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# LOW REGULARITY *A PRIORI* BOUNDS FOR THE MODIFIED KORTEWEG-DE VRIES EQUATION

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Abstract. We study the local well-posedness in the Sobolev space $H^s(\mathbb{R})$ for the modified Korteweg-de Vries (mKdV) equation $\partial_t u + \partial_x^3 u \pm \partial_x u^3 = 0$ on $\mathbb{R}$. Kenig-Ponce-Vega [10] and Christ-Colliander-Tao [1] established that the data-to-solution map fails to be uniformly continuous on a fixed ball in $H^s(\mathbb{R})$ when $s < \frac{1}{4}$. In spite of this, we establish that for $-\frac{1}{8} < s < \frac{1}{4}$, the solution satisfies global in time $H^s(\mathbb{R})$ bounds which depend only on the time and on the $H^s(\mathbb{R})$ norm of the initial data. This result is weaker than global well-posedness, as we have no control on differences of solutions. Our proof is modeled on recent work by Christ-Colliander-Tao [2] and Koch-Tataru [11] employing a version of Bourgain's Fourier restriction spaces adapted to time intervals whose length depends on the spatial frequency.

## 1. Introduction

We study the well-posedness of the initial-value problem for the modified Korteweg-de Vries (mKdV) on $\mathbb{R}$:

$$\partial_t u + \partial_x^3 u \pm \partial_x u^3 = 0, \qquad u(0) = u_0 \tag{1.1}$$

where $u = u(x,t) \in \mathbb{R}$ with $(x,t) \in \mathbb{R}^{1+1}$. This equation has scaling

$$u(x,t) \mapsto \lambda u(\lambda x, \lambda^3 t)$$

and the scale invariant homogeneous Sobolev norm is $\dot{H}^{-\frac{1}{2}}$. The equation is globally well-posed in $H^s$ for $s \geq \frac{1}{4}$. Specifically, given initial data in $H^s$, a solution exists in $C([0,+\infty); H^s) \cap X$, where $X$ is a certain auxiliary function space; this solution is unique among all solutions that reside in this function class; and for any $T > 0$, the data-to-solution map from a fixed ball in $H^s$ to $C([0,T]; H^s)$ is uniformly continuous. The local result was proved by Kenig-Ponce-Vega [8] by the contraction method in a function space where several dispersive estimates for the linear flow hold. An alternate proof in the setting of the Fourier restriction norm spaces was given later in Tao [15]. Colliander-Keel-Staffilani-Takaoka-Tao [3] proved that this local solution extends to a global solution by studying the almost conservation of of the norm of a high frequency-damped copy of the solution (the $I$-method). On the other hand, for $s < \frac{1}{4}$, (1.1) on $\mathbb{R}$ is ill-posed in the sense that the data-to-solution map fails to be *uniformly* continuous on a fixed ball in $H^s$. This was established by Kenig-Ponce-Vega [10] for the focusing equation (+ sign in front of the nonlinearity; Theorem

1.3 on p. 623 of their paper), and by Christ-Colliander-Tao [1] for the defocusing equation ($-$ sign in front of the nonlinearity; Theorem 4 on p. 1240 of their paper)[1]. This leaves open the question as to whether or not there is a well-posedness result for $s < \frac{1}{4}$ giving only the continuity (as opposed to *uniform* continuity) of the data-to-solution map. One result in this direction is Kato [6], where global weak solutions for $s = 0$ are constructed. We will here prove another result in this direction, giving an *a priori* bound in $H^s$ for $-\frac{1}{8} < s < \frac{1}{4}$ in terms of the $H^s$ norm of the initial data but establishing no continuity. Our method is analogous to that in Christ-Colliander-Tao [2] and Koch-Tataru [11] dealing with the nonlinear Schrödinger equation (NLS) on $\mathbb{R}$. The related problem for the mKdV equation was considered by Liu [12].

**Theorem 1.1.** *Let $-\frac{1}{8} < s < \frac{1}{4}$. Then for any $R > 0$ and $T > 0$ there exists*[2] *$C = C(R,T) > 0$ so that for any initial data $u_0 \in \mathcal{S}$ satisfying*

$$\|u_0\|_{H^s} \le R\,,$$

*the unique solution $u \in C([0,T];\mathcal{S})$ to (1.1) (focusing or defocusing) satisfies*

$$\|u\|_{L^\infty_{[0,T]}H^s_x} \le C\|u_0\|_{H^s}\,.$$

We note that our proof also applies for $s = -\frac{1}{8}$, but with a $C$ which depends on the full $H^{-\frac{1}{8}}$ frequency envelope of $u$. This dependence is likely nonoptimal, and it would simplify once the $-1/8$ threshold is crossed.

We also note that in the process of establishing the above result we also prove that the solutions belong to a smaller space $X^s$ defined later in the paper.

An easy consequence of our result is the existence of weak solutions for $H^s$ data:

**Corollary 1.2.** *Given any initial data $u_0 \in H^s$, there exists a global solution $u$ to* (1.1) *which solves the equation in the sense of distributions and satisfies*

$$\|u(t)\|_{H^s} \lesssim C(t, \|u_0\|_{H^s})$$

*with $C$ as in the theorem above.*

The weak solution is constructed as a weak limit of strong solutions. The uniform local $H^s$ bound does not suffice in order to verify that the equation is verified in the sense of distributions. Instead, this is true due the uniform $X^s$ bound, which is also implicit in the construction. We refer to these solutions as weak solutions as we currently do not have any uniqueness or continuous dependence result in $H^s$ for $s < -\frac{1}{4}$.

Currently the analogous problem for the periodic mKdV ((1.1) with $(x,t) \in \mathbb{T}\times\mathbb{R}$) is better understood. The threshold of $s = \frac{1}{4}$ for mKdV on $\mathbb{R}$ is replaced by $s = \frac{1}{2}$

[1] The proof given by [1] holds for $-\frac{1}{4} < s < \frac{1}{4}$, but the authors remark that the restriction to $s > -\frac{1}{4}$ is likely an artifact of their method.

[2] The proof actually yields $C = \max\{1, R^{-\frac{8s}{1+8s}}T^{-\frac{s}{1+8s}}\}$ but this is very likely nonoptimal.

for mKdV on $\mathbb{T}$. Kappeler-Topalov [5] construct, via inverse scattering theory, global solutions in $L^2$. Tsutsumi-Takaoka [14] construct solutions for data in $H^s$ for $\frac{3}{8} < s < \frac{1}{2}$ via Fourier restriction norm estimates and a nonlinear ansatz. Both of these results assert the continuity of the data-to-solution map.

Regarding our result, we believe that in principle, by adding another correction term (or maybe more) to the modified energy in §5, we could improve the lower threshold to $s \geq -\frac{1}{6}$ since the trilinear $\ell^2 U_A^{s,2}$ estimate in §4 is valid down to this threshold. It seems that to push to $s < -\frac{1}{6}$ would require a better understanding of "diagonal" or "resonant" frequency interactions. We do not know if there is any significance to the number $s = -\frac{1}{6}$ in regard to the actual behavior of solutions or whether it is just an artifact of our method.

An outline of the paper is as follows. In §2, we define the function spaces employed in the analysis. We use the $U^p$ and $V^p$ spaces, originally introduced to this subject in unpublished work of Tataru and then in Koch-Tataru [11], since they are ideally suited to time-truncations. In §3, we discuss the fundamental dispersive estimates employed in the proofs of the trilinear estimate and the energy bound. These include the Strichartz estimates, local smoothing and maximal function estimates, and Bourgain's bilinear "refined Strichartz" estimates. In §4, the trilinear estimate is proved along the lines of Christ-Colliander-Tao [2] and Koch-Tataru [11]. In §5, an energy bound is obtained on a high-frequency-damped energy functional. The method here is essentially an adaptation of the $I$-method of Colliander-Keel-Staffilani-Takaoka-Tao [3]. Our method does not establish any analogue of this energy bound for *differences* of solutions, which is the reason we cannot obtain a full well-posedness result in $H^s$, $-\frac{1}{8} < s < \frac{1}{4}$. Finally, in §6, the components are brought together to give a proof of Theorem 1.1.

In the conclusion of the introduction we give a heuristic that explains why, when $s < \frac{1}{4}$, we expect a piece of the solution at frequency $N \gg 1$ to propagate according to *linear* dynamics for at least a time $N^{4s-1} \ll 1$. Solutions to the linear equation satisfy the Strichartz estimate (see Lemmas 3.3, 3.4 below)

$$\|D_x^{1/6} e^{-t\partial_x^3}\phi\|_{L_t^6 L_x^6} \lesssim \|\phi\|_{L^2} \,. \tag{1.2}$$

Now suppose $u$ is a solution to (1.1) which is localized at frequency $N \gg 1$, and suppose $u \approx e^{-t\partial_x^3}\phi$ on $[0,T]$, with $\|\phi\|_{H^s} \sim 1$. In the integral equation,

$$u(t) = e^{-t\partial_x^3}\phi \mp \int_0^t e^{-(t-t')\partial_x^3}\partial_x u(t')^3 \, dt' \,,$$

we need to have

$$\left\| \int_0^t e^{-(t-t')\partial_x^3}\partial_x u(t')^3 \, dt' \right\|_{L^\infty_{[0,T]} H_x^s} \ll 1 \,. \tag{1.3}$$

We estimate this term as

$$\left\| \int_0^t e^{-(t-t')\partial_x^3} \partial_x u(t')^3 \, dt' \right\|_{L^\infty_{[0,T]} H^s_x} \le N^{1+s} \|u^3\|_{L^1_{[0,T]} L^2_x} \le T^{\frac{1}{2}} N^{1+s} \|u\|^3_{L^6_{[0,T]} L^6_x} .$$

Making the heuristic substitution $u(t) \approx e^{-t\partial_x^3}\phi$ and applying the Strichartz estimate (1.2),

$$\|u\|_{L^6_{[0,T]} L^6_x} \approx \|e^{-t\partial_x^3}\phi\|_{L^6_{[0,T]} L^6_x} \lesssim N^{-\frac{1}{6}} \|\phi\|_{L^2} \approx N^{-\frac{1}{6}-s} ,$$

we see that to achieve (1.3), we need $T \lesssim N^{4s-1}$. Motivated by this, our main function spaces $X^s_M$ defined in the next section are constructed by using linear type norms at frequency $N$ on the timescale $N^{4s-1}$.

