

Lawrence Berkeley National Laboratory

LBL Publications

Title

Controlling volume fluctuations for studies of critical phenomena in nuclear collisions

Permalink

<https://escholarship.org/uc/item/1521100t>

Authors

Holzmann, Romain

Koch, Volker

Rustamov, Anar

et al.

Publication Date

2024-10-01

DOI

10.1016/j.nuclphysa.2024.122924

Peer reviewed

1 Controlling volume fluctuations for studies of critical phenomena in nuclear collisions

2 Romain Holzmann,^{1,*} Volker Koch,^{2,3,†} Anar Rustamov,^{1,‡} and Joachim Stroth^{4,1,5,§}

3 ¹*GSI Helmholtzzentrum für Schwerionenforschung, 64291 Darmstadt, Germany*

4 ²*Nuclear Science Division Lawrence Berkeley National Laboratory Berkeley, CA, 94720, USA*

5 ³*ExtreMe Matter Institute EMMI, GSI, 64291 Darmstadt, Germany*

6 ⁴*Institut für Kernphysik, Goethe-Universität, 60438 Frankfurt am Main, Germany*

7 ⁵*Helmholtz Research Academy Hesse for FAIR (HFHF),*

8 *Campus Frankfurt , 60438 Frankfurt am Main, Germany*

9 We generalize and extend the recently proposed method [1] to account for contributions of system size (or volume/participant) fluctuations to the experimentally measured moments of particle multiplicity distributions. We find that in the general case there are additional biases which are not directly accessible to experiment. These biases are, however, parametrically suppressed if the multiplicity of the particles of interest is small compared to the total charged-particle multiplicity, e.g., in the case of proton number fluctuations at top RHIC and LHC energies. They are also small if the multiplicity distribution of charged particles per wounded nucleon is close to the Poissonian limit, which is the case at low energy nuclear collisions, e.g., at GSI/SIS18. We further find that mixed events are not necessarily needed to extract the correction for volume fluctuations. We provide the formulas to correct pure and mixed cumulants of particle multiplicity distributions up to any order together with their associated biases.

10
11
12
13
14
15
16
17
18
19

* r.holzmann@gsi.de

† vkoch@lbl.gov

‡ a.rustamov@gsi.de

§ j.stroth@gsi.de

I. INTRODUCTION

One of the main goals of studying relativistic heavy-ion collisions is to explore the structure of the QCD phase diagram. Fluctuations of observed particles carrying quantum numbers of conserved charges, baryon number (B), electric charge and strangeness, represent a powerful tool for this endeavor as the cumulants of their distributions measure the derivatives of the grand-canonical partition function, and thus the pressure (P), with respect to the associated chemical potentials. For example, for a thermal system of volume V and temperature T , the cumulants of the net baryon number distribution, within the Grand Canonical Ensemble (GCE), are given by [2]

$$\kappa_n[B] = \frac{\partial^n (\ln Z)}{\partial (\mu_B/T)^n} = \frac{V}{T} \frac{\partial^n P}{\partial (\mu_B/T)^n},$$

where Z is a GCE partition function and μ_B is a baryon chemical potential. Any nontrivial structures in the equation of state such as a possible phase transition [3–6] will result in potentially large derivatives of the pressure and thus in large values of the cumulants of conserved charges. In addition, as cumulants are derivatives of the pressure, they are accessible (at vanishing or small values of chemical potential) to Lattice QCD calculations [7, 8], which in principle enables a direct comparison of results from ab initio QCD calculations with experiment. For example, as pointed out in Ref. [9], the measurement of higher-order cumulants close to vanishing chemical potential may test the remnants of chiral criticality.

Measurements of fluctuations have meanwhile been carried out by many experiments. The STAR collaboration has measured cumulants of the net-proton number up to sixth order over the entire energy range available at RHIC [10, 11]. The HADES experiment has measured cumulants of proton number up to fourth order at the low energy of $\sqrt{s_{\text{NN}}} = 2.4$ GeV [12] and ALICE has measured the second- and third-order net-proton number cumulants at $\sqrt{s_{\text{NN}}} = 2.76$ and 5.02 TeV [13, 14].

When comparing cumulants measured in experiment with those obtained from lattice QCD or other field theoretical calculations [15] one needs to be aware of several key differences. While theoretical calculations are typically done in the grand canonical ensemble where charges can be exchanged with a heat bath and are only conserved on the average, in experiment charges are explicitly conserved on event by event basis and one has to account for global as well as local charge conservation [16–20]. Also, in experiments one usually is restricted to the measurement of net protons whereas theory can only calculate cumulants of the net baryon number. In the presence of many pions this difference can be corrected for [21]. Finally, and this will be the topic of the present paper, in experiment the size of the particle emitting system is not constant. Even under the tightest centrality selection criteria, this gives rise to so-called volume fluctuations [22] or, equivalently, fluctuations of the number of wounded nucleons [23]. Moreover, there is a strong correlation between the event activity, i.e., charged particle multiplicity and the size of the system, e.g., within the model of independent particle sources [24]. To ensure accurate measurement of the fluctuation of conserved charges, it is crucial to separate the particles used for determining these fluctuations from those used for centrality determination. This separation is necessary to avoid autocorrelations, which can introduce artificial modifications to the fluctuation signals being measured [23]. One effective method to achieve this separation is to perform centrality selection using detectors that cover different rapidity intervals than those used for fluctuation analysis [12, 13]. Alternatively, another approach involves excluding the particles of interest, such as protons in net-proton analysis, from the centrality determination process [25]. However, even with this exclusion, some autocorrelation effects may persist. This persistence is partly due to strong correlations between pions and protons, which arise from the decays of baryonic resonances [23]. Moreover, volume fluctuations can significantly impact the measurements, particularly at lower energies. At these energies, the multiplicity of charged particles is predominantly made up of primordial protons, which can limit the resolution of centrality selection that can be achieved. This limitation underscores the importance of carefully considering the effects of volume fluctuations in the analysis. In Ref. [1] a novel and promising method based on event mixing has been proposed to experimentally determine and subtract the contributions to the cumulants caused by volume fluctuations. In the present work we will further elaborate on this topic, generalize the results, and provide the formulas for corrections of any higher-order cumulants.

This paper is organized as follows. In the next section we define the notation. We then present an analytical formulation of event mixing as proposed in Ref. [1]. We find that the cumulants of the the mixed events have additional bias terms which were assumed to vanish in the original work of [1], and we discuss the magnitude of these corrections for various scenarios. Next we extend our study to cumulants of higher order before we discuss and summarize our results.

II. NOTATION

In this paper we will mostly work within the wounded-nucleon model [24] to discuss volume or participant fluctuations. However, as we shall show later, the formalism can be easily applied also to the situation where one has generic volume fluctuations, as for example discussed in Refs. [22, 23]. Let us start with the expression of the particle number cumulants $\kappa_j[N]$ in the presence of wounded-nucleon fluctuations (for details see Appendix A):

$$\kappa_1[N] = \langle N_w \rangle \kappa_1[n] = \langle N_w \rangle \langle n \rangle = \langle N \rangle \quad (1)$$

$$\kappa_2[N] = \langle N_w \rangle \kappa_2[n] + \langle n \rangle^2 \kappa_2[N_w] = \bar{\kappa}_2[N] + \langle N \rangle^2 \frac{\kappa_2[N_w]}{\langle N_w \rangle^2} \quad (2)$$

$$\kappa_3[N] = \langle N_w \rangle \kappa_3[n] + 3 \langle n \rangle \kappa_2[n] \kappa_2[N_w] + \langle n \rangle^3 \kappa_3[N_w] = \bar{\kappa}_3[N] + 3 \langle N \rangle \bar{\kappa}_2[N] \frac{\kappa_2[N_w]}{\langle N_w \rangle^2} + \langle N \rangle^3 \frac{\kappa_3[N_w]}{\langle N_w \rangle^3} \quad (3)$$

$$\begin{aligned} \kappa_4[N] &= \langle N_w \rangle \kappa_4[n] + 4 \langle n \rangle \kappa_3[n] \kappa_2[N_w] + 3 \kappa_2^2[n] \kappa_2[N_w] + 6 \langle n \rangle^2 \kappa_2[n] \kappa_3[N_w] + \langle n \rangle^4 \kappa_4[N_w] \\ &= \bar{\kappa}_4[N] + 4 \langle N \rangle \bar{\kappa}_3[N] \frac{\kappa_2[N_w]}{\langle N_w \rangle^2} + 3 \bar{\kappa}_2^2[N] \frac{\kappa_2[N_w]}{\langle N_w \rangle^2} + 6 \langle N \rangle^2 \bar{\kappa}_2[N] \frac{\kappa_3[N_w]}{\langle N_w \rangle^3} + \langle N \rangle^4 \frac{\kappa_4[N_w]}{\langle N_w \rangle^4} \end{aligned} \quad (4)$$

Here N refers to the particles of interest, say protons, and n to the number of these particles arising from one wounded nucleon; thus $\langle n \rangle$ is the average number of particles per wounded nucleon. The cumulants of the wounded-nucleon distribution are denoted by $\kappa_j[N_w]$ while the cumulants for the distribution of particles stemming from one wounded nucleon are $\kappa_j[n]$. The corresponding relations for cumulants of any order can be obtained with the provided software package [26].

The cumulants of interest are those at a fixed number of wounded nucleons. They reflect the true density fluctuations in a system at constant volume. We denote these cumulants for a system with fixed, i.e. non-fluctuating, number of $\langle N_w \rangle$ wounded nucleons as

$$\bar{\kappa}_j[N] = \langle N_w \rangle \kappa_j[n],$$

Below we will also deal with factorial cumulants, which we shall denote by C_j . Factorial cumulants, which measure the deviation from Poisson statistics, tell us about the true correlations in the system. As discussed in the Appendix B, they are linear combinations of the regular cumulants. For the first four orders we have

$$\begin{aligned} C_1[N] &= \kappa_1[N] = \langle N \rangle, \\ C_2[N] &= -\kappa_1[N] + \kappa_2[N], \\ C_3[N] &= 2\kappa_1[N] - 3\kappa_2[N] + \kappa_3[N], \\ C_4[N] &= -6\kappa_1[N] + 11\kappa_2[N] - 6\kappa_3[N] + \kappa_4[N]. \end{aligned}$$

The expressions for the particle number factorial cumulants are similar to Eqs. 1- 4

$$C_1[N] = \langle N_w \rangle C_1[n] = \langle N_w \rangle \langle n \rangle = \langle N \rangle, \quad (5)$$

$$C_2[N] = \bar{C}_2[N] + \langle N \rangle^2 \frac{\kappa_2[N_w]}{\langle N_w \rangle^2}, \quad (6)$$

$$C_3[N] = \bar{C}_3[N] + 3 \langle N \rangle \bar{C}_2[N] \frac{\kappa_2[N_w]}{\langle N_w \rangle^2} + \langle N \rangle^3 \frac{\kappa_3[N_w]}{\langle N_w \rangle^3}, \quad (7)$$

$$C_4[N] = \bar{C}_4[N] + 4 \langle N \rangle \bar{C}_3[N] \frac{\kappa_2[N_w]}{\langle N_w \rangle^2} + 3 \bar{C}_2^2[N] \frac{\kappa_2[N_w]}{\langle N_w \rangle^2} + 6 \langle N \rangle^2 \bar{C}_2[N] \frac{\kappa_3[N_w]}{\langle N_w \rangle^3} + \langle N \rangle^4 \frac{\kappa_4[N_w]}{\langle N_w \rangle^4}. \quad (8)$$

Similar to the cumulants, we denote by

$$\bar{C}_k[N] = \langle N_w \rangle C_k[n]$$

the factorial cumulants for a system at constant volume or number of wounded nucleons, $\langle N_w \rangle$.

III. MIXED EVENTS

In Ref. [1] a mixed event is constructed such that it has the same total multiplicity as a given real event but each particle (track) is drawn from a different event, so that, by construction, the mixed events follow the same total

84 multiplicity distribution as the original events. This is done in order to preserve volume fluctuations as in real events.
 85 Since each particle (track) is chosen randomly from a random event, the distribution of particle species will follow a
 86 multinomial distribution with the Bernoulli probabilities $p_i = \langle N_i \rangle / \langle M \rangle$ for particles of type i . Here $\langle N_i \rangle$ denotes
 87 the mean number of particles of type i and $\langle M \rangle$ the mean total multiplicity. Hence, the probability to find A particles
 88 (successes) of type A and B particles of type B is given by the trinomial probability $B_3(A, B, M; p_A, p_B)$ and so on.
 89 Here M denotes the multiplicity of the event under consideration. Thus the distribution, $P_{mix}(A, B)$, of particles of
 90 species A and B in the mixed events is obtained by folding the multiplicity distribution $P_M(M)$ with a trinomial (in
 91 general multinomial) distribution:

$$P_{mix}(A, B) = \sum_M B_3(A, B, M; p_A, p_B) P_M(M)$$

92 with

$$p_A = \frac{\langle A \rangle}{\langle M \rangle}, \quad p_B = \frac{\langle B \rangle}{\langle M \rangle}.$$

and

$$B_3(A, B, M; p_A, p_B) = \frac{M!}{A!B!(M-A-B)!} p_A^A p_B^B (1-p_A-p_B)^{M-A-B} \quad (9)$$

The factorial cumulant-generating function for this distribution is

$$\begin{aligned} g_{F,mix}(z_A, z_B) &= \ln \left[\sum_{A,B} P_{mix}(A, B) (z_A)^A (z_B)^B \right] \\ &= \ln \left[\sum_M [h_3(z_A, z_B)]^M P_M(M) \right] \\ &= G_{F,M}(h_3(z_A, z_B)) \end{aligned} \quad (10)$$

93 where z_A and z_B are auxiliary variables and $h_3(z_A, z_B)$ defined as

$$h_3(z_A, z_B) = \sum_{A,B} B_3(A, B; M=1; p_A, p_B) (z_A)^A (z_B)^B = (1-p_A-p_B+p_A z_A + p_B z_B)$$

94 is the factorial moment-generating function for the trinomial distribution with one trial ($M=1$).

