On non-linear cell bundles

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A useful property of the orthogonal group $O(n)$ is that it leaves invariant the unit ball and unit sphere of R^n. Consequently one may pass freely from R^n bundles to D^n and S^{n-1} bundles. This convenient coincidence does not occur in the topological category $\mathcal{T}$, or in the piecewise linear (PL) category $\mathcal{P}$. William Browder showed [1] that some R^n bundles do not contain any D^n subbundles. The purpose of this paper is to prove that stably the situation is better. The following result is proved, valid in both $\mathcal{T}$ and $\mathcal{P}$.

THEOREM A. *Let B be a polyhedron.*

(a) *Let ξ be an R^n bundle over B, and ε^1 the trivial R^1 bundle. Then there exists a D^{n+1} bundle η such that $\xi \oplus \varepsilon^1 = \operatorname{int} \eta$.*

(b) *Let η_0 and η_1 be D^n bundles over B, and δ^1 the trivial D^1 bundle. If $\operatorname{int} \eta_0 \sim \operatorname{int} \eta_1$, then $\eta_0 \oplus \delta^1 \sim \eta_1 \oplus \delta^1$.*

Part (a) in the category $\mathcal{T}$ is due to Barry Mazur (unpublished).

In the proof of Theorem A the following result is needed.

THEOREM B. *Let ξ be an R^n bundle over B. There exists an S^n bundle $\zeta = (p, E, B)$ with a section $f: B \to E$ such that $\xi = (p \mid E - f(B), E - f(B), B)$. Moreover the pair (ζ, f) is unique up to isomorphism.*

All definitions and lemmas will apply to both $\mathcal{P}$ and $\mathcal{T}$ unless the contrary is explicitly indicated.

Outline of the proof of Theorem A

Let KY be the functor assigning to B the set of isomorphism classes of Y bundles over B. Let $K_0 Y$ assign the set of isomorphism classes of bundles-plus-sections.

It will be proved that the natural transformation COMP: $K_0 S^n \to KR^n$, arising from an identification $R^n = S^n - x_0$, is an isomorphism. The commutativity of the

MAIN DIAGRAM:

$$
\begin{tikzcd}
KD^n \arrow[rr, "\text{INT}"] \arrow[dd, "\oplus\delta_1"'] & & KR^n \arrow[dd, "\oplus\varepsilon^1"] \\
 & K_0 S^n \arrow[ur, "\text{COMP}", "\approx"'] \arrow[dl, "\text{CYL}"] & \\
KD^{n+1} \arrow[rr, "\text{INT}"'] & & KR^{n+1}
\end{tikzcd}
$$

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will be established, where CYL is defined (intuitively) by assigning to an S^n bundle its mapping cylinder. Part (a) of Theorem A follows from commutativity in the lower quadrilateral, and (b) from the upper.

Bundles with semi-simplicial structural groups

Let $\mathbf{G} = \{G_k\}$ be a semi-simplicial (s.s.) group of homeomorphisms of a space Y. That is, each element of G_k is a homeomorphism $f\colon \Delta_f \times Y \to \Delta_k \times Y$ of the form $f(x, y) = (x, f_x(y))$. Call such a map a Y *twist*. Here Δ_k is the standard k simplex. If $\lambda\colon \Delta_j \to \Delta_k$ is a simplicial map, the corresponding s.s. operator $\lambda\colon G_k \to G_j$ is defined by setting

$$(\lambda f)_x = f_{\lambda(x)} .$$

Let $p\colon E \to B$ be a map. A *Y trivialization* of $\xi = (p, E, B)$ is a homeomorphism $h\colon E \to B \times Y$ that makes the diagram

$$\begin{tikzcd} E \arrow[rr, "h"] \arrow[dr, "p"'] & & B \times Y \arrow[dl, "p_1"] \\ & B & \end{tikzcd}$$

commute. (Projection on the first factor is always denoted by p_1.) Given this trivialization and a map $f\colon B' \to B$, the *induced trivialization* $f^*h = h'$ of the *induced map* $f^*\xi = (p', E', B')$ is defined like this. Put

$$E' = \{(x, y) \in B' \times E \mid f(x) = p(y)\} ,$$

and $p'(x, y) = x$. Then $h'\colon E' \to B' \times Y$ is given by $h'(x, y) = (x, h(y))$.

A *local Y trivialization* of a map (p, E, B) is a pair (U, h) where h is a Y trivialization of $(p \mid p^{-1}U, U)$, and $U \subset B$ is open. A *Y atlas* for (p, E, B) is a family $\{(U_i, h_i)\}$ of local Y trivializations such that $B = \bigcup U_i$. If a Y atlas exists, then (p, E, B) is called a *Y bundle.*

The *trivial* Y bundle over B is $(p_1, B \times Y, B)$.

In order to deal with polyhedra rather than simplicial complexes, we shall consider only s.s. groups of homeomorphisms of Y having the following property. If $f\colon \Delta_k \times Y \to \Delta_k \times Y$ is in the group, and $g\colon \Delta_j \to \Delta_k$ is a PL map, then $g^*f\colon \Delta_j \times Y \to \Delta_j \times Y$ is also in the group, where $(g^*f)_x = f_{g(x)}$.

