s (N e^{-t D_{\tau h}^2}) - \Sigma_0 \right) dt + \int_0^1 t^{s-1} \Sigma_0 dt \\ & \quad - \sum_{q=1}^{n+k} (-1)^q q \int_0^1 t^{s-1} \dim(\ker D_q^2) dt. \end{aligned}$$

By Remark 4.1.4, the first integral above is holomorphic for $s \in \mathbb{C}$ satisfying $\text{Re}(s) > -1$.

Next, notice that each term in

$$\int_0^1 t^{s-1} \Sigma_0 dt = \sum_{p=-k}^r \int_0^1 t^{s-1} \frac{1}{(4\pi t)^{\frac{n+k}{2}}} a_p t^{\frac{p}{2}} dt$$

creates a single pole at $s = -\frac{k}{2}, -\frac{k+1}{2}, \dots$. Also, we see that

$$-\sum_{q=1}^{n+k} (-1)^q q \int_0^1 t^{s-1} \dim(\ker D_q^2) dt = -\frac{1}{s} \sum_{q=1}^{n+k} (-1)^q q \dim(\ker D_q^2)$$

has a simple pole at $s = 0$. Thus the proof is completed by noticing that $\frac{1}{\Gamma(s)}$ is an entire function with a simple zero at $s = 0$. ■

Remark 4.1.6 *The results above can be extended easily to the following setting. Assume the metric $\tilde{g}_E$ still takes the form $g_M + \tilde{g}_Y$ as (2.3). Instead of a family of Euclidean metrics, we now assume that $\tilde{g}_Y$ is a smooth family of asymptotically flat metrics with bounded geometry. In this case, we consider the metric g_E constructed in the beginning of this section as a background metric, and define the potential function h using the distance function of g_E . Under these assumptions, all the results in this section continue to hold, with only minor modifications to the proofs.*

4.2 Comparison with other regularization

In this section, we compare our treatment with other regularizations of analytic torsions over noncompact manifolds.

In [8], F. Calegari and A. Venkatesh used the torsion in the homology of arithmetic groups to study the arithmetic Langlands program. Their main tool is analytic torsion and a Cheeger-Müller type theorem. Due to the noncompact nature of certain arithmetic manifolds, they were forced to develop a regularization theory for analytic torsion on a

3-dimensional complete noncompact hyperbolic manifold (the idea goes back to early works of others, including W. Müller [28] and R. Melrose [29]). In their situation, the noncompactness comes from the existence of toric cusps. Roughly speaking, a toric cusp is a finite volume end at infinity which looks like $\mathbb{T}^2 \times [0, \infty)$. For the simplicity of discussion, assume M^3 is a finite-volume hyperbolic manifold with exactly one cusp C . Then their idea was to create a truncation $M(Y)$ by cutting off the cusp at a large "height" Y . As $M(Y)$ is a compact manifold with boundary, there is a nicely behaved heat kernel $K_Y(t, x, y)$.

As a consequence of the Selberg trace formula, they obtained the following asymptotic formula as Y approaches infinity:

$$\int_{M(Y)} \text{Tr } K_Y(t, x, x) dx \sim k_0 \log Y + k_1 + (\text{rapidly decaying terms in } Y), \quad (4.9)$$

where k_0 and k_1 are constants. In particular, they define the regularized heat trace to be the constant term k_1 .

A similar local regularization also appears in the work of W. Müller and J. Pfaff [30], where they considered a general noncompact hyperbolic manifold M^n . They pointed out that $\text{Tr}^*(e^{-t\Delta})$ actually equals the spectral side of the Selberg trace formula applied to $e^{-t\Delta}$. In addition, they computed the following small-time asymptotic expansion:

$$\text{Tr}^*(e^{-t\Delta}) \sim a_{-n/2} t^{-\frac{n}{2}} + \cdots + a_{-1/2} t^{-\frac{1}{2}} + b t^{-\frac{1}{2}} \log t + O(t^{1/2}). \quad (4.10)$$

The disadvantage of the regularization above is twofold. First, compared to the compact situation, there is an extra logarithm singularity in the asymptotic expansion (4.10). This will lead to a technical difficulty in the analyze of the small time behavior. Second, as the regularized heat trace is only the constant term in (4.9), if one varies the

metric smoothly, there may not be a nice variation formula for the regularized analytic torsion. In contrast to their treatment, our regularization given by Witten deformation yields an asymptotic expansion (4.8) that contains only polynomial singularities. Besides, as will be seen in the next section, the regularized analytic torsion admits the same variation formula as in the compact case, which following a standard argument implies the metric independence of the Quillen metric.

4.3 Quillen Metric and Topological Invariance

In this section, we introduce a corresponding Quillen metric on the determinant line associated with $E^{n+k} \rightarrow M^n$, and show that it is independent of the metric chosen. In order to obtain a nontrivial analytic torsion with a nice asymptotic expansion, we impose here an additional assumption that the dimensions of E and M are both odd. Let $g(\sigma)$ be a family of Riemannian metrics on E that depends on $\sigma \in [0, 1]$ smoothly. Assume for each σ ,

$$g(\sigma) = g_M(\sigma) + g_Y(\sigma) \quad (4.11)$$

takes the form as in Section 3.2. For each σ , let $h_\sigma = |g|_{g(\sigma)}^2$ be the function defined in (3.22). Let $L(E) = \bigotimes_{i=0}^{\dim E} (\det H_{dR}^i(E))^{(-1)^i}$ denote the determinant line of E , and $|\cdot|_{L(E), g(\sigma)}$ denote the L^2 -norm on $L(E)$ induced by $g(\sigma)$. Following the compact case, we introduce a Quillen metric on E .

Definition 4.3.1 *The Quillen metric of E associated with the metric $g(\sigma)$ is defined to be*

$$\|\cdot\|_{L(E), \tau h, g(\sigma)} = T_{\tau h}(E, g(\sigma)) |\cdot|_{L(E), g(\sigma)}$$

To address a variation formula for the Quillen metric, we first compute the derivative of the Witten Laplacian. In the following we will alternatively use a dot to stand for the

derivative with respect to the parameter σ . Sometimes we will omit the dependence on σ for convenience. For instance, we will write $\Delta_{\tau h}(\sigma)$ as $\Delta_{\tau h}$. Let $*_{\sigma}$ be the Hodge star operator associated with $g(\sigma)$, and $\alpha := *_{\sigma}^{-1} \dot{*}_{\sigma}$.

Lemma 4.3.2 $\dot{\Delta}_{\tau h} = A + B$, where

$$\begin{aligned} A &= -d_{\tau h} \alpha d_{\tau h}^* + d_{\tau h} d_{\tau h}^* \alpha - \alpha d_{\tau h}^* d_{\tau h} + d_{\tau h}^* \alpha d_{\tau h}, \\ B &= [d_{\tau h}, \tau \dot{h}] d_{\tau h}^* + d_{\tau h}^* [d_{\tau h}, \tau \dot{h}] - [d_{\tau h}^*, \tau \dot{h}] d_{\tau h} - d_{\tau h} [d_{\tau h}^*, \tau \dot{h}]. \end{aligned}$$

Proof: First, we observe that

$$\dot{d}_{\tau h} = \tau d\dot{h} \wedge = [d_{\tau h}, \tau \dot{h}], \quad (4.12)$$

where $[\cdot, \cdot]$ is the supercommutator. On the other hand, recall that $d_{\tau h}^* = e^{\tau h} d^* e^{-\tau h}$. Taking derivative on σ yields

$$\begin{aligned} \dot{d}_{\tau h}^* &= \tau \dot{h} e^{\tau h} d^* e^{-\tau h} + e^{\tau h} (-\alpha d^* + d^* \alpha) e^{-\tau h} - e^{\tau h} d^* (\tau \dot{h}) e^{-\tau h} \\ &= -\alpha d_{\tau h}^* + d_{\tau h}^* \alpha - [d_{\tau h}^*, \tau \dot{h}]. \end{aligned} \quad (4.13)$$

Thus the assertion follows from a straightforward computation. ■

Now we extend the classic variation formula of the heat trace to the noncompact case.

Proposition 4.3.3 *For each $t > 0$, the heat operator $e^{-t\Delta_{\tau h}(\sigma)}$ depends differentiably on σ , and there holds*

$$\frac{\partial}{\partial \sigma} \text{Tr}_s (N e^{-t\Delta_{\tau h}}) = -t \text{Tr}_s (N \dot{\Delta}_{\tau h} e^{-t\Delta_{\tau h}}). \quad (4.14)$$

Proof: Without loss of generality, in the following we will only provide estimates for operators restricting to the space of functions. The estimates for differential forms

are very similar. To prove (4.14), write $\Delta_{\tau h}(\sigma) = \Delta(\sigma) + V(\sigma)$, where V is a smooth function coming from Witten deformation. As clearly V has a finite lower bound, by shifting V if necessary, we may assume that $V > 0$. Notice that for each $\sigma > 0$ fixed, $(E, g(\sigma))$ has bounded geometry, and Δ is a C^∞ -bounded uniformly elliptic differential operator. It follows from Lemma 1.2 in [31, Appendix 1] that for any $u \in C_0^\infty(E)$,

$$\|u\|_{H^2(\sigma)} \leq C_1 (\|\Delta u\|_{H^0(\sigma)} + \|u\|_{H^0(\sigma)}) \quad (4.15)$$

for some uniform constant $C_1 > 0$. Here we use $H^i(\sigma)$ to denote the i -th Sobolev space associated with the metric $g(\sigma)$. On the other hand, as $\dot{\Delta}$ is a C^∞ -bounded second order differential operator, it induces a bounded linear operator $\dot{\Delta} : H^2(\sigma) \rightarrow H^0(\sigma)$. Consequently, one has

$$\|\dot{\Delta}u\|_{H^0(\sigma)} \leq C_2 \|u\|_{H^2(\sigma)} \leq C_3 (\|\Delta u\|_{H^0(\sigma)} + \|u\|_{H^0(\sigma)}), \quad (4.16)$$

for any $u \in C_0^\infty(E)$. In particular, this implies that $\dot{\Delta}$ is Δ -bounded. On the other hand, as $\dot{V}$ also grows quadratically in y , there is a constant $C_4 > 0$ such that

$$\|\dot{V}u\|_{H^0(\sigma)} \leq C_4 (\|Vu\|_{H^0(\sigma)} + \|u\|_{H^0(\sigma)}). \quad (4.17)$$

Thus $\dot{V}$ is also V -bounded.

To show that the right hand side of (4.14) is well-defined, we need to prove that $\dot{\Delta}_{\tau h}$ is $\Delta_{\tau h}$ -bounded. Pick $u \in C_0^\infty(E)$. First we make the following observation

$$\|\Delta_{\tau h}u\|_{H^0(\sigma)}^2 = \|\Delta u\|_{H^0(\sigma)}^2 + \|Vu\|_{H^0(\sigma)}^2 + 2(\Delta u, Vu)_{H^0(\sigma)}. \quad (4.18)$$

Using the Bochner formula, we can write the Hodge Laplacian as $\Delta = \nabla^* \nabla + R$, where

R is a bounded zeroth order operator that depends on the curvature. Regarding the last term in (4.18), one has

$$\begin{aligned} (\Delta u, Vu)_{H^0(\sigma)} &= \int_E \langle \nabla^* \nabla u, Vu \rangle d\text{vol}_{g(\sigma)} + \int_E \langle Ru, Vu \rangle d\text{vol}_{g(\sigma)} \\ &= \int_E \langle \nabla u, (\nabla V)u \rangle d\text{vol}_{g(\sigma)} + \int_E V |\nabla u|^2 d\text{vol}_{g(\sigma)} + \int_E \langle Ru, Vu \rangle d\text{vol}_{g(\sigma)}. \end{aligned} \quad (4.19)$$

By Cauchy-Schwarz inequality, we have

$$\left| \int_E \langle \nabla u, (\nabla V)u \rangle d\text{vol}_{g(\sigma)} \right| \leq \frac{1}{2} \int_E |V| |\nabla u|^2 d\text{vol}_{g(\sigma)} + \frac{1}{2} \int_E \frac{|\nabla V|^2}{|V|} |u|^2 d\text{vol}_{g(\sigma)}. \quad (4.20)$$

Observe that as a function, $|\nabla V|^2(x, y)$ can be written as a second order polynomial in y . Therefore there exists a $C_5 > 0$ satisfying

$$|\nabla V|^2 \leq C_5 |V|. \quad (4.21)$$

Combining (4.19), (4.20) and (4.21), we infer that

$$(\Delta u, Vu)_{H^0(\sigma)} \geq -C_6 \|u\|_{H^0(\sigma)}^2. \quad (4.22)$$

Therefore (4.18) yields

$$\|\Delta u\|_{H^0(\sigma)}^2 + \|Vu\|_{H^0(\sigma)}^2 \leq \|\Delta_{\tau h} u\|_{H^0(\sigma)}^2 + C_6 \|u\|^2. \quad (4.23)$$

Using (4.16), (4.17), and (4.23), we deduce that

$$\|\dot{\Delta}_{\tau h} u\|_{H^0(\sigma)} \leq \|\dot{\Delta} u\|_{H^0(\sigma)} + \|\dot{V} u\|_{H^0(\sigma)} \leq C_7 (\|\Delta_{\tau h} u\|_{H^0(\sigma)} + \|u\|_{H^0(\sigma)}). \quad (4.24)$$

Consequently, the operator $\dot{\Delta}_{\tau h}(\Delta_{\tau h} + I)^{-1}$ is bounded, and $\dot{\Delta}_{\tau h}e^{-t\Delta_{\tau h}} = \dot{\Delta}_{\tau h}(\Delta_{\tau h} + I)^{-1} \circ (\Delta_{\tau h} + I)e^{-t\Delta_{\tau h}}$ is of trace class.

Using Duhamel's principle, we see that the heat operator $e^{-t\Delta_{\tau h}}$ is differentiable with respect to σ , and its derivative is given by

$$\frac{\partial}{\partial \sigma} e^{-t\Delta_{\tau h}} = - \int_0^t e^{-(t-s)\Delta_{\tau h}} \dot{\Delta}_{\tau h} e^{-s\Delta_{\tau h}} ds. \quad (4.25)$$

By our discussion above, we can apply traces to both sides to obtain

$$\begin{aligned} \frac{\partial}{\partial \sigma} \text{Tr} (e^{-t\Delta_{\tau h}}) &= \text{Tr} \left(\frac{\partial}{\partial \sigma} e^{-t\Delta_{\tau h}} \right) \\ &= - \text{Tr} \left(\int_0^t e^{-(t-s)\Delta_{\tau h}} \dot{\Delta}_{\tau h} e^{-s\Delta_{\tau h}} ds \right) \\ &= - \int_0^t \text{Tr} \left(e^{-(t-s)\Delta_{\tau h}} \dot{\Delta}_{\tau h} e^{-s\Delta_{\tau h}} \right) ds \\ &= - \int_0^t \text{Tr} \left(\dot{\Delta}_{\tau h} e^{-t\Delta_{\tau h}} \right) ds \\ &= -t \text{Tr} \left(\dot{\Delta}_{\tau h} e^{-t\Delta_{\tau h}} \right). \end{aligned} \quad (4.26)$$

As obviously the number operator N is bounded, this completes the proof. $\blacksquare$

To obtain a variation formula for the analytic torsion, we shall compute the right hand side of (4.14) explicitly.

Proposition 4.3.4 $\frac{\partial}{\partial \sigma} \text{Tr}_s (N e^{-t\Delta_{\tau h}}) = t \frac{\partial}{\partial t} \text{Tr}_s (\alpha e^{-t\Delta_{\tau h}}).$

Proof: It follows from a Hodge-type theorem (see [12, Theorem 1.3]), Lemma 4.3.2, and Proposition 4.3.3 that

$$\begin{aligned} \frac{\partial}{\partial \sigma} \text{Tr}_s (N e^{-t\Delta_{\tau h}}) &= \frac{\partial}{\partial \sigma} \text{Tr}_s (N e^{-t\Delta_{\tau h}}) \\ &= -t \text{Tr}_s \left(N \dot{\Delta}_{\tau h} e^{-t\Delta_{\tau h}} \right) \\ &= -t \text{Tr}_s (N A e^{-t\Delta_{\tau h}}) - t \text{Tr}_s (N B e^{-t\Delta_{\tau h}}). \end{aligned} \quad (4.27)$$

Let $\Delta_{\tau h, i}$ denote the restriction of $\Delta_{\tau h}$ to $\Omega^i(E)$. For the A -part, we have

$$\begin{aligned}
-t \operatorname{Tr}_s (N A e^{-t \Delta_{\tau h}}) &= -t \operatorname{Tr}_s (N (-d_{\tau h} \alpha d_{\tau h}^* + d_{\tau h} d_{\tau h}^* \alpha - \alpha d_{\tau h}^* d_{\tau h} + d_{\tau h}^* \alpha d_{\tau h}) e^{-t \Delta_{\tau h}}) \\
&= -t \sum_{i=1}^{n+k} (-1)^i i \left[-\operatorname{Tr} (d_{\tau h} \alpha d_{\tau h}^* e^{-t \Delta_{\tau h, i}}) + \operatorname{Tr} (d_{\tau h} d_{\tau h}^* \alpha e^{-t \Delta_{\tau h, i}}) \right. \\
&\quad \left. - \operatorname{Tr} (\alpha d_{\tau h}^* d_{\tau h} e^{-t \Delta_{\tau h, i}}) + \operatorname{Tr} (d_{\tau h}^* \alpha d_{\tau h} e^{-t \Delta_{\tau h, i}}) \right].
\end{aligned} \tag{4.28}$$

Using the fact that $d_{\tau h} \Delta_{\tau h, i} = \Delta_{\tau h, i+1} d_{\tau h}$ and $d_{\tau h}^* \Delta_{\tau h, i} = \Delta_{\tau h, i-1} d_{\tau h}^*$, we find that

$$\begin{aligned}
-\sum_{i=1}^{n+k} (-1)^i i \operatorname{Tr} (d_{\tau h} \alpha d_{\tau h}^* e^{-t \Delta_{\tau h, i}}) &= -\sum_{i=1}^{n+k} (-1)^i i \operatorname{Tr} (d_{\tau h}^* d_{\tau h} \alpha e^{-t \Delta_{\tau h, i-1}}) \\
&= \sum_{i=0}^{n+k} (-1)^i (i+1) \operatorname{Tr} (d_{\tau h}^* d_{\tau h} \alpha e^{-t \Delta_{\tau h, i}}),
\end{aligned} \tag{4.29}$$

and

$$\begin{aligned}
\sum_{i=1}^{n+k} (-1)^i i \operatorname{Tr} (d_{\tau h}^* \alpha d_{\tau h} e^{-t \Delta_{\tau h, i}}) &= \sum_{i=1}^{n+k} (-1)^i i \operatorname{Tr} (d_{\tau h} d_{\tau h}^* \alpha e^{-t \Delta_{\tau h, i+1}}) \\
&= -\sum_{i=1}^{n+k} (-1)^i (i-1) \operatorname{Tr} (d_{\tau h} d_{\tau h}^* \alpha e^{-t \Delta_{\tau h, i}}),
\end{aligned} \tag{4.30}$$

Joining (4.28) with (4.29) and (4.30), we conclude that

$$\begin{aligned}
-t \operatorname{Tr}_s (N A e^{-t \Delta_{\tau h}}) &= -t \sum_{i=0}^{n+k} (-1)^i [(i+1) \operatorname{Tr} (d_{\tau h}^* d_{\tau h} \alpha e^{-t \Delta_{\tau h, i}}) - i \operatorname{Tr} (d_{\tau h}^* d_{\tau h} \alpha e^{-t \Delta_{\tau h, i}})] \\
&\quad - t \sum_{i=1}^{n+k} (-1)^i [i \operatorname{Tr} (d_{\tau h} d_{\tau h}^* \alpha e^{-t \Delta_{\tau h, i}}) - (i-1) \operatorname{Tr} (d_{\tau h} d_{\tau h}^* \alpha e^{-t \Delta_{\tau h, i}})] \\
&= -t \operatorname{Tr}_s (\Delta_{\tau h} \alpha e^{-t \Delta_{\tau h}}) \\
&= t \frac{\partial}{\partial t} \operatorname{Tr}_s (\alpha e^{-t \Delta_{\tau h}}).
\end{aligned} \tag{4.31}$$