95 The variables z_A and z_B facilitate the computation of factorial cumulants by taking the appropriate derivatives of
 96 the factorial cumulant-generating function for the multiplicity distribution $G_{F,M}(h_3(z_A, z_B))$ (see Eq.(B2))

97

$$C_{i,j}^{mix}[mix] = \frac{\partial^{(i+j)}}{\partial (z_A)^i \partial (z_B)^j} G_{F,M}(h_3(z_A, z_B)) \Big|_{z_A=z_B=1} = p_A^i p_B^j C_{i+j}[M],$$

with $C_k[M]$ being the k^{th} -order factorial cumulant. Using the expression for the factorial cumulants of the multiplicity distribution derived in Appendix C, Eq. C5, we get within the wounded-nucleon model

$$\begin{aligned} C_1^{mix}[A] &= \kappa_1^{mix}[A] = p_A \langle N_w \rangle \langle m \rangle \\ C_2^{mix}[A] &= p_A^2 C_2[M] = p_A^2 \left[\kappa_2[N_w] \langle m \rangle^2 + \langle N_w \rangle C_2[m] \right] \\ C_{1,1}^{mix}[A, B] &= p_A p_B C_2[M] = p_A p_B \left[\kappa_2[N_w] \langle m \rangle^2 + \langle N_w \rangle C_2[m] \right]. \end{aligned} \quad (11)$$

For the corresponding cumulants up to second order we get accordingly

$$\begin{aligned} \kappa_1^{mix}[A] &= C_1^{mix}[A] = p_A \langle N_w \rangle \langle m \rangle \\ \kappa_2^{mix}[A] &= C_2^{mix}[A] + C_1^{mix}[A] = p_A^2 \left[\kappa_2[N_w] \langle m \rangle^2 + \langle N_w \rangle (\kappa_2[m] - \kappa_1[m]) \right] + p_A \langle N_w \rangle \langle m \rangle \\ &= p_A^2 \left[\kappa_2[N_w] \langle m \rangle^2 + \langle N_w \rangle (\kappa_2[m] - \langle m \rangle) \right] + p_A \langle N_w \rangle \langle m \rangle \\ cov^{mix}[A, B] &= C_{1,1}^{mix}[A, B] = p_A p_B \left[\kappa_2[N_w] \langle m \rangle^2 + \langle N_w \rangle (\kappa_2[m] - \langle m \rangle) \right]. \end{aligned} \quad (12)$$

98 With $\langle m \rangle$ denoting the mean number of total particles emitted by a wounded nucleon, we get $\langle a \rangle = p_A \langle m \rangle$ and
 99 $\langle b \rangle = p_B \langle m \rangle$ for the mean number of particles per wounded nucleon of type A and B, respectively, and recover the
 100 results of Ref. [1]. For that we have to assume that the multiplicity distribution per wounded nucleon is Poissonian,
 101 i.e. that $C_2[m] = \kappa_2[m] - \langle m \rangle = 0$. This has been an implicit assumption in Ref. [1], which however is not valid in
 102 general as we shall discuss below.

103 The main benefit of the event mixing is to be able to relate the factorial cumulants of the various multiplicity
 104 distributions, as can be seen from Eq.11. All that enters is the second-order factorial cumulant, $C_2[M]$. The binomial
 105 probabilities, p_A and p_B , are in the sense trivial as they can be determined without any mixed events. Thus we
 106 may express the fluctuations of the wounded nucleons in terms of the factorial cumulant of the track multiplicity
 107 distribution

$$\langle N \rangle^2 \frac{\kappa_2[N_w]}{\langle N_w \rangle^2} = \frac{\langle N \rangle^2}{\langle M \rangle^2} (C_2[M] - \langle N_w \rangle C_2[m]) = \frac{\langle N \rangle^2}{\langle M \rangle^2} (C_2[M] - \bar{C}_2[M]), \quad (13)$$

108 where $\bar{C}_2[M] = \langle N_w \rangle C_2[m]$ is the second-order factorial cumulant for a system of $\langle N_w \rangle$ wounded nucleons *without*
 109 wounded nucleon fluctuations and N stands now for the multiplicity of the particles of interest, i.e., either A or
 110 B . While the factorial cumulant of the multiplicity distribution, $C_2[M]$, is accessible to experiment, that of a non-
 111 fluctuating system, $\bar{C}_2[M]$, is not. Let us, therefore define a bias term, Δ_2 , as

$$\Delta_2 \equiv \frac{\langle N \rangle^2}{\langle M \rangle^2} \bar{C}_2[M]. \quad (14)$$

112 In case of a Poissonian multiplicity distribution for one wounded nucleon the bias term vanishes, i.e., $\Delta_2 = 0$, since
 113 $\bar{C}_2[M] = \langle N_w \rangle C_2[m] = 0$ in this case, and we recover the results of Ref. [1]. Let us furthermore define the corrected
 114 cumulant, $\kappa_2^{corr}[N]$, which is based on measurable quantities only

$$\kappa_2^{corr}[N] = \kappa_2[N] - \frac{\langle N \rangle^2}{\langle M \rangle^2} C_2[M]. \quad (15)$$

Following Eq. 2 and using Eq. 6, the cumulant of the system *without* wounded nucleon fluctuations, $\bar{\kappa}_2[N]$, is given
 by

$$\bar{\kappa}_2[N] = \kappa_2[N] - \langle N \rangle^2 \frac{\kappa_2[N_w]}{\langle N_w \rangle^2} = \kappa_2^{corr}[N] + \Delta_2. \quad (16)$$

115 The bias, Δ_2 , while not directly measurable, may be constrained by a fit to the track multiplicity distribution
 116 within the wounded-nucleon model [24], as it is commonly done [11, 27, 28]. In addition, we note that for protons at
 117 very high collision energies we have $\langle N_p \rangle \ll \langle M \rangle$ so that Δ_2 is suppressed parametrically. This behavior can indeed
 118 be illustrated with simulations as presented in Sec. V. Since cumulants scale with the system size, or in our case with
 119 the number of wounded nucleons, $\langle N_w \rangle$, it is instructive to scale the (factorial) cumulants with the mean number of
 120 particles

$$\frac{\bar{\kappa}_2[N]}{\langle N \rangle} = \frac{\kappa_2[N]}{\langle N \rangle} - \frac{\langle N \rangle}{\langle M \rangle} \left(\frac{C_2[M]}{\langle N \rangle} - \frac{\bar{C}_2[M]}{\langle N \rangle} \right) = \frac{\kappa_2^{corr}[N]}{\langle N \rangle} + \frac{\Delta_2}{\langle N \rangle} \quad (17)$$

121 The scaled bias is then given by

$$\frac{\Delta_2}{\langle N \rangle} = \frac{\langle N \rangle}{\langle M \rangle} \bar{c}_2[M], \quad (18)$$

122 where $\bar{c}_2[M] = \bar{C}_2[M]/\langle M \rangle$ is a scaled factorial cumulant. Typically, the scaled cumulants are of order unity,
 123 $\kappa_j[N]/\langle N \rangle \sim \mathcal{O}(1)$. In addition, the scaled factorial cumulants, $c_k[N] = C_k[N]/\langle N \rangle$, are expected to depend only
 124 weakly on the multiplicity. Therefore, the scaled bias should be much smaller than one, $\Delta_2/\langle N \rangle \ll 1$, for the volume
 125 correction to be reliable.

Finally, one may express the bias term Δ_2 also in terms of cumulants by using the relation between cumulants and
 factorial cumulants (see Appendix B), $C_2[M] = \kappa_2[M] - \langle M \rangle$ and so forth. This gives,

$$\Delta_2 = \frac{\langle N \rangle^2}{\langle M \rangle^2} (\bar{\kappa}_2[M] - \langle M \rangle) \quad (19)$$

126 A note of caution may be useful in this context. One might be inclined to express the fluctuations of the wounded
 127 nucleon directly using the cumulants of the multiplicity distribution, in which case one would get

$$\langle N \rangle^2 \frac{\kappa_2[N_w]}{\langle N_w \rangle^2} = \frac{\langle N \rangle^2}{\langle M \rangle^2} (\kappa_2[M] - \langle N_w \rangle \kappa_2[m]).$$

128 And since $\langle N_w \rangle \kappa_2[m] = \bar{\kappa}_2[M]$ is not directly accessible to experiment, one may further assign the bias to be
 129 $\Delta_2 = \frac{\langle N \rangle^2}{\langle M \rangle^2} \bar{\kappa}_2[M]$. This, however, would considerably overestimate its true value, Eq. 19, as cumulants always contain
 130 a ‘‘trivial’’ component proportional to the number of particles, which in principle is measurable.

131

IV. HIGHER-ORDER RESULTS

Let us now discuss the corrections for volume fluctuations up to fourth order. Given the discussion in the previous section the strategy is straightforward. First we express the fluctuations of the wounded nucleons in terms of factorial cumulants of the multiplicity distribution. Then we identify the parts which are experimentally accessible and those which are not. The latter will be the bias while the former will be subtracted from the expression for the cumulants in order to remove most of the effect of volume fluctuations. The terms involving cumulants of the wounded-nucleon distribution as they appear in the expressions for the cumulants as $\kappa_j[N_w]/\langle N_w \rangle^j$, see Eqs. (6-8) are:

$$\frac{\kappa_2[N_w]}{\langle N_w \rangle^2} = \frac{C_2[M] - \bar{C}_2[M]}{\langle M \rangle^2} \quad (20)$$

$$\frac{\kappa_3[N_w]}{\langle N_w \rangle^3} = -3 \frac{\bar{C}_2[M]}{\langle M \rangle^2} \frac{\kappa_2[N_w]}{\langle N_w \rangle^2} + \frac{C_3[M] - \bar{C}_3[M]}{\langle M \rangle^3} \quad (21)$$

$$\frac{\kappa_4[N_w]}{\langle N_w \rangle^4} = -6 \frac{\bar{C}_2[M]}{\langle M \rangle^2} \frac{\kappa_3[N_w]}{\langle N_w \rangle^3} - \frac{4\bar{C}_3[M]\langle M \rangle + 3\bar{C}_2[M]^2}{\langle M \rangle^4} \frac{\kappa_2[N_w]}{\langle N_w \rangle^2} + \frac{C_4[M] - \bar{C}_4[M]}{\langle M \rangle^4} \quad (22)$$

132 We note that binomial efficiency corrections do not affect the results as both, numerators and denominators of the
 133 right hand side of the above expressions, scale with the same power of the efficiency.

134 Inserting these expressions into Eqs. (2-4) for the cumulants $\kappa_j[N]$, we can solve for the cumulants of the system
 135 with fixed number of wounded nucleons, namely the $\bar{\kappa}_j[N]$. The results are given in the following general form

$$\bar{\kappa}_j[N] = \kappa_j^{corr}[N] + \Delta_j[N] \quad (23)$$

with $\bar{\kappa}_j[N]$ the cumulant of order j for a system with fixed N_w nucleons, $\kappa_j^{corr}[N]$ the cumulant including the measurable corrections for volume fluctuations, and Δ_j the corresponding bias due to quantities that are not measurable. The second-order result we already derived in Sec. A, Eqs. (15) and (14), namely

$$\begin{aligned} \kappa_2^{corr}[N] &= \kappa_2[N] - \frac{\langle N \rangle^2}{\langle M \rangle^2} C_2[M] \\ \Delta_2 &= \frac{\langle N \rangle^2}{\langle M \rangle^2} \bar{C}_2[M]. \end{aligned} \quad (24)$$

For the third order we have

$$\begin{aligned} \kappa_3^{corr}[N] &= \kappa_3[N] - \frac{3C_2[M]\kappa_2[N]\langle N \rangle}{\langle M \rangle^2} + \frac{3C_2[M]^2\langle N \rangle^3}{\langle M \rangle^4} - \frac{C_3[M]\langle N \rangle^3}{\langle M \rangle^3} \\ \Delta_3 &= \bar{C}_2[M] \left(\frac{3\kappa_2[N]\langle N \rangle}{\langle M \rangle^2} - \frac{3C_2[M]\langle N \rangle^3}{\langle M \rangle^4} \right) + \frac{\bar{C}_3[M]\langle N \rangle^3}{\langle M \rangle^3}. \end{aligned} \quad (25)$$

And the fourth order result reads

$$\begin{aligned}
\kappa_4^{corr}[N] &= \kappa_4[N] - \left(\frac{6\kappa_2[N]\langle N \rangle^2 (C_3[M]\langle M \rangle - 3C_2[M]^2)}{\langle M \rangle^4} + \frac{4C_2[M]\kappa_3[N]\langle N \rangle}{\langle M \rangle^2} \right. \\
&\quad \left. + \frac{3C_2[M]\kappa_2[N]^2}{\langle M \rangle^2} + \frac{\langle N \rangle^4 (-10C_3[M]C_2[M]\langle M \rangle + 15C_2[M]^3)}{\langle M \rangle^6} + \frac{C_4[M]\langle N \rangle^4}{\langle M \rangle^4} \right) \\
\Delta_4 &= \bar{C}_2[M] \left(-\frac{18C_2[M]\kappa_2[N]\langle N \rangle^2}{\langle M \rangle^4} + \frac{15C_2[M]^2\langle N \rangle^4}{\langle M \rangle^6} - \frac{4C_3[M]\langle N \rangle^4}{\langle M \rangle^5} + \frac{4\kappa_3[N]\langle N \rangle}{\langle M \rangle^2} + \frac{3\kappa_2[N]^2}{\langle M \rangle^2} \right) \\
&\quad + \bar{C}_3[M] \left(\frac{6\kappa_2[N]\langle N \rangle^2}{\langle M \rangle^3} - \frac{6C_2[M]\langle N \rangle^4}{\langle M \rangle^5} \right) + \frac{\bar{C}_4[M]\langle N \rangle^4}{\langle M \rangle^4}. \tag{26}
\end{aligned}$$

136 The corresponding relations for correction and bias terms of any order can be obtained with the provided software
137 package [26]. A Python package is provided to derive analytical formulas for both mixed and pure cumulants of
138 multiplicity distributions, including participant/volume fluctuations. The correction formulas and their bias terms
139 can be derived as well. The package, including a dedicated graphical user interface, can be downloaded via Ref. [26].

