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Publication Date

2022-05-31

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The importance of randomization in resource assignment problems

Keith Paarporn, Rahul Chandan, Mahnoosh Alizadeh, and Jason R. Marden

***Abstract*— In this paper, we consider problems involving a central commander that must assign a pool of available resources to two separate competitions. In each competition, a sub-colonel allocates its endowed resources from the assignment against an opponent. We consider General Lotto games as the underlying model of competition. Here, we also take into account that the commander's randomized resource assignments cause the opponents to have uncertainty about the sub-commanders' actual assigned endowments. We find that randomized assignments, which induce General Lotto games of incomplete and asymmetric information in the component competitions, do not offer strategic advantages over deterministic ones when the opponents have fixed resource endowments. However, this is not the case when the opponents have per-unit costs to utilize resources. We find the optimal randomized assignment strategy can actually improve the commander's payoff two-fold when compared to optimal deterministic assignments.**

I. Introduction

The problem of how to allocate a limited amount of resources is central to problems involving strategy in adversarial environments. For example, the scheduling of security patrols in airports, border control, and wildlife conservation constitute a diverse set of resource allocation problems that are consequential to the security of modern infrastructures [14], [17], [19]. Strategies for military defense are also largely based on resource allocation. Here, a single decision-making entity is often not entirely responsible for how all resources are to be allocated. This can be due to several reasons: it may not have the capability to operate the entire pool of resources due to computational or complexity constraints. Moreover, a hierarchical decision-making structure may exist where this is simply not possible. Instead, the responsibility of allocating resources against adversaries is delegated to multiple decision-makers.

In this paper, we focus on the problem of how a commander should assign a pool of resources to multiple sub-colonels that engage in competition against their opponents, so as to maximize cumulative payoffs. The role of randomizing such assignments is significant, as evidenced by the development and implementation of algorithms in the aforementioned application domains. Indeed, randomizing the assignments causes opponents to hold uncertainty about the actual resource endowments of the sub-colonels. We take this feature into account, where we consider General Lotto games as the underlying model in each competition. Specifically, a randomized assignment induces General Lotto games of incomplete information as the component competitions. Figure 1 depicts the scenario under consideration.

We completely characterize the commander's optimal assignment strategies when there are two sub-colonels. Our main conclusions show that randomized assignments do not offer any strategic advantages over deterministic assignments when the opponents are known to have fixed resource endowments. This, however, is not the case in settings where the opponents instead have a per-unit cost to invest in resources. In fact, the optimal (randomized) assignment strategy can improve the commander's payoff two-fold in comparison to the best deterministic strategies.

The Colonel Blotto game models two competing opponents that strategically allocate their endowed resources across a set of battlefields. It is known that such solutions are notoriously difficult to derive, and characterizations for the most general settings are still unknown [2], [5], [15], [16], [18]. The General Lotto game is a popular variant of the Colonel Blotto game. It offers more analytical tractability because it relaxes budget support constraints that are required for mixed strategies in Blotto games [8], [11]. For this reason, Lotto games are versatile models of competition for which to study more complex adversarial settings [4], [6], [7], [10].

Settings with incomplete information have received attention in the recent literature [1], [3], [9], [12], [13]. For the commander assignment problem under consideration in this paper, randomized assignments induce incomplete information Lotto competitions where the opponent is uncertain about the sub-colonel's resource budget. We will leverage equilibrium characterizations of such interactions from [13] to address the commander assignment problem.

The paper is organized as follows. In Section II, we provide preliminaries on General Lotto games and two models of incomplete information. In Section III, we define the commander assignment problems and summarize the main results. Section IV provides characterizations of optimal assignment strategies for a parameter regime of interest that highlights the importance of randomized assignments. The full characterizations are provided in the Appendix.

II. Preliminaries

We begin with a brief description of the commander assignment problem (detailed formulation given in Section III), and give preliminary details about General Lotto games of complete and incomplete information.

This work is supported by UCOP Grant LFR-18-548175, ONR grant #N00014-20-1-2359, AFOSR grants #FA9550-20-1-0054 and #FA9550-21-1-0203, and the Army Research Lab through the ARL DCIST CRA #W911NF-17-2-0181. A full version of this paper is currently available online as a pre-print [13].

