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Borel Graphs: Measurable Consequences of their Geometry and Complexity of Labeling Problems

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Borel Graphs: Measurable Consequences of their Geometry and Complexity of Labeling Problems

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Doctor of Philosophy in Mathematics

by

Alexander Sebastien Kastner

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ABSTRACT OF THE DISSERTATION

Borel Graphs: Measurable Consequences of their Geometry
and Complexity of Labeling Problems

by

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Doctor of Philosophy in Mathematics
University of California, Los Angeles, 2025
Professor Andrew Marks, Co-Chair
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This dissertation investigates Borel graphs on Polish spaces. It explores two main directions: the relationship between a Borel graph's geometry and its measurable properties, and the projective complexity of labeling problems in Borel combinatorics.

Chapter 2 contains joint work with Clark Lyons on the Baire measurable combinatorics of Borel graphs with non-amenable connected components. In this setting, we show that a Baire measurable perfect matching exists when the graph is vertex transitive, and that a Baire measurable balanced orientation exists when all degrees are even.

Chapter 3 presents a proof that Borel graphs of subexponential growth are measure hyperfinite, and includes a discussion of recent advances concerning hyperfiniteness under growth rate constraints.

Chapter 4 introduces a problem of Kechris and Chen about the σ-structurability of compressible countable Borel equivalence relations. We provide a proof in non-probabilistic language that a locally finite vertex transitive connected graph has a realization as a

probability-measure-preserving graph if and only if it is unimodular. We also generalize this result to the case of countable relational structures with compact stabilizers.

Chapter 5 explores the projective complexity of characterizing which Borel graphs admit Borel solutions to labeling problems. We introduce a formal notion of gadget reduction and use this notion to lift NP-completeness results in finite combinatorics to $\mathbf{\Sigma}^1_2$-completeness results for their Borel analogues. We then give several concrete examples where this idea is applied.

Chapter 6 contains joint work with Clark Lyons to provide a classical proof that for a Borel family of games, the set of games where player II wins is Baire measurable, universally measurable, and Ramsey measurable.

The dissertation of Alexander Sebastien Kastner is approved.

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2025

To my friends and family.

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VITA

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CHAPTER 1

Introduction

The study of Borel graphs on Polish spaces has blossomed in recent years. There have been two general directions to this study, with much overlap between them. The first direction is *descriptive combinatorics*, which explores definable solutions to combinatorial problems on Borel graphs. For example, we may ask whether a given Borel graph admits a proper coloring or a perfect matching which is Borel, measurable, or Baire measurable.

The second direction is their use in the study of *countable Borel equivalence relations (CBERs)*, which are Borel equivalence relations on Polish spaces whose equivalence classes are all countable. Borel equivalence relations originally arose in the study of the complexity of classification problems in mathematics, as well as in the study of Borel actions of (countable) groups on Polish spaces in ergodic theory. A fruitful approach for understanding a CBER E has been to study Borel graphs G whose connectedness relation is E. This approach has introduced many geometric arguments into the study of CBERs. A good example is the introduction and study of the Borel asymptotic dimension of Borel graphs (see [7]), which has led to progress on hyperfiniteness and Weiss' question.

This dissertation touches on both these directions, restricting to locally finite Borel graphs as is customary. On the one hand, Chapter 3 explores (measure) hyperfiniteness for slow-growing Borel graphs and Chapter 4 explores the question of when a Borel graph admits an invariant probability measure – both hyperfiniteness and the presence of an invariant probability measure are really properties of the induced CBER. On the other hand, Chapter 2 explores Baire measurable combinatorics and Chapter 5 explores the connection between

NP-complete decision problems for finite graphs (e.g. existence of proper colorings) and the projective complexity of the Borel version of those decision problems for Borel graphs (e.g. existence of Borel proper colorings).

1.1 From Graph Geometry to Measurable Properties

Independent of the division above, there is a common theme that permeates Chapters 2, 3, and 4, and much of the recent work on Borel graphs. Namely, we make various *geometric assumptions* about the countable graphs that form the connected components of a Borel graph G, and study the consequences of these assumptions for the CBER induced by G and the descriptive combinatorics of G. A countable, bounded degree graph is called **amenable** if for every $\varepsilon > 0$, there is a finite set F of vertices whose boundary ∂F satisfies $|\partial F| < \varepsilon|F|$. In Chapter 2, we show that Borel graphs with *non-amenable* connected components do admit Baire measurable solutions to certain combinatorial problems, such as having Baire measurable perfect matchings and Baire measurable balanced orientations.

Similarly, in Chapter 3, we present a proof that a Borel graph with a subexponential upper bound on the growth rate of balls (for the usual graph metric in each countable connected component) is hyperfinite after discarding a set of measure zero. This relates to a long-standing open problem posed by Weiss in [36]. Specifically: if a countable amenable group acts in a Borel way on a Polish space, is the orbit equivalence relation hyperfinite (that is, expressible as a countable increasing union of Borel equivalence relations with finite classes)? A special case of Weiss' problem, which is also open, asks whether the orbit equivalence relation of the Borel action of a group of subexponential growth is hyperfinite. This shifts the emphasis away from the algebraic structure of the group to focus instead on softer geometric properties. In fact, we can ask whether *all* Borel graphs of subexponential growth are hyperfinite. For instance, while Jackson, Kechris, and Louveau showed in [14] that groups of polynomial growth induce hyperfinite CBERs, Bernshteyn and Yu recently showed in [3] that *all* Borel graphs of polynomial growth are hyperfinite. In the same vein, a recent breakthrough result by Grebík, Marks, Rozhon, and Shinko (as yet unpublished) establishes that Borel graphs with r-balls of size at most $O(\exp(n^{0.15229}))$ are hyperfinite.

Finally, in Chapter 4, we study which locally finite, connected graphs can appear as the connected components of a Borel graph equipped with an invariant probability measure. This connects the geometric notion of unimodularity to the descriptive set theoretic condition of admitting an invariant probability measure. Motivation for this study comes in part from the following problem posed by Chen and Kechris, when the σ below is the Scott sentence for a locally finite graph.

Question 1 (Problem 9.12 in [6])**.** Find "natural" examples of $\mathscr{L}_{\omega_1,\omega}$-sentences σ such that all σ-structurable CBERs are compressible (that is, admit no invariant probability measure).

In the spirit of [6], we also generalize this study to characterize which countable relational structures one can assign in a Borel way to each equivalence class of a CBER admitting an invariant probability measure.

1.2 Complexity of Borel Labeling Problems

In Chapters 5 and 6, we shift our attention to the descriptive set theoretic complexity of characterizing which Borel graphs admit a Borel solution to a given labeling problem. For example, how hard is it to decide whether a given locally finite Borel graph admits a proper Borel 3-coloring? In a breakthrough result, Todorčević and Vidnyánszky showed in [33] that the set of (codes for) locally finite Borel graphs admitting a proper Borel 3-coloring is $\mathbf{\Sigma}^1_2$-complete. Assuming $\mathsf{P} \neq \mathsf{NP}$, Thornton generalized the Todorčević-Vidnyánszky result to show that NP-complete constraint satisfaction problems all have $\mathbf{\Sigma}^1_2$-complete Borel versions. Inspired by the Appendix in [32], we show in Chapter 5 how an appropriately formalized notion of "gadget reductions" allows one to establish the $\mathbf{\Sigma}^1_2$-completeness of the Borel version of all the usual NP-complete decision problems on finite graphs (and structured hypergraphs).

In [4], the authors generalize the Todorčević-Vidnyánszky in a different direction by showing that even the problem of deciding whether a d-regular acyclic graph admits a proper Borel d-coloring is $\mathbf{\Sigma}^1_2$-complete. They do this by generalizing the celebrated determinacy method of Marks (see [26]) and in the process the authors encounter certain Borel families of games. In order to ensure the measurability of the set of games where player II has a winning strategy, the authors initially appealed to certain metamathematical results involving weakly provable Δ^1_2 sets. In Chapter 6, we provide an alternative classical proof of the required measurability properties.

1.3 Background and Notation

For all undefined notions we refer to [16] and [34] for descriptive set theory, to [28] for descriptive combinatorics, and to [17] for countable Borel equivalence relations. Moreover, many definitions will be introduced or recalled in subsequent chapters, in proximity to their actual use.

The central objects of study in this dissertation are **Borel graphs** and **countable Borel equivalence relations (CBERs)** on Polish spaces. A graph on a vertex set $V(G)$ is usually identified with its set of edges $E(G) \subseteq V(G) \times V(G)$, where $E(G)$ is assumed to be irreflexive (no loops) and symmetric (no orientation). So a Borel graph has $V(G)$ a Polish space, and $E(G)$ a Borel subset of $V(G) \times V(G)$. We often restrict to Borel graphs that are locally finite/countable (i.e. all degrees are finite/countable) or have bounded degrees, in large part so that we can perform counting arguments with finite numbers.

As mentioned earlier, a CBER E on a Polish space X is a Borel subset $E \subseteq X \times X$ that is an equivalence relation and has countable classes. The CBER E is called **hyperfinite** if E is an increasing union of CBERs that all have finite classes. Hyperfinite CBERS form the simplest nontrivial class of CBERs in the Borel reducibility hierarchy (see [11]). A key tool in the study of CBERs and locally finite (or locally countable) graphs is the following theorem.

Theorem 2 (Lusin-Novikov, see Section 18.3 in [16]). *If X, Y are Polish spaces and $R \subseteq X \times Y$ is Borel with countable vertical sections, then there exist countably many partial Borel functions $f_n : X \to Y$ such that*

$$R = \bigcup_n \mathrm{graph}(f_n),$$

where $\mathrm{graph}(f_n) = \{(x, f(x)) : x \in \mathrm{dom}(f_n)\}$.

This theorem is so foundational to the theory of locally countable Borel graphs and CBERs that it is customary to use it without explicit mention in order to justify that constructions are Borel. One important application is that if G is a locally countable Borel graph on X, then the connectedness equivalence relation, E_G, is a (countable) Borel equivalence relation. If we instead considered a Borel graph with uncountable degrees, then the Lusin-Novikov theorem wouldn't apply and we could only say that E_G is an analytic equivalence relation. Obviously, every CBER E is the connectedness relation of some locally countable Borel graph G (e.g., trivially, $E(G) = E \setminus \{(x, x) : x \in X\}$). But actually we even have the following:

Theorem 3 (Theorem 3.12 in [14] and the following remark). *Every CBER is the connectedness relation of a locally finite Borel graph.*

This theorem gives additional motivation for the approach of studying CBERs by studying locally finite Borel graphs that induce them as connectedness relations, which was mentioned earlier on.

When working with connected graphs, we will very often invoke the graph metric $d(x, y)$, which is the shortest length for a path between x and y. We use $B_r(x)$ to denote the ball of radius r around x, that is, the set of all vertices y such that $d(x, y) \leq r$. So, if G is a Borel graph and $x \in V(G)$, then

$$B_r(x) = \{y \in [x]_{E_G} : d(x, y) \leq r\}.$$

A graph G is *vertex transitive* if there is an automorphism of the graph taking any vertex to any other vertex; G is called *quasi-transitive* if there are finitely many orbits for the action $\mathrm{Aut}(G) \curvearrowright G$.

If E is a CBER on X, then a probability Borel measure μ on X is called **E-invariant** if whenever $A, B \subseteq X$ are Borel, and $f : A \to B$ is an injective Borel function with $\mathrm{graph}(f) \subseteq E$, we have $\mu(B) = \mu(A)$. In this case, we also say E is a **probability-measure-preserving (pmp)** CBER on (X, μ). A locally countable Borel graph on (X, μ) called pmp if its connectedness relation E_G is pmp on (X, μ). The Lusin-Novikov theorem implies the following folkore result (a reference is [35], Proposition 5.3):

Theorem 4 (Mass Transport Principle characterization of pmp)**.** *If E is a CBER on X, then a Borel probability measure μ is invariant if and only if for all Borel $f : E \to [0, \infty]$ we have*

$$\int \sum_{y \in [x]_E} f(x, y)\, d\mu(x) = \int \sum_{y \in [x]_E} f(y, x)\, d\mu(x).$$

CHAPTER 2

Non-Amenability and Baire Measurable Combinatorics

The results of this chapter are part of joint work with Clark Lyons.