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## 2. Function spaces

We first recall from Koch-Tataru [11] (see also the careful exposition in Hadac-Herr-Koch [4, §2]) the space-time function spaces $U^p(I)$ (atomic-space) and $V^p(I)$ (space of functions of bounded $p$-variation), $1 \le p \le \infty$. These are defined on a time interval $I = [a, b)$, where $-\infty \le a < b \le +\infty$ and take values in $L^2(\mathbb{R})$ or any other Hilbert space. Given a partition $a = t_0 < t_1 < \cdots < t_K = b$ of $I$ and a sequence $\{\phi_k\}_{k=0}^{K-1} \subset L^2_x$ such that $\phi_0 = 0$ and $\sum_{k=1}^K \|\phi_{k-1}\|^p_{L^2_x} = 1$, the function

$$a(t) = \sum_{k=1}^K \phi_{k-1} \chi_{[t_{k-1}, t_k)}(t)$$

is called a $U^p(I)$ atom. The space $U^p(I)$ is then the collection of functions $u(t)$ on $I$ of the form

$$u(t) = \sum_{\ell=0}^{+\infty} \lambda_\ell a_\ell \, , \tag{2.1}$$

where $a_\ell$ are $U^p(I)$ atoms, with norm

$$\|u(t)\|_{U^p(I)} = \inf_{\text{representations (2.1)}} \sum_{\ell=0}^{+\infty} |\lambda_\ell| \, .$$

It follows that elements $u(t)$ of $U^p(I)$ are right-continuous and satisfy the boundary conditions

$$u(a) = \lim_{t \searrow a} u(t) = 0 \quad \text{and} \quad u(b) \stackrel{\text{def}}{=} \lim_{t \nearrow b} u(t) \text{ exists} . \tag{2.2}$$

To define the space $V^p(I)$, we consider functions $v : I \to L^2_x$ such that

$$v(a) = \lim_{t \searrow a} v(t) \text{ exists} \quad \text{and} \quad v(b) \stackrel{\text{def}}{=} \lim_{t \nearrow b} v(t) = 0\,, \tag{2.3}$$

and for such functions $v(t)$ define the norm

$$\|v\|_{V^p(I)} = \sup_{\{t_k\}} \left( \sum_{k=1}^{K} \|v(t_k) - v(t_{k-1})\|_{L^2_x}^p \right)^{1/p} \,,$$

where the supremum is taken over partitions $a = t_0 < \cdots < t_K = b$. The fact that the requirement (2.3) is preserved in the limit under the $V^p(I)$ norm follows from [4, Prop 2.4(i)].

Note that for $I = [a, b)$, $-\infty < a < b < \infty$, we have

$$\|u\|_{U^p(I)} = \|\chi_I u\|_{U^p([-\infty,+\infty))}$$

provided $u(a) = 0$. If $u(a) \neq 0$, then the left-side is not defined (i.e. $u \notin U_p(I)$), while the right-side is defined. Also,

$$\|v\|_{V^p(I)} + \|v(a)\|_{L^2_x} = \|\chi_I v\|_{V^p([-\infty,+\infty))}$$

provided $v(b) = 0$. If $v(b) \neq 0$, then the left-side is not defined (i.e. $v \notin V_p(I)$), while the right-side is defined. Note that a consequence of (2.4) is that for any $v$ with $v(b) = 0$, we have

$$\|\chi_I v\|_{V^p([-\infty,+\infty))} \le 2\|v\|_{V^p(I)}\,. \tag{2.4}$$

**Lemma 2.1** ($U$-$V$ embeddings)**.** *Fix an interval $I = [a, b)$.*

1. *If $1 \le p \le q < \infty$, then $\|u\|_{U^q} \le \|u\|_{U^p}$ and $\|u\|_{V^q} \le \|u\|_{V^p}$.*
2. *If $1 \le p < \infty$ and $u(b) = 0$, then $\|u\|_{V^p} \lesssim \|u\|_{U^p}$.*
3. *If $1 \le p < q < \infty$, $u(a) = 0$, and $u \in V^p$ is right-continuous, then $\|u\|_{U^q} \lesssim \|u\|_{V^p}$.*
4. *Suppose that $1 \le p < q < \infty$, and $T$ is a linear operator with the boundedness properties:*

   $$\|Tu\|_E \le C_q \|u\|_{U^q_A}\,, \qquad \|Tu\|_E \le C_p \|u\|_{U^p_A}\,, \qquad \text{with } 0 < C_p \le C_q\,,$$

   *for some Banach space $E$. Then*

   $$\|Tu\|_E \lesssim \langle \ln \frac{C_q}{C_p} \rangle \|u\|_{V^p_A}\,,$$

   *with implicit constant depending only on the proximity of $q$ and $p$.*

The first three statements are from Koch-Tataru [11], while the last originates in Hadac–Herr–Koch [4]. The precise references in [4] for all four parts are: for (1), see Prop. 2.2(ii) and Prop. 2.4(iv); for (2), see Prop. 2.4(iii); for (3), see Cor. 2.6; for (4) Prop. 2.17. We emphasize that in (3), (4), we have *strict* inequality $p < q$. We also remark that (4) should be thought of as a quantitative version of (3).

We now define the space

$$DU^2(I) = \{\, \partial_t u \mid u \in U^2(I)\, \}\,,$$

where the derivative is taken in the sense of distributions. Given $f \in DU^2(I)$, a $u \in U^2(I)$ such that $\partial_t u = f$ is in fact *unique* (recall $u(a) = 0$). Hence we can define

$$\|f\|_{DU^2(I)} = \|u\|_{U^2(I)}\,,$$

which makes $DU^2(I)$ a Banach space. For example, if $u$ is an atom, i.e. $u = \sum_{k=1}^K \phi_{k-1}\chi_{[t_{k-1},t_k)}$ with $a = t_0 < \cdots < t_K = b$, $\phi_0 = 0$ and $\sum_{k=1}^K \|\phi_{k-1}\|_{L^2_x}^2 = 1$, then

$$f = \partial_t u = \sum_{k=1}^K (\phi_k - \phi_{k-1})\delta_{t_k}\,,$$

(where $\delta_{t_k}$ is the Dirac mass at $t_k$ and we take $\phi_K \overset{\text{def}}{=} 0$) is an element of $DU^2(I)$ with $\|f\|_{DU^2(I)} = 1$. Note that in this $f$, there is no Dirac mass at position $a$ but there is one at position $b$ (namely $-\phi_{K-1}\delta_b$).

**Lemma 2.2** ($DU$-$V$ duality)**.** *We have $(DU^2(I))^* = V^2(I)$ with respect to the usual pairing $\langle f, v\rangle = \int_a^b \langle f(t), v(t)\rangle_x\, dt = \int_a^b \int_x f\bar{v}\, dx\, dt$.*

*Proof.* First, we show that if $u \in U^2$ is such that $\partial_t u = f$, $u(a) = 0$, then $|\langle f, v\rangle| \le \|\chi_I u\|_{U^2(I)}\|v\|_{V^2(I)}$ for all $v \in V^2(I)$. Indeed, it suffices to show this for $u$ an atom, i.e $u = \sum_{k=1}^K \phi_{k-1}\chi_{[t_{k-1},t_k)}$, where $a = t_0 < \cdots < t_K = b$ and $\phi_0 = 0$ and $\sum_{k=1}^K \|\phi_k\|_{L^2_x}^2 = 1$. Since $u(a) = 0$ and $v(b) = 0$, we have

$$\begin{aligned}\langle f, v\rangle = \langle \partial_t u, v\rangle = -\langle u, \partial_t v\rangle &= -\sum_{k=1}^K \int_a^b \chi_{[t_{k-1},t_k)}\langle \phi_{k-1}, \partial_t v\rangle_x \\ &= -\sum_{k=1}^K \langle \phi_{k-1}, (v(t_k) - v(t_{k-1}))\rangle\end{aligned}$$

By Cauchy-Schwarz,

$$|\langle f, v\rangle| \le \left(\sum_{k=1}^K \|\phi_{k-1}\|_{L^2_x}^2\right)^{1/2} \left(\sum_{k=1}^K \|v(t_k) - v(t_{k-1})\|_{L^2_x}^2\right)^{1/2} \le \|v\|_{V^2}\,.$$

Next we show that $\sup_{\|f\|_{DU^2(I)}\le 1} |\langle f, v\rangle| = \|v\|_{V^2(I)}$. Pick a partition $a = t_0 < \cdots < t_K = b$ and define $\phi_0 = 0$, and for $2 \le k \le K$ define

$$\phi_{k-1} = \frac{v(t_k) - v(t_{k-1})}{\left(\sum_{j=2}^K \|v(t_j) - v(t_{j-1})\|_{L^2_x}^2\right)^{1/2}}$$

Then, defining $u = \sum_{k=1}^{K} \phi_{k-1}\chi_{[t_{k-1},t_k)}$ and $f = \partial_t u$ and arguing as above, $u$ is an atom and

$$\langle f, v\rangle = \left(\sum_{j=2}^{K} \|v(t_j) - v(t_{j-1})\|_{L^2_x}^2\right)^{1/2}.$$

Taking the supremum over all partitions and using that $\lim_{t\searrow a} v(t) = v(a)$, we obtain the claim.