Suppose B is a polyhedron, $\xi = (p, E, B)$ a Y bundle, and $\mathbf{G} = \{G_k\}$ an s.s. group of homeomorphisms of Y. A $\mathbf{G}$ *structure* on ξ is an atlas $\Phi = \{(U_i, h_i)\}_{i \in A}$ for ξ having the following property, and maximal in this property: for each $i, j \in A$, there is an ordered triangulation T of the polyhedron $U_i \cap U_j$ such that, if $t\colon \Delta_k \to U_i \cap U_j$ is an ordered simplex of T, then the following twist belongs to G_k:

$$(t^*h_i)\cdot(t^*h_j)^{-1}\colon \Delta_k \times Y \longrightarrow \Delta_k \times Y\,.$$

Here t^*h_i is the trivialization of the bundle $t^*(\xi \mid U_i)$ over Δ_k induced from h_i by $t\colon \Delta_k \to U_i$. The pair (ξ, Φ) is called a $(\mathbf{G}, Y)$ *bundle.*

It is clear that if $\Phi = \{(U_i, h_i)\}$ is a $\mathbf{G}$ structure for ξ, and if $f\colon B' \to B$ is a PL map, then the $\mathbf{G}$ atlas $f^*\Phi = \{(f^{-1}U_i, f^*h_i)\}$ is contained in a unique $\mathbf{G}$ structure for $f^*\xi$. Therefore if ξ denotes a $(\mathbf{G}, Y)$ bundle, then $f^*\xi$ has a well defined $\mathbf{G}$ structure; thus $f^*\xi$ will also denote a $(\mathbf{G}, Y)$ bundle.

If (ξ, Φ) and (ξ', Φ') are $(\mathbf{G}, Y)$ bundles over B, an *isomorphism* $g\colon \xi \to \xi'$ is a homeomorphism $g\colon E\xi \to E\xi'$ covering 1_B such that if $(U, h) \in \Phi'$, then $(U, hg) \in \Phi$. If such an isomorphism exists, we write $\xi \sim \xi'$.

The *covering homotopy theorem* is true in the sense that if $f_0, f_1\colon B' \to B$ are homotopic PL maps, and ξ is a $(\mathbf{G}, Y)$ bundle over B, then $f_0^*\xi \sim f_1^*\xi$. (See [3].)

The s.s. group of all Y twists (in the proper category $\mathcal{P}$ or $\mathcal{T}$) is denoted by $\mathbf{G}(Y)$. A $(\mathbf{G}(Y), Y)$ bundle is the same thing as a Y bundle over a polyhedron. Let $K(\mathbf{G}, Y)$ denote the contravariant functor (from $\mathcal{P}$ to sets) that assigns to the polyhedron B the set of isomorphism classes of $(\mathbf{G}, Y)$ bundles over B. Put $K(G(Y), Y) = KY$. If Y is endowed with a base point y_0, let $\mathbf{G}(Y, y_0)$ denote the s.s. subgroup of $\mathbf{G}(Y)$ whose k simplices are the twists $\Delta_k \times Y \to \Delta_k \times Y$ leaving $\Delta_k \times y_0$ fixed. Put $K(\mathbf{G}(Y, y_0), Y) = K_0Y$.

Semi-simplicial fibrations

If $A \subset Y$ is a subspace, let $\mathbf{G}(Y, A)$ denote the subgroup of $\mathbf{G}(Y)$ leaving A invariant. (That is, $G_k(Y, A)$ consists of those twists carrying $\Delta_k \times A$ onto itself.) Put $K(\mathbf{G}(Y, A), Y) = K(Y, A)$. If $\mathbf{G}$ is any subgroup of $\mathbf{G}(Y, A)$, there is defined the restriction homomorphism $\mathbf{G} \to \mathbf{G}(A)$ and $\mathbf{G} \to \mathbf{G}(Y - A)$, and the inclusion homomorphism $\mathbf{G} \to \mathbf{G}(Y)$. Each $(\mathbf{G}, Y)$ bundle has associated to it a natural A subbundle; there is defined a morphism $K(\mathbf{G}, Y) \to KA$. Examples of these morphisms are:

∂: $KD^n \to KS^{n-1}$, where $S^{n-1} = \partial D^n$;

INT: $KD^n \to KR^n$, where $R^n = \operatorname{int} D^n$;

COMP: $K_0S^n \to KR^n$, where $R^n = S^n - x_0$.

These morphisms are special cases of the following general fact.

LEMMA 1. *Let* $\mathbf{G}$ *and* $\mathbf{H}$ *be* s.s. *groups of homeomorphisms of spaces* Y *and* Z *respectively. An* s.s. *homomorphism* $\varphi\colon \mathbf{G} \to \mathbf{H}$ *induces a morphism* $\varphi_*\colon K(\mathbf{G}, Y) \to K(\mathbf{H}, Z)$. *If* φ *induces isomorphisms of homotopy groups, then* φ_* *is an isomorphism.*

PROOF. This theorem is a semi-simplicial analogue of a well known fact about topological transformations groups. It may be proved directly through a simplex-by-simplex approach (compare [2]), or classifying spaces can be used, as in [3].