We say a connected graph G of bounded degree is **non-amenable** if there exists $\delta > 0$ such that whenever $F \subseteq V(G)$ is finite, the set of edges $E(F, V(G) \setminus F)$ between F and $V(G) \setminus F$ satisfies $|E(F, V(G) \setminus F)| \geq \delta|F|$. For example, the Cayley graphs of finitely generated non-amenable groups with respect to any finite symmetric generating set (not containing the identity) are non-amenable graphs. There are many other examples that are not Cayley graphs. For example, non-amenable quasi-transitive unimodular graphs have been studied in [22] and [2]. And some graphs that are not unimodular, such as the grandparent graph, are also covered by this definition.

In this paper, we consider non-amenable Borel graphs on Polish spaces, and prove that certain classical combinatorial problems can be solved Baire measurably, that is, on a Borel comeager invariant set. Our main theorem concerns the existence of perfect matchings:

Theorem 5. *Let G be a Borel graph such that each component is an infinite, bounded degree, non-amenable vertex transitive graph. Then G admits a Borel perfect matching on a Borel comeager invariant set.*

Corollary 6. *Every Schreier graph of a free Borel action of a finitely generated non-amenable group admits a Borel perfect matching on a Borel comeager invariant set.*

In [25], Marks and Unger studied Baire measurable matchings in the context of *bipartite* Borel graphs, with a view towards applications to Baire measurable equidecompositions.

Though not explicitly stated in their paper, their Theorem 1.3 implies that every bipartite Borel graph whose components are bounded degree, regular, and non-amenable has a Baire measurable perfect matching. Thus, our theorem can be viewed as an extension of their result to the non-bipartite setting. The existence of regular, quasi-transitive, non-amenable graphs without perfect matchings, (see Remark 22 of [2]) leads us to assume vertex transitivity, not just regularity.

Another theorem we prove concerns balanced orientations. Given a graph with only even degrees, a **balanced orientation** is an orientation of the edges so that every vertex has in-degree equal to out-degree. Euler's classical theorem about Euler circuits and a compactness argument shows that every locally finite graph with only even degrees admits a balanced orientation. For non-amenable bounded degree Borel graphs we have the following:

Theorem 7. *Let G be a bounded degree non-amenable Borel graph with only even degrees. Then G admits a Borel balanced orientation on a Borel comeager invariant set.*

Our arguments draw inspiration from the study of factor of i.i.d. combinatorial structures for Cayley graphs of non-amenable groups, or quasi-transitive unimodular non-amenable graphs more generally; see [21], [10], [2]. By studying the spectrum of the Markov operator associated with random walks on these graphs, one shows a measure expansion property for the associated Bernoulli graphings. This measure expansion property is then used to establish the existence of a combinatorial structure (say, a perfect matching or balanced orientation) for the Bernoulli graphing on a Borel conull invariant set. While our arguments generally differ from those used to obtain factors of i.i.d., several of the ideas we use were inspired from that setting. It seems likely that many of the results pertaining to factors of i.i.d. for non-amenable graphs will have Baire measurable analogues.

2.1 Perfect Matchings

The classical theorem of Tutte, repeated below, characterizes when a locally finite graph admits a perfect matching.

Theorem 8 (Tutte's theorem)**.** *A locally finite graph G admits a perfect matching if and only if whenever $X \subseteq V(G)$ is finite, the graph $G - X$ has at most $|X|$ many finite components of odd size.*

By **Tutte's condition** we will mean the condition that "$G - X$ has at most $|X|$ many odd components for each finite $X \subseteq V(G)$".

The proof of Theorem 5 consists in two steps. First, we establish a Baire measurable variant of Tutte's theorem which gives a sufficient condition for a locally finite Borel graph to admit a perfect matching on a Borel comeager invariant set (Theorem 10). Second, we show that non-amenable vertex transitive graphs satisfy this sufficient condition (Lemma 12).

Definition 9. If G is a locally finite graph and $X \subseteq G$ is finite, define

$$\mathcal{C}_{\text{fin}}(X) := \{\text{finite components of } G - X\}$$

and

$$\mathcal{C}_{\text{odd}}(X) := \{\text{odd components of } G - X\}.$$

Also let

$$\text{hull}_{\text{fin}}(X) := X \cup \bigcup \mathcal{C}_{\text{fin}}(X),$$

and

$$\text{hull}_{\text{odd}}(X) := X \cup \bigcup \mathcal{C}_{\text{odd}}(X).$$

We sometimes add superscripts to indicate the ambient graph when there is ambiguity.

Theorem 10. *Let G be a locally finite Borel graph on a Polish space $V(G)$, and suppose there exists $\varepsilon > 0$ such that for every finite set $X \subseteq V(G)$, we have*

$$|X| \geq |\mathcal{C}_{\text{odd}}(X)| + \varepsilon |\text{hull}_{\text{odd}}(X)|.$$

Then G admits a Borel perfect matching on a Borel comeager invariant set.

For the proof, we say a locally finite graph G satisfies $\text{Tutte}_{\varepsilon,k}$ if (i) Tutte's condition holds, and (ii) whenever $X \subseteq V(G)$ is finite such that $\text{hull}_{\text{odd}}(X)$ is connected and has size at least k,

$$|X| \geq |\mathcal{C}_{\text{odd}}(X)| + \varepsilon|\text{hull}_{\text{odd}}(X)|.$$

Observe that the condition in Theorem 10 is equivalent to $\text{Tutte}_{\varepsilon,1}$. This is an analogue of $\text{Hall}_{\varepsilon,k}$ in the proof of Theorem 1.3 in [25]. Our proof of Theorem 10 follows the same general strategy as the proof in [25]. In particular, we will need the following lemma from that paper.

Lemma 11. *Let G be a locally finite Borel graph on a Polish space $V(G)$, and let $f : \mathbb{N} \to \mathbb{N}$. Then there exist Borel sets $A_n \subseteq V(G)$, $n \in \mathbb{N}$, such that $\bigcup_n A_n$ is a Borel comeager invariant set and $d_G(x, y) > f(n)$ whenever x, y are distinct vertices in A_n.*

Proof of Theorem 10. Let $f : \mathbb{N} \to \mathbb{N}$ be a sufficiently fast-growing increasing function so that

1. $\sum_n \frac{4}{f(n)} < \varepsilon$;
2. letting $\varepsilon_n = \varepsilon - \sum_{m\leq n} \frac{4}{f(n)}$, we have $\varepsilon_{n-1}f(n) > 4$ for each n.

For convenience, we write $\varepsilon_{-1} = \varepsilon$. Let A_n be the Borel sets given by Lemma 11 for this f. Given a matching M, we write $G - M$ for the graph obtained from G by removing all the *vertices* covered by M (that is, $G - M$ is the induced subgraph on the set of vertices not covered by M). We define increasing Borel matchings M_n such that their union will be a perfect matching of the Borel comeager invariant set $\bigcup_n A_n$. We will ensure that M_n covers the vertices in A_n and $G - M_n$ satisfies $\text{Tutte}_{\varepsilon_n,f(n)}$. We can take M_{-1} to be the empty matching, and the hypothesis of the theorem implies that $G - M_{-1}$ satisfies $\text{Tutte}_{\varepsilon_{-1},1}$.

Assume M_{n-1} has been defined. For each vertex $x \in A_n \cap V(G - M_{n-1})$, let e_x be the least edge not in M_{n-1} such that $(G - M_{n-1}) - e_x$ satisfies Tutte's condition, equivalently such that $(G - M_{n-1}) - e_x$ admits a perfect matching. We know such an edge exists as the hypothesis that $\mathrm{Tutte}_{\varepsilon_{n-1}, f(n-1)}$ holds for $G - M_{n-1}$ implies in particular that Tutte's condition holds for $G - M_{n-1}$, hence $G - M_{n-1}$ has a perfect matching. If we pick an edge e_x that belongs to a perfect matching of $G - M_{n-1}$, then $(G - M_{n-1}) - e_x$ will still satisfy Tutte's condition. Since Tutte's condition quantifies over finite sets, the matching

$$M_n := M_{n-1} \cup \{e_x : x \in A_n \cap V(G - M_{n-1})\}$$

is Borel.

We verify that $G - M_n$ satisfies $\mathrm{Tutte}_{\varepsilon_n, f(n)}$. As a first step, we show that $G - M_n$ has no odd component (this is verifying Tutte's condition for $X = \emptyset$). Assume for contradiction that C is an odd component of $G - M_n$, and let X' denote the set of endpoints of edges $e_x \in M_n - M_{n-1}$ such that e_x is adjacent to C. Since $G - M_{n-1}$ had no odd component, $X' \neq \emptyset$ and $\mathrm{hull}_{\mathrm{odd}}^{G - M_{n-1}}(X')$ must be connected.

Case 1: Suppose $|X'| \geq 4$, so that there are at least two distinct edges $e_x \in M_n - M_{n-1}$ that are adjacent to C. Since $C \cup X'$ is connected and the vertices in X' corresponding to distinct edges are a distance of at least $f(n)$ from one another, we have

$$|\mathrm{hull}_{\mathrm{odd}}^{G - M_{n-1}}(X')| = |C \cup X'| \geq \frac{|X'|}{2} \cdot \frac{f(n)}{2} \geq \frac{f(n)}{4} |X'|.$$

In particular, $|\mathrm{hull}_{\mathrm{odd}}^{G - M_{n-1}}(X')| \geq f(n) \geq f(n-1)$. So, applying the inductive assumption of $\mathrm{Tutte}_{\varepsilon_{n-1}, f(n-1)}$ to $G - M_{n-1}$ and X', we obtain

$$|X'| \geq \varepsilon_{n-1} |\mathrm{hull}_{\mathrm{odd}}^{G - M_{n-1}}(X')| + |\mathcal{C}_{\mathrm{odd}}^{G - M_{n-1}}(X')| \geq \varepsilon_{n-1} \frac{f(n)}{4} |X'|$$

Since f was chosen so that $\varepsilon_{n-1} f(n) > 4$, this is impossible.

Case 2: Suppose $|X'| = 2$, so that there is a single edge $e_x \in M_n - M_{n-1}$ that is adjacent to C. But this case is impossible as we chose e_x specifically so that $M_{n-1} \cup \{e_x\}$ extends to a perfect matching, so the appearance of the odd component C in $G - M_n$ cannot only be due to e_x.

So far we have proved that $G - M_n$ has no odd component. Let $X \subseteq V(G - M_n)$ be a finite set such that $\mathrm{hull}_{\mathrm{odd}}(X)$ is connected. Let E_X be the set of edges $e_x \in M_n - M_{n-1}$ such that at least one of the endpoints of e_x is adjacent to $\mathrm{hull}_{\mathrm{odd}}^{G-M_n}(X)$ in G.

<u>Case 1</u>: Suppose that $|E_X| \geq 2$. Since $\mathrm{hull}_{\mathrm{odd}}^{G-M_n}(X)$ is connected and distinct edges in E_x are a distance of at least $f(n)$ from one another, we have

$$|\mathrm{hull}_{\mathrm{odd}}^{G-M_n}(X)| \geq |E_x| \frac{f(n)}{2}.$$

In particular, $|\mathrm{hull}_{\mathrm{odd}}^{G-M_{n-1}}(X')| \geq f(n) \geq f(n-1)$. So, applying the inductive assumption of $\mathrm{Tutte}_{\varepsilon_{n-1}, f(n-1)}$ to $G - M_{n-1}$ and

$$X' = X \cup \{v \in V(G) : v \text{ is an endpoint of some } e \text{ in } E_X\},$$

yields

$$|X'| \geq |\mathcal{C}_{\mathrm{odd}}^{G-M_{n-1}}(X')| + \varepsilon_{n-1} |\mathrm{hull}_{\mathrm{odd}}^{G-M_{n-1}}(X')|.$$

Therefore

$$\begin{aligned}
|\mathcal{C}_{\mathrm{odd}}^{G-M_n}(X)| &= |\mathcal{C}_{\mathrm{odd}}^{G-M_{n-1}}(X')| \\
&\leq |X'| - \varepsilon_{n-1} |\mathrm{hull}_{\mathrm{odd}}^{G-M_{n-1}}(X')| \\
&\leq |X'| - \varepsilon_{n-1} |\mathrm{hull}_{\mathrm{odd}}^{G-M_n}(X)| \\
&= |X| + 2|E_X| - \varepsilon_{n-1} |\mathrm{hull}_{\mathrm{odd}}^{G-M_n}(X)| \\
&\leq |X| + \frac{4}{f(n)} |\mathrm{hull}_{\mathrm{odd}}^{G-M_n}(X)| - \varepsilon_{n-1} |\mathrm{hull}_{\mathrm{odd}}^{G-M_n}(X)| \\
&= |X| - \varepsilon_n |\mathrm{hull}_{\mathrm{odd}}^{G-M_n}(X)|.
\end{aligned}$$