Finally, we must show that if $\tilde{v} \in (DU^2(I))^*$, then there exists $v \in V^2(I)$ such that $\tilde{v}(f) = \langle f, v\rangle$ for all $f \in DU^2(I)$. Fix $a < t < b$, and we first define $w(t)$ as follows. The functional $\phi \mapsto \tilde{v}(\phi \cdot \delta_t)$ (where $\delta_t$ is the Dirac mass at $t$) is a bounded linear mapping $L^2_x \to \mathbb{C}$. Hence there exists $w(t) \in L^2_x$ such that $\langle \phi, w(t)\rangle_x = \tilde{v}(\phi \cdot \delta_t)$. It follows from [4, Prop. 2.4(i)] that $w(a) \stackrel{\text{def}}{=} \lim_{t\searrow a} w(t)$ exists and $w(b) \stackrel{\text{def}}{=} \lim_{t\nearrow b} w(t)$ exists. Set $v(t) = w(t) - w(b)$. Then if $u$ is an atom in $U^2(I)$ (taking $\phi_K \stackrel{\text{def}}{=} 0$ for notational convenience in the summations) and $f = \partial_t u$,

$$\begin{aligned}\langle f, v\rangle &= \left\langle \sum_{k=1}^{K} (\phi_k - \phi_{k-1})\delta_{t_k}, v\right\rangle = \sum_{k=1}^{K} \langle (\phi_k - \phi_{k-1}), v(t_k)\rangle_x = \sum_{k=1}^{K} \langle (\phi_k - \phi_{k-1}), w(t_k)\rangle_x \\ &= \sum_{k=1}^{K} \tilde{v}((\phi_k - \phi_{k-1})\delta_{t_k}) = \tilde{v}\left(\sum_{k=1}^{K} (\phi_k - \phi_{k-1})\delta_{t_k}\right) = \tilde{v}(f)\end{aligned}$$

□

Now we use the $U^p$ and $V^p$ spaces defined above to construct similar spaces adapted to the Airy flow. As base Hilbert spaces in which functions in $U^p$ and $V^p$ take values, we will use $L^2$, $H^s$, as well as a different norm $H^s_M$ on $H^s$ defined by

$$\|\phi\|_{H^s_M} = \|(|\xi|^2 + M)^{\frac{s}{2}} \hat{\phi}\|_{L^2}, \qquad M \geq 1$$

Finally, for a positive smooth even symbol $a$ satisfying $|a_\xi(\xi)| \lesssim a(\xi)$ we define the space $H^a$ with norm

$$\|\phi\|_{H^a}^2 = \langle \phi, a(D)\phi\rangle$$

If the $L^2$ space in the definition of $U^p(I)$, $V^p(I)$ and $DU^2$ spaces is replaced by another Hilbert space $H \in \{L^2, H^s, H^s_M, H^a\}$, we denote the corresponding spaces by $U^2(I;H)$, $V^2(I;H)$, respectively $DU^2(I;H)$. Finally, pulling back by the Airy group $e^{-t\partial_x^3}$ gives the spaces

$$\|u\|_{U^p_A(I;H)} \stackrel{\text{def}}{=} \|e^{t\partial_x^3} u\|_{U^p(I;H)}, \quad \|u\|_{V^p_A(I;H)} \stackrel{\text{def}}{=} \|e^{t\partial_x^3} u\|_{V^p(I;H)},$$

$$\|u\|_{DU^2_A(I;H)} \stackrel{\text{def}}{=} \|e^{t\partial_x^3} u\|_{DU^2(I;H)}$$

The properties in Lemmas 2.1,2.2 are easily transferred to this setting.

Consider a dyadic partition of frequencies ($N = 2^k$ for some $k = 0, 1, \ldots$), $E_N = \{\xi : N/2 \leq |\xi| \leq 2N\}$, and let $E_0 = [-1, 1]$. Fix consideration to the time interval $[0, 1)$. Consider a smooth Littlewood-Paley partition of unity in frequency $1 = \sum P_N$

where each multiplier $P_N$ is localized to the corresponding set $E_N$. For $H$ as above let

$$\|u\|_{\ell^2 L^\infty_{[0,1)} H} \stackrel{\text{def}}{=} \Big[ \sum_N \Big( \|P_N u(t)\|_{L^\infty_{[0,1)} H} \Big)^2 \Big]^{1/2}.$$

Clearly $\|u\|_{L^\infty_{[0,1)} H} \le \|u\|_{\ell^2 L^\infty_{[0,1)} H}$, but the converse is not true.

To measure the solutions to the mKdV equation we define the spaces $X^s_M$ with the norm

$$\|u\|_{X^s_M} \stackrel{\text{def}}{=} \Big( \sup_{|I|=M^{4s-1}} \|\chi_I P_{\le M} u\|^2_{U^2_A H^s_M} + \sum_{N>M} \sup_{|I|=N^{4s-1}} \|\chi_I P_N u\|^2_{U^2_A H^s_M} \Big)^{1/2},$$

where[3] the supremum is taken over all half-open subintervals $I = [a,b) \subset [0,1)$ of length $N^{1-4s}$.

To measure the nonlinearity in the mKdV equation we define the spaces $Y^s_M$ with the norm

$$\|f\|_{Y^s_M} \stackrel{\text{def}}{=} \Big( \sup_{|I|=M^{4s-1}} \|P_{\le M} f\|^2_{DU^2_A H^s_M} + \sum_{N>M} \sup_{|I|=N^{4s-1}} \|P_N f\|^2_{DU^2_A(I;H^s_M)} \Big)^{1/2},$$

Similarly we define the space $X^a_M$ and $Y^a_M$.

## 3. Basic estimates

**Lemma 3.1.** *Suppose* $\partial_t u + \partial_x^3 u = f$ *on* $[0,1)$*. Then*

$$\|u\|_{X^s_M} \lesssim \|u\|_{\ell^2 L^\infty_{[0,1)} H^s_M} + \|f\|_{Y^s_M}$$

*Proof.* Reduce to the case of a single frequency $N$ by applying $P_N$ to the equation, and then consider a fixed time interval $I = [t_0, t_1)$. We need to show

$$\|\chi_I u\|_{U^2_A H} \le \|u(t_0)\|_H + \|f\|_{DU^2_A(I;H)}.$$

But $\partial_t [e^{t\partial_x^3} u(t)] = e^{t\partial_x^3} f(t)$, and thus

$$\|f\|_{DU^2_A(I;H)} = \|e^{t\partial_x^3} f(t)\|_{DU^2(I;H)} = \|\chi_I (e^{t\partial_x^3} u(t) - u(a))\|_{U^2 H}.$$

Hence

$$\begin{aligned} \|\chi_I u\|_{U^2_A H} &= \|\chi_I e^{t\partial_x^3} u(t)\|_{U^2 H} \\ &\le \|\chi_I (e^{t\partial_x^3} u(t) - u(t_0))\|_{U^2 H} + \|\chi_I u(t_0)\|_{U^2 H} \\ &= \|u(t_0)\|_H + \|f\|_{DU^2_A(I;H)}. \end{aligned}$$

□

**Lemma 3.2** (Bernstein inequality)**.** *For* $1 \le p \le q \le \infty$,

$$\|P_N f\|_{L^q} \lesssim N^{\frac{1}{q} - \frac{1}{p}} \|f\|_{L^p}$$

[3] Note that here we have written $\|\chi_I P_N u\|_{U^{s,2}_A}$ and not $\|P_N u\|_{U^{s,2}_A(I)}$. Naturally, we are not assuming $u$ vanishes at the left endpoint of each of these intervals.

3.1. **Strichartz, local smoothing, and maximal function estimates.** A pair $(p,q)$ of Hölder exponents will be called admissible if

$$\frac{2}{p}+\frac{1}{q}=\frac{1}{2}, \qquad 4\le p\le\infty, \quad 2\le q\le\infty\,. \tag{3.1}$$

In particular, we note that the following pairs $(p,q)$ of indices are admissible: $(\infty,2)$, $(6,6)$, $(4,\infty)$.