Let D^n denote a topological or PL closed n cell.

LEMMA 2. *Let* $0 \in \operatorname{int} D^n$. *The inclusion homomorphism* $i: \mathbf{G}(D^n, 0) \to \mathbf{G}(D^n)$ *is a homotopy equivalence. Therefore the induced morphism* $i_* K_0 D^n \to KD^n$ *is an isomorphism.*

PROOF. Let $\mathbf{S}(\operatorname{int} D^n)$ denote the singular complex of $\operatorname{int} D^n$ (using PL singular simplices for the category $\mathcal{P}$). Define an s.s. map $e: \mathbf{G}(D^n) \to \mathbf{S}(\operatorname{int} D^n)$ as follows. Given $f \in G_k(D^n)$, define $e(f): \Delta_k \to \operatorname{int} D^n$ by $e(f)(x) = f_x(0)$. Since e is constant on each coset of $\mathbf{G}(D^n, 0)$, there is induced a map

$$\tilde{e}: \mathbf{G}(D^n)/\mathbf{G}(D^n, 0) \longrightarrow \mathbf{S}(\operatorname{int} D^n) .$$

It is easy to see that $\tilde{e}$ is injective; we shall show that e, and hence $\tilde{e}$, is surjective.

In the category $\mathcal{T}$ this is quite easy. Consider D^n as the unit ball in R^n. Given $x \in \operatorname{int} D^n$, let $g^x: D^n \to D^n$ be the homeomorphism taking 0 to x, and mapping each segment $0z$ linearly onto xz for each $z \in \partial D^n$. Given $u: \Delta_k \to \operatorname{int} D^n$ in $S_k(\operatorname{int} D^n)$, define $v: \Delta_k \times D^n \to \Delta_k \times Y$ by $v_x = g^{u(x)}$. Since $e(v) = u$, this proves that $\tilde{e}$ is surjective.

Surjectivity in the PL case follows from Hudson [4]; alternatively, the proof just given for $\mathcal{T}$ can be adapted.

We have proved that $\tilde{e}: \mathbf{G}(D^n)/\mathbf{G}(D^n, 0) \to \mathbf{S}(\operatorname{int} D^n)$ is an isomorphism of s.s. complexes. The projection $\mathbf{G}(D^n) \to \mathbf{G}(D^n)/\mathbf{G}(D^n, 0)$ is an s.s. fibration. Since $\mathbf{S}(\operatorname{int} D^n)$ is contractible, the lemma follows.

Let A and B be subspaces of Y. Let $\mathbf{G}_A(Y) \subset \mathbf{G}(Y)$ and $\mathbf{G}_A(Y, B) \subset G(Y, B)$ be the s.s. subgroups leaving A (pointwise) fixed. Let $r: \mathbf{G}(Y, A) \to \mathbf{G}(A)$ denote the restriction homomorphism. Let *bdry* A denote the boundary of A in Y.

LEMMA 3. *Suppose that* bdry A *is invariant under every homeomorphism of* A, *and has a collar neighborhood in* $Y - \operatorname{int} A$. *Then the sequence*

$$\mathbf{G}_A(Y) \xrightarrow{\ i\ } \mathbf{G}(Y, A) \xrightarrow{\ r\ } \mathbf{G}(A)$$

is an s.s. *fibration.*

PROOF. Let $\mathbf{C}$ = image of $\mathbf{G}(Y, A)$ in $G(A)$ under r. It is easy to see that r induces an isomorphism $\mathbf{G}(Y, A)/\mathbf{G}_A(Y) \to \mathbf{C}$. Therefore it suffices to prove that $\mathbf{C}$ is the union of arc components of $\mathbf{G}(A)$. It must be proved that if $f \in G_k(A)$ has a vertex in $\mathbf{C}$, then $f \in \mathbf{C}$. In other words, given a twist $f: \Delta_k \times A \to \Delta_k \times A$ such that $f \mid \Delta_0 \times A$ extends to a homeomorphism of $\Delta_0 \times Y$, then f extends to a twist $g: \Delta_k \times Y \to \Delta_k \times Y$. Put $A' = \operatorname{bdry} A$, and let $\theta\colon A' \times [0, 2] \to Y - \operatorname{int} A$ be a collaring, that is, an embedding such that $\theta(a, 0) = a$. Let $\Delta_0 = z \in \Delta_k$, and let $f_z\colon A \to A$ extend to a homeomorphism $F\colon Y \to Y$. Let $\varphi\colon \Delta_k \times I \to \Delta_k$ be a strong deformation retraction to z. Now define $g\colon \Delta_k \times \theta(A' \times [0, 1]) \to \Delta_k \times \theta(A' \times [0, 1])$ by:

$$g(x, \theta(a, t)) = (x, \theta(f_{\varphi(x,t)}(a), t)) .$$

Since A' is invariant under $f_{\varphi(x,t)}$, clearly g belongs to $G_k(\theta(A' \times [0,1])$ and leaves invariant $\Delta_k \times \theta(A' \times 0)$ and $\Delta_k \times \theta(A' \times 1)$. Moreover, on $\Delta_k \times \theta(A' \times 0)$, g reduces to f, while on $\Delta_k \times \theta(A' \times 1)$, g is given by

$$g(x, \theta(a, 1)) = (x, \theta(F(a), 1)) .$$

Let $\psi_0\colon A' \times [1, 2] \to A' \times [0, 2]$ be the homeomorphism taking each segment $a \times [1, 2]$ linearly onto $a \times [0, 2]$, preserving orientation. Define

$$\psi_1 = \theta\psi_0\theta^{-1}\colon \theta(A' \times [1, 2]) \longrightarrow \theta(A' \times [0, 2]) .$$