<u>Case 2</u>: Suppose that $|E_X| \leq 1$. If E_X is empty, then the fact that X does not violate $\mathrm{Tutte}_{\varepsilon_n, f(n)}$ simply follows from the fact that $G - M_{n-1}$ satisfies the (stronger) $\mathrm{Tutte}_{\varepsilon_{n-1}, f(n-1)}$. So suppose that E_X consists of a single edge e_x, for some $x \in A_n \cap V(G - M_{n-1})$. We chose e_x so that Tutte's condition holds for $(G - M_{n-1}) - e_x$, so in particular

$$|\mathcal{C}_{\mathrm{odd}}^{G-M_{n-1}-e_x}(X)| \leq |X|.$$

But e_x is the only edge adjacent to $\text{hull}_{\text{odd}}^{G-M_n}(X)$ in $G - M_n$, so the odd components of $(G - M_{n-1} - e_x) - X$ are precisely the same as the odd components of $(G - M_n) - X$. Hence, X does not violate Tutte's condition in $G - M_n$. Suppose now that $|\text{hull}_{\text{odd}}^{G-M_n}(X)| \geq f(n) \geq f(n-1)$, and as in Case 1 let

$$X' = X \cup \{v \in V(G) : v \text{ is an endpoint of some } e \text{ in } E_X\}.$$

Applying $\text{Tutte}_{\varepsilon_{n-1}, f(n-1)}$ to $G - M_{n-1}$ and X' yields

$$\begin{aligned}
|\mathcal{C}_{\text{odd}}^{G-M_n}(X)| &= |\mathcal{C}_{\text{odd}}^{G-M_{n-1}}(X')| \\
&\leq |X'| - \varepsilon_{n-1}|\text{hull}_{\text{odd}}^{G-M_{n-1}}(X')| \\
&\leq |X| + 2 - \varepsilon_{n-1}|\text{hull}_{\text{odd}}^{G-M_n}(X)| \\
&\leq |X| + \frac{2}{f(n)}|\text{hull}_{\text{odd}}^{G-M_n}(X)| - \varepsilon_{n-1}|\text{hull}_{\text{odd}}^{G-M_n}(X)| \\
&\leq |X| + \varepsilon_n|\text{hull}_{\text{odd}}^{G-M_n}(X)|.
\end{aligned}$$

So X does not violate $\text{Tutte}_{\varepsilon_n, f(n)}$ in Case 2 either. □

Next, we show that non-amenable vertex transitive graphs satisfy the condition in Theorem 10.

Lemma 12. *Let G be an infinite, connected, locally finite, non-amenable, vertex transitive graph. Then there exists $\varepsilon > 0$ such that for all finite $X \subseteq V(G)$,*

$$|X| \geq |\mathcal{C}_{\text{fin}}(X)| + \varepsilon|\text{hull}_{\text{fin}}(X)|.$$

In particular, there exists $\varepsilon > 0$ such that for all finite $X \subseteq V(G)$,

$$|X| \geq |\mathcal{C}_{\text{odd}}(X)| + \varepsilon|\text{hull}_{\text{odd}}(X)|.$$

Proof. Fix a finite set $X \subseteq V(G)$. By Lemma 2.3 of [10], the assumption that G is a (connected, infinite) d-regular, vertex transitive graph implies that each element of $\mathcal{C}_{\text{fin}}(X)$ has at least d many edges in its boundary. And so

$$\left|E\left(X, \bigcup \mathcal{C}_{\text{fin}}(X)\right)\right| = \sum_{F \in \mathcal{C}_{\text{fin}}(X)} \left|E(X, F)\right| \geq d|\mathcal{C}_{\text{fin}}(X)|.$$

Also by the expansion property

$$\left|E(X, V(G) \setminus \text{hull}_{\text{fin}}(X))\right| \geq \delta|\text{hull}_{\text{fin}}(X)|,$$

where δ is the expansion constant of the graph. Therefore

$$d|X| \geq \left|E\left(X, \bigcup \mathcal{C}_{\text{fin}}(X)\right)\right| + \left|E\Big(X, V(G) \setminus \text{hull}_{\text{fin}}(X)\Big)\right| \geq d|\mathcal{C}_{\text{fin}}(X)| + \delta|\text{hull}_{\text{fin}}(X)|.$$

And so

$$|X| \geq |\mathcal{C}_{\text{fin}}(X)| + \varepsilon|\text{hull}_{\text{fin}}(X)|,$$

where $\varepsilon = \frac{\delta}{d}$. □

As discussed earlier, combining Theorem 10 and Lemma 12 immediately yields Theorem 5.

2.2 Balanced Orientations

The proof of Theorem 7 is quite straightforward and is an adaptation of the ideas in Section 5 of [2]. Given any graph G with only even degrees, we define an auxiliary bipartite graph G^* such that perfect matchings of G^* induce balanced orientations of G. The following definition is taken essentially verbatim from [2] and is repeated here for the convenience of the reader.

Definition 13. Let G be a graph with only even degrees. The graph G^* has a vertex for every edge e of G and $\deg(v)/2$ many vertices for every vertex v of G, i.e.

$$V(G^*) = \{x_e : e \in E(G)\} \cup \{v_i : v \in V(G), i \in [\deg(v)/2]\}.$$

Then every vertex corresponding to a former edge is joined to all copies of its former endpoints:

$$E(G^*) = \{x_{uv}v_i : uv \in E(G), i \in [\deg(v)/2]\}.$$

Observe that any perfect matching of G^* induces a balanced orientation of G by orienting an edge $e \in E(G)$ toward its endpoint v if and only if x_e and v_i are matched in G^* for some

$i \in [\deg(v)/2]$. In the case when G is a Borel graph, it is also straightforward to put an appropriate Polish topology on $V(G^*)$ so that Baire measurable perfect matchings of G^* yield Baire measurable balanced orientations of G. In order to show that the Borel bipartite graph G^* has a Baire measurable perfect matching, we will apply Theorem 1.3 of [25] (which is an analogue of our Theorem 5 in the bipartite setting).

Theorem 14 (Theorem 1.3 of [25])**.** *Let G be a locally finite bipartite Borel graph with bipartition $V(G) = B_0 \sqcup B_1$ (the sets B_0 and B_1 need not be Borel). Suppose there exists $\varepsilon > 0$ such that whenever F is a finite set contained in either B_0 or B_1, we have*

$$|N(F)| \geq (1+\varepsilon)|F|.$$

Then G admits a Borel perfect matching on a Borel comeager invariant set.

Proof of Theorem 7. The proof is an adaptation of the proof of Lemma 25 from [2]. Write $\pi : V(G^*) \to V(G) \cup E(G)$ for the projection function. Let $\delta > 0$ be the expansion constant for the non-amenable graph G, and let d be a bound on the degrees. Suppose that $F \subseteq V(G^*)$ is a finite set of vertex-type vertices. Then

$$\begin{aligned}
|N_{G^*}(F)| &= \frac{1}{2}\sum_{u\in\pi(F)} \deg(u) + \frac{1}{2}|E(\pi(F), V(G) \setminus \pi(F))| \\
&\geq \sum_{u\in\pi(F)} \frac{\deg(u)}{2} + \frac{\delta}{2}|\pi(F)| \\
&\geq |F| + \frac{\delta}{d}|F|.
\end{aligned}$$

Now suppose that $F \subseteq V(G^*)$ is a finite set of edge-type vertices (that is, $F \subseteq E(G)$), and let S denote the set of vertices $u \in V(G)$ which are incident to some edge $e \in F$. Then

$$\begin{aligned}
|N_{G^*}(F) &= \sum_{u \in S} |\pi^{-1}(u)| \\
&= \sum_{u \in S} \frac{\deg(u)}{2} \\
&= |E(S,S)| + \frac{1}{2}|E(S, V(G) \setminus S)| \\
&\geq |F| + \frac{\delta}{2}|S| \\
&\geq |F| + \frac{\delta}{2d}|F|.
\end{aligned}$$

If we choose $\varepsilon > 0$ such that $\varepsilon < \frac{\delta}{2d}$, then the hypotheses for Theorem 14 hold. So G^* has a Baire measurable perfect matching, and this implies that G has a Baire measurable balanced orientation. $\square$

CHAPTER 3

Subexponential Growth and Hyperfiniteness

We say a locally finite graph G has **subexponential growth** if for all $x \in V(G)$ and $\varepsilon > 0$,

$$|B_r(x)| = o((1+\varepsilon)^r).$$

Note that we don't assume G is *uniformly* subexponential, i.e. there may not exist a subexponential function $h(r)$ such that $|B_r(x)| \leq h(r)$ for all $x \in V(G)$ and $r > 0$. In this short chapter, we show that all Borel graphs of subexponential growth on a standard probability space (X, μ) are μ-hyperfinite.

We do not assume μ is invariant. And we do not assume the graph is vertex transitive; in fact the main difficulty is proving the result even for graphs that may be highly non-uniform. In [14], Jackson, Kechris, and Louveau proved that the orbit equivalence relations induced by Borel actions of groups of polynomial growth are hyperfinite. Although not explicitly stated, their argument also works for vertex transitive Borel graphs of polynomial growth, and even for Borel graphs admitting $C > 0$ and d such that for all $r \geq 1$,

$$\frac{1}{C} r^d \leq |B_r(x)| \leq C r^d.$$

However we would like results where only an *upper* bound is assumed on the growth rate of balls. Indeed, in the study of Borel graphs and CBERs, one often restricts to Borel subgraphs or sub-equivalence relations. When we do this, we might only get an upper bound on the growth rate, not a lower bound.

In [3], Bernshteyn and Yu proved that *all* Borel graphs of polynomial growth are hyperfinite by adapting a randomized local construction from [19] using the Borel Lovász Local Lemma for Borel graphs of subexponential growth from [9]. Recently, a deterministic "ball-carving" construction that avoids using a Borel LLL was given by Grebík, Marks, Rozhon, and Shinko in [24]. The authors also show that Borel graphs with r-balls of size at most $O(\exp(n^{0.15229}))$ are hyperfinite.

Conjecture 15. All Borel graphs of subexponential growth are hyperfinite.

Although phrased in a different context, work by Tessera ([31], Proposition 3.5) implies that this conjecture holds up to discarding a null set. We provide a somewhat different proof for Tessera's result, discovered independently. This proof was inspired by a suggestion from Felix Weilacher.

We follow the terminology from [23].

Definition 16. A CBER E on X is **Borel amenable** if there are Borel functions $\lambda^n : E \to [0, \infty)$ such that

1. for each $x \in X$, $\sum_{z \in [x]_E} \lambda^n(x, z) = 1$;
2. for each $(x, y) \in E$, $\sum_{z \in [x]_E} |\lambda^n(x, z) - \lambda^n(y, z)| \to 0$ as $n \to \infty$.

Note that if E is the connectedness relation of a (locally countable) Borel graph, then it suffices to check condition (2) whenever x, y are adjacent. For fixed x and n, we will call the function $\lambda^n(x, \cdot)$ a **Reiter function**.

Remark 17. It's important to distinguish between a Borel graph having Borel amenable connectedness relation and having connected components that are all amenable graphs (in the sense of Chapter 2).

