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Birational Geometry of Blowups of Toric Projective Varieties

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Noble Williamson

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Dissertation Committee:

Dr. José González, Chairperson
Dr. Patricio Gallardo Candela
Dr. Ziv Ran

The Dissertation of Noble Williamson is approved:

Committee Chairperson

University of California, Riverside

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ABSTRACT OF THE DISSERTATION

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Noble Williamson

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The Cox ring of an algebraic variety encodes important information on the birational geometry of the variety. When its Cox ring is finitely generated, a variety admits particularly desirable properties in the context of the Minimal Model Program and is called a Mori dream space. For example, all toric varieties are known to be Mori dream spaces so a natural next step in the problem is to study the birational geometry of projective varieties that can be constructed as blowups of toric varieties by studying their pseudoeffective cones and Cox rings. In this dissertation, we present a concrete criterion for the finite generation of the Cox ring of toric projective surfaces of Picard number one blown up at a smooth point using the coordinates of a polytope of the toric variety. We also present a criterion for the irreducibility of an effective divisor of the moduli space of n-pointed stable rational curves.

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Chapter 1

Introduction

One of the dreams of algebraic geometry is the classification of projective varieties up to birational equivalence. To this end, for a given projective variety X defined over a field k, we want to find the simplest possible variety X' that is birational to X called a minimal model of X. This process is known as the minimal model program or MMP. The hunt for minimal models is approached by studying the divisors on a variety and the birational maps that arise from them. Recall that a prime divisor on a normal variety X is an irreducible codimension-one subvariety of X and a Weil divisor on X is an element of the free abelian group generated by the prime divisors on X. To make this group more manageable, we study suitable equivalence classes on X. For a nonzero rational function f on X, we can define a Weil divisor div(f) called a principal divisor and we consider two Weil divisors D_1 and D_2 on X to be linearly equivalent, denoted $D_1 \sim D_2$, if there exists a nonzero rational function f on X such that $D_1 - D_2 = \mathrm{div(f)}$. We call the group of linear equivalence classes of X the class group of X denoted $\mathrm{Cl}(X)$. We define a Cartier divisor on X to be a Weil divisor that is locally principal. For a Weil divisor D on X, If there exists a positive integer m such that mD is a Cartier divisor, we say that D is $\mathbb{Q}$-Cartier and if all Weil divisors on X are $\mathbb{Q}$-Cartier, we say that X is $\mathbb{Q}$-factorial. Finally, from a divisor D on X, we get a vector space of global sections defined by $H^0(X, D) = \{f \in k(X)^* : \mathrm{div(f)} + \mathrm{D} \geq 0\}$.

The structure of a variety's divisors is often studied using techniques from convex geometry and intersection theory. For Cartier divisor D and a curve C on X, we can define an integer $D \cdot C$ called the intersection product of D and C. We say that two Cartier divisors D_1 and D_2 are numerically equivalent, denoted $D_1 \equiv D_2$, if $D_1 \cdot C = D_2 \cdot C$ for all irreducible curves C on X. The group of numerical equivalence classes of divisors on X, denoted $N^1(X)$, is called the Néron-Severi group of X and it is a finitely generated abelian group whose rank we define to be the Picard number of X.

The vector spaces of global sections of divisors on a normal projective $\mathbb{Q}$-factorial variety X are used to define an important invariant called the Cox ring which encodes the birational geometry of the variety. For a normal projective $\mathbb{Q}$-factorial variety X defined over a field k, the Cox ring of X is defined by

$$\mathrm{Cox(X)} = \bigoplus_{\mathrm{D \in Cl(X)}} \mathrm{H^0(X, D)}.$$

If Cox(X) is finitely generated as a k-algebra, X is a Mori dream space (MDS) which is defined by Hu and Keel in [Hu00] to be a variety that behaves particularly well in the context of the minimal model program. The first varieties that were shown to be Mori dream spaces are called toric projective varieties (see [Cox95]) which are particularly concrete varieties that arise from combinatorial objects called fans and polytopes. These combinatorics provide valuable tools to study the divisors of these varieties and allowed Cox to prove that their Cox rings are all polynomial rings in finitely many variables so they are finitely generated. Even when a variety is constructed by blowing up a toric projective variety at a single smooth point, its birational geometry is unknown in general. In this dissertation, we take this natural next step and study the birational geometry of certain projective varieties that can be constructed as blowups of toric projective varieties by studying the structure of their divisors and Cox rings. Our main result involves blowups of toric projective surfaces whose Picard number is one. Varieties called weighted projective surfaces, which are defined in a similar fashion to the projective plane as a quotient of affine space, are examples of such toric projective surfaces. The Mori dream space property of weighted projective surfaces blown up at a point have been studied extensively (see [Hun82], [Cut91], [Sri91], [Hau18], [He21] for examples of such Mori dream spaces and [Got94] and [Gon16] for examples that are not Mori dream spaces). Many of these results use concrete criteria involving the weights of the weighted projective space. In this dissertation, we sought to develop similarly concrete criteria involving some fundamental aspect of weighted projective surfaces of Picard number one given that these weights are not available to us in general and we succeeded in doing so by developing a sufficient condition for the MDS property of toric projective surfaces of Picard number one blown up at a point using the coordinates of the vertices for a strategically chosen polytope associated with the variety (see Lemma 3.1.1 and Theorem 3.1.8).

A related problem to determining the Mori dream space property is to determine when a surface admits an irreducible curve with non-positive self-intersection that is not the exceptional divisor of the blowup. For the sake of simplicity, we call such a curve a negative curve. If X is a toric projective surface of Picard number one blown up at a point and if Cox(X) is a finitely generated k-algebra, then there necessarily exists an irreducible curve on X with non-positive self-intersection. Hence, Theorem 3.1.8 gives a sufficient condition for the existence of a negative curve on X.

A moduli space is a space that parameterizes objects in a manner that mirrors the structure of the objects it represents. It is used in the classification of geometric objects like algebraic varieties and stacks. Let $M_{g,n}$ be the moduli space of smooth genus g curves with n marked points. This space is not compact in general because one could construct a sequence in which two of the marked points converge. In order to find a compactification, we have the notion of a stable curve which is a curve whose singularities are at worst double points and whose automorphism group is finite. We define the compactification $\overline{M}_{g,n}$ to be the moduli space of stable genus g curves with n marked points. When $g = 0$, this is the moduli space parameterizing nodal trees of projecive lines such that each component contains three points that are either marked points or nodes. In [Kap92], Kapranov proved that $\overline{M}_{0,n}$ can be realized as the iterated blowup of $\mathbb{P}^{n-3}$ at $n-1$ points in general position and along the linear subspaces spanned by those points.

Doran, Giansiracusa, and Jensen utilized Kapranov's construction in [Dor17] to exhibit a bijection between the homogeneous elements of $\mathrm{Cox}(\overline{\mathrm{M}}_{0,\mathrm{n}})$ that are not divisible by any exceptional divisor section and weighted, pure-dimensional simplicial complexes admitting a balancing condition. González, Gunther, and Zheng built upon this result in [Gon20] to simplify the balancing condition and classify all irreducible elements of the monoid of effective divisors on $\overline{M}_{0,n}$ that arise from non-singular simplicial complexes. We build upon these results to develop a concrete criterion for the irreducibility of such elements that, when implemented in a computer program, allowed us to rediscover all extremal elements of $\overline{\mathrm{Eff}}(\overline{M}_{0,n})$ that appear in the literature at the time of this writing.

Chapter 2

Preliminaries

We will focus our attention throughout on normal algebraic varieties defined over an algebraically closed field k of characteristic zero. Recall that a variety X is *normal* if for all $x \in X$, the local ring $\mathcal{O}_{X,x}$ is an integrally closed domain. Recall that a *prime divisor* Y on a variety X is an irreducible subvariety of codimension-one. A *Weil divisor* D on X is a finite formal linear combination

$$D = \sum a_i Y_i$$

where Y_i is a prime divisor on X and a_i is an integer. The additive group of all Weil divisors on X is denoted $\mathrm{Div(X)}$. We say that a Weil divisor $D = \sum a_i Y_i$ is effective, denoted, $D \geq 0$, if $a_i \geq 0$ for all i. An *algebraic r-cycle* on X is a formal combination with integer coefficients of r-dimensional subvarieties on X. The additive group of r-cycles on X is denoted $Z_r(X)$ so $\mathrm{Div(X) = Z_{n-1}(X)}$.

Theorem 2.0.1. *Let X be a normal variety and let Y be an irreducible divisor on X then the local ring $\mathcal{O}_{X,Y}$ is a discrete valuation ring.*

The previous theorem allows us to define a valuation $\mathrm{ord_Y} : \mathcal{O}_{\mathrm{X,Y}} \to \mathbb{Z}$ on $\mathcal{O}_{X,Y}$. Given a nonzero rational function f on X, we define the *principal divisor* associated with f to be

$$\mathrm{div(f)} = \sum_{\mathrm{Y}} \mathrm{ord_Y(f)Y}$$

where the sum is taken over all prime divisors of X. Let D_1 and D_2 be Weil divisors on X. We say that D_1 and D_2 are *linearly equivalent*, written $D_1 \sim D_2$, if they differ by a principal divisor, that is, if there exists a nonzero rational function f such that $D_1 - D_2 = \mathrm{div(f)}$. We define the *divisor class group* to be $\mathrm{Cl(X)} \coloneqq \mathrm{Div(X)}/\sim$.

A variety X is *complete* if, for any variety Y, the projection morphism

$$X \times Y \longrightarrow Y$$

is a closed morphism. Every projective variety is complete.

For a Weil divisor $D = \sum a_i Y_i$ on a normal variety X, we define the restriction of D to a Zariski open subset $U \subset X$ to be

$$D\big|_U = \sum_{Y_i \cap U \neq \emptyset} a_i(Y_i \cap U).$$

A *Cartier divisor* on X is a Weil divisor D such that there exists an open cover $X = \cup_i U_i$ and $f_i \in k(U_i)^*$ such that

$$D\big|_{U_i} = \mathrm{div(f_i)}$$

. That is, a Cartier divisor is a Weil divisor that is *locally principal.* Let D be a Weil divisor, we say D is $\mathbb{Q}$-*Cartier* if there exists a natural number m such that mD is a Cartier divisor. A variety X such that every Weil divisor D on X is $\mathbb{Q}$-Cartier is said to be $\mathbb{Q}$-*factorial.*

2.1 Intersection Theory

Recall Bézout's theorem which states that if V and W are two projective plane curves defined over an algebraically closed field k then the number of intersection points of V and W is either infinite or equal to the product of the degrees of the two curves. Significant effort has been dedicated to generalizing this intuitive notion of intersection as much as possible. The simplest case is when, for two algebraic cycles V and W on a variety X with set-theoretic intersection $V \cap W$, we have

$$\dim(V \cap W) = \dim V + \dim W - \dim X \tag{2.1}$$

which is the case in Bézout's theorem where $\dim V = \dim W = 1$ and $\dim X = 2$. We call intersections where equation 2.1 holds *proper intersections.* In the case of a proper intersection, the intersection product $V \cdot W$ of an r-cycle V and an s-cycle W on an n-dimensional variety X is simply an $(r+s-n)$-cycle given by a linear combination of the irreducible components of $V \cap W$.

One obstacle for the generalization of this notion is that cycles can be positioned in inconvenient ways. For example, two cycles may be parallel lines in a plane or a plane curve and the plane that contains it in higher dimensional space. For this reason, we need to be able to "move" our cycles in a suitable way in order to generalize Bézout's theorem in cases like these. To do so, we define a suitable equivalence so that, if cycles V and W are in inconvenient positions, we can instead consider the intersection of equivalent cycles V' and W' which are in better positions.

An adequate equivalence relation allowing us to define a well-defined intersection product of algebraic cycles is rational equivalence which is a generalization of linear equivalence defined above. Let X be an n-dimensional variety. Given a subvariety Y of X of dimension $r+1$ and a nonzero rational function φ on Y, one can associate a cycle $\mathrm{div}_{\mathrm{X}}(\varphi)$ in $Z_r(X)$ generalizing the construction of the divisor associated to a nonzero rational function on X (see [Ful13, Chapter 1] for details). This cycle $\mathrm{div}_{\mathrm{X}}(\varphi)$ is equal to the push-forward to $Z_r(X)$ of a cycle in $Z_r(Y)$ of the form

$$\mathrm{div(f)} \coloneqq \sum_{\mathrm{Z}} \mathrm{ord_Z(f)Z}$$

where the sum is taken over all irreducible divisors Z of Y, and this can be constructed even when Y is not normal (see [Ful13, chapter 1]). Let $B_r(X)$ be the subgroup of $Z_r(X)$ generated by all r-cycles of the form $\mathrm{div}_{\mathrm{X}}(\varphi)$ for $\varphi \in k(Y)^*$ for all subvarieties Y of dimension

$r+1$ on X. Then we define the groups of r-cycle classes on X to be the *Chow groups* on X given by $A_r(X) := Z_r(X)/B_r(X)$ and we denote the r-cycle class of an r-cycle V by $[V] \in A_r(X)$. We say two r-cycles V and V' on X are *rationally equivalent* if they define the same cycle class in $A_r(X)$. For any subvariety Y of X with inclusion morphism, $i : Y \to X$, rational equivalence is respected by the pushforward maps $i_* : A_r(Y) \to A_r(X)$ defined by $[V] \mapsto [V]$. Let U be an open subset of X then rational equivalence is also respected by the pullback maps $i^* : A_r(X) \to A_r(U)$ defined by $[V] \mapsto [V \cap U]$ where $i^*\mathrm{div(f)} = \mathrm{div}(\mathrm{f}\big|_{\mathrm{U}})$.

Now let D be a Cartier divisor on X and let V be an r-dimensional subvariety of X with inclusion map $i : V \to X$. Then we can define the *intersection product* $D \cdot V \in A_{r-1}(X)$ of D and V to be the

$$D \cdot [V] = i_*[i^*(\mathcal{O}_X(D)].$$

This definition is bilinear and depends only on the divisor class of D. In the sequel, we will usually intersect divisors with curves where the intersection product is zero-dimensional in which case we normally associate the intersection product with its degree to get an integer as our output. In other words, if D is a Cartier divisor on X and C is a curve on X with inclusion $i : C \to X$, then

$$D \cdot C = \sum_i a_i \in \mathbb{Z}$$

where $i_*[i^*\mathcal{O}_X(D)] = \deg_C(i^*(\mathcal{O}_X(D))) = \sum_i a_i[P_i]$ where $\{P_i\}$ are the finitely many points of the set-theoretic intersection of D and C.