Extend ψ_1 to a homeomorphism $\psi\colon Y - \theta(A' \times [0, 1)) \to Y$ by making ψ the identity on $Y - \theta(A' \times [0, 2])$. Now extend g over $\Delta_k \times (Y - \theta(A' \times [0, 1]))$ by $g = 1 \times \psi F \psi^{-1}$. This completes the proof of Lemma 3.

COROLLARY 4. *Suppose that Y is a manifold such that*

(a) *the restriction $r\colon \mathbf{G}(Y) \to \mathbf{G}(\partial Y)$ is surjective;*

and

(b) *$G_{\partial Y}(Y)$ is contractible.*

Then $r\colon \mathbf{G}(Y) \to \mathbf{G}(\partial Y)$ is a homotopy equivalence. Therefore the morphism $r_*\colon KY \to K\partial Y$ is bijective.

Cones

For any space Y, let CY denote the cone on Y. That is, CY is obtained from $Y \times I$ by collapsing $Y \times 1$ to a point. If Y is a compact polyhedron embedded in R^k, then CY is the join of Y with a point of $R^{k+1} - R^k$; thus CY is also a compact polyhedron.

Our next objective is to define a morphism CYL: $KY \to KCY$ which assigns to the Y bundle $\xi = (p, E, B)$ the CY bundle whose total space is the mapping cylinder $M(p)$ and whose projection is the natural map $M(p) \to B$. In the topological category the definition just given is sufficient; it is easy to see that the isomorphism class of CYL(ξ) depends only on that of ξ. In fact, given an isomorphism $f\colon E \to E'$ of Y bundles $\xi = (p, E, B)$ and $\xi' = (p', E', B)$, there is an obvious extension of f to a homeomorphism $M(p) \to M(p')$ which is an isomorphism CYL $(\xi) \to$ CYL (ξ') of CY bundles. If ξ and ξ' are PL bundles, however, g may fail to be PL, as simple examples illustrate. For this reason, CYL must be defined more carefully for the PL category.

Identify Y with a subspace of CY in the natural way.

LEMMA 5. *Let Y be compact. Then the restriction homomorphism $r: \mathbf{G}(CY, Y) \to \mathbf{G}(Y)$ is a homotopy equivalence.*

PROOF. Lemma 3 will be used, with (Y, A) of Lemma 2 replaced by the present (CY, Y). It must be shown that

(i) $\mathbf{G}_Y(CY)$ is contractible,

and

(ii) $r: \mathbf{G}(CY, Y) \to \mathbf{G}(Y)$ is surjective.

The identification

(iii) $CP \times CQ = C((CP \times Q) \cup (P \times CQ))$ will be used.

To prove (ii), assume inductively that $r_i: G_i(CY, Y) \to G_i(Y)$ is surjective for $i < k$; it must be proved that r_k is then surjective. Given $f: \Delta_k \times Y \to \Delta_k \times Y$ in $G_k(Y)$, an extension of f to a CY twist $g: \Delta_k \times CY \to \Delta_k \times CY$ must be found. We may assume that g is already defined over Δ_k, by the inductive assumption. That is, we assume a homeomorphism h of $\Delta_k \times CY$ which extends $f \mid \partial\Delta_k$, and which commutes with projection on $\partial\Delta_k$. Now $\Delta_k = C(\partial\Delta_k)$. Therefore, by (iii), we have

(iv) $\Delta_k \times CY = C((\Delta_k \times Y) \cup (\partial\Delta_k \times CY))$.

The homeomorphism $f \cup h$ is defined on $(\Delta_k \times Y) \cup (\partial\Delta_k \times CY)$; the cone on $f \cup h$ is a homeomorphism of $\Delta_k \times CY$ which extends $f \cup h$. The identification (iv) has the property that projection on Δ_k is preserved by $C(f \cup h)$. Since the cone on a PL map is again PL, (ii) is proved for $\mathcal{P}$ as well as $\mathcal{T}$.