Theorem 18 (Tessera, Proposition 3.5 in [31])**.** *If G is a Borel graph of subexponential growth on a standard Borel space X, then E_G is Borel amenable.*

Corollary 19. *If G is a Borel graph of subexponential growth on a standard probability space (X, μ), then G is μ-hyperfinite.*

Proof. This follows from the Connes-Feldman-Weiss theorem, which says that an amenable CBER is μ-hyperfinite (even when μ is not necessarily invariant). See [23] for more on the Connes-Feldman-Weiss theorem. □

Proof of Theorem 18. The Reiter functions we use to prove a Borel graph of subexponential growth is amenable are similar to the probability measure on a countable subexponential graph used by Conley and Tamuz in [8] to prove all Borel graphs of subexponential growth have Borel unfriendly colorings. For fixed $x \in X$ and $\varepsilon > 0$, the subexponential growth assumption implies that $K(x, \varepsilon) := \sum_{z \in [x]_G} (1-\varepsilon)^{d(x,z)}$ converges (this is the only place where the subexponential growth assumption is used). Define

$$\lambda^\varepsilon(x, z) := \frac{(1-\varepsilon)^{d(x,z)}}{K(x, \varepsilon)}.$$

We need to check that for a fixed pair x, y of adjacent vertices,

$$\sum_{z \in [x]_G} \left| \frac{(1-\varepsilon)^{d(x,z)}}{K(x,\varepsilon)} - \frac{(1-\varepsilon)^{d(y,z)}}{K(y,\varepsilon)} \right| \to 0, \quad \text{as } \varepsilon \to 0.$$

Writing $K(x)$ and $K(y)$ instead of $K(x, \varepsilon)$ and $K(y, \varepsilon)$, we have

$$\begin{aligned}
&\sum_{z \in [x]_G} \left| \frac{(1-\varepsilon)^{d(x,z)}}{K(x)} - \frac{(1-\varepsilon)^{d(y,z)}}{K(y)} \right| \\
&= \frac{1}{K(x)K(y)} \sum_{z \in [x]_G} \left| K(y)(1-\varepsilon)^{d(x,z)} - K(x)(1-\varepsilon)^{d(y,z)} \right| \\
&= \frac{1}{K(x)K(y)} \sum_{z \in [x]_G} \left| (K(y) - K(x))(1-\varepsilon)^{d(x,z)} + K(x)\left((1-\varepsilon)^{d(x,z)} - (1-\varepsilon)^{d(y,z)}\right) \right| \\
&\leq \frac{1}{K(x)K(y)} \left(|K(x) - K(y)|\, K(x) + K(x) \sum_{z \in [x]_G} \varepsilon(1-\varepsilon)^{d(x,z)} + K(x) \sum_{z \in [x]_G} \varepsilon(1-\varepsilon)^{d(y,z)} \right) \\
&= \left| \frac{K(x)}{K(y)} - 1 \right| + \varepsilon \cdot \frac{K(x)}{K(y)} + \varepsilon.
\end{aligned}$$

To justify the inequality step above, observe that if $d(y, z) = d(x, z) + 1$, then

$$(1-\varepsilon)^{d(x,z)} - (1-\varepsilon)^{d(y,z)} = (1-\varepsilon)^{d(x,z)}(1 - (1-\varepsilon)) = \varepsilon(1-\varepsilon)^{d(x,z)},$$

and something similar happens when $d(x,z) = d(y,z) + 1$. It remains to show that

$$\frac{K(x)}{K(y)} = \frac{\sum_{z \in [x]_G} (1-\varepsilon)^{d(x,z)}}{\sum_{z \in [x]_G} (1-\varepsilon)^{d(y,z)}}$$

is close to 1 when ε is small. But this follows from the following inequalities:

$$1 - \varepsilon = \frac{\sum_{z \in [x]_G} (1-\varepsilon)^{d(y,z)+1}}{\sum_{z \in [x]_G} (1-\varepsilon)^{d(y,z)}} \leq \frac{K(x)}{K(y)} \leq \frac{\sum_{z \in [x]_G} (1-\varepsilon)^{d(x,z)}}{\sum_{z \in [x]_G} (1-\varepsilon)^{d(x,z)+1}} = \frac{1}{1-\varepsilon}.$$

□

CHAPTER 4

Measure-Preserving Realizations of Graphs

Many results in measurable combinatorics only hold in the presence of an invariant probability measure. If G is a countable graph, we say that a pmp Borel graph H is a **pmp realization** of G if every connected component of H is isomorphic to G. In this chapter, we explore the question of which countable graphs admit pmp realizations, which connects to the theory of unimodular graphs (see [22] and [20]). Although some of this material is discussed in probability contexts (see Chapter 18 in [20] for the case of vertex transitive graphs), we reframe the presentation in descriptive set theoretic terms.

This topic also connects to the theory of *structurable CBERs*, as introduced by Jackson, Kechris, and Louveau in [14] (Definition 2.17) and further developed by Chen and Kechris in [6]. If σ is an $\mathscr{L}_{\omega_1,\omega}$-sentence, we say a CBER E is **σ-structurable** if there is a Borel way of putting a countable model of σ on every E-class. Chen and Kechris pose the following question:

Question 20 (Problem 9.12 in [6])**.** Find "natural" examples of σ such that all σ-structurable CBERs are compressible (that is, admit no invariant probability measure).

We answer this question in the case where σ is the Scott sentence of a locally finite graph, or more generally where σ is the Scott sentence of a relational structure whose automorphism group has compact stabilizers.

4.1 Vertex Transitive Graphs

To simplify the presentation, we start by considering vertex transitive graphs. In the next section, we generalize the discussion to relational structures with compact stabilizers. Recall that if H is a graph, $E(H)$ denotes the edge set of H and E_H denotes the connectedness relation of H.

Definition 21 (see [22], Section 8.2)**.** A vertex transitive, locally finite, connected graph G is **unimodular** if and only if it satisfies the **Mass Transport Principle**, namely whenever $F : V(G) \times V(G) \to [0, \infty]$ is $\mathrm{Aut}(G)$-invariant, we have

$$\sum_{v \in V(G)} F(o, v) = \sum_{v \in V(G)} F(v, o)$$

for some (equivalently, all) $o \in V(G)$.

Intuitively, the Mass Transport Principle says that if F is defined in an $\mathrm{Aut}(G)$-invariant way, then the F-flow leaving a vertex equals the F-flow coming into a vertex. Recall the following characterization of a pmp Borel graph:

Proposition 22 (Mass Transport Principle characterization of pmp)**.** Let E be a CBER on a standard probability space (X, μ). Then E is pmp if and only if for all measurable $f : E \to [0, \infty]$, we have

$$\int_x \sum_{y \in [x]_E} f(x, y)\, d\mu(x) = \int_x \sum_{y \in [x]_E} f(y, x)\, d\mu(x).$$

In view of this proposition, the following result should not be too surprising. The result also appears in Section 18.3 of [20] using probabilistic language.

Theorem 23. *Let G be a vertex transitive, locally finite, connected graph. Then G has a pmp realization if and only if G is unimodular.*

Remark 24. Since locally finite, vertex transitive, amenable graphs (in the sense of Chapter 2) are unimodular (see Proposition 8.14 in [22]), all such graphs have pmp realizations.

Proof of the forward direction of Theorem 23. Suppose H is a pmp realization of G. To verify that G satisfies the Mass Transport Principle, fix an automorphism invariant function $F : V(G) \times V(G) \to [0, \infty]$, and $o \in V(G)$. Define $f : E_H \to [0, \infty]$ by

$$f(x, y) := F(o, \varphi(y)),$$

where φ is an isomorphism from $H \upharpoonright [x]_H$ to G satisfying $\varphi(x) = o$. By automorphism invariance of F, the value $F(o, \varphi(y))$ does not depend on φ. The locally finite assumption on G implies that there is an automorphism taking one vertex of G to another if and only if the r-balls around each vertex are all isomorphic; hence, f is a Borel function. By the Mass Transport Characterization of a pmp equivalence relation, we obtain

$$\int_x \sum_{y \in [x]_H} f(x, y)\, d\mu(x) = \int_x \sum_{y \in [x]_H} f(y, x)\, d\mu(x).$$

But the integrands, $\sum_{y \in [x]_H} f(x, y)$ and $\sum_{y \in [x]_H} f(y, x)$, are actually constant functions of x. So for any fixed $x \in V(H)$ we have

$$\sum_{y \in [x]_H} f(x, y) = \sum_{y \in [x]_H} f(y, x),$$

which implies

$$\sum_{v \in V(G)} F(o, v) = \sum_{v \in V(G)} F(v, o).$$

This verifies the Mass Transport Principle for G, so we conclude that G must be unimodular.

□

We next introduce Bernoulli graphings using non-probabilistic language, with the goal of proving that the Bernoulli graphing of a (vertex transitive, locally finite, connected) unimodular graph is pmp.

Let G be an arbitrary vertex transitive, locally finite, connected graph, with a distinguished vertex/root $o \in V(G)$. Define

$$X := \{x \in [0, 1]^{V(G)} : x \text{ injective}\},$$

which is a compact space. We consider the free continuous action $\mathrm{Aut}(G) \curvearrowright X$ defined by shifting the labels. More explicitly, if $\varphi \in \mathrm{Aut}(G)$ and $x \in [0,1]^{V(G)}$, the action is defined by

$$(\varphi \cdot x)(g) := x(\varphi^{-1}(g)).$$

Using the assumption that G is locally finite, we see that the group $\mathrm{Aut}(G)$ is locally compact, while the stabilizer $\mathrm{Aut}_o(G)$ is compact. It follows that the orbit equivalence relation of $\mathrm{Aut}_o(G)$ is a compact subset of $X \times X$, and so has a Borel transversal $\tilde{X}$ (see [16]). (In this case, one can easily describe a continuous selector function s for the orbit equivalence relation of the action of $\mathrm{Aut}_o(G)$ directly. Fix an enumeration $V(G) = \{o, v_1, v_2, \dots\}$. Given $x \in X$, let $s(x)(o) = x(o)$, let $s(x)(v_1)$ be the least label possible given $s(x)(o) = x(o)$, let $s(x)(v_2)$ be the least label possible given the previous choices for $s(x)(o)$ and $s(x)(v_1)$, and so on. This gives a compact transversal.)

The quotient space $\hat{X} = X/\mathrm{Aut}_o(G)$, which can be identified with the compact transversal $\tilde{X}$, is a standard Borel space. We define an edge relation on $\hat{X}$ by

$$[x], [y] \text{ adjacent} \quad \Leftrightarrow_{\text{def}} \quad (\exists \varphi \in \mathrm{Aut}(G))[\varphi \cdot x = y \text{ and } \varphi \text{ moves } o \text{ to one of its neighbors}].$$

Using the locally finite assumption of G again, it is not hard to check that this defines a Borel graph on $\hat{X}$. We denote this graph by $\hat{G}$, and call it the **Bernoulli graphing** of G. It is also not hard to see that each component of $\hat{G}$ is isomorphic to G, and we omit the proof. Intuitively, the neighbors of $[x]$ are just obtained by "moving the root to a neighbor of the root", and the vertices in the connected component of $[x]$ in $\hat{G}$ are obtained from the labeling x by just designating other vertices $v \in V(G)$ as the root.

Let μ denote the usual product measure on $[0,1]^{V(G)}$, so that $\mu(X) = 1$. The transversal $\tilde{X}$ satisfies $\mu(\tilde{X}) = 0$. However, we can define a probability measure $\hat{\mu}$ on $\hat{X}$ by

$$\hat{\mu}(A) := \mu\left(\bigcup A\right).$$

We now prove the harder direction of Theorem 23. The proof was sketched for me by László Tóth at the 2023 Workshop on Measurable Combinatorics at the Fields Institute.

Proposition 25. Let G be a vertex transitive, locally finite, connected, unimodular graph. Then $\hat{\mu}$ is an invariant measure for the Bernoulli graphing $\hat{G}$.