Two Cartier divisors D_1 and D_2 on a variety X are said to be numerically equivalent, written $D_1 \equiv D_2$ if

$$D_1 \cdot C = D_2 \cdot C$$

for all irreducible curves C on X. Note that, by the linearity of the intersection product, we could equivalently consider the intersections over all 1-cycles on X. For projective varieties, intersection products are invariant under numerical equivalence. Since all of the varieties we will study are projective, we will write $[D]$ for the numerical equivalence class of D

Proposition 2.1.1. *Let X be a complete irreducible variety, let $D_1, D_2, \ldots, D_k$ be Cartier divisors on X, and let V be a k-cycle on X. Let $D_1', D_2', \ldots, D_k'$ be Cartier divisors on X such that $D_i \equiv D_i'$ then*

$$(D_1 \cdot D_2 \cdot \cdots \cdot D_k) \cdot [V] = (D_1' \cdot D_2' \cdot \cdots \cdot D_k') \cdot [V].$$

Let $\mathrm{Num(X)} \subset \mathrm{Div(X)}$ be the subgroup of divisors on X that are numerically equivalent to zero. Then we define the *Néron-Severi group* to be the group

$$N^1(X) = \mathrm{Div(X)/Num(X)}$$

of all numerical equivalence classes of divisors on X. The rank of $N^1(X)$ is called the *Picard number* of X. Often, for the purpose of being able to employ concepts from convex geometry, we consider the vector space $N^1(X) \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^{\rho(X)}$.

2.2 Positive Cones and Cox Rings

Let D be a Cartier divisor on a variety X over a field k. We define $H^0(X, \mathcal{O}_X(D))$ to be the vector space of global sections of $\mathcal{O}_X(D)$ given by

$$H^0(X, \mathcal{O}_X(D)) = \{f \in k(X)^* : \mathrm{div(f) + D} \geq 0\} \cup \{0\}$$

whose dimension is denoted $h^0(X, \mathcal{O}_X(D))$. Often, we will simply write $H^0(X, D)$ in place of $H^0(X, \mathcal{O}_X(D))$. A Cartier divisor $\{(f_i, U_i)\}$ is effective if f_i is regular on U_i for all i. When X is a normal variety, the group of Cartier divisors on X form the subgroup of the group of Weil divisors given by Weil divisors D such that there exists an open cover $\{U_i\}$ of X and nonzero rational functions $f_i \in k(U_i)^*$ such that

$$D\Big|_{U_i} = \mathrm{div(f_i)}.$$

Note that, if a Cartier divisor $\{(f_i, U_i)\}$ is effective, its corresponding Weil divisor is effective. For a complete variety X, a Cartier divisor D on X is effective if and only if $h^0(X, D) > 0$ in which case we can define a rational map

$$\varphi_D : X \dashrightarrow \mathbb{P}(H^0(X, D)^\vee) \cong \mathbb{P}^n$$

where $\mathbb{P}(H^0(X, L)^\vee)$ is the projective space of codimension one subspaces of $H^0(X, L)$. Moreover, choosing a basis $f_0, f_1, \ldots, f_n$ of $H^0(X, L)$ gives a morphism

$$X \setminus V(f_0, \ldots, f_n) \longrightarrow \mathbb{P}(H^0(X, D)^\vee) \cong \mathbb{P}^n \tag{2.2}$$

$$x \longmapsto [f_0(x) : f_1(x) : \cdots : f_n(x)] \tag{2.3}$$

from the subset of X where at least one section does not vanish to projective space.

For a Cartier divisor D on X, let m be a positive integer and let φ_m be the rational map to projective space associated with the divisor mD. If mD is effective for some positive integer m, we define the *Iitaka dimension* of D to be

$$\kappa(D) = \max_m(\dim(\varphi_m(X))).$$

If there does not exist a positive integer m such that mD is effective, the image of X under φ_m is empty for all m so we define $\kappa(D) = -\infty$. We say that D is *big* if $\kappa(D) = \dim(X)$.

We say that a Cartier divisor D is *base point free* if there does not exist a point, called a *base point*, where all global sections $f_0, f_1, \ldots, f_n$ simultaneously vanish, so base point free line bundles give rise to morphisms from X to projective space. A Cartier divisor D is said to be *globally generated* if there exists an index set I such that the map of $\mathcal{O}_X$-modules

$$\bigoplus_{i \in I} \mathcal{O}_X \longrightarrow \mathcal{O}_X(D)$$

is surjective. In fact, D is globally generated if and only if it is base point free. Moreover, we say D is *semi-ample* if there exists a positive integer r such that rD is globally generated. We say D is *very ample* if the corresponding map $X \to \mathbb{P}^n$ is a closed immersion and D is said to be *ample* if there exists a positive integer multiple rD that is very ample. Finally, we say that D is *nef* if $D \cdot C \geq 0$ for all irreducible curves C on X.

Clearly the sum of any two effective divisors and any positive scalar multiple of an effective divisor is still effective, then the convex psotive cone in $N^1(X)_{\mathbb{R}}$ generated by all effective divisors is called the *effective cone* of X, denoted Eff(X). In general, those cone may not be closed so we define its closure $\overline{\mathrm{Eff}}(X)$ to be the *pseudoeffective cone* of X. Similarly, we denote the positive cone in $N^1(X)_{\mathbb{R}}$ of big divisors on X by $\mathrm{Big}(X)$, the positive cone in $N^1(X)_{\mathbb{R}}$ of nef divisors on X by Nef(X), and the positive cone in $N^1(X)_{\mathbb{R}}$ of ample divisors on X by Ample(X). Note that for a positive cone $C \subset N^1(X)_{\mathbb{R}}$, the closure $\overline{C}$ and the interior C° are psotive cones in $N^1(X)_{\mathbb{R}}$ as well. We will employ the following theorem, which synthesizes results from [Laz17, Theorem 1.4.23] and [Laz17, Theorem 2.2.26] describing the relationships among these positive cones extensively in the sequel.

Theorem 2.2.1. *Let X be a projective variety.*

1. $\mathrm{Nef(X)} = \overline{\mathrm{Ample}}\mathrm{(X)}$ *and* $\mathrm{Ample(X)} = \mathrm{Nef(X)}^\circ$
2. $\overline{\mathrm{Eff}}(X) = \overline{\mathrm{Big}(X)}$ *and* $\mathrm{Big}(X) = \overline{\mathrm{Eff}}(X)^\circ$
3. *Nef(X) is the dual cone to the cone of effective numerical equivalence classes of curves on X. Hence if X is a surface then Nef(X) $= \overline{\mathrm{Eff}}(X)^*$.*

We call a ray R of a positive cone C in $N^1(X)_{\mathbb{R}}$ *extremal* if for all $v, w \in C$ such that $v + w \in R$, we have that $v, w \in R$. The following theorem allows allows us to guarantee that the extremal rays of $\overline{\mathrm{Eff}}(X)$ are all generated by irreducible curves under certain conditions.

Proposition 2.2.2. *Let X be a normal algebraic surface and let A be an ample divisor on X. Then for all $\varepsilon > 0$ there exist rational curves $l_1, \ldots, l_r$ such that*

$$\overline{\mathrm{Eff}}(X) = \mathbb{R}_+[l_1] + \cdots + \mathbb{R}_+[l_r] + \overline{\mathrm{Eff}}_\varepsilon(X, A)$$

where $\overline{\mathrm{Eff}}_\varepsilon(X, A) = \{C \in \overline{\mathrm{Eff}}(X) : \ (K_X + \varepsilon A) \cdot C \geq 0\}$.

Proof. Sakai proves this for normal Moishezon surfaces in [FS85, Proposition 4.8] using the more general result by Mori from [Mor06, Theorem 1.2]. Since all algebraic surfaces are Moishezon, the result follows. □

The Cox ring Cox(X) of a projective variety X is an important invariant that captures the birational geometry of X. The Cox ring of a projective variety X is given by

$$\mathrm{Cox(X)} = \bigoplus_{\mathrm{D} \in \mathrm{Cl(X)}} \mathrm{H}^0(\mathrm{X}, \mathrm{D}).$$

When X is a normal, projective, $\mathbb{Q}$-factorial variety and Cox(X) is a finitely generated k-algebra, we say that X is a *Mori dream space* because it admits particularly favorable properties in the context of the minimal model program. Additionally, if Cox(X) is a finitely generated k-algebra then there exists finitely many irreducible curves $C_1, C_2, \ldots, C_n$ such that $C_i^2 \leq 0$ and

$$\overline{\mathrm{Eff}}(X) = \mathbb{R}_+[C_1] + \mathbb{R}_+[C_2] + \cdots + \mathbb{R}_+[C_n].$$

The following lemmas which predate the notion of Cox rings or Mori dream spaces, will prove to be invaluable as we seek to determine the Mori dream space property of projective varieties. The first was proven by Zariski in [Zar62, Theorem 4.2].

Lemma 2.2.3 (Zariski)**.** *Let H and D be semi-ample Cartier divisors on a normal projective surface X that is proper over an algebraically closed field k, then*

$$\bigoplus_{m,n \geq 0} H^0(X, mH + nD)$$

is a finitely generated k-algebra.

Lemma 2.2.4. *Suppose that D_1 and D_2 are Weil divisors of a variety X. Then for any integers $a, b > 0$,*

$$\bigoplus_{m,n \geq 0} H^0(mD_1 + nD_2)$$

is a finitely generated k-algebra if and only if

$$\bigoplus_{m,n\geq 0} H^0(amD_1 + bnD_2)$$

is a finitely generated k-algebra.

The following result was proven by Cutkosky in [Cut91, Lemma 6].

Lemma 2.2.5 (Cutkosky). *Let S be a normal projective surface. Suppose that C is a $\mathbb{Q}$-Cartier divisor and an irreducible curve on S such that $(C \cdot (-K_S)) > 0$ and $C^2 = 0$. Then C is semi-ample.*

2.3 Analytic Geometry and Algebraic Geometry

The geometric theory of several complex variables, or simply complex analytic geometry, has many parallels with algebraic geometry and it is often advantageous to take inspiration from one subject in order to study the other. Similarly to algebraic schemes, we define an *analytic space* to be a locally ringed space which is locally isomorphic to an open subset of the vanishing locus of finitely many holomorphic functions. Note that, unlike algebraic varieties, an analytic scheme need not be integral, reduced, or irreducible but since normal points are irreducible and irreducible points are reduced, a normal analytic space is locally irreducible and reduced. Moreover, we have the following purity criterion, the details of which can be found in [Gra94, Chapter I §10].

Theorem 2.3.1. *A reduced analytic space is pure-dimensional at all of its irreducible points.*

From this, we can conclude that every normal analytic space is locally irreducible and pure-dimensional.

As in modern algebraic geometry, sheaves play an critical role in the study of analytic spaces. Fix a ringed space $(X, \mathcal{O}_X)$ (such as an algebraic variety or an analytic space), and let $\mathcal{F}$ and $\mathcal{G}$ be coherent $\mathcal{O}_X$-modules. Recall that, for a sheaf $\mathcal{F}$ of abelian groups on a topological space (for example, the underlying topological space of a variety or analytic space), the *sheaf cohomology groups* $H^i(X, \mathcal{F})$ are defined as the right derived functors of the functor of global sections $\mathcal{F} \to \mathcal{F}(X)$. In algebraic geometry, a duality theorem for sheaf cohomology known as "Serre duality" is a valuable tool for the classification of algebraic varieties. There is a similar theorem, which we will also call Serre duality, for analytic spaces having certain additional properties. One can find a detailed exposition of the theory in [Gra94, Chapter III §4].

Theorem 2.3.2 (Serre)**.** *Let X be a compact Cohen-Macaulay analytic space of pure dimension n, let $\mathcal{F}$ be a coherent sheaf on X, and let ω_X be the dualizing sheaf on X. Then*

$$H^q(X, \mathcal{F}) \cong Ext^{n-q}(\mathcal{F}, \omega_X)$$

for all $0 \leq q \neq n$.

Let X be an algebraic variety defined over $\mathbb{C}$. By observing that the regular functions on X which are locally defined by polynomials can be considered to be holomorphic functions on X which are locally defined by power series, we can associate to X a complex analytic space X_h. This allows us to apply techniques from complex geometry to algebraic varieties over $\mathbb{C}$. Moreover, if $\mathcal{F}$ is a sheaf on X then we can define a corresponding analytic sheaf $\mathcal{F}_h$ on X_h. When X is projective, the relationship between X and X_h is even closer. In his celebrated paper [Ser56] affectionately known as "GAGA," Serre proved the following two theorems that will be useful to us later.

Theorem 2.3.3 (Serre)**.** *Let X be a projective variety and let X_h be the corresponding analytic space. For all coherent sheaves $\mathcal{F}$ on X and for all $q \geq 0$, we have*

$$H^q(X, \mathcal{F}) \cong H^q(X_h, \mathcal{F}_h).$$

Theorem 2.3.4 (Serre). *Let X be a projective variety and let X_h be the corresponding analytic space. For all coherent analytic sheaves $\mathcal{G}$ on X_h, there exists a coherent sheaf $\mathcal{F}$ on X such that $\mathcal{F}_h \cong \mathcal{G}$.*

Using these theorems, we can associate to any morphism of analytic spaces between projective varieties, a morphism of varieties.

Proposition 2.3.5. *Let X and Y be projective varieties defined over $\mathbb{C}$ and let X_h and Y_h be the associated complex analytic spaces. Let $f : X_h \to Y_h$ be a morphism of analytic spaces then there exists a morphism of varieties $\varphi : X \to Y$ such that $\varphi_h = f$.*

Proof. First we consider the case where $Y_h = \mathbb{P}^n$ for some n. Since the invertible analytic sheaf $f^*(\mathcal{O}(1))$ on X_h is very ample, it is generated by the global sections $s_0, s_1, \ldots, s_n$ such that $s_i = f^*(x_i)$ where x_i is the i-th homogeneous coordinate of $\mathbb{P}^n$. Since X is projective, there exists an invertible sheaf L on X such that $L_h = f^*(\mathcal{O}(1))$ by 2.3.4. Then since

$$H^0(X, L) \cong H^0(X_h, L_h),$$

there exist global sections $t_0, t_1, \ldots, t_n$ such that $(t_i)_h = s_i$ that generate L so L is very ample and determines a closed embedding of algebraic varieties $\varphi : X \to \mathbb{P}^n$ such that $\varphi_h = f$.

Now let Y_h be any projective complex analytic space. Since Y_n is projective, there exists an embedding of complex analytic spaces $g : Y_h \to \mathbb{P}^n$ so we get a map $(g \circ f) : X_h \to \mathbb{P}^n$. By the previous paragraph, we can consider both g and $(g \circ f)$ to be morphisms of algebraic varieties so the same must be true for f. $\square$

The following proposition will allow us to consider varieties defined over an algebraically closed field k of characteristic zero instead as a variety defined over $\mathbb{C}$ without losing generality.