To prove (i), suppose a k simplex σ of $\mathbf{G}_Y(CY)$ is given whose boundary is at the identity $1 \in \mathbf{G}_Y(CY)$. We must find a homotopy of σ to 1 rel $\partial\sigma$. Passing to homeomorphisms, we are given a twist

$$f: \Delta_k \times CY \longrightarrow \Delta_k \times CY$$

such that $f \mid \Delta_k \times Y = 1$, and $f \mid \partial\Delta_k \times CY = 1$. We must define

$$F: \Delta_k \times CY \times I \longrightarrow \Delta_k \times CY \times I$$

which preserves projection on $\Delta_k \times I$, such that

$$F = 1 \qquad \text{on } (\Delta_k \times Y \times I) \cup (\partial\Delta_k \times CY \times I) \cup (\Delta_k \times CY \times 1)$$

and

$$F = f \qquad \text{on } \Delta_k \times CY \times 0\ .$$

Thus we are given a homeomorphism G of

$$(\Delta_k \times Y \times I) \cup (\partial\Delta_k \times CY \times I) \cup (\Delta_k \times CY \times \partial I)$$

that must be extended over $\Delta_k \times CY \times I$, preserving projection on $\Delta_k \times I$. Put $\Delta_k = C(\partial\Delta_k)$ and $I = C(\partial I)$. Then iterating (iii) gives

$$\Delta_k \times CY \times I$$
$$= C\big((\Delta_k \times Y \times I) \cup (\partial\Delta_k \times CY \times I) \cup (\Delta_k \times CY \times \partial I)\big) .$$

Hence we define $F = CG$. If G is PL, so is F. This completes the proof of Lemma 5.

Now we define the morphism CYL: $KY \to KCY$ to be the composition

$$KY \xrightarrow{r_*^{-1}} K(CY, Y) \xrightarrow{i_*} KCY$$

where i: $\mathbf{G}(CY, Y) \to \mathbf{G}(CY)$ is the inclusion. (It can be shown that the total space of CYL (ξ) can be chosen to be the mapping cylinder of the projection of ξ, but we do not need this fact).

In the special case $Y = S^{n-1}$, identify $CY = D^n$, and observe that

$$\mathbf{G}(D^n, S^{n-1}) = \mathbf{G}(D^n) .$$

LEMMA 6. *The morphisms* CYL: $KS^{n-1} \to KD^n$ *and* $r_* = \partial$: $KD^n \to KS^{n-1}$ *are inverse to each other.*

Commutativity relations

If $\eta = (p, E, B)$ is a D^n bundle, define a $D^n \times I$ bundle

$$\eta \oplus \delta^1 = (p', E \times I, B) ,$$

where p' is the composition $E \times I \xrightarrow{p_1} E \xrightarrow{p} B$.

LEMMA 7. $\eta \oplus \delta^1 \sim$ CYL (η).

PROOF. To make sense of Lemma 7, identify $(D^n \times I, D^n \times 0)$ with (CD^n, D^n) by a homeomorphism taking $(x, 0)$ to x for $x \in D^n$. Now both $\eta \oplus \delta^1$ and CYL (η) are CD^n bundles. Moreover, each has a $\mathbf{G}(CD^n, D^n)$ structure that, under the restriction homomorphism r: $\mathbf{G}(CD^n, D^n) \to \mathbf{G}(D^n)$, gives η. Since, by Lemma 5, r_*: $K(CD^n, D^n) \to KD^n$ is an isomorphism, it follows that $\eta \oplus \delta^1 \sim$ CYL (η).

Identify S^n with the union $D^n \cup D^n_-$ of two n cells having a common boundary and disjoint interiors. We have $D^n \subset S^n \subset CS^n$. Now identify the pair (CS^n, D^n) with (CD^n, D^n) by a homeomorphism fixed on D^n. Further, identify D^{n+1} with CD^n as in the proof of Lemma 7.

Let d: $\mathbf{G}(D^n) \to \mathbf{G}(S^n)$ be the homomorphism that doubles every homeomorphism of D^n.

LEMMA 8. *Under the identifications just described, the following diagram of morphisms commutes*:

$$\begin{tikzcd} KD^n \arrow[d, "\text{CYL}"'] \arrow[dr, "d_*"] & \\ KD^{n+1} & KS^n \arrow[l, "\text{CYL}"] \end{tikzcd}$$

PROOF. Let ξ be a D^n bundle. By Lemma 7 we replace $\oplus\,\delta^1$ by CYL: $KD^n \to KCD^n = KD^{n+1}$. Both CYL (ξ) and CYL $(d_*\xi)$ are CD^n bundles with group $\mathbf{G}(CD^n, D^n)$ which give rise to the same D^n bundle under the restriction isomorphism $r_*: K(CD^n, D^n) \to KD^n$. Lemma 8 follows.

If $f: Y \to X$ is a homeomorphism, let $f_*: KY \to KX$ denote the obvious isomorphism of functors.

An n cell $E^n \subset \operatorname{int} D^n$ is called *concentric* if there exists a homeomorphism $D^n - \operatorname{int} E^n \approx \partial D^n \times I$. This condition is always satisfied in the PL case.

LEMMA 9. *Let $E^n \subset \operatorname{int} D^n$ be a concentric n cell. There exists a homomorphism $t: \mathbf{G}(D^n) \to \mathbf{G}(D^n, E^n)$ and a homeomorphism $v: D^n \to E^n$ making this diagram commutative*:

$$\begin{tikzcd} & K(D^n, E^n) \arrow[d, "i_*"] \arrow[dr, "r_*"] & \\ KD^n \arrow[ur, "t_*"] \arrow[r, equal] & KD^n \arrow[r, "\approx", "v_*"'] & KE^n \end{tikzcd}$$

PROOF. Let $u: \partial D^n \times I \to D^n - \operatorname{int} E^n$ be a homeomorphism such that $u(x, 0) = x$. Let $v: D^n \to E^n$ be a homeomorphism such that, for $x \in \partial D^n$, $v(x) = u(x, 1)$.