Proof. For each $v \in V(G)$, fix an automorphism $\varphi_v \in \mathrm{Aut}(G)$ satisfying $\varphi_v(v) = o$. Let $f : E_{\hat{G}} \to [0, \infty]$ be a measurable function. We want to verify the Mass Transport Principle characterization of pmp for f, namely show that

$$\int_{\hat{x} \in \hat{X}} \sum_{\hat{y} \in [\hat{x}]_{\hat{G}}} f(\hat{x}, \hat{y})\, d\hat{\mu}(\hat{x}) = \int_{\hat{x} \in \hat{X}} \sum_{\hat{y} \in [\hat{x}]_{\hat{G}}} f(\hat{y}, \hat{x})\, d\hat{\mu}(\hat{x}).$$

By definition of $\hat{\mu}$, this is the same as

$$\int_{x \in X} \sum_{\hat{y} \in [[x]_{\mathrm{Aut}_o(G)}]_{\hat{G}}} f([x]_{\mathrm{Aut}_o(G)}, \hat{y})\, d\mu(x) = \int_{x \in X} \sum_{\hat{y} \in [[x]_{\mathrm{Aut}_o(G)}]_{\hat{G}}} f(\hat{y}, [x]_{\mathrm{Aut}_o(G)})\, d\mu(x).$$

This, in turn, is equivalent to

$$\int_{x \in X} \sum_{v \in V(G)} f([x]_{\mathrm{Aut}_o(G)}, [\varphi_v \cdot x]_{\mathrm{Aut}_o(G)})\, d\mu(x) = \int_{x \in X} \sum_{v \in V(G)} f([\varphi_v \cdot x]_{\mathrm{Aut}_o(G)}, [x]_{\mathrm{Aut}_o(G)})\, d\mu(x).$$

Now that the infinite sum does not depend on x, we can interchange the integral and the sum. So the above is equivalent to

$$\sum_{v \in V(G)} \int_{x \in X} f([x]_{\mathrm{Aut}_o(G)}, [\varphi_v \cdot x]_{\mathrm{Aut}_o(G)})\, d\mu(x) = \sum_{v \in V(G)} \int_{x \in X} f([\varphi_v \cdot x]_{\mathrm{Aut}_o(G)}, [x]_{\mathrm{Aut}_o(G)})\, d\mu(x). \tag{4.1}$$

Define $F : V(G) \times V(G) \to [0, \infty]$ by

$$F(u, w) := \int_{x \in X} f([\varphi_u \cdot x]_{\mathrm{Aut}_o(G)}, [\varphi_w \cdot x]_{\mathrm{Aut}_o(G)})\, d\mu(x).$$

Note that for any fixed labeling $x \in X$ and any fixed $u \in V(G)$, the equivalence class $[\varphi_u \cdot x]_{\mathrm{Aut}_o(G)}$ will be the same for any $\varphi_u \in \mathrm{Aut}(G)$ such that $\varphi_u(u) = o$. In particular, $F(u, w)$ does not depend on our choice of φ_u and φ_w at the start of the proof. We claim that

F is automorphism invariant. Fix $\psi \in \mathrm{Aut}(G)$. Then

$$\begin{aligned} F(\psi(u), \psi(w)) &= \int_{x \in X} f([\varphi_{\psi(u)} \cdot x]_{\mathrm{Aut}_o(G)}, [\varphi_{\psi(w)} \cdot x]_{\mathrm{Aut}_o(G)})\, d\mu(x) \\ &= \int_{x \in X} f([\varphi_u \cdot (\psi^{-1} \cdot x)]_{\mathrm{Aut}_o(G)}, [\varphi_w \cdot (\psi^{-1} \cdot x)]_{\mathrm{Aut}_o(G)})\, d\mu(x) \\ &= \int_{x \in X} f([\varphi_u \cdot x]_{\mathrm{Aut}_o(G)}, [\varphi_w \cdot x]_{\mathrm{Aut}_o(G)})\, d\mu(x) \\ &= F(u, w), \end{aligned}$$

where we used that the action of $\mathrm{Aut}(G)$ on X preserves the measure μ. Hence, by the unimodularity of G, we get

$$\sum_{v \in V(G)} F(o, v) = \sum_{v \in V(G)} F(v, o).$$

But

$$\begin{aligned} F(o, v) &= \int_{x \in X} f([x]_{\mathrm{Aut}_o(G)}, [\varphi_v \cdot x]_{\mathrm{Aut}_o(G)})\, d\mu(x) \\ F(v, o) &= \int_{x \in X} f([\varphi_v \cdot x]_{\mathrm{Aut}_o(G)}, [x]_{\mathrm{Aut}_o(G)})\, d\mu(x), \end{aligned}$$

so we deduce that (4.1) holds. □

4.2 Structures with Compact Stabilizers

In this section, we generalize the results from the previous section to relational structures M such that the stabilizers of elements are compact in $\mathrm{Aut}(M)$. Let $\mathcal{L}$ be a countable relational signature, and write $S(x)$ for the stabilizer of $x \in M$. Note that M has compact stabilizers if and only if $|S(x)y|$ is finite for all $x, y \in M$.

Recall that two elements $x, y \in M$ have the same **type** if and only if there is an automorphism $\varphi \in \mathrm{Aut}(M)$ such that $\varphi(x) = y$.

Definition 26. A countable $\mathcal{L}$-structure M with compact stabilizers is **unimodular** if and only if

$$|S(x)y| = |S(y)x|$$

whenever $x, y \in M$ have the same type.

One can show this definition agrees with the previous definition for vertex transitive graphs in terms of the Mass Transport Principle (see Theorem 30 and the arguments in Section 8.2 in [22] for graphs which generalize to this setting too).

Proposition 27. Let E be a pmp CBER on (X, μ) with a Borel assignment of an $\mathcal{L}$-structure with compact stabilizers to each E-class. Then for μ-a.e. E-class, the $\mathcal{L}$-structure assigned to that E-class is unimodular.

Proof. For each $k, \ell \in \mathbb{N}$, consider the Borel set

$$A_{k,\ell} := \{(x, y) \in E : x \text{ and } y \text{ have the same type, } |S(x)y| = k, \text{ and } |S(y)x| = \ell\}.$$

Note that $\pi_1[A_{k,\ell}] = \pi_2[A_{k,\ell}]$, and this is a Borel set, which we may call $B_{k,\ell}$. Since E is pmp, we have

$$\int_X \sum_{y \in [x]_H} 1_{A_{k,\ell}}(x, y)\, d\mu(x) = \int_X \sum_{y \in [x]_H} 1_{A_{k,\ell}}(y, x)\, d\mu(x),$$

which can be rewritten as

$$\int_{B_{k,\ell}} k\, d\mu(x) = \int_{B_{k,\ell}} \ell\, d\mu(x).$$

Thus, if $\mu(B_{k,\ell}) > 0$, then we must have $k = \ell$. This proves that μ-a.e. E-class is assigned a unimodular $\mathcal{L}$-structure. $\square$

We next quote two results from Section 8.2 in [22] adapted to our setting. The proofs are analogous to the proofs for graphs.

Theorem 28 (Theorem 8.10 in [22])**.** *Let M be a countable $\mathcal{L}$-structure with compact stabilizers. Then there exist positive numbers μ_x for $x \in M$ such that for all $x, y \in M$,*

$$\frac{\mu_x}{\mu_y} = \frac{|S(x)y|}{|S(y)x|}.$$

Moreover, the μ_x are unique up to a constant multiple.

Proof idea. Fix $o \in M$. Then check that $\mu_x := \frac{|S(x)o|}{|S(o)x|}$ works. $\square$

Remark 29. A countable $\mathcal{L}$-structure with compact stabilizers is unimodular if and only if $\mu_x = \mu_y$ whenever x, y have the same type.

Theorem 30 (Mass Transport Principle, Corollary 8.11 in [22])**.** *Suppose that M is a countable unimodular $\mathcal{L}$-structure, and choose a complete set S of representatives for the types in M (S could be finite or countably infinite). If $F : M \times M \to [0, \infty]$ is* $\mathrm{Aut}(M)$*-invariant, then*

$$\sum_{o \in S} \frac{1}{\mu_o} \sum_{x \in M} F(o, x) = \sum_{o \in S} \frac{1}{\mu_o} \sum_{x \in M} F(x, o).$$

Given this theorem we always choose the weights μ_o to satisfy $\sum_{o \in S} \frac{1}{\mu_o} = 1$, provided this sum converges. It then makes sense to think of $o \in S$ as being picked randomly with probability $\frac{1}{\mu_o}$. If we denote such a random root by $\hat{o}$, then we can rewrite the conclusion of the Mass Transport Principle as

$$\mathbb{E}\left[\sum_{x \in M} f(\hat{o}, x)\right] = \mathbb{E}\left[\sum_{x \in M} f(x, \hat{o})\right].$$

Next we define the **Bernoulli structuring** $\hat{M}$ of a countable unimodular $\mathcal{L}$-structure M with compact stabilizers, where we assume that

$$\sum_{o \in S} \frac{1}{\mu_o} = 1 < \infty.$$

It will be a Borel way of assigning a structure isomorphic to M to each E-class of a pmp CBER E. Let

$$X := \{x \in [0, 1]^M : x \text{ injective}\},$$

and

$$Y := X \times S.$$

Let μ be the usual product measure on X, and consider the probability measure ν on Y given by

$$\nu \restriction X \times \{o\} = \frac{\mu}{\mu_o}.$$

The space for the Bernoulli structuring $\hat{M}$ is

$$\hat{Z} := \bigsqcup_{o \in S} (X \times \{o\}/\mathrm{Aut}_o(M)).$$

Let R be an n-ary relation symbol in $\mathcal{L}$. The elements $[x_1, o_1], [x_2, o_2], \ldots, [x_n, o_n]$ are R-related in $\hat{M}$ if and only if

(i) all the x_i are in the same $\mathrm{Aut}(M)$-orbit, and

(ii) there are elements $v_2, \ldots, v_n \in M$ of the same type as $o_2, \ldots, o_n$, respectively, such that $(o_1, v_2, \ldots, v_n) \in R^M$ and $x_2(o_2) = x_1(v_2), \ldots, x_n(o_n) = x_1(v_n)$.

The associated CBER $E = E_{\hat{M}}$ is defined by condition (i) as for the case of vertex transitive graphs: $[x_1, o_1]$ and $[x_2, o_2]$ are E-related when x_1 and x_2 are in the same $\mathrm{Aut}(M)$-orbit. We define the measure $\hat{\nu}$ on $\hat{Z}$ by $\hat{\nu}(A) := \nu(\bigcup A)$.

Claim: $\hat{\nu}$ is an invariant probability measure for $\hat{M}$.

Proof. The proof is quite similar to the proof for vertex transitive graphs. Again, we will verify the Mass Transport Principle characterization of pmp for E by using the Mass Transport Principle for M. Let $f : E \to [0, \infty]$ be a measurable function. We need to show

$$\sum_{o \in S} \frac{1}{\mu_o} \int_{x \in X} \sum_{[y,o'] E [x,o]} f([x,o],[y,o'])\, d\mu(x) = \sum_{o \in S} \frac{1}{\mu_o} \int_{x \in X} \sum_{[y,o'] E [x,i]} f([y,o'],[x,o])\, d\mu(x).$$

For each $v \in M$, choose $\varphi \in \mathrm{Aut}(M)$ such that φ_v sends v to the representative in S of the same type as v. Then we need to show

$$\sum_{o \in S} \frac{1}{\mu_o} \int_{x \in X} \sum_{v \in M} f([x,o],[\varphi_v \cdot x, \mathrm{tp}(v)])\, d\mu(x) = \sum_{o \in S} \frac{1}{\mu_o} \int_{x \in X} \sum_{v \in M} f([\varphi_v \cdot x, \mathrm{tp}(v)],[x,o])\, d\mu(x).$$

Now that the index of summation (for the inner sum) does not depend on $x \in X$, we can interchange the integral and the sum. So this is equivalent to

$$\sum_{o \in S} \frac{1}{\mu_o} \sum_{v \in M} \int_{x \in X} f([x,o],[\varphi_v \cdot x, \mathrm{tp}(v)])\, d\mu(x) = \sum_{o \in S} \frac{1}{\mu_o} \sum_{v \in M} \int_{x \in X} f([\varphi_v \cdot x, \mathrm{tp}(v)],[x,o])\, d\mu(x). \tag{4.2}$$

Define $F : M \times M \to [0, \infty]$ by

$$F(u, w) := \int_{x \in X} f([\varphi_u \cdot x, \mathrm{tp}(u)], [\varphi_w \cdot x, \mathrm{tp}(w)])\, d\mu(x).$$

Note that for any fixed labeling $x \in X$ and any fixed $u \in M$, the equivalence class $[\varphi_u \cdot x, \mathrm{tp}(u)]$ will be the same for any $\varphi_u \in \mathrm{Aut}(M)$ which sends u to the representative in u's orbit. In particular, $F(u, w)$ does not depend on our earlier choice of φ_u and φ_w. We claim that F is automorphism invariant. Fix $\psi \in \mathrm{Aut}(M)$. Then

$$\begin{aligned} F(\psi(u), \psi(w)) &= \int_{x \in X} f([\varphi_{\psi(u)} \cdot x, \mathrm{tp}(\psi(u))], [\varphi_{\psi(w)} \cdot x, \mathrm{tp}(\psi(w))])\, d\mu(x) \\ &= \int_{x \in X} f([\varphi_u \cdot (\psi^{-1} \cdot x), \mathrm{tp}(u)], [\varphi_w \cdot (\psi^{-1} \cdot x), \mathrm{tp}(w)])\, d\mu(x) \\ &= \int_{x \in X} f([\varphi_u \cdot x, \mathrm{tp}(u)], [\varphi_w \cdot x, \mathrm{tp}(w)])\, d\mu(x) \\ &= F(u, w), \end{aligned}$$

where we used that the action of $\mathrm{Aut}(M)$ on X preserves the product measure μ. By the Mass Transport Principle applied to M, we conclude that

$$\sum_{o \in S} \frac{1}{\mu_o} \sum_{v \in M} F(o, v) = \sum_{o \in S} \frac{1}{\mu_o} \sum_{v \in M} F(v, o).$$

But

$$\begin{aligned} F(o, v) &= \int_{x \in X} f([x, o], [\varphi_v \cdot x, \mathrm{tp}(v)])\, d\mu(x) \\ F(v, o) &= \int_{x \in X} f([\varphi_v \cdot x, \mathrm{tp}(v)], [x, o])\, d\mu(x), \end{aligned}$$

so we conclude that (4.2) holds. □

Proposition 31. Let M be a countable unimodular $\mathcal{L}$-structure with compact stabilizers such that $\sum_{o \in S} \frac{1}{\mu_o} = \infty$. Then M has no pmp realization.