Proposition 2.3.6. *Let X_k be a variety defined over an algebraically closed field k of characteristic zero and let $X_{\mathbb{C}}$ be the same variety defined over $\mathbb{C}$. Moreover, let D_k be a divisor on X_k and $D_{\mathbb{C}}$ the corresponding divisor on $X_{\mathbb{C}}$. Then*

1. *$D_{\mathbb{C}}$ is effective if and only if D_k is,*
2. *$\mathcal{O}_{X_{\mathbb{C}}}(X_{\mathbb{C}}, D_{\mathbb{C}})$ is globally generated if and only if $\mathcal{O}_{X_k}(X_k, D_k)$ is.*

Proof. Let X be a variety defined over an algebraically closed field k of characteristic zero and let D be a divisor of X. Let $\tilde{k}$ be the minimal field of definition for X. In what follows, for a field F over which X is defined, let X_F, D_F, etc. denote the objects X, D, etc. defined over F. Since $\tilde{k}$ is a finite extension of a field isomorphic to $\mathbb{Q}$, it is an algebraic and separable extension of $\mathbb{Q}$ so we can embed it into $\mathbb{C}$. Let L be a field extension of $\tilde{k}$ considered as a subfield of $\mathbb{C}$ then L is a faithfully flat $\tilde{k}$-module. Then

$$H^0(X_L, \mathcal{O}_{X_L}(D_L)) \cong H^0(X_{\tilde{k}}, \mathcal{O}_{X_{\tilde{k}}}(D_{\tilde{k}})) \otimes_{\tilde{k}} L$$

so D_L is effective if and only if $D_{\tilde{k}}$ is effective. Recall that $\mathcal{O}_{X_{\tilde{k}}}(D_{\tilde{k}})$ is globally generated if and only if there exists an index set I such that the map of $\mathcal{O}_{X_{\tilde{k}}}$-modules

$$\varphi_{\tilde{k}} : \bigoplus_{i \in I} \mathcal{O}_{X_{\tilde{k}}} \longrightarrow \mathcal{O}_{X_{\tilde{k}}}(D_{\tilde{k}})$$

is surjective. By taking the tensor product of $\varphi_{\tilde{k}}$ with L, we get the map

$$\varphi_L : \bigoplus_{i \in I} \mathcal{O}_{X_L} \longrightarrow \mathcal{O}_{X_L}(D_L)$$

and since L is a faithfully flat $\tilde{k}$-module, φ_L is surjective if and only if $\varphi_{\tilde{k}}$ is surjective. Finally, φ_L is surjective if and only if D_L is globally generated so D_L is globally generated if and only if $D_{\tilde{k}}$ is globally generated. □

2.4 Toric Varieties

Definition 2.4.1. The *algebraic torus* is the affine variety $T = (k^*)^n$. A *toric variety* X is a normal variety that contains the torus T as an open subset such that the group action of T on itself extends to an action of T on all of X.

Example 2.4.2. Clearly affine space k^n contains T as an open subset and T acts on k^n by $(a_1, a_2, \dots, a_n) \cdot (t_1, t_2, \dots, t_n) = (t_1 a_1, t_2 a_2, \dots, t_n a_n)$ so k^n is a toric variety. Hence, since every affine patch U_i of projective space $\mathbb{P}^n$ is a toric variety, $\mathbb{P}^n$ is as well.

Example 2.4.3. Given positive integers $w_0, w_1, \dots, w_n$, we can define the weighted projective space $\mathbb{P}(w_0, \dots, w_n)$ to be

$$\mathbb{P}(w_0, w_1, \dots, w_n) \coloneqq (k^{n+1} \setminus \{0\})/\sim$$

where $(p_0, p_1, \dots, p_n) \sim (q_0, q_1, \dots, q_n)$ if there exists $\lambda \in k^*$ such that

$$(q_0, q_1, \dots, q_n) = (\lambda^{w_0} p_0, \lambda^{w_1} p_1, \dots, \lambda^{w_n} p_n).$$

Then the image of $(k^*)^{n+1} \subset k^{n+1} \setminus \{0\}$ in $\mathbb{P}(w_0, w_1, \dots, w_n)$ is $(k^*)^{n+1}/(k^*) \cong (k^*)^n$ where $k^* \hookrightarrow (k^*)^{n+1}$ by $t \mapsto (t^{w_0}, t^{w_1}, \dots, t^{w_n})$. Then the action of $(k^*)^{n+1}$ on $k^{n+1} \setminus \{0\}$ by component-wise multiplication gives rise to an action on $\mathbb{P}(w_0, w_1, \dots, w_n)$ making weighted projective space a toric variety.

Toric varieties are closely related to two lattices. A *lattice* N is a free abelian group of finite rank so that, after picking a $\mathbb{Z}$-basis for N, we have $N \cong \mathbb{Z}^n$ for some n. From N we get the *dual lattice*

$$M = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{N}, \mathbb{Z})$$

and we denote by $\langle m, u \rangle$ the image of $u \in N$ under the morphism $m \in M$. By picking a $\mathbb{Z}$-basis for N we get a dual basis for M such that $M \cong \mathbb{Z}^n$. Then, tensoring by k^* gives

$$T(N) \coloneqq N \otimes_{\mathbb{Z}} k^* \cong (k^*)^n$$

which we call the *torus* of N.

Net $u \in N$ and suppose $u \mapsto (a_1, \ldots, a_n)$ under the isomorphism $N \cong \mathbb{Z}^n$. Then we define

$$\lambda^u : \mathbb{C}^* \longrightarrow T(N)$$

by $\lambda^u(t) = (t^{a_1}, \ldots, t^{a_n})$. Now let $m \in M$ such that $m \mapsto (b_1, \ldots, b_n)$ under the isomorphism $M \cong \mathbb{Z}^n$ then we define the *character* χ^m of $T(N)$ by $\chi^m : T(n) \longrightarrow k^*$ where $\chi^m(t_1, \ldots, t_n) = t_1^{b_1} \cdot \cdots \cdot t_n^{b_n}$. We call $t_1^{b_1} \cdot \cdots \cdot t_n^{b_n}$ a *Laurent monomial* which lies in the ring $k[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}]$ of *Laurent polynomials.*

The Toric Variety of a Fan

Let N be a lattice with dual lattice M as above. Tensoring by $\mathbb{R}$ gives

$$N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^n$$

and $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^n$. Let S be a finite subset of $N_{\mathbb{R}}$ then the *convex polyhedral cone* or just polyhedral cone generated by S is

$$\mathrm{Cone}(S) = \left\{ \sum_{u \in S} a_u u : a_u \geq 0 \right\}$$

where we set $\mathrm{Cone}(\emptyset) = \{0\}$. We say $\mathrm{Cone}(S)$ is *rational* if we can take $S \subset N$. Given a convex polyhedral cone $\sigma \subset N_{\mathbb{R}}$, we define the *dual cone* to σ by

$$\sigma^\vee = \{ m \in M_{\mathbb{R}} : \langle m, u \rangle \geq 0 \; \forall \, u \in \sigma \}.$$

For all $m \in M_{\mathbb{R}}$, we can define a hyperplane

$$H_m = \{u \in N_{\mathbb{R}} : \langle m, u \rangle = 0\}$$

and a closed half plane

$$H_m^+ = \{u \in N_{\mathbb{R}} : \langle m, u \rangle \geq 0\}.$$

For $m \in M_{\mathbb{R}} \setminus \{0\}$ and a polyhedral cone $\sigma \subset H_m^+$, we define a *face* of σ to be the set $\tau = H_m \cap \sigma$. A *facet* of σ is a face of codimension 1 and an *edge* of σ is a one-dimensional face.

A convex polyhedral cone σ is *strongly convex* if $\sigma \cap -\sigma = \{0\}$. An edge of a strongly convex cone is called a ray. If σ is a strongly convex rational polyhedral cone and if ρ is a ray of σ then there is a unique primitive element $n_\rho \in \rho \cap N$ called a *ray generator*. The set $\{n_\rho\}$ of all ray generators form a minimal generating set for the cone σ. Recall that a semigroup algebra of a semigroup S over a field k is given by

$$k[S] = \{a_1\chi^{u_1} + a_2\chi^{u_2} + \cdots + a_r\chi^{u_r} : \; u_i \in S \text{ and } a_i \in k\}$$

with multiplicative structure given by $\chi^{u_i} \cdot \chi^{u_j} = \chi^{u_i + u_j}$.

Example 2.4.4. Clearly $k[\mathbb{Z}_{\geq 0}] = \{\sum_1^r a_i\chi^{u_i} : \; u_i \in \mathbb{Z}_{\geq 0} \text{ and } a_i \in k\} \cong k[x]$ and $k[\mathbb{Z}] \cong k[x, x^{-1}]$. By identifying

$$\chi^{(a_1, a_2, \ldots, a_n)} \mapsto \prod_1^n x_i^{a_i}$$

we see that $k[\mathbb{Z}^n] \cong k[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$.

For a rational polyhedral cone $\sigma \subset N_{\mathbb{R}}$, the lattice points $S_\sigma = \sigma^\vee \cap M$ form a finitely generated semigroup with semigroup algebra $k[S_\sigma]$. From this, we get an affine toric variety $U_\sigma = \operatorname{Spec}(k[S_\sigma])$ associated to σ. Moreover, if σ is strongly convex then U_σ is a normal affine toric variety.

A *fan* Σ is a set of strongly convex rational polyhedral cones such that

1. if $\sigma \in \Sigma$ and τ is a face of σ then $\tau \in \Sigma$,
2. if $\sigma, \tau \in \Sigma$ then $\sigma \cap \tau \in \Sigma$ is a face of both σ and τ.

These properties allow us to define a (separated) normal toric variety X_Σ by gluing together the normal affine toric varieties U_σ for $\sigma \in \Sigma$.

Example 2.4.5. Let $\{u_0, u_1, \ldots, u_n\} \subset N_\mathbb{R}$ be a set of vectors that generate the rays of the fan associated with the weighted projective space $\mathbb{P}(w_0, w_1, \ldots, w_n)$. In [Cox11, Example 5.1.4], Cox, Little, and Schenck characterize $\mathbb{P}(w_0, w_1, \ldots, w_n)$ by the following properties.

1. The images of $\{u_0, u_1, \ldots, u_n\}$ under $N \cong \mathbb{Z}^n$ generate $\mathbb{Z}^n$,
2. $w_0u_0 + w_1u_1 + \cdots + w_nu_n = 0$.

Theorem 2.4.6. *Let X_Σ be the (separated) normal toric variety associated to a fan Σ. If the minimal generating set for each $\sigma \in \Sigma$ is a linearly independent set over $\mathbb{R}$ then X_Σ is $\mathbb{Q}$-factorial.*

Lattice Polytopes

A *lattice polytope* $P \subset M_\mathbb{R} \cong \mathbb{R}^n$ is the convex hull of a finite subset of M. Let S be a finite subset of M and let

$$P = \mathrm{Conv}(S) = \left\{ \sum_{m \in S} c_m m \ : \ c_m \geq 0, \ \sum_{m \in S} c_m = 1 \right\} \subset M_\mathbb{R}.$$

The dimension of P is the dimension of the smallest affine subspace of $M_\mathbb{R}$ containing P. Given $u \in N_\mathbb{R}$ and $b \in \mathbb{R}$ we can define a *affine hyperplane*

$$H_{u,b} = \{m \in M_\mathbb{R} \ : \ \langle m, u \rangle = b\}$$

and a *closed half-space*

$$H_{u,b}^+ = \{m \in M_\mathbb{R} \ : \ \langle m, u \rangle \geq b\}.$$

A subset $Q \subset P$ is a *face* of P if there exists $u \in N_{\mathbb{R}}$ and $b \in \mathbb{R}$ such that $P \subset H_{u,b}^{+}$ and $Q = P \cap H_{u,b}$. A *facet* of P is a face of codimension-one.

Let $M_{\mathbb{R}} \cong \mathbb{R}^n$ and let $P \subset M_{\mathbb{R}}$ be a lattice polytope of dimension n. Then, for each facet F of P, there exists a unique normal vector $n_F \in N_{\mathbb{R}}$ that is primitive in N and points toward the interior of P. We call such a vector n_F the unique *facet normal vector* of F. Now let $\mathcal{F}$ be any face of P and define

$$\sigma_{\mathcal{F}} = \mathrm{Cone}(\{n_F : \ F \text{ is a facet of } P \text{ containing } \mathcal{F}\})$$

then we can define a fan called the *normal fan* Σ_P of P by $\Sigma_P = \{\sigma_{\mathcal{F}} : \ \mathcal{F} \text{ is a face of } P\}$. Finally, for a lattice polytope P with normal fan Σ, we can define a normal toric variety $X_P \coloneqq X_{\Sigma_P}$.

Theorem 2.4.7. *A normal toric variety X_Σ of a fan Σ in $N_{\mathbb{R}} \cong \mathbb{R}^n$ is projective if and only if Σ is the normal fan of an n-dimensional lattice polytope P in $M_{\mathbb{R}}$.*

Example 2.4.8. Let $P = \mathrm{Conv}((0,0),(0,1),(1,0))$ then $X_P = \mathbb{P}^2$. To see this, let $F_1 = \mathrm{Conv}((0,0),(0,1))$, $F_2 = \mathrm{Conv}((0,0),(1,0))$, and $F_3 = \mathrm{Conv}((0,1),(1,0))$ be the facets of P with inner normal vectors

$$u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad , u_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad , u_3 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

which generate the fan Σ_P. Since $B_1 = \{u_1, u_2\}$, $B_2 = \{u_1, u_3\}$, and $B_3 = \{u_2, u_3\}$ all form bases for $\mathbb{Z}_2$, the affine toric variety associated with the 2-dimensional cone σ_i spanned by B_i for all i is $X_{\sigma_i} \cong \mathbb{C}^2$ so we construct $\mathbb{P}^2$ by gluing $\{X_{\sigma_i}\}$ along $\{X_{\sigma_i \cap \sigma_j}\}$.

For an n-dimensional lattice polytope P with normal fan Σ_P, observe that there is a one-to-one correspondence between between the d-dimensional faces of P and the $(n-d)$-dimensional cones of Σ_P.

Divisors on Toric Varieties

Let $P \subset M_{\mathbb{R}}$ be an n-dimensional lattice polytope with normal fan $\Sigma_P \subset N_{\mathbb{R}}$ and let σ be a cone in Σ_P. An orbit O of the torus $T(N)$ corresponds with σ if and only if $\lim_{t\to 0} \lambda^u(t) \subset O$ exists for all u in the interior of σ in which case we write $O = \operatorname{orb}(\sigma)$. For all cones $\sigma \in \Sigma$, we have

$$\dim \sigma + \dim \operatorname{orb}(\sigma) = n.$$

Let $\mathcal{F}$ be a d-dimensional face of P so, there exists a corresponding $(n-d)$-dimensional cone $\sigma_{\mathcal{F}}$. We have

$$\dim \sigma_{\mathcal{F}} + \dim \mathcal{F} = n$$

so there exists a one-to-one correspondence between the facets of P and torus-invariant irreducible divisors of X_P.