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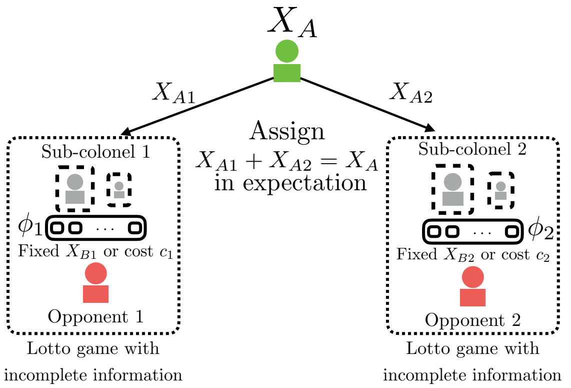

Fig. 1: A commander must decide how to assign available resources to two sub-colonels competing over distinct sets of battlefields. Here, the assignment strategy of the commander allocates a total of X_A resources to two sub-colonels in expectation. The sub-colonels' endowments X_{Ai} ($i = 1, 2$) are drawn from the commander's (possibly randomized) assignment strategy. Opponent i thus engages in an incomplete information Lotto game where it does not observe sub-colonel i's true endowment. They compete over a set of battlefields worth ϕ_i in total. Here, we consider two possible models for the opponents. Both either have fixed resource endowments X_{Bi}, or unlimited available resources but pay per-unit costs c_i to utilize them. This paper identifies the commander's assignment strategy that maximizes the cumulative payoff associated with the two sub-colonels.

A. Commander assignment problem

The task of the commander is to assign resources to two sub-colonels $A1$ and $A2$ that engage in separate competitions against their opponents. We will consider General Lotto games as the underlying model of competition in this paper. The commander has a limited pool of total resources to assign and seeks to maximize the cumulative payoff from both competitions (see Figure 1). We assume that sub-colonels and their opponents will play equilibrium strategies in their respective competitions. The opponents' beliefs about the sub-colonels' resource endowments can be shaped by the commander's assignment distribution. In particular, randomized assignments induce opponents to engage in Lotto games of incomplete information, whereas deterministic assignments induce complete information Lotto games.

B. Background on General Lotto games

A General Lotto game (GL) consists of two players A and B, that have limited resource endowments X_A, X_B, who compete over n battlefields with values $v_j \geq 0$, $j \in [n]$. A player wins battlefield j and its value v_j if it allocates more resources than its opponent. The losing player on j gets zero value. An allocation for player $\ell \in \{A, B\}$ is any vector $\boldsymbol{x}_\ell = \{x_{\ell,j}\}_{j\in[n]} \in \mathbb{R}^n_+$ (non-negative real vectors). An admissible strategy for player $\ell \in \{A, B\}$ is any n-variate distribution F_ℓ over allocations $\boldsymbol{x}_\ell \in \mathbb{R}^n_+$ that satisfies the condition

$$\mathbb{E}_{\boldsymbol{x}_\ell \sim F_\ell}\left[\sum_{j\in[n]} x_{\ell,j}\right] \leq X_\ell. \tag{1}$$

In words, a player can randomize its allocation, as long as it does not exceed its endowment *in expectation*. We denote $\mathcal{L}(X_\ell)$ as the set of all admissible strategies that satisfy (1). The GL game is a two-player simultaneous move game with constant-sum payoffs. The payoff function is given by

$$U_\ell(F_\ell, F_{-\ell}) := \sum_{j\in[n]} v_j \int_0^\infty F_{-\ell,j}(x_{\ell,j}) dF_\ell(x_{\ell,j}) \tag{2}$$

where the integral term is the probability that ℓ allocates more resources to battlefield j. The equilibrium characterization of all GL games are widely known [8], [10], [11], which admit unique payoffs.

C. Incomplete information Lotto games

To address the commander assignment problem, equilibrium characterizations of General Lotto games with incomplete information are required. Accordingly, we consider settings where player A's endowment is assigned randomly before play. Player B does not observe the true realization of A's endowment. Note that player A here plays the role of a "sub-colonel" and B is the opponent. Two models of incomplete information Lotto games are defined below. In the first model, player B has a fixed resource endowment. In the second model, player B has a per-unit cost to invest in resources.

Definition 1. *In the* Bayesian General Lotto game *(BL), player A's endowment is assigned as X_1 with probability p, and X_2 with probability $1 - p$ where $X_1 \geq X_2$. A strategy for player A is a pair $F_A := (F_A(t_1), F_A(t_2)) \in \mathcal{L}(X_1) \times \mathcal{L}(X_2)$, where t_1 indicates the "high" endowment type X_1, and t_2 is the "low" endowment type X_2. Player B does not observe which type is realized, and itself has a fixed endowment X_B that is common knowledge. A strategy for player B is any $F_B \in \mathcal{L}(X_B)$. The expected payoffs to each player are given by*

$$\begin{aligned}\Pi_A &:= p \cdot U_A(F_A(t_1), F_B) + (1-p) \cdot U_A(F_A(t_2), F_B) \\ \Pi_B &:= p \cdot U_B(F_B, F_A(t_1)) + (1-p) \cdot U_B(F_B, F_A(t_2))\end{aligned} \tag{3}$$

The tuple $G = (X_1, X_2, p, X_B)$ represents one instance of this game. The class of all such game instances is denoted $\mathcal{G}$ with arbitrary member G.