Proof. Assume for contradiction that E is a pmp CBER on (X, μ) with a Borel way of putting the structure M on each E-class. For each $o \in S$ consider

$$A_o := \{x \in X : \mathrm{tp}(x) = \mathrm{tp}(o)\}.$$

Define $F : E \to [0,1]$ by

$$F(x,y) := 1_{\{\mathrm{tp}(x,y)=\mathrm{tp}(o_1,o_2)\}}.$$

Since E is pmp,

$$\int_{x \in X} \sum_{y \in [x]_E} F(x,y)\, d\mu(x) = \int_{x \in X} \sum_{y \in [x]_E} F(y,x)\, d\mu(x).$$

This reduces to

$$\mu(A_{o_1})|S(o_1)o_2| = \mu(A_{o_2})|S(o_2)o_1|,$$

in other words

$$\mu(A_{o_2}) = \frac{\mu_{o_1}}{\mu_{o_2}} \mu(A_{o_1}).$$

If we fix o_1, then this implies that $\mu(X) = \sum_{o_2 \in S} \mu(A_{o_2}) = \infty$, a contradiction. □

Remark 32. More generally, the proof of the above theorem shows that for a pmp realization of a countable unimodular $\mathcal{L}$-structure with compact stabilizers with $\sum_{o \in S} \frac{1}{\mu_o} = 1$, we have

$$\mu(A_{o_1}) = \mu\left(\{x : \mathrm{tp}(x) = o_1\}\right) = \frac{1}{\mu_{o_1}}.$$

Putting everything together, we have the following characterization.

Theorem 33. *Let M be a countable $\mathcal{L}$-structure with compact stabilizers. Then M has a pmp realization if and only if M is unimodular and $\sum_{o \in S} \frac{1}{\mu_o} < \infty$.*

Remark 34. All countable amenable graphs with finitely many vertex types are unimodular (see Exercise 8.30 in [22]), and of course satisfy $\sum_{o \in S} \frac{1}{\mu_o} < \infty$ since this sum is finite, so they have pmp realizations.

CHAPTER 5

Borel Decision Problems and Gadget Reductions

In previous chapters, we discussed the question of when a *particular* Borel graph admits a Borel, measurable, or Baire measurable solution to some labeling problem. For example, we studied Baire measurable matchings and orientations, and witnesses for hyperfiniteness can also be described as certain labelings. In this chapter, we turn our attention to **decision problems** in Borel combinatorics. That is, given a fixed labeling problem $\mathcal{P}$ (like the existence of a 3-coloring, a perfect matching, or a balanced orientation), we ask if there is a simple characterization of when a Borel graph admits a Borel solution to $\mathcal{P}$. More precisely, we investigate the projective complexity of

$$\{c \in \mathbb{N}^{\mathbb{N}} : c \text{ is the code of a Borel graph admitting a Borel solution to } \mathcal{P}\}.$$

We refer to Section 35.B in [16] and Section 3H in [27] for detailed discussions of Borel codes. The real c is supposed to code a Borel set, and there are many ways to set up this encoding, all of which are equivalent and can be translated between one another. The most intuitive is perhaps to have c encode a well-founded labeled tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$ that indicates a construction of the Borel set, where the leaves of the tree are labeled with basic open sets, and the other nodes are labeled with symbols denoting countable unions, countable intersections, and set complements. Since we need to say that codes are *well-founded* trees, we get the following proposition.

Proposition 35. The set of Borel codes is a $\mathbf{\Pi}^1_1$. Further, the set of codes for Borel graphs is also $\mathbf{\Pi}^1_1$.

We can also consider Borel graphs with additional structure (like oriented edges, edge-colorings, etc.) or Borel hypergraphs as inputs to the decision problems. The set of their Borel codes will also be $\mathbf{\Pi}^1_1$ if the added structure is simple enough. It also turns out the set of codes for Borel functions is $\mathbf{\Pi}^1_1$ (rather than just $\mathbf{\Sigma}^1_2$).

Proposition 36 (see [13], Lemma A.3)**.** If X, Y are Polish spaces, the set of Borel codes c for graphs of Borel functions $X \to Y$ is $\mathbf{\Pi}^1_1$.

Thus, for example, the set

$$\{(c_1, c_2) \in \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} : c_1 \text{ codes a Borel graph } G, \\ \text{and } c_2 \text{ codes a Borel proper countable coloring of } G\}$$

is $\mathbf{\Pi}^1_1$. Since the set of Borel codes is $\mathbf{\Pi}^1_1$ to begin with, we see that saying that a certain Borel function is a proper coloring is as simple as possible in this setup.

Recall the following dichotomy of Kechris, Solecki, and Todorčević:

Theorem 37 (G_0-dichotomy, [15])**.** *For a Borel graph G, exactly one of the following holds:*

1. *G has a countable Borel coloring;*
2. *there is a Borel graph homomorphism from G_0 to G.*

There is an effective strengthening of the G_0-dichotomy (see Theorem 6.4 in [15]), which says that a lightface Δ^1_1 graph has a countable lightface Δ^1_1 coloring. This has the following consequence:

Proposition 38. The set

$$\{c \in \mathbb{N}^{\mathbb{N}} : c \text{ codes a Borel graph with a countable Borel coloring}\}$$

is $\mathbf{\Pi}^1_1$.

Proof. By the effective strengthening of the G_0 dichotomy mentioned above,

$$\{c \in \mathbb{N}^{\mathbb{N}} : c \text{ codes a Borel graph with a countable Borel coloring}\}$$
$$= \{c \in \mathbb{N}^{\mathbb{N}} : c \text{ codes a Borel graph with a countable } \Delta^1_1(c) \text{ coloring}\},$$

and the latter set is $\mathbf{\Pi}^1_1$. □

Similarly, there is an L_0-dichotomy theorem (see [5]) that characterizes when a Borel graph admits a Borel 2-coloring, which also has an effective version. By the same argument as in the proof of Proposition 38, we also get:

Proposition 39. The set

$$\{c \in \mathbb{N}^{\mathbb{N}} : c \text{ codes a Borel graph with a Borel 2-coloring}\}$$

is $\mathbf{\Pi}^1_1$.

So, despite the existence of 2-regular Borel graphs with no Borel 2-coloring which show that Borel chromatic numbers can differ from standard chromatic numbers, the *decision problem* of determining whether a given (Borel code for a) Borel graph is Borel 2-colorable is as simple as one can hope for.

On the other hand, determining if a Borel graph is Borel 3-colorable is hard, as the following groundbreaking result of Todorčević and Vidnyánszky shows. Recall that a set $B \subseteq Y$ is $\mathbf{\Sigma}^1_2$-complete if B is $\mathbf{\Sigma}^1_2$ and for every Σ^1_2 set $A \subseteq X$ there is a Borel function $f : X \to Y$ such that $[x \in A$ if and only if $f(x) \in B]$.

Theorem 40 ([33], 2021)**.** *The set*

$$\{c \in \mathbb{N}^{\mathbb{N}} : c \textit{ codes a locally finite Borel graph with a Borel proper } 3\textit{-coloring}\}$$

is $\mathbf{\Sigma}^1_2$*-complete.*

So there is no simpler characterization of Borel 3-colorability than "*There exist* Borel sets Red, Blue, Green partitioning the vertex set such that *all* adjacent vertices x, y are in different color classes." The theorem also rules out characterizations of Borel 3-colorability akin to the G_0 and L_0 dichotomies, since such a dichotomy would give a complexity upper bound of $\mathbf{\Delta}^1_2$. More generally, the theorem rules out the existence of a countable set of Borel graphs with no Borel 3-coloring such that any Borel graph with no Borel 3-coloring would admit a Borel graph homomorphism from a graph in this set.

Recently, the Todorčević-Vidnyánszky result was strengthened by combining the apparatus of [33] with the game method of Marks to obtain:

Theorem 41 ([4]). *For $d \geq 3$, the set of Borel d-regular acyclic graphs admitting a Borel d-coloring is $\mathbf{\Sigma}^1_2$-complete.*

In the proof, the authors need to consider certain Borel families of games. While Martin's Borel determinacy theorem ensures each individual game in this family is determined, the authors need to know that the set of games where, say, player II has a winning strategy satisfies certain regularity properties (like measurability, Baire measurability, Ramsey measurability). The original argument used by the authors to establish these regularity properties was metamathematical, involving the notion of weakly provable Δ^1_2 sets. In Chapter 6 (need to cite), we give a streamlined classical proof of this fact, just using the (general) Borel determinacy theorem.

5.1 Gadget Reductions

In this section, we show that the gadget reductions arising in computational complexity theory can often be adapted to show Borel decision problems are $\mathbf{\Sigma}^1_2$-complete. The motivation for this work comes from Thornton's work in [32] exploring the connection between NP-completeness and Borel combinatorics for constraint satisfaction problems. For a fixed relational structure H, the associated constraint satisfaction problem $\mathsf{CSP}(H)$ is the problem of deciding if a structure (over the same signature) admits a homomorphism into H. For example, deciding whether a graph is k-colorable corresponds to $\mathsf{CSP}(K_k)$, where K_k is the complete graph on k vertices. Other notable examples of constraint satisfaction problems are k-SAT and deciding if a linear system is consistent. The celebrated CSP dichotomy theorem characterizes when a CSP is in P (i.e. solvable by a polynomial time algorithm) and when it is NP-complete:

Theorem 42 (CSP dichotomy theorem, [37])**.** *Let H be a finite relational structure.*

1. *If there is a homomorphism $f : H^4 \to H$ such that for all $a, e, r \in H$, $f(r, a, r, e) = f(a, r, e, a)$, then $\mathsf{CSP}(H)$ is in P;*

2. *If there is no such homomorphism, then $\mathsf{CSP}(H)$ is NP-complete.*

Thornton's main result in [32] is the following:

Theorem 43. *If H is a finite relational structure and there is no homomorphism $f : H^4 \to H$ as above, then the Borel constraint satisfaction problem $\mathsf{CSP}_B(H)$ is $\mathbf{\Sigma}^1_2$-complete. Hence, assuming $\mathsf{P} \neq \mathsf{NP}$, this shows all NP-complete CSPs have $\mathbf{\Sigma}^1_2$-complete Borel versions.*

We would like to show that other NP-complete labeling problems have $\mathbf{\Sigma}^1_2$-complete Borel versions. After all, most labeling problems we're accustomed to – involving matchings, orientations, edge-colorings, k-factors, etc. – are not constraint satisfaction problems. In an appendix to [32], Thornton shows that Borel 3-edge-colorability of Borel graphs is $\mathbf{\Sigma}^1_2$-complete. He does this by adapting to the Borel setting the gadget reduction from 3-SAT used to show 3-edge colorability of finite graphs is NP-complete.

In what follows, we generalize this method to show that *all the usual NP-complete decision problems on graphs (or structured hypergraphs) are $\mathbf{\Sigma}^1_2$-complete*. While the definition of NP-completeness refers to polynomial-time algorithms, in practice the reductions that arise in when reducing an NP-complete problem to another NP-complete problem are considerably simpler. An NC^0 function taking finite binary strings to finite binary strings is one defined by a uniform family of circuits of polynomial size and constant depth consisting of NOT-gates, pairwise AND-gates, and pairwise OR-gates. An AC^0 function is defined similarly except that the AND-gates and OR-gates can take any number of inputs. In [1], Agrawal, Allender, and Rucich first observe that all the well-known NP-complete problems seem to be complete with NC^0-reductions, and go on to prove that NP-completeness for AC^0-reductions implies NP-completeness for NC^0-reductions. NC^0 reductions are very close to the informal notion of gadget reductions that comes up in typical NP-completeness proofs, where we systematically replace parts of the input problem instance with parts of the output problem instance (e.g. when reducing from 3-SAT, we usually have gadgets in the output problem instance representing the clauses and the variables). With an NC^0 function, each bit of the output string only depends on a bounded number of bits from the input string. On the other hand, bits of the input string can still influence an unbounded number of bits in the output string. This last part is a bit too general for the methods we will use. But we can put forward the following conjectures, where the second one is weaker than the first.