**Lemma 3.3** (Strichartz estimates)**.** *Let* $(p,q)$ *satisfy the admissibility condition* (3.1)*. Then*

$$\|D_x^{\frac{1}{p}} e^{-t\partial_x^3}\phi\|_{L_t^pL_x^q} \lesssim \|\phi\|_{L^2}\,. \tag{3.2}$$

In particular, we have, for $N\ge 1$,

$$\begin{aligned}\|P_N e^{-t\partial_x^3}\phi\|_{L_t^\infty L_x^2} &\lesssim \|\phi\|_{L^2},\\ \|P_N e^{-t\partial_x^3}\phi\|_{L_t^6L_x^6} &\lesssim N^{-\frac{1}{6}}\|\phi\|_{L^2},\\ \|P_N e^{-t\partial_x^3}\|_{L_t^4L_x^\infty} &\lesssim N^{-\frac{1}{4}}\|\phi\|_{L^2}\,.\end{aligned}$$

*Proof.* In Kenig-Ponce-Vega [7] Lemma 2.4 / Kenig-Ponce-Vega [8] Lemma 3.18(i), the estimate

$$\|D_x^{\frac{1}{4}} e^{-t\partial_x^3}\phi\|_{L_t^4L_x^\infty} \lesssim \|\phi\|_{L_x^2}$$

is proved. On the other hand, we have trivially,

$$\|e^{-t\partial_x^3}\phi\|_{L_t^\infty L_x^2} = \|\phi\|_{L_x^2}\,.$$

Now we can apply Stein's theorem on analytic interpolation [13] to obtain (3.2). ☐

**Lemma 3.4** (Local smoothing/maximal function estimates)**.** *Let* $(p,q)$ *satisfy the admissibility condition* (3.1)*. Then*

$$\|D_x^{1-\frac{5}{p}} e^{-t\partial_x^3}\phi\|_{L_x^pL_t^q} \lesssim \|\phi\|_{L^2}\,. \tag{3.3}$$

In particular, we note the following estimates, for $N\ge 1$:

$$\begin{aligned}\|P_N e^{-t\partial_x^3}\phi\|_{L_x^\infty L_t^2} &\le cN^{-1}\|\phi\|_{L^2},\\ \|P_N e^{-t\partial_x^3}\phi\|_{L_x^6L_t^6} &\le cN^{-\frac{1}{6}}\|\phi\|_{L^2},\\ \|P_N e^{-t\partial_x^3}\phi\|_{L_x^4L_t^\infty} &\le cN^{\frac{1}{4}}\|\phi\|_{L^2}\,.\end{aligned}$$

*Proof.* The local smoothing estimate (Kenig-Ponce-Vega [8], Theorem 3.5(i)) is

$$\|\partial_x e^{-t\partial_x^3}\phi\|_{L_x^\infty L_t^2} \lesssim \|\phi\|_{L^2}\,.$$

It is basically reducible to Plancherel in $t$. On the other hand, we have the maximal function estimate (Kenig-Ponce-Vega [8], Theorem 3.7(i) on p. 556)

$$\|D_x^{-\frac{1}{4}} e^{-t\partial_x^3}\phi\|_{L_x^4L_t^\infty} \lesssim \|\phi\|_{L^2}\,.$$

It is proved by reducing by duality and a $TT^*$ argument to an estimate that is proved by the theorem on fractional integration and a pointwise Airy function estimate. We now apply Stein's theorem on analytic interpolation [13] to obtain (3.3). □

The next two corollaries are consequences of these estimates, and relate the Strichartz space-time norms to the Airy-atomic norm $U^{s,2}_A$ norm of *any* function $u(x,t)$ (not necessarily a solution to the linear Airy equation).

**Corollary 3.5.** *If $I = [a,b)$ is any interval, and $u = u(x,t)$ any function, then for $(p,q)$ satisfying the admissibility condition* (3.1)*, we have, for $N \geq 1$,*

$$\|P_N u\|_{L^p_I L^q_x} \lesssim N^{-\frac{1}{p}} \|\chi_I u\|_{U^p_A L^2}\,, \tag{3.4}$$

*and we have the dual relation for $p > 2$*

$$\|P_N u\|_{DU^2_A(I;L^2)} \lesssim N^{-\frac{1}{p}} \|u\|_{L^{p'}_I L^{q'}_x}\,, \tag{3.5}$$

*where $(p', q')$ denotes the Hölder dual pair.*

*Proof.* To prove (3.4), it suffices to assume $I = [-\infty, +\infty)$, since $\chi_I$ can be inserted. It also suffices to consider a $U^p_A$-atom

$$u(t,x) = \sum_{k=1}^K \chi_{[t_{k-1},t_k)}(t) e^{-t\partial_x^3} \phi_{k-1}(x), \quad \sum_{k=1}^K \|\phi_{k-1}\|^p_{L^2_x} = 1\,, \quad \phi_0 = 0\,, \tag{3.6}$$

and prove that

$$\|P_N u\|_{L^p_t L^q_x} \lesssim N^{-\frac{1}{p}}\,. \tag{3.7}$$

But (3.7) follows directly from (3.2), as follows:

$$\begin{aligned} \|P_N u\|^p_{L^p_t L^q_x} &= \sum_{k=1}^K \|\chi_{[t_{k-1},t_k)}(t) P_N e^{-t\partial_x^3} \phi_{k-1}\|^p_{L^p_t L^q_x} \\ &\lesssim N^{-1} \sum_{k=1}^K \|\phi_{k-1}\|^p_{L^2_x} = N^{-1}\,. \end{aligned}$$

To prove (3.5), note that since $(DU^2(I;L^2))^* = V^2(I;L^2)$, we have

$$\|P_N u\|_{DU^2_A(I;L^2)} = \sup_{\|v\|_{V^2_A(I;L^2)} \leq 1} \int_I \int_x P_N u \, \bar{v} \, dx \, dt\,.$$

But

$$|\langle P_N u, v \rangle| \leq \|u\|_{L^{p'}_{I'} L^{q'}_x} \|P_N v\|_{L^p_I L^q_x}\,,$$

and by (3.4) and Lemma 2.1(3) (applied on the interval $[-\infty, +\infty)$), we have, for $p > 2$,

$$\|P_N v\|_{L^p_I L^q_x} \lesssim \|\chi_I v\|_{U^p_A L^2} \lesssim \|\chi_I v\|_{V^2_A L^2}\,.$$

Apply (2.4) ($\|\chi_I v\|_{V^2_A L^2} \leq 2\|v\|_{V^2_A(I;L^2)}$) to complete the proof. □

**Corollary 3.6.** *If $(p, q)$ is admissible according to* (3.1) *and $p, q \geq r$, then*

$$\|P_N u\|_{L^p_x L^q_I} \lesssim N^{\frac{5}{p}-1} \|\chi_I u\|_{U^r_A L^2} . \tag{3.8}$$

*for any interval $I = [a, b)$. We also have the dual relation for $q > 2$,*

$$\|P_N u\|_{DU^2_A(I;L^2)} \lesssim N^{\frac{5}{p}-1} \|u\|_{L^{p'}_x L^{q'}_I} , \tag{3.9}$$

*where $(p', q')$ is the Hölder dual pair.*

*Proof.* As we argued in the proof of Cor. 3.5, it suffices to prove (3.8) for $u$ an atom of the form (3.6) (with $p$ replaced by $q$) on $I = [-\infty, +\infty)$. For such $u$ we write

$$u = \sum u_k, \qquad u_k = \chi_{[t_{k-1}, t_k)}(t) P_N e^{-t\partial_x^3} \phi_{k-1}$$

Applying (3.3) for each $u_k$, it remains to show that

$$\|u\|^r_{L^p_x L^q_t} \lesssim \sum_k \|u_k\|^r_{L^p_x L^q_t}$$

or equivalently

$$\||u|^r\|_{L^{\frac{p}{r}}_x L^{\frac{q}{r}}_t} \lesssim \sum_k \||u_k|^r\|_{L^{\frac{p}{r}}_x L^{\frac{q}{r}}_t}$$

But $u_k$ have disjoint supports therefore $|u|^r = \sum |u_k|^r$ and the last relation follows by the triangle inequality.

For (3.9), we note that since $(DU_A(I; L^2))^* = V^2_A(I; L^2)$

$$\|P_N u\|_{DU^2_A(I;L^2)} = \sup_{\|v\|_{V^2_A(I;L^2)}=1} \left| \int_I \int_x P_N u \, \bar{v} \, dx \, dt \right| .$$

But by Hölder,

$$\left| \int_I \int_x P_N u \, \bar{v} \, dx \, dt \right| \leq \|u\|_{L^{p'}_I L^{q'}_x} \|P_N v\|_{L^p_I L^q_x} ,$$

and by (3.8) and for $q > 2$, we have

$$\|P_N v\|_{L^p_I L^q_x} \leq \|\chi_I v\|_{U^q_A L^2} \leq \|\chi_I v\|_{V^2_A L^2} .$$

Finally apply (2.4) to obtain the bound by $\|v\|_{V^2_A(I;L^2)}$. □

### 3.2. **Bilinear estimate.**

**Lemma 3.7** (Bilinear estimate)**.** *Suppose $E_1, E_2 \subset \mathbb{R}$ and $M_1, M_2 > 0$ are dyadic values (no restriction to $\geq 1$) such that*

$$\forall\, \xi_1 \in E_1 \text{ and } \xi_2 \in E_2, \qquad |\xi_1 + \xi_2| \sim M_1 \text{ and } |\xi_1 - \xi_2| \sim M_2 .$$