Equivalent expressions for the factorial cumulants, $C_n[N]$, and their related biases, $\Delta_{n,F}$, may then be obtained by using the relation between factorial cumulants and regular cumulants

$$\begin{aligned}
C_n^{corr}[N] &= \sum_{j=1}^n s(n, j) \kappa_j^{corr}[N], \\
\Delta_{n,F} &= \sum_{j=1}^n s(n, j) \Delta_j,
\end{aligned}$$

140 with $s(n, j)$ denoting Stirling numbers of the first kind (cf. Eq. B6)

141 The results for the corrected factorial cumulants are given in Appendix E.

142

V. SIMULATIONS

143 Experimental data are usually analyzed in centrality percentiles by introducing selection criteria on, e.g., the energy
144 deposited in a forward detector system covering typically the projectile (target) spectator region or the multiplicity
145 of charged particles emitted from the mid-rapidity region, with an acceptance reaching close to the projectile (target)
146 rapidity regions in case of low beam energies [27, 28]. For the latter, care must be taken to ensure that the evaluated
147 particles are not simultaneously used to determine the critical fluctuations [25]. The respective distributions, like
148 e.g., the forward energy deposit or the charged particle multiplicity, are commonly modelled using the Glauber
149 Monte Carlo Model [29]. The model provides event by event and for a given impact parameter the number of
150 projectile/target nucleons which are “wounded” and responsible for the event activity (multiplicity), and those, which
151 proceed nearly undisturbed into the phase space region covered by the forward detectors. To determine centrality using
152 charged particle multiplicity the respective distribution is generally modelled assuming that particles are “produced”
153 independently from distinct sources following a negative binomial distribution (NBD). Its probability mass function
154 is defined as
155
156

$$P(n; \mu, k) = \frac{\Gamma(n+k)}{\Gamma(n+1)\Gamma(k)} \left(\frac{\mu}{k}\right)^n \left(\frac{\mu}{k} + 1\right)^{-(n+k)}, \tag{27}$$

157 where μ denotes the mean of the NBD, while the combination of μ and k determines its higher-order cumulants

$$\kappa_n^{NBD} = \left. \frac{\partial^n \ln M(t)}{\partial t^n} \right|_{t=0}, \tag{28}$$

158 with

$$M(t) = \sum_{n=0}^{\infty} e^{tn} P(n; \mu, k) = \left(\frac{k}{k + \mu(1 - e^t)} \right)^k \tag{29}$$

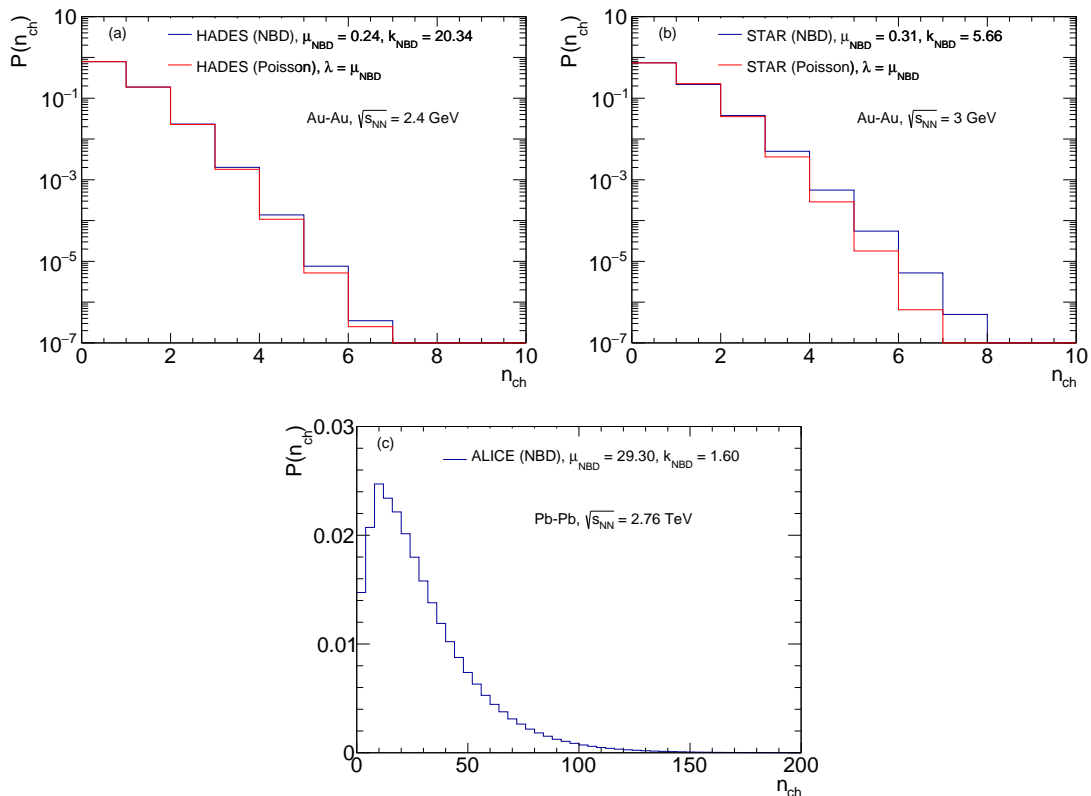


Figure 1. The NBD of charged particles per source (n_{ch}) adjusted to HADES (a) [27], STAR (b) [11], and ALICE (c) [28] is presented with blue lines. For the HADES and STAR data, corresponding Poisson distributions are also presented (red lines). The parameters used are listed in Table I.

being the moment-generating function of the NBD. The first four cumulants read

$$\kappa_1^{\text{NBD}} = \mu, \quad (30)$$

$$\kappa_2^{\text{NBD}} = \frac{\mu(k + \mu)}{k}, \quad (31)$$

$$\kappa_3^{\text{NBD}} = \frac{\mu(k + \mu)(k + 2\mu)}{k^2}, \quad (32)$$

$$\kappa_4^{\text{NBD}} = \frac{\mu(k + \mu)(k^2 + 6k\mu + 6\mu^2)}{k^3}, \quad (33)$$

159 The parameters of the NBD are fixed in each experiment by the fitting procedure. In a first step the number of
 160 particle-emitting sources n_s is determined according to [11, 28]

$$n_s = fN_w + (1 - f)N_{coll}, \quad (34)$$

161 where N_w and N_{coll} are the numbers of wounded nucleons and binary collisions, respectively. Sampling impact
 162 parameters according to $d\sigma = b db$ a list of number of sources, n_s^i , is generated, with $i \in [1, \dots, N_{\text{Event}}]$. Then, for each
 163 event i , NBD is sampled n_s^i times and the parameters of the NBD, μ , k and f , are adjusted such that the obtained
 164 multiplicity distribution agrees with the corresponding experimental one. The mixing parameter f is introduced to
 165 improve the description by accounting also particles produced in hard (prompt) processes.

Panel (a) of Fig. 1 represents the NBD distribution as observed by the HADES experiment for Au+Au collisions at $\sqrt{s_{NN}} = 2.4$ GeV, with parameters $\mu = 0.24$, $k = 20.34$, and $f = 1$ taken from [27]. For comparison, we also present a Poisson distribution with the same mean, μ . Similar distributions from the STAR [11] and ALICE [28] experiments are presented in panels (b) and (c). Figure 1 shows that at the HADES energy the fitted NBDs are very close to Poisson distributions. Quantitatively this can be seen by evaluating the cumulants of the HADES NBD ($\mu = 0.24$,

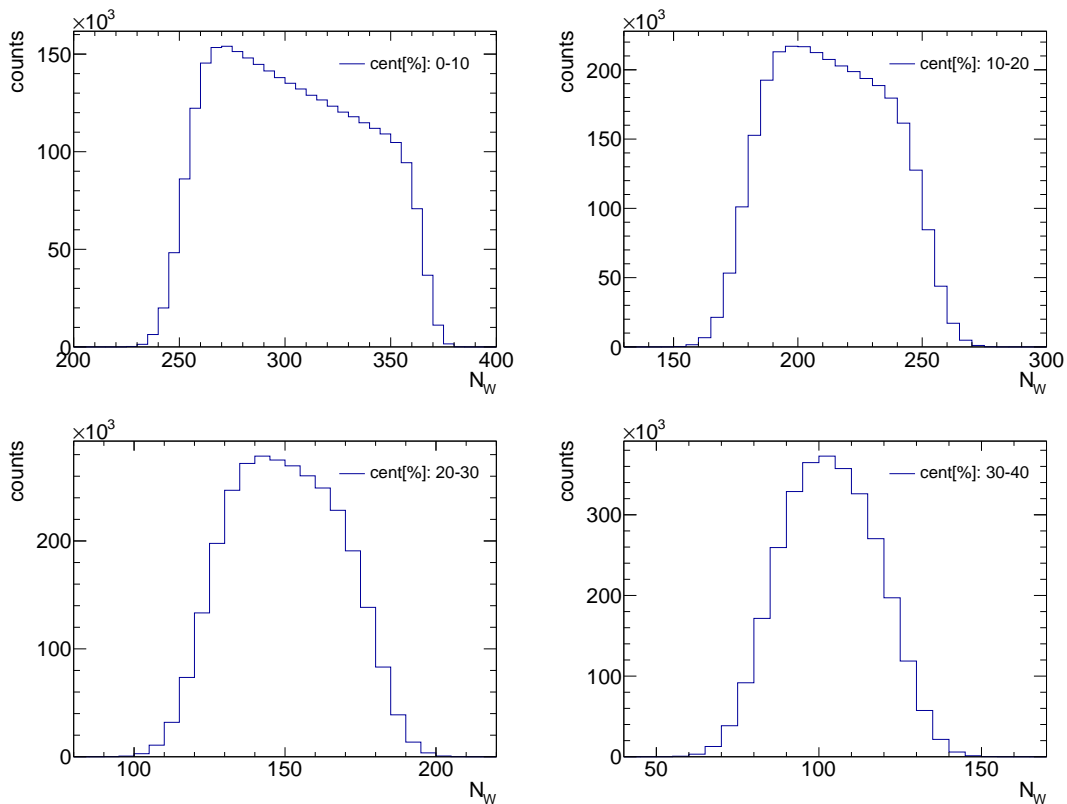


Figure 2. Distribution of wounded nucleons in Au+Au collisions at $\sqrt{s_{\text{NN}}} = 2.4$ GeV for four selected centrality classes, as obtained from the Glauber Monte Carlo simulations.

$k = 20.34$)

$$\kappa_1^{\text{NBD}}(\text{HADES}) = 0.24 \quad (35)$$

$$\kappa_2^{\text{NBD}}(\text{HADES}) = 0.2428, \quad (36)$$

$$\kappa_3^{\text{NBD}}(\text{HADES}) = 0.2486, \quad (37)$$

$$\kappa_4^{\text{NBD}}(\text{HADES}) = 0.2602. \quad (38)$$

167 For a Poisson distribution all cumulants are equal to its mean and the HADES data are indeed close to fulfilling this
 168 condition. The statement, to a lesser extent, is also valid for the STAR Au+Au data at 3 GeV (see Fig. 1). The
 169 corresponding ALICE distribution, however, is much wider compared to the Poissonian baseline, but the ALICE NBD
 170 is obtained for very different acceptance than that used for the cumulant analysis. In Table I we also provide the NBD
 171 parameters as obtained by the STAR and ALICE collaborations for Au+Au and Pb+Pb collisions at $\sqrt{s_{\text{NN}}} = 3$ GeV
 172 and 2.76 TeV, respectively.

experiments	μ	k	f
HADES	0.24	20.34	1
STAR	0.31	5.66	0.94
ALICE	29.3	1.6	0.8

Table I. NBD parameters as extracted from Glauber fits to particle distributions observed in different experiments. For simulations, the distributions should be folded within the experimental acceptance in which the cumulants are measured.

173 In the following we test the proposed method using two different simulations referred to as Model A and Model B.
 174 While the sampling of wounded nucleons is the same for both models, in Model A we sample different particle species
 175 independently while in Model B we introduce correlations between pions and protons via resonance production and
 176 decay. For both models A and B, we use the Glauber model to extract the distributions of wounded nucleons

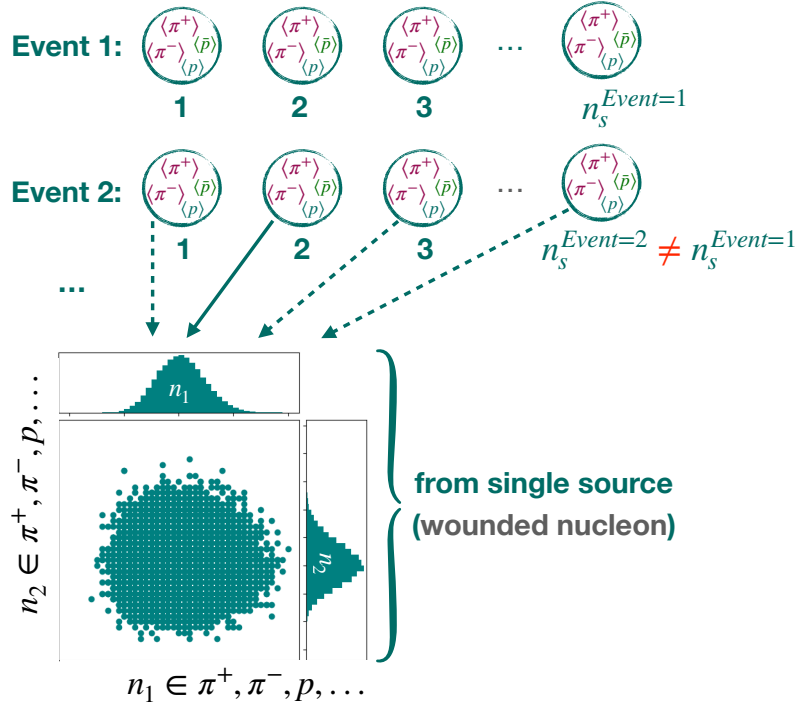


Figure 3. Simulation of particle production within the model of independent sources. The circles indicate the individual sources sampled according to Eq. 34. n_1 and n_2 show particle species used to sample emission from a single source. The distributions per single source can be chosen arbitrarily. See models A and B discussed below.

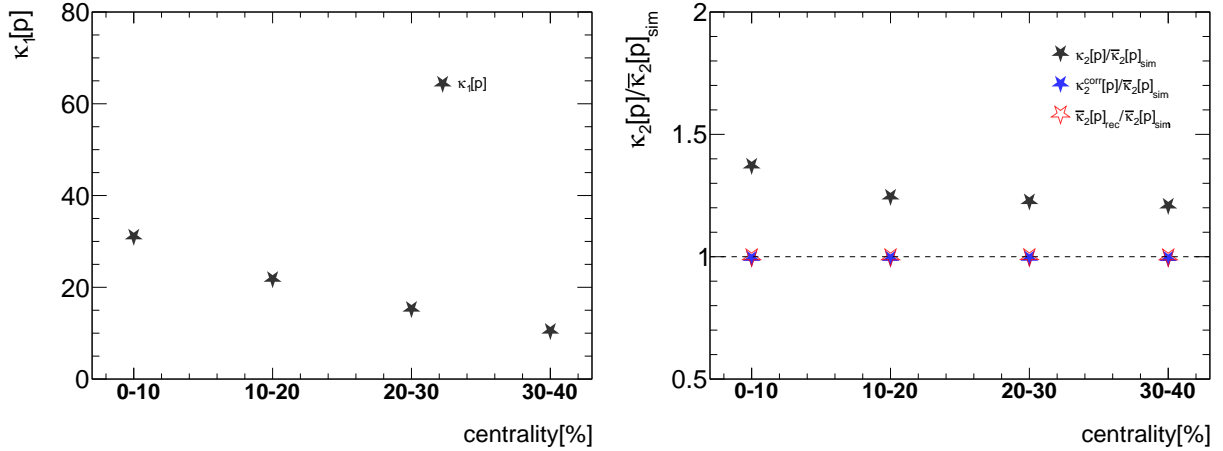


Figure 4. Left panel: Mean number of simulated protons used in model A as a function of centrality, adjusted to multiplicities measured by HADES in Au-Au collisions at $\sqrt{s_{NN}} = 2.4$ GeV [12]. Right panel: Second-order cumulants of protons in model A including volume fluctuations (black stars), corrected with Eq. 24 (blue stars) and reconstructed with Eq. 23 (open red stars). The results are normalized to $\bar{\kappa}_2[p]_{sim}$, corresponding to the second-order cumulants of protons in the absence of volume fluctuations.

177 corresponding to four different centrality classes in Au+Au collisions at $\sqrt{s_{NN}} = 2.4$ GeV. They are presented in
 178 Figure 2. Particles are produced by independent sources. (cf. Eq. 34). We note that other sources of correlation,
 179 such as those stemming from the reconstruction of closely spaced tracks and/or non-binomial efficiencies, were not
 180 considered in these simulations. The simulation process is schematically illustrated in Fig. 3.