Definition 2. *In the* Bayesian General Lotto game with costs *(BLC), the model for player A remains the same as in Definition 1. However, player B has an unlimited resource endowment but pays a per-unit cost $c \geq 0$ to allocate resources. That is, a strategy for player B is any $F_B \in \mathcal{L}(X_B)$ where X_B is any positive number. While player A's payoff function from* (3) *remains the same, player B's payoff here is given by*

$$\Pi_B - c \cdot \mathbb{E}_{\boldsymbol{x}_B \sim F_B}\left[\sum_{j\in[n]} x_{B,j}\right]. \tag{4}$$

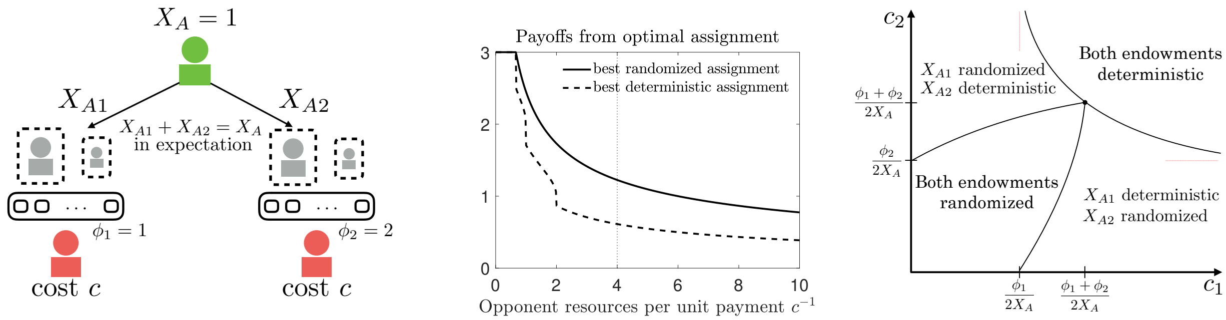

Fig. 2: (Left) An example setup where the commander must assign $X_A = 1$ resources to the two sub-colonels in expectation against two opponents with the same per-unit cost $c_1 = c_2 = c$ to utilize resources. Here, $\phi_1 = 1$ and $\phi_2 = 2$. (Center) This plot compares the commander's payoffs from using optimal randomized assignment strategies versus optimal deterministic ones. The x-axis indicates the amount of resources an opponent obtains from a unit payment, c^{-1}. We note that the best randomized assignment improves upon the best deterministic assignment two-fold for $c^{-1} > 2$. For the particular case $c^{-1} = 6$, the commander's optimal payoff is 0.5 for deterministic and 1 for randomized assignments. The optimal deterministic assignment is $X_{A1} = 1/3$ (from Theorem 4.1). The optimal randomized assignment strategy (calculated from Theorem 4.2) is to assign (X_{A1}, X_{A2}) as $\{(0,0),(0,2),(1,0),(1,2)\}$ with probabilities $\{\frac{4}{9},\frac{2}{9},\frac{2}{9},\frac{1}{9}\}$. (Right) This diagram shows parameter regions where the optimal commander assignment strategy takes on varying degrees of randomization for the endowments of each sub-colonel.

The tuple $\hat{G} = (X_1, X_2, p, c)$ represents one instance of this game. The class of all such game instances is denoted $\hat{\mathcal{G}}$ with arbitrary element $\hat{G}$.

Complete equilibrium characterizations of both BL and BLC games are available in [13]. Indeed, these results allow us to investigate the problem of how a commander should (randomly) assign resources to two sub-colonels. We are interested in whether randomized assignments offer advantages over deterministic ones.

III. Problem formulation and summary of results

The BL and BLC games represent two possible models of engagement between a sub-colonel and its opponent. In this paper, we consider the problem in which the commander must assign, in expectation, a total amount X_A among two sub-colonels (see Figure 1). The assignment strategies available to the commander are the set of probability distributions P on $\mathbb{R}^2_+$ such that each marginal P_i, $i = 1,2$ has at most two values in its support. Furthermore, we require the commander assigns no more than X_A resources to the two sub-colonels in expectation:

$$\sum_{(X_{A1},X_{A2})\in\text{supp}(P)} P(X_{A1},X_{A2}) \cdot (X_{A1} + X_{A2}) \leq X_A \quad (5)$$

where X_{Ai} is the resource endowment assigned to sub-colonel i. Let us denote $\mathcal{P}(X_A)$ as the set of all such distributions P. As such, the marginal P_i is associated with a probability $p_i := P_i(X_{i,1})$ on a high endowment $X_{i,1}$, and the probability $1 - p_i$ on a low endowment $X_{i,2}$, for sub-colonel i. A pair (X_{A1}, X_{A2}) is drawn according to P, upon which sub-colonel i selects an allocation strategy belonging to $\mathcal{L}(X_{Ai})$.