Now we turn to the B -part. Using a similar trick, we deduce that

$$\begin{aligned}
&-t \operatorname{Tr}_s (N B e^{-t \Delta_{\tau h}}) \\
&= -t \sum_{i=1}^{n+k} (-1)^i i \left[\operatorname{Tr} \left([d_{\tau h}, \tau \dot{h}] d_{\tau h}^* e^{-t \Delta_{\tau h, i}} \right) + \operatorname{Tr} \left(d_{\tau h}^* [d_{\tau h}, \tau \dot{h}] e^{-t \Delta_{\tau h, i}} \right) \right. \\
&\quad \left. - \operatorname{Tr} \left([d_{\tau h}^*, \tau \dot{h}] d_{\tau h} e^{-t \Delta_{\tau h, i}} \right) - \operatorname{Tr} \left(d_{\tau h} [d_{\tau h}^*, \tau \dot{h}] e^{-t \Delta_{\tau h, i}} \right) \right] \\
&= -t \sum_{i=1}^{n+k} (-1)^i \left[- (i+1) \operatorname{Tr} \left(d_{\tau h}^* [d_{\tau h}, \tau \dot{h}] e^{-t \Delta_{\tau h, i}} \right) + i \operatorname{Tr} \left(d_{\tau h}^* [d_{\tau h}, \tau \dot{h}] e^{-t \Delta_{\tau h, i}} \right) \right. \\
&\quad \left. + (i-1) \operatorname{Tr} \left(d_{\tau h} [d_{\tau h}^*, \tau \dot{h}] e^{-t \Delta_{\tau h, i}} \right) - i \operatorname{Tr} \left(d_{\tau h} [d_{\tau h}^*, \tau \dot{h}] e^{-t \Delta_{\tau h, i}} \right) \right] \\
&= t \left[\operatorname{Tr}_s \left(d_{\tau h} [d_{\tau h}^*, \tau \dot{h}] e^{-t \Delta_{\tau h}} \right) + \operatorname{Tr}_s \left(d_{\tau h} [d_{\tau h}^*, \tau \dot{h}] e^{-t \Delta_{\tau h}} \right) \right] \\
&= t \left[\operatorname{Tr}_s \left(d_{\tau h} d_{\tau h}^* \tau \dot{h} e^{-t \Delta_{\tau h}} \right) - \operatorname{Tr}_s \left(d_{\tau h} \tau \dot{h} d_{\tau h}^* e^{-t \Delta_{\tau h}} \right) \right. \\
&\quad \left. + \operatorname{Tr}_s \left(d_{\tau h} d_{\tau h}^* \tau \dot{h} e^{-t \Delta_{\tau h}} \right) - \operatorname{Tr}_s \left(d_{\tau h} \tau \dot{h} d_{\tau h}^* e^{-t \Delta_{\tau h}} \right) \right] \\
&= t \left[\operatorname{Tr}_s \left(d_{\tau h} d_{\tau h}^* \tau \dot{h} e^{-t \Delta_{\tau h}} \right) + \operatorname{Tr}_s \left(d_{\tau h}^* d_{\tau h} \tau \dot{h} e^{-t \Delta_{\tau h}} \right) \right. \\
&\quad \left. + \operatorname{Tr}_s \left(d_{\tau h} d_{\tau h}^* \tau \dot{h} e^{-t \Delta_{\tau h}} \right) + \operatorname{Tr}_s \left(d_{\tau h}^* d_{\tau h} \tau \dot{h} e^{-t \Delta_{\tau h}} \right) \right] \\
&= 2t \operatorname{Tr}_s \left(\Delta_{\tau h} \tau \dot{h} e^{-t \Delta_{\tau h}} \right) \\
&= t \frac{\partial}{\partial t} \operatorname{Tr}_s \left(-2 \tau \dot{h} e^{-t \Delta_{\tau h}} \right).
\end{aligned} \tag{4.32}$$

However, as E is odd-dimensional, we must have $\text{Tr}_s \left(-2\tau \dot{h} e^{-t\Delta_{\tau h}} \right) \equiv 0$. This completes the proof. ■

Now we are ready to show a variation formula for the analytic torsion and the Quillen metric. Recall that by [12] and Theorem 3.3.2, one has the following identification for the null space of the Witten Laplacian for any $\tau > 0$:

$$\ker \Delta_{\tau h} \cong H_{(2)}^*(E, d_{\tau h}) \cong H_{dR}^*(M). \quad (4.33)$$

In particular, $\ker \Delta_{\tau h}$ doesn't depend on the Riemannian metric of E .

Theorem 4.3.5 *Let $H : L^2(\Lambda^* E) \rightarrow \ker \Delta_{\tau h}$ denote the orthogonal projection onto the space of Witten-harmonic forms. Then*

$$\frac{\partial}{\partial \sigma} \log T_{\tau h}(E, g(\sigma)) = -\frac{1}{2} \text{Tr}_s(\alpha H).$$

In terms of the Quillen metrics,

$$\frac{\partial}{\partial \sigma} \|\cdot\|_{\lambda, \tau h, g(\sigma)} = 0,$$

In particular, the Quillen metric $\|\cdot\|_{\lambda, \tau h, g(\sigma)}$ doesn't depend on the choice of $g(\sigma)$.

Proof: In view of Proposition 4.3.4, the assertion directly follows from a computation as in [2, Section 7]. We provide a simple proof for completeness.

It follows from Remark 4.1.4 and the assumption on the dimension that for $t > 0$ small enough,

$$\text{Tr}_s \left(\alpha e^{-t\Delta_{\tau h}} \right) \sim a_{-\frac{n+2k}{2}} t^{-\frac{n+2k}{2}} + \cdots + a_{-\frac{1}{2}} t^{-\frac{1}{2}} + a_{\frac{1}{2}} t^{\frac{1}{2}} + \cdots. \quad (4.34)$$

In particular, there is no constant term in this expansion. Recall that the analytic torsion

is defined as the derivative of the torsion zeta function at the origin. For $s \in \mathbb{C}$ with $\operatorname{Re}(s) \gg 1$, by Proposition 4.3.4 and using integration by parts, we find that

$$\begin{aligned}
\frac{\partial}{\partial \sigma} \zeta_{\tau h}(s) &= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{\partial}{\partial \sigma} \operatorname{Tr}_s (N e^{-t \Delta_{\tau h}}) dt \\
&= \frac{1}{\Gamma(s)} \int_0^\infty t^s \frac{\partial}{\partial t} \operatorname{Tr}_s (\alpha e^{-t \Delta_{\tau h}}) dt \\
&= -\frac{s}{\Gamma(s)} \int_0^\infty t^{s-1} \operatorname{Tr}_s (\alpha e^{-t \Delta_{\tau h}}) dt \\
&= -\frac{s}{\Gamma(s)} \int_1^\infty t^{s-1} \operatorname{Tr}_s (\alpha e^{-t \Delta_{\tau h}}) dt - \frac{s}{\Gamma(s)} \int_0^1 t^{s-1} (\alpha e^{-t \Delta_{\tau h}}) dt \\
&\quad + \frac{s}{\Gamma(s)} \int_0^1 t^{s-1} \operatorname{Tr}_s (\alpha H) dt \\
&:= -I - J + K.
\end{aligned} \tag{4.35}$$

Due to the uniqueness of meromorphic extension of the torsion zeta function, the calculation above also holds true for $s \in \mathbb{C}$ with $|s|$ small. Using (4.34), it is easy to see that

$$\left. \frac{\partial}{\partial s} \right|_{s=0} I = \left. \frac{\partial}{\partial s} \right|_{s=0} J = 0. \tag{4.36}$$

For the last term, we have

$$\left. \frac{\partial}{\partial s} \right|_{s=0} K = \left. \frac{\partial}{\partial s} \right|_{s=0} \left(\frac{1}{\Gamma(s)} \operatorname{Tr}_s (\alpha H) \right) = \operatorname{Tr}_s (\alpha H). \tag{4.37}$$

This verifies the first equation. To show the last equation, for each p , we pick a basis $[c_1], \dots, [c_{b_p}]$ of $L^2(H_{dR}^p E)$. For any $\sigma \geq 0$, let $\omega_i(\sigma)$ be the unique $g(\sigma)$ -harmonic form in the class $[c_i]$. It is easy to see that its derivative is exact, i.e. $\dot{\omega}_i = d\eta_i$ for some $\eta_i \in \Omega^{p-1}(E)$. Up to a rescaling, we may assume that $\{[c_i]\}$ is chosen such that $\{\omega_i(\sigma_0)\}$ is orthonormal for some σ_0 . Let $G_p = (G_{ij})$ be the matrix given by $G_{ij} = (\omega_i(\sigma), \omega_j(\sigma))_{g(\sigma)}$.

Then

$$\begin{aligned}
\dot{G}_{ij} &= \frac{\partial}{\partial \sigma} \int_E \omega_i \wedge *_\sigma \omega_j \\
&= (\dot{\omega}_i, \omega_j) + (\omega_i, \alpha \omega_j) + (\omega_i, \dot{\omega}_j) \\
&= (d\eta_i, \omega_j) + (\omega_i, \alpha \omega_j) + (\omega_i, d\eta_j) \\
&= (\omega_i, \alpha \omega_j),
\end{aligned} \tag{4.38}$$

which implies $\dot{G}_p = \alpha H|_{L^2(\wedge^p E)}$. Consequently, by Jacobi's formula we have

$$\begin{aligned}
\frac{\partial}{\partial \sigma} \Big|_{\sigma=\sigma_0} \log \|\cdot\|_{\lambda, g(\sigma)}^2 &= \frac{\partial}{\partial \sigma} \Big|_{\sigma=\sigma_0} \sum_p (-1)^p \det G_p \\
&= \sum_p (-1)^p \frac{\partial}{\partial \sigma} \Big|_{\sigma=\sigma_0} \det G_p \\
&= \sum_p (-1)^p \text{Tr } \dot{G}_p \\
&= \text{Tr}_s(\alpha H)
\end{aligned} \tag{4.39}$$

for any $\sigma_0 \in [0, 1]$. It follows that

$$\frac{\partial}{\partial \sigma} \log \|\cdot\|_{\lambda, \tau h, g(\sigma)}^2 = 2 \frac{\partial}{\partial \sigma} \log T_{\tau h}(E) + \frac{\partial}{\partial \sigma} \log \|\cdot\|_{\lambda, g(\sigma)}^2 = 0. \tag{4.40}$$

■

Remark 4.3.6 Suppose $\tilde{g}(\sigma)$ is a smooth family of Riemannian metrics on E that all satisfy the asymptotically flat condition in Remark 4.1.6. Then, by an argument similar to the one above, one can show that the corresponding Quillen metrics are still independent of σ .

Chapter 5

Equivalence of Quillen metrics

5.1 Main theorem and several intermediate results

As in the last chapter, let (M, g_M) be a closed n -dimensional Riemannian manifold, and $E \rightarrow M$ be a vector bundle over M with rank k . Suppose E is equipped with the metric g_E constructed in Section 3.2. As the analytic torsion twisted with an orthogonal representation is always trivial on even-dimensional manifolds, we further impose a parity assumption on the dimensions: suppose both n and $n + k$ are odd.

Now we explain the construction of Quillen metrics. In view of (3.39), we have the following isomorphisms between determinant line bundles:

$$L(M) := \det M \cong \det(\ker \tilde{D}_{T, \tau h}^2) \cong \det E_2 = \det E_\infty \cong \det E =: L(E). \quad (5.1)$$

The isomorphisms above provide us with two different L^2 -norms on L , which we denote by $|\cdot|_{L(M)}$ and $|\cdot|_{L(E)}$, induced by g_M and g_E respectively. Let $\|\cdot\|_{L(M)} = T(M)|\cdot|_{L(M)}$ and $\|\cdot\|_{L(E), \tau h} = T_{\tau h}(E)|\cdot|_{L(E)}$ denote the corresponding Quillen metrics.

The main result of this paper is

Theorem 5.1.1 *Suppose M^n is a closed manifold, and $E^{n+k} \rightarrow M^n$ is a vector bundle of rank k . Assume n is odd and k is even. Then for any $\tau \in (0, \infty)$,*

$$\log T_{\tau h}(E) - \log \left(\frac{|\cdot|_{L(M)}}{|\cdot|_{L(E)}} \right) = \log T(M). \quad (5.2)$$

In particular, $\|\cdot\|_{L(E), \tau h} = \|\cdot\|_{L(M)}$.

The proof of Theorem 5.1.1 relies on a family of constructions and estimates. For reader's convenience, in this section we collect all technical results needed for the proof, and in the next section we provide a proof of the main theorem. Some of these results will be proved in later sections. In particular, the proof of Theorem 5.1.4 will be postponed to section 3.2, Theorem 5.1.9 to section 5.3.1, Theorem 5.1.10 to section 5.3.2, Theorem 5.1.11 to section 5.3.3, and Theorem 5.1.12 to section 5.4.

To begin with, we equip E with the an adiabatic metric $g_\varepsilon = \frac{1}{\varepsilon^2}g_M + g_Y$. Let us first compute the difference of the Levi-Civita connection $\nabla^{L, \varepsilon}$ and the connection ∇^E constructed in section 2.1. Recall that ∇^L denotes the Levi-Civita connection for $g_E = g_1$. Let $S^\varepsilon = \nabla^{L, \varepsilon} - \nabla^E$ be the difference tensor.

Proposition 5.1.2 (1) $S^\varepsilon = \varepsilon^2 P^H S + P^V S$.

$$(2) \quad \nabla^{L, \varepsilon} = \nabla^E + S^\varepsilon = \nabla^E + \varepsilon^2 P^H S + P^V S$$

(3) $g_E(S(\cdot), \cdot, \cdot)$ is independent of g^M . In particular, $g_\varepsilon(S^\varepsilon(X)Y, Z) = g_E(S(X)Y, Z)$ for any $X, Y, Z \in \Gamma(TE)$.

For $a > 0$, we use $P_\varepsilon^{(0, a]}$ to denote the orthogonal projection from $L^2(\Lambda^* E)$ onto the eigenspaces $\oplus_{0 < |\lambda'_\varepsilon| \leq a} E(\lambda'_\varepsilon)$ of $\frac{1}{\varepsilon^2} D_{\varepsilon, \tau h}^2$ corresponding to small eigenvalues. We define $P_\varepsilon^{[0, a]}$ and $P_\varepsilon^{(a, \infty)}$ accordingly.

Theorem 5.1.3 (1) *There exists $\lambda'_0 > 0$ such that for any $\varepsilon > 0$ sufficiently small, we have the following uniform bound for non-zero eigenvalues λ'_ε of $\frac{1}{\varepsilon}D_{\varepsilon,\tau h}$:*

$$|\lambda'_\varepsilon| \geq \lambda'_0, \quad \forall \lambda'_\varepsilon \neq 0 \in \text{spec} \left(\frac{1}{\varepsilon} D_{\varepsilon,\tau h} \right).$$

(2) *Let $a = \frac{\lambda'_0}{2}$. Then $\lim_{\varepsilon \rightarrow 0} \text{Tr}_s \left(\frac{1}{\varepsilon^2} D_{\varepsilon,\tau h}^2 P_\varepsilon^{[0,a]} \right) = 0$.*

Proof: (1) has been proved in Proposition 3.2.1, and (2) is a direct consequence of (1). ■

The next three theorems describe the large time behavior of the supertrace and the L^2 -norm on the determinant line bundle.

Theorem 5.1.4 *For any $t > 0$ fixed,*

$$(1) \lim_{\varepsilon \rightarrow 0} \text{Tr}_s \left(N e^{-\frac{t}{\varepsilon^2} D_{\varepsilon,\tau h}^2} \right) = \text{Tr}_s \left(N_M e^{-t D_M^2} \right).$$

$$(2) \left| \text{Tr}_s \left(N_Y e^{-\frac{t}{\varepsilon^2} D_{\varepsilon,\tau h}^2} \right) \right| < C\varepsilon.$$

$$(3) \left| \text{Tr}_s \left(e^{-\frac{t_0}{\varepsilon^2} D_{\varepsilon,\tau h}^2} \right) \right| < C, \text{ for any } t_0 \geq t \text{ and } 0 < \varepsilon < 1.$$

Proof: The proof is essentially the same as Theorem 3.3.1, and is omitted here. ■

Definition 5.1.5 *We define the secondary Euler characteristics of M and E to be*

$$\begin{aligned} \chi_2(M) &:= \sum_{p=0}^n (-1)^p p b_p(M), \\ \chi_2(E) &:= \sum_{p=0}^{n+k} (-1)^p p b_p(E), \end{aligned} \tag{5.3}$$

where $b_p(M)$ and $b_p(E)$ denote the p -th Betti numbers of M and E respectively.

Since the total space E is homotopically equivalent to the image of the zero section, which is diffeomorphic to M , we see that

$$b_p(E) = b_p(M), \quad \chi_2(E) = \chi_2(M). \quad (5.4)$$

Theorem 5.1.6 *Let a be the constant given in Theorem 5.1.3. For any $t > 0$ fixed,*

$$(1) \lim_{\varepsilon \rightarrow 0} \text{Tr}_s \left(N e^{-\frac{t}{\varepsilon^2} D_{\varepsilon, \tau h}^2} P_{\varepsilon}^{(a, \infty)} \right) = \text{Tr}_s \left(N_M e^{-t \underline{D}_M^2} \right).$$

$$(2) \left| \text{Tr}_s \left(N e^{-\frac{t}{\varepsilon^2} D_{\varepsilon, \tau h}^2} P_{\varepsilon}^{(a, \infty)} \right) \right| < C e^{-Ct}.$$

Here $\underline{D}_M^2$ denotes the Hodge Laplacian with the null space removed.

Proof: It follows from the proof of Theorem 5.1.3 that for ε small enough, the only eigenvalue of $\frac{1}{\varepsilon^2} D_{\varepsilon, \tau h}^2$ that lies in $[0, a]$ is 0. From this and (5.4), we deduce

$$\begin{aligned} \text{Tr}_s \left(N e^{-\frac{t}{\varepsilon^2} D_{\varepsilon, \tau h}^2} P_{\varepsilon}^{(a, \infty)} \right) &= \text{Tr}_s \left(N e^{-\frac{t}{\varepsilon^2} D_{\varepsilon, \tau h}^2} \right) - \sum_{p=0}^{n+k} (-1)^p p \dim \ker D_{\varepsilon, \tau h}^2 \\ &= \text{Tr}_s \left(N e^{-\frac{t}{\varepsilon^2} D_{\varepsilon, \tau h}^2} \right) - \chi_2(E) \\ &= \text{Tr}_s \left(N e^{-\frac{t}{\varepsilon^2} D_{\varepsilon, \tau h}^2} \right) - \chi_2(M). \end{aligned}$$

Therefore (1) follows from a similar argument as Theorem 5.1.4.

Since a is the uniform lower bound of all non-zero eigenvalues of $\frac{1}{\varepsilon^2} D_{\varepsilon, \tau h}^2$, we obtain the exponential decay in (2). ■

Before stating the rest of the results, we need to introduce a new metric that is essentially conformal to g_{ε} .

Definition 5.1.7 *For any $T > 0$, we define a metric on E by*

$$\tilde{g}_T = g_M + \frac{1}{T^2} g_Y.$$

Let $\tilde{D}_T$ be the corresponding Dirac operator, and

$$\tilde{D}_{T,\tau h} := \tilde{D}_T + \tau \hat{c}(\tilde{\nabla}^T h)$$

be the Witten-Dirac operator, where $\tilde{\nabla}^L$ is the Levi-Civita connection of $\tilde{g}_T$.

Let $|\cdot|_{L(E),T}$ denote the L^2 -norm on $\det E$ naturally induced by $\tilde{g}_T$. When $T = 1$, we see that $|\cdot|_{L(E),1} = |\cdot|_{L(E)}$.