Recall that a facet F of a lattice polytope P with inner normal vector u_F lies on a hyperplane defined by

$$H_F = \{m \in M : \langle m, u_F\rangle = -a_F\}.$$

Each facet F of P corresponds with an irreducible torus invariant divisor D_F on X_P. These divisors are related to the principal divisors $\mathrm{div}(\chi^{\mathrm{m}})$ where $M \in m$ by

$$\mathrm{div}(\chi^{\mathrm{m}}) = \sum_{\mathrm{F}} \langle \mathrm{m}, \mathrm{u_F}\rangle \mathrm{D_F}$$

where the sum is taken over all facets F of P and u_F is the inner normal vector associated with F. This allows us to define an exact sequence

$$M \xrightarrow{\alpha} \bigoplus_F \mathbb{Z}D_F \xrightarrow{\beta} \mathrm{Cl}(X_P) \longrightarrow 0$$

where $\alpha : M \longrightarrow \bigoplus_F \mathbb{Z}D_F$ is defined by $m \mapsto \mathrm{div}(\chi^{\mathrm{m}})$ and $\beta : \bigoplus_F \mathbb{Z}D_F \longrightarrow \mathrm{Cl}(X_P)$ sends each Weil divisor in the image to its linear equivalence class. Moreover, when the normal vectors n_F span $N_{\mathbb{R}}$, α is injective, so we get a short exact sequence.

The coefficients a_F defining the supporting hyperplane of each facet of P allow us to define the *divisor associated with* P by

$$D_P = \sum_F a_F D_F$$

where the sum is taken over all facets F of P. This divisor has a number of favorable properties which we will mention here. For a more thorough description, see [Cox11].

Proposition 2.4.9. *For an n-dimensional lattice polytope $P \subset M_{\mathbb{R}}$ with facets $\{F\}$ each of which lie on a hyperplane in $M_{\mathbb{R}}$ defined by $H_F = \{m \in M : \langle m, u_F \rangle = -a_F\}$ where u_F is the inner normal vector of F, the divisor*

$$D_P = \sum_F a_F D_F$$

is an ample Cartier divisor.

Proof. Let F be a facet of P and let $m \in F$ be a lattice point with corresponding n-dimensional cone $\sigma_m \in \Sigma_P$ then $D_F \cap U_{\sigma_m} \neq 0$. Let χ^m be the character associated with m then

$$D_P\Big|_{U_{\sigma_m}} = \sum_{F \ni m} a_F D_F = -\sum_{F \ni m} \langle m, n_F \rangle D_F = -\mathrm{div}(\chi^{\mathrm{m}})\Big|_{\mathrm{V}_{\sigma_{\mathrm{m}}}}$$

so D_P is locally principal and therefore Cartier.

To see that D_P is ample, we use the fact that

$$H^0(X, D_P) = \bigoplus_{m \in P \cap M} k \cdot \chi^m$$

and define a map $\varphi : (k^*)^n \to \mathbb{P}^{l-1}$ by $\varphi(t_1, \ldots, t_n) = [\chi^{m_1} : \cdots : \chi^{m_l}]$ where $m_1, \ldots, m_l$ are the lattice points of P. Let $Y_{P \cap M}$ be the projective toric variety given by the image of this map. Since $X_P = X_{\nu P}$ for any positive integer ν and $D_{\nu P} = \nu D_P$ then $H^0(X_P, \nu D_P)$ defines a map

$$\varphi_\nu : (k^*)^n \to \mathbb{P}^{l_\nu - 1}$$

where $l_\nu = |\nu P \cap M|$ such that $X_P \cong Y_{\nu P \cap M}$ for $\nu >> 0$ so since, $Y_{\nu P \cap M}$ is embedded in projective space, we see that a sufficiently large multiple of D_P defines an embedding of X_P into projective space. In other words, D_P is ample. $\square$

Consider an n-dimensional normal variety X with singular locus Y so the codimension of Y in X is at least 2 by normality. Let $U = X \setminus Y$ be the smooth locus of X with the inclusion map $i : U \to X$. Recall that the canonical sheaf ω_U on U is defined to be the the n-th exterior power of the sheaf of differential 1-forms on U. That is, $\omega_U = \wedge^n \Omega^1_U$. Then the canonical sheaf ω_X on X is the defined by $\omega_X = i_*(\omega_U)$. This is a reflexive sheaf of rank one so it is divisorial which means there exists a Weil divisor K_X on X such that $\omega_X = \mathcal{O}_X(K_X)$. For any two divisors K_X and K'_X such that $\omega_X = \mathcal{O}_X(K_X) = \mathcal{O}_X(K'_X)$, we have that K_X and K'_X are linearly equivalent so there exists a unique canonical class in the class group. We often informally refer to any representative of the canonical class as the *canonical divisor* on X. For toric varieties, there is a standard representative for the canonical class given by the following theorem.

Theorem 2.4.10. *Let $\Sigma \subset N_\mathbb{R} \cong \mathbb{R}^n$ be a fan with corresponding toric variety X_Σ. The canonical sheaf ω_{X_Σ} is given by*

$$\omega_{X_\Sigma} = \mathcal{O}_{X_\Sigma}\left(-\sum_\rho D_\rho\right)$$

where the sum $\sum_\rho D_\rho$ is taken over all rays in Σ and D_ρ is the divisor on X_Σ associated with ρ. Hence $K_{X_\Sigma} = -\sum_\rho D_\rho$ is a torus-invariant canonical divisor on X_Σ.

Corollary 2.4.11. *Let $P \subset M_\mathbb{R} \cong \mathbb{R}^n$ be a lattice polytope with corresponding projective toric variety X_P then*

$$K_X = -\sum_F D_F,$$

where the sum is taken over all facets F of P, is a torus-invariant canonical divisor on X_P.

2.5 Moduli of Curves

A *moduli problem* is a functor that assigns to each scheme a set that parameterizes certain mathematical objects up to a suitable equivalence. A moduli space, intuitively speaking, is then a solution to a moduli problem. Specifically, a *fine moduli space* is a moduli space that fully classifies and parameterizes all equivalence classes of the objects under consideration by providing a one-to-one correspondence between these equivalence classes and points on the moduli space. Fine moduli spaces can be elusive so we also consider *course moduli spaces* which is a moduli space only captures the essential properties of the objects under consideration in a way that allows multiple equivalence classes to lie on the same point on the moduli space. One of the main obstacles to the existence of a fine moduli space is the existence of nontrivial automorphisms among the objects being parameterized. One approach to overcoming such obstacles is to add marked points to the objects in question.

In what follows, we define a curve to be a one-dimensional projective variety defined over a field k of characteristic zero. We define $\mathcal{M}_g$ to be the moduli space of isomorphism classes of smooth curves of genus g. This is a coarse moduli space since such curves admit nontrivial automorphisms. For example, all smooth curves of genus zero are isomorphic to $\mathbb{P}^1$ which has an automorphism group isomorphic to the projective general linear group of degree 2 over k. That is, each element of $\mathcal{M}_g$ admits infinitely many nontrivial automorphisms.

To reduce the number of automorphisms on these curves, we often study the moduli space $\mathcal{M}_{g,n}$ of smooth genus g curves with n distinct ordered marked points. This is still a course moduli space but it is, in some sense, more fine that $\mathcal{M}_g$. Moreover, $\mathcal{M}_{g,n}$ is not compact because, for example, we could construct a sequence of curves where two of their marked points converge. Hence, we define the Deligne-Mumford compactification of $\mathcal{M}_{g,n}$, denoted $\overline{\mathcal{M}}_{g,n}$, to be the coarse moduli space of stable curves of genus g with n distinct ordered marked points where a curve is *stable* if its singularities are at worst double points and whose automorphism group is finite. Since these moduli spaces of curves are course, we cannot generally expect them to be varieties themselves. However, in [Kap92], Kapranov proved that the moduli space $\overline{\mathcal{M}_{0,n}}$ of stable rational curves with n distinct ordered marked points can be constructed as the iterated blowup of $\mathbb{P}^{n-3}$ at $n-1$ general points then along all linear subspaces passing through those points. Consequently, $\overline{\mathcal{M}_{0,n}}$ is a $(n-3)$-dimensional smooth projective variety so we often dispense with the calligraphic font in this case and write this projective variety as $\overline{M}_{0,n}$.

Chapter 3

Blowups of Toric Projective Varieties of Picard Number One

Let $N \cong \mathbb{Z}^n$ be a lattice with dual lattice $M = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{N}, \mathbb{Z}) \cong \mathbb{Z}^{\mathrm{n}}$. Let us denote $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ and $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$. Recall from Chapter 2, Section 4 that for an n-dimensional toric projective variety X_P arising from a lattice polytope $P \subset M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$ with facets $\{F\}$, we have the following short exact sequence

$$0 \longrightarrow M \xrightarrow{\alpha} \bigoplus_F \mathbb{Z} D_F \xrightarrow{\beta} \mathrm{Cl}(X_P) \longrightarrow 0.$$

Tensoring with $\mathbb{R}$ gives

$$0 \longrightarrow \mathbb{R}^n \longrightarrow \mathbb{R}^{|\{F\}|} \longrightarrow \mathbb{R}^{\rho(X_P)} \longrightarrow 0$$

where $\rho(X_P)$ is the Picard number of X_P.

Recall that a simplex is a d-dimensional polytope that is the convex hull of $d+1$ points. Notice that a simplex has $d+1$ facets and that a d dimensional polytope with $d+1$ facets is a simplex. Hence from the previous short exact sequence, we can make the following observation.

Proposition 3.0.1. *Let X be a toric projective variety of Picard number one, then there exists a simplex Δ of dimension $\dim X$ such that $X = X_\Delta$.*

In [Cox95], Cox proves that that the Cox ring of a toric variety is a polynomial ring in finitely many variables. So if X is a normal projective $\mathbb{Q}$-factorial toric variety, X is a Mori dream space but the birational geometry of such a toric variety blown up at a single point in its torus is already unknown in general. Hence, determining whether the Cox ring of a toric projective surface of Picard number one blown up at a single point on its torus is finitely generated is a natural next step in this problem.

Recall that weighted projective spaces $\mathbb{P}(w_0, w_1, \ldots, w_n)$ are examples of toric projective varieties of Picard number one. The finite generation of the Cox rings of weighted projective surfaces blown up at a general point has attracted significant attention. In [Hun82], Huneke proved that $\mathrm{Cox}(\mathrm{Bl_p}(\mathbb{P}(w_0, w_1, w_2)))$ is a finitely generated k-algebra if $w_0 = 3$ or $w_0 = 4$. In [Sri91], Srinivasan proved that $\mathrm{Cox}(\mathrm{Bl_p}(\mathbb{P}(w_0, w_1, w_2)))$ is a finitely generated k-algebra if $w_0 = 6$; or $w_2 > w_1(w_0 - 3)$; or $w_2 > w_1(w_0 - 4)$ and $2|w_0$. In [Cut91], Cutkosky proved that if $(w_0 + w_1 + w_2)^2 > w_0 w_1 w_2$ then $\mathrm{Cox}(\mathrm{Bl_p}(\mathbb{P}(w_0, w_1, w_2)))$ is a finitely generated k-algebra and he presented some potential obstructions to the finite generation of this Cox ring. Then, in [Got94], Goto, Nishida, and Watanabe proved that if $n \in \mathbb{Z}^+$, $n \geq 4$, $3 \nmid n$ then $\mathrm{Cox}(\mathrm{Bl_p}\, \mathbb{P}(7n - 3, 8n - 3, 5n^2 - 2n))$ is not a finitely generated k-algebra and if $n \in \mathbb{Z}^+$, $n \geq 5$, $3 \nmid n - 1$, $59 \nmid n + 7$ then $\mathrm{Cox}(\mathrm{Bl_{t_0}}\, \mathbb{P}(7n - 10, 8n - 3, 5n^2 - 7n + 1))$ is not a finitely generated k-algebra.

We would like to develop a similarly concrete numerical criterion for the Mori dream space property of a toric projective surface of Picard number one blown up at a point in its torus but one obstacle is that, when generalizing out of the case of weighted projective surfaces, we no longer have access to the natural parameters provided by the weights w_0, w_1, and w_2. Nonetheless, we will see in the next section that we can overcome this obstacle using the coordinates of a carefully selected polytope that gives rise to the toric variety.

3.1 Blowups of Toric Projective Surfaces of Picard Number One

Let $\Delta = \text{Conv}(v_1, v_2, v_3) \subset M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$ be a lattice polytope given by the convex hull of $v_1, v_2, v_3 \in M$, let X_Δ be the associated normal projective toric surface. Recall that the canonical divisor of X_Δ is

$$K_{X_\Delta} = -\sum_F D_F$$

where the sum is taken over all facets F of Δ and D_F is the torus-invariant divisor associated with the facet F. Note that for any two-dimensional lattice polytope Δ, we can pick a basis $\{m_1, m_2\}$ for M such that one of its facets is horizontal. Since the variety X_Δ only depends on the inner normal vectors for the facets of Δ, translating Δ preserves X_Δ. So without loss of generality we can assume that one of the sides of the polytope Δ is horizontal and one vertex of the horizontal side lies at the origin. Notice also that reflections with respect to either of the coordinate axes and the *shear transformation* $(x, y) \mapsto (x + y, y)$ are automorphisms of $\mathbb{Z}^2$. Hence applying any of these translations, reflections on the coordinate axes, and shears to the polytope Δ preserves the toric variety up to isomorphism. It follows that by applying translations, reflections along the coordinate axes, and shear transformations to the lattice triangle $\Delta \subset M_{\mathbb{R}}$ we can arrive to a lattice triangle $\Delta' \subset M_{\mathbb{R}}$ with vertices of the form $(0, 0)$, (b, a), and $(b + c, 0)$ for some $a, b, c \in \mathbb{Z}$ with $a > 0$, $b \geq 0$ and $b + c \geq 0$, such that X_Δ and $X_{\Delta'}$ are isomorphic as toric varieties. Hence, we can assume without loss of generality that our lattice triangle has the form $\Delta = \text{Conv}((0, 0), (b, a), (b + c, 0))$ with a, b, c integers as described.

Let F_1, F_2, and F_3 be the facets of Δ with inward normal vectors u_1, u_2, and u_3 respectively and let D_1, D_2, and D_3 be the associated torus-invariant divisors.