Given an element $f: \Delta_k \times D^n \to \Delta_k \times D^n$ of $G_k(D^n)$, define another element $t_k(f)$ $g: \Delta_k \times D^n \to \Delta_k \times D^n$ as follows. For each $x \in \Delta_k$, $g_x: D^n \to D^n$ is given by

$$g_x \mid E^n = v f_x v^{-1}$$

and

$$g_x \mid D^n - \operatorname{int} E^n = u(f_x \mid \partial D^n \times 1_I)u^{-1} .$$

That is, if $(z, y) \in \partial D^n \times I$, then $g_x u(z, y) = u(f_x(z), y)$. It is easy to see that g is a well defined homomorphism $G_k(D^n) \to G_k(D^n, E^n)$, and that the sequence $\{t_k\}$ is an s.s. homomorphism $t: \mathbf{G}(D^n) \to \mathbf{G}(D^n, E^n)$. Moreover, it is clear that the following diagram commutes:

$$\begin{tikzcd} \mathbf{G}(D^n) \arrow[r, "t"] \arrow[d, "r"] & \mathbf{G}(D^n, E^n) \arrow[d, "i"] \\ \mathbf{G}(\partial D^n) & \mathbf{G}(D^n) \arrow[l, "r"] \end{tikzcd}$$

Since $r_*: KD^n \to K\partial D^n$ is bijective (Lemma 6), the triangle

$$\begin{tikzcd} & K(D^n, E^n) \arrow[d, "r_*"] \\ KD^n \arrow[ur, "t_*"] \arrow[r, equal] & KD^n \end{tikzcd}$$

commutes. The construction of t gives commutativity in the triangle

$$\begin{tikzcd} K(D^n, E^n) \arrow[d, "i_*"'] \arrow[dr, "r_*"] & \\ KD^n \arrow[r, "v_*"'] & KE^n \end{tikzcd},$$

which completes the proof of Lemma 9.

LEMMA 10. *Let $E^n \subset \operatorname{int} D^n$ be a concentric n cell. The restriction morphism r_*: $K(\operatorname{int} D^n, E^n) \to KE^n$ is bijective.*

PROOF. By Lemma 3, it suffices to prove that

(i) $\mathbf{G}_{E^n}(\operatorname{int} D^n)$ is contractible.

and

(ii) r: $\mathbf{G}(D^n, E^n) \to \mathbf{G}(E^n)$ is surjective.

The restriction homomorphism $\mathbf{G}_{E^n}(\operatorname{int} D^n) \to \mathbf{G}_{\partial D^n}(\operatorname{int} D^n - \operatorname{int} E^n)$ is clearly an isomorphism. Let $R_+ = [0, \infty)$, and identify $\operatorname{int} D^n - \operatorname{int} E^n$ with $S^{n-1} \times R_+$. From [2, Lem. 6] it follows that $\mathbf{G}_{S^{n-1}\times 0}(S^{n-1} \times R_+)$ is contractible, which proves (i). And (ii) is obvious from the identification above.

Recall the identification $S^n = D^n \cup D^n_-$, and the doubling homomorphism d: $\mathbf{G}(D^n) \to \mathbf{G}(S^n)$. Let $x_0 \in \operatorname{int} D^n$, and let $x_1 \in \operatorname{int} D^n_-$ be the corresponding point. Let $E^n \subset \operatorname{int} D^n$ be a concentric n cell whose interior contains x_0. Let w: $(S^n - x_1, D^n) \approx (\operatorname{int} D^n, E^n)$ be a homeomorphism.

From now on, unlabeled arrows denote inclusion or restriction morphisms.

LEMMA 11. *Under the choice of base points $x_0 \in D^n$ and $x_1 \in S^n$, the following diagram commutes*:

$$\begin{tikzcd} KD^n \arrow[r] \arrow[d] & K(\operatorname{int} D^n) \\ K_0D^n \arrow[d, "d_*"'] & \\ K_0S^n \arrow[r] & K(S^n - x_1) \arrow[uu, "w_*"'] \end{tikzcd}$$

PROOF. The idea behind the proof is this. We must prove two $\operatorname{int} D^n$ bundles isomorphic. One, say ξ, is the complement of a section of an S^n bundle which is the double of a D^n bundle η. Thus ξ contains an E^n subbundle isomorphic to η. The other $\operatorname{int} D^n$ bundle is just $\operatorname{int} \eta$. It follows from Lemma 9 that η is isomorphic to a D^n bundle containing an E^n subbundle isomorphic to η. Therefore the problem is this: prove that if two $\operatorname{int} D^n$ bundles contain isomorphic E^n subbundles, they are themselves isomorphic. This is proved by Lemma 10. The technical details follow.