Theorem 5.1.8

$$\lim_{T \rightarrow \infty} \log \left(\frac{|\cdot|_{L(E),T}}{|\cdot|_{L(M)}} \right) = \log \left(\frac{|\cdot|_{L(E)}}{|\cdot|_{L(M)}} \right).$$

Proof: See [24, section 6(e)] for the proof. ■

In the end, we discuss the contribution of supertrace when the time is small. Recall that by assumption, M is compact and odd-dimensional. It follows from [32, Theorem 5.1] that

$$\mathrm{Tr}_s \left(N_M e^{-tD_M^2} \right) \sim b_{-1/2} t^{-\frac{1}{2}} + \sum_{j \geq 1, \text{odd}} b_{j/2} t^{\frac{j}{2}}, \quad (5.5)$$

and

$$b_{-1/2} e^1 \wedge \cdots \wedge e^n = 2i(-1)^{\frac{n+1}{2}} (16\pi)^{-\frac{n}{2}} \sum_{k=1}^n (-1)^k \mathrm{Pf}(R_k) \wedge e^k, \quad (5.6)$$

where $R_k = \left(\sum_{p,q \neq k} R_{ijkl} e^p \wedge e^q \right)_{i,j \neq k}$ is the matrix 2-form obtained by deleting the k -th row and column from the curvature matrix 2-form.

Theorem 5.1.9 *As $t \rightarrow 0$, the asymptotic expansion of $\mathrm{Tr}_s \left(N e^{-tD_{\tau h}^2} \right)$ starts with $a_{-1/2} t^{-\frac{1}{2}}$ and does not contain a constant term. Moreover, for the leading coefficient we have*

$$a_{-1/2} = b_{-1/2}. \quad (5.7)$$

Proof: See section 5.3.1. ■

Theorem 5.1.10 *For any $T > 0$ fixed,*

$$\lim_{\sigma \rightarrow 0} \mathrm{Tr}_s \left(N_Y e^{-\sigma^2 \tilde{D}_{T/\sigma, \tau h}^2} \right) = 0.$$

Proof: See section 5.3.2. ■

Theorem 5.1.11 *There exists a constant $C > 0$ such that for any $\sigma \in (0, 1]$ and $T \in [\sigma, 1]$,*

$$\left| \frac{1}{T} \mathrm{Tr}_s \left(N_Y e^{-\sigma^2 \tilde{D}_{T/\sigma, \tau h}^2} \right) \right| \leq C.$$

Proof: See section 5.3.3. ■

Theorem 5.1.12 *There exist $\delta \in (0, 1]$ and $C > 0$ such that for any $\sigma \in (0, 1]$ and $T \in [1, \infty)$,*

$$\left| \frac{1}{T} \mathrm{Tr}_s \left(N_Y e^{-\sigma^2 \tilde{D}_{T/\sigma, \tau h}^2} \right) \right| \leq \frac{C}{T^{1+\delta}}.$$

Proof: See section 5.4. ■

5.2 Proof of main theorem

In this section we provide a proof of Theorem 5.1.1. Here we use the idea of adiabatic limit in an essential way. Historically, the adiabatic limit of analytic torsion was studied by X. Dai and R. B. Melrose in [32], where they considered a compact fibration $Y \hookrightarrow E \xrightarrow{\pi} M$ over a closed manifold M . Employing the technique of microlocal analysis, the adiabatic limit of the analytic torsion of the total space is related to those of the base and the fiber, as well as certain topological invariants of the fibration.

Let us first discuss the idea of the proof. Intuitively, in the case of vector bundle, analyzing the behavior of the analytic torsion under the adiabatic limit may link

$\log T_{\tau h}(E, g_\varepsilon)$ to $\log T(M)$. Unfortunately, there are two main issues. First, since the torsion zeta function is an integral from 0 to infinity, when the time t is large, the supertrace behaves differently by nature than when t is small. This forces us to deal with large and small time separately. If t is large, we decompose $\frac{1}{\varepsilon}D_{\varepsilon, \tau h}$ under the splitting $L^2(\Lambda^*E) = L^2(\ker \tilde{D}_Y)^\perp \oplus L^2(\ker \tilde{D}_Y)$, where $\tilde{D}_Y$ is the Witten-Dirac operator on the fiber Y . Then a careful study of the resolvent yields the desired estimate. On the other hand, local index theory plays an important role in the small time case. Motivated by the rescaling technique of E. Getzler [33], we introduce several different rescalings that fit into various situations, and use them to derive the corresponding local index theorems as well as some small time estimates.

Second, due to the lack of a uniform estimate, the two limits $\lim_{\varepsilon \rightarrow 0}$ and $\lim_{t \rightarrow \infty}$ may not commute. To solve this issue, we follow the idea of A. Berthomieu and J.-M. Bismut [24] (see also [34]) to construct a closed 1-form $\alpha_{t,T}$ on $\mathbb{R}_+ \times \mathbb{R}_+$ and integrate it over a closed rectangular contour. We analyze the asymptotic behavior of the integral of $\alpha_{t,T}$ over each side as we push two sides to infinity and one side to the x -axis. Then Theorem 5.1.1 follows from matching the divergent terms that appear in the above limiting process.

We will proceed as in [24, Section 4] to prove Theorem 5.1.1. Set $T = \frac{1}{\varepsilon}$ for later convenience.

Definition 5.2.1 (1) For any $t, T > 0$, we define a Riemannian metric $\tilde{g}_{t,T}$ on E by

$$\tilde{g}_{t,T} = \frac{1}{t^2}\tilde{g}_T = \frac{1}{t^2} \left(g_M + \frac{1}{T^2}g_Y \right).$$

Let $\tilde{*}_{t,T}$ and $\tilde{D}_{t,T}$ denote the corresponding Hodge star operator and Dirac operator. Moreover, let

$$\tilde{D}_{t,T,\tau h} := \tilde{D}_{t,T} + \tau \hat{c}(\nabla^{t,T} h)$$

denote the associated Witten-Dirac operators.

(2) Let $\alpha_{t,T,\tau h}$ be a 1-form on $\mathbb{R}_+^* \times \mathbb{R}_+$ defined by

$$\alpha_{t,T} = dt \operatorname{Tr}_s \left[\tilde{*}_{t,T}^{-1} \partial_t \tilde{*}_{t,T} e^{-\tilde{D}_{t,T,\tau h}^2} \right] + dT \operatorname{Tr}_s \left[\tilde{*}_{t,T}^{-1} \partial_T \tilde{*}_{t,T} e^{-\tilde{D}_{t,T,\tau h}^2} \right].$$

Remark 5.2.2 Under the identification $T = \frac{1}{\varepsilon}$, we have the following relations between metrics:

$$\tilde{g}_T = \varepsilon^2 g_\varepsilon, \quad \tilde{g}_{t,T} = \left(\frac{\varepsilon}{t} \right)^2 g_\varepsilon.$$

Proposition 5.2.3 (1) $\alpha_{t,T}$ is closed.

(2) $\tilde{*}_{t,T}^{-1} \partial_t \tilde{*}_{t,T} = \frac{1}{t} (2N - n - k)$, $\tilde{*}_{t,T}^{-1} \partial_T \tilde{*}_{t,T} = \frac{1}{T} (2N_Y - k)$. Therefore

$$\alpha_{t,T} = \frac{2dt}{t} \operatorname{Tr}_s \left[N e^{-\tilde{D}_{t,T,\tau h}^2} \right] + \frac{2dT}{T} \operatorname{Tr}_s \left[N_Y e^{-\tilde{D}_{t,T,\tau h}^2} \right]. \quad (5.8)$$

(3) If $\tilde{g} = \frac{1}{\sigma^2} g$, then $\sigma^N \tilde{D} \sigma^{-N} = \sigma D$, and

$$\operatorname{Tr}_s \left(N e^{-\tilde{D}^2} \right) = \operatorname{Tr}_s \left(N e^{-\sigma^2 D^2} \right).$$

In particular,

$$\operatorname{Tr}_s \left(N e^{-\tilde{D}_{t,T,\tau h}^2} \right) = \operatorname{Tr}_s \left(N e^{-\left(\frac{t}{\varepsilon} \right)^2 D_{\varepsilon,\tau h}^2} \right).$$

Proof: (1) follows from [24, Theorem 4.3].

For (2), recall that $\{e^1, \dots, e^n, f^1, \dots, f^k\}$ denotes an orthonormal coframe of $g = g_1$.

Let $\{\bar{e}^1, \dots, \bar{e}^n, \bar{f}^1, \dots, \bar{f}^k\}$ be the corresponding orthonormal basis with respect to $g_{t,T}$.

Then clearly we have $\bar{e}^i = \frac{1}{t}e^i$ and $\bar{f}^j = \frac{1}{tT}f^j$. Now if

$$\tilde{*}_{t,T}(\bar{e}^{i_1} \dots \bar{e}^{i_p} \wedge \bar{f}^{j_1} \dots \bar{f}^{j_q}) = \bar{e}^{i_{p+1}} \dots \bar{e}^{i_n} \wedge \bar{f}^{j_{q+1}} \dots \bar{f}^{j_k},$$

then equivalently we have

$$\begin{aligned} \tilde{*}_{t,T}(e^{i_1} \dots e^{i_p} f^{j_1} \dots f^{j_q}) \\ = t^{2(p+q)-(n+k)} T^{2q-k} e^{i_{p+1}} \dots e^{i_n} f^{j_{q+1}} \dots f^{j_k}. \end{aligned}$$

This proves the first line of (2). (5.8) follows from the fact that the dimension of E is odd and thus

$$\mathrm{Tr}_s(e^{-\tilde{D}_{t,T,\tau h}^2}) = \chi(E, d_{\tau h}) = 0.$$

We observe that the first part in (3) follows from a straightforward calculation. Using Remark 5.2.2 we obtain the last equality. ■

Consider the following loop $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$ in $\mathbb{R}^2$ with counterclockwise orientation, where

$$\Gamma_1 = \{(T_0, t) \mid \sigma \leq t \leq A\},$$

$$\Gamma_2 = \{(T, A) \mid 1 \leq T \leq T_0\},$$

$$\Gamma_3 = \{(1, t) \mid \sigma \leq t \leq A\},$$

$$\Gamma_4 = \{(T, \sigma) \mid 1 \leq T \leq T_0\}.$$

Denote $I_k^0 := \int_{\Gamma_k} \alpha_{t,T}$ for $k = 1, 2, 3, 4$. Then Proposition 5.2.3 implies that

$$I_1^0 + I_2^0 + I_3^0 + I_4^0 = 0. \tag{5.9}$$

The main theorem will then follow from letting $A \rightarrow \infty$, $T_0 \rightarrow \infty$ (or equivalently $\varepsilon_0 \rightarrow 0$) and $\sigma \rightarrow 0$ sequentially for each I_k^0 (subtracting divergent terms if necessary), and comparing the limits.

5.2.1 I_1^0

Observe that

$$\begin{aligned} I_1^0 &= 2 \int_{\sigma}^A \operatorname{Tr}_s \left(N e^{-\tilde{D}_{t,T}^2} \right) \frac{dt}{t} \\ &= 2 \int_{\sigma}^A \operatorname{Tr}_s \left(N e^{-\left(\frac{t}{\varepsilon_0}\right)^2 D_{\varepsilon_0, \tau h}^2} \right) \frac{dt}{t} \\ &= \int_{\sigma^2}^{A^2} \operatorname{Tr}_s \left(N e^{-\frac{\tilde{t}}{\varepsilon_0^2} D_{\varepsilon_0, \tau h}^2} \right) \frac{d\tilde{t}}{\tilde{t}}, \end{aligned}$$

where in the last equality we made a substitution $\tilde{t} = t^2$. We will still write the last line as $\int_{\sigma^2}^{A^2} \operatorname{Tr}_s \left(N e^{-\frac{t}{\varepsilon_0^2} D_{\varepsilon_0, \tau h}^2} \right) \frac{dt}{t}$ when there is no confusion of notation.

(1) $A \rightarrow \infty$: To cancel the term which diverges at infinity, we subtract

a term $2\chi_2(E) \log A$ to obtain

$$\begin{aligned} I_1^0 - 2\chi_2(E) \log A &\rightarrow \\ I_1^1 &:= \int_{\sigma^2}^1 \operatorname{Tr}_s \left(N e^{-\frac{t}{\varepsilon_0^2} D_{\varepsilon_0, \tau h}^2} \right) \frac{dt}{t} + \int_1^{\infty} \operatorname{Tr}_s \left(N e^{-\frac{t}{\varepsilon_0^2} D_{\varepsilon_0, \tau h}^2} \right) \frac{dt}{t}. \end{aligned} \tag{5.10}$$

(2) $\varepsilon_0 \rightarrow 0$: Theorem 5.1.4 shows that

$$\int_{\sigma^2}^1 \operatorname{Tr}_s \left(N e^{-\frac{t}{\varepsilon_0^2} D_{\varepsilon_0, \tau h}^2} \right) \frac{dt}{t} \rightarrow \int_{\sigma^2}^1 \operatorname{Tr}_s \left(N_M e^{-t D_M^2} \right) \frac{dt}{t}.$$

On the other hand,

$$\begin{aligned} & \int_1^\infty \text{Tr}_s \left(N e^{-\frac{t}{\varepsilon_0^2} D_{\varepsilon, \tau h}^2} \right) \frac{dt}{t} \\ &= \int_1^\infty \text{Tr}_s \left(N e^{-\frac{t}{\varepsilon_0^2} D_{\varepsilon_0, \tau h}^2} P_{\varepsilon_0}^{(a, \infty)} \right) \frac{dt}{t} + \int_1^\infty \text{Tr}_s \left(N e^{-\frac{t}{\varepsilon_0^2} D_{\varepsilon_0, \tau h}^2} P_{\varepsilon_0}^{(0, a]} \right) \frac{dt}{t}. \end{aligned}$$

For the first term, it follows from Theorem 5.1.6 that

$$\int_1^\infty \text{Tr}_s \left(N e^{-\frac{t}{\varepsilon_0^2} D_{\varepsilon_0, \tau h}^2} P_{\varepsilon_0}^{(a, \infty)} \right) \frac{dt}{t} \rightarrow \int_1^\infty \text{Tr}_s \left(N_M e^{-t D_M^2} \right) \frac{dt}{t}$$

For the second term, Theorem 5.1.3 implies that $P_{\varepsilon_0}^{(0, a]} = 0$ for ε_0 small enough. Therefore if we let $u = \frac{t}{\varepsilon_0^2} D_{\varepsilon_0, \tau h}^2$, then

$$\begin{aligned} & \int_1^\infty \text{Tr}_s \left(N e^{-\frac{t}{\varepsilon_0^2} D_{\varepsilon_0, \tau h}^2} P_{\varepsilon_0}^{(0, a]} \right) \frac{dt}{t} = \text{Tr}_s \left(N \int_{\frac{1}{\varepsilon_0^2} D_{\varepsilon_0, \tau h}^2}^\infty e^{-u} \frac{du}{u} P_{\varepsilon_0}^{(0, a]} \right) \\ &= \text{Tr}_s \left(N \int_{\frac{1}{\varepsilon_0^2} D_{\varepsilon_0, \tau h}^2}^1 (e^{-u} - 1) \frac{du}{u} P_{\varepsilon_0}^{(0, a]} \right) - \text{Tr}_s \left(N (\log \frac{1}{\varepsilon_0^2} D_{\varepsilon_0, \tau h}^2) P_{\varepsilon_0}^{(0, a]} \right) \\ &\quad + \text{Tr}_s \left(N \int_1^\infty e^{-u} \frac{du}{u} P_{\varepsilon_0}^{(0, a]} \right) \\ &\rightarrow 0. \end{aligned} \tag{5.11}$$

Consequently,

$$I_1^1 \rightarrow I_1^2 := \int_{\sigma^2}^1 \text{Tr}_s \left(N_M e^{-t D_M^2} \right) \frac{dt}{t} + \int_1^\infty \text{Tr}_s \left(N_M e^{-t D_M^2} \right) \frac{dt}{t}. \tag{5.12}$$

(3) $\sigma \rightarrow 0$: Recall that for t small,

$$\text{Tr}_s \left(N_M e^{-t D_M^2} \right) = b_{-\frac{1}{2}} t^{-\frac{1}{2}} + o(t^{\frac{1}{2}}).$$

A straightforward calculation shows

$$\begin{aligned} I_1^2 - 2b_{-\frac{1}{2}} \frac{1}{\sigma} &\rightarrow I_1^3 := \int_0^1 \left[\text{Tr}_s \left(N_M e^{-tD_M^2} \right) - b_{-\frac{1}{2}} t^{-\frac{1}{2}} \right] \frac{dt}{t} \\ &\quad + \int_1^\infty \text{Tr}_s \left(N_M e^{-tD_M^2} \right) \frac{dt}{t} - 2b_{-\frac{1}{2}}. \end{aligned} \quad (5.13)$$

5.2.2 I_2^0

Let $P_\varepsilon : L^2(\Lambda^* E) \rightarrow L^2(\ker D_{\varepsilon, \tau h}^2)$ be the orthogonal projection onto the kernel.

Notice that

$$I_2^0 = - \int_1^{T_0} \text{Tr}_s \left(\tilde{*}_T^{-1} \partial_T \tilde{*}_T e^{-A^2 T^2 D_{\varepsilon, \tau h}^2} \right) \frac{dT}{T}. \quad (5.14)$$

(1) $A \rightarrow \infty$: Clearly we have

$$I_2^0 \rightarrow I_2^1 := - \int_1^{T_0} \text{Tr}_s \left(\tilde{*}_T^{-1} \partial_T \tilde{*}_T P_\varepsilon \right) \frac{dT}{T}. \quad (5.15)$$

Then by [24, Proposition 4.18], we have

$$I_2^1 = \log \left(\frac{|\cdot|_{L(E), T_0}}{|\cdot|_{L(M)}} \right)^2. \quad (5.16)$$

(2) $T_0 \rightarrow \infty$: It follows from Theorem 5.1.8 that

$$I_2^1 \rightarrow I_2^2 := \log \left(\frac{|\cdot|_{L(E)}}{|\cdot|_{L(M)}} \right). \quad (5.17)$$

(3) $\sigma \rightarrow 0$: $I_2^3 = I_2^2 = \log \left(\frac{|\cdot|_{L(E)}}{|\cdot|_{L(M)}} \right).$

5.2.3 I_3^0

Proposition 5.2.3 implies

$$\begin{aligned} I_3^0 &= -2 \int_{\sigma}^A \operatorname{Tr}_s \left(N e^{-\tilde{D}_{t,1,\tau h}^2} \right) \frac{dt}{t} \\ &= - \int_{\sigma^2}^{A^2} \operatorname{Tr}_s \left(N e^{-t D_{\tau h}^2} \right) \frac{dt}{t}. \end{aligned} \quad (5.18)$$

(1) $A \rightarrow \infty$: As before,

$$\begin{aligned} I_3^0 + 2\chi_2(E) \log A &\rightarrow \\ I_3^1 &:= - \int_{\sigma^2}^1 \operatorname{Tr}_s \left(N e^{-t D_{\tau h}^2} \right) \frac{dt}{t} - \int_1^{\infty} \operatorname{Tr}_s \left(N e^{-t D_{\tau h}^2} \right) \frac{dt}{t} \end{aligned} \quad (5.19)$$

(2) $T_0 \rightarrow \infty$: Since I_3^1 is independent of T_0 , we have $I_3^2 = I_3^1$.

(3) $\sigma \rightarrow 0$: Recall that by Theorem 5.1.9,

$$\operatorname{Tr}_s \left(N e^{-t D_{\tau h}^2} \right) = a_{-1/2} t^{-\frac{1}{2}} + o(t^{\frac{1}{2}}).$$

Thus

$$\begin{aligned} I_3^2 + 2a_{-1/2} \frac{1}{\sigma} &\rightarrow I_3^3 := - \int_0^1 \left[\operatorname{Tr}_s \left(N e^{-t D_{\tau h}^2} \right) - a_{-1/2} t^{-\frac{1}{2}} \right] \frac{dt}{t} \\ &\quad - \int_1^{\infty} \operatorname{Tr}_s \left(N e^{-t D_{\tau h}^2} \right) \frac{dt}{t} + 2a_{-1/2}. \end{aligned} \quad (5.20)$$

5.2.4 I_4^0

To begin with,

$$I_4^0 = 2 \int_1^{T_0} \operatorname{Tr}_s \left(N_Y e^{-\sigma^2 \tilde{D}_{T,\tau h}^2} \right) \frac{dT}{T}. \quad (5.21)$$

(1) $A \rightarrow \infty$: Obviously $I_4^1 = I_4^0$.