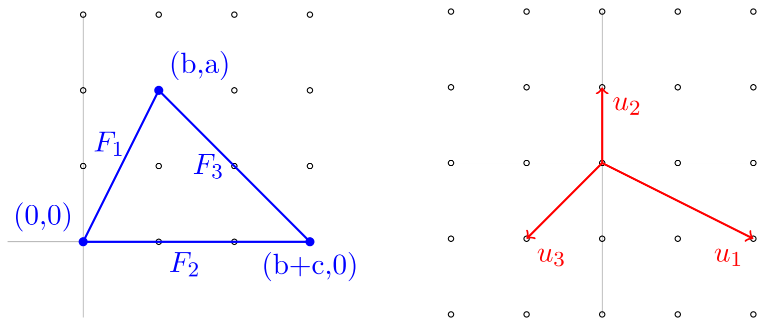

Figure 3.1: A triangular lattice polytope (left) and its inner normal vectors (right)

Lemma 3.1.1 (Williamson)**.** *Let $\Delta \subset M_{\mathbb{R}}$ be a lattice polytope with vertices $(0,0)$, (b,a), and $(b+c,0)$ with respect to a basis $\{m_1, m_2\}$ of M so $a, b, c \in \mathbb{Z}$. Then*

$$[-K_{X_\Delta}] = \left(\frac{b+c+\gcd(a,b)+\gcd(a,c)}{a(b+c)}\right)[D_\Delta].$$

Proof. Let $F_1 = \mathrm{Conv}((0,0),(b,a))$, $F_2 = \mathrm{Conv}((0,0),(b+c,0))$ and $F_3 = \mathrm{Conv}((b,a),(b+c,0))$ be the facets of Δ. Then the primitive inner normal vectors for F_1, F_2 and F_3 are

$$u_1 = \begin{pmatrix} \frac{a}{\gcd(a,b)} \\ \frac{-b}{\gcd(a,b)} \end{pmatrix}, \qquad u_2 = e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad u_3 = \begin{pmatrix} \frac{-a}{\gcd(a,c)} \\ \frac{-c}{\gcd(a,c)} \end{pmatrix}$$

respectively. Since the supporting hyperplanes for F_1 and F_2 pass through the origin and since the supporting hyperplane for F_3 is given by

$$\left\{ m \in M_{\mathbb{R}} : \langle m, u_3 \rangle = \frac{-a(b+c)}{\gcd(a,c)} \right\},$$

we have $D_\Delta = \left(\frac{a(b+c)}{\gcd(a,c)}\right) D_3$. Moreover, $\mathrm{div}(\chi^{\mathrm{m_1}}) = \left(\frac{\mathrm{a}}{\mathrm{gcd(a,b)}}\right)\mathrm{D_1} + \left(\frac{-\mathrm{a}}{\mathrm{gcd(a,c)}}\right)\mathrm{D_3}$ and $\mathrm{div}(\chi^{\mathrm{m_2}}) = \left(\frac{-\mathrm{b}}{\mathrm{gcd(a,b)}}\right)\mathrm{D_1} + \mathrm{D_2} + \left(\frac{-\mathrm{c}}{\mathrm{gcd(a,c)}}\right)\mathrm{D_3}$. Then

$$\begin{aligned}\left(\frac{b+c+\gcd(a,b)+\gcd(a,c)}{a(b+c)}\right)&D_\Delta + \left(\frac{b+\gcd(a,b)}{a}\right)\mathrm{div}(\chi^{\mathrm{e_1}}) + \mathrm{div}(\chi^{\mathrm{e_2}})\\ &= D_1 + D_2 + D_3 = -K_{X_\Delta}\end{aligned}$$

so

$$\left(\frac{b+c+\gcd(a,b)+\gcd(a,c)}{a(b+c)}\right)[D_\Delta] = [-K_{X_\Delta}].$$

□

Example 3.1.2. Let $\Delta \subset M_\mathbb{R}$ be a right triangle with vertices $(0,0)$, $(0,a)$, and $(c,0)$ with respect to a basis $\{m_1, m_2\}$ of M so $a, c \in \mathbb{Z}$. Then in the notation above, $b = 0$ and $\gcd(a,b) = a$ so we have

$$[-K_{X_\Delta}] = \left(\frac{a+c+\gcd(a,c)}{ac}\right)[D_\Delta].$$

Let $\pi_p : X \to X_\Delta$ be the blow-up of X_Δ at a point p in the torus of X_Δ. Then X is also a normal projective surface and $K_X = K_{X_\Delta} + E = -\sum_F D_F + E$ where E is the exceptional divisor of the blowup. Let D_Δ be the ample divisor of X_Δ determined by Δ and let $H = \pi^*(D_\Delta)$. Then $N^1(X) \otimes \mathbb{R}$ is generated by H and E. Note that since $D_\Delta^2 = 2 \cdot \mathrm{area}(\Delta)$ then

$$(H^2) = \deg(\pi_p) D_\Delta^2 = D_\Delta^2 = 2 \cdot \mathrm{area}(\Delta).$$

The pseudoeffective cone $\overline{\mathrm{Eff}}(X)$ is a cone contained in the half space $\{aA + bE : a \geq 0\}$ and since $E^2 = -1 < 0$, one of the two extremal rays of $\overline{\mathrm{Eff}}(X)$ is $\mathbb{R}_+[E]$. Let R be the other extremal ray. We will develop a criterion for the finite generation of Cox(X) using only the coordinates of the polytope Δ. Recall that, since X is a normal, projective, $\mathbb{Q}$-factorial variety, we say that X is a Mori dream space if Cox(X) is a finitely generated k-algebra.

First we note that if X is the blowup of a toric projective surface of Picard number one, then X is not necessarily a Mori dream space.

Example 3.1.3. It is well known that weighted projective surfaces $\mathbb{P}(w_1, w_2, w_3)$ are toric projective surfaces of Picard number one. The first infinite family of counterexamples for the finite generation of $\mathrm{Cox}(\mathrm{Bl_p}(\mathbb{P}(w_1, w_2, w_3)))$ was given by Goto, Nishida, and Watanabe in [Got94] who showed that the blowups of weighted projective surfaces of the form $\mathbb{P}(7N-3, (5N-2)N, 8N-3)$ for integers $N \geq 4$ at a general point are not Mori dream spaces. Then, in [Gon16], Gonzalez and Karu developed a criterion extending Goto, Nishida, and Watanabe's result to a much larger family of non-examples including some with smaller weights than those found previously, the smallest of which is $\mathbb{P}(7, 15, 26)$.

Our goal is to find sufficient conditions to guarantee that X is a Mori dream space. First we consider the case when there exists a negative curve C on X aside from the exceptional curve E. In the following lemma we study $\overline{\mathrm{Eff}}(X)$ in this case.

Lemma 3.1.4. *Suppose that $-K_X$ is big and $R = \mathbb{R}_+[C]$ for some irreducible curve C such that $C^2 < 0$. Let $S = R^\perp \cap \mathrm{Eff(X)}$ then $S = \mathbb{R}_+[D]$ for an effective divisor D such that $|D|$ is base-point free.*

Proof. By Proposition 2.3.6, we can assume without loss of generality that $k = \mathbb{C}$. Since X is a projective algebraic surface over $\mathbb{C}$, we can associate to it a projective analytic surface X_h. Let C be as above and let C_h be the corresponding curve on X_h then by a variant of Grauert's contraction theorem due to Sakai [Sak84], there exists a contraction morphism of analytic spaces $f : X_h \to Y_h$ such that Y_h is a normal analytic surface and $f(C_h) = q$ is a normal point on Y_h. Note that since $f\big|_{X_h \setminus C_h} : X_h \setminus C_h \to Y_h \setminus \{q\}$ is an isomorphism, the meromorphic function fields of X_h and Y_h are isomorphic so the transcendence degrees of these fields over $\mathbb{C}$ are equal. Since X is a projective variety and the meromorphic function field of X_h is isomorphic to the rational function field of X, the transcendence degree of the meromorphic function field over $\mathbb{C}$ of X_h is 2 so the same is true for Y_h. Hence, Y_h is a Moishezon surface. By Brenton [Bre77], a Moishezon surface Y_h whose geometric genus $p_g(Y_h) = h^2(Y_h, \mathcal{O}_{Y_h}) = 0$ is projective algebraic so we want to show that $p_g(Y_h) = 0$.

Suppose for the sake of contradiction that there exists a nonzero meromorphic function

$$g_h \in H^0(Y_h, K_{Y_h}) \cong H^0(Y_h \setminus \{q\}, K_{Y_h}) \cong H^0(X_h \setminus C_h, K_{X_h}\big|_{X_h \setminus C_h}).$$

Then there exists a nonzero meromorphic function $\tilde{g}_h \in H^0(X_h, K_{X_h} + mC_h)$ for some non-negative integer m such that $\tilde{g}_h\big|_{X_h \setminus C_h} = g_h$. Since X_h is the analytic space associated with a projective variety X, by Theorem 2.3.3 there exists a nonzero rational function

$$g \in H^0(X, K_X + mC) \cong H^0(X_h, K_{X_h} + mC_h)$$

so $H^0(X, K_X + mC) \neq 0$. This is a contradiction because we assumed $-K_X$ to be big which implies that it lies in the interior of $\overline{\mathrm{Eff}}(X)$ and C spans an extremal ray of $\overline{\mathrm{Eff}}(X)$ so $K_X + nC \notin \overline{\mathrm{Eff}}(X)$ for any $n \in \mathbb{Z}$. Hence, $H^0(Y_h, K_{Y_h}) = 0$. Since Y_h is a normal analytic surface, it is Cohen-Macaulay and pure dimensional so by Theorem 2.3.2,

$$p_g(Y_h) = H^2(Y_h, \mathcal{O}_{Y_h}) \cong \mathrm{Ext}^0_{\mathcal{O}_{Y_h}}(\mathcal{O}_{Y_h}, K_{Y_h}) \cong H^0(Y_h, K_{Y_h}) = 0.$$

We conclude that Y_h must be projective algebraic so there exists a projective algebraic space Y over $\mathbb{C}$ whose corresponding analytic space is Y_h. By Lemma 2.3.5, we can consider $f : X \to Y$ to be a morphism of algebraic spaces. Then, we get a morphism from X into projective space $\mathbb{P}^n$ by composing f with an embedding from Y to $\mathbb{P}^n$. This morphism from X into $\mathbb{P}^n$ must be induced by the base-point free complete linear system $|D|$ of an effective divisor D on X. Moreover, we have the following commutative diagram

$$\begin{tikzcd} X \arrow[r, "j"] & \mathbb{P}^n \\ C \arrow[u, hook, "i"] \arrow[r, "f\big|_C"'] & \{q\} \arrow[u, hook, "\alpha"'] \end{tikzcd}$$

which gives

$$\begin{aligned} D.C &= \deg_C(i^*(\mathcal{O}(D))) \\ &= \deg_C(i^*j^*(\mathcal{O}_{\mathbb{P}^n}(1))) \\ &= \deg_C((j \circ i)^*(\mathcal{O}_{\mathbb{P}^n}(1))) \\ &= \deg_C((\alpha \circ f\big|_C)^*(\mathcal{O}_{\mathbb{P}^n}(1))) \\ &= \deg_C(f\big|_C^*\alpha^*(\mathcal{O}_{\mathbb{P}^n}(1))) \\ &= \deg_C(f\big|_C^*\mathcal{O}_p) = \deg_C \mathcal{O}_C = 0. \end{aligned}$$

so $S = \mathbb{R}_+[D] = R^\perp \cap \text{Eff(X)}$. $\square$

The previous lemma tells us that, when there exists an irreducible curve on X that is not equal to the exceptional divisor of the blowup and has negative self-intersection, the nef cone of X is generated by semi-ample divisors.

Lemma 3.1.5. *Let F be an effective divisor on X and let H, E, C, and D be as above. Then*

1. *Let a and b be positive integers then $H^0(X, aH + bE) = H^0(X, aH)$,*

2. *If $F \cdot C < 0$ then $H^0(X, F - C) \cong H^0(X, F)$.*

Proof. First we will prove *(1)*. It is enough to show that $H^0(X, aH + E) = H^0(X, aH)$. Clearly, if $g \in H^0(X, aH)$ then $g \in H^0(X, aH + E)$ since if div(g) + aH ≥ 0 then div(g) +

$\mathrm{aH} + \mathrm{E} \geq 0$ as well. On the other hand, suppose $g \in H^0(X, aH + E)$ so $\mathrm{div(g)} + \mathrm{aH} + \mathrm{E} \geq 0$. Since $(aH) \cdot E = 0$,

$$(\mathrm{div(g)} + \mathrm{aH} + \mathrm{E}) \cdot \mathrm{E} = (\mathrm{div(g)} \cdot \mathrm{E}) + ((\mathrm{aH}) \cdot \mathrm{E}) + \mathrm{E}^2 = \mathrm{E}^2 = -1 < 0$$

so E is a component of $\mathrm{div(g)}+\mathrm{aH}+\mathrm{E}$. Hence, we have $(\mathrm{div(g)}+\mathrm{aH}+\mathrm{E})-\mathrm{E} = \mathrm{div(g)}+\mathrm{aH} \geq 0$ so $g \in H^0(X, aH)$. So we can conclude that

$$H^0(X, aH + E) = H^0(X, aH).$$

To prove *(2)*, first we notice that $C \sim aH + bE$ for some $a \geq 0$ and $b \leq 0$. Let $g \in k(X)^*$ such that $C = \mathrm{div(g)}+\mathrm{aH}+\mathrm{bE}$. Let F be an effective divisor on X such that $F \cdot C < 0$. Then, $[F]$ must lie in the positive cone spanned by $\mathbb{R}_+[D]$ and $\mathbb{R}_+[C]$. So, $F \sim mH + nE$ where $m \geq 0$ and $n \leq 0$ then $F - C \sim (m-a)H + (n-b)E$. Consider the linear transformation

$$\begin{aligned} H^0(X, (m-a)H + (n-b)E) &\to H^0(X, mH + nE) \\ f &\longmapsto fg \end{aligned}$$

which is well-defined since

$$\begin{aligned} \mathrm{div(fg)} + \mathrm{mH} + \mathrm{nE} &= \mathrm{div(g)} + \mathrm{aH} + \mathrm{bE} + \mathrm{div(f)} + (\mathrm{m} - \mathrm{a})\mathrm{H} + (\mathrm{n} - \mathrm{b})\mathrm{E} \\ &= C + \mathrm{div(f)} + (\mathrm{m} - \mathrm{a})\mathrm{H} + (\mathrm{n} - \mathrm{b})\mathrm{E} \geq 0 \end{aligned}$$

and it is clearly linear. We now show that this linear transformation is an isomorphism. If $fg = f'g$ for f and f' in $H^0(X, (m-a)H + (n-b)E)$ then $f = f'$ since $H^0(X, mH + nE) \subset k(X)^*$ which is a domain so the map is injective. To see surjectivity, let $\varphi \in H^0(X, mH+nE)$ then

$$W = \mathrm{div}(\varphi) + \mathrm{mH} + \mathrm{nE} \sim \mathrm{div}(\varphi) + (\mathrm{F} - \mathrm{C}) + \mathrm{C} \geq 0.$$