181 We will concentrate on the HADES data but also briefly discuss the high-energy limits, which concern the ALICE
 182 and STAR experiments. For the HADES data the extracted number of binary collision is zero ($f = 1$, see Table I),
 183 the number of sources per event are therefore sampled exclusively from the wounded nucleon distributions presented

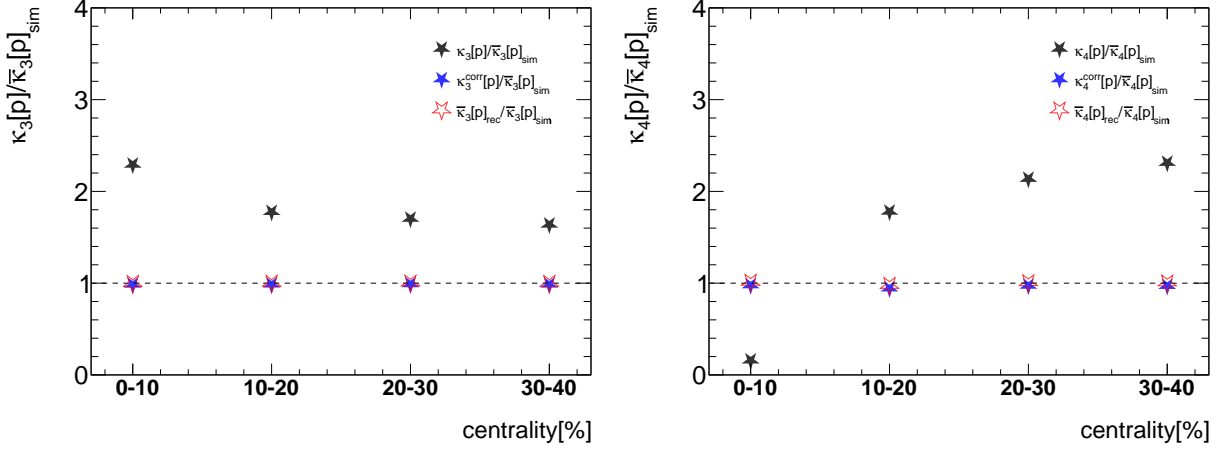


Figure 5. Left panel: Third order cumulants of protons in model A including volume fluctuations (black stars), corrected with Eq. 25 (blue stars) and reconstructed with Eq. 23 (open red stars). The results are normalized to $\bar{\kappa}_3[p]_{\text{sim}}$, corresponding to the second-order cumulants of protons in the absence of volume fluctuations. Right panel: Fourth order cumulants of protons in model A including volume fluctuations (black stars), corrected with Eq. 26 (blue stars) and reconstructed with Eq. 23 (open red stars). The results are normalized to $\bar{\kappa}_4[p]_{\text{sim}}$, corresponding to the fourth order cumulants of protons in the absence of volume fluctuations.

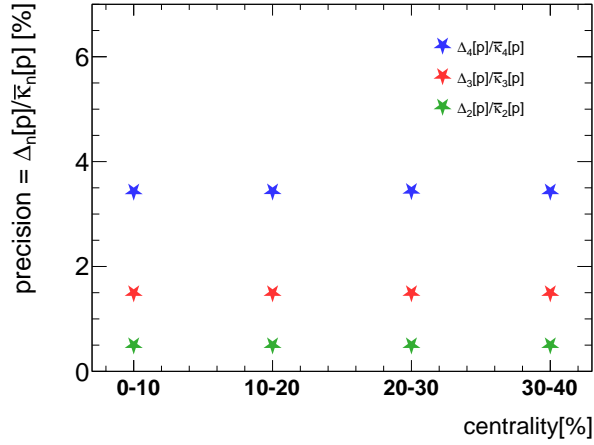


Figure 6. The normalised bias terms in model A.

184 in Fig. 2.

185

VI. MODEL A

186 In model A we first generate the charged-particle multiplicity for individual events using the NDB distribution as
 187 extracted by experimental measurements. In doing so we sample the NDB distribution n_s times. Different particle
 188 species are then taken as fractions of the total number of charged particles. For example, from a randomly sampled
 189 NDB distribution a respective fraction is assigned to protons. From the remaining charged particles another fraction
 190 is assigned to positively charged pions and the rest is taken as negatively charged pions. These fractions are chosen
 191 such that the overall probability of having protons, positively and negatively charged pions correspond to 75%, 9%
 192 and 16 % of all charged particles, respectively, based on the actual HADES measurement in Au+Au collisions (see [30]
 193 and references therein). In addition, we account for acceptance effects, because the NDB parameters are obtained in
 194 different acceptance than that used for the fluctuation analyses. We therefore fold the entire NDB distribution with
 195 a binomial distribution such that the experimentally measured mean multiplicities of particles in the acceptance used
 196 for fluctuation analysis are reproduced. Volume fluctuations are naturally accounted for as for each event the number
 197 of sources n_s are randomly sampled from the corresponding distributions.

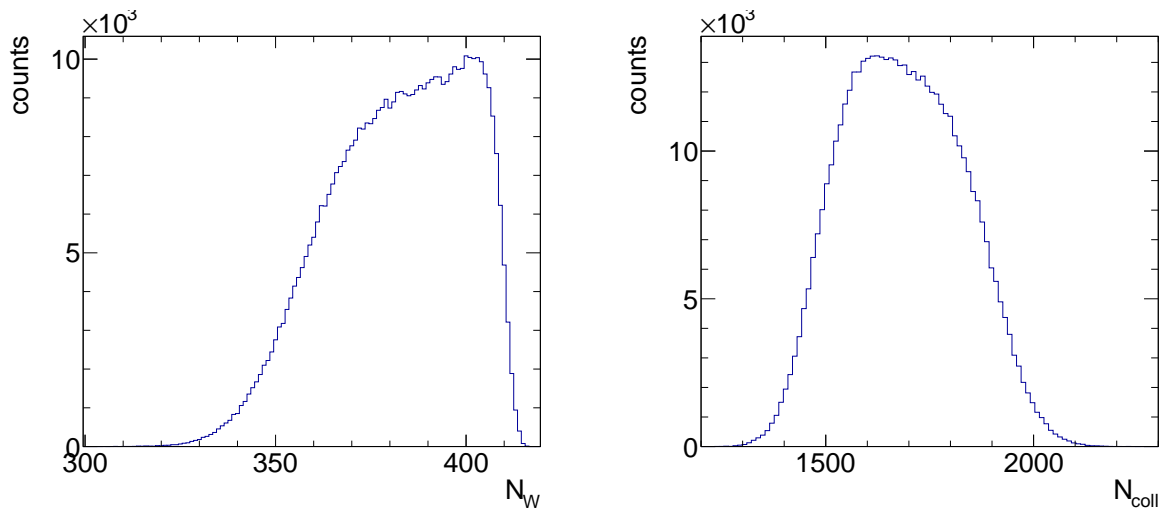


Figure 7. Number of wounded nucleons (left panel) and binary collisions (right panel) as generated with a Glauber Monte Carlo simulation using input from the ALICE experiment [28]. The selection corresponds to the 5% most central Pb-Pb collisions at $\sqrt{s_{NN}} = 2.76$ TeV.

198 The simulated mean numbers of protons are shown in the left panel of Fig. 4 for the four centrality classes. In the
 199 right panel of Fig. 4 the reconstructed second-order cumulants of protons are presented, normalized to the expected
 200 true cumulant, $\bar{\kappa}_2[p]_{sim}$. The black stars represent the results which include volume fluctuations. The values κ_2^{corr}
 201 as calculated using Eq. 24 are shown with blue stars, while the red stars are evaluated using Eq. 23. The results for the
 202 third and fourth order cumulants are shown in Fig. 5. The corresponding normalized biases $\Delta_n/\bar{\kappa}_n$ are presented in
 203 Fig. 6. We find the normalized biases to be very small, of the order of a few percent, so that the corrected cumulants,
 204 κ_n^{corr} are very close to their expected true values, $\bar{\kappa}_n$. As already discussed, this is to be expected since the multiplicity
 205 distribution per wounded nucleon in the present model is close to Poisson.

206 To investigate the high-energy limit, we also apply the method to ALICE data. In doing so we first run Glauber
 207 Monte Carlo simulations for Pb-Pb collisions at $\sqrt{s_{NN}} = 2.76$ TeV. The input parameters are taken from Ref. [28],
 208 which selects different centrality classes by introducing sharp cuts on the charged-particle distributions. The distribu-
 209 tions of wounded nucleons and binary collisions corresponding to the 5% most central collisions are presented in
 210 Fig. 7 [23]. The reconstructed mean number of wounded nucleons and binary collisions, corresponding to the 5%
 211 most central collisions are $\langle N_W \rangle \approx 382$ and $\langle N_{coll} \rangle \approx 1685$, respectively, consistent with the numbers given in [28].
 212 With these numbers one can estimate a mean number of particle emitting sources, yielding $\langle n_s \rangle \approx 642$ (see Eq. 34).
 213 The corresponding mean number of charged particles can be estimated as $\langle N_{ch} \rangle = \langle n_s \rangle \times \mu_{NBD} \approx 18811$. On the
 214 other hand, the total number of charged particles measured inside the ALICE acceptance is about 1601 [31]. We thus
 215 folded the ALICE NBD distribution with a binomial with the acceptance factor of $\epsilon = 1601/18811 \approx 8.5\%$ to obtain
 216 the distribution within the experimental acceptance.¹ The so obtained NDB distribution from ALICE is presented
 217 in Fig. 8. We further note that only the acceptance in rapidity is considered. Fluctuation analyses are performed
 218 within a finite momentum range. Inclusion of the latter will further reduce the discrepancy between NBD and the
 219 corresponding Poisson distribution shown with the red histogram in Fig. 8. Finally using the NBD distribution pre-
 220 sented in Fig. 8, and measured proton number, $\langle N_p \rangle \approx 35$ [31], we estimated $\Delta_2[p] \approx 1.2$. This corresponds to a bias
 221 of $\Delta_2[p]/\bar{\kappa}_2[p] \approx 3.3\%$ (cf. Eq. 24).
 222

223 VII. MODEL B

224 In model B we introduce correlations between charged particles, specifically pions and protons by generating res-
 225 onances. Especially for HADES energies most of the observed pions are believed to originate from decays of Delta
 226 resonances. Therefore, the effect of such decay correlations, while not treated quantitatively here, needs to be taken

¹ We note that binomial folding of the NBD distribution, with the acceptance factor ϵ , changes only the parameter μ of the original NBD distribution ($\mu \rightarrow \epsilon\mu$), while the parameter k stays unchanged.

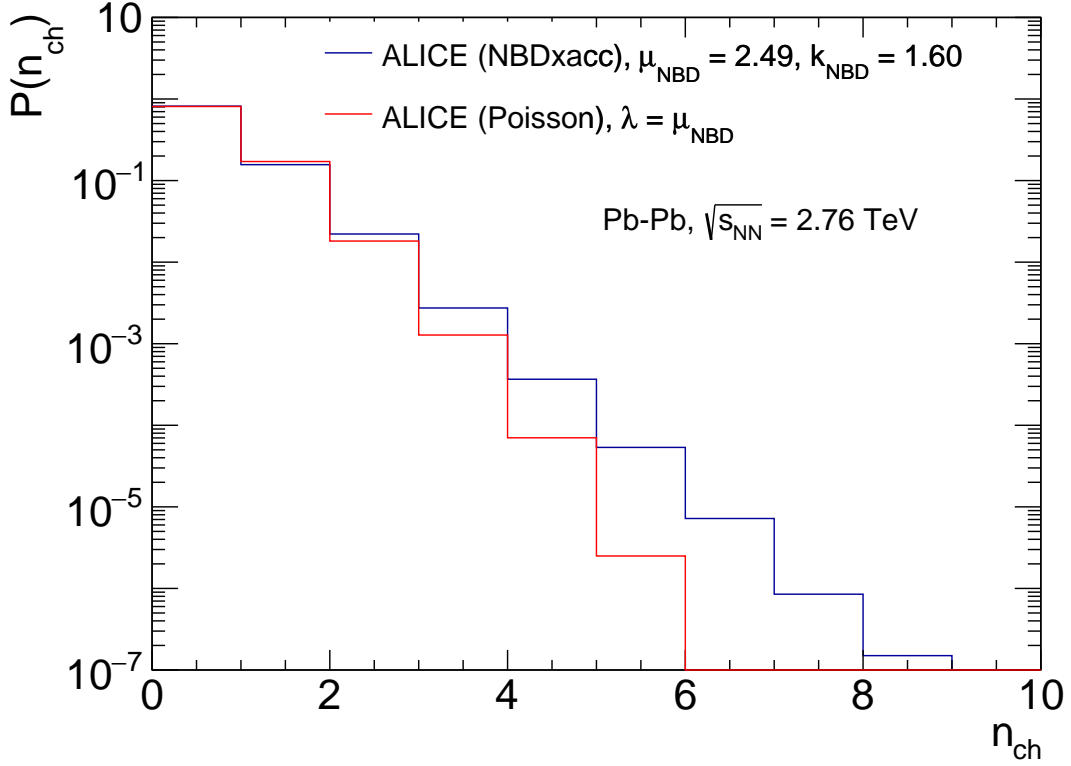


Figure 8. NBD distributions from ALICE as measured by fitting the signal amplitude in VZEROs (left panel) and after folding within the experimental acceptance 8.5% (see the text for details). The red histogram corresponds to a Poisson distribution with the same mean, μ as the folded NBD distribution. The original NBD distribution from ALICE is presented in the panel (c) of Fig. 1.

227 into account. Specifically, this is done by generating resonances from each source and letting them decay into two
 228 different particle species. Moreover, the resonances are generated from a Poisson distribution. In addition we produce
 229 independent particles from each source as well, sampled also from a Poisson distribution. Schematic illustration of
 230 the model for a single source is given in Fig. 9. Fluctuations of sources are introduced like in the model A. The input
 231 parameters for model B are given in Table II.

particles	mean numbers per source
resonances	0.03
independent protons	0.23
other particles	0.21

Table II. Parameters for model B are mean numbers of different particles species per source (see [30] and references therein). In addition, each resonance decays into one proton and one pion. Numbers of resonances, independent protons and other particles are sampled from independent Poisson distributions.

232 By construction the simulated protons, pions and resonances follow a Poisson distribution, however, the distribution
 233 of the total number of particles does not, due to the correlation between pions and protons introduced via the resonance
 234 decay. Indeed, particle production through resonance decays enhances their fluctuations (see Appendix F).

235 In experiments measurements are performed inside the finite acceptance by imposing selection criteria in momentum
 236 space, e.g., on rapidity and/or transverse momentum of particles. Moreover, such conditions typically lead to different
 237 acceptances for different particle species. In order to study the impact of the finite acceptance on the presented
 238 formalism, we introduce arbitrary rapidity distributions for protons, pions and other particles as illustrated in the right
 239 panel of Fig. 9. To this end we generate rapidity values for protons, pions and other particles from the corresponding
 240 distributions presented in Fig. 9.