(2) $T_0 \rightarrow \infty$: It follows from Proposition 5.1.4 that

$$\begin{aligned}
I_4^1 \rightarrow I_4^2 &:= 2 \int_1^\infty \text{Tr}_s \left(N_Y e^{-\sigma^2 \tilde{D}_{T, \tau h}^2} \right) \frac{dT}{T} \\
&= 2 \int_\sigma^1 \text{Tr}_s \left(N_Y e^{-\sigma^2 \tilde{D}_{T/\sigma, \tau h}^2} \right) \frac{dT}{T} + 2 \int_1^\infty \text{Tr}_s \left(N_Y e^{-\sigma^2 \tilde{D}_{T/\sigma, \tau h}^2} \right) \frac{dT}{T},
\end{aligned} \tag{5.22}$$

where in the last equality we make a change of variable $\tilde{T} = \sigma T$ and still write the new variable as T .

(3) $\underline{\sigma \rightarrow 0}$: For the first term in the last line of (5.22), using Theorem

5.1.10, Theorem 5.1.11, and the dominated convergence theorem, we obtain

$$2 \int_\sigma^1 \text{Tr}_s \left(N_Y e^{-\sigma^2 \tilde{D}_{T/\sigma, \tau h}^2} \right) \frac{dT}{T} \rightarrow 0. \tag{5.23}$$

For the second term, similarly we use Theorem 5.1.10, Theorem 5.1.12 and dominated convergence theorem to deduce that

$$2 \int_1^\infty \text{Tr}_s \left(N_Y e^{-\sigma^2 \tilde{D}_{T/\sigma, \tau h}^2} \right) \frac{dT}{T} \rightarrow 0. \tag{5.24}$$

Therefore

$$I_4^2 \rightarrow I_4^3 = 0. \tag{5.25}$$

5.2.5 Matching the divergences

By Theorem 5.1.9, we have the following cancellation of the divergent terms appearing in section 5.2.1-5.2.4:

$$-2\chi'(E) \log A - 2b_{-1/2} \frac{1}{\sigma} + 2\chi'(E) \log A + 2a_{-1/2} \frac{1}{\sigma} = 0. \tag{5.26}$$

Therefore

$$\begin{aligned}
\sum_{i=0}^4 I_i^3 &= \lim_{\substack{A \rightarrow \infty \\ T_0 \rightarrow \infty \\ \sigma \rightarrow 0}} \left[\sum_{i=0}^4 I_i^0 - 2\chi'(E) \log A - 2b_{-1/2} \frac{1}{\sigma} + 2\chi'(E) \log A + 2a_{-1/2} \frac{1}{\sigma} \right] \\
&= \lim_{\substack{A \rightarrow \infty \\ T_0 \rightarrow \infty \\ \sigma \rightarrow 0}} [I_1^0 + I_2^0 + I_3^0 + I_4^0] = 0.
\end{aligned} \tag{5.27}$$

This completes the proof of Theorem 5.1.1.

5.3 Local index theorems and small time behavior

From now on, we will assume n is odd and k is even.

To evaluate $\int_0^\infty t^{s-1} \text{Tr}_s(Ne^{-t\Delta}) dt$, we still need to understand the behavior of the supertrace when t is small. The key to this is the local index theory, which is usually obtained by applying a rescaling technique introduced by Getzler [33], and studying the limit of the supertrace of the heat kernel.

The standard Getzler's rescaling technique amounts to a rescale operator G acting on functions, differential forms, as well as differential operators over M . More precisely:

- (1) For $f(t, x) \in C^\infty(\mathbb{R}_+^1 \times M)$, we define $(Gf)(t, x) = f(\varepsilon^2 t, \varepsilon x)$. One can easily verify that

$$G \circ \frac{\partial}{\partial x_i} \circ G^{-1} = \frac{1}{\varepsilon} \frac{\partial}{\partial x_i}, \quad G \circ \frac{\partial}{\partial t} \circ G^{-1} = \frac{1}{\varepsilon^2} \frac{\partial}{\partial t}.$$

- (2) Suppose $e^1, \dots, e^n$ is a local orthonormal frame for T^*M . Then G acts on $\Lambda^p M$ as a rescale by its degree:

$$G(e^{i_1} \wedge \dots \wedge e^{i_p}) = \varepsilon^{-p} e^{i_1} \wedge \dots \wedge e^{i_p}.$$

Thus it acts on Clifford multiplications by conjugation as

$$G \circ c(e_i) \circ G^{-1} = \frac{1}{\varepsilon} e^i \wedge -\varepsilon \iota_{e_i}.$$

In this section, we will introduce several modified rescalings that fit into different settings. Then we will use them to prove Theorem 5.1.9, Theorem 5.1.10, and Theorem 5.1.11.

5.3.1 Matching the leading terms: Proof of Theorem 5.1.9

By Theorem 4.1.3, for t small enough $\text{Tr}_s \left(N e^{-t D_{\tau h}^2} \right)$ admits an asymptotic expansion as

$$\text{Tr}_s \left(N e^{-t D_{\tau h}^2} \right) \sim \sum_{p=-n-2k}^{\infty} \sum_{q \in \mathbb{Z}} a_{p/2, q} t^{\frac{p}{2}} \tau^q. \quad (5.28)$$

By the construction in [11], we immediately see that $a_{p/2, q} = 0$ for p even, i.e. the expansion only consists of terms with half-integer powers in t .

Now, we study the convergence of the coefficients in the asymptotic expansion. We adopt the construction in [18] and consider a rescaling G_1 defined as follows:

- (1) For $f(t, x, y) \in C^\infty(\mathbb{R}^1 \times E)$, $(G_1 f)(t, x, y) = f(t, \sqrt{t}x, \sqrt{t}y)$. It can be checked easily that as operators acting on functions,

$$G_1 \circ e_i \circ G_1^{-1} = \frac{1}{\sqrt{t}} e_i, \quad G_1 \circ f_j \circ G_1^{-1} = \frac{1}{\sqrt{t}} f_j.$$

- (2) The action on differential forms is defined accordingly. For our purpose, we rescale

both the left and right Clifford multiplications. As a result, we have

$$\begin{aligned} G_1 \circ c(e^i) \circ G_1^{-1} &= t^{-\frac{1}{4}} \left(e^i \wedge -t^{\frac{1}{2}} \iota_{e_i} \right) := t^{-\frac{1}{4}} c_t(e^i), \\ G_1 \circ \hat{c}(e^i) \circ G_1^{-1} &= t^{-\frac{1}{4}} \left(e^i \wedge +t^{\frac{1}{2}} \iota_{e_i} \right) := t^{-\frac{1}{4}} \hat{c}_t(e^i). \end{aligned}$$

We define the actions on $c(f^j)$ and $\hat{c}(f^j)$ similarly. A useful observation is that c_t and $\hat{c}_t$ converge to wedge products as $t \rightarrow 0$.

- (3) In analogy to the treatment in [11], we rescale the time parameter of Witten deformation by $G_1(\tau) = t^{-\frac{1}{2}}\tau$.

Remark 5.3.1 *In fact, G_1 amounts to localizing the calculation from (x, y) to any fixed point (x_0, y_0) . In the definition above we set $(x_0, y_0) = (0, 0)$ for convenience.*

Next, we derive the corresponding Bochner-Lichnerowicz-Weitzenböck formula (or BLW formula for short). In what follows, we will write the index corresponding to f_i as $\bar{i}$, and use the notation $\overline{\sum_{ijkl}}$ to denote the summation over all possible combinations of $\{i, j, k, l\}$, while allowing each index to be either with or without a "bar". For instance,

$$\begin{aligned} & \overline{\sum_{i,j,k,l}} R_{ijkl} c(e_i) c(e_j) \hat{c}(e_k) \hat{c}(e_l) \\ &:= \sum_{i,j,k,l} R_{ijkl} c(e_i) c(e_j) \hat{c}(e_k) \hat{c}(e_l) + \sum_{\bar{i},j,k,l} R_{\bar{i}jkl} c(f_{\bar{i}}) c(e_j) \hat{c}(e_k) \hat{c}(e_l) + \cdots \\ &+ \sum_{\bar{i},\bar{j},k,l} R_{\bar{i}\bar{j}kl} c(f_{\bar{i}}) c(f_{\bar{j}}) \hat{c}(e_k) \hat{c}(e_l) + \cdots + \sum_{\bar{i},\bar{j},\bar{k},l} R_{\bar{i}\bar{j}\bar{k}l} c(f_{\bar{i}}) c(f_{\bar{j}}) \hat{c}(f_{\bar{k}}) \hat{c}(e_l) \\ &+ \sum_{\bar{i},\bar{j},\bar{k},\bar{l}} R_{\bar{i}\bar{j}\bar{k}\bar{l}} c(f_{\bar{i}}) c(f_{\bar{j}}) \hat{c}(f_{\bar{k}}) \hat{c}(f_{\bar{l}}). \end{aligned}$$

Lemma 5.3.2 *Let R^E denote the scalar curvature of E with respect to $g_1 = g_E$. Then*

$$\begin{aligned} D_{\tau h}^2 = & - \sum_i \left[e_i + \frac{1}{8} \overline{\sum_{j,k,l} (R_{ijkl}^L(0)x^l + O(|x|^2))} (c(e^j)c(e^k)) - \hat{c}(e^j)\hat{c}(e^k) \right]^2 \\ & - \sum_i \left[f_i + \frac{1}{8} \overline{\sum_{j,k,l} (R_{ijkl}^L(0)x^l + O(|x|^2))} (c(e^j)c(e^k)) - \hat{c}(e^j)\hat{c}(e^k) \right]^2 \\ & + \frac{R^E}{4} + \frac{1}{8} \overline{\sum_{i,j,k,l} R_{ijkl}^L} c(e^i)c(e^j)\hat{c}(e^k)\hat{c}(e^l) + \tau \sum_i c(f^i)\hat{c}(f^i) + \tau^2|y|^2. \end{aligned}$$

Proof: For the Witten Laplacian, since $h = \frac{1}{2}\|y\|^2$, one can easily verify that

$$D_{\tau h}^2 = D^2 + \tau \sum_i c(f^i)\hat{c}(f^i) + \tau^2|y|^2. \quad (5.29)$$

By this and Corollary 2.3.6, we complete the proof. ■

Next we consider the conjugation action of G_1 on $-tD_{\tau h}^2$ and compute the limit.

Lemma 5.3.3

$$\begin{aligned} L_1 &:= \lim_{t \rightarrow 0} G_1 \circ (-tD_{\tau h}^2) \circ G_1^{-1} \\ &= -\Delta_0 - \frac{1}{8} \overline{\sum_{i,j,k,l} R_{ijkl}^L} (x_0, y_0) e^i \wedge e^j \wedge \hat{e}^k \wedge \hat{e}^l \wedge -\tau \sum_i f^i \wedge \hat{f}^i - \tau^2|y_0|^2, \end{aligned}$$

where $\Delta_0 = -\sum_i e_i^2 - \sum_i f_i^2$ is the Euclidean Laplacian.

Proof: To apply Getzler's rescaling technique, for any $p_0 = (x_0, y_0) \in E$, we trivialize ΛT^*E over a normal coordinate neighborhood of p_0 via parallel transport, with respect to ∇^L , along radical geodesics. Thus if $q \in E$ is sufficiently close to p_0 , we can identify $T_q E$ with $T_{p_0} E$.

Notice that under the conjugate action of G_1 , each Clifford multiplication c (or $\hat{c}$) comes with a factor $t^{-\frac{1}{4}}$. The rest of the proof follows from a straightforward computation. ■

Corollary 5.3.4

$$\lim_{t \rightarrow 0} \sqrt{t} G_1 \circ \left(N e^{-t D_{\tau h}^2} \right) \circ G_1^{-1} = \left(\frac{1}{2} \sum_r e^r \wedge \hat{e}^r \wedge + \frac{1}{2} \sum_r f^r \wedge \hat{f}^r \wedge \right) e^{L_1}.$$

Proof: Recall that the number operator decomposes as $N = N_M \otimes 1 + 1 \otimes N_Y$, where

$$N_M = e^i \wedge \iota_{e_i} = \frac{n}{2} + \frac{1}{2} \sum_i c(e^i) \hat{c}(e^i),$$

$$N_Y = f^i \wedge \iota_{f_i} = \frac{k}{2} + \frac{1}{2} \sum_i c(f^i) \hat{c}(f^i)$$

are the horizontal and vertical number operators. Therefore by the definition of G_1 and Lemma 5.3.3, we complete the proof. $\blacksquare$

The above limit should be understood in the sense that the coefficients of $\sqrt{t} G_1 \circ \left(N e^{-t D_{\tau h}^2} \right) \circ G_1^{-1}$ converge to those of the limit operator as $t \rightarrow 0$. In order to see the effect on the supertraces, we recall that the heat kernel $K_\tau(t, x, y)$ (along the diagonal) of $N e^{-t D_{\tau h}^2}$ admits an asymptotic expansion of the form

$$K_\tau(t, x, y) \sim \sum_{j,l} t^j \tau^l A_{j,l;I,J,I',J'}(x, y) c(e^I f^J) \hat{c}(e^{I'} f^{J'}), \quad (5.30)$$

while for $\text{Tr}_s \left(N e^{-t D_{\tau h}^2} \right)$ we have

$$\text{Tr}_s \left(N e^{-t D_{\tau h}^2} \right) \sim \sum_{p \text{ odd}} \sum_q a_{p/2,q} t^{p/2} \tau^q. \quad (5.31)$$

Notice that the corresponding heat kernel of the rescaled operator $G_1 \circ N e^{-t D_{\tau h}^2} \circ G_1^{-1}$

is $t^{\frac{n+k}{2}} G_1(K_\tau)$. By the definition of G_1 , one has

$$\begin{aligned} & t^{\frac{n+k}{2}} G_1(K_\tau)(t, x, y) \\ & \sim t^{\frac{n+k}{2}} \sum_{j,l} t^j \cdot t^{-\frac{l}{2}} \tau^l A_{j,l;I,I',J'}(\sqrt{t}x, \sqrt{t}y) t^{-\frac{|I|+|I'|+|J|+|J'|}{4}} c_t(e^I f^J) \hat{c}_t(e^{I'} f^{J'}). \end{aligned}$$

Since only terms with $|I| = |I'| = n$ and $|J| = |J'| = k$ can survive under the supertrace, the two factors $t^{\frac{n+k}{2}}$ and $t^{-\frac{|I|+|I'|+|J|+|J'|}{4}}$ are canceled. This leads to

$$\mathrm{Tr}_s \left(t^{\frac{n+k}{2}} G_1(K_\tau)(t, 0, 0) \right) \sim (-1)^{\frac{(n+k)(n+k+1)}{2}} 2^{n+k} \sum_{j,l} t^{j-\frac{l}{2}} \tau^l A_{j,l;\mathrm{top}}(0, 0).$$

In particular,

$$\mathrm{Tr}_s \left(G_1 \circ N e^{-tD_{\tau h}^2} \circ G_1^{-1} \right) \sim \sum_{p \text{ odd}} \sum_q a_{p/2,q} t^{\frac{p-q}{2}} \tau^q. \quad (5.32)$$

Recall that by assumption, the base manifold M is compact and odd-dimensional. It follows from [32, Theorem 5.1] that

$$\mathrm{Tr}_s \left(N_M e^{-tD_M^2} \right) \sim b_{-1/2} t^{-\frac{1}{2}} + \sum_{j \geq 1, \text{odd}} b_{j/2} t^{\frac{j}{2}}, \quad (5.33)$$

and

$$b_{-1/2} e^1 \wedge \cdots \wedge e^n = 2i(-1)^{\frac{n+1}{2}} (16\pi)^{-\frac{n}{2}} \sum_{k=1}^n (-1)^k \mathrm{Pf}(R_k) \wedge e^k, \quad (5.34)$$

where $R_k = \left(\sum_{p,q \neq k} R_{ijkl} e^p \wedge e^q \right)_{i,j \neq k}$ is the matrix 2-form obtained by deleting the k -th row and column from the curvature matrix 2-form.

Now we are ready to prove the main theorem of this section.

Proof: [Proof of Theorem 5.1.9] Let $\tilde{K}_\tau(x, y; x, y) \in \Gamma(\Lambda^* E \hat{\otimes} \Lambda^* \hat{E})$ denote the diag-

onal heat kernel of the limiting operator e^{L_1} in Lemma 5.3.3. It is easy to see that

$$\tilde{K}_\tau = \frac{1}{(4\pi)^{\frac{n+k}{2}}} e^{-\tau^2|y|^2} e^{-\frac{1}{8}\overline{\sum_{ijkl} R_{ijkl}^L e^i \wedge e^j \wedge \hat{e}^k \wedge \hat{e}^l - \tau \sum_i f^i \wedge \hat{f}^i}}. \quad (5.35)$$

Recall that as an element of $\Lambda^* E \hat{\otimes} \Lambda^* \hat{E}$, the curvature $R^L := (\nabla^L)^2$ is given by

$$R^L = \frac{1}{4} \overline{\sum_{i,j,k,l} R_{ijkl}^L e^i \wedge e^j \wedge \hat{e}^k \wedge \hat{e}^l}. \quad (5.36)$$

To deal with the mixed curvature terms in (5.35), we consider another connection $\tilde{\nabla}$ on TE defined by

$$\tilde{\nabla}_{e_i} e_j = \nabla_{e_i}^M e_j, \quad \tilde{\nabla}_{e_i} f_j = \tilde{\nabla}_{f_i} e_j = \tilde{\nabla}_{f_i} f_j = 0. \quad (5.37)$$

Clearly $\tilde{\nabla}$ is the extension of the pullback connection of ∇^M , therefore its curvature form $\tilde{R}$ is

$$\tilde{R} = \frac{1}{4} \sum_{i,j,k,l} R_{ijkl}^M e^i \wedge e^j \wedge \hat{e}^k \wedge \hat{e}^l = R^M. \quad (5.38)$$

By Chern-Weil theory, there is an $\omega \in \Omega^{n-1}(E)$ such that

$$\int^B e^{-\frac{1}{2}R^L} = \int^B e^{-\frac{1}{2}R^M} + d\omega. \quad (5.39)$$

Moreover, ω can be written as polynomials of g^E and g^M as well as their derivatives. Since (E, g^E) clearly has bounded geometry, both $e^{-\tau^2|y|^2} \omega$ and $d(e^{-\tau^2|y|^2} \omega)$ are integrable. It follows from a generalized Stokes theorem (see [35]) that

$$\int_E d(e^{-\tau^2|y|^2} \omega) = 0. \quad (5.40)$$

Notice that

$$d\left(e^{-\tau^2|y|^2}\omega\right) = \sum_{i=0}^k 2\tau^2 y_i e^{-\tau^2|y|^2} dy^i \wedge \omega + e^{-\tau^2|y|^2} d\omega. \quad (5.41)$$

By the construction above we see that ω is independent of y . Since $y_i e^{-\tau^2|y|^2}$ is an odd function in y , we have $\int_E \sum_{i=0}^k 2y_i e^{-\tau^2|y|^2} dy^i \wedge \omega = 0$, and

$$\int_E e^{-\tau^2|y|^2} d\omega = \int_E d\left(e^{-\tau^2|y|^2}\omega\right) = 0. \quad (5.42)$$

In particular, this shows that in (5.35), only the curvature terms coming from the base manifold can make a contribution to the integration over E .