Since $F \cdot C < 0$ then

$$W \cdot C = (\mathrm{div}(\varphi) \cdot \mathrm{C}) + \mathrm{F} \cdot \mathrm{C} = (\mathrm{F} \cdot \mathrm{C}) < 0$$

so C is a component of W which means $W - C$ is effective. Since $W \sim mH + nE$ and $C \sim aH+bE$ then $W-C \sim (m-a)H+(n-b)E$ so there exists $h \in H^0(X,(m-a)H+(n-b)E)$ such that $W - C = \mathrm{div(h) + (m - a)H + (n - b)E}$, but then

$$W = \mathrm{div(h) + (m - a)H + (n - b)E + div(g) + aH + bE = div(hg) + mH + nE}$$

which means $\mathrm{div}(\varphi) = \mathrm{div(hg)}$ for $h \in H^0(X,(m-a)H+(n-b)E)$. Hence, $\varphi = \lambda hg$ for a nonzero constant λ so $\lambda h \in H^0(X,(m-a)H+(n-b)E)$ maps to $\varphi \in H^0(X, mH + nE)$. Finally, we have shown that

$$H^0(X, F - C) \cong H^0(X,(m-a)H+(n-b)E) \cong H^0(X, mH + nE) \cong H^0(X, F).$$

□

Lemma 3.1.6 (Williamson)**.** *Suppose that $R = \mathbb{R}_+[C]$ for an irreducible curve C on X such that $C^2 < 0$. Recall that by Lemma 3.1.4 there exists an effective divisor D such that $|D|$ is base-point free and $S = R^\perp \cap \mathrm{Eff(X)} = \mathbb{R}_+[\mathrm{D}]$. Then X is a Mori dream space.*

Proof. We will split $\mathrm{Cl}(X)$ into regions and consider each region separately. Let ① be the positive cone spanned by $\mathbb{R}_+[H]$ and $\mathbb{R}_+[E]$, let ② be the positive cone spanned by $\mathbb{R}_+[H]$ and S, let ③ be the positive cone spanned by S, and R and let ④ $= N^1(X) \setminus \overline{\mathrm{Eff}}(X)$.

First we show that if

$$\bigoplus_{F \in ②} H^0(X, F)$$

is a finitely generated k-algebra then

$$\mathrm{Cox(X)} = \bigoplus_{\mathrm{F \in Cl(X)}} \mathrm{H^0(X, F)}$$

is a finitely generated k-algbera. To see this, note that if $F \in ①$ then $F \sim aH + bE$ where a and b are positive integers but then

$$H^0(X, F) \cong H^0(X, aH + bE) \cong H^0(X, aH)$$

by Lemma 3.1.5 and the isomorphism is induced by multiplication b times by a section of $H^0(X, E)$. Since $aH \in ②$, then the summands $H^0(X, F)$ for $F \in ①$ are generated by those with $F \in ②$ and the sections in $H^0(X, E)$.

If $F \in ③$ then

$$H^0(X, F) \cong H^0(X, F - C)$$

by Lemma 3.1.5 and the isomorphism is given by multiplication by a section in $H^0(X, C)$. This can be repeated until we obtain an effective divisor $F - mC$ for some $m > 0$ such that $H^0(X, F) \cong H^0(X, F - mC)$ and such that $F - mC$ is in ②. Therefore the summands $H^0(X, F)$ with $F \in ③$ are generated by the summands $H^0(X, F)$ with $F \in ②$ and the sections in $H^0(X, C)$. Finally, if $F \in ④$, $H^0(X, F) = 0$ since F is not effective.

Next, we observe that since H is the pullback of D_Δ on X_Δ and since D_Δ is ample, H is semiample and we showed in Lemma 3.1.4 that D is semiample. Hence, by Lemma 2.2.3 and Lemma 2.2.4, $\bigoplus_{F \in ②} H^0(X, F)$ is a finitely generated k-algebra, so Cox(X) is as well. Therefore, X is a Mori dream space. □

Theorem 3.1.7 (Williamson)**.** *If $-K_X$ is big then X is a Mori dream space.*

Proof. If there exists an irreducible curve $C \neq E$ such that $(C^2 < 0)$ then $R = \mathbb{R}_+[C]$ since curves with negative self-intersection necessarily span extremal rays of $\overline{\mathrm{Eff}}(X)$. Then, Cox(X) is a finitely generated k-algebra by Lemma 3.1.6.

Now suppose there does not exist such a curve C then for all effective divisors D of X, we must have $(D^2) \geq 0$. Recall that the nef cone Nef(X) is contained in the pseudoeffective cone $\overline{\mathrm{Eff}}(X)$ and Nef(X) is dual to $\overline{\mathrm{Eff}}(X)$. Let γ be a divisor class generating the extremal ray R. We claim that $\gamma^2 = 0$. Suppose by contradiction that $\gamma^2 > 0$. We can write

$\gamma \equiv H - aE \in R$ for some $b \in \mathbb{R}^+$. Then $\overline{\mathrm{Eff}}(X)$ is the cone spanned by E and γ. Hence Nef(X) is the cone spanned by H and a class δ such that $\delta \cdot E > 0$ and $\delta \cdot \gamma = 0$. So, we can write $\delta = H - bE \in R$ for some $b \in \mathbb{R}^+$. Since $0 = \gamma \cdot \delta = H^2 - ab$, then $ab = H^2$. Then,

$$\delta^2 = H^2 - b^2 = \frac{1}{H^2}[(H^2)^2 - H^2b^2] = \frac{1}{H^2}[a^2b^2 - H^2b^2] = \frac{b^2}{H^2}[a^2 - H^2] = -\frac{b^2}{H^2}\gamma^2 < 0.$$

We have then that $b^2 > H^2 > a^2 > 0$. Hence $b > a$ and therefore the class δ is outside $\overline{\mathrm{Eff}}(X)$ but this is a contradiction since δ is in the cone Nef(X) and Nef(X) $\subseteq \overline{\mathrm{Eff}}$(X). Therefore $\gamma^2 = 0$ and Nef(X) $=$ [H]$\mathbb{R}_+ +$ R

Now, since $-K_X = rH - E$, it is nef and since $-K_X$ is big, it lies in the interior of $\overline{\mathrm{Eff}}(X)$ which means it must lie in the interior of Nef(X) so $-K_X$ is ample. By Proposition 2.2.2, for all $\varepsilon > 0$, there exist rational curves l_1 and l_2 such that

$$\overline{\mathrm{Eff}}(X) = \mathbb{R}_+[l_1] + \mathbb{R}_+[l_2] + \overline{\mathrm{Eff}}_\varepsilon(X, H)$$

where $\overline{\mathrm{Eff}}_\varepsilon(X, H) = \{C \in \overline{\mathrm{Eff}}(X) : (K_X - \varepsilon H) \cdot C \geq 0\}$. We want to show that $\overline{\mathrm{Eff}}_\varepsilon(X, H) = \{0\}$. Fix $0 \neq C \in \overline{\mathrm{Eff}}(X)$ then, $(-K_X) \cdot C > 0$ and $H \cdot C > 0$ by Kleiman's ampleness criterion since $-K_X$ and H are ample so $K_X \cdot C < 0$ and $-\varepsilon H \cdot C < 0$ from which we can conclude that

$$(K_X - \varepsilon H) \cdot C = K_X \cdot C - \varepsilon H \cdot C < 0$$

for all $\varepsilon > 0$. Hence, $\overline{\mathrm{Eff}}_\varepsilon(X, H) = \{0\}$ so we conclude that that there exists an irreducible curve γ such that $\gamma^2 = 0$ and $R = \mathbb{R}_+[\gamma]$.

Since $\gamma^2 = 0$ and $-K_X \equiv aH + b\gamma$ for $a, b > 0$, $(-K_X) \cdot \gamma = a(H \cdot \gamma) + b\gamma^2 = a(H \cdot \gamma) > 0$ since H is ample by Kleiman's criterion so by Lemma 2.2.5, γ is semi-ample. Since Nef(X) $= \mathbb{R}_+$[H] $+ \mathbb{R}_+[\gamma]$,

$$\bigoplus_{D \in \mathrm{Nef(X)}} H^0(X, D)$$

is a finitely generated k-algebra by Lemma 2.2.3 and by Lemma 2.2.4. Finally, by Lemma 3.1.5, the fact that $\bigoplus_{D\in \mathrm{Nef(X)}} H^0(X,D)$ is a finitely generated k-algebra implies that

$$\mathrm{Cox(X)} = \bigoplus_{\mathrm{D}\in \mathrm{Cl(X)}} \mathrm{H^0(X,D)}$$

is a finitely generated k-algebra so X is a Mori dream space. $\square$

Now we see that Theorem 3.1.7 together with Lemma 3.1.1 give us a concrete numerical criterion for the Mori dream space property of X.

Theorem 3.1.8 (Williamson)**.** *Let $\Delta = \mathrm{Conv((0,0),(b,a),(b+c,0))}$, let X_Δ be the normal projective toric variety arising from Δ and let D_Δ be the ample divisor of X_Δ arising from Δ. Let X be the blowup of X_Δ at a point p. If*

$$(b+c+\gcd(a,b)+\gcd(a,c))^2 > a(b+c)$$

then X is a Mori dream space.

Proof. Note that

$$[-K_{X_\Delta}] = \left(\frac{b+c+\gcd(a,b)+\gcd(a,c)}{a(b+c)}\right)[D_\Delta]$$

by Lemma 3.1.1. Then,

$$[-K_X] = \left(\frac{b+c+\gcd(a,b)+\gcd(a,c)}{a(b+c)}\right)[D_\Delta] - [E]$$

then $[-K_X]$ lies in the fourth quadrant of $N^1(X) \otimes \mathbb{R} \cong \mathbb{R}^2$. Moreover,

$$\begin{aligned}
(-K_X)^2 &= \left(\frac{b+c+\gcd(a,b)+\gcd(a,c)}{a(b+c)}\right)^2 D_\Delta^2 + E^2 \\
&= \left(\frac{b+c+\gcd(a,b)+\gcd(a,c)}{a(b+c)}\right)^2 2\cdot \text{area}(\Delta) + E^2 \\
&= \left(\frac{b+c+\gcd(a,b)+\gcd(a,c)}{a(b+c)}\right)^2 2\left(\frac{a(b+c)}{2}\right) + E^2 \\
&= \left(\frac{(b+c+\gcd(a,b)+\gcd(a,c))^2}{a(b+c)}\right) - 1
\end{aligned}$$

so if $(b+c+\gcd(a,b)+\gcd(a,c))^2 > a(b+c)$ then $(-K_X)^2 > 0$. We know from the proof of Theorem 3.1.7 that if a curve C lies on the extremal ray R of $\overline{\text{Eff}}(X)$ then $C^2 \leq 0$ which means $-K_X$ lies on a ray strictly above R. Hence $-K_X$ is big so Cox(X) is a finitely generated k-algebra by Theorem 3.1.7. $\square$

Next we will present an example of a blowup of a toric projective surface of Picard number one that is known to not be a Mori dream space and we will see that it fails to satisfy the assumptions of Theorem 3.1.8.

Example 3.1.9. Recall Example 3.1.3 where we exhibited examples given by Goto, Nishida, and Watanabe in [Got94], and by Ganzalez and Karu in [Gon16] of weighted projective spaces $\mathbb{P}(w_1, w_2, w_3)$ such that $\text{Bl}_p(\mathbb{P}(w_1, w_2, w_2))$ is not a Mori dream space. The example with the smallest weights was $\mathbb{P}(7, 15, 26)$ given by Gonzalez and Karu.

It is well-known that the primitive vectors $u_0, u_1, \ldots, u_n$ generating the rays of the fan of a weighted projective space $\mathbb{P}(w_0, w_1, \ldots, w_n)$ have the following two properties:

1. The vectors $u_0, \ldots, u_n$ generate the lattice $N \cong \mathbb{Z}^n$,

2. $w_0u_0 + w_1u_1 + \cdots + w_nu_n = 0$.

Consider Gonzalez and Karu's example, $\mathbb{P}(7, 15, 26)$. Using the two properties above, we can find primitive generators of the rays of the fan to be

$$u_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} -26 \\ 29 \end{pmatrix}, \text{ and } u_3 = \begin{pmatrix} 15 \\ -17 \end{pmatrix}.$$

If Δ is a lattice polytope such that $X_\Delta = \mathbb{P}(7, 15, 29)$, we know that the facets F_1, F_2, and F_3 of Δ must be perpendicular to u_1, u_2, and u_3 respectively and the vertices of Δ must have integer coordinates. We see that $\Delta = \text{Conv}((0, 0), (442, 390)(7, 0))$ satisfies these conditions and, in the notation of Theorem 3.1.8, we have $a = 390$, $b = 442$, $c = -435$, $\gcd(a, b) = 26$, and $\gcd(a, c) = 15$ so

$$(b + c + \gcd(a, b) + \gcd(a, c))^2 = 2304 < 2730 = a(b + c)$$

so $\mathbb{P}(7, 15, 26)$ fails to satisfy the assumption of our theorem, as we would expect.

Theorem 3.1.8 allows us plentiful flexibility when looking for examples of polytopes Δ such that $X = \text{Bl}_p(X_\Delta)$ is a Mori dream space. We either want $b + c$ to be large compared to a or we want $\gcd(a, b)$ and $\gcd(a, c)$ to be large. Next we will use this flexibility to find some examples of families of blowups of toric projective surfaces of Picard number one that are Mori dream spaces. As we have seen, the weighted projective surfaces $\mathbb{P}(w_1, w_2, w_3)$ are examples of toric projective surfaces of Picard number one. In [Cut91], Cutkosky proved that for $Y' = \mathbb{P}(w_1, w_2, w_3)$ and $Y = \text{Bl}_p(Y')$, if $(-K_Y)^2 > 0$ then Y is a Mori dream space. Our method allows us to find many new examples of Mori dream spaces that are not blowups of weighted projective surfaces.

Example 3.1.10. Let $\Delta = \text{Conv}((0, 0), (r, 2r), (2r, 0))$ for a positive integer r so $a = 2r$, $b = c = r$, and $\gcd(a, b) = \gcd(a, c) = r$ then

$$(b + c + \gcd(a, b) + \gcd(a, c))^2 = 16r^2 > 4r^2 = a(b + c)$$

so if $X = \mathrm{Bl}_p(X_\Delta)$ then Cox(X) is a finitely generated k-algebra by Theorem 3.1.8. Moreover, since the inner normal vectors

$$u_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \qquad u_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad u_3 = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$$

do not span the lattice N, X_Δ is not a weighted projective space. However, all such choices for r result in isomorphic toric varieties so this only gives one example.