The commander's objective is to select $P \in \mathcal{P}(X_A)$ that maximizes the expected cumulative equilibrium payoff that the sub-colonels derive in their respective engagements. We first consider the problem where the engagements are specified by BL games.

Definition 3. *The* commander assignment problem with respect to BL games *is defined by the optimization problem*

$$\max_{P\in\mathcal{P}(X_A)} \{\pi_A(X_{1,1}, X_{1,2}, p_1, X_{B1}|\phi_1) + \pi_A(X_{2,1}, X_{2,2}, p_2, X_{B2}|\phi_2)\} \quad (6)$$

where $\pi_A(G|\phi)$ indicates the equilibrium payoff to player A in a BL game $G \in \mathcal{G}$ (derived in [13]) with total battlefield value ϕ. Here, the relevant parameters are specified by a total pool of resources $X_A > 0$, fixed opponent budgets X_{B1} and X_{B2}, and total values ϕ_1, $\phi_2 > 0$ for each set of battlefields.

We find that randomized assignment strategies offer no payoff benefits in comparison to the optimal deterministic assignment reported in [10]. Indeed, we show that the optimization problem (6) can be reduced to deterministic assignment strategies. We next consider a second scenario where the engagements are specified by BLC games.

Definition 4. *The* commander assignment problem with respect to BLC games *is defined by the optimization problem*

$$\max_{P\in\mathcal{P}(X_A)} \{\hat{\pi}_A(X_{1,1}, X_{1,2}, p_1, c_1|\phi_1) + \hat{\pi}_A(X_{2,1}, X_{2,2}, p_2, c_2|\phi_2)\} \quad (7)$$

where $\hat{\pi}_A(\hat{G}|\phi)$ denotes the (best-case) equilibrium payoff to player A in a BLC game $\hat{G} \in \hat{\mathcal{G}}$ with total battlefield value ϕ (derived in [13]). Here, c_i is the per-unit cost for sub-colonel i's opponent to utilize resources.

Under BLC engagements, we find the central commander's cumulative payoff can improve two-fold when utilizing an optimal randomized assignment compared to an optimal deterministic assignment. An example of this comparison is shown in Figure 2, which highlights the payoff improvements the commander attains. A two-fold improvement is attainable for a particular subset of parameters which we highlight in Theorems 4.1 and 4.2. The full characterization, i.e. for all possible parameters, of the commander's optimal deterministic and randomized assignment strategies are provided in Appendix B.

IV. Main results

In this section, we address the commander assignment problems (6) and (7) for selected subset of parameters in order to highlight the importance of randomization. Full characterizations are given in the Appendix.

A. Assignment problem with respect to BL engagements

Let us consider the commander's problem (6). Here, the opponents have fixed resource endowments X_{B1} and X_{B2} and thus the engagements are BL games. The commander's optimal assignment strategy is actually completely deterministic – randomized assignment strategies do not provide any strict benefits over deterministic ones. Any assignment P that assigns X_{A1} to sub-colonel 1 in expectation can do no better than the strategy that simply assigns X_{A1} to sub-colonel 1 (and $X_A - X_{A1}$ to sub-colonel 2) with probability one. This result follows from the characterization of equilibrium payoffs in BL games:

Fact 4.1. *For any BL game $G \in \mathcal{G}$, it holds that $\pi_A(G) \leq \pi_A^{CI}(G)$, where*

$$\pi_A^{CI}(G) := \phi \cdot \begin{cases} \frac{\bar{X}}{2X_B}, & \text{if } \bar{X}/X_B < 1 \\ 1 - \frac{X_B}{2\bar{X}}, & \text{if } \bar{X}/X_B \geq 1 \end{cases} \tag{8}$$

is the equilibrium payoff from the complete information Lotto game in which player A's assigned the endowment $\bar{X} := pX_1 + (1-p)X_2$ with probability one. Here, ϕ is the sum of battlefield valuations.

The proof of the above fact is reported in [13]. Thus, any randomized assignment strategy $P \in \mathcal{P}$ weakly underperforms its corresponding deterministic assignment when the opponents have fixed resource endowments. We can then restrict attention to deterministic assignment strategies. Indeed, the optimal deterministic assignment strategies been characterized in the literature – see [10].