It follows from Proposition 2.2.4, Corollary 5.3.4, and the standard McKean-Singer formula that

$$\begin{aligned} & \lim_{t \rightarrow 0} \sqrt{t} \operatorname{Tr}_s \left(G_1 \circ N e^{-tD_{\tau^h}^2} \circ G_1^{-1} \right) \quad (5.43) \\ &= (-1)^{\frac{(n+k)(n+k+1)}{2}} 2^{n+k} \int_E \int^B \left[\left(\frac{1}{2} \sum_r e^r \wedge \hat{e}^r \wedge + \frac{1}{2} \sum_r f^r \wedge \hat{f}^r \wedge \right) e^{L_1} \right] d\operatorname{vol}_E \\ &= (-1)^{\frac{n+k+1}{2}} 2^{n+k-1} \frac{1}{(4\pi)^{\frac{n+k}{2}}}. \\ & \int_E e^{-\tau^2|y|^2} \left(\int^B \sum_r e^r \wedge \hat{e}^r \wedge e^{-\frac{1}{8} \sum_{ijkl} R_{ijkl}^L e^i \wedge e^j \wedge \hat{e}^k \wedge \hat{e}^l - \tau \sum_i f^i \wedge \hat{f}^i} \right) d\operatorname{vol}_E \\ &= (-1)^{\frac{n+k+1}{2}} 2^{n+k-1} \frac{1}{(4\pi)^{\frac{n+k}{2}}}. \\ & \left(\int_{\mathbb{R}^k} (-1)^{\frac{k(k-1)}{2}} \tau^k e^{-\tau^2|y|^2} dy \right) \cdot \int_M \left(\int^B \sum_r e^r \wedge \hat{e}^r \wedge e^{-\frac{1}{2} R^M} \right) d\operatorname{vol}_M \\ &= b_{-1/2}. \end{aligned}$$

On the other hand, equation (5.32) implies

$$\lim_{t \rightarrow 0} \sqrt{t} \operatorname{Tr}_s \left(G_1 \circ N e^{-t D_{\tau h}^2} \circ G_1^{-1} \right) = \sum_{p-q+1=0} a_{p/2,q} \tau^q. \quad (5.44)$$

However by (5.43), the right hand side should be independent of τ . Therefore $a_{p/2,q} = 0$ for all $q = p + 1 \neq 0$ and

$$a_{-1/2,0} = \lim_{t \rightarrow 0} \sqrt{t} \operatorname{Tr}_s \left(G_1 \circ N e^{-t D_{\tau h}^2} \circ G_1^{-1} \right) = b_{-1/2}. \quad (5.45)$$

■

5.3.2 A Vanishing result: Proof of Theorem 5.1.10

We consider a new metric $g_{\sigma,T}$ on E given by

$$g_{\sigma,T} := \frac{1}{\sigma^2} g_M + \frac{1}{T^2} g_Y. \quad (5.46)$$

One should notice that this metric depends on two parameters $\sigma, T > 0$ in a non-trivial way. Let $D_{\sigma,T,\tau h}$ denote the Witten-Dirac operator associated with $g_{\sigma,T}$. Observe that $g_{\sigma,T} = \frac{1}{\sigma^2} \tilde{g}_{T/\sigma}$. Therefore by Proposition 5.2.3,

$$\operatorname{Tr}_s \left(N_Y e^{-D_{\sigma,T,\tau h}^2} \right) = \operatorname{Tr}_s \left(N_Y e^{-\sigma^2 \tilde{D}_{T/\sigma,\tau h}^2} \right). \quad (5.47)$$

Let $\{e^1, \dots, e^n, f^1, \dots, f^k\}$ be a locally orthonormal coframe for the metric $g_E = g_{1,1}$. Set $\tilde{e}^i = \sigma e^i$ and $\tilde{f}^i = T f^i$. Then $\{\tilde{e}^1, \dots, \tilde{e}^n, \tilde{f}^1, \dots, \tilde{f}^k\}$ forms an orthonormal coframe for $g_{\sigma,T}$. We will use $c(\tilde{e}^{i_1 \dots i_p} \tilde{f}^{j_1 \dots j_q})$ to denote $c(\tilde{e}^{i_1}) \dots c(\tilde{e}^{i_p}) c(\tilde{f}^{j_1}) \dots c(\tilde{f}^{j_q})$.

Inspired by [17], we consider a new rescaling G_2 that only involves the base direction:

(1) For $f(x, y) \in C^\infty(E)$, $(G_2 f)(x, y) = f(\sigma x, y)$. Thus

$$G_2 \circ \tilde{e}_i \circ G_2^{-1} = e_i.$$

(2) We only rescale the left Clifford multiplications along the base direction, i.e.

$$G_2 \circ c(\tilde{e}^i) \circ G_2^{-1} = \frac{1}{\sigma} (\tilde{e}^i \wedge -\sigma^2 \iota_{\tilde{e}_i}) := \frac{1}{\sigma} c_\sigma(\tilde{e}^i).$$

As an analogue of Lemma 5.3.2, we have:

Lemma 5.3.5 *Let $R^{E, \sigma, T}$ be the scalar curvature of E with respect to $g_{\sigma, T}$. Then*

$$\begin{aligned} D_{\sigma, T, \tau h}^2 = & -\sigma^2 \sum_i \left[e_i + \frac{1}{8} \overline{\sum_{j, k, l}} \left(R_{ijkl}^{L, \sigma, T}(0) x^l + O(|x|^2) \right) (c(\tilde{e}^j) c(\tilde{e}^k) - \hat{c}(\tilde{e}^j) \hat{c}(\tilde{e}^k)) \right]^2 \\ & - T^2 \sum_i \left[\nabla_{f_i}^E + \frac{1}{2} \sum_{j, k} \langle S^{\sigma, T}(f_i) \tilde{f}_j, \tilde{e}_k \rangle_{\sigma, T} \left(c(\tilde{f}^j) c(\tilde{e}^k) - \hat{c}(\tilde{f}^j) \hat{c}(\tilde{e}^k) \right) \right. \\ & \left. + \frac{1}{4} \sum_{j, k} \langle S^{\sigma, T}(f_i) \tilde{e}_j, \tilde{e}_k \rangle_{\sigma, T} (c(\tilde{e}^j) c(\tilde{e}^k) - \hat{c}(\tilde{e}^j) \hat{c}(\tilde{e}^k)) \right]^2 + \frac{1}{4} R^{E, \sigma, T} \\ & + \frac{1}{8} \overline{\sum_{i, j, k, l}} R_{ijkl}^{L, \sigma, T} c(\tilde{e}^i) c(\tilde{e}^j) \hat{c}(\tilde{e}^k) \hat{c}(\tilde{e}^l) + \tau \sum_i c(\tilde{f}^i) \hat{c}(\tilde{f}^i) + \tau^2 T^2 |y|^2. \end{aligned}$$

Proof: Recall that $\nabla_{f_i}^{L, \sigma, T} = \nabla_{f_i}^E + S^{\sigma, T}(f_i)$. By [18, (4.31)], since $S^{\sigma, T}(f_i)$ is anti-symmetric as a derivation acting on $\Lambda^*(T^*E)$, its action is given by

$$\frac{1}{4} \overline{\sum_{j, k}} \langle S^{\sigma, T}(f_i) \tilde{e}_j, \tilde{e}_k \rangle_{\sigma, T} (c(\tilde{e}^j) c(\tilde{e}^k) - \hat{c}(\tilde{e}^j) \hat{c}(\tilde{e}^k)).$$

Applying Proposition 2.1.3, Proposition 5.1.2, and Lemma 5.3.2, we get the desired result. ■

Let $S^{\sigma, T} = \nabla^{L, \sigma, T} - \nabla^E$, where $\nabla^{L, \sigma, T}$ is the Levi-Civita connection of $g_{\sigma, T}$ and ∇^E is the connection in Definition 2.1.2.

Lemma 5.3.6 $S^{\sigma,T} = \frac{\sigma^2}{T^2} P^H S + P^V S$.

Proof: Let $U_i, U \in \Gamma(TM), W_i, W \in \Gamma(VE)$. Using Proposition 2.1.3, one can easily verify that

$$\begin{aligned} S^{\sigma,T}(W_1)W_2 &= \frac{\sigma^2}{T^2} S(W_1)W_2, & P^V S^{\sigma,T}(W)U &= P^V S(W)U, \\ S^{\sigma,T}(U)W &= \frac{\sigma^2}{T^2} S(U)W, & S^{\sigma,T}(U_1)U_2 &= S(U_1)U_2, \\ P^H S^{\sigma,T}(W)U &= \frac{\sigma^2}{T^2} S(W)U, \end{aligned}$$

which completes the proof. ■

Now we calculate the limit of the rescaled Dirac operator.

Lemma 5.3.7 *Let R^E denote the curvature tensor of ∇^E . Then*

$$\begin{aligned} L_2 &:= \lim_{\sigma \rightarrow 0} G_2 \circ (-D_{\sigma,T,\tau h}^2) \circ G_2^{-1} \\ &= \sum_i e_i^2 + T^2 \sum_i \left[\nabla_{f_i}^E + \frac{1}{2} \langle S^T(f_i) f_j, e_k \rangle_T c(\tilde{f}_j) e^k \wedge \right. \\ &\quad \left. + \frac{1}{4} \langle S^T(f_i) e_j, e_k \rangle_T e^j \wedge e^k \wedge \right]^2 - \frac{1}{8} \sum_{ijkl} R_{ijkl}^M e^i \wedge e^j \wedge \hat{e}^k \wedge \hat{e}^l \\ &\quad - \frac{1}{8} \sum_{ijkl} R_{ijkl}^E e^i \wedge e^j \wedge \hat{c}(\tilde{f}^k) \hat{c}(\tilde{f}^l) - \frac{T}{4} \sum_{ijkl} R_{ijkl}^E e^i \wedge c(\tilde{f}^j) \hat{c}(\tilde{f}^k) \hat{c}(\tilde{f}^l) \\ &\quad - \tau \sum_i c(\tilde{f}^i) \hat{c}(\tilde{f}^i) - \tau^2 T^2 |y|^2. \end{aligned}$$

Proof: First, we calculate the (σ, T) -dependence of the curvature components. By Lemma 5.3.6, the Levi-Civita connection decomposes as

$$\nabla^{L,\sigma,T} = \nabla^E + \frac{\sigma^2}{T^2} P^H S + P^V S.$$

By a direct calculation, we have

$$\begin{aligned}
R^{L,\sigma,T}(e_i, e_j)e_k &= \nabla_{e_i}^{L,\sigma,T} \nabla_{e_j}^{L,\sigma,T} e_k - \nabla_{e_j}^{L,\sigma,T} \nabla_{e_i}^{L,\sigma,T} e_k - \nabla_{[e_i, e_j]}^{L,\sigma,T} e_k \\
&= \left[R^M(e_i, e_j)e_k + \frac{\sigma^2}{T^2} \left(S(e_i)S(e_j)e_k - S(e_j)S(e_i)e_k - P^H S([e_i, e_j])e_k \right) \right] \\
&\quad + \left[S(e_i) \nabla_{e_j}^M e_k + \nabla_{e_i}^E S(e_j)e_k - S(e_j) \nabla_{e_i}^M e_k - \nabla_{e_j}^E S(e_i)e_k - P^V S([e_i, e_j])e_k \right], \\
R^{L,\sigma,T}(e_i, e_j)f_k &= \nabla_{e_i}^{L,\sigma,T} \nabla_{e_j}^{L,\sigma,T} f_k - \nabla_{e_j}^{L,\sigma,T} \nabla_{e_i}^{L,\sigma,T} f_k - \nabla_{[e_i, e_j]}^{L,\sigma,T} f_k \\
&= \frac{\sigma^2}{T^2} \left[S(e_i) \nabla_{e_j}^E f_k + \nabla_{e_i}^M S(e_j)f_k - S(e_j) \nabla_{e_i}^E f_k - \nabla_{e_j}^M S(e_i)f_k - S([e_i, e_j])f_k \right] \\
&\quad + \left[R^E(e_i, e_j)f_k + \frac{\sigma^2}{T^2} \left(S(e_i)S(e_j)f_k - S(e_j)S(e_i)f_k \right) \right], \\
R^{L,\sigma,T}(e_i, f_j)e_k &= \nabla_{e_i}^{L,\sigma,T} \nabla_{f_j}^{L,\sigma,T} e_k - \nabla_{f_j}^{L,\sigma,T} \nabla_{e_i}^{L,\sigma,T} e_k - \nabla_{[e_i, f_j]}^{L,\sigma,T} e_k \\
&= \frac{\sigma^2}{T^2} \left[\nabla_{e_i}^M P^H S(f_j)e_k + S(e_i)P^V S(f_j)e_k - P^H S(f_j) \nabla_{e_i}^M e_k - S(f_j)S(e_i)e_k \right. \\
&\quad \left. - P^H S([e_i, f_j])e_k \right] + \left[\frac{\sigma^2}{T^2} S(e_i)P^H S(f_j)e_k + \nabla_{e_i}^E P^V S(f_j)e_k \right. \\
&\quad \left. - P^V S(f_j) \nabla_{e_i}^M e_k - \nabla_{f_j}^E S(e_i)e_k - P^V S([e_i, f_j])e_k \right], \\
R^{L,\sigma,T}(e_i, f_j)f_k &= \nabla_{e_i}^{L,\sigma,T} \nabla_{f_j}^{L,\sigma,T} f_k - \nabla_{f_j}^{L,\sigma,T} \nabla_{e_i}^{L,\sigma,T} f_k - \nabla_{[e_i, f_j]}^{L,\sigma,T} f_k \\
&= \frac{\sigma^2}{T^2} \left[S(e_i)S(f_j)f_k + \nabla_{e_i}^M S(f_j)f_k - \frac{\sigma^2}{T^2} P^H S(f_j)S(e_i)f_k - S(f_j) \nabla_{e_i}^E f_k \right. \\
&\quad \left. - S([e_i, f_j])f_k \right] + \left[R^E(e_i, f_j)f_k + \frac{\sigma^2}{T^2} S(e_i)S(f_j)f_k - \frac{\sigma^2}{T^2} P^V S(f_j)S(e_i)f_k \right], \\
R^{L,\sigma,T}(f_i, f_j)f_k &= \nabla_{f_i}^{L,\sigma,T} \nabla_{f_j}^{L,\sigma,T} f_k - \nabla_{f_j}^{L,\sigma,T} \nabla_{f_i}^{L,\sigma,T} f_k - \nabla_{[f_i, f_j]}^{L,\sigma,T} f_k \\
&= \frac{\sigma^2}{T^2} \left[S(f_i) \nabla_{f_j}^E f_k + \frac{\sigma^2}{T^2} P^H S(f_i)S(f_j)f_k - S(f_j) \nabla_{f_i}^E f_k \right. \\
&\quad \left. - \frac{\sigma^2}{T^2} P^H S(f_j)S(f_i)f_k - S([f_i, f_j])f_k \right] + \left[R^E(f_i, f_j)f_k + \frac{\sigma^2}{T^2} P^V S(f_i, f_j)f_k \right].
\end{aligned}$$

Recall that for the metric $g_{\sigma,T}$, $\{\sigma e_1, \dots, \sigma e_n, T f_1, \dots, T f_k\}$ is a local orthonormal frame.

We let $R_{ijkl}^{L,\sigma,T} = Rm^{L,\sigma,T}(\sigma e_i, \sigma e_j, \sigma e_k, \sigma e_l)$, $R_{ijkl}^M = Rm^M(e_i, e_j, e_k, e_l)$, etc. Then

$$\begin{aligned} R_{ijkl}^{L,\sigma,T} &= \sigma^2 R_{ijkl}^M + O\left(\frac{\sigma^4}{T^2}\right), \quad R_{ijkl}^{L,\sigma,T} = O\left(\frac{\sigma^3}{T}\right), \quad R_{ijkl}^{L,\sigma,T} = \sigma^2 R_{ijkl}^E + O\left(\frac{\sigma^4}{T^2}\right), \\ R_{ijkl}^{L,\sigma,T} &= O(\sigma^2), \quad R_{ijkl}^{L,\sigma,T} = \sigma T R_{ijkl}^E + O\left(\frac{\sigma^3}{T}\right), \quad R_{ijkl}^{L,\sigma,T} = T^2 R_{ijkl}^E + O(\sigma^2). \end{aligned}$$

It is easy to see that if we apply the conjugate $G_2 \circ \bullet \circ G_2^{-1}$ to Lemma 5.3.5, the first two terms become $\sum_i e_i^2 + T^2 \sum_i \left[\nabla_{f_i}^E + \frac{1}{2} \langle S^T(f_i) f_j, e_k \rangle_T c(\tilde{f}_j) e^k \wedge + \frac{1}{4} \langle S^T(f_i) e_j, e_k \rangle_T e^j \wedge e^k \right]^2 + o(\varepsilon^2)$. On the other hand, the scalar curvature is only a function of x and y , and the conjugation acts as a scaling on the variables. Notice that

$$R^{E,\sigma,T} = T^2 \sum_{\bar{i}\bar{j}\bar{k}\bar{l}} R_{\bar{i}\bar{j}\bar{k}\bar{l}}^E + O(\sigma^2),$$

and $R_{\bar{i}\bar{j}\bar{k}\bar{l}}^E$ is actually the Riemannian curvature of the flat fiber. Therefore the scalar curvature term drops out in the limit.

Now we calculate the rest of the curvature terms. Under the conjugate action, each left Clifford multiplication along the base direction, like $c(\tilde{e}^i)$, comes with a factor σ^{-1} . Therefore by the (σ, T) -dependence of the curvature, the only terms that can survive in the limit are R_{ijkl}^M , R_{ijkl}^E , and R_{ijkl}^E . This completes the proof. $\blacksquare$

As a consequence, we have

Corollary 5.3.8

$$\lim_{\sigma \rightarrow 0} G_2 \circ N_Y e^{-D_{\sigma,T,\tau h}^2} \circ G_2^{-1} = N_Y e^{L_2}, \quad (5.48)$$

where L_2 is the limit operator in Lemma 5.3.7.

In the end, we prove Theorem 5.1.10.

Proof: [Proof of Theorem 5.1.10]

By (5.47), Corollary 5.3.8, and arguing as in [33], we have

$$\lim_{\sigma \rightarrow 0} \mathrm{Tr}_s \left(N_Y e^{-\sigma^2 \tilde{D}_{T/\sigma, \tau h}^2} \right) = \lim_{\sigma \rightarrow 0} \mathrm{Tr}_s \left(N_Y e^{-D_{\sigma, T, \tau h}^2} \right) = \mathrm{Tr}_s \left(N_Y e^{L_2} \right).$$

Since the horizontal differential forms, i.e. those generated by e^i and $\hat{e}^i$, in L_2 are all of even degree, but the dimension of M is odd, we see that

$$\mathrm{Tr}_s \left(N_Y e^{L_2} \right) = 0.$$

This completes the proof. ■

5.3.3 Remainder estimate: Proof of Theorem 5.1.11

Inspired by [24, section 8], we consider a new parameter $a = \frac{T}{\sigma}$. Then

$$\sigma^2 \tilde{D}_{T/\sigma, \tau h}^2 = \frac{T^2}{a^2} \tilde{D}_{a, \tau h}^2.$$

Since the ranges for σ and T are $\sigma \in (0, 1]$ and $T \in [\sigma, 1]$, $a \in [1, \frac{1}{\sigma}]$ and thus Theorem 5.1.11 amounts to a uniform upper bound when $a \rightarrow \infty$.

Proof: [Proof of Theorem 5.1.11] Using the notations in section 5.3.2, we see that

$$\begin{aligned} \mathrm{Tr}_s \left(N_Y e^{-\sigma^2 \tilde{D}_{T/\sigma, \tau h}^2} \right) &= \mathrm{Tr}_s \left(N_Y e^{-\frac{T^2}{a^2} \tilde{D}_{a, \tau h}^2} \right) \\ &= \mathrm{Tr}_s \left(N_Y e^{-T^2 D_{1/a, 1, \tau h}^2} \right), \end{aligned} \tag{5.49}$$

where $D_{1/a, 1, \tau h}$ is the Witten-Dirac operator of

$$g_{1/a, 1} = a^2 g_M + g_Y.$$

Consider another rescaling G_3 constructed from G_2 by replacing σ with $\frac{1}{a}$. A similar argument as in the previous section shows that $G_3 \circ N_Y e^{-T^2 D_{1/a,1,\tau h}^2} \circ G_3^{-1}$ approaches a finite limit $N_Y e^{T^2 L_3}$ as $a \rightarrow \infty$. Moreover, it is easy to see that L_3 is uniformly elliptic.