Example 3.1.11. Since $\gcd(a,b) + \gcd(a,c) \geq 2$, we have

$$(b + c + \gcd(a,b) + \gcd(a,c))^2 - a(b+c) \geq (b+c)^2 + (4-a)(b+c) + 4$$

which is greater than zero whenever $a < 8$ so Cox(X) is a finitely generated k-algebra whenever $a < 8$.

Example 3.1.12. Let $a > 1$ be any integer and choose b and c such that $b + c > a - 4$ then since $\gcd(a,b) + \gcd(a,c) > 2$, we have

$$\begin{aligned}(b + c + \gcd(a,b) + \gcd(a,c))^2 &> (b+c)^2 + 4(b+c) \\ &> (a-4)(b+c) + 4(b+c) = a(b+c)\end{aligned}$$

so if $\Delta = \mathrm{Conv}((0,0),(b,a),(b+c,0))$ then Cox(X) is a finitely generated k-algebra where $X = \mathrm{Bl}_p(X_\Delta)$. Moreover, if $\gcd(a,b) = \gcd(a,c) = 1$ then the inner normal vectors

$$u_1 = \begin{pmatrix} a \\ -c \end{pmatrix}, \qquad u_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad u_3 = \begin{pmatrix} -a \\ -b \end{pmatrix}$$

will not span the lattice N so X_Δ is not a weighted projective space.

3.2 Blowups of Toric Projective Threefolds of Picard Number One

Note that toric threefolds of Picard number one arise from tetrahedral lattice polytopes. As in the case of surfaces, for any lattice tetrahedron $\Delta \subset M_{\mathbb{R}} \cong \mathbb{R}^3$ we can pick a basis $\{m_1, m_2, m_3\}$ of the lattice M such that one facet of Δ lies on the horizontal plane and since X_Δ depends only on the direction of the inner normal vectors we can shift Δ freely while maintaining X_Δ. By an analogous argument to the case of surfaces, we see that, we can set $\Delta = \mathrm{Conv}((0,0,0),(a,b,c),(a+f,0,0),(a+d,b+e,0))$ where $a,b,c,a+d,a+f,b+e > 0$ without loss of generality.

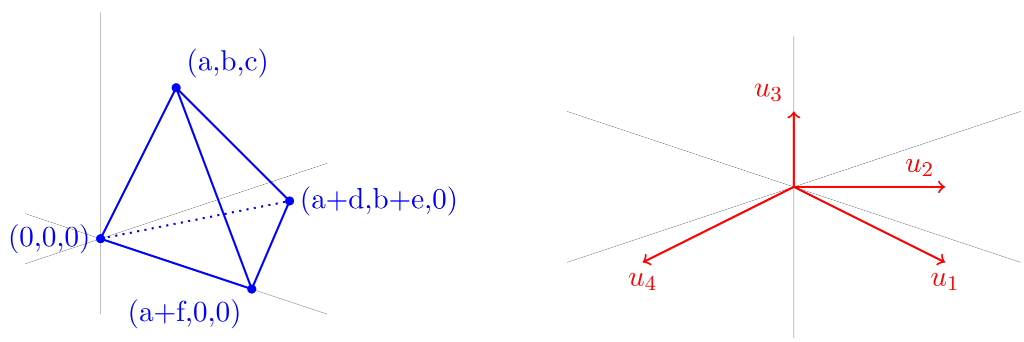

Figure 3.2: A tetrahedral lattice polytope (right) and its set of inner normal vectors (right)

Set F_1 to be the facet of Δ given by $F_1 = \mathrm{Conv}((0,0,0),(a,b,c),(a+d,b+e,0))$, F_2 to be the facet of Δ given by $F_2 = \mathrm{Conv}((0,0,0),(a,b,c),(a+f,0,0))$, F_3 to be the facet of Δ given by $F_3 = \mathrm{Conv}((0,0,0),(a+d,b+e,0),(a+f,0,0)$, and finally F_4 to be the facet of Δ given by $F_4 = \mathrm{Conv}((a,b,c),(a+d,b+e,0),(a+f,0,0))$. Then we can compute the inner normal vectors u_1, u_2, u_3, and u_4 by taking the cross product of the linearly independent vectors in the supporting hyperplanes of each facet and possibly rescaling to ensure primitivity. Before rescaling, the inner normal vector u_2 lies on the ray given by

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \times \begin{pmatrix} a+d \\ b+e \\ 0 \end{pmatrix} = \begin{pmatrix} c(b+e) \\ -c(a+d) \\ c(a+d) - a(b+e) \end{pmatrix}$$

so

$$u_1 = \frac{\begin{pmatrix} c(b+e) \\ -c(a+d) \\ c(a+d)-a(b+e) \end{pmatrix}}{\gcd(c(b+e), c(a+d), c(a+d)-a(b+e))}.$$

Similarly, u_2 lies on the ray given by

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \times \begin{pmatrix} a+f \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ c(a+f) \\ -b(a+f) \end{pmatrix}$$

so

$$u_2 = \frac{\begin{pmatrix} 0 \\ c(a+f) \\ -b(a+f) \end{pmatrix}}{\gcd(c(a+f), b(a+f))}.$$

Since F_3 lies on the $x-y$ plane, clearly,

$$u_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

which leaves u_4 which lies on the ray given by

$$\left(\begin{pmatrix} a+d \\ b+e \\ 0 \end{pmatrix} - \begin{pmatrix} a \\ b \\ c \end{pmatrix}\right) \times \left(\begin{pmatrix} a+f \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} a \\ b \\ c \end{pmatrix}\right) = \begin{pmatrix} d \\ e \\ -c \end{pmatrix} \times \begin{pmatrix} f \\ -b \\ -c \end{pmatrix} = \begin{pmatrix} -c(b+e) \\ c(d-f) \\ bd+ef \end{pmatrix}$$

so

$$u_4 = \frac{\begin{pmatrix} -c(e+b) \\ c(d-f) \\ -bd-ef \end{pmatrix}}{\gcd(c(e+b), c(d-f), bd+ef)}.$$

Finally, since the facets F_1, F_2, and F_3 all lie on hyperplanes that pass through the origin, we have that

$$D_\Delta = -\left\langle \begin{pmatrix} a+f \\ 0 \\ 0 \end{pmatrix}, u_4 \right\rangle D_4 = \left(\frac{-c(a+f)(b+e)}{\gcd(c(b+e), c(d-f), bd+ef)} \right) D_4.$$

Now, let $\{m_1, m_2, m_3\}$ be our basis for M. Recall that the divisor associated with the character χ^m for $m \in M$ is

$$\mathrm{div}(\chi^{\mathrm{m}}) = \sum_{\mathrm{F}} \langle \mathrm{m}, \mathrm{u_F} \rangle \mathrm{D_F}$$

so

$$\begin{aligned} \mathrm{div}(\chi^{\mathrm{m_1}}) = &\left(\frac{\mathrm{c(b+e)}}{\mathrm{gcd(c(b+e), c(a+d), c(a+d) - a(b+e))}} \right) \mathrm{D_1} \\ &\qquad - \left(\frac{c(b+e)}{\gcd(c(b+e), c(d-f), bd+ef)} \right) D_4. \end{aligned}$$

Similarly,

$$\begin{aligned} \mathrm{div}(\chi^{\mathrm{m_2}}) = &\left(\frac{\mathrm{-c(a+d)}}{\mathrm{gcd(c(b+e), c(a+d), c(a+d) - a(b+e))}} \right) \mathrm{D_1} \\ &+ \left(\frac{c(a+f)}{\gcd(c(a+f), b(a+f))} \right) D_2 + \left(\frac{c(d-f)}{\gcd(c(b+e), c(d-f), bd+ef)} \right) D_4, \end{aligned}$$

and

$$\operatorname{div}(\chi^{\mathrm{m}_3}) = \left(\frac{\mathrm{c(a+d)-a(b+e)}}{\mathrm{gcd(c(b+e),c(a+d),c(a+d)-a(b+e))}}\right)\mathrm{D}_1 \\ -\left(\frac{b(a+f)}{\gcd(c(a+f),b(a+f))}\right)D_2 + D_3 + \left(\frac{bd+ef}{\gcd(c(b+e),c(d-f),bd-ef)}\right)D_4.$$

Using these equations, we would like find a rational number w such that $-K_{X_\Delta} = wD_\Delta$ like what we have in the case of surfaces. To do so, we set

$$-K_{X_\Delta} = D_1 + D_2 + D_3 + D_4 = wD_\Delta + x\operatorname{div}(\chi^{\mathrm{m}_1}) + \mathrm{y}\operatorname{div}(\chi^{\mathrm{m}_2}) + \mathrm{z}\operatorname{div}(\chi^{\mathrm{m}_3})$$

which allows us the gather the coefficients of each D_i into a system of equations. Since D_3 only appears in $\operatorname{div}(\chi^{\mathrm{m}_3})$, we see that $z = 1$. Then, the coefficients on D_2 give

$$1 = \frac{yc(a+f) - zb(a+f)}{\gcd(c(a+f),b(a+f))}$$

so

$$y = \frac{b(a+f) + \gcd(c(a+f),b(a+f))}{c(a+f)}.$$

The coefficients on D_1 give

$$1 = \frac{xc(b+e) - yc(a+d) + z(c(a+d) - a(b+e))}{\gcd(c(b+e),c(a+d),c(a+d)-a(b+e))}$$

so

$$x = \frac{(a+d)b}{a(b+e)} + \frac{(a+d)\gcd(c(a+f),b(a+f)}{c(a+f)(b+e)} - \frac{a+d}{b+e} + \frac{a}{c} \\ + \frac{\gcd(c(b+e),c(a+d),c(a+d)-a(b+e))}{c(b+e)}.$$

Finally, the coefficients of D_4 give

$$\begin{aligned}1 = w(&\frac{-c(a+f)(b+e)}{\gcd(c(b+e),c(d-f),bd+ef)}) + x\left(\frac{c(b+e)}{\gcd(c(b+e),c(d-f),bd+ef)}\right)\\ &+y\left(\frac{c(d-f)}{\gcd(c(b+e),c(d-f),bd+ef)}\right) + z\left(\frac{bd+ef}{\gcd(c(b+e),c(d-f),bd-ef)}\right)\end{aligned}$$

so

$$\begin{aligned}w = &\frac{(a+d)b}{a(a+f)(b+e)} + \frac{(a+d)c\gcd(c(a+f),b(a+f))}{(a+f)^2(b+e)c} + \frac{a+d}{a+f} + \frac{a}{(a+f)c}\\ &+\frac{\gcd(c(b+e),c(a+d),c(a+d)-a(b+e)}{(a+f)(b+e)c} + \frac{(b(d-f)}{(a+f)(b+e)c}\\ &+\frac{(d-f)\gcd(c(a+f),b(a+f))}{(a+f)^2(b+e)c} + \frac{(bd+ef)}{(a+f)(b+e)c} - \frac{\gcd(c(b+e),c(d-f),bd-ef)}{(a+f)(b+e)c}.\end{aligned}$$

Hence, we can conclude the following lemma.

Lemma 3.2.1. *Let* $\Delta = \mathrm{Conv}((0,0,0),(a,b,c),(a+f,0,0),(a+d,b+e,0))$ *where* $a,b,c,a+d,a+f,b+e>0$ *then for* $w \in \mathbb{Q}$ *as above, we have*

$$-K_{X_\Delta} \equiv wD_\Delta.$$

We can see that these expressions rapidly become more complicated as we increase dimension. However, for any n-dimensional simplex Δ with vertices in M with corresponding normal toric projective variety X_Δ, this procedure allows us to express $-K_{X_\Delta}$ as a rational multiple of D_Δ up to numerical equivalence and, perhaps with the help of a computer, we believe this result is still useful.

Chapter 4

The Moduli Space $\overline{\mathbf{M}}_{\mathbf{0,n}}$

In this section we present ongoing joint work with José González, Connor Halleck-Dubé, Jocelyn Wang, and Nicholas Wawrykow to study effectiveness of divisors on the moduli space $\overline{M}_{0,n}$ via simplicial complexes.

Define $\overline{M}_{g,n}$ to be the moduli space of stable genus g curves with n marked points. Recall that a curve is stable if its singularities are at worst double points and its automorphism group is finite. One of the first steps to studying the birational geometry of these spaces is to study whether or not the pseudoeffective cone $\overline{\mathrm{Eff}}(\overline{M}_{g,n})$ is finitely generated. For large g and n, it has been shown that $\overline{\mathrm{Eff}}(\overline{M}_{g,n})$ is not finitely generated. For $g = 0$ it has been shown that it is finitely generated for $n \leq 6$ and not finitely generated for $n \geq 10$. So the simplest case that remains unknown is when $g = 0$ and $7 \leq n \leq 9$. Kapranov showed in [Kap92] that $\overline{M}_{0,n}$ can be constructed as the iterated blowup of $\mathbb{P}^{n-3}$ at $n-1$ general points and then along the strict transforms of the linear subspaces spanned by those points. This construction allows us to employ some combinatorial methods to study the structure of $\overline{M}_{0,n}$.

4.1 Simplicial Complexes

Given a nonnegative integer d and a set S, a *d-simplex* σ on S is a multiset of elements of S with cardinality $|\sigma| = d+1$. We call the set of elements of a d-simplex σ the *support* of σ and if the cardinality of the support of σ is less than the cardinality of σ, that is, if σ includes repeated elements, we say σ is *singular.* The number of times an element i appears in a simplex σ is called the *multiplicity* of i in σ and is denoted $m(i \in \sigma)$. Similarly, the multiplicity of a multiset $T \subseteq S$ in σ is

$$m(T \subseteq S) = \prod_{i \in S} \binom{m(i \in S)}{m(i \in T)}.$$

A *d-complex* Δ on S is a finite set of d-simplices. Given a ring R and a d-complex Δ, we define a *weighting* on Δ to be a function $w : \Delta \to R$. Finally, we say Δ is *balanced in degree* j if, we have

$$\sum_{T \subseteq \sigma \in \Delta} w(\sigma) \cdot m(T \subseteq \sigma) = 0$$

for each multiset S such that $|T| = j$. We say Δ is *balanced* if it is balanced for all $0 \leq j \leq d$. A d-complex Δ is *minimal* if it does not properly contain another balanced d-complex.

In [Dor17], Doran, Giansiracusa, and Jensen proved the following.