B. Assignment problem with respect to BLC engagements

Let us now consider the commander's problem (7). Here, the opponents have no limit on the amount of resources available, but pay the per-unit costs c_1, c_2, respectively, to utilize them. In the following analyses, we first consider deterministic assignment strategies. Then, we consider randomized strategies $P \in \mathcal{P}(X_A)$. We derive the optimal strategies and payoffs in both cases for a specific parameter regime, and provide some comparisons. The full characterizations for all parameters, for both settings, are given in the Appendix.

Deterministic assignments: A deterministic assignment strategy is any $X_{A1} \in [0, X_A]$, which assigns X_{A1} resources to sub-colonel 1 and $X_A - X_{A1}$ to sub-colonel 2. The commander's objective is written as

$$\begin{aligned} &\max_{X_{A1} \in [0,X_A]} U_A(X_{A1}) \\ &:= \max_{X_{A1} \in [0,X_A]} \{\hat{\pi}_A^{\text{CI}}(X_{A1}, c_1|\phi_1) + \hat{\pi}_A^{\text{CI}}(X_A - X_{A1}, c_2|\phi_2)\} \end{aligned} \tag{9}$$

where $\hat{\pi}_A^{\text{CI}}(X, c|\phi)$ is the equilibrium payoff from a BLC game in which A is assigned the endowment X w.p. one (provided in Appendix A). The following result highlights the commander's optimal deterministic assignment X_{A1}^* for a subset of parameters. The particular parameter subset highlighted in the result below is of interest because, as we will see, randomized assignments offer a two-fold payoff gain over the best deterministic assignments.

Theorem 4.1. *Suppose $c_1 < \frac{\phi_1}{2X_A}$ and $c_2 < \frac{\phi_2}{2X_A}$. Then the optimal deterministic assignment is given by $X_{A1}^* = \frac{c_1\phi_1}{c_1\phi_1 + c_2\phi_2} X_A$, which gives a payoff $U_A(X_{A1}^*) = \sqrt{\frac{X_A(c_1\phi_1 + c_2\phi_2)}{2}}$.*

Randomized assignments: Suppose now the commander is able to implement randomized assignments, i.e. any distribution $P \in \mathcal{P}(X_A)$. The following result explicitly characterizes the optimal assignment strategy P^* for the commander for a subset of the parameters. This subset contains the parameter region specified in Theorem 4.1. The optimal randomized assignment of a fixed budget X_A under all other parameters are characterized in Appendix B.

Theorem 4.2. *Define for $i = 1, 2$,*

$$r_i(c_{-i}) := \frac{1}{2}\left[\frac{\phi_i}{2X_A} + \sqrt{\left(\frac{\phi_i}{2X_A}\right)^2 + \frac{2c_{-i}\phi_{-i}}{X_A}}\right]. \tag{10}$$

Suppose $c_1 \leq r_1(c_2)$ and $c_2 \leq r_2(c_1)$. Let $X_{A1}^* = \frac{c_1\phi_1 X_A}{c_1\phi_1 + c_2\phi_2}$, $X_{A2}^* = \frac{c_2\phi_2 X_A}{c_1\phi_1 + c_2\phi_2}$, $p_1^* = \sqrt{2c_1 X_{A1}^*/\phi_1}$, and $p_2^* = \sqrt{2c_2 X_{A2}^*/\phi_2}$. Then the optimal assignment $P^* \in \mathcal{P}(X_A)$ is given by: $P^*(0,0) = (1-p_1^*)(1-p_2^*)$, $P^*\left(0, \sqrt{\frac{X_{A2}^*\phi_2}{2c_2}}\right) = (1-p_1^*)p_2^*$, $P^*\left(\sqrt{\frac{X_{A1}^*\phi_1}{2c_1}}, 0\right) = p_1^*(1-p_2^*)$, and $P^*\left(\sqrt{\frac{X_{A1}^*\phi_1}{2c_1}}, \sqrt{\frac{X_{A2}^*\phi_2}{2c_2}}\right) = p_1^* p_2^*$. The resulting payoff to the commander is $U_A(P^*) = \sqrt{2X_A(c_1\phi_1 + c_2\phi_2)}$.

This result indicates the commander can improve its payoff two-fold over the best deterministic assignment by randomizing both sub-colonels' endowments. In other parameter regimes, the optimal assignment ranges from randomizing a single sub-colonel's endowment to using a completely deterministic assignment for both (see Appendix B).