241 In the left panel of Fig. 10 we present mean multiplicities of protons produced via resonances (red circles) and

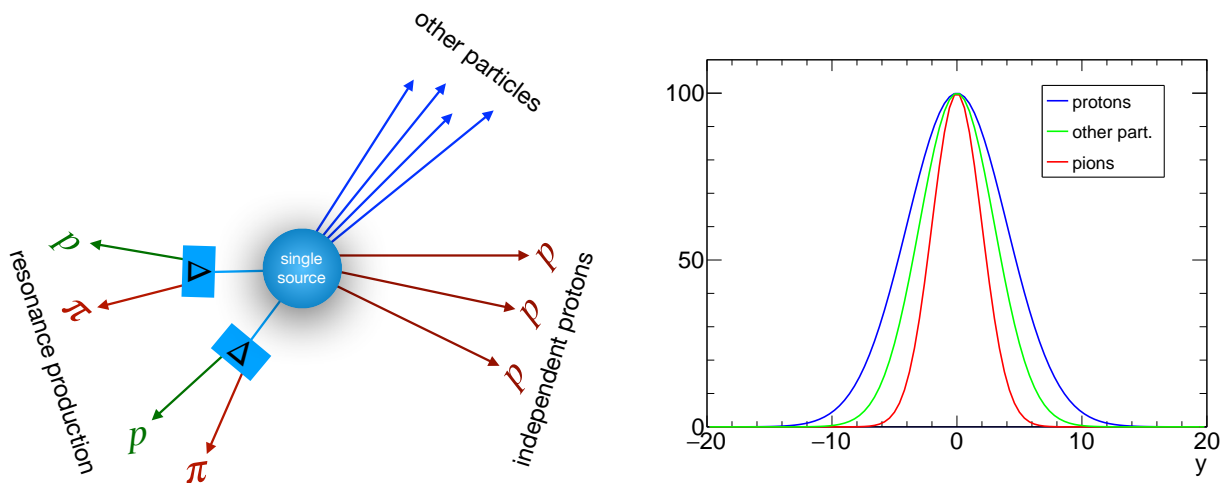


Figure 9. Left panel: Schematic illustration of model B. From a single source two resonances and three protons are produced. The resonances further decay to protons and pions. In simulations both, resonances and independent protons are sampled from Poisson distributions. The corresponding mean values of resonances and independent protons are provided in Table II. Right panel: Schematic rapidity distributions for different particles in order to study acceptance effects.

242 independently (blue circles), while the black circles correspond to the total number of protons. In addition, we
 243 produce pions from resonances, and, by construction, their mean values are equal to those of protons from resonances.
 244 The right panel of Fig. 10 shows the second-order cumulants of protons divided by the expected value $\bar{\kappa}_2[p]_{sim}$.
 245 The black stars represent those including participant (volume) fluctuations, $\kappa_2[p]/\bar{\kappa}_2[p]_{sim}$. The corrected cumulants
 246 $\kappa_2^{corr}[p]/\bar{\kappa}_2[p]_{sim}$ (see Eq. 24) are shown with blue symbols, while the open red stars represent the true reconstructed
 247 values of fluctuations of protons $\bar{\kappa}_2[p]/\bar{\kappa}_2[p]_{sim}$ as calculated using Eq. 23. Similar results for the third and fourth
 248 order cumulants of protons are presented in Fig. 11 (see Eqs. 25, 26, 23). In Fig. 12 the normalized cumulants as a
 249 function of cumulant order are presented for the 10% most central collisions. The right panel of Fig. 12 corresponds
 250 to the full acceptance, while in the right panel the results in the finite acceptance, delimited as $|y| < 1$, are presented.
 251 One clearly observes that in the finite acceptance the precision of the method is significantly better. In Fig. 13
 252 we show the magnitude of the corresponding normalized biases, $\Delta_n/\bar{\kappa}_n$, for the full acceptance (left panel) and for
 253 $|y| < 1$ (right panel). While the biases for the full acceptance may at first sight appear rather large ($\sim 40\%$) one
 254 should realize that for the most central events the uncorrected fourth order cumulant is more than a factor of 50
 255 larger in magnitude than the true cumulants. In other words the proposed corrections, while not perfect are a huge
 256 improvement of the measurement. The situation gets better for the limited acceptance.

257 VIII. DISCUSSION AND SUMMARY

- 258
- 259 • We have shown that using mixed events to determine the contributions of wounded nucleon or volume fluctua-
 260 tions is equivalent to extracting the latter from the track multiplicity distribution. However event mixing
 261 may offer an advantage since it allows to generate an almost arbitrarily large event ensemble with the same
 262 multiplicity distribution, and thus eliminate possible constraints due to limited event statistics.
 263 In either case, not all contributions can be accessed by direct measurement. The remaining terms lead to biases,
 264 Δ_k , which depend on the multiplicity distribution per wounded nucleon. These biases are, however, paramet-
 265 rically suppressed by powers of $\langle N \rangle / \langle M \rangle$ depending on the order of the cumulants. The biases are also small
 266 if the multiplicity distribution per wounded nucleon is close to Poisson. We suggest constraining these biases
 267 in experiments with fits to the observed multiplicity distribution within the wounded-nucleon model. Here, we
 268 refer to the Glauber Monte Carlo fits performed by each experiment. One of the outcomes of such a fitting
 269 procedure is the distribution of charged particles per source, as presented in Fig. 1 for the HADES case, which
 can be used to estimate the bias terms.
 - 270 • We have worked here within the wounded nucleon model to formulate volume fluctuations. Alternatively, one
 271 may introduce generic volume fluctuations as done e.g. in [22]. It is easy to show (see Appendix D) that the
 272 resulting expressions for the corrected cumulants, κ_j^{corr} , and the biases, Δ_j , are identical to those derived here,
 273 i.e. Eqs. (24-26).

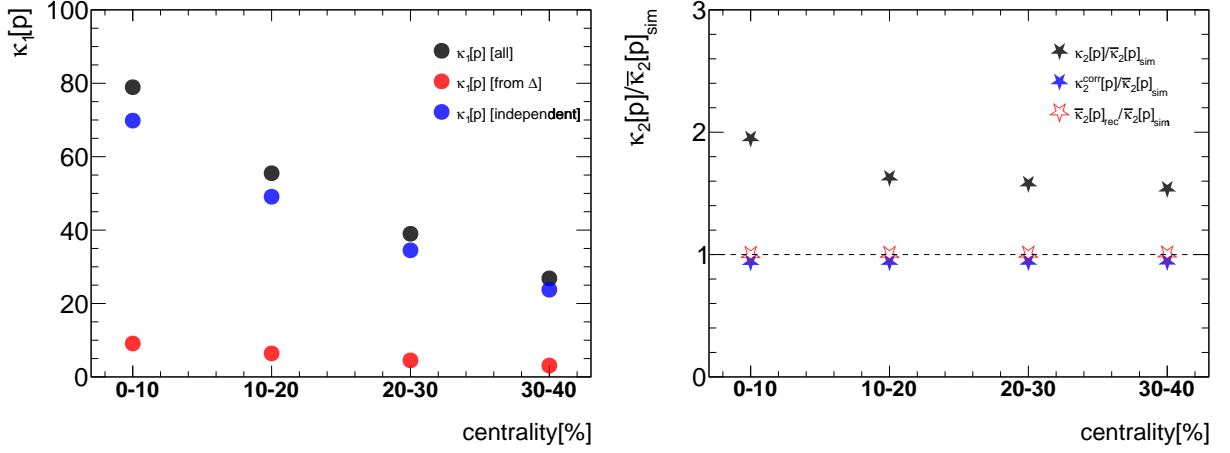


Figure 10. Left panel: Mean number of protons produced via resonances and independently are presented with the red and blue circles respectively. The black circles represent total multiplicity of protons. Pions are produced only via resonances. Right panel: Reconstructed second-order cumulants of protons including participant fluctuations (black stars). Corrected values for cumulants $\kappa_2^{corr}[p]$, i.e., without the bias term $\Delta_2[p]$ are presented with blue stars, while red stars represent fully corrected, against volume fluctuation. The results are normalized to the true second order cumulant, $\bar{\kappa}_2[p]_{sim}$.

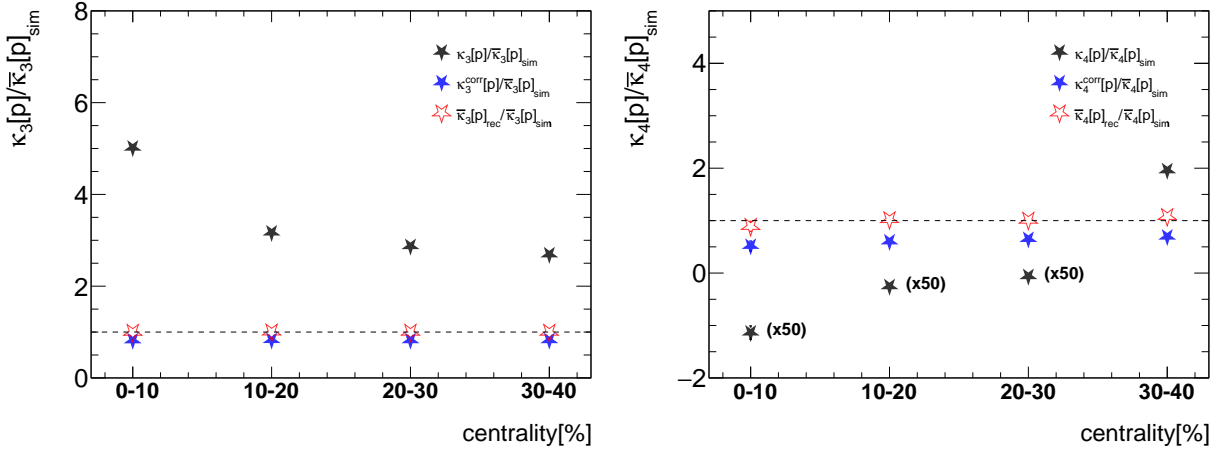


Figure 11. Left panel: Reconstructed third-order cumulants of protons including participant fluctuations (black stars) for model B. Corrected values for cumulants $\kappa_3^{corr}[p]$, i.e. without the bias term $\Delta_3[p]$ are presented with blue stars, while red stars represent fully corrected, against volume fluctuations, values $\bar{\kappa}_3[p] = \kappa_3^{corr}[p] + \Delta_3[p]$. Right panel: Similar to the left panel but for the fourth-order cumulants. Note that the values for the fourth-order cumulants with volume fluctuations (black stars) need to be multiplied by 50 for the first three centrality classes. The results are normalized to the true third or fourth order cumulant, $\bar{\kappa}_3[p]_{sim}$, $\bar{\kappa}_4[p]_{sim}$, respectively

- 274
- 275
- 276
- 277
- 278
- 279
- 280
- 281
- 282
- 283
- 284
- We note that one gets similar expressions for the fluctuations from the wounded nucleons, Eqs. (20-22), in terms of the factorial cumulants of, for example, pions instead of the total track multiplicity. This has the advantage that the corrections do not involve the particles of interest, protons, in our case.
 - We have checked that the proposed method also works if the multiplicity distribution is determined for a different acceptance than the particle distribution of interest. In this case all quantities in the expression for the corrected cumulants, Eqs.(24-26) involving the multiplicity should be evaluated in the multiplicity acceptance while all quantities involving the particles of interest, such as $\langle N \rangle$ or $\kappa_j[N]$ should be determined in the particle acceptance.
 - We have verified that the proposed method is not affected by potentially different rapidity distributions for different particle types.
 - We did not consider separate contributions from produced and stopped protons in the finally measured fluctu-

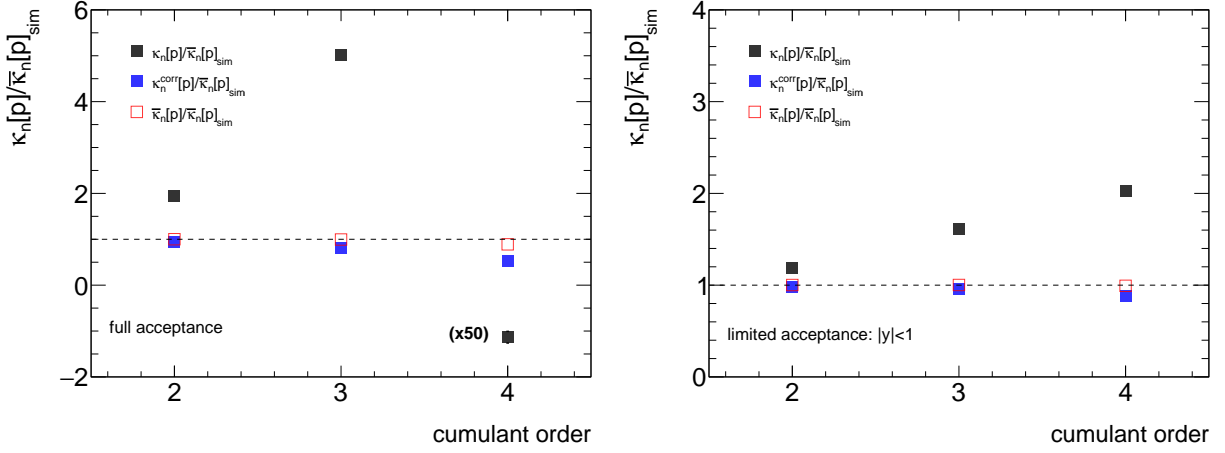


Figure 12. Left panel: Cumulants of protons in the full acceptance, presented for the 10% most central collisions (cf. Figs. 10 and 11). Right panel: Similar plot for the 10% most central collisions, but inside the finite acceptance delimited with the $|y| < 1$ criterion. The results are normalized to the true cumulants, $\bar{\kappa}_n[p]_{sim}$.

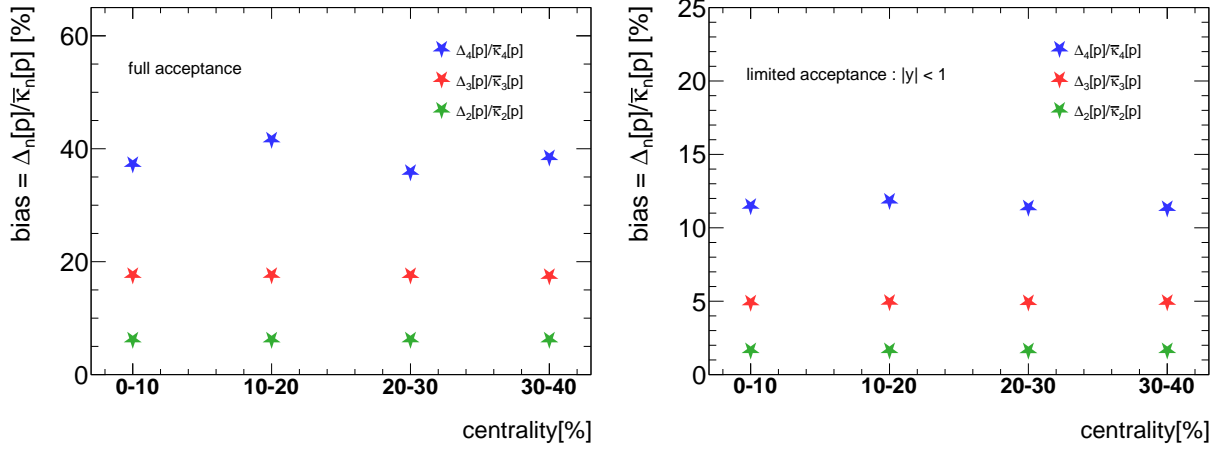


Figure 13. The bias terms for model B, in the full (left panel) and finite (right panel) acceptances. The acceptance, $|y| < 1$ is introduced using rapidity distributions of pions and protons as shown in Fig. 9.