*Let $P_j$ be the $x$-frequency projection operators defined as $\widehat{P_j f}(\xi) = \chi_{E_j}(\xi) \hat{f}(\xi)$ for a function $f = f(x)$. Then,*

$$\|P_1 e^{-t\partial_x^3} \phi \; P_2 e^{-t\partial_x^3} \psi\|_{L^2_t L^2_x} \lesssim (M_1 M_2)^{-\frac{1}{2}} \|P_1 \phi\|_{L^2} \|P_2 \psi\|_{L^2} . \tag{3.10}$$

*Proof.*

$$[P_1 e^{-t\partial_x^3}\phi \; P_2 e^{-t\partial_x^3}\psi]\widehat{\ }(\xi,t) = \int_{\substack{\xi_1\in E_1\\ \xi_2\in E_2\\ \xi=\xi_1+\xi_2}} e^{it\xi_1^3}\hat\phi(\xi_1)e^{it\xi_2^3}\hat\psi(\xi_2)$$

and thus

$$\begin{aligned}[P_1 e^{-t\partial_x^3}\phi \; P_2 e^{-t\partial_x^3}\psi]\widehat{\ }(\xi,\tau) &= \int_{\substack{\xi_1\in E_1\\ \xi_2\in E_2\\ \xi=\xi_1+\xi_2}} \delta(\tau-\xi_1^3-\xi_2^3)\hat\phi(\xi_1)\hat\psi(\xi_2)\\ &= \frac{\chi_{E_1}(\xi_1)\chi_{E_2}(\xi_2)\hat\phi(\xi_1)\hat\psi(\xi_2)}{3(\xi_1^2-\xi_2^2)}\end{aligned}$$

where, in the last line, $(\xi_1,\xi_2)$ is the solution to

$$\tau = \xi_1^3+\xi_2^3, \qquad \xi=\xi_1+\xi_2\,.$$

[In fact, there could be 0, 1, or 2 solutions $(\xi_1,\xi_2)$ depending upon the particular $(\xi,\tau)$; a proper argument would exhibit these regions separately, etc.] The Jacobian for the change of variable $(\xi,\tau)\mapsto(\xi_1,\xi_2)$ is

$$d\tau d\xi = 3|\xi_1^2-\xi_2^2|d\xi_1 d\xi_2\,.$$

The result then follows from Plancherel's theorem and this change of variable. ☐

**Corollary 3.8.** *Under the hypothesis of Lemma 3.7, if $u=u(x,t)$, $v=v(x,t)$ are any functions, then*[4]

$$\|P_1u\;P_2v\|_{L_I^2L_x^2} \lesssim (M_1M_2)^{-\frac12}\|\chi_IP_1u\|_{U_A^2L^2}\|\chi_IP_2v\|_{U_A^2L^2} \tag{3.11}$$

$$\|P_1u\;P_2v\|_{L_I^2L_x^2} \lesssim (M_1M_2)^{-\frac12}\left\langle \ln\frac{M_1}{M_2}\right\rangle^2\|\chi_IP_1u\|_{V_A^2L^2}\|\chi_IP_2v\|_{V_A^2L^2} \tag{3.12}$$

*Proof.* It clearly suffices to prove the estimates for $I=[-\infty,+\infty)$, since we can insert $\chi_I$ cutoffs on $u$ and $v$. We begin noting that if we fix $u=e^{-t\partial_x^3}\psi$, and $v$ a $U_A^2$ atom, i.e.

$$v(x,t)=\sum_{k=1}^K \chi_{[t_{k-1},t_k)}(t)e^{-t\partial_x^3}\phi_{k-1}\,, \quad \phi_0=0\,, \quad \sum_{k=1}^K\|\phi_{k-1}\|_{L_x^2}^2=1\,,$$

then it follows from Lemma 3.7 that

$$\|P_1u\;P_2v\|_{L_I^2L_x^2} \lesssim (M_1M_2)^{-\frac12}\|\psi\|_{L^2}\,. \tag{3.13}$$

By linearity in $u$, we obtain the estimate (3.11) when both $u$ and $v$ are $U_A^2$ atoms. The general case of (3.11) follows by linearity and density. The estimate (3.12) follows

[4] Note the use of the truncation functions $\chi_I$ and then evaluation in $U_A^{0,2}([-\infty,+\infty))$ or $V_A^{0,2}([-\infty,+\infty))$ on the right-side. We are not using the norms $U_A^{0,2}(I)$ or $V_A^{0,2}(I)$, since they require vanishing at the left and right endpoints of $I$, respectively. We do not want to impose such a condition for finite-length intervals $I$.

from (3.11) by the argument in [4, Cor. 2.18] which appeals to their Prop. 2.17 (our Lemma 2.1(4)). □

## 4. Trilinear estimate

**Proposition 4.1** (Trilinear estimate)**.** *For all* $-\frac{1}{7} < s < \frac{1}{4}$ *and* $M \geq 1$ *we have*

$$\|\partial_x(u_1 u_2 u_3)\|_{Y^s_M} \lesssim \|u_1\|_{X^s_M} \|u_2\|_{X^s_M} \|u_3\|_{X^s_M} .$$

*Proof.* We insert frequency projections $P_{N_j}$, $P_N$ where $N, N_j \geq M$. Denoting the truncated functions by $u_{N_j} = P_{N_j} u_j$ for $N_j > M$ while $u_{N_j} = P_{<M} u_j$ for $N_j = M$, we reduce matters to proving, for an interval $|J| = N^{4s-1}$ with $N > M$, a bound of the type

$$\|P_N \partial_x (u_{N_1}\ u_{N_2}\ u_{N_3})\|_{DU^2_A(J;H^s_M)} \leq \alpha(N, N_1, N_2, N_3) \prod_{j=1}^{3} \sup_{|I_j| = N_j^{4s-1}} \|\chi_{I_j} u_{N_j}\|_{U^2_A H^s_M} \tag{4.1}$$

as well as the similar bound with $P_N$ replaced by $P_{<M}$. This can be rewritten as

$$\|P_N \partial_x (u_{N_1}\ u_{N_2}\ u_{N_3})\|_{DU^2_A(J;L^2)} \leq \alpha(N, N_1, N_2, N_3) \frac{N_1^s N_2^s N_3^s}{N^s} \prod_{j=1}^{3} \sup_{|I_j| = N_j^{4s-1}} \|\chi_{I_j} u_{N_j}\|_{U^2_A L^2}$$

Here $\alpha$ should have certain summability properties. As a general rule, we need at least that $|\alpha(N, N_1, N_2, N_3)| \lesssim 1$, and in some cases, need a slight power decay in $N$ and/or $N_j$ to insure the summation with respect to all indices.

*Case 1.* $N_1, N_2, N_3 \lesssim N$. We can assume that $N_1 \leq N_2 \leq N_3 \sim N$. In this case, all $I_j$ have length $\geq |J|$ and can be neglected. We distribute the derivative, which in the worst case applies to $u_{N_3}$. By (3.5) and (3.8),

$$\begin{aligned}
\|P_N(u_{N_1}\ u_{N_2}\ \partial_x u_{N_3})\|_{DU^2_A(J;L^2)} &\lesssim \|u_{N_1}\ u_{N_2}\ \partial_x u_{N_3}\|_{L^1_J L^2_x} \\
&\lesssim |J|^{\frac{1}{2}} \|u_{N_1}\ u_{N_2}\ \partial_x u_{N_3}\|_{L^2_J L^2_x} \\
&\lesssim N^{2s-\frac{1}{2}} \|u_{N_1}\|_{L^4_x L^\infty_J} \|u_{N_2}\|_{L^4_x L^\infty_J} \|\partial_x u_{N_3}\|_{L^\infty_x L^2_J} \\
&\lesssim N^{2s-\frac{1}{2}} N_1^{\frac{1}{4}} \|\chi_J u_{N_1}\|_{U^2_A L^2} N_2^{\frac{1}{4}} \|\chi_J u_{N_2}\|_{U^2_A L^2} \|\chi_J u_{N_3}\|_{U^2_A L^2}
\end{aligned}$$

Thus we have (4.1) with $\alpha = N^{2s-\frac{1}{2}} N_1^{\frac{1}{4}-s} N_2^{\frac{1}{4}-s}$, which suffices for all $s$.

*Case 2.* $N_1 \lesssim N \ll N_2 \sim N_3$. The $u_2, u_3$ terms need to be evaluated in norms restricted to intervals $I$ of size $|I| = N_3^{4s-1}$. We divide $J$ into $|J|/|I| = (N_3/N)^{1-4s} \gg 1$ intervals of size $|I| = N_3^{4s-1}$. For $u \in V_A^2(J;L^2)$ we estimate by duality (Lemma 2.2)

$$\Big| \int_J \int_x u_{N_1} u_{N_2}\, u_{N_3}\, u_N \, dx\, dt \Big| \leq \left(\frac{N_3}{N}\right)^{1-4s} \sup_{\substack{I \subset J \\ |I| = N_3^{4s-1}}} \Big| \int_I \int_x u_{N_1}\, u_{N_2}\, u_{N_3}\, u_N \, dx\, dt \Big|$$
$$\leq \left(\frac{N_3}{N}\right)^{1-4s} \sup_{\substack{I \subset J \\ |I| = N_3^{4s-1}}} \|u_{N_1}\, u_{N_2}\|_{L_I^2 L_x^2} \|u_N\, u_{N_3}\|_{L_I^2 L_x^2} .$$

Using the bilinear estimate (3.11),(3.12) we bound the above by

$$\left(\frac{N_3}{N}\right)^{1-4s} N_3^{-2} \left\langle \ln \frac{N_3}{N} \right\rangle^2 \sup_{\substack{I \subset J \\ |I| = N_3^{4s-1}}} \|\chi_I u_{N_1}\|_{U_A^2 L^2} \|\chi_I u_{N_2}\|_{U_A^2 L^2} \|\chi_I u_{N_3}\|_{U_A^2 L^2} \|\chi_I u_N\|_{V_A^2 L^2} .$$

Finally, we apply (2.4) ($\|\chi_I P_N u\|_{V_A^2} \leq 2\|P_N u\|_{V_A^2(J)}$). Adding a factor of $N$ to account for the derivative in (4.1) we obtain

$$\alpha = N_3^{-1-6s} N^{5s} N_1^{-s} \left\langle \ln \frac{N_3}{N} \right\rangle^2$$

so this case is handled if $s \geq -\frac{1}{6}$.