Let $t\colon \mathbf{G}(D^n) \to \mathbf{G}(D^n, E^n)$ be the homomorphism of Lemma 9. Since the composition $KD^n \xrightarrow{t_*} K(D^n, E^n) \xrightarrow{i_*} KD^n$ is the identity, it suffices to prove commutativity in

$$\begin{tikzcd}
KD^n \arrow[r, "t_*"] & K(D^n, E^n) \arrow[r] & K(\operatorname{int} D^n, E^n) \arrow[d] \\
K_0D^n \arrow[u] \arrow[d, "d_*"'] & & K(\operatorname{int} D^n) \\
K_0S^n \arrow[rr] & & K(S^n - x_1) \arrow[u, "w_*"'] .
\end{tikzcd}$$

Next, observe that $d_*\colon K_0D^n \to K_0S^n$ factors through $K(S^n, D^n \cup x_1)$. Therefore it suffices to prove commutativity in

$$\begin{tikzcd}
KD^n \arrow[r, "t_*"] & K(D^n, E^n) \arrow[r] & K(\operatorname{int} D^n, E^n) \\
K_0D^n \arrow[u] \arrow[d, "d_*"'] & & \\
K(S^n, D^n \cup x_1) \arrow[rr] & & K(S^n - x_1, D^n) \arrow[uu, "w_*"']
\end{tikzcd}$$

By Lemma 10, the restriction morphism $K(\operatorname{int} D^n, E^n) \to KE^n$ is bijective. Therefore it suffices to prove commutativity in

$$\begin{tikzcd}
KD^n \arrow[r, "t_*"] \arrow[dd, "d_*"'] & K(D^n, E^n) \arrow[d] \\
& KE^n \\
K(S^n, D^n) \arrow[r] & KD^n \arrow[u, "w_{0*}"']
\end{tikzcd}$$

where $w_0 = w \mid D^n$. The composition

$$KD^n \xrightarrow{d_*} K(S^n, D^n) \xrightarrow{r_*} KD^n$$

is obviously the identity. The required commutativity therefore follows from Lemma 9. Lemma 11 is proved.

For any spaces Y, Z, let m (perhaps with subscripts) denote the multiplication homomorphism $\mathbf{G}(Y) \to \mathbf{G}(Y \times Z)$ defined by $m(f) = f \times 1_Z$.

LEMMA 12. *Let Y be a manifold without boundary. Let $h\colon (0, \infty) \to R$ be a homeomorphism fixed on $[1, \infty)$. Then this diagram commutes:*

$$\begin{array}{ccc} K(Y \times [0, \infty)) & \xrightarrow{r_*} & K(Y \times 0) = KY \\ \downarrow & & \swarrow m_* \\ K(Y \times (0, \infty)) & \xrightarrow[h_*]{} & K(Y \times R) \end{array}$$

PROOF. (Note first that the restriction $K(Y \times [0, \infty)) \to K(Y \times (0, \infty))$ exists because $\partial Y = \varnothing$.) By [2, Lem. 6] and Lemma 3, the multiplication morphism $m_{1*}: KY \to K(Y \times [0, \infty))$ is inverse to r_*, which is an isomorphism. Therefore it suffices to prove the commutativity of

$$\begin{array}{ccc} K(Y \times (0, \infty)) & \xleftarrow{m_{1_*}} & KY \\ \downarrow & & \downarrow m^* \\ K(Y \times (0, \infty)) & \xrightarrow[h_*]{} & K(Y \times R)\ . \end{array}$$

Clearly m_{1*} factors through $K(Y \times [0, \infty), Y \times [1, \infty))$, and similarly for m_*. It suffices, therefore, to prove commutative the diagram

$$\begin{array}{ccc} K(Y \times [0, \infty), Y \times [1, \infty)) & \xleftarrow{m_{1_*}} & KY \\ \downarrow r_* & & \downarrow m_* \\ K(Y \times (0, \infty), Y \times [1, \infty)) & \xrightarrow[h_*]{} & K(Y \times R, Y \times [1, \infty))\ . \end{array}$$

Let $s_*: K(Y \times R, Y \times [1, \infty)) \to K(Y \times [1, \infty))$ be the restriction morphism. It is obvious that $s_* m_* = s_* h_* r_* m_{1_*}: KY \to K(Y \times [1, \infty))$. Since s_* is an isomorphism. Lemma 12 is proved.

Identify $R^{n+1} = R^n \times R$, and let $\oplus\, \varepsilon^1: KR^n \to KR^{n+1}$ denote the multiplication morphism m_*.

LEMMA 13. *Let $y_0 \in S^n$ be the base point of D^{n+1} and also of $\partial D^{n+1} = S^n$. Given a homeomorphism $\varphi: S^n - y_0 \approx R^n$, there exists a homeomorphism ψ: $\operatorname{int} D^{n+1} \to R^{n+1}$ such that the diagram*

$$\begin{array}{ccccc} K_0 S^n & \longrightarrow & K(S^n - y_0) & \xrightarrow{\varphi_*} & KR^n \\ \text{CYL} \downarrow & & & & \downarrow \oplus\, \varepsilon^1 \\ K_0 D^{n+1} & \longrightarrow & K(\operatorname{int} D^{n+1}) & \xrightarrow[\psi_*]{} & KR^{n+1} \end{array}$$

commutes.