285 ation signals. This is an important topic which goes beyond our studies presented in this work.

286 • We note that the correction term and bias for the second-order cumulant depends on the properties of the
 287 multiplicity distribution only while those for higher-order cumulants also involve the (uncorrected) cumulants
 288 of interest (at a lower order), $\kappa_n[N]$ (see Eq. (25,26).

289 • The proposed method is also applicable for mixed cumulants, such as the covariance between two particle species.
 290 The relevant formulas for mixed cumulants between species A and B up to $\kappa_{2,2}[A, B]$ are given in Appendix G.

291 The corresponding relations for correction and bias terms for mixed cumulants of any order can also be obtained
 292 with the provided software package [26].

• Here we have not explicitly discussed corrections for net proton cumulants. Since the expressions for the cumulants of the net-particle distribution, in the presence of wounded nucleon fluctuations, are the same as Eqs. (1-4) with $N \rightarrow N - \bar{N}$ [22, 23] our formulas above can be readily applied. For example for the second

order cumulant of the net-particle number distribution, we obtain using Eq.(24)

$$\begin{aligned}\kappa_2^{corr}[N - \bar{N}] &= \kappa_2[N - \bar{N}] - \frac{(\langle N \rangle - \langle \bar{N} \rangle)^2}{\langle M \rangle^2} C_2[M] \\ \Delta_{2,N-\bar{N}} &= \frac{(\langle N \rangle - \langle \bar{N} \rangle)^2}{\langle M \rangle^2} \bar{C}_2[M].\end{aligned}\quad (39)$$

Alternatively and as a cross check, one can use the expression for the mixed cumulants provided in Appendix G:

$$\kappa_2[N - \bar{N}] = \kappa_2[N] + \kappa_2[\bar{N}] - 2\kappa_{1,1}[N, \bar{N}] \quad (40)$$

so that (see Eq. (G8))

$$\kappa_2^{corr}[N - \bar{N}] = \kappa_2[N - \bar{N}] - \frac{(\langle N \rangle - \langle \bar{N} \rangle)^2}{\langle M \rangle^2} C_2[M] \quad (41)$$

with the bias

$$\Delta_{2,N-\bar{N}} = \frac{(\langle N \rangle - \langle \bar{N} \rangle)^2}{\langle M \rangle^2} \bar{C}_2[M] \quad (42)$$

293 For systems at vanishing baryon chemical potential, such as those created at very high collision energies around
294 mid-rapidity, $\langle N \rangle = \langle \bar{N} \rangle$, the corrected second-order cumulant is identical to the measured one (as discussed in
295 [22, 23]) and the bias vanishes.

296 In a similar way the correction formulas for mixed cumulants and the corresponding bias terms can be expressed
297 for net particles. In this case the replacements $A \rightarrow A - \bar{A}$ and $B \rightarrow B - \bar{B}$ should be performed (see Eqs. G8-
298 G14).

299 In summary, we have presented a method to correct experimentally measured particle number cumulants for the
300 effect of volume fluctuations. The essential idea is to extract the contribution from the volume fluctuations from the
301 distribution of charged particles which, after appropriate re-scaling, may be subtracted from the measured cumulants
302 of interest. Our proposed method is not exact as there remains a bias or remnant which can not be accessed directly
303 from experiment. However, we have shown by model calculations that this bias is very small compared to the
304 contribution from participant fluctuations and we hence consider our method an important step towards measuring
305 the true dynamical fluctuations of the system.

306 ACKNOWLEDGMENTS

307 V.K would like to thank GSI and the Institute for Nuclear Theory at the University of Washington for their kind
308 hospitality and stimulating research environment. V.K. has been supported by the U.S. Department of Energy, Office
309 of Science, Office of Nuclear Physics, under contract number DE-AC02-05CH11231, by the INT's U.S. Department of
310 Energy grant No. DE-FG02-00ER41132, and by the ExtreMe Matter Institute EMMI at the GSI Helmholtzzentrum
311 für Schwerionenforschung, Darmstadt, Germany.

312 Appendix A: Wounded-nucleon model

313 Here we briefly discuss the wounded-nucleon model following the Appendix of Ref. [32]. The wounded-nucleon model
314 assumes that particles are produced by independent sources, called wounded nucleons or participants. Therefore, the
315 probability to find A particles of type A and B particles of type B can be written as

$$P(A, B) = \sum_w W(w) \sum_{a_1, \dots, a_w} \sum_{b_1, \dots, b_w} p(a_1, b_1) \cdots p(a_w, b_w) \delta_{A, \sum_{k=1}^w a_k} \delta_{B, \sum_{k=1}^w b_k}. \quad (A1)$$

Here $W(w)$ denotes the probability to have w wounded nucleons, and $p(a, b)$ is the probability to have a particles of type A and b particles of type B from one wounded nucleon. The moment-generating function, $h(t_A, t_B)$ is then

$$\begin{aligned}
H(t_A, t_B) &= \sum_{A, B} e^{t_A A} e^{t_B B} P(A, B) \\
&= \sum_w W(w) \sum_{a_1, \dots, a_w} \sum_{b_1, \dots, b_w} p(a_1, b_1) \cdots p(a_w, b_w) e^{t_A \sum_{k=1}^w a_k} e^{t_B \sum_{k=1}^w b_k} \\
&= \sum_w W(w) \sum_{a_1, b_1} p(a_1, b_1) e^{t_A a_1 + t_B b_1} \cdots \sum_{a_w, b_w} p(a_w, b_w) e^{t_A a_w + t_B b_w} \\
&= \sum_w W(w) \left[\sum_{a, b} p(a, b) e^{t_A a + t_B b} \right]^w \\
&= \sum_w W(w) [h_w(t_A, t_B)]^w \\
&= \sum_w W(w) e^{w g_w(t_A, t_B)}
\end{aligned} \tag{A2}$$

316 where $h_w(t_A, t_B) = \sum_{a, b} p(a, b) e^{t_A a + t_B b}$ is the moment-generating function and $g_w(t_A, t_B) = \ln[h_w(t_A, t_B)]$ the
317 cumulant-generating function for one wounded nucleon, respectively. The cumulant-generating function, $G(t_A, t_B) =$
318 $\ln[H(t_A, t_B)]$, is then given by

$$G(t_A, t_B) = \ln[H(t_A, t_B)] = \ln \left[\sum_w W(w) e^{w g_w(t_A, t_B)} \right] = G_W(g_w(t_A, t_B)) \tag{A3}$$

319 where $G_W(t) = \ln[\sum_w W(w) e^{w t}]$ is the cumulant-generating function for the wounded-nucleon distribution, $W(w)$.
320 We note, that $g_w(0, 0) = G_W(0) = 0$ by construction. The cumulants are then obtained as

$$\kappa_{j, k}[A, B] = \left. \frac{\partial^{(j+k)}}{\partial t_A^j \partial t_B^k} G(t_A, t_B) \right|_{t_A=t_B=0}. \tag{A4}$$

321 For example, denoting $\kappa_{i, j}[a, b]$ as the cumulants of the distribution from one wounded nucleon, we have

$$\begin{aligned}
\kappa_1[A] &= \left. \frac{\partial}{\partial t_A} G(t_A, 0) \right|_{t_A=0} = \left. \frac{dG_w}{dg_w} \frac{dg_w(t_A, 0)}{dt_A} \right|_{t_A=0} = \left. \frac{dG_w}{dg_w} \right|_{g_w=0} \left. \frac{dg_w(t_A, 0)}{dt_A} \right|_{t_A=0} \\
&= \kappa_1[N_w] \kappa_1[a] = \langle N_w \rangle \langle a \rangle
\end{aligned} \tag{A5}$$

322 where $\kappa_1[n] = \langle a \rangle$ denotes the mean number of particles of type A per wounded nucleon and $\kappa_1[w] = \langle N_w \rangle$ the mean
323 number of wounded nucleons. The second-order cumulant is

$$\begin{aligned}
\kappa_2[A] &= \left. \frac{\partial^2}{\partial t_A^2} G(t_A, 0) \right|_{t_A=0} \\
&= \left. \frac{d^2 G_w}{dg_w^2} \left(\frac{dg_w(t_A, 0)}{dt_A} \right)^2 \right|_{t_A=0} + \left. \frac{dG_w}{dg_w} \frac{d^2 g_w(t_A, 0)}{dt_A^2} \right|_{t_A=0} \\
&= \kappa_2[N_w] \kappa_1[a]^2 + \kappa_1[N_w] \kappa_2[a] = \kappa_2[N_w] \langle a \rangle^2 + \langle N_w \rangle \kappa_2[a]
\end{aligned} \tag{A6}$$

324 The covariance is

$$\begin{aligned}
cov[A, B] &= \frac{\partial^2}{\partial t_A \partial t_B} G(t_A, t_B) \Big|_{t_A, t_B=0} = \frac{\partial}{\partial t_B} \left(\frac{dG_w}{dg_w} \frac{\partial g_w(t_A, t_B)}{\partial t_A} \right) \Big|_{t_A, t_B=0} \\
&= \frac{d^2 G_w}{dg_w^2} \frac{\partial g_w(t_A, t_B)}{\partial t_A} \frac{\partial g_w(t_A, t_B)}{\partial t_B} \Big|_{t_A, t_B=0} + \left(\frac{dG_w}{dg_w} \frac{\partial^2 g_w(t_A, t_B)}{\partial t_A \partial t_B} \right) \Big|_{t_A, t_B=0} \\
&= \kappa_2[N_w] \kappa_1[a] \kappa_1[b] + \kappa_1[N_w] cov[a, b] \\
&= \kappa_2[N_w] \langle a \rangle \langle b \rangle + \langle N_w \rangle cov[a, b]
\end{aligned} \tag{A7}$$

325 Using the relation between the cumulant and factorial cumulant-generating function, Eq. B3, the factorial cumulant
326 generating function is given by

$$G_F(z_A, z_B) = G(\ln(z_A), \ln(z_B)) = G_W(g_w(\ln(z_A), \ln(z_B))) = G_W(g_{F,w}(z_A, z_B)), \tag{A8}$$

327 with

$$g_{F,w}(z_A, z_B) = \ln \left[\sum_{a,b} p(a, b) z_A^a z_B^b \right]$$

the factorial cumulant-generating function for the distribution of one wounded nucleon, $p(a, b)$. The structure is the same as for the cumulant-generating function, except that now the argument of the wounded-nucleon cumulant-generating function is the factorial cumulant-generating function, $g_{F,w}$. Thus the factorial cumulants are easily obtained by simply replacing all the cumulants of the particle distribution for a given wounded nucleon, $\kappa_{i,j}[a, b]$ with the corresponding factorial cumulants, with $C_1[X] = \kappa_1[X] = \langle X \rangle$

$$\begin{aligned}
C_1[A] &= \frac{\partial}{\partial z_A} G_F(z_A, 1) \Big|_{z_A=1} = \kappa_1[N_w] C_1[a] = \langle N_w \rangle \langle a \rangle \\
C_2[A] &= \kappa_2[N_w] \langle a \rangle^2 + \langle N_w \rangle C_2[a] \\
C_{1,1}[A, B] &= cov[A, B]
\end{aligned} \tag{A9}$$

328 Appendix B: Cumulant and factorial cumulant-generating functions

329 Given a multiplicity distribution for particles of type A and B , $P(A, B)$ the generating functions for cumulants,
330 $g(t_A, t_B)$, and factorial cumulants, $g_F(z_A, z_B)$ are given by

$$g(t_A, t_B) = \ln \left[\sum_{A,B} P(A, B) e^{t_A A} e^{t_B B} \right] \tag{B1}$$

$$g_F(z_A, z_B) = \ln \left[\sum_{A,B} P(A, B) (z_A)^A (z_B)^B \right]. \tag{B2}$$

331 By construction, $g(t_A = 0, t_B = 0) = 0$ and $g_F(z_A = 1, z_B = 1) = 0$. Cumulants of order (i, j) , $\kappa_{i,j}[A, B]$, are then
332 obtained through

$$\kappa_{j,k}[A, B] = \frac{\partial^{(j+k)}}{\partial t_A^j \partial t_B^k} g(t_A, t_B) \Big|_{t_A=t_B=0},$$

333 while the factorial cumulants, $C_{j,k}[A, B]$, are given by

$$C_{j,k}[A, B] = \frac{\partial^{(j+k)}}{\partial z_A^j \partial z_B^k} g_F(z_A, z_B) \Big|_{z_A=z_B=1}.$$

334 The generating functions are related through

$$g_F(z_A, z_B) = g[\ln(z_A), \ln(z_B)] \quad (\text{B3})$$

335 or vice versa

$$g(t_A, t_B) = g_F(e^{t_A}, e^{t_B}) \quad (\text{B4})$$

336 These relations may also be used to convert cumulants into factorial cumulants and vice versa [33]. For example,
337 for the diagonal cumulants, $\kappa_n[A]$ we have

$$\kappa_n[A] = \sum_{j=1}^n S(n, j) C_j[A] \quad (\text{B5})$$

338 where $S(n, j)$ denotes the Stirling numbers of the second kind. The inverse relation is

$$C_n[A] = \sum_{j=1}^n s(n, j) \kappa_j[A], \quad (\text{B6})$$

with $s(n, j)$ denoting the Stirling numbers of the first kind. For the first four orders this evaluates to

$$\begin{aligned} \kappa_1 &= C_1 \\ \kappa_2 &= C_1 + C_2 \\ \kappa_3 &= C_1 + 3C_2 + C_3 \\ \kappa_4 &= C_1 + 7C_2 + 6C_3 + C_4 \end{aligned} \quad (\text{B7})$$

and

$$\begin{aligned} C_2 &= \kappa_2 - \kappa_1 \\ C_3 &= 2\kappa_1 - 3\kappa_2 + \kappa_3 \\ C_4 &= -6\kappa_1 + 11\kappa_2 - 6\kappa_3 + \kappa_4 \end{aligned} \quad (\text{B8})$$

339

Appendix C: Multiplicity Distribution

The multiplicity distribution, $P(M)$, is given by summing over all (charged) particles,

$$P_M(M) = \sum_{A, B, X} P(A, B, X) \delta_{M, A+B+X}, \quad (\text{C1})$$

340 where

$$P(A, B, X) = \sum_w W(w) \sum_{a_1, \dots, a_w} \sum_{b_1, \dots, b_w} \sum_{x_1, \dots, x_w} p(a_1, b_1, x_1) \cdots p(a_w, b_w, x_w) \delta_{A, \sum_{k=1}^w a_k} \delta_{B, \sum_{k=1}^w b_k} \delta_{X, \sum_{k=1}^w x_k}$$

341 is the distribution of particles of type A , B and all others, denoted by X . The distribution for particles A and B
342 defined in Appendix A are then given by $P(A, B) = \sum_{X=0}^{\infty} P(A, B, X)$ while that for the particles per wounded
343 nucleons are given by $p(a, b) = \sum_{x=0}^{\infty} p(a, b, x)$