V. Conclusions

This paper considers the problem of how limited resources should be assigned to multiple decision-makers in adversarial environments. We considered a class of resource allocation problems under the framework of General Lotto games, where a commander decides how to split assets to two sub-colonels. We identified settings where the commander can exploit informational asymmetry against opponents by randomizing its resource assignments to the sub-colonels. Specifically, randomization induces the opponents to hold uncertainty about the sub-colonels' resource endowments. When the opponents have fixed resource budgets, probabilistic assignments offer no benefits over completely deterministic assignments. When the opponents have per-unit costs, we find optimal assignments can improve the commander's payoff two-fold in comparison to the optimal deterministic assignment.

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APPENDIX

This appendix contains the complete characterizations and proofs for the commander's optimal assignment strategies in all parameter regimes. The results are given for both deterministic and randomized assignments with respect to BLC games. As such, Theorems 4.1 and 4.2 are subsumed as special cases.

A. Commander's optimal deterministic assignments

The optimal deterministic assignment, i.e. the X_{A1}^* that solves (9), is given below for any set of parameters $X_A, \{c_i, \phi_i\}_{i=1,2}$. We note the result of Theorem 4.1 is subsumed as Case 1 below.

Lemma A.1. *The optimal deterministic assignment is given in the following cases. Denote* $Q_i := \sqrt{\frac{c_i\phi_i(X_A - \frac{\phi_{-i}}{2c_{-i}})}{2}}$ *for* $i = 1,2$ *and define* $r_i(c_{-i}) := \frac{1}{2}\left[\frac{\phi_i}{2X_A} + \sqrt{\left(\frac{\phi_i}{2X_A}\right)^2 + \frac{2c_{-i}\phi_{-i}}{X_A}}\right]$.

Case 1: $c_1 < \frac{\phi_1}{2X_A}$, $c_2 < \frac{\phi_2}{2X_A}$. *Then* $X_{A1}^* = \frac{c_1\phi_1}{c_1\phi_1 + c_2\phi_2}X_A$ *and* $U_A(X_{A1}^*) = \sqrt{\frac{X_A(c_1\phi_1+c_2\phi_2)}{2}}$.

Case 2: $c_1 \geq \frac{\phi_1}{2X_A}$, $c_2 < \frac{\phi_2}{2X_A}$. *If* $c_1 \geq r_1(c_2)$, *then* $X_{A1}^* = \frac{\phi_1}{2c_1}$ *and* $U_A(X_{A1}^*) = \phi_1 + Q_2$. *If* $c_1 < r_1(c_2)$, *then* X_{A1}^* *is given by*

$$\begin{cases} \frac{\phi_1}{2c_1}, & \text{if } \phi_1 + Q_2 \geq \sqrt{\frac{X_A(c_1\phi_1+c_2\phi_2)}{2}} \\ \frac{c_1\phi_1}{c_1\phi_1+c_2\phi_2}X_A, & \text{if } \phi_1 + Q_2 < \sqrt{\frac{X_A(c_1\phi_1+c_2\phi_2)}{2}} \end{cases} \tag{11}$$

and the payoff is

$$U_A(X_{A1}^*) = \max\left\{\phi_1 + Q_2, \sqrt{\frac{X_A(c_1\phi_1+c_2\phi_2)}{2}}\right\}. \tag{12}$$

Case 2': $c_1 < \frac{\phi_1}{2X_A}$, $c_2 \geq \frac{\phi_2}{2X_A}$. *If* $c_2 \geq r_2(c_1)$, *then* $X_{A1}^* = X_A - \frac{\phi_2}{2c_2}$ *and* $U_A(X_{A1}^*) = \phi_2 + Q_1$. *If* $c_2 < r_2(c_1)$, *then* X_{A1}^* *is given by*

$$\begin{cases} X_A - \frac{\phi_2}{2c_2}, & \text{if } \phi_2 + Q_1 \geq \sqrt{\frac{X_A(c_1\phi_1+c_2\phi_2)}{2}} \\ \frac{c_1\phi_1}{c_1\phi_1+c_2\phi_2}X_A, & \text{if } \phi_2 + Q_1 < \sqrt{\frac{X_A(c_1\phi_1+c_2\phi_2)}{2}} \end{cases} \tag{13}$$

and $U_A(X_{A1}^*) = \max\left\{\phi_2 + Q_1, \sqrt{\frac{X_A(c_1\phi_1+c_2\phi_2)}{2}}\right\}$.

Case 3: $c_i > \frac{\phi_i}{2X_A}$ $(i = 1,2)$ *and* $X_A - \frac{\phi_2}{2c_2} < \frac{\phi_1}{2c_1}$. *Then*

$$X_{A1}^* = \begin{cases} X_A - \frac{\phi_2}{2c_2}, & \text{if } \phi_2 + Q_1 \geq \phi_1 + Q_2 \\ \frac{\phi_1}{2c_1}, & \text{if } \phi_2 + Q_1 < \phi_1 + Q_2 \end{cases} \tag{14}$$

and $U_A(X_{A1}^*) = \max_{i=1,2}\{\phi_i + Q_{-i}\}$.