Conjecture 44 ([32])**.** Every NP-complete locally checkable labeling decision problem on graphs (more generally, structured hypergraphs) has a Borel version that is $\mathbf{\Sigma}^1_2$-complete.

Conjecture 45. Every NP-complete locally checkable labeling decision problem on graphs that is already complete for NC^0-reductions has a Borel version that is $\mathbf{\Sigma}^1_2$-complete.

We introduce a formal notion of gadget reduction in the spirit of LOCAL algorithms. All gadget reductions between (structured hyper-)graph problems from the NP-completeness literature that the author is aware of satisfy this definition. This will let us conclude $\mathbf{\Sigma}^1_2$-completeness results for the Borel versions of all NP-complete graph decision problems commonly found in the literature.

Definition 46 (Gadget reduction)**.** Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two labeling problems on classes of countable structured hypergraphs $\mathcal{K}_1$ and $\mathcal{K}_2$, respectively. For simplicity, we assume $\mathcal{K}_2$ consists of graphs. A map $G \mapsto H$ from $\mathcal{K}_1$ to $\mathcal{K}_2$ is a **gadget reduction** from $\mathcal{P}_1$ to $\mathcal{P}_2$ if there exists $r > 0$ and finitary functions f, g, h, k (whose domains will be clear from context) taking isomorphic inputs (rooted structured hypergraphs and partial labelings) to isomorphic outputs such that

1. the vertices of H are obtained by creating a finite set of vertices $f(B_G(v,r))$, for each $v \in V(G)$;

2. the edges of H are obtained by creating edges $g(B_G(v,2r))$ between a vertex in $f(B_G(v,r))$ and a vertex in $\bigcup_{w\in N_G(v)} f(B_G(w,r))$, for each $v \in V(G)$;

3. given a $\mathcal{P}_1$-labeling ℓ_1 of G, the labeling ℓ_2 of H given by

$$\ell_2 \restriction f(B_G(v,r)) = h(B_G(v,r), \ell_1 \restriction B_G(v,r)), \quad \forall v \in V(G)$$

   is a $\mathcal{P}_2$-labeling;

4. given a $\mathcal{P}_2$-labeling ℓ_2 of H, the labeling ℓ_1 of G given by

$$\ell_1(v) = k\left(B_G(v,2r), \ell_2 \restriction \left(\bigcup_{w\in B_G(v,r)} f(B_G(w,r)\right)\right), \quad \forall v \in V(G)$$

   is a $\mathcal{P}_1$-labeling.

Although the definition is a little tedious to write down, it's just trying to formally capture the intuitive notion of gadget reductions that arises for decision problems on structured hypergraphs. Items (1) and (2) are a general way of expressing that each part of the input G locally gives rise to a finite portion of H. Items (3) and (4) are not needed explicitly in computational complexity theory but are needed for our Borel setting. Indeed, reductions in computational complexity don't need to explicitly come with an algorithm for converting a solution to $\mathcal{P}_1$ into a solution to $\mathcal{P}_2$, and vice versa. But in order to conclude the existence of *Borel* solutions to $\mathcal{P}_2$ from the existence of Borel solutions to $\mathcal{P}_1$ (and vice versa) we need some guarantees that the former can be easily obtained from the latter.

The proof of the next theorem is just about unfolding the formal definition above, and we don't give full details.

Theorem 47. *Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two labeling problems on classes of countable structured hypergraphs $\mathcal{K}_1$ and $\mathcal{K}_2$, respectively. If there is a gadget reduction from $\mathcal{P}_1$ to $\mathcal{P}_2$ and the Borel version of $\mathcal{P}_1$ is $\mathbf{\Sigma}^1_2$-complete, then the Borel version of $\mathcal{P}_2$ is also $\mathbf{\Sigma}^1_2$-complete.*

Proof. Again, we assume for simplicity that $\mathcal{K}_2$ is a class of graphs. We need to explain how to get a Borel graph H with connected components in $\mathcal{K}_2$ from a Borel structured hypergraph G with connected components in $\mathcal{K}_1$. It is then routine to convert these instructions into operations on Borel codes. The vertex set $V(H)$ will be the Borel subset of $V(G) \times \mathbb{N}$ where

$$(v, i) \in V(H) \quad \Longleftrightarrow_{\text{def}} \quad 1 \leq i \leq |f(B_G(v, r))|.$$

For simplicity assume that we had $f(B_G(v, r)) = \{(v, 1), \ldots, (v, |f(B_G(v, r))|)\}$ all along. Then the edge set $E(H)$ of H is given by

$$(v_1, i_1)E(H)(v_2, i_2) \quad \Longleftrightarrow_{\text{def}} \quad ((v_1, i_1), (v_2, i_2)) \in g(B_G(v_1, 2r))$$
$$\text{or } ((v_1, i_1), (v_2, i_2)) \in g(B_G(v_2, 2r)).$$

Items 3 and 4 Definition 46 easily imply that a Borel solution to $\mathcal{P}_1$ for G can be converted to a Borel solution to $\mathcal{P}_2$ for H, and conversely a Borel solution to $\mathcal{P}_2$ for H can be converted to a Borel solution to $\mathcal{P}_1$ for G. □

Remark 48. In order to break symmetry, we often need to use a proper Borel coloring of $G^{\leq r}$ (where we put an edge between vertices that are distance r or less apart) in order to construct H. But this coloring can be part of the additional structure of G in the definition of the labeling problem $\mathcal{P}_1$ on $\mathcal{K}_1$. It plays the role that an enumeration of the vertices of a finite graph would play in the finitary theory.

5.2 Examples

We give some examples of $\mathbf{\Sigma}^1_2$ complete labeling problems by adapting the gadget reductions from computational complexity to the Borel setting using Definition 46 and Theorem 47.

Again, as far as I'm aware, the usual NP-complete labeling problems on structured hypergraphs can all be shown to have $\boldsymbol{\Sigma}^1_2$-complete Borel version via this method. But it's still open to prove a formal result to this effect for, say, all NP-complete locally checkable labeling problems.

Example 49 (3-colorability for graphs of degree at most 4)**.** The original $\boldsymbol{\Sigma}^1_2$-completeness result of Todorčević and Vidnyánszky showed that Borel 3-colorability is $\boldsymbol{\Sigma}^1_2$-complete for *locally finite* Borel graphs. The later result in [4] shows that Borel 3-colorability is still $\boldsymbol{\Sigma}^1_2$-complete for 3-regular acyclic graphs, but the proof is quite involved. However, one can easily reduce general Borel 3-colorability to Borel 3-colorability for Borel graphs of degree at most 4 via the elementary gadget reductions shown in the two figures below. We note that this method won't prove $\boldsymbol{\Sigma}^1_2$-completeness for Borel 3-colorability for Borel graphs of degree at most 3. This is because Brooks' theorem implies that determining if a finite graph of degree at most 3 is 3-colorable is in P, so there's of course no gadget reduction to adapt to the Borel setting.

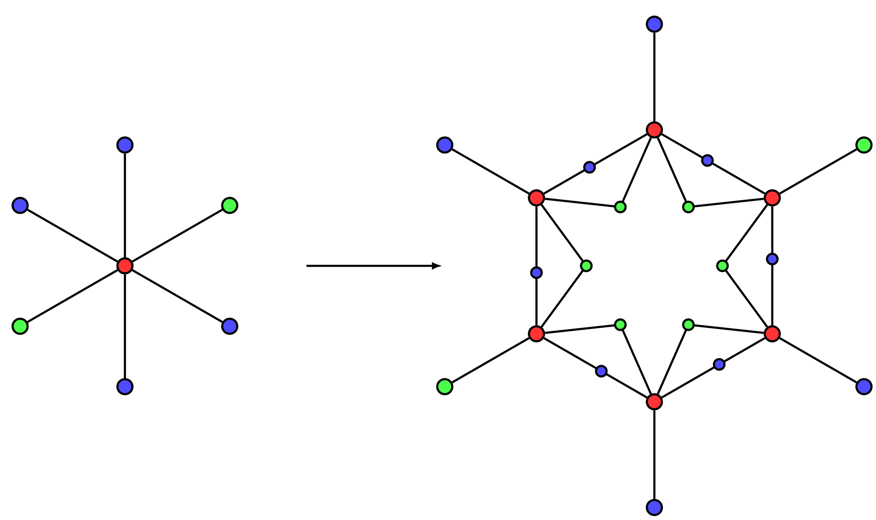

Figure 5.1: Replacing a vertex with a gadget whose vertices all have degree ≤ 5.

Example 50 (3-SAT)**.** The fact that Borel 3-SAT is $\boldsymbol{\Sigma}^1_2$ complete follows from Thornton's work on CSPs in [32]. One can also easily reduce Borel 3-colorability to Borel 3-SAT via a

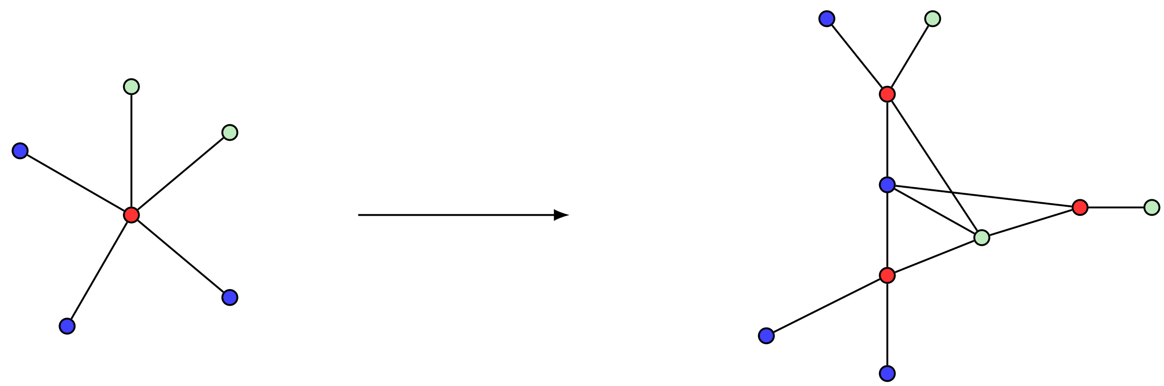

Figure 5.2: Replacing a degree-5 vertex with a gadget whose vertices all have degree ≤ 4.

gadget reduction. Here we think of 3-SAT as a labeling problem on a hypergraph, where the vertices correspond to the variables and each clause is encoded as a certain hyperedge containing three vertices. Given a graph G, for each vertex v of G, we create variables $x_{v,R}$, $x_{v,B}$, $x_{v,G}$ that will respectively encode whether the vertex v is colored Red, Blue, or Green. For each vertex v, we create clauses that say v is colored with exactly one color, and for each edge of G, we create clauses that say the endpoints are assigned different colors. By Example 49, we also get $\mathbf{\Sigma}^1_2$-completeness for 3-SAT where each variable only appears in a bounded number of clauses.

The problem of determining whether a finite graph admits a perfect matching is in P, both in the bipartite setting and the non-bipartite setting (using the augmenting chains algorithm and Edmonds' blossom-contraction algorithm, respectively). It is an open problem what the complexity is in the Borel setting.

Question 51. What is the projective complexity of the set of Borel bipartite graphs admitting a Borel perfect matching?

On the other hand, the analogue of the perfect matching problem with hyperedges of size 3 is $\mathbf{\Sigma}^1_2$-complete.

Example 52 (3D-perfect matching)**.** Consider a hypergraph where every hyperedge is a set containing exactly three elements. The 3D-perfect matching problem asks whether we can select a subset of these hyperedges such that each vertex is covered by exactly one hyperedge in that subset. The finite version of 3D-perfect matching is known to be NP-complete by reducing 3-SAT to it using a gadget reduction (see the pictures in [18]). So the Borel 3D-perfect matching problem is also $\mathbf{\Sigma}^1_2$-complete.

Example 53 (Efficient dominating set)**.** We define an *efficient dominating set* for a graph to be a subset of the vertices with the property that every ball of radius 1 intersects that subset in exactly one vertex. Deciding whether a finite graph admits an efficient dominating set is known to be NP-complete using a gadget reduction from 3D-perfect matching, and so the Borel version of the efficient dominating set problem is also $\mathbf{\Sigma}^1_2$-complete. Here's a brief description of the gadget reduction. Suppose G is an instance of the 3D-perfect matching problem, i.e. G has hyperedges that all contain exactly three vertices. Each vertex v in G has a copy of itself in H, and each hyperedge e in H gives rise to two adjacent vertices in H which we can call x_e and y_e. Moreover, for each hyperedge e, connect each x_e to the three vertices v_1, v_2, v_3 that e contained. It is easy to check that H has an efficient dominating set if and only if G has a 3D-perfect matching.