*Case 3.* $N \ll N_1 \leq N_2 = N_3$. We can assume that $\xi_1$, $\xi_2$ have the same sign and that $\xi_3$ has the opposite sign. [Indeed, if $N_1 \ll N_2$, then this is achieved by permuting $N_2$ and $N_3$ if necessary, and if $N_1 \sim N_2 \sim N_3$, then this can be arranged by permuting the indices.] Note that then obviously we have $|\xi_1 - \xi_3| \sim N_3$, but also since $N \ll N_2 \sim N_3$, we have $|\xi_1 + \xi_3| = |\xi + \xi_2| \sim N_3$ and $|\xi - \xi_2| \sim N_3$.

We again argue by duality (Lemma 2.2) and divide into subintervals of size $|I| = N_3^{4s-1}$. For $v \in V_A^2(J;L^2)$,

$$\Big| \int_{t\in J} \int_x u_{N_1}\, u_{N_2}\, u_{N_3}\, u_N \, dx\, dt \Big| \leq \left(\frac{N_3}{N}\right)^{1-4s} \sup_{\substack{I \subset J \\ |I| = N_3^{4s-1}}} \Big| \int_{t\in I} \int_x u_{N_1}\, u_{N_2}\, u_{N_3}\, u_N u \, dx\, dt \Big|$$
$$\leq \left(\frac{N_3}{N}\right)^{1-4s} \sup_{\substack{I \subset J \\ |I| = N_3^{4s-1}}} \|u_N\, u_{N_2}\|_{L_I^2 L_x^2} \|u_{N_1}\, u_{N_3}\|_{L_I^2 L_x^2} .$$

We then apply the bilinear estimate (3.11), (3.12) to bound the above by

$$\leq \left(\frac{N_3}{N}\right)^{1-4s} N_3^{-2} \left\langle \ln \frac{N_3}{N} \right\rangle^2 \sup_{\substack{I \subset J \\ |I| = N_3^{4s-1}}} \|\chi_I u_{N_1}\|_{U_A^2 L^2} \|\chi_I u_{N_2}\|_{U_A^2 L^2} \|\chi_I u_{N_3}\|_{U_A^2 L^2} \|\chi_I u_N\|_{V_A^2 L^2}$$

Finally, we apply (2.4). Thus we have $\alpha = N_3^{-1-6s} N^{5s} N_1^{-s}$, which is satisfactory if $s > -\frac{1}{7}$.

□

## 5. Energy bound

For expositional convenience, in this section, we will assume that we are in the more difficult case $s \leq 0$. We study the almost conservation of the $H^s$ norm using a variant of the $I$-method of Colliander-Keel-Staffilani-Takaoka-Tao [3]. The main result of this section is as follows:

**Proposition 5.1** (Energy bound). *For all $-\frac{1}{8} \leq s \leq 0$, $M > 0$ and $u$ solving* (1.1) *we have the following bound in the time interval* $[0, 1]$*:*

$$\|u\|^2_{\ell^2 L^\infty H^s_M} \leq c(\|u\|^4_{\ell^2 L^\infty H^s_M} + \|u\|^6_{X^s_M}) \tag{5.1}$$

Due to the $l^2$ dyadic summation on the left we cannot simply obtain a uniform in time bound for the $H^s$ norm of $u$. Instead for small $\epsilon > 0$ we introduce a class $S_M$ of real smooth positive even symbols $a(\xi)$ which have the following properties:

(i) $a(\xi)$ is constant for $|\xi| \leq M$.

(ii) Regularity:

$$|\partial^\alpha_\xi a(\xi)| \leq c_\alpha a(\xi) \langle \xi \rangle^{-\alpha} \tag{5.2}$$

(iii) Decay properties

$$-\frac{1}{2} \leq \frac{d \log a(\xi)}{d \log(1 + \xi^2)} \leq 0 \tag{5.3}$$

The latter property implies that $a(\xi)$ is nonincreasing but decays no faster than[5] $|\xi|^{-\frac{1}{2}}$. For $a \in S_M$ we will prove the uniform bound

$$\|u\|^2_{L^\infty H^a} \leq \|u(0)\|^2_{H^a} + c(\|u\|^2_{L^\infty H^s_M} \|u\|^2_{L^\infty H^a} + \|u\|^4_{X^s_M} \|u\|^2_{X^a_M}) \tag{5.4}$$

which implies the desired bound (5.1). To see this, for each dyadic $N \geq M$ we consider a symbol $a_N \in S_M$ such that

$$a_N(\xi) \stackrel{\text{def}}{=} \begin{cases} N^{2s} & \text{if } |\xi| \leq N \\ N^{\frac{1}{2}+2s} |\xi|^{-\frac{1}{2}} & \text{if } |\xi| \geq 2N \end{cases}.$$

Then (5.1) follows from (5.4) applied to $a_N$ due to the obvious relations

$$\|u\|^2_{\ell^2 L^\infty H^s_M} \approx \sum_{N \geq M} \|u\|^2_{L^\infty H^{a_N}},$$

$$\|u\|^2_{X^s_M} \approx \sum_{N \geq M} \|u\|^2_{X^a_N}$$

It remains to prove the bound (5.4). We define the energy functional

$$E_0(u) \stackrel{\text{def}}{=} \langle A(D)u, u \rangle = \|u\|^2_{H^a}$$

[5] In effect decay rates up to $|\xi|^{-1}$ are still acceptable, but not needed here.

and compute its derivative along the flow. Since $a(\xi)$ is even and $u$ is real, $A(D)u$ is real. Also, $A(D)$ is self-adjoint since $a(\xi)$ is real. Thus, substituting (1.1),

$$\frac{d}{dt}E_0(u) = R_4(u) \stackrel{\text{def}}{=} \pm 2\langle A(D)\partial_x u, u^3\rangle\,.$$

Using the fact that $u$ is a real valued function, which implies that $\overline{\hat{u}(-\xi)} = \hat{u}(\xi)$, we write $R_4$ as a multilinear operator in Fourier space:

$$R_4(u) = \pm 2\int_{P_4} i\xi_1 a(\xi_1)\;\hat{u}(\xi_1)\hat{u}(\xi_2)\hat{u}(\xi_3)\hat{u}(\xi_4)\,d\sigma\,,$$

where $P_4 = \{\,(\xi_1,\xi_2,\xi_3,\xi_4)\in\mathbb{R}^4 \mid \xi_1+\xi_2+\xi_3+\xi_4 = 0\,\}$. This expression for $R_4$ can be symmetrized as

$$R_4(u) = \pm\frac{1}{2}\int_{P_4} i(\xi_1 a(\xi_1)+\xi_2 a(\xi_2)+\xi_3 a(\xi_3)+\xi_4 a(\xi_4))\hat{u}(\xi_1)\hat{u}(\xi_2)\hat{u}(\xi_3)\hat{u}(\xi_4)\,d\sigma\,.$$

We seek to cancel this term by perturbing the energy to $E_0+E_1$, where $E_1$ has the form

$$E_1(u) = \int_{P_4} b_4(\xi_1,\xi_2,\xi_3,\xi_4)\hat{u}(\xi_1)\hat{u}(\xi_2)\hat{u}(\xi_3)\hat{u}(\xi_4)\,d\sigma\,.$$

To determine the proper choice for $b_4$, we compute

$$\frac{d}{dt}E_1(u) = \int_{P_4} b_4(\xi_1,\xi_2,\xi_3,\xi_4) i(\xi_1^3+\xi_2^3+\xi_3^3+\xi_4^3)\;\hat{u}(\xi_1)\hat{u}(\xi_2)\hat{u}(\xi_3)\hat{u}(\xi_4)\,d\sigma + R_6(u)\,,$$

where $R_6(u)$ has the form (if we for convenience go ahead and assume that $b_4$ is symmetric under exchange of any pair from $\xi_1$, $\xi_2$, $\xi_3$ and $\xi_4$)

$$R_6(u) = \mp\frac{1}{4}\int_{P_4} b_4(\xi_1,\xi_2,\xi_3,\xi) i\xi\widehat{u^3}(\xi)\hat{u}(\xi_1)\hat{u}(\xi_2)\hat{u}(\xi_3)\,d\sigma\,.$$

Now we see that the proper choice of $b_4$ to cancel the term $R_4$ is

$$b_4(\xi_1,\xi_2,\xi_3,\xi_4) = \pm\frac{1}{2}\frac{\xi_1 a(\xi_1)+\xi_2 a(\xi_2)+\xi_3 a(\xi_3)+\xi_4 a(\xi_4)}{\xi_1^3+\xi_2^3+\xi_3^3+\xi_4^3}$$

In conclusion, we have

$$\frac{d}{dt}(E_0+E_1)(u) = R_6(u)\,.$$

Hence in order to prove (5.4) we need to establish the following two bounds:

$$E_1(u) \lesssim \|u\|_{H^s_M}^2\|u\|_{H^a}^2 \tag{5.5}$$

respectively

$$\int_0^1 R_6(u(t))dt \lesssim \|u\|_{X^s_M}^4\|u\|_{X^a}^2 \tag{5.6}$$

In order to do this we need to study the size and regularity of $b_4$.