Case 4: $X_A - \frac{\phi_2}{2c_2} \geq \frac{\phi_1}{2c_1}$. *Then* $X_{A1}^* \in [\frac{\phi_1}{2c_1}, X_A - \frac{\phi_2}{2c_2})$ *and* $U_A(X_{A1}^*) = \phi_1 + \phi_2$.

Proof. The proof relies on utilizing the characterization of equilibrium payoffs in sub-colonel i's BLC game with complete information, i.e. sub-colonel i has the resource endowment $X_{Ai} \geq 0$ with probability 1. It is given as follows:

$$\hat{\pi}_A^{\text{CI}}(X_{Ai}, c_i|\phi_i) = \begin{cases} \sqrt{\frac{c_i X_{Ai}}{2}}, & \text{if } c_i < \frac{\phi_i}{2X_{Ai}} \\ \phi_i, & \text{if } c_i > \frac{\phi_i}{2X_{Ai}} \end{cases} \quad (15)$$

for $i = 1, 2$. This allows us to explicitly pose the commander assignment problem (9). Its solution is given as the four cases presented in the statement. For space concerns, we provide the proof for Case 1 only. The proof for other cases are reported in [13].

When $c_1 < \frac{\phi_1}{2X_A}$ and $c_2 < \frac{\phi_2}{2X_A}$, the objective of (9) can be written as $U_A(X_{A1}) = \sqrt{\frac{c_1\phi_1 X_{A1}}{2}} + \sqrt{\frac{c_2\phi_2(X_A - X_{A1})}{2}}$ for all $X_{A1} \in [0, X_A]$. This function is concave and the derivative is zero at $X_{A1}^* = \frac{c_1\phi_1}{c_1\phi_1 + c_2\phi_2} X_A \in [0, X_A]$. ■

B. Commander's optimal randomized assignments

We now provide the complete characterization of the commander's optimal randomized assignment strategies for any set of parameters $c_1, c_2, \phi_1, \phi_2 \geq 0$ (7). The optimal joint assignment strategy P^* ranges from randomizing both sub-colonels' endowments (Case 1), to using a completely deterministic assignment (Case 4). The result of Theorem 4.2 is subsumed as Case 1 below.

Lemma B.1. *Consider the assignment problem* (7)*. The optimal distribution $P^* \in \mathcal{P}(X_A)$ on resource assignments and the resulting payoff $U_A(P^*)$ are given by the following four cases.*

Case 1 *(Theorem 4.2): Suppose $c_1 \leq r_1(c_2)$ and $c_2 \leq r_2(c_1)$. Let $X_{A1}^* = \frac{c_1\phi_1 X_A}{c_1\phi_1 + c_2\phi_2}$, $X_{A2}^* = \frac{c_2\phi_2 X_A}{c_1\phi_1 + c_2\phi_2}$, $p_1^* = \sqrt{2c_1 X_{A1}^*/\phi_1}$, and $p_2^* = \sqrt{2c_2 X_{A2}^*/\phi_2}$. We have $P^*(0,0) = (1-p_1^*)(1-p_2^*)$, $P^*\left(0, \sqrt{\frac{X_{A2}^*\phi_2}{2c_2}}\right) = (1-p_1^*)p_2^*$, $P^*\left(\sqrt{\frac{X_{A1}^*\phi_1}{2c_1}}, 0\right) = p_1^*(1-p_2^*)$, and $P^*\left(\sqrt{\frac{X_{A1}^*\phi_1}{2c_1}}, \sqrt{\frac{X_{A2}^*\phi_2}{2c_2}}\right) = p_1^* p_2^*$. The resulting payoff to the commander is $U_A(P^*) = \sqrt{2X_A(c_1\phi_1 + c_2\phi_2)}$.*

Case 2: *Suppose $c_1 > r_1(c_2)$ and $c_2 < \frac{c_1\phi_2}{2X_A c_1 - \phi_1}$. Let $X_{A1}^* = \frac{\phi_1}{2c_1}$, $X_{A2}^* = X_A - \frac{\phi_1}{2c_1}$, and $p_2^* = \sqrt{2c_2 X_{A2}^*/\phi_2}$. Then $P^*(X_{A1}^*, 0) = 1 - p_2^*$ and $P^*\left(X_{A1}^*, \sqrt{\frac{X_{A2}^*\phi_2}{2c_2}}\right) = p_2^*$. The resulting payoff to the commander is $U_A(P^*) = \phi_1 + \sqrt{2c_2\phi_2\left(X_A - \frac{\phi_1}{2c_1}\right)}$.*