Example 54 (Hamiltonian path)**.** For finite graphs, the Hamiltonian path problem asks whether the input graph admits a path that visits every vertex exactly once. A natural Borel version of this problem asks whether the input Borel graph admits a Borel choice of a bi-infinite path for each connected component that passes through each vertex exactly once. We can still use the gadget reduction from 3-SAT of the NP-completeness proof of the finite Hamiltonian path problem; see p.314-319 in [30]. However, this gadget reduction uses an enumeration of the variables of the input instance of 3-SAT, and using a Borel distance coloring for a Borel instance of 3-SAT isn't enough. Instead, we can use the fact that the original Todorčević and Vidnyánszky result shows that Borel 3-colorability is actually $\mathbf{\Sigma}^1_2$-complete for *hyperfinite* Borel graphs. Therefore, all the other Borel labeling problems we've shown to be $\mathbf{\Sigma}^1_2$-complete by reducing from Borel 3-colorability are also $\mathbf{\Sigma}^1_2$-complete for hyperfinite graphs. When reducing Borel 3-SAT to Borel Hamiltonian path, we can therefore make use of a Borel discrete linear ordering of each connected component. This discrete linear ordering is sufficient to now translate the gadget reduction from the finite context to the Borel version of the problem.

CHAPTER 6

Measurability of Winning Sets in Families of Games

The results of this chapter are part of joint work with Clark Lyons.

A *family of games* is a set $B \subseteq X \times \mathbb{N}^{\mathbb{N}}$, where $\mathbb{N}^{\mathbb{N}}$ is the Baire space. We think of each vertical section $B_x \subseteq \mathbb{N}^{\mathbb{N}}$ as the payoff set *for player II*. In this chapter, we give an elementary proof of the following theorem, which is related to recent work on homomorphism graphs in [4]. The theorem also follows from arguments in [29], but we give a much more streamlined proof. The result can also be obtained as a consequence of Feng, Magidor, and Woodin's work on universally Baire set of reals [12]. We refer to Kechris's book [16] for definitions of all undefined concepts.

Theorem 55. *Let X be a Polish space, and suppose that $B \subseteq X \times \mathbb{N}^{\mathbb{N}}$ is a Borel family of games. Then*

$$W = \{x \in X : \textit{player II has a winning strategy in } B_x\}$$

is Baire measurable and universally measurable (that is, μ-measurable for every Borel probability measure μ). In the case where $X = [\mathbb{N}]^{\aleph_0}$, the set W is also completely Ramsey.

The following game and theorem, in the case when $(X, \mathcal{T})$ is Polish and the U_i, V_i come from a countable basis, appear to be due to Solovay in unpublished notes.

Definition 56. Let $(X, \mathcal{T})$ be a Choquet space, and let d be a metric whose open balls are in $\mathcal{T}$. Suppose that σ_C is a winning strategy for player II in the Choquet game for X. We may assume that σ_C only depends on the most recent move of player I. If $B \subseteq X \times \mathbb{N}^{\mathbb{N}}$, we define the game $\mathcal{G}(X, B)$ as follows:

I	(U_0, m_0)		(U_1, m_1)		
					$\dots$
II		(V_0, n_0)		(V_1, n_1)	

where the m_i, n_i are natural numbers, U_i, V_i are open sets in $\mathcal{T}$, $\mathrm{diam}_d(U_i), \mathrm{diam}_d(V_i) < 2^{-i}$, $V_i \subseteq U_i$, and $U_{i+1} \subseteq \sigma_C(V_i)$. This ensures that $\bigcap_i U_i = \bigcap_i V_i = \{x\}$ is a singleton. Player II wins if and only if $(x, (m_0, n_0, m_1, n_1, \dots)) \in B$. Note that we can always choose to restrict the U_i and V_i to any weak basis for $(X, \mathcal{T})$.

Theorem 57. *Let $(X, \mathcal{T})$ be a Choquet space whose topology refines the topology induced by some metric d, and let $B \subseteq X \times \mathbb{N}^{\mathbb{N}}$.*

(a) If player II has a winning strategy in $\mathcal{G}(X, B)$, then

$$W = \{x \in X : \textit{player II has a winning strategy in } B_x\}$$

is comeager.

(b) If player I has a winning strategy in $\mathcal{G}(X, B)$, then

$$L = \{x \in X : \textit{player I has a winning strategy in } B_x\}$$

is comeager in some nonempty open set $U \in \mathcal{T}$.

Proof. (a) Suppose that σ is a winning strategy for player II. For each m_0, Zorn's lemma yields a collection $\mathcal{U}_{m_0}$ of open sets $U_0 \in \mathcal{T}$ such that

$$\mathcal{V}_{m_0} := \{\sigma(U_0, m_0)_0 : U_0 \in \mathcal{U}_{m_0}\}$$

is a disjoint collection of open sets in $\mathcal{T}$ whose union is dense in $(X, \mathcal{T})$. (Here the subscript 0 to the right of $\sigma(U_0, m_0)_0$ denotes the left entry of the pair $\sigma(U_0, m_0)$; the subscript 1 to the right will denote the right entry.) Now for each $U_0 \in \mathcal{U}_{m_0}$ and m_1, Zorn's lemma yields a collection $\mathcal{U}_{U_0, m_0, m_1}$ of open sets $U_1 \subseteq \sigma(U_0, m_0)_0$ such that

$$\mathcal{V}_{U_0, m_0, m_1} := \{\sigma(U_0, m_0, U_1, m_1)_0 : U_1 \in \mathcal{U}_{U_0, m_0, m_1}\}$$

is a disjoint collection of open sets in $\mathcal{T}$ whose union is dense in $\sigma(U_0, m_0)_0$. In particular, for each fixed m_0, m_1, the collection

$$\mathcal{V}_{m_0,m_1} := \bigcup_{U_0 \in \mathcal{U}_0} \mathcal{V}_{U_0,m_0,m_1}$$

has dense union in $(X, \mathcal{T})$. By continuing the construction above, we get collections $\mathcal{V}_{m_0,\ldots,m_k}$ of disjoint open sets in $\mathcal{T}$, all of which have dense union in $(X, \mathcal{T})$. We claim that the dense G_δ

$$\bigcap_{k \in \mathbb{N}} \bigcap_{m_0,\ldots,m_k} \left(\bigcup \mathcal{V}_{m_0,\ldots,m_k} \right)$$

is contained in W. Fix x in this set. We define a winning strategy for player II for the game B_x. If player I first plays m_0, let U_0 be the unique element of $\mathcal{U}_{m_0}$ such that $x \in \sigma(U_0, m_0)_0$. Player II should play $\sigma(U_0, m_0)_1$. Now suppose that player I's next move is m_1. Let U_1 be the unique element of $\mathcal{U}_{U_0,m_0,m_1}$ such that $x \in \sigma(U_0, m_0, U_1, m_1)_0$. Player II should then play $\sigma(U_0, m_0, U_1, m_1)_1$. Keep going in this way. By construction, we will have

$$(x, (m_0, n_0, m_1, n_1, \ldots)) \in B,$$

so this describes a winning strategy for player II in B_x. The proof of part (b) is entirely analogous. □

Recall that a *Baire space* is a topological space where countable intersections of dense open sets are dense (that is, where the Baire category theorem holds).

Proposition 58. Let X be a Baire space, and let $A \subseteq X$. Then A is Baire measurable if and only if for every nonempty open set $U \subseteq X$, either A is comeager in U or there exists a nonempty open $V \subseteq U$ such that A is meager in V.

Proof. The forward direction is just the well-known Baire alternative applied to U. For the backward direction, consider the open set

$$U(A^c) := \bigcup \{U : U \text{ is a nonempty open set in which } A \text{ is meager}\}.$$

By Theorem 8.29 in cite Kechris, A is meager in $U(A^c)$. The space X can be partitioned as

$$X = U(A^c) \sqcup \partial U(A^c) \sqcup (X \setminus \overline{U(A^c)}).$$

The boundary $\partial U(A^c)$ is meager, and we claim that A is comeager in the open set $X \setminus \overline{U(A^c)}$. Otherwise, there is a nonempty open set $V \subseteq X \setminus \overline{U(A^c)}$ in which A is meager. But this would contradict the definition of $U(A^c)$. Let's summarize: A is meager in $U(A^c)$, $\partial U(A^c)$ is meager, and A is comeager in $X \setminus \overline{U(A^c)}$. It follows that A is Baire measurable. □

In the corollary below, we write $\mathcal{G}(U, B)$ for $\mathcal{G}(U, B \cap U \times \mathbb{N}^{\mathbb{N}})$ to simplify notation.

Corollary 59. *Let $(X, \mathcal{T})$ be a Choquet space whose topology refines the topology induced by some metric d, and let $B \subseteq X \times \mathbb{N}^{\mathbb{N}}$. Suppose that for every open set $U \in \mathcal{T}$, the game $\mathcal{G}(U, B)$ is determined. Then*

$$W = \{x \in X : \text{player II has a winning strategy in } B_x\}$$

is Baire measurable in $(X, \mathcal{T})$.

Theorem 60 (General Borel Determinacy)**.** *Let A be a discrete topological space (possibly uncountable). Then any Borel $B \subseteq A^{\mathbb{N}}$ corresponds to a determined game.*

Theorem 61. *Let X be a Polish space with compatible metric d, and suppose that $B \subseteq X \times \mathbb{N}^{\mathbb{N}}$ is a Borel family of games. Suppose that $\mathcal{T}$ is a Choquet topology on X that refines the Polish topology given by d. Then*

$$W = \{x \in X : \text{player II has a winning strategy in } B_x\}$$

is Baire measurable in $(X, \mathcal{T})$.

Proof. By Corollary 59, we need to show the game $\mathcal{G}(U, B)$ is determined for every $U \in \mathcal{T}$. So fix $U \in \mathcal{T}$, and let T be the tree of legal positions in $\mathcal{G}(U, B)$. Writing $[T]$ for the set of branches as usual, consider the map

$$\varphi : [T] \to X \times \mathbb{N}^{\mathbb{N}}$$

$$((U_0, m_0), (V_0, n_0), (U_1, m_1), \dots) \mapsto (x, (m_0, n_0, m_1, \dots))$$

where $\{x\} = \bigcap_i U_i = \bigcap_i V_i$. If we can show that $\varphi : [T] \to (X, d) \times \mathbb{N}^{\mathbb{N}}$ is continuous (where $[T]$ is viewed as a closed subspace of a product of uncountable discrete spaces), then this will imply that the payoff set for player II in $\mathcal{G}(U, B)$ is Borel as a subset of $[T]$. Hence, the General Borel Determinacy Theorem would imply that $\mathcal{G}(U, B)$ is determined.

Suppose that $((U_0, m_0), (V_0, n_0), (U_1, m_1), \dots) \in [T]$, and let

$$\varphi\left((U_0, m_0), (V_0, n_0), (U_1, m_1), \dots\right) = (x, (m_0, n_0, m_1, \dots)).$$

A basic open neighborhood of $(x, (m_0, n_0, m_1, \dots))$ will have the form

$$B_d(x; 2^{-k}) \times N_{m_0, n_0, \dots, m_k, n_k}.$$

Our requirement that $\mathrm{diam}_d(U_{k+1}), \mathrm{diam}_d(V_{k+1}) < 2^{-(k+1)}$ ensures that

$$\varphi[N_{(U_0,m_0),(V_0,n_0),\dots,(U_{k+1},m_{k+1}),(V_{k+1},m_{k+1})}] \subseteq B_d(x; 2^{-k}) \times N_{m_0,n_0,\dots,m_k,n_k}.$$

This verifies that $\varphi : [T] \to (X, d) \times \mathbb{N}^{\mathbb{N}}$ is a continuous function, and we are done. □

Our main theorem, Theorem 55, now easily follows.

Proof of Theorem 55. We apply Theorem 61 for different choices of $\mathcal{T}$. To show W is Baire measurable, take $\mathcal{T}$ to be the given Polish topology. To show W is universally measurable, take $\mathcal{T}$ to be the density topology (after reducing to the case where the measure is Lebesgue measure on $(0, 1)$). To show W is completely Ramsey, take $\mathcal{T}$ to be the Ellentuck topology. □

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