**Lemma 5.2.** *Let $a \in S_M$. Then there exists a symbol $b_4$ in $\mathbb{R}^4$ so that*

$$\xi_1 a(\xi_1) + \xi_2 a(\xi_2) + \xi_3 a(\xi_3) + \xi_4 a(\xi_4) = b_4(\xi_1^3 + \xi_2^3 + \xi_3^3 + \xi_4^3) \qquad \text{on } P_4 \tag{5.7}$$

*with the following size and regularity in dyadic regions $\{|\xi_j| \sim N_j > M\}$ respectively $\{\xi_j \lesssim N_j = M\}$:*

$$|\partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2}\partial_{\xi_3}^{\alpha_3}\partial_{\xi_4}^{\alpha_4} b_4(\xi_1,\xi_2,\xi_3,\xi_4)| \leq c_\alpha b_4(N_1,N_2,N_3,N_4) N_1^{-\alpha_1} N_2^{-\alpha_2} N_3^{-\alpha_3} N_4^{-\alpha_4}, \tag{5.8}$$

*where*

$$b_4(N_1,N_2,N_3,N_4) = a(N_2)N_4^{-2} \qquad \text{when } N_1 \leq N_2 \leq N_3 \sim N_4$$

*Proof.* On $P_4$, we have the factorization

$$\xi_1^3 + \xi_2^3 + \xi_3^3 + \xi_4^3 = (\xi_1+\xi_2)(\xi_1+\xi_3)(\xi_2+\xi_3)\,.$$

Let $N_j \geq M$ denote the dyadic zone of $|\xi_j|$ (as before the $M$ dyadic zone includes all frequencies below $M$). On $P_4$ we necessarily have $N_3 \sim N_4$. If all $|\xi_j| \leq M$, then the left hand side of (5.7) is zero since $a(\xi) = \text{const}$ for $|\xi| \leq M$, we have that . Therefore, we take $b_4 = 0$ there and assume $N_4 \geq M$ in the remainder of the proof. We consider several cases.

*Case 1.* $N_1 \ll N_2 \leq N_3 \sim N_4$. Then we define

$$b_4(\xi) = -\frac{\xi_1 a(\xi_1) + \xi_2 a(\xi_2)}{(\xi_1+\xi_2)(\xi_1+\xi_3)(\xi_1+\xi_4)} - \frac{1}{(\xi_1+\xi_3)(\xi_1+\xi_4)}\frac{\xi_3 a(\xi_3) + \xi_4 a(\xi_4)}{\xi_3+\xi_4}$$

Since $|\xi_1+\xi_2| \sim N_2$ and $|\xi_1+\xi_3|, |\xi_1+\xi_4| \sim N_4$, the conclusion easily follows by taking advantage of the cancellation in the last fraction when $\xi_3+\xi_4 = 0$.

*Case 2.* $N_1 \sim N_2 \ll N_3 \sim N_4$. Then we have $|\xi_1+\xi_3|, |\xi_2+\xi_3|, |\xi_1+\xi_2+\xi_3| \sim N_4$. Hence we define

$$b_4(\xi) = \frac{\xi_1 a(\xi_1) + \xi_2 a(\xi_2) + \xi_3 a(\xi_3) - (\xi_1+\xi_2+\xi_3)a(-\xi_1-\xi_2-\xi_3)}{(\xi_1+\xi_2)(\xi_1+\xi_3)(\xi_2+\xi_3)} \tag{5.9}$$

and the only difficulty comes from the division by $\xi_1+\xi_2$. We rewrite $b_4$ as

$$b_4(\xi) = \frac{1}{(\xi_1+\xi_3)(\xi_2+\xi_3)}(g(\xi_1,\xi_2) - g(\xi_3, -\xi_1-\xi_2-\xi_3))$$

where the function $g$ is defined by

$$g(\xi,\eta) = \frac{\xi a(\xi) + \eta a(\eta)}{\xi+\eta}.$$

Since $a$ is even and satisfies (5.2), it follows that $g$ is smooth on the dyadic scale and has size $\lesssim a(N)$ when $|\xi| \sim |\eta| \sim N$. The conclusion again follows.

*Case 3.* $N_1 \sim N_2 \sim N_3 \sim N_4 \sim N$. Using a partition of unit on the $N$ scale and permuting the indices we can assume that we localized the problem to a region where

$|\xi_2+\xi_3|, |\xi_1+\xi_2+\xi_3| \sim N$. Then we define $b_4$ using again (5.9), and rewrite it in the form

$$b_4(\xi) = \frac{1}{\xi_2+\xi_3}\frac{g(\xi_1,\xi_2) - g(\xi_3, -\xi_1-\xi_2-\xi_3))}{(\xi_1+\xi_3)}$$

Now the first factor is elliptic, and in the second factor the numerator vanishes on $\{\xi_1+\xi_3 = 0\}$ therefore we have again a smooth division on the dyadic scale. □

The next result implies the bound (5.5):

**Corollary 5.3.** *Let $a \in S_M$ and $b_4$ as in Lemma 5.2. Then*

$$|E_1(u)| \lesssim \|u\|_{H^a}^2 \|u\|^2_{H_M^{-\frac{1}{2}}} \tag{5.10}$$

*Proof.* Given the expression of $b_4$, it suffices to prove this when $\hat{u}$ is positive and $b_4$ is estimated pointwise by (5.8). Using again the notation $u_N = P_N u$ for $N > M$ and $u_M = P_{\leq M} u$, by Bernstein's inequality we have

$$\begin{aligned} |E_1(u)| &\lesssim \sum_{M \leq N_1 \leq N_2 \leq N_3 \sim N_4} \frac{a(N_2)}{N_4^2} \|u_{N_1} u_{N_2} u_{N_3} u_{N_4}\|_{L^1} \\ &\lesssim \sum_{M \leq N_1 \leq N_2 \leq N_3 \sim N_4} \frac{a(N_2) N_1^{\frac{1}{2}} N_2^{\frac{1}{2}}}{N_4^2} \|u_{N_1}\|_{L^2} \|u_{N_2}\|_{L^2} \|u_{N_3}\|_{L^2} \|u_{N_4}\|_{L^2} \\ &= \sum_{M \leq N_1 \leq N_2 \leq N_3 \sim N_4} \left(\frac{a(N_2)N_1}{a(N_1)N_2}\right)^{\frac{1}{2}} \frac{N_2}{N_4} \|u_{N_1}\|_{H^a} \|u_{N_2}\|_{H^a} \|u_{N_3}\|_{\dot{H}^{-\frac{1}{2}}} \|u_{N_4}\|_{\dot{H}^{-\frac{1}{2}}} \end{aligned}$$

and the summation with respect to the $N_i$'s is now straightforward. □

We conclude the proof of Proposition 5.1 with

*Proof of the estimate 5.6.* Writing $\xi = \xi_4 + \xi_5 + \xi_6$ as the frequency decomposition in the cubic product we write $R_6(u)$ in the form

$$R_6(u) = \mp \frac{1}{4} \int_{P_6} i\xi b_4(\xi, \xi_1, \xi_2, \xi_3) \hat{u}(\xi_1)\hat{u}(\xi_2)\hat{u}(\xi_3)\hat{u}(\xi_4)\hat{u}(\xi_5)\hat{u}(\xi_6)\, d\sigma\,.$$

where $P_6 = \{\xi_1+\xi_2+\xi_3+\xi_4+\xi_5+\xi_6 = 0\}$. For $b_4$ we use the extension given by Lemma 5.2. Since this extension is smooth in all variables on the dyadic scales, without any restriction we can separate variables and reduce the problem to the case when $b_4$ is of product type. Then we can return to the physical space and rewrite

$$R_6(u) = \sum_{N, N_2, \cdots, N_7} N b_4(N, N_1, N_2, N_3) \int u_{N_1} u_{N_2} u_{N_3} P_N(u_{N_4} u_{N_5} u_{N_6}) dx, \quad u_{N_i} = P_{N_i} u$$

where the factors in $b_4$ are harmlessly included in the spectral projectors. This is allowed because $L^2$ bounded multipliers are also bounded in $U_A^2$ spaces.

By symmetry we can assume that $N_1 \leq N_2 \leq N_3$, as well as $N_4 \leq N_5 \leq N_6$. We also take an increasing rearrangement

$$\{N_1, N_2, N_3, N_4, N_5, N_6\} = \{M_1, M_2, M_3, M_4, M_5, M_6\}$$

where we must always have $N \lesssim M_5 \sim M_6$.