The moment-generating function is then given by (proceeding analogously to Eq. A2):

$$\begin{aligned}
H_M(t) &= \sum_M P_M(M) e^{tM} = \sum_{M,A,B,X} P(A,B,X) \delta_{M,A+B+X} e^{tM} = \sum_{A,B,X} P(A,B,X) e^{t(A+B+X)} \\
&= \sum_w W(w) \sum_{a_1, \dots, a_w} \sum_{b_1, \dots, b_w} \sum_{x_1, \dots, x_w} p(a_1, b_1, x_1) \cdots p(a_w, b_w, x_w) e^{t \sum_{k=1}^w a_k} e^{t \sum_{k=1}^w b_k} e^{t \sum_{k=1}^w x_k} \\
&= \sum_w W(w) \sum_{a_1} \sum_{b_1} \sum_{x_1} p(a_1, b_1, x_1) e^{t(a_1+b_1+x_1)} \cdots \sum_{a_w} \sum_{b_w} \sum_{x_w} p(a_w, b_w, x_w) e^{t(a_w+b_w+x_w)} \\
&= \sum_w W(w) \left[\sum_a \sum_b \sum_x p(a, b, x) e^{t(a+b+x)} \right]^w \\
&= \sum_w W(w) [h_{m,w}(t)]^w \\
&= \sum_w W(w) e^{w g_{m,w}(t)}
\end{aligned}$$

344 where, $h_{m,w}(t)$ the moment-generating function and $g_{m,w}(t) = \ln [h_{m,w}(t)]$ the cumulant-generating function of the
345 multiplicity distribution for *one* wounded nucleon, $p(m) = \sum_{a,b,x} p(a, b, x) \delta_{m,a+b+x}$. The cumulant-generating func-
346 tion, $G_M(t)$, for the multiplicity distribution, $P(M)$, is then given by

$$G_M(t) = \ln [H_M(t)] = \ln \left[\sum_w W(w) e^{w g_{m,w}(t)} \right] = G_W(g_{m,w}(t)) \quad (\text{C2})$$

The cumulants of the multiplicity distribution are given by (following the analogous Eqs. A5 and A6)

$$\begin{aligned}
\kappa_1[M] &= \langle N_w \rangle \langle m \rangle \\
\kappa_2[M] &= \kappa_2[N_w] \langle m \rangle^2 + \langle N_w \rangle \kappa_2[m]
\end{aligned} \quad (\text{C3})$$

347 where $\kappa_i[m]$ denote the cumulants of the multiplicity distribution of *one* wounded nucleon and $\kappa_i[N_W]$ those of the
348 wounded nucleon distribution. Analogous to Eq. A8 the factorial cumulant-generating function is readily obtained

$$G_{F,M}(z) = G_M(\ln(z)) = G_W(g_{m,w}(\ln(z))) = G_W(g_{F,m,w}(z)), \quad (\text{C4})$$

349 with

$$g_{F,m,w}(z) = \ln \left[\sum_m p(m) z^m \right],$$

the factorial cumulant-generating function for the multiplicity distribution of one nucleon, $p(m)$. Again, the factorial cumulants are obtained by replacing the cumulants of the distribution $p(m)$, $\kappa_i[m]$ with the corresponding factorial cumulants, $C_i[m]$, in Eq. C3 by the factorial cumulants

$$\begin{aligned}
C_1[M] &= \langle N_w \rangle \langle m \rangle \\
C_2[M] &= \kappa_2[N_w] \langle m \rangle^2 + \langle N_w \rangle C_2[m]
\end{aligned} \quad (\text{C5})$$

350 Appendix D: Wounded Nucleon vs Volume Fluctuations

351 Here we will discuss the relation between wounded nucleon fluctuations [23] and so-called volume fluctuations as
352 they are discussed e.g. in [22]. Following Ref. [22] the cumulant-generating function is given by

$$\Phi(t) = \ln \left[\int dV P(V) e^{V \xi(t)} \right] = \chi^V(\xi(t)) \quad (\text{D1})$$

353 with $\chi^V(t)$ the cumulant-generating function for the distribution of volumes, $P(V)$, and

$$\xi(t) = \frac{1}{V} \ln \left[\sum_N p(N; V) e^{Nt} \right] \quad (\text{D2})$$

354 the generating function for scaled cumulants, κ/V , given for the distribution of particles at fixed volume, $p(N;V)$.
 355 Then, for a fixed volume V , the scaled cumulants are given by

$$\frac{\kappa_j}{V} = \frac{\partial^j}{\partial t^j} \xi(t)|_{t=0}. \quad (\text{D3})$$

356 For the wounded nucleon model we have (see Appendix A)

$$G(t) = \ln \left[\sum_w W(w) e^{w g_w(t)} \right] = G_W(g_w(t)) \quad (\text{D4})$$

357 with $G_W(t)$ the cumulant-generating function for the wounded nucleon distribution, $W(w)$, and

$$g_w(t) = \ln \left[\sum_n p(n) e^{nt} \right] \quad (\text{D5})$$

358 the generating function for the distribution of particles for one wounded nucleon. The cumulants per wounded nucleons
 359 for a fixed number of wounded nucleons, N_w , are given by

$$\frac{\kappa_j[N]}{N_w} = \kappa_j[n] = \frac{\partial^j}{\partial t^j} g_w(t)|_{t=0}. \quad (\text{D6})$$

Comparing the above expressions, one finds that the cumulants for volume fluctuations can be obtained from those for the wounded-nucleon number by the following replacements

$$\begin{aligned} \kappa_j[N_w] &\rightarrow \kappa_j[V] \\ \bar{\kappa}_j[N] = \langle N_w \rangle \kappa_j[N] &\rightarrow \langle V \rangle \frac{\kappa_j}{V} \end{aligned}$$

Indeed comparing the second-order cumulants for both scenarios we have

$$\begin{aligned} \kappa_2[N] &= \langle N_w \rangle \kappa_2[n] + \langle n \rangle^2 \kappa_2[N_w] = \bar{\kappa}_2[N] + \langle N \rangle^2 \frac{\kappa_2[N_w]}{\langle N_w \rangle^2} \\ \kappa_2[N] &= \langle V \rangle \frac{\kappa_2}{V} + \left(\frac{\kappa_1}{V} \right)^2 \kappa_2[V] = \bar{\kappa}_2[N] + \langle N \rangle^2 \frac{\kappa_2[V]}{\langle V \rangle^2} \end{aligned}$$

360 where in the second line we used $\kappa_1 = \langle N \rangle$ and $\bar{\kappa}_2 = \langle V \rangle \frac{\kappa_2}{V}$. Obviously, analogous replacements also hold for the
 361 factorial cumulants

$$\bar{C}_j[N] = \langle N_w \rangle C_j[n] \rightarrow \langle V \rangle \frac{C_j}{V}$$

362 with C_j/V the volume scaled factorial cumulants.

363 Appendix E: Results for factorial cumulants

364 Here we provide the formulas for the corrected factorial cumulants, C_k^{corr} and c_k^{corr} , and the associated biases,
 365 $\Delta_{k,F}$ and $\delta_{k,F}$. Both the factorial cumulants and the biases are related to the corresponding cumulants via the linear
 366 relation Eqs. B7 and B8. The corrected factorial cumulants and the associated biases are

$$C_2^{corr} = C_2[N] - \frac{\langle N \rangle^2}{\langle M \rangle^2} C_2[M] \quad (\text{E1})$$

$$C_3^{corr} = C_3[N] - \frac{3C_2[M]C_2[N]\langle N \rangle}{\langle M \rangle^2} + \frac{3C_2[M]^2\langle N \rangle^3}{\langle M \rangle^4} - \frac{C_3[M]\langle N \rangle^3}{\langle M \rangle^3} \quad (\text{E2})$$

$$C_4^{corr} = C_4[N] - \left(\frac{6C_2[N]\langle N \rangle^2 (C_3[M]\langle M \rangle - 3C_2[M]^2)}{\langle M \rangle^4} + \frac{4C_2[M]C_3[N]\langle N \rangle}{\langle M \rangle^2} \right) \quad (\text{E3})$$

$$+ \frac{3C_2[M]C_2[N]^2}{\langle M \rangle^2} + \frac{\langle N \rangle^4 (-10C_3[M]C_2[M]\langle M \rangle + C_4[M]\langle M \rangle^2 + 15C_2[M]^3)}{\langle M \rangle^6} \quad (\text{E4})$$

$$\Delta_{2,F} = \frac{\langle N \rangle^2}{\langle M \rangle^2} \bar{C}_2[M] \quad (\text{E5})$$

$$\Delta_{3,F} = \bar{C}_2[M] \left(\frac{3C_2[N]\langle N \rangle}{\langle M \rangle^2} - \frac{3C_2[M]\langle N \rangle^3}{\langle M \rangle^4} \right) + \frac{\bar{C}_3[M]\langle N \rangle^3}{\langle M \rangle^3} \quad (\text{E6})$$

$$\Delta_{4,F} = \bar{C}_2[M] \left(\frac{4C_3[N]\langle N \rangle}{\langle M \rangle^2} + \frac{3C_2[N]^2}{\langle M \rangle^2} - \frac{18C_2[M]C_2[N]\langle N \rangle^2}{\langle M \rangle^4} - \frac{4C_3[M]\langle N \rangle^4}{\langle M \rangle^5} + \frac{15C_2[M]^2\langle N \rangle^4}{\langle M \rangle^6} \right) \quad (\text{E7})$$

$$+ \bar{C}_3[M] \left(\frac{6C_2[N]\langle N \rangle^2}{\langle M \rangle^3} - \frac{6C_2[M]\langle N \rangle^4}{\langle M \rangle^5} \right) + \frac{\bar{C}_4[M]\langle N \rangle^4}{\langle M \rangle^4} \quad (\text{E8})$$

367 For the scaled factorial cumulants we have

$$c_2^{corr} = c_2[N] - \frac{\langle N \rangle}{\langle M \rangle} c_2[M] \quad (\text{E9})$$

$$c_3^{corr} = c_3[N] - 3 \frac{\langle N \rangle}{\langle M \rangle} c_2[M] c_2[N] + \left(\frac{\langle N \rangle}{\langle M \rangle} \right)^2 (3c_2[M]^2 - c_3[M]) \quad (\text{E10})$$

$$c_4^{corr} = c_4[N] - \frac{\langle N \rangle}{\langle M \rangle} (3c_2[M]c_2[N]^2 + 4c_2[M]c_3[N]) \quad (\text{E11})$$

$$+ \left(\frac{\langle N \rangle}{\langle M \rangle} \right)^2 (18c_2[M]^2c_2[N] - 6c_3[M]c_2[N]) \quad (\text{E12})$$

$$+ \left(\frac{\langle N \rangle}{\langle M \rangle} \right)^3 (-15c_2[M]^3 + 10c_2[M]c_3[M] - c_4[M]) \quad (\text{E13})$$

$$\delta_{2,F} = \frac{\langle N \rangle}{\langle M \rangle} \bar{c}_2[M] \quad (\text{E14})$$

$$\delta_{3,F} = 3 \frac{\langle N \rangle}{\langle M \rangle} c_2[N] \bar{c}_2[M] + \left(\frac{\langle N \rangle}{\langle M \rangle} \right)^2 (\bar{c}_3[M] - 3c_2[M] \bar{c}_2[M]) \quad (\text{E15})$$

$$\delta_{4,F} = \frac{\langle N \rangle}{\langle M \rangle} (3c_2[N]^2 \bar{c}_2[M] + 4c_3[N] \bar{c}_2[M]) \quad (\text{E16})$$

$$+ \left(\frac{\langle N \rangle}{\langle M \rangle} \right)^2 (6c_2[N] \bar{c}_3[M] - 18c_2[M] c_2[N] \bar{c}_2[M]) \quad (\text{E17})$$

$$+ \left(\frac{\langle N \rangle}{\langle M \rangle} \right)^3 (15c_2[M]^2 \bar{c}_2[M] - 6c_2[M] \bar{c}_3[M] - 4c_3[M] \bar{c}_2[M] + \bar{c}_4[M]) \quad (\text{E18})$$

368 Appendix F: Particle production through cluster decays

369 Let us assume that particles are produced via clusters and that each cluster further decays into two particles.
 370 Moreover, clusters are generated from a Poisson distribution. As each cluster decays into two particles the probability
 371 of measuring k particles is equivalent to measuring $k/2$ clusters and can be presented as:

$$p(k; \langle N_{cl} \rangle) = e^{-\langle N_{cl} \rangle} \frac{\langle N_{cl} \rangle^{k/2}}{(k/2)!} \quad (\text{F1})$$

372 The corresponding moment-generating function reads:

$$M(t) = \sum_{k/2=0}^{\infty} e^{tk} e^{-\langle N_{cl} \rangle} \frac{\langle N_{cl} \rangle^{k/2}}{(k/2)!} = e^{\langle N_{cl} \rangle (e^{2t} - 1)}, \quad (\text{F2})$$

where $\langle N_{cl} \rangle$ denotes mean number of clusters produced.
 The cumulants of total particle number k can be computed as:

$$\kappa_n[k] = \left. \frac{\partial^n}{dt^n} \ln(M(t)) \right|_{t=0} \quad (\text{F3})$$

For the first two cumulants one gets:

$$\kappa_1[k] = 2\langle N_{cl} \rangle \quad (\text{F4})$$

$$\kappa_2[k] = 4\langle N_{cl} \rangle \quad (\text{F5})$$

One clearly sees from Eqs. F4 and F5 that $\kappa_1(k) \neq \kappa_2(k)$, i.e the total number of particles does not follow a Poisson distribution, although the clusters do. Moreover, one observes that particle production through clusters enhances fluctuations. In general, for clusters following a Poisson distribution and decaying into m particles, the cumulants of total particle number can be written as:

$$\kappa_n[k] = m^n \langle N_{cl} \rangle \quad (\text{F6})$$

We note that Eq. F6 also applies to particle production through resonance decays.