Case 3: *Suppose $c_2 > r_2(c_1)$ and $c_1 < \frac{\phi_1}{2X_A}$, or $r_2(c_1) < c_2 < \frac{c_1\phi_2}{2X_A c_1 - \phi_1}$ and $c_1 \geq \frac{\phi_1}{2X_A}$. Let $X_{A1}^* = X_A - \frac{\phi_2}{2c_2}$, $X_{A2}^* = \frac{\phi_2}{2c_2}$, and $p_1^* = \sqrt{2c_1 X_{A1}^*/\phi_1}$. Then $P^*(0, X_{A1}^*) = 1 - p_1^*$ and $P^*\left(\sqrt{\frac{X_{A1}^*\phi_1}{2c_1}}, X_{A2}^*\right) = p_1^*$. The resulting payoff to the commander is $U_A(P^*) = \phi_2 + \sqrt{2c_1\phi_1\left(X_A - \frac{\phi_2}{2c_2}\right)}$.*

Case 4: *Suppose $X_A - \frac{\phi_2}{2c_2} \geq \frac{\phi_1}{2c_1}$. Then any P^* that satisfies $P^*(X_{A1}^*, X_{A2}^*) = 1$ for some $X_{A1}^* \in [\frac{\phi_1}{2c_1}, X_A - \frac{\phi_2}{2c_2}]$ and $X_{A2}^* = X_A - X_{A1}^*$ is an optimal assignment. The resulting payoff to the commander is $U_A(P^*) = \phi_1 + \phi_2$.*

Proof. Note that any feasible $P \in \mathcal{P}(X_A)$ induces expected endowments $\bar{X}_{Ai} = p_i X_{1,i} + (1-p_i)X_{2,i}$ for each sub-colonel, where $\bar{X}_{A1} + \bar{X}_{A2} = X_A$ is satisfied. For given expected endowments $\bar{X}_{A1}, \bar{X}_{A2}$, the distribution P that maximizes the commander's objective (7) is one whose marginals optimize the sub-colonels' equilibrium payoff in their respective BLC games. The optimal marginals are characterized in [13], and are given by

$$(X_{1,i}^*, X_{2,i}^*, p_i^*) = \begin{cases} \left(\sqrt{\frac{\bar{X}_{Ai}}{2c_i}}, 0, \sqrt{2c_i\bar{X}_{Ai}}\right) & \text{if } c_i < \frac{\phi_i}{2\bar{X}_{Ai}} \\ \left(\bar{X}_{Ai}, \bar{X}_{Ai}, \times\right), & \text{if } c_i \geq \frac{\phi_i}{2\bar{X}_{Ai}} \end{cases} \quad (16)$$

and the resulting payoff to Ai in its BLC game is

$$\Pi_A^*(\bar{X}_{Ai}, c_i|\phi_i) := \begin{cases} \sqrt{2c_i\phi_i\bar{X}_{Ai}} & \text{if } c_i < \frac{\phi_i}{2\bar{X}_{Ai}} \\ \phi_i, & \text{if } c_i \geq \frac{\phi_i}{2\bar{X}_{Ai}} \end{cases}. \quad (17)$$

Thus, the optimization (7) is equivalent to determining the optimal expected endowment to sub-colonel 1:

$$\max_{\bar{X}_{A1} \in [0, X_A]} \{\Pi_A^*(\bar{X}_{A1}, c_1|\phi_1) + \Pi_A^*(X_A - \bar{X}_{A1}, c_2|\phi_2)\} \quad (18)$$

The objective function above takes distinct forms depending on the values of c_1, c_2, ϕ_1, ϕ_2. For space concerns, we provide the proof for Case 1 only. The proof for other cases are reported in [13].

If $c_1 < \frac{\phi_1}{2X_A}$ and $c_2 < \frac{\phi_2}{2X_A}$, the objective can be written as $\sqrt{2c_1\phi_1\bar{X}_{A1}} + \sqrt{2c_2\phi_2(X_A - \bar{X}_{A1})}$, $\forall \bar{X}_{A1} \in [0, X_A]$. This function is concave. Setting the derivative (w.r.t. $\bar{X}_{A1}$) to zero, we obtain $\bar{X}_{A1}^* = \frac{c_1\phi_1}{c_1\phi_1 + c_2\phi_2} X_A$ and the optimal commander's payoff is $U_A(P^*) = \sqrt{2X_A(c_1\phi_1 + c_2\phi_2)}$. Here, the distribution $P^* \in \mathcal{P}(X_A)$ is given by substituting the solution $\bar{X}_{A1}^*$ into the characterization of optimal marginal distributions (16) for each sub-colonel. ■