By a similar argument as in Section 5.3.1 we know for T small, $\text{Tr}_s \left(N_Y e^{-T^2 D_{1/a,1,\tau h}^2} \right)$ admits an expansion of the form

$$\text{Tr}_s \left(N_Y e^{-T^2 D_{1/a,1,\tau h}^2} \right) \sim d_1(a)T + d_3(a)T^3 + \dots \quad (5.50)$$

Since $\text{Tr}_s \left(N_Y e^{T^2 L_3} \right) = 0$ and the asymptotic expansion depends only on the local symbol, all the coefficients $d_i(a)$ converge to 0 as $a \rightarrow \infty$.

On the other hand, by an argument as in [17, P. 64] (see also [36, P. 194-197]), each $d_i(a)$ admits an expansion like $\sum d_{i,j} a^j$ for a sufficiently large. In particular, the powers of a can only be integers. This implies that $d_i(a) = O(\frac{1}{a})$ for each i . Thus by (5.49) and (5.50) we have

$$\left| \frac{1}{T} \text{Tr}_s \left(N_Y e^{-\sigma^2 \tilde{D}_{T/\sigma,\tau h}^2} \right) \right| = \left| \frac{1}{T} \text{Tr}_s \left(N_Y e^{-T^2 D_{1/a,1,\tau h}^2} \right) \right| \leq \frac{C}{a} \leq C.$$

■

5.4 Proof of Theorem 5.1.12

In this section, we provide a proof of Theorem 5.1.12 that essentially follows the argument in [24, section 9] and [34, section 11, section 13].

5.4.1 Localization of the problem

Let $\alpha > 0$ be a constant with $\alpha \ll \frac{1}{4} \text{injad}(M)$, $f : \mathbb{R} \rightarrow [0, 1]$ be a smooth even cut-off function such that

$$f(t) = \begin{cases} 1, & \text{if } |t| \leq \frac{\alpha}{2}, \\ 0, & \text{if } |t| \geq \alpha. \end{cases}$$

Let $g(t) = 1 - f(t)$. For any $v > 0$, let

$$\begin{aligned} F_v(a) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{it\sqrt{2}a - t^2/2} f(vt) dt, \\ G_v(a) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{it\sqrt{2}a - t^2/2} g(vt) dt, \\ H_v(a) &= \frac{1}{\sqrt{2\pi}v} \int_{-\infty}^{\infty} e^{it\sqrt{2}a - t^2/2v^2} g(t) dt. \end{aligned}$$

In the following we collect properties about these functions. The proof is standard and is omitted here.

Proposition 5.4.1 (1) $G_v(a) = H_v(\frac{a}{v})$.

(2) $F_v(a) + G_v(a) = e^{-a^2}$. Therefore

$$F_\sigma(\sigma \tilde{D}_{T/\sigma, \tau h}) + G_\sigma(\sigma \tilde{D}_{T/\sigma, \tau h}) = e^{-\sigma^2 \tilde{D}_{T/\sigma, \tau h}^2}.$$

(3) There exist holomorphic functions $\tilde{F}_v(a)$ and $\tilde{H}_v(a)$ such that

$$F_v(a) = \tilde{F}_v(a^2), \quad H_v(a) = \tilde{H}_v(a^2).$$

(4) For any $c > 0$, $m \in \mathbb{N}$, there exist constants $C_1, C_2 > 0$ such that for any $v \in$

$(0, 1], a \in \mathbb{C}$ with $|\operatorname{Im} a| \leq c$,

$$\sup_a |a^m H_v(a)| \leq C_1 e^{-\frac{C_2}{v^2}}.$$

(5) For any $c > 0 \in \mathbb{R}$, $m \in \mathbb{N}$, there exists a constant $C > 0$ such that for any $v \in$

$(0, 1], a \in \mathbb{C}$ with $\operatorname{Re} a \geq \frac{(\operatorname{Im} a)^2}{4c^2} - c^2$,

$$\sup_a |a^m F_v(a)| \leq C.$$

It is useful to observe that for the metric $g_{\sigma,T} = \frac{1}{\sigma^2}g_M + \frac{1}{T^2}g_Y$ considered in section 5.3.2, we have

$$\operatorname{Tr}_s \left(N_Y e^{-D_{\sigma,T,\tau h}^2} \right) = \operatorname{Tr}_s \left(N_Y e^{-\sigma^2 \tilde{D}_{T/\sigma,\tau h}^2} \right).$$

It turns out that for $G_\sigma(\sigma \tilde{D}_{T/\sigma,\tau h})$, the computation would be easier if we consider the original metric $\tilde{g}_{T/\sigma,\tau h}$, while for $F_\sigma(\sigma \tilde{D}_{T/\sigma,\tau h})$ it is better to use the metric $g_{\sigma,T}$.

Using Lemma 5.3.6, we obtain the following analogue of (3.23).

Lemma 5.4.2

$$\begin{aligned} D_{\sigma,T} &= \sigma c(\tilde{e}^i) \nabla_{e_i}^u + T c(\tilde{f}^j) \nabla_{f_j}^E + T \tau \tilde{c}(dh) - \frac{\sigma^2}{4} c(\tilde{e}^\alpha) c(\tilde{e}^\beta) \tilde{c}(\operatorname{Tor})(e_\alpha, e_\beta) \\ &:= \sigma \tilde{D}_M + T \tilde{D}_{Y,T} - \frac{\sigma^2}{4} \operatorname{Tor}, \end{aligned}$$

where $\tilde{c}(dh) = \frac{1}{T} dh \wedge + T \iota_{\nabla h}$.

One can easily verify that $T^2 \tilde{D}_{Y,T}^2 = T^2 \tilde{D}_Y^2$. Thus (3.29) implies that

$$\ker T^2 \tilde{D}_{Y,T}|_{\Lambda^* Y} = \ker \tilde{D}_Y|_{\Lambda^* Y} = \operatorname{span}_{\mathbb{R}} \left\{ e^{-\frac{\tau |y|^2}{2}} \right\}$$

is independent of σ and T .

Proposition 5.4.1 shows that

$$\begin{aligned}
\mathrm{Tr}_s \left(N_Y e^{-\sigma^2 \tilde{D}_{T/\sigma, \tau h}^2} \right) & \quad (5.51) \\
&= \mathrm{Tr}_s \left(N_Y F_\sigma(\sigma \tilde{D}_{T/\sigma, \tau h}) \right) + \mathrm{Tr}_s \left(N_Y G_\sigma(\sigma \tilde{D}_{T/\sigma, \tau h}) \right) \\
&= \mathrm{Tr}_s \left(N_Y F_\sigma(D_{\sigma, T, \tau h}) \right) + \mathrm{Tr}_s \left(N_Y H_\sigma(\tilde{D}_{T/\sigma, \tau h}) \right).
\end{aligned}$$

First we deal with the second term. Let $H^p(\Lambda^* E)$ be the p -th Sobolev space of sections, and $\|\cdot\|_p$ be the corresponding Sobolev norm.

Proposition 5.4.3 *There exist constants $C > 0$ and $T_0 \geq 1$ such that for any $\sigma \in (0, 1]$, $T \in [T_0, \infty)$,*

$$\left| \mathrm{Tr}_s \left(N_Y H_\sigma(\tilde{D}_{T/\sigma, \tau h}) \right) \right| \leq \frac{C}{T}.$$

Proof: It follows from Lemma 5.4.2 that

$$\begin{aligned}
D_{\sigma, T} &= \tilde{D}_M - \frac{1}{4} \mathrm{Tor} + \frac{T}{\sigma} \tilde{D}_{Y, T/\sigma} \\
&:= E + \frac{T}{\sigma} \tilde{D}_{Y, T/\sigma},
\end{aligned}$$

Obviously we have

$$D_{\sigma, T}^2 = E^2 + \frac{T}{\sigma} [E, \tilde{D}_{Y, T/\sigma}] + \frac{T^2}{\sigma^2} \tilde{D}_{Y, T/\sigma}^2.$$

Since the pair $(E, \tilde{D}_{Y, T/\sigma})$ is uniformly elliptic, [12, Lemma 2.1] and the standard elliptic estimate imply for any $s \in H^1(\Lambda^* E)$,

$$\begin{aligned}
\langle (E^2 + \frac{T^2}{\sigma^2} \tilde{D}_{Y, T/\sigma}^2) s, s \rangle_0 &= \langle (E^2 + \tilde{D}_{Y, T/\sigma}^2) s, s \rangle_0 + \left(\frac{T^2}{\sigma^2} - 1 \right) \langle \tilde{D}_{Y, T/\sigma}^2 s, s \rangle_0 \\
&\geq C (\|s\|_1 - \|s\|_0) + \left(\frac{T^2}{\sigma^2} - 1 \right) \|\tilde{D}_{Y, T/\sigma} s\|_0^2.
\end{aligned}$$

On the other hand, from the proof of [17, Proposition 4.41] one has

$$\frac{T}{\sigma} \langle [E, \tilde{D}_{Y,T/\sigma}]s, s \rangle_0 \leq \frac{CT}{\sigma} \|\tilde{D}_{Y,T/\sigma}s\|_0^2 + \frac{CT}{\sigma} \|s\|_0^2.$$

Here we have an extra term $\|s\|_0^2$ since we no longer assume s to be perpendicular to $\ker \tilde{D}_{Y,T/\sigma}$. This implies

$$\|D_{\sigma,T}s\|_0^2 \geq C \left(\|s\|_1^2 - \|s\|_0^2 - \frac{T}{\sigma} \|s\|_0^2 \right) + \left(\frac{T^2}{\sigma^2} - 1 - \frac{CT}{\sigma} \right) \|\tilde{D}_{Y,T/\sigma}s\|_0^2.$$

Since we can always choose T large enough so that $\frac{T^2}{\sigma^2} - 1 - \frac{CT}{\sigma} \geq T^2 - 1 - CT > 0$, it follows that

$$\|s\|_1^2 \leq C \left(\|D_{\sigma,T}s\|_0^2 + \frac{T}{\sigma} \|s\|_0^2 \right). \quad (5.52)$$

Using (5.52), Proposition 5.4.1 (4) and arguing as in [24, Section 8] (see also [24, Section 5]), we complete the proof. ■

For the rest of the section, we aim to show that the estimate of the first term in (5.51) can be localized. Let $p_0 \in \pi^{-1}(x_0) \subset E$ be a fixed point. Notice that $e^{it\sqrt{2}D_{\sigma,T,\tau h}}$ is a wave type operator. By the finite propagation speed of solutions of hyperbolic equations and the fact that $f(t) \equiv 0$ for $|t| \geq \alpha$, the heat kernel of $F_\sigma(D_{\sigma,T,\tau h})$ depends only on the restriction of $D_{\sigma,T,\tau h}$ to $B_\alpha(p_0)$. For this reason, we introduce the following construction (see [13, section 4.2], [34, section 13(e)]):

- (1) Set $M_0 := T_{x_0}M \cong \mathbb{R}^n$. Let $\{e_i\}$ be an orthonormal basis of M_0 , $X = \sum_{i=1}^n x_i e_i$ be the radial vector field on M_0 . We will also use X to denote an arbitrary point in M_0 .
- (2) Pick $\varepsilon > 0$ such that $4\varepsilon < \alpha$. We identify $B_{4\varepsilon}^{M_0}(0) \subset M_0$ with $B_{4\varepsilon}^M(0) \subset M$ by the exponential map $\exp_{x_0}$. Then locally E can be viewed as a vector bundle over

$B_{4\varepsilon}^{M_0}(0)$. Moreover, we trivialize $E|_{B_{4\varepsilon}^{M_0}(0)}$ by identifying E_X for $X \in B_{4\varepsilon}^{M_0}(0)$ with E_{x_0} by parallel transport along radial geodesics.

(3) Let $\rho : \mathbb{R} \rightarrow [0, 1]$ be a smooth cutoff function such that

$$\rho(t) = \begin{cases} 1, & \text{if } |t| < 2; \\ 0, & \text{if } |t| > 4. \end{cases}$$

We define a map $\rho_\varepsilon : M_0 \rightarrow M_0$ by $\rho_\varepsilon(X) = \rho\left(\frac{|X|}{\varepsilon}\right)X$. Let g_{M_0} be a Riemannian metric on M_0 defined by $g_{M_0}(X) = g_M(\rho_\varepsilon(X))$. Since g_{M_0} coincides with g_M on $B_{2\varepsilon}^{M_0}(0)$, in the following we will replace (M, g_M) with (M_0, g_{M_0}) .

- (4) Using ρ_ε again, we obtain the extended trivial vector bundle $(E_0 = M_0 \times Y, g_{E_0})$ over M_0 , satisfying $(E_0, g_{E_0})|_{B_{2\varepsilon}^{M_0}(0)} = (E, g_E)|_{B_{2\varepsilon}^{M_0}(0)}$. The family of metrics $g_{\sigma, T}$ also induces metrics $g_{E_0, \sigma, T}$ on E_0 . Similarly, using parallel transport and ρ_ε , for any $(X, y) \in E_0$ we can identify $(\Lambda^* E_0)_{(X, y)} = (\Lambda^* M_0)_X \otimes (\Lambda^* Y)_y$ with $(\Lambda^* M)_{x_0} \otimes (\Lambda^* Y)_0$.
- (5) Similarly, we extend $\ker \tilde{D}_Y$ to $K := \Lambda^* M_0 \otimes \text{span}_{\mathbb{R}} \left\{ e^{-\frac{\tau|y|^2}{2}} \right\}$. Then K is a subbundle of $\Lambda^* E_0$ satisfying

$$K = \begin{cases} \Lambda^* M \otimes \ker \tilde{D}_Y|_{\Lambda^* Y}, & \text{on } B_{2\varepsilon}^{M_0}(0) \times Y; \\ (\Lambda^* M)_{x_0} \otimes \ker \tilde{D}_Y|_{\Lambda^* Y} \cong (\Lambda^* M)_{x_0}, & \text{on } E_0 - B_{4\varepsilon}^{M_0}(0) \times Y. \end{cases}$$

Let $\tilde{P} : L^2(\Lambda^* E_0) \rightarrow L^2(K)$ denote the orthogonal projection with respect to the inner product induced by g_{E_0} . Set $\tilde{P}^\perp = 1 - \tilde{P}$.

(6) Set $\varphi(X) = \rho(\frac{|X|}{\varepsilon})$. Define

$$L_{\sigma,T} := \varphi^2 D_{\sigma,T,\tau h}^2 + (1 - \varphi^2) \left(\sigma^2 D_{M_0}^2 + T^2 \tilde{P}^\perp \tilde{D}_Y^2 \tilde{P}^\perp \right), \quad (5.53)$$

where $\tilde{D}_{Y,x_0}^2$ is the Witten Laplacian on the fiber $\{x_0\} \times Y$.

(7) From Lemma 5.3.5, we see that $L_{\sigma,T}$ fails to be uniformly elliptic as $\sigma \rightarrow 0$. To correct this, we consider

$$\tilde{L}_{\sigma,T} := G_2 \circ L_{\sigma,T} \circ G_2^{-1}, \quad (5.54)$$

where G_2 is the rescaling defined in Section 5.3.2.

If A is a heat type operator on E_0 , we use $A(p)$ to denote its diagonal heat kernel at $p \in E_0$. As a result of the construction above,

$$\begin{aligned} \text{Tr}_s (N_Y F_\sigma(D_{\sigma,T,\tau h})) (p_0) &= \text{Tr}_s \left(N_Y \tilde{F}_\sigma(D_{\sigma,T,\tau h}^2) \right) (p_0) \\ &= \text{Tr}_s \left(N_Y \tilde{F}_\sigma(\tilde{L}_{\sigma,T}) \right) (p_0). \end{aligned} \quad (5.55)$$

Again we decompose $\tilde{L}_{\sigma,T}$ with respect to the splitting $L^2(\Lambda^* E_0) = L^2(K)^\perp \oplus L^2(K)$ as

$$\tilde{L}_{\sigma,T} = \begin{pmatrix} \tilde{L}_{\sigma,T,1} & \tilde{L}_{\sigma,T,2} \\ \tilde{L}'_{\sigma,T,2} & \tilde{L}_{\sigma,T,3} \end{pmatrix}. \quad (5.56)$$

Theorem 5.4.4 *Let $D_{Y,\sigma}$ denote the standard Dirac operator on the fiber $Y_{(\sigma x,y)}$. Then*

as $T \rightarrow \infty$,

$$\begin{aligned}\tilde{L}_{\sigma,T,1} &= T^2 L_1 + O(T), \\ \tilde{L}_{\sigma,T,2} &= T L_2 + O(1), \\ \tilde{L}'_{\sigma,T,2} &= T L'_2 + O(1), \\ \tilde{L}_{\sigma,T,3} &= L_3 + O\left(\frac{1}{T}\right),\end{aligned}$$

where

$$\begin{aligned}L_1 &= \tilde{P}^\perp \tilde{D}_Y^2 \tilde{P}^\perp, \\ L_2 &= \tilde{P}^\perp \left\{ -\frac{\varphi^2(\sigma X)}{2} \left[\nabla_{f_i}^E, \frac{\sigma}{2} \sum_{j,k} \langle S(f_i) f_j, e_k \rangle \left(c(\tilde{f}^j) c(\tilde{e}^k) - \hat{c}(\tilde{f}^j) \hat{c}(\tilde{e}^k) \right) \right] \right. \\ &\quad \left. + \frac{\varphi^2(\sigma X)}{4} R_{i\bar{j}k\bar{l}}^E c(\tilde{e}^i) c(\tilde{f}^j) \hat{c}(\tilde{f}^k) \hat{c}(\tilde{f}^l) \right\} \tilde{P}, \\ L'_2 &= L_2 \text{ with } \tilde{P} \text{ and } \tilde{P}^\perp \text{ interchanged,} \\ L_3 &= \tilde{P} \left\{ D_{M_0}^2 + B(\sigma) \right\} \tilde{P}.\end{aligned}$$

Here $B(\sigma)$ is a first order differential operator, for which the differential is only along the base direction.

Proof: The proof follows directly from Lemma 5.3.5, Lemma 5.3.6, (5.53), and (5.54). ■

5.4.2 Uniform estimate on heat kernel

Let $H^p(\Lambda^* M_0 \otimes \Lambda^* Y) = H^p(\Lambda^* E_0)$ denote the p -th Sobolev space of sections of $\Lambda^* E_0$. Obviously $H^0(\Lambda^* M_0 \otimes \Lambda^* Y) = L^2(\Lambda^* E_0)$.

To keep track of the dependence of σ and T , we introduce weighted Sobolev norms

that fit into our construction.

Definition 5.4.5 Let $\nabla = \nabla^{M_0} + \nabla^Y$ be a connection on E_0 , where ∇^{M_0} and ∇^Y are the pullbacks of the Levi-Civita connections on M_0 and Y respectively. Set

$$\psi_\sigma(X) := 1 + (1 + |X_0|^2 + |X|^2)^{\frac{1}{2}} \varphi\left(\frac{\sigma X}{2}\right). \quad (5.57)$$

For any $s \in H^0(\Lambda^q M_0 \otimes \Lambda^* Y)$ and $s', s'' \in H^1(\Lambda^* E_0)$, we define

$$\begin{aligned} |s|_{\sigma;0}^2 &= \frac{1}{(2\pi)^n} \int_{E_0} |s|^2 \psi_\sigma^{2(n-q)} d\text{vol}_{E_0}, \\ |s'|_{\sigma,T;1}^2 &= T^2 |\tilde{P}^\perp s'|_{\sigma;0}^2 + |\tilde{P} s'|_{\sigma;0}^2 + \sum_{i=1}^n |\nabla_{e_i} s'|_{\sigma;0}^2 + T^2 \sum_{j=1}^k |\nabla_{f_j} \tilde{P}^\perp s'|_{\sigma;0}^2, \\ |s''|_{\sigma,T;-1} &= \sup_{v \neq 0 \in H^1(\Lambda^* E_0)} \frac{|\langle s'', v \rangle_{\sigma;0}|}{|v|_{\sigma,T;1}}, \end{aligned}$$

where $\langle \cdot, \cdot \rangle_{\sigma;0}$ denotes the Hermitian inner product associated with $|\cdot|_{\sigma;0}^2$.