PROOF. Identify R^n with $S^n - y_0 = \partial(D^{n+1} - y_0)$ by φ. Let

$$\theta: D^{n+1} - y_0 \approx R^n \times [0, \infty)$$

be a homeomorphism such that $\theta(x) = (x, 0)$ for $x \in R^n$. Let $h: R^n \times (0, \infty) \approx R^n \times R$ be a homeomorphism fixed on $R^n \times [0, 1)$. The diagram

$$\begin{tikzcd}
K_0S^n \arrow[r] \arrow[d, "\mathrm{CYL}"'] & KR^n \arrow[r, "\oplus\,\varepsilon^1"] & K(R^n \times R) & \\
K_0D^{n+1} \arrow[r] & K(D^{n+1} - y_0) \arrow[r, "\theta_*"'] & K(R^n \times [0, \infty)) \arrow[r] \arrow[ul] & K(R^n \times (0, \infty)) \arrow[ul, "h_*"']
\end{tikzcd}$$

commutes by Lemma 12. It follows that setting $\psi = h\theta \mid \operatorname{int} D^{n+1}$ proves Lemma 13.

Now the commutativity of the main diagram can be established.

LEMMA 14. *There exist identifications* $\operatorname{int} D^n \approx R^n$, $\operatorname{int} D^{n+1} \approx R^{n+1}$, *and* $S^n - x_0 \approx R^n$, *that make the following diagram commutative*:

$$\begin{tikzcd}
KD^n \arrow[rr, "\mathrm{INT}"] \arrow[dr, "d'"'] \arrow[dd, "\oplus\,\delta^1"'] & & KR^n \arrow[dd, "\oplus\,\varepsilon^1"] \\
& K_0S^n \arrow[ur, "\mathrm{COMP}"'] \arrow[dl, "\mathrm{CYL}"] & \\
KD^{n+1} \arrow[rr, "\mathrm{INT}"'] & & KR^{n+1}
\end{tikzcd}$$

Here INT *and* COMP *are restriction morphisms, and* d' *is the composition* $KD^n \xrightarrow{i_*^{-1}} K_0D^n \xrightarrow{d_*} K_0S^n$.

PROOF. Apply Lemmas 13, 11 and 8.

The proof of the main theorems will be completed as soon as it is proved that COMP: $K_0S^n \to KR^n$ is bijective. This is trivial in the topological category, since every homeomorphism of $\Delta_k \times R^n$ extends uniquely to one of $\Delta_k \times S^n$. (If we were to use topological structure groups instead of semi-simplicial ones, the proof would be harder; it would require at least a proof of the fact [6], well known but not trivial, that the group of homeomorphisms of R^n is indeed a topological group.)

The next lemma completes the proof of the main theorems.

LEMMA 15. *The restriction homomorphism* $\mathbf{G}(S^n, x_0) \to \mathbf{G}(S^n - x_0)$ *is a homotopy equivalence.*

PROOF. We need consider only the PL case. Put $R^n = S^n - x^0$, and let $D^n \subset R^n$ be a closed n cell. Let $\mathbf{E}(D^n, R^n)$ be the s.s. complex of embeddings $\Delta_k \times D^n \to \Delta_k \times R^n$ that preserve projection on Δ_k. Let π: $\mathbf{G}(S^n, x_0) \to \mathbf{E}(D^n, R^n)$ and π': $\mathbf{G}(R^n) \to \mathbf{E}(D^n, R^n)$ denote the restriction maps. Then this diagram commutes:

$$\begin{tikzcd}
\mathbf{G}(S^n, x_0) \arrow[rr] \arrow[dr, "\pi"'] & & \mathbf{G}(R^n) \arrow[dl, "\pi'"] \\
& \mathbf{E}(D^n, R^n) &
\end{tikzcd}$$

The lemma is proved by showing that π and π' are surjective fibrations with contractible fibres. That they are s.s. fibrations is a special case of a theorem of Hudson [4]. The base space has only two components, since two PL embeddings $D^n \to R^n$ are isotopic if and only if either both of them preserve, or both reverse orientation. Clearly each component meets the images of both π and π'. Since these maps are fibrations, surjectivity follows.

Let $B^n = S^n - \operatorname{int} D^n$, and $\partial D^n = \partial B^n = S^{n-1}$. Then B^n is also a PL n cell. The fibre of π is isomorphic to $\mathbf{G}_{S^{n-1}}(B^n, x_0)$. The proof of (i) in the proof of Lemma 5 shows also that $\mathbf{G}_Y(CY, v)$ is contractible, where $v \in CY$ is the vertex of the cone. Therefore $\mathbf{G}_{S^{n-1}}(B^n, x_0)$ is contractible. The fibre of π' is isomorphic to $\mathbf{G}_{S^{n-1}\times 0}(S^{n-1} \times [0, \infty))$, which is contractible by [2, Lem. 6].

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(Received August 14, 1964)
(Revised February 16, 1966)