Appendix G: Mixed Cumulants

Here we provide the relevant formulas for mixed cumulants. Given the generating function, Eq. A3, the mixed cumulants for particles of type A and B are given by (see Eq. A4)

$$\kappa_{j,k}[A, B] = \left. \frac{\partial^{(j+k)}}{\partial t_A^j \partial t_B^k} g(t_A, t_B) \right|_{t_A=t_B=0} \quad (\text{G1})$$

The explicit formulas for the four lowest-order mixed cumulants are:

$$\kappa_{1,1}[A, B] = \bar{\kappa}_{1,1}[A, B] + \langle A \rangle \langle B \rangle \frac{\kappa_2[N_W]}{\langle N_W \rangle^2} \quad (\text{G2})$$

$$\kappa_{2,1}[A, B] = \bar{\kappa}_{2,1}[A, B] + (2\langle A \rangle \bar{\kappa}_{1,1}[A, B] + \langle B \rangle \bar{\kappa}_{2,0}[A, B]) \frac{\kappa_2[N_W]}{\langle N_W \rangle^2} + \langle A \rangle^2 \langle B \rangle \frac{\kappa_3[N_W]}{\langle N_W \rangle^3} \quad (\text{G3})$$

$$\kappa_{1,2}[A, B] = \bar{\kappa}_{1,2}[A, B] + (\langle A \rangle \bar{\kappa}_{0,2}[A, B] + 2\langle B \rangle \bar{\kappa}_{1,1}[A, B]) \frac{\kappa_2[N_W]}{\langle N_W \rangle^2} + \langle A \rangle \langle B \rangle^2 \frac{\kappa_3[N_W]}{\langle N_W \rangle^3} \quad (\text{G4})$$

$$\begin{aligned} \kappa_{2,2}[A, B] = & \bar{\kappa}_{2,2}[A, B] + (\langle A \rangle^2 \bar{\kappa}_{0,2}[A, B] + 4\langle A \rangle \langle B \rangle \bar{\kappa}_{1,1}[A, B] + \langle B \rangle^2 \bar{\kappa}_{2,0}[A, B]) \frac{\kappa_3[N_W]}{\langle N_W \rangle^3} \\ & + (2\langle A \rangle \bar{\kappa}_{1,2}[A, B] + 2\langle B \rangle \bar{\kappa}_{2,1}[A, B] + 2\bar{\kappa}_{1,1}[A, B]^2 + \bar{\kappa}_{0,2}[A, B] \bar{\kappa}_{2,0}[A, B]) \frac{\kappa_2[N_W]}{\langle N_W \rangle^2} \\ & + \langle A \rangle^2 \langle B \rangle^2 \frac{\kappa_4[N_W]}{\langle N_W \rangle^4} \end{aligned} \quad (\text{G5})$$

where, analogous to the notation for the regular cumulants, $\bar{\kappa}_{j,k}[A, B]$ denotes the mixed cumulant for constant number of wounded nucleons $\langle N_W \rangle$. Note, that $\kappa_{j,0}[A, B] = \kappa_j[A]$ and $\kappa_{0,j}[A, B] = \kappa_j[B]$ correspond to the regular cumulant for particles of type A and B respectively. The first order mixed cumulant, $\kappa_{1,1}[A, B] = \text{cov}[A, B]$ is also referred to as the covariance between the distributions of particles A and B . In order to obtain the corrected mixed cumulants we proceed in the same fashion as for the regular cumulant. We express the terms involving cumulants of the wounded nucleons, $\kappa_i[N_W] / \langle N_W \rangle^i$ in terms of the factorial cumulants of the multiplicity distribution (See Eqs.

20-22) and solve for the the mixed cumulants with fixed number of wounded nucleons, $\bar{\kappa}_{j,k}[A, B]$. Again, the results are given in the form

$$\bar{\kappa}_{j,k}[A, B] = \kappa_{j,k}^{corr}[A, B] + \Delta_{j,k} \quad (\text{G6})$$

where $\kappa_{j,k}^{corr}[A, B]$ are the cumulants including the measurable corrections and $\Delta_{j,k}$ are the corresponding biases due to quantities which are not directly measurable.

$$\kappa_{1,1}^{corr}[A, B] = \kappa_{1,1}[A, B] - \frac{\langle A \rangle \langle B \rangle}{\langle M \rangle^2} C_2[M] \quad (\text{G7})$$

$$\Delta_{1,1} = \frac{\langle A \rangle \langle B \rangle}{\langle M \rangle^2} \bar{C}_2[M] \quad (\text{G8})$$

$$\kappa_{2,1}^{corr}[A, B] = \kappa_{2,1}[A, B] - \frac{\langle B \rangle C_2[M] \bar{\kappa}_{2,0}[A, B]}{\langle M \rangle^2} - \frac{2\langle A \rangle C_2[M] \kappa_{1,1}[A, B]}{\langle M \rangle^2} + \frac{\langle A \rangle^2 \langle B \rangle (2C_2[M]^2 - \langle M \rangle C_3[M])}{\langle M \rangle^4} \quad (\text{G9})$$

$$\Delta_{2,1} = \frac{1}{\langle M \rangle^4} [2\langle A \rangle \langle M \rangle^2 \bar{C}_2[M] \kappa_{1,1}[A, B] + \langle B \rangle \langle M \rangle^2 \bar{C}_2[M] \bar{\kappa}_{2,0}[A, B] - \langle A \rangle^2 \langle B \rangle (\bar{C}_2[M] (\bar{C}_2[M] + C_2[M]) - \langle M \rangle \bar{C}_3[M])] \quad (\text{G10})$$

$$\kappa_{1,2}^{corr}[A, B] = \kappa_{1,2}[A, B] - \frac{\langle A \rangle C_2[M] \bar{\kappa}_{0,2}[A, B]}{\langle M \rangle^2} - \frac{2\langle B \rangle C_2[M] \kappa_{1,1}[A, B]}{\langle M \rangle^2} + \frac{\langle B \rangle^2 \langle A \rangle (2C_2[M]^2 - \langle M \rangle C_3[M])}{\langle M \rangle^4} \quad (\text{G11})$$

$$\Delta_{1,2} = \frac{1}{\langle M \rangle^4} [2\langle B \rangle \langle M \rangle^2 \bar{C}_2[M] \kappa_{1,1}[A, B] + \langle A \rangle \langle M \rangle^2 \bar{C}_2[M] \bar{\kappa}_{0,2}[A, B] - \langle A \rangle \langle B \rangle^2 (\bar{C}_2[M] (\bar{C}_2[M] + C_2[M]) - \langle M \rangle \bar{C}_3[M])] \quad (\text{G12})$$

$$\begin{aligned} \kappa_{2,2}^{corr}[A, B] = & \kappa_{2,2}[A, B] + \frac{2\langle A \rangle^2 C_2[M]^2 \bar{\kappa}_{0,2}[A, B]}{\langle M \rangle^4} + \frac{2\langle B \rangle^2 C_2[M]^2 \bar{\kappa}_{2,0}[A, B]}{\langle M \rangle^4} \\ & - \frac{C_2[M] \bar{\kappa}_{0,2}[A, B] \bar{\kappa}_{2,0}[A, B]}{\langle M \rangle^2} - \frac{\langle A \rangle^2 C_3[M] \bar{\kappa}_{0,2}[A, B]}{\langle M \rangle^3} - \frac{\langle B \rangle^2 C_3[M] \bar{\kappa}_{2,0}[A, B]}{\langle M \rangle^3} \\ & + \frac{12\langle A \rangle \langle B \rangle C_2[M]^2 \kappa_{1,1}[A, B]}{\langle M \rangle^4} - \frac{2C_2[M] \kappa_{1,1}[A, B]^2}{\langle M \rangle^2} - \frac{2\langle A \rangle C_2[M] \kappa_{1,2}[A, B]}{\langle M \rangle^2} \\ & - \frac{2\langle B \rangle C_2[M] \kappa_{2,1}[A, B]}{\langle M \rangle^2} - \frac{4\langle A \rangle \langle B \rangle C_3[M] \kappa_{1,1}[A, B]}{\langle M \rangle^3} - \frac{10\langle A \rangle^2 \langle B \rangle^2 C_2[M]^3}{\langle M \rangle^6} \\ & + \frac{8\langle A \rangle^2 \langle B \rangle^2 C_3[M] C_2[M]}{\langle M \rangle^5} - \frac{\langle A \rangle^2 \langle B \rangle^2 C_4[M]}{\langle M \rangle^4} \end{aligned} \quad (\text{G13})$$

$$\begin{aligned} \Delta_{2,2} = & \bar{C}_2[M]^2 \left(-\frac{\langle A \rangle^2 \bar{\kappa}_{0,2}[A, B]}{\langle M \rangle^4} - \frac{\langle B \rangle^2 \bar{\kappa}_{2,0}[A, B]}{\langle M \rangle^4} + \frac{3C_2[M] \langle A \rangle^2 \langle B \rangle^2}{\langle M \rangle^6} \right) \\ & + \bar{C}_2[M] \left(-\frac{C_2[M] \langle A \rangle^2 \bar{\kappa}_{0,2}[A, B]}{\langle M \rangle^4} - \frac{C_2[M] \langle B \rangle^2 \bar{\kappa}_{2,0}[A, B]}{\langle M \rangle^4} + \frac{\bar{\kappa}_{0,2}[A, B] \bar{\kappa}_{2,0}[A, B]}{\langle M \rangle^2} \right. \\ & - \frac{2\bar{C}_3[M] \langle A \rangle^2 \langle B \rangle^2}{\langle M \rangle^5} - \frac{12C_2[M] \langle A \rangle \langle B \rangle \kappa_{1,1}[A, B]}{\langle M \rangle^4} + \frac{2\langle A \rangle \kappa_{1,2}[A, B]}{\langle M \rangle^2} + \frac{2\kappa_{1,1}[A, B]^2}{\langle M \rangle^2} \\ & \left. + \frac{2\langle B \rangle \kappa_{2,1}[A, B]}{\langle M \rangle^2} + \frac{6C_2[M]^2 \langle A \rangle^2 \langle B \rangle^2}{\langle M \rangle^6} - \frac{2C_3[M] \langle A \rangle^2 \langle B \rangle^2}{\langle M \rangle^5} \right) \\ & + \bar{C}_3[M] \left(\frac{\langle A \rangle^2 \bar{\kappa}_{0,2}[A, B]}{\langle M \rangle^3} + \frac{\langle B \rangle^2 \bar{\kappa}_{2,0}[A, B]}{\langle M \rangle^3} + \frac{4\langle A \rangle \langle B \rangle \kappa_{1,1}[A, B]}{\langle M \rangle^3} - \frac{4C_2[M] \langle A \rangle^2 \langle B \rangle^2}{\langle M \rangle^5} \right) \\ & + \frac{\bar{C}_2[M]^3 \langle A \rangle^2 \langle B \rangle^2}{\langle M \rangle^6} + \frac{\bar{C}_4[M] \langle A \rangle^2 \langle B \rangle^2}{\langle M \rangle^4} \end{aligned} \quad (\text{G14})$$

-
- 384 [1] A. Rustamov, J. Stroth, and R. Holzmann, Nucl. Phys. A **1034**, 122641 (2023), arXiv:2211.14849 [nucl-th].
385 [2] F. Gross *et al.*, Eur. Phys. J. C **83**, 1125 (2023), arXiv:2212.11107 [hep-ph].
386 [3] M. Stephanov, Phys.Rev.Lett. **102**, 032301 (2009), arXiv:0809.3450 [hep-ph].

- 387 [4] M. Stephanov, Phys.Rev.Lett. **107**, 052301 (2011), arXiv:1104.1627 [hep-ph].
- 388 [5] A. Bzdak, S. Esumi, V. Koch, J. Liao, M. Stephanov, and N. Xu, Phys. Rept. **853**, 1 (2020), arXiv:1906.00936 [nucl-th].
- 389 [6] M. Asakawa, S. Ejiri, and M. Kitazawa, Phys. Rev. Lett. **103**, 262301 (2009), arXiv:0904.2089 [nucl-th].
- 390 [7] A. Bazavov *et al.* (HotQCD), Phys. Rev. D **96**, 074510 (2017), arXiv:1708.04897 [hep-lat].
- 391 [8] S. Borsanyi, Z. Fodor, J. N. Guenther, S. K. Katz, K. K. Szabo, A. Pasztor, I. Portillo, and C. Ratti, JHEP **10**, 205
392 (2018), arXiv:1805.04445 [hep-lat].
- 393 [9] B. Friman, F. Karsch, K. Redlich, and V. Skokov, Eur.Phys.J. **C71**, 1694 (2011), arXiv:1103.3511 [hep-ph].
- 394 [10] M. Abdallah *et al.* (STAR), Phys. Rev. C **104**, 024902 (2021), arXiv:2101.12413 [nucl-ex].
- 395 [11] M. Abdallah *et al.* (STAR), Phys. Rev. C **107**, 024908 (2023), arXiv:2209.11940 [nucl-ex].
- 396 [12] J. Adamczewski-Musch *et al.* (HADES), Phys. Rev. (2020), arXiv:2002.08701 [nucl-ex].
- 397 [13] S. Acharya *et al.* (ALICE), Phys. Lett. B **807**, 135564 (2020), arXiv:1910.14396 [nucl-ex].
- 398 [14] S. Acharya *et al.* (ALICE), Phys. Lett. B **844**, 137545 (2023), arXiv:2206.03343 [nucl-ex].
- 399 [15] W.-j. Fu, J. M. Pawłowski, F. Rennecke, and B.-J. Schaefer, Phys. Rev. D **94**, 116020 (2016), arXiv:1608.04302 [hep-ph].
- 400 [16] M. Bleicher, S. Jeon, and V. Koch, Phys. Rev. **C62**, 061902 (2000), arXiv:hep-ph/0006201 [hep-ph].
- 401 [17] A. Bzdak, V. Koch, and V. Skokov, Phys. Rev. **C87**, 014901 (2013), arXiv:1203.4529 [hep-ph].
- 402 [18] P. Braun-Munzinger, A. Rustamov, and J. Stachel, Nucl. Phys. A **982**, 307 (2019), arXiv:1807.08927 [nucl-th].
- 403 [19] P. Braun-Munzinger, B. Friman, K. Redlich, A. Rustamov, and J. Stachel, Nucl. Phys. A **1008**, 122141 (2021),
404 arXiv:2007.02463 [nucl-th].
- 405 [20] P. Braun-Munzinger, K. Redlich, A. Rustamov, and J. Stachel, (2023), arXiv:2312.15534 [nucl-th].
- 406 [21] M. Kitazawa and M. Asakawa, Phys. Rev. **C85**, 021901 (2012), arXiv:1107.2755 [nucl-th].
- 407 [22] V. Skokov, B. Friman, and K. Redlich, Phys. Rev. **C88**, 034911 (2013), arXiv:1205.4756 [hep-ph].
- 408 [23] P. Braun-Munzinger, A. Rustamov, and J. Stachel, Nucl. Phys. **A960**, 114 (2017), arXiv:1612.00702 [nucl-th].
- 409 [24] A. Bialas, M. Bleszynski, and W. Czyz, Nucl. Phys. B **111**, 461 (1976).
- 410 [25] J. Adam *et al.* (STAR), Phys. Rev. Lett. **126**, 092301 (2021), arXiv:2001.02852 [nucl-ex].
- 411 [26] <https://github.com/anarrustamov/VolumeFluctuations>.
- 412 [27] J. Adamczewski-Musch *et al.* (HADES), Eur. Phys. J. A **54**, 85 (2018), arXiv:1712.07993 [nucl-ex].
- 413 [28] B. Abelev *et al.* (ALICE), Phys. Rev. C **88**, 044909 (2013), arXiv:1301.4361 [nucl-ex].
- 414 [29] C. Loizides, J. Nagle, and P. Steinberg, SoftwareX **1-2**, 13 (2015), arXiv:1408.2549 [nucl-ex].
- 415 [30] S. Harabasz, J. Kolaś, R. Ryblewski, W. Florkowski, T. Galatyuk, M. Gumberidze, P. Salabura, J. Stroth, and H. P.
416 Zbroszczyk, Phys. Rev. C **107**, 034917 (2023), arXiv:2210.07694 [nucl-th].
- 417 [31] B. Abelev *et al.* (ALICE), Phys. Rev. C **88**, 044910 (2013), arXiv:1303.0737 [hep-ex].
- 418 [32] A. Bzdak, V. Koch, and V. Skokov, Eur. Phys. J. **C77**, 288 (2017), arXiv:1612.05128 [nucl-th].
- 419 [33] B. Friman and K. Redlich, (2022), arXiv:2205.07332 [nucl-th].