Our next contention is that we can harmlessly discard the projector $P_N$ by separating variables. To see this we use the Fourier representation of the symbol

$$p_N(\xi_1 + \xi_2 + \xi_3) = \int e^{i\lambda\xi_1} e^{i\lambda\xi_2} e^{i\lambda\xi_3} f(\lambda) d\lambda, \qquad f_N(\lambda) = \int e^{-i\lambda\xi} p_N(\xi) d\xi$$

The complex exponentials are bounded symbols and thus bounded on $U_A^2 L^2$, while $\|f_N\|_{L^1} \lesssim 1$ uniformly in $N$.

Assuming now that we have separated variables, we can sum the coefficient in $R_6$ with respect to $N$

$$\sum_{N \leq N_3} N b_4(N, N_1, N_2, N_3) \sim a(N_2) N_3^{-1} \lesssim a(M_2) M_3^{-1}$$

and we are left with having to estimate

$$I = \sum_{M_1 \leq \cdots \leq M_5 = M_6} a(M_2) M_3^{-1} \int_0^1 \int_{\mathbb{R}} u_{M_1} u_{M_2} u_{M_3} u_{M_4} u_{M_5} u_{M_6} dx dt$$

We divide the time interval $[0, 1]$ in $M_6^{1-4s}$ subintervals of size $M_6^{-1+4s}$ corresponding to the highest frequency factor. We estimate the integral in each such subinterval, taking a loss of $M_6^{1-4s}$ due to the interval summation. Depending on how many frequency $M_6$ factors there are we split into several cases:

*Case (a).* $M_4 \ll M_6$. Then we can use two bilinear $L^2$ bounds for the products $u_3 u_5$ and $u_4 u_6$ and Bernstein to derive a pointwise bound for $u_{N_1}$ and $u_{N_2}$. We obtain

$$\begin{aligned} |I_{(i)(a)}| &\lesssim \sum_{M_1 \leq \cdots \leq M_5 = M_6} a(M_2) M_3^{-1} M_6^{1-4s} M_6^{-2} M_1^{\frac{1}{2}} M_2^{\frac{1}{2}} \sup_{|I| = M_6^{-1+4s}} \prod_{j=1}^{6} \|\chi_I u_{M_j}\|_{U_A^2 L^2} \\ &\lesssim \sum_{M_1 \leq \cdots \leq M_5 = M_6} \left( \frac{a(M_2) M_1}{a(M_1) M_2} \right)^{\frac{1}{2}} \frac{M_2}{M_3} \frac{M_6^s}{M_3^s} \frac{M_6^s}{M_4^s} M_6^{-1-8s} \prod_{j=1}^{2} \|u_{M_j}\|_{X^a} \prod_{j=3}^{6} \|u_{M_j}\|_{X_M^s} \end{aligned}$$

where the factors were reorganized to make clear the summation with respect to the $M_j$'s. It is also transparent here that the total balance of exponents can only be favorable if $s \geq -\frac{1}{8}$.

*Case (b).* $M_3 \ll M_4 \sim M_6$. The same argument as above applies after observing that two of the frequencies $\xi_4$, $\xi_5$ and $\xi_6$ must have an $M_6$ separation, therefore the bilinear $L^2$ estimate can be applied.

*Case (c).* $M_3 \sim M_6$. As in the previous case we can apply the $L^2$ bilinear estimate for two of the high frequency factors, say $u_{M_5} u_{M_6}$. Then we use the $L^4_x L^\infty_t$ bound for $u_{M_2}$ and $u_{M_3}$, the $L^\infty_x L^2_t$ for $u_{M_4}$ as well as the $L^\infty$ bound for $u_{M_1}$. We obtain

$$\begin{aligned}
|I_{(i)(a)}| &\lesssim \sum_{M_1 \le M_2 \le M_3 = \cdots = M_6} a(M_2) M_6^{-1} M_6^{1-4s} M_6^{-1} M_1^{\frac{1}{2}} M_2^{\frac{1}{4}} M_6^{-\frac{3}{4}} \sup_{|I| = M_6^{-1+4s}} \prod_{j=1}^{6} \|\chi_I u_{M_j}\|_{U^2_A L^2} \\
&\lesssim \sum_{M_1 \le M_2 \le M_3 = \cdots = M_6} \left( \frac{a(M_2) M_1}{a(M_1) M_2} \right)^{\frac{1}{2}} \left( \frac{M_2}{M_3} \right)^{\frac{3}{4}} M_6^{-1-8s} \prod_{j=1}^{2} \|u_{M_j}\|_{X^a} \prod_{j=3}^{6} \|u_{M_j}\|_{X^s_M}
\end{aligned}$$

Again the summability with respect to $M_j$'s is straightforward.

□

## 6. Proof of Theorem 1.1

For expositional convenience, in this section, we will assume again that we are in the more difficult case $s < 0$. We first establish a short time small data result:

**Proposition 6.1.** *Let $M \ge 1$ and $-\frac{1}{4} \le s < 0$. For any initial data $u_0 \in \mathcal{S}$ with*

$$\|u_0\|_{H^s_M} \ll 1\,,$$

*the unique solution $u \in C([0,1]; \mathcal{S})$ to* (1.1) *(focusing or defocusing) satisfies*

$$\|u\|_{L^\infty_{[0,1]} H^s_M} \le C \|u_0\|_{H^s_M}\,.$$

*Proof.* For $h \in [0,1]$ let $u_h$ be the global solution to (1.1) with initial data $u_{0h} = h u_0$. By Lemma 3.1 and the trilinear estimate (Prop. 4.1),

$$\|u_h\|_{X^s_M} \lesssim \|u_h\|_{\ell^2 L^\infty_{[0,1)} H^s_M} + \|u_h\|^3_{X^s_M}\,. \tag{6.1}$$

By the energy bound in Prop. 5.1 we have

$$\|u_h\|^2_{l^2 L^\infty_{[0,1)} H^s_M} \lesssim \|u_{h0}\|^2_{H^s_M} + \|u_h\|^4_{X^s_M} + \|u_h\|^6_{X^s_M}\,. \tag{6.2}$$

Combining (6.1) and (6.2), we obtain

$$\|u_h\|_{X^s_M} \le C(h \|u_0\|_{H^s_M} + \|u_h\|^3_{X^s_M})$$

Since $\|u_0\|_{H^s_M} \ll 1$ and $\|u_h\|_{X^s_M}$ is a continuous function of $h$ vanishing at $h = 0$, we conclude via a continuity argument that

$$\|u_h\|_{X^s_M} \lesssim h \|u_0\|_{H^s_M}, \qquad h \in [0,1]$$

Returning to (6.2), it follows that

$$\|u\|_{L^\infty H^s_M} \lesssim \|u_0\|_{H^s_M}$$

The proof is concluded. □

Given Proposition 6.1, we can conclude the proof of Theorem 1.1 using a scaling argument. Let $0 \geq s > -\frac{1}{8}$ and $u_0 \in H^s$ with $\|u_0\|_{H^s} \leq R$. Then we have

$$\|u_0\|_{H_M^{-\frac{1}{8}}} \leq RM^{-\frac{1}{8}-s}, \qquad M \geq 1$$

Let $u_{0\lambda}(x) = \lambda u_0(\lambda x)$ and $u_\lambda(x,t) = \lambda u(\lambda x, \lambda^3 t)$. Then $u_\lambda$ solves (1.1) with initial data $u_{0\lambda}$. We consider $u_\lambda$ on the time interval $[0,1)$, with $\lambda$ to be chosen below. We have

$$\|u_{\lambda 0}\|_{H_{\lambda M}^{-\frac{1}{8}}} \leq \lambda^{\frac{3}{8}} \|u_0\|_{H_M^{-\frac{1}{8}}} \leq \lambda^{\frac{3}{8}} RM^{-\frac{1}{8}-s}, \qquad M, \lambda M > 1$$

Taking $\lambda$ such that $\lambda^{\frac{3}{8}} RM^{-\frac{1}{8}-s} \ll 1$ we can apply Proposition 6.1 to conclude that

$$\|u_\lambda\|_{L^\infty_{[0,1]} H_{\lambda M}^{-\frac{1}{8}}} \lesssim \|u_{0\lambda}\|_{H_{\lambda M}^{-\frac{1}{8}}}.$$

Scaling back to the interval $[0,T]$ with $T = \lambda^3$ we obtain

$$\|u\|_{L^\infty_{[0,T]} H_M^{-\frac{1}{8}}} \lesssim \|u_0\|_{H_M^{-\frac{1}{8}}}, \qquad T^{\frac{1}{8}} RM^{-\frac{1}{8}-s} \ll 1$$

The last restriction gives a bound from below on $M$,

$$M \gg M(R,T) \stackrel{\text{def}}{=} (RT^{\frac{1}{8}})^{(\frac{1}{8}+s)^{-1}}$$

Taking a weighted square sum with respect to such $M$ in the previous relation we obtain

$$\|u\|_{L^\infty_{[0,T]} H^s_{M(R,T)}} \lesssim \|u_0\|_{H^s_{M(R,T)}}$$

This in turn shows that

$$u\|_{L^\infty_{[0,T]} H^s} \lesssim M(R,T)^{-s} \|u_0\|_{H^s}$$

concluding the proof of the theorem.

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