If $L : (H^m(\Lambda^* E_0), |\cdot|_{\sigma,T;m}) \rightarrow (H^{m'}(\Lambda^* E_0), |\cdot|_{\sigma,T;m'})$ is a bounded linear operator, then we use $\|L\|_{\sigma,T;m,m'}$ to denote its operator norm.

The next result follows from [34, Theorem 13.27].

Lemma 5.4.6 There exist constants $C, C' > 0$ such that for any $\sigma \in (0, 1]$, $T \in [1, \infty)$, and $s, s' \in H^1(\Lambda^* E_0)$, we have

$$\begin{aligned} \text{Re} \langle \tilde{L}_{\sigma,T} s, s \rangle_{\sigma;0} &\geq C |s|_{\sigma,T;1}^2 - C' |s|_{\sigma;0}^2, \\ \left| \text{Im} \langle \tilde{L}_{\sigma,T} s, s \rangle_{\sigma;0} \right| &\leq C |s|_{\sigma,T;1} |s|_{\sigma;0}, \\ |\langle \tilde{L}_{\sigma,T} s, s' \rangle_{\sigma;0}| &\leq C |s|_{\sigma,T;1} |s'|_{\sigma,T;1}. \end{aligned}$$

In particular, $\|\tilde{L}_{\sigma,T}\|_{\sigma,T;1,-1} \leq C$.

Proposition 5.4.7 *Set $\tilde{\Gamma} := \{\lambda = a + ib \in \mathbb{C} \mid a = \delta b^2 - A\}$. Then there exist constants $C > 0$, $0 < \delta \ll 1$, $A \gg 1$ such that for any $\sigma \in (0, 1]$, $T \in [1, \infty)$, and $\lambda \in \tilde{\Gamma}$,*

(1) *the resolvent $(\lambda - \tilde{L}_{\sigma,T})^{-1}$ exists;*

(2) $\|(\lambda - \tilde{L}_{\sigma,T})^{-1}\|_{\sigma,T;0,0} \leq C$,

$\|(\lambda - \tilde{L}_{\sigma,T})^{-1}\|_{\sigma,T;-1,1} \leq C(1 + |\lambda|^2)$.

Proof: By Lemma 5.4.6, we find that for any $s \in H^1(\Lambda^* E_0)$ and $\lambda = a + ib \in \tilde{\Gamma}$,

$$\begin{aligned} & \left| \langle (\tilde{L}_{\sigma,T} - \lambda)s, s \rangle_{\sigma;0} \right| \\ &= \left| \left(\operatorname{Re} \langle \tilde{L}_{\sigma,T}s, s \rangle_{\sigma;0} - a|s|_{\sigma;0}^2 \right) + i \left(\operatorname{Im} \langle \tilde{L}_{\sigma,T}s, s \rangle_{\sigma;0} - b|s|_{\sigma;0}^2 \right) \right| \\ &\geq \max \left\{ \operatorname{Re} \langle \tilde{L}_{\sigma,T}s, s \rangle_{\sigma;0} - a|s|_{\sigma;0}^2, \left| \operatorname{Im} \langle \tilde{L}_{\sigma,T}s, s \rangle_{\sigma;0} - b|s|_{\sigma;0}^2 \right| \right\} \\ &\geq \max \left\{ C|s|_{\sigma,T;1}^2 - (C' + a)|s|_{\sigma;0}^2, -C|s|_{\sigma,T;1}|s|_{\sigma;0} + |b||s|_{\sigma;0}^2 \right\} \\ &= \max \left\{ C\alpha^2 - (C' + a), -C\alpha + |b| \right\} \cdot |s|_{\sigma;0}^2, \end{aligned}$$

where $\alpha = \frac{|s|_{\sigma,T;1}}{|s|_{\sigma;0}} \geq 1$. It is easy to see that if $\delta > 0$ is sufficiently small and $A > 1$ is sufficiently large, there exists a constant $C'' > 0$ such that

$$\max \left\{ C\alpha^2 - (C' + a), -C\alpha + |b| \right\} \geq C'', \quad \forall \lambda \in \tilde{\Gamma}.$$

Using this and the Cauchy-Schwarz inequality, we see that

$$|(\tilde{L}_{\sigma,T} - \lambda)s|_{\sigma;0} \geq C''|s|_{\sigma;0}. \quad (5.58)$$

This proves (1) as well as the first inequality in (2).

Now, by the first inequality in Lemma 5.4.6, for $\lambda_0 \in \mathbb{R}$ with $\lambda_0 \leq -2C'$ and $s \in$

$H^1(\Lambda^*E)$, we have

$$\begin{aligned}
\left| \langle (\tilde{L}_{\sigma,T} - \lambda_0)s, s \rangle_{\sigma;0} \right| &\geq \operatorname{Re} \langle \tilde{L}_{\sigma,T}s, s \rangle_{\sigma;0} - \lambda_0 |s|_{\sigma;0}^2 \\
&\geq C |s|_{\sigma,T;1}^2 - (C' + \lambda_0) |s|_{\sigma;0}^2 \\
&\geq C |s|_{\sigma,T;1}^2.
\end{aligned}$$

As a consequence,

$$\|(\lambda_0 - \tilde{L}_{\sigma,T})^{-1}\|_{\sigma,T;-1,1} \leq \frac{1}{C}. \quad (5.59)$$

For any $\lambda \in \tilde{\Gamma}$, the first resolvent identity yields

$$(\lambda - \tilde{L}_{\sigma,T})^{-1} = (\lambda_0 - \tilde{L}_{\sigma,T})^{-1} - (\lambda - \lambda_0)(\lambda - \tilde{L}_{\sigma,T})^{-1}(\lambda_0 - \tilde{L}_{\sigma,T})^{-1}. \quad (5.60)$$

Notice that (5.58) implies

$$\|(\lambda - \tilde{L}_{\sigma,T})^{-1}\|_{\sigma,T;1,0} \leq \frac{1}{C''}. \quad (5.61)$$

Using (5.59), (5.60) and (5.61), we find that

$$\|(\lambda - \tilde{L}_{\sigma,T})^{-1}\|_{\sigma,T;-1,0} \leq \frac{1}{C} + \frac{1}{CC''} |\lambda - \lambda_0|. \quad (5.62)$$

Now we interchange the last two factors $(\lambda - \tilde{L}_{\sigma,T})^{-1}$ and $(\lambda_0 - \tilde{L}_{\sigma,T})^{-1}$ in (5.60) and apply (5.59) and (5.62) to obtain

$$\|(\lambda - \tilde{L}_{\sigma,T})^{-1}\|_{\sigma,T;-1,1} \leq \frac{1}{C} + \frac{|\lambda - \lambda_0|}{C} \left(\frac{1}{C} + \frac{1}{CC''} |\lambda - \lambda_0| \right) \leq C'''(1 + |\lambda|^2). \quad (5.63)$$

■

Proposition 5.4.8 *Let $m \in \mathbb{N}$, $\mathcal{Q} = \{\tilde{P}\nabla_{e_i}\tilde{P}, \tilde{P}^\perp\nabla_{e_i}\tilde{P}^\perp, \tilde{P}^\perp\nabla_{f_j}\tilde{P}^\perp\}_{j=1,\dots,k}^{i=1,\dots,n}$. Then there exists a constant $C_m > 0$ such that for any $\sigma \in (0, 1]$, $T \in [1, \infty)$, and $Q_1, \dots, Q_m \in \mathcal{Q}$,*

$$\left\| [Q_1, [Q_2, \dots, [Q_m, \tilde{L}_{\sigma,T}] \dots]] \right\|_{\sigma,T;1,-1} \leq C_m. \quad (5.64)$$

Proof: We first assume $m = 1$. By multiplying by a constant factor if necessary, we may assume that

$$\int_Y e^{-\tau|y|^2} dy = 1.$$

Then for $s \in L^2(\Lambda^*E_0)$, the projection $\tilde{P}s$ is given by

$$\tilde{P}s = e^{-\frac{\tau|y|^2}{2}} \int_Y e^{-\frac{\tau|y|^2}{2}} s(X, y) d\text{vol}_Y. \quad (5.65)$$

By Definition 5.4.5 and (5.65), we see that

$$[\nabla_{e_i}, \tilde{P}] = 0, \quad \tilde{P}\nabla_{f_i}\tilde{P} = 0, \quad [\varphi^2, \tilde{P}] = 0. \quad (5.66)$$

Let $Q_1 = \tilde{P}\nabla_{e_i}\tilde{P}$, $Q_2 = \tilde{P}^\perp\nabla_{e_i}\tilde{P}^\perp$, $Q_3 = \tilde{P}^\perp\nabla_{f_j}\tilde{P}^\perp$. It follows from (5.66) that $[Q_1, \tilde{L}_{\sigma,T}]$ can be expressed as

$$\sum_{k,l} a_{k,l}(\sigma, T) \nabla_{e_k} \nabla_{e_l} + \sum_k b_k(\sigma, T) \nabla_{e_k} + T \sum_{\bar{l}} b'_l(\sigma, T) \nabla_{f_l} \tilde{P}^\perp + c(\sigma, T), \quad (5.67)$$

and $a_{k,l}, b_k, b'_l, c$, together with their derivatives, are uniformly bounded for any $(X, y) \in E_0, \sigma \in (0, 1]$ and $T \in [1, \infty)$. Therefore by Definition 5.4.5 and integration by parts, (5.64) is verified for $[Q_1, \tilde{L}_{\sigma,T}]$.

Similarly, for $i = 2, 3$ we have

$$\begin{aligned}
[Q_i, \tilde{L}_{\sigma, T}] &= \sum_{k, l} a_{k, l}(\sigma, T) \nabla_{e_k} \nabla_{e_l} + T^2 \sum_{\bar{k}, \bar{l}} a'_{k, l}(\sigma, T) \tilde{P}^\perp \nabla_{f_k} \nabla_{f_l} \tilde{P}^\perp \\
&\quad + \sum_k b_k(\sigma, T) \nabla_{e_k} + T \sum_l b'_l(\sigma, T) \nabla_{f_l} \tilde{P}^\perp + c(\sigma, T).
\end{aligned} \tag{5.68}$$

The coefficients are also uniformly bounded as before. Thus we have proved (5.64) for $m = 1$.

In general, using an iteration argument as in the proof of [13, Proposition 4.9], $[Q_1, [Q_2, \dots, [Q_m, \tilde{L}_{\sigma, T}] \dots]]$ has the same type (5.68) as $[Q_i, \tilde{L}_{\sigma, T}]$. This completes the proof. ■

Definition 5.4.9 *Let $m \in \mathbb{N}$. We define a norm $\| \cdot \|_{\sigma; m}^2$ on $H^m(\Lambda^* E_0)$ by*

$$\|s\|_{\sigma; m}^2 = \sum_{l=0}^m \sum_{Q_i \in \mathcal{Q}} |Q_1 \cdots Q_l s|_{\sigma; 0}^2.$$

If $A : (H^m(\Lambda^ E_0), \| \cdot \|_{\sigma; m}^2) \rightarrow (H^{m'}(\Lambda^* E_0), \| \cdot \|_{\sigma; m'}^2)$ is a bounded linear operator, we use $\| \| A \| \|_{\sigma; m, m'}$ to denote the corresponding operator norm.*

Proposition 5.4.10 *For any $m \in \mathbb{N}$, $\sigma \in (0, 1]$, $T \in [1, \infty)$, and $\lambda \in \tilde{\Gamma}$, the resolvent $(\lambda - \tilde{L}_{\sigma, T})^{-1}$ maps $H^m(\Lambda^* E_0)$ to $H^{m+1}(\Lambda^* E_0)$. Moreover, there exist $N \in \mathbb{N}$ and $C_m > 0$ such that*

$$\left\| (\lambda - \tilde{L}_{\sigma, T})^{-1} \right\|_{\sigma; m, m+1} \leq C_m (1 + |\lambda|)^N.$$

Proof: First, we notice that for $Q_i \in \mathcal{Q}$, the operator $Q_1 \cdots Q_m (\lambda - \tilde{L}_{\sigma, T})^{-1}$ can be expressed, using commutators, as a linear combination of operators of the type

$$[Q_1, [Q_2, \dots, [Q_l, (\lambda - \tilde{L}_{\sigma, T})^{-1}] \dots]] Q_{l+1} \cdots Q_m$$

for some $l \leq m$. Let $\mathcal{C} = \{[Q_{i_1}, [Q_{i_1}, \dots [Q_{i_p}, (\lambda - \tilde{L}_{\sigma,T})^{-1}] \dots]]\}$ be the set of commutators. Then each $[Q_1, [Q_2, \dots [Q_l, (\lambda - \tilde{L}_{\sigma,T})^{-1}] \dots]]$ has the form

$$(\lambda - \tilde{L}_{\sigma,T})^{-1} R_1 (\lambda - \tilde{L}_{\sigma,T})^{-1} R_2 \cdots R_l (\lambda - \tilde{L}_{\sigma,T})^{-1},$$

where $R_i \in \mathcal{C}$.

By Proposition 5.4.7 and Proposition 5.4.8, there exists $C > 0$ such that

$$\|R_i\|_{\sigma,T;1,-1} \leq C, \quad \|(\lambda - \tilde{L}_{\sigma,T})^{-1}\|_{\sigma,T;-1,1} \leq C(1 + |\lambda|^2), \quad \|Q_i\|_{\sigma,T;1,0} \leq C.$$

Since we obviously have $\|s\|_{\sigma;1} \leq C|s|_{\sigma,T;1}$, the assertion follows. ■

Let $\tilde{K}_{\sigma,T}$ denote the heat kernel of $\tilde{F}_\sigma(\tilde{L}_{\sigma,T})$. Using the results above, we obtain the following uniform estimate for $\tilde{K}_{\sigma,T}$.

Theorem 5.4.11 *For any $\sigma \in (0, 1]$ and $T \in [1, \infty)$, the heat kernel $\tilde{K}_{\sigma,T}$, together with all its derivatives, is uniformly bounded on compact subsets.*

Proof: In the whole proof, C (or C_m) will denote a constant that is independent of σ, T , and may vary from line to line.

Let $\lambda \in \tilde{\Gamma}$. By Proposition 5.4.7, we can express $\tilde{F}_\sigma(\tilde{L}_{\sigma,T})$ as

$$\tilde{F}_\sigma(\tilde{L}_{\sigma,T}) = \frac{(-1)^{m-1}(m-1)!2^{m-1}}{2\pi i} \int_{\tilde{\Gamma}} \tilde{F}_\sigma(\lambda) (\lambda - \tilde{L}_{\sigma,T})^{-m} d\lambda \quad (5.69)$$

for any $m \in \mathbb{N}$. Using (5.69) and applying Proposition 5.4.10 for m times, we find that

$$\left\| \left\| \tilde{F}_\sigma(\tilde{L}_{\sigma,T}) \right\| \right\|_{\sigma,T;0,m} \leq C_m.$$

This implies

$$\left\| \left\| \Delta^l \tilde{F}_\sigma(\tilde{L}_{\sigma,T}) \right\| \right\|_{\sigma,T;0,0} \leq C_l, \quad \forall l \in \mathbb{N}. \quad (5.70)$$

By an adjoint argument, we find that

$$\left\| \left\| \tilde{F}_\sigma(\tilde{L}_{\sigma,T}) \Delta^{l'} \right\| \right\|_{\sigma,T;0,0} \leq C_{l'}, \quad \forall l' \in \mathbb{N}. \quad (5.71)$$

As a result, for any $l, l' \in \mathbb{N}$ we have

$$\left\| \left\| \Delta^l \tilde{F}_\sigma(\tilde{L}_{\sigma,T}) \Delta^{l'} \right\| \right\|_{\sigma,T;0,0} \leq C_{l,l'}. \quad (5.72)$$

Now we restrict our discussion to $U = \{|X| \leq 1, |y| \leq 1\}$. Let $\|\cdot\|_U$ denote the L^2 -norm of $L^2(\Lambda^* E_0|_U)$, and $\|\cdot\|_U$ denote the corresponding operator norm. Then using the trivial embedding $L^2(\Lambda^* E_0|_U) \hookrightarrow L^2(\Lambda^* E_0)$, we see that for $s \in L^2(\Lambda^* E_0|_U)$,

$$|s|_U \leq |s|_{\sigma;0} \leq C |s|_U. \quad (5.73)$$

(5.72) and (5.73) imply an L^2 -bound in the standard sense:

$$\|\Delta^l \tilde{F}_\sigma(\tilde{L}_{\sigma,T}) \Delta^{l'}\|_U \leq C_{l,l'}. \quad (5.74)$$

Thus by Sobolev's inequalities, we find that for any $q, q' \in \mathbb{N}$,

$$\sup_{(p,p') \in U \times U} |\Delta_p^q \Delta_{p'}^{q'} \tilde{K}_{\sigma,T}(p, p')| \leq C_{q,q'}. \quad (5.75)$$

This completes the proof. ■

5.4.3 Uniform estimate on supertrace

Now we are ready to prove Theorem 5.1.12.

In view of (5.51), Proposition 5.4.3, and (5.55), Theorem 5.1.12 is a direct consequence of the following theorem, for which the proof will be postponed to the end of this section.

Theorem 5.4.12 *There exist constants $\delta > 0$, $C > 0$, such that for any $\sigma \in (0, 1]$, $T \in [1, \infty)$,*

$$\left| \frac{1}{T} \operatorname{Tr}_s \left(N_Y \tilde{F}_\sigma(\tilde{L}_{\sigma,T}) \right) \right| \leq \frac{C}{T^{1+\delta}}. \quad (5.76)$$

Recall that under the splitting $L^2(\Lambda^* E_0) = L^2(K)^\perp \oplus L^2(K)$, the operator $\tilde{L}_{\sigma,T}$ decomposes as

$$\tilde{L}_{\sigma,T} = \begin{pmatrix} \tilde{L}_{\sigma,T,1} & \tilde{L}_{\sigma,T,2} \\ \tilde{L}'_{\sigma,T,2} & \tilde{L}_{\sigma,T,3} \end{pmatrix}. \quad (5.77)$$

If we write $(\lambda - \tilde{L}_{\sigma,T})^{-1} = \begin{pmatrix} U_1 & U_2 \\ U'_2 & U_3 \end{pmatrix}$, then

$$U_3 = \left((\lambda - \tilde{L}_{\sigma,T,3}) - \tilde{L}'_{\sigma,T,2}(\lambda - \tilde{L}_{\sigma,T,1})^{-1} \tilde{L}_{\sigma,T,2} \right)^{-1}. \quad (5.78)$$

Definition 5.4.13 *Let $\Theta_\sigma : L^2(\Lambda^* E_0) \rightarrow L^2(\Lambda^* E_0)$ be the operator*

$$\Theta_\sigma = \begin{pmatrix} 0 & 0 \\ 0 & L_3 - L'_2 L_1^{-1} L_2 \end{pmatrix}, \quad (5.79)$$

where L_i 's are the operators given in Theorem 5.4.4.

Remark 5.4.14 By Lemma 5.4.2, if we let $E = \tilde{D}_M - \frac{1}{4}\text{Tor}$, then

$$\begin{aligned} \tilde{L}_{\sigma,T}^2 &= \varphi^2(\sigma X) \left[E^2 + T(E\tilde{D}_{Y,T} + \tilde{D}_{Y,T}E) + T^2\tilde{D}_{Y,T}^2 \right] \\ &\quad + (1 - \varphi^2(\sigma X)) \left(D_{M_0}^2 + T^2\tilde{P}^\perp\tilde{D}_Y^2\tilde{P}^\perp \right). \end{aligned}$$

It is clear that $\tilde{D}_{Y,T} = \tilde{D}_Y^2 + O(T)$. Suppose $E = E_1 + O(T)$. Therefore, compared to Theorem 5.4.4, we find that

$$\begin{aligned} L_1 &= \tilde{P}^\perp\tilde{D}_Y^2\tilde{P}^\perp, \\ L_2 &= \tilde{P}^\perp \left[\varphi^2(\sigma X)\tilde{D}_Y E_1 \right] \tilde{P}, \\ L'_2 &= \tilde{P} \left[\varphi^2(\sigma X)E_1\tilde{D}_Y \right] \tilde{P}^\perp, \\ L_3 &= \tilde{P} \left[\varphi^2(\sigma X)E_1^2 \right] \tilde{P}. \end{aligned}$$

When $|X| \leq \frac{\varepsilon}{\sigma}$, $\varphi^2(\sigma X) \equiv 1$, and thus

$$L_3 - L'_2 L_1^{-1} L_2 = \tilde{P} E_1^2 \tilde{P} - \tilde{P} E_1 \tilde{P}^\perp E_1 \tilde{P} = (\tilde{P} E_1 \tilde{P})^2. \quad (5.80)$$

Moreover, if we view D_M^2 as an operator acting on $\Lambda^* M \otimes \ker \tilde{D}_Y|_{\Lambda^* Y}$, then $(\tilde{P} E_1 \tilde{P})^2 = G_2 \circ (\sigma D_M)^2 \circ G_2^{-1}$ for $X \in B_{\varepsilon/\sigma}^{M_0}(0)$.

Proposition 5.4.15 There exists a constant $C > 0$ such that for any $\sigma \in (0, 1]$ and $T \in [1, \infty)$, we have

- (1) $\|\tilde{P}^\perp \tilde{F}_\sigma(\tilde{L}_{\sigma,T}) \tilde{P}^\perp\|_{\sigma;0,0} \leq \frac{C}{T};$
- (2) $\|\tilde{P}^\perp \tilde{F}_\sigma(\tilde{L}_{\sigma,T}) \tilde{P}\|_{\sigma;0,0} \leq \frac{C}{T};$
- (3) $\|\tilde{P} \tilde{F}_\sigma(\tilde{L}_{\sigma,T}) \tilde{P}^\perp\|_{\sigma;0,0} \leq \frac{C}{T};$

$$(4) \quad \|\tilde{P}\tilde{F}_\sigma(\tilde{L}_{\sigma,T})\tilde{P} - \tilde{P}\tilde{F}_\sigma(\Theta_\sigma)\tilde{P}\|_{\sigma;0,0} \leq \frac{C}{T^{1/2}}.$$

$$\text{In particular, } \|\tilde{F}_\sigma(\tilde{L}_{\sigma,T}) - \tilde{P}\tilde{F}_\sigma(\Theta_\sigma)\tilde{P}\|_{\sigma;0,0} \leq \frac{C}{T^{1/2}}.$$

Proof: By Definition 5.4.5 and Proposition 5.4.7, we find that for any $\lambda \in \tilde{\Gamma}$ and $s \in L^2(\Lambda^*E_0)$,

$$\begin{aligned} |\tilde{P}^\perp(\lambda - \tilde{L}_{\sigma,T})^{-1}s|_{\sigma;0} &\leq \frac{C}{T}|(\lambda - \tilde{L}_{\sigma,T})^{-1}s|_{\sigma,T;1} \\ &\leq \frac{C}{T}(1 + |\lambda|^2)|s|_{\sigma,T;-1} \\ &\leq \frac{C}{T}(1 + |\lambda|^2)|s|_{\sigma;0}. \end{aligned} \tag{5.81}$$

Using Proposition 5.4.1(5), (5.81), and the fact that $\tilde{F}_\sigma(\tilde{L}_{\sigma,T}) = \frac{1}{2\pi i} \int_{\tilde{\Gamma}} \tilde{F}_\sigma(\lambda)(\lambda - \tilde{L}_{\sigma,T})^{-1}d\lambda$, we obtain the first three inequalities.

To derive the last inequality, notice that

$$\begin{aligned} &\tilde{P}\tilde{F}_\sigma(\tilde{L}_{\sigma,T})\tilde{P} - \tilde{P}\tilde{F}_\sigma(\Theta_\sigma)\tilde{P} \\ &= \frac{1}{2\pi i} \int_{\tilde{\Gamma}} \tilde{F}_\sigma(\lambda) \left[\tilde{P}(\lambda - \tilde{L}_{\sigma,T})^{-1}\tilde{P} - \tilde{P}(\lambda - \Theta_\sigma)^{-1}\tilde{P} \right] d\lambda, \end{aligned} \tag{5.82}$$

and

$$\begin{aligned} &\tilde{P}(\lambda - \tilde{L}_{\sigma,T})^{-1}\tilde{P} - \tilde{P}(\lambda - \Theta_\sigma)^{-1}\tilde{P} \\ &= U_3 - \tilde{P}(\lambda - \Theta_\sigma)^{-1}\tilde{P} \\ &= -U_3 \left[\tilde{L}_{\sigma,T,3} + \tilde{L}'_{\sigma,T,2}(\lambda - \tilde{L}_{\sigma,T,1})^{-1}\tilde{L}_{\sigma,T,2} - L_3 + L_2L_1^{-1}L_2 \right] \tilde{P}(\lambda - \Theta_\sigma)^{-1}\tilde{P}. \end{aligned} \tag{5.83}$$

Now we estimate the operators in the last line of (5.83) separately.

(a) Since $U_3 = \tilde{P}(\lambda - \tilde{L}_{\sigma,T})^{-1}\tilde{P}$, by Proposition 5.4.7 we deduce that

$$\|U_3\|_{\sigma,T;0,0} \leq C, \quad \|U_3\|_{\sigma,T;-1,1} \leq C(1 + |\lambda|^2). \tag{5.84}$$

(b) By a similar argument as in the proof of Theorem 5.4.7, if $\delta, A, \sigma, T, \lambda$

satisfy the conditions therein, then

$$\begin{aligned}\|\tilde{P}(\lambda - \Theta_\sigma)^{-1}\tilde{P}\|_{\sigma,T;0,0} &\leq C, \\ \|\tilde{P}(\lambda - \Theta_\sigma)^{-1}\tilde{P}\|_{\sigma,T;-1,1} &\leq C(1 + |\lambda|^2).\end{aligned}\tag{5.85}$$

(c) It is easy to see that $\tilde{L}_{\sigma,T,3} - L_3 = \frac{1}{T}R$, where R is a first order

differential operator for which the differential only involves ∇_{e_i} . Thus for any $s, s' \in L^2(\Lambda^*E_0)$, we have

$$\left| \langle (\tilde{L}_{\sigma,T,3} - L_3)s, s' \rangle_{\sigma;0} \right| \leq \frac{1}{T} |s|_{\sigma,T;1} |s'|_{\sigma,T;1},$$

which implies

$$\|\tilde{L}_{\sigma,T,3} - L_3\|_{\sigma,T;1,-1} \leq \frac{C}{T}\tag{5.86}$$

(d) Let $\tilde{P}_\sigma : L^2(\Lambda^*E_0) \rightarrow L^2(K)^\perp$ be the projection map with respect to the inner product $\langle \cdot, \cdot \rangle_{\sigma;0}$. Define $\tilde{L}'_{\sigma,T,1} = \tilde{P}_\sigma^\perp \tilde{L}_{\sigma,T} \tilde{P}_\sigma^\perp$. Then clearly we have

$$\tilde{L}'_{\sigma,T,1} - \tilde{L}_{\sigma,T,1} = (\tilde{P}_\sigma^\perp - \tilde{P}^\perp) \tilde{L}_{\sigma,T} \tilde{P}_\sigma^\perp.\tag{5.87}$$

For any $s, s' \in L^2(\Lambda^*E_0)$,

$$\langle \tilde{L}'_{\sigma,T,1} \tilde{P}_\sigma^\perp s, \tilde{P}_\sigma^\perp s' \rangle_{\sigma;0} = \langle \tilde{L}_{\sigma,T} \tilde{P}_\sigma^\perp s, \tilde{P}_\sigma^\perp s' \rangle_{\sigma;0}.\tag{5.88}$$

Using (5.88) and arguing as in Proposition 5.4.7, one can show that for $\lambda \in \tilde{\Gamma}$, $(\lambda - \tilde{L}'_{\sigma,T,1})^{-1}$ maps $H^{-1}(\Lambda^*E_0)$ to $H^1(\Lambda^*E_0)$, and

$$\|(\lambda - \tilde{L}'_{\sigma,T,1})^{-1}\|_{\sigma,T;-1,1} \leq C(1 + |\lambda|^2).\tag{5.89}$$

On the other hand, if we let $\tilde{P}^{*,\sigma}$ denote the adjoint of $\tilde{P}$ with respect to $\langle \cdot, \cdot \rangle_{\sigma;0}$, then by Proposition 5.4.6 we have

$$\begin{aligned}
& \left| \langle (\tilde{L}'_{\sigma,T,1} - \tilde{L}_{\sigma,T,1}) \tilde{P}^\perp s, \tilde{P}^\perp s' \rangle_{\sigma;0} \right| \\
&= \left| \langle (1 - \tilde{P}^\perp) \tilde{L}_{\sigma,T} \tilde{P}^\perp s, \tilde{P}^\perp s' \rangle_{\sigma;0} \right| \\
&= \left| \langle \tilde{P} \tilde{L}_{\sigma,T} \tilde{P}^\perp s, \tilde{P}^\perp s' \rangle_{\sigma;0} \right| \\
&= \left| \langle \tilde{L}_{\sigma,T} \tilde{P}^\perp s, \tilde{P}^{*,\sigma} \tilde{P}^\perp s' \rangle_{\sigma;0} \right| \\
&= \left| \langle \tilde{L}_{\sigma,T} \tilde{P}^\perp s, (\tilde{P}^{*,\sigma} - \tilde{P}) \tilde{P}^\perp s' \rangle_{\sigma;0} \right| \\
&\leq C |\tilde{P}^\perp s|_{\sigma,T;1} |(\tilde{P}^{*,\sigma} - \tilde{P}) \tilde{P}^\perp s|_{\sigma,T;1}.
\end{aligned} \tag{5.90}$$

In view of (5.57) and Definition 5.4.5, one has $\tilde{P}^{*,\sigma} = \psi_\sigma^{-2(n-q)} \tilde{P} \psi_\sigma^{2(n-q)}$. Since

$$\begin{aligned}
\left| \frac{\psi_\sigma^{2(n-q)}(X + X')}{\psi_\sigma^{2(n-q)}(X)} - 1 \right| &= \left| \left(\frac{1 + \sqrt{1 + |X + X'|^2} \varphi(\frac{\sigma(X+X')}{2})}{1 + \sqrt{1 + |X|^2} \varphi(\frac{\sigma X}{2})} \right)^{2(n-q)} - 1 \right| \\
&\leq C(1 + |X'|)^{2(n-q)},
\end{aligned}$$

we find that

$$|(\tilde{P}^{*,\sigma} - \tilde{P}) \tilde{P}^\perp s|_{\sigma,T;1} \leq C |\tilde{P}^\perp s|_{\sigma,T;1}. \tag{5.91}$$

Combining (5.89), (5.90) and (5.91), we see that if $|\lambda| \leq T^{\frac{1}{8}}$,

$$\|(\lambda - \tilde{L}_{\sigma,T,1})^{-1}\|_{\sigma,T;-1,1} \leq C(1 + |\lambda|^2) \leq CT^{\frac{1}{4}}. \tag{5.92}$$

Since $(\lambda - \tilde{L}_{\sigma,T,1})^{-1}$ maps $L^2(K)^\perp$ to $L^2(K)^\perp$, (5.92) implies

$$\|(\lambda - \tilde{L}_{\sigma,T,1})^{-1}\|_{\sigma,T;-1,0} \leq \frac{C}{T^{3/4}}. \tag{5.93}$$

Now, we notice that

$$\begin{aligned}
& \tilde{L}'_{\sigma,T,2}(\lambda - \tilde{L}_{\sigma,T,1})^{-1}\tilde{L}_{\sigma,T,2} - L'_2(-L_1)^{-1}L_2 \\
&= TL'_2(\lambda - \tilde{L}_{\sigma,T,1})^{-1}(\lambda - \tilde{L}_{\sigma,T,1} + T^2L_1)(-T^2L_1)^{-1}TL_2 \\
&+ TL'_2(\lambda - \tilde{L}_{\sigma,T,1})^{-1}(\tilde{L}_{\sigma,T,2} - TL_2) + (\tilde{L}'_{\sigma,T,2} - TL'_2)(\lambda - \tilde{L}_{\sigma,T,1})^{-1}TL_2 \\
&+ (\tilde{L}'_{\sigma,T,2} - TL'_2)(\lambda - \tilde{L}_{\sigma,T,1})^{-1}(\tilde{L}_{\sigma,T,2} - TL_2)
\end{aligned} \tag{5.94}$$

By proceeding as above, we find that

- i) $\|\tilde{L}_{\sigma,T,2}\|_{\sigma,T;1,-1} \leq C$, $\|TL_2\|_{\sigma,T;1,-1} \leq C$, $\|TL_2\|_{\sigma,T;0,-1} \leq C$.
- ii) $\|\tilde{L}_{\sigma,T,1}^{-1}\|_{\sigma,T;-1,1} \leq C$, $\|(T^2L_1)^{-1}\|_{\sigma,T;-1,1} \leq C$.
- iii) $\|\tilde{L}_{\sigma,T,2} - TL_2\|_{\sigma,T;1,0} \leq C$, $\|\tilde{L}_{\sigma,T,2} - TL_2\|_{\sigma,T;1,-1} \leq \frac{C}{T}$.
- iv) For $\lambda \in \tilde{\Gamma}$ with $|\lambda| < T^{\frac{1}{8}}$, $\|\lambda - \tilde{L}_{\sigma,T,1} + L_1\|_{\sigma,T;1,-1} \leq C(1 + |\lambda|) \leq CT^{\frac{1}{8}}$.
- v) $\tilde{L}'_{\sigma,T,2}$ and L'_2 satisfy the exact same estimate as $\tilde{L}_{\sigma,T,2}$ and L_2 .

As a consequence, we have

$$\|\tilde{L}'_{\sigma,T,2}(\lambda - \tilde{L}_{\sigma,T,1})^{-1}\tilde{L}_{\sigma,T,2} - L'_2(-L_1)^{-1}L_2\|_{\sigma,T;1,-1} \leq \frac{C}{T^{1/2}}. \tag{5.95}$$

Therefore by (5.82), (5.83), and (5.95),

$$\left\| \frac{1}{2\pi i} \int_{\tilde{\Gamma} \cap \{|\lambda| \leq T^{\frac{1}{8}}\}} \tilde{F}_\sigma(\lambda) \left[\tilde{P}(\lambda - \tilde{L}_{\sigma,T})^{-1}\tilde{P} - \tilde{P}(\lambda - \Theta_\sigma)^{-1}\tilde{P} \right] d\lambda \right\|_{\sigma,0,0} \leq \frac{C}{T^{1/2}}. \tag{5.96}$$

On the other hand, using Proposition 5.4.1, (5.81) and (5.4.3), it is not hard to see

$$\begin{aligned}
& \left\| \frac{1}{2\pi i} \int_{\tilde{\Gamma} \cap \{|\lambda| \geq T^{\frac{1}{8}}\}} \tilde{F}_\sigma(\lambda) \left[\tilde{P}(\lambda - \tilde{L}_{\sigma,T})^{-1} \tilde{P} - \tilde{P}(\lambda - \Theta_\sigma)^{-1} \tilde{P} \right] d\lambda \right\|_{\sigma;0,0} \\
& \leq \left\| \frac{1}{2\pi i} \int_{\tilde{\Gamma} \cap \{|\lambda| \geq T^{\frac{1}{8}}\}} \tilde{F}_\sigma(\lambda) \left[\tilde{P}(\lambda - \tilde{L}_{\sigma,T})^{-1} \tilde{P} \right] d\lambda \right\|_{\sigma;0,0} \\
& + \left\| \frac{1}{2\pi i} \int_{\tilde{\Gamma} \cap \{|\lambda| \geq T^{\frac{1}{8}}\}} \tilde{F}_\sigma(\lambda) \left[\tilde{P}(\lambda - \Theta_\sigma)^{-1} \tilde{P} \right] d\lambda \right\|_{\sigma;0,0} \\
& \leq \frac{C}{T},
\end{aligned} \tag{5.97}$$

from which the last inequality follows. ■

Proof: [Proof of Theorem 5.4.12] In the proof we will use dp to denote the volume element $dvol_{E_0}(p)$. Recall that we use $\tilde{K}_{\sigma,T}$ to denote the heat kernel of $\tilde{F}_\sigma(\tilde{L}_{\sigma,T})$. Let K'_σ be the heat kernel of $\tilde{P}\tilde{F}_\sigma(\Theta_\sigma)\tilde{P}$. Set $f_{\sigma,T}(p, p') = \tilde{K}_{\sigma,T}(p, p') - K'_\sigma(p, p')$. Let ξ be a smooth nonnegative function on E_0 such that

$$\text{supp } \xi \subset B_1^{E_0}(0), \text{ and } \int_{E_0} \xi(p) dp = 1. \tag{5.98}$$

Let $a \in (0, 1]$ be a constant. Since $f_{\sigma,T}$ is $\text{End}(\Lambda^* E_0)$ -valued, we take $U, U' \in \Lambda^* E_0$. By Proposition 5.4.11, for any $p, p' \in E_0$,

$$\begin{aligned}
& \left| \langle f_{\sigma,T}(0, 0)U, U' \rangle - \iint_{E_0 \times E_0} \langle f_{\sigma,T}(p, p')U, U' \rangle \frac{\xi(p/a)}{a^{2(n+k)}} \frac{\xi(p'/a)}{a^{2(n+k)}} dp dp' \right| \\
& = \left| \iint_{E_0 \times E_0} \langle (f_{\sigma,T}(p, p') - f_{\sigma,T}(0, 0))U, U' \rangle \frac{\xi(p/a)}{a^{2(n+k)}} \frac{\xi(p'/a)}{a^{2(n+k)}} dp dp' \right| \\
& = \left| \iint_{E_0 \times E_0} \langle (f_{\sigma,T}(ap, ap') - f_{\sigma,T}(0, 0))U, U' \rangle \xi(p) \xi(p') dp dp' \right| \\
& \leq Ca \cdot \iint_{E_0 \times E_0} \xi(p) \xi(p') dp dp' = Ca.
\end{aligned} \tag{5.99}$$

On the other hand, using Proposition 5.4.15 we deduce that

$$\begin{aligned} & \left| \iint_{E_0 \times E_0} \langle f_{\sigma,T}(p, p') U, U' \rangle \frac{\xi(p/a)}{a^{2(n+k)}} \frac{\xi(p'/a)}{a^{2(n+k)}} dp dp' \right| \\ & \leq \left[\iint_{E_0 \times E_0} \langle f_{\sigma,T}(p, p') U, U' \rangle^2 dp dp' \right]^{\frac{1}{2}} \cdot \|\xi\|_0^2 \leq \frac{C}{T^{1/2}}. \end{aligned} \quad (5.100)$$

Take $a = \frac{1}{T^{1/2}} \leq 1$. By (5.99) and (5.100), we find that

$$|\langle f_{\sigma,T}(0, 0) U, U' \rangle| \leq \frac{C}{T^{1/2}}. \quad (5.101)$$

The exact same estimate holds for any point $(p, p) \in E_0 \times E_0$ on the diagonal. Using the McKean-Singer formula, we have

$$\left| \text{Tr}_s \left(N_Y \tilde{F}_\sigma(\tilde{L}_{\sigma,T}) \right) - \text{Tr}_s \left(N_Y \tilde{P} \tilde{F}_\sigma(\Theta_\sigma) \tilde{P} \right) \right| \leq \frac{C}{T^{1/2}}. \quad (5.102)$$

By Remark 5.4.14 and the finite propagation speed property, we find that

$$\text{Tr}_s \left(N_Y \tilde{P} \tilde{F}_\sigma(\Theta_\sigma) \tilde{P} \right) = \text{Tr}_s \left(N_Y G_2 \circ (\sigma D_M)^2 \circ G_2^{-1} \right) = 0, \quad (5.103)$$

where the second equality is due to a parity argument as in the proof of Theorem 5.1.10.

The proof is completed. ■

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