Theorem 4.1.1 (Doran-Giansiracusa-Jensen)**.** *Let $d \geq 0$ and $n \geq 5$ be positive integers. Then,*

- *There is a bijection between degree $d+1$ multihomogeneous elements of $\mathrm{Cox}(\overline{M}_{0,n})$, not divisible by any exceptional divisor section, and nondegenerately balanced d-complexes on $\{1, 2, \ldots, n-1\}$. Let Δ be a complex on $\{1, 2, \ldots, n-1\}$ then the associated divisor $D_\Delta \in \mathrm{Pic}(\overline{M}_{0,n})$ is given by*

$$D_\Delta = (d+1)H - \sum_I \left(d + 1 - \max_{\sigma \in \Delta} \left\{ \sum_{i \in I} m(i \in \sigma) \right\} \right) E_I$$

where $I \subseteq \{1, 2, \ldots, n-1\}$ and $1 \leq |I| \leq n-4$. Moreover, all nondegenerate balancings on Δ correspond with elements in that class.

- *Let $D \in \mathrm{Pic}(\overline{M}_{0,n})$ be a class such that $D - E_I$ is not effective for any I. Then D is effective if and only if there is a balanceable complex Δ with $D_\Delta = D$.*

- *Let Δ be a nonsingular, balanceable, minimal d-complex on $\{1, 2, \ldots, n-1\}$, with $d \leq n-5$, over a field, and which is not a product. Then D_Δ is irreducible in $M(\overline{M}_{0,n})$, $h^0(\overline{M}_{0,n}, D_\Delta)$=1, and every generating set for $\mathrm{Cox}(\overline{M}_{0,n})$ includes the unique up to scalar section of D_Δ.*

In [Gon20], González, Gunther, and Zhang proved that if a d-complex Δ is balanced on its facets, the set of $(d-1)$-simplices contained in the d-simplices of Δ, then it is balanced and they classified all irreducible divisors of $\overline{M}_{0,7}$ arising from non-singular simplicial complexes, that is, simplicial complexes whose simplices have no repeated elements. Hence, in order to find new extremal effective divisors of $\overline{M}_{0,n}$, we need to study the case of singular complexes.

In order for an effective divisor to be extremal in $\overline{\mathrm{Eff}}(\overline{M}_{0,n})$ it must be irreducible, that is, it must not be equal to the sum of two effective divisors. We first observe that products of complexes result in reducible divisors.

Definition 4.1.2. Given a d_1-complex Δ_1 and a d_2-complex Δ_2, the *product* of Δ_1 and Δ_2 is the $(d_1 + d_2 + 1)$-complex $\Delta_1 \cdot \Delta_2 = \{\sigma_1 \uplus \sigma_2 \mid \sigma_1 \in \Delta_1, \sigma_2 \in \Delta_2\}$ where $\uplus$ is a multiset sum of the simplices which takes the union of their elements with multiplicity.

Example 4.1.3. The 1-complex forming the sides of a square $\{\langle 1,2\rangle, \langle 1,4\rangle, \langle 2,3\rangle, \langle 3,4\rangle\}$ is the product $\{\langle 1\rangle, \langle 3\rangle\} \cdot \{\langle 2\rangle, \langle 4\rangle\}$ and the octahedron

$$\{\langle 1,2,3\rangle, \langle 1,2,5\rangle, \langle 1,3,4\rangle, \langle 1,3,5\rangle, \langle 2,3,6\rangle, \langle 2,5,6\rangle, \langle 3,4,6\rangle, \langle 3,5,6\rangle\}$$

is the product $\{\langle 1\rangle, \langle 6\rangle\} \cdot \{\langle 2,3\rangle, \langle 2,5\rangle, \langle 3,4\rangle, \langle 4,5\rangle\}$.

This leads to the following lemma.

Lemma 4.1.4. *For any nonempty complexes Δ_1 and Δ_2 on $\{1, 2, \ldots, n-1\}$, of degrees d_1 and d_2,*

$$D_{\Delta_1 \cdot \Delta_2} = D_{\Delta_1} + D_{\Delta_2}.$$

Proof. Let Δ_1 be a d_1-complex and let Δ_2 be a d_2-complex. We have that $D_{\Delta_1} = (d_1 + 1)H - \Sigma a_I E_I$, $D_{\Delta_2} = (d_2+1)H - \Sigma b_I E_I$ and $D_{\Delta_1 \cdot \Delta_2} = (d_1 + d_2 + 2)H - \Sigma c_I E_I$, where for each I,

$$a_I = d_1 + 1 - \max_{\sigma_1 \in \Delta_1} \left\{ \sum_{i \in I} m(i \in \sigma_1) \right\}, \quad b_I = d_2 + 1 - \max_{\sigma_2 \in \Delta_2} \left\{ \sum_{i \in I} m(i \in \sigma_2) \right\},$$
$$c_I = d_1 + d_2 + 2 - \max_{\sigma \in \Delta_1 \cdot \Delta_2} \left\{ \sum_{i \in I} m(i \in \sigma) \right\}.$$

Hence it is enough to fix an index I and show that

$$\max_{\sigma \in \Delta_1 \cdot \Delta_2} \left\{ \sum_{i \in I} m(i \in \sigma) \right\} = \max_{\sigma_1 \in \Delta_1} \left\{ \sum_{i \in I} m(i \in \sigma_1) \right\} + \max_{\sigma_2 \in \Delta_2} \left\{ \sum_{i \in I} m(i \in \sigma_2) \right\}.$$

But this is clear, since the simplices $\sigma \in \Delta_1 \cdot \Delta_2$ are precisely those of the form $\sigma = \sigma_1 \uplus \sigma_2$ for some $\sigma_1 \in \Delta_1$ and $\sigma_2 \in \Delta_2$, and $m(i \in \sigma_1 \uplus \sigma_2) = m(i \in \sigma_1) + m(i \in \sigma_2)$ for any $i \in I$. $\square$

This gives rise to the following irreducibility criterion obtained in joint work with José González, Connor Halleck-Dubé, Jocelyn Wang, and Nicholas Wawrykow.

Theorem 4.1.5. *Suppose that D is an effective divisor with positive degree on $\overline{M}_{0,n}$ such that $D - E_I$ is not effective for all I, for all $I \subseteq \{1, 2, \ldots, n-1\}$, with $1 \leq |I| \leq n-4$. We will refer to this as D being strictly effective. Then $D = D_{\Delta(D)}$, where $\Delta(D)$ is the collection of simplices σ such that $D_\sigma \leq D$. There exists a non empty complex $\Delta \subseteq \Delta(D)$ that admits a nondegenerate balancing such that $D = D_\Delta = D_{\Delta(D)}$. Let $\Delta_1, \Delta_2, \ldots, \Delta_r \subseteq \Delta(D)$ be all such nonempty complexes admitting nondegenerate balancings and with $D = D_{\Delta_1} = D_{\Delta_2} = \ldots = D_{\Delta_r} = D_{\Delta(D)}$. Assume that for all $1 \leq i \leq r$ there is no product $\Delta_i \subseteq \Sigma_1 \cdot \Sigma_2 \subseteq \Delta(D)$,* *with Σ_1 and Σ_2 nonempty lower dimensional complexes that admit nondegenerate balancings. Then D is irreducible in the monoid of effective divisor classes of $\overline{M}_{0,n}$.*

Proof. Suppose D were reducible. Since it is strictly effective, the only way in which it could decompose is as a sum $D = D_1 + D_2$ where D_1 and D_2 are also strictly effective divisors of positive degree. By Theorem A, there exist complexes Σ_1 and Σ_2 that admit nondegenerate balancings such that $D_1 = D_{\Sigma_1}$ and $D_2 = D_{\Sigma_2}$. Furthermore, by Lemma 4.1.4, we have $D_{\Sigma_1} + D_{\Sigma_2} = D_{\Sigma_1 \cdot \Sigma_2}$. So

$$D = D_1 + D_2 = D_{\Sigma_1} + D_{\Sigma_2} = D_{\Sigma_1 \cdot \Sigma_2}.$$

Since Σ_1 and Σ_2 are balanceable complexes, their product $\Sigma_1 \cdot \Sigma_2$ inherits a possibly degenerate balancing. Let $\Sigma \subseteq \Sigma_1 \cdot \Sigma_2$ be the support of that balancing so $D_\Sigma \leq D_{\Sigma_1 \cdot \Sigma_2} = D$. Since D_Σ has the same degree as D and D is strictly effective, then we must have $D_\Sigma = D$. Hence, Σ is a complex that admits a nondegenerate balancing such that $D_\Sigma = D$ which means $\Sigma = \Delta_i$ for some $1 \leq i \leq r$ and then

$$\Delta_i \subseteq \Sigma_1 \cdot \Sigma_2 \subseteq \Delta(D)$$

which contradicts our assumptions. Therefore, D must be irreducible in the monoid of effective divisor classes of $\overline{M}_{0,n}$. $\square$

Using this irreducibility criterion, we wrote an efficient computer program to find new irreducible effective divisors of $\overline{M}_{0,n}$ and rediscovered all of those in degree at most three that have been shown to be effective in [Cas13], [Dor17], [Opi16], [Gon20], and [DS22] as well as some that do not yet appear in the literature to our knowledge.

We started our search in the case of degree three divisors arising from singular complexes which is the simplest unsolved case. However, we observe that if a degree three divisor arises from a 2-complex that is a product then the two complex must be a product of a 0-complex and a 1-complex. In [Dor17, Lemma 3.8], Doran, Giansiracusa, and Jensen show that the divisor D_Δ in $\overline{M}_{0,n}$ arising from a complex Δ which contains a simplex of the form $\{i, i, \cdot, i\}$ is the pullback of a divisor in $\overline{M}_{0,m}$ for some $m < n$. Then we can focus our attention on complexes which do not contain simplices of the form $\{i, i, \cdot, i\}$, that is with all their entries equal to each other. This leads us to the following characterization of products of minimal balanceable 0-complexes and 1-complexes.

Lemma 4.1.6. *Let Σ be a 2-complex not containing simplices of the form $\langle i, i, i\rangle$. Then Σ is the product of a minimal 0-complex $\Sigma_0 = \{\langle a\rangle, \langle b\rangle\},\ a \neq b$ and a 1-complex $\Sigma_1 = \{\sigma_1, \ldots, \sigma_r\}$ if and only if*

$$\Sigma = \left\langle \bigcup_{i=1}^{r} (\langle a\rangle \uplus \sigma_i) \cup \bigcup_{i=1}^{r} (\langle b\rangle \uplus \sigma_i) \right\rangle \setminus \bigcup_{c \in C} \langle a, b, c\rangle$$

where the left hand side of the equality is a multiset and C is the set of vertices c such that $\langle a, a, c\rangle$, $\langle a, b, c\rangle$, and $\langle b, b, c\rangle$ all appear in Σ.

Proof. First suppose that a 2-complex Σ is the product of a minimal 0-complex $\Sigma_0 = \{\langle a\rangle, \langle b\rangle,\ a \neq b\}$ and a 1-complex $\Sigma_1 = \{\sigma_1, \ldots, \sigma_r\}$. Then

$$\begin{aligned} \Sigma &= \Sigma_0 \cdot \Sigma_1 \\ &= \{\langle a\rangle, \langle b\rangle\} \cdot \{\sigma_1, \ldots, \sigma_r\} \\ &= \left(\bigcup_{i=1}^{r} (\{a\} \uplus \sigma_i) \cup \bigcup_{i=1}^{r} (\{b\} \uplus \sigma_i) \right). \end{aligned}$$

If the set C of vertices c such that $\langle a, a, c\rangle$, $\langle a, b, c\rangle$, and $\langle b, b, c\rangle$ all appear in Σ is empty,

$$\left(\bigcup_{i=1}^{r} (\{a\} \uplus \sigma_i) \cup \bigcup_{i=1}^{r} (\{b\} \uplus \sigma_i) \right) = \left\langle \bigcup_{i=1}^{r} (\{a\} \uplus \sigma_i) \cup \bigcup_{i=1}^{r} (\{b\} \uplus \sigma_i) \right\rangle$$

and we are finished. If not, then there exists a vertex $c \neq a, b$ such that $\langle a, a, c\rangle$, $\langle a, b, c\rangle$, and $\langle b, b, c\rangle$ all appear in Σ but then $\langle a, c\rangle$ and $\langle b, c\rangle$ must appear in Σ_1 so $\langle a\rangle \uplus \langle b, c\rangle = \langle b\rangle \uplus \langle a, c\rangle$ so that simplex will appear twice in the multiset

$$\left\langle \bigcup_{i=1}^{r} (\{a\} \uplus \sigma_i) \cup \bigcup_{i=1}^{r} (\{b\} \uplus \sigma_i) \right\rangle$$

and must be removed. Removing all such duplicate simplices gives the desired equality.

Now suppose

$$\Sigma = \left\langle \bigcup_{i=1}^{r} (\langle a \rangle \uplus \sigma_i) \cup \bigcup_{i=1}^{r} (\langle b \rangle \uplus \sigma_i) \right\rangle \setminus \bigcup_{c \in C} \langle a, b, c \rangle$$

where C is the set of vertices c such that $\langle a, a, c \rangle$, $\langle a, b, c \rangle$, and $\langle b, b, c \rangle$ all appear in Σ. Then

$$\begin{aligned}
\Sigma &= \left\langle \bigcup_{i=1}^{r} (\langle a \rangle \uplus \sigma_i) \cup \bigcup_{i=1}^{r} (\langle b \rangle \uplus \sigma_i) \right\rangle \setminus \bigcup_{c \in C} \{a, b, c\} \\
&= \bigcup_{i=1}^{r} (\{a\} \uplus \sigma_i) \cup \bigcup_{i=1}^{r} (\{b\} \uplus \sigma_i) \\
&= \{\langle a \rangle, \langle b \rangle\} \cdot \{\sigma_1, \dots, \sigma_r\} \\
&= \Sigma_0 \cdot \Sigma_1
\end{aligned}$$

by definition. □

Example 4.1.7. To determine if the 2-complex

$$\Delta = \{\langle 1,1,2 \rangle, \langle 1,1,3 \rangle, \langle 1,1,4 \rangle, \langle 1,2,2 \rangle, \langle 1,2,3 \rangle, \langle 1,2,4 \rangle, \langle 2,2,3 \rangle\}$$

is a product, we will attempt to find a 0-complex Σ_0 that could be a factor. If such a factor exists, $\Sigma_0 = \{\langle a \rangle, \langle b \rangle\}$ where every simplex $\delta \in \Delta$ contains either a or b as a vertex and for every simplex $\langle a, c, d \rangle \in \Delta$, there must also be a simplex $\langle b, c, d \rangle \in \Delta$. Note that we can't have $\Sigma_0 = \{\langle 1 \rangle, \langle 3 \rangle\}$ since $\langle 1,1,3 \rangle \in \Delta$ but $\langle 1,3,3 \rangle \notin \Delta$. Similarly, we see that $\Sigma_0 \neq \{\langle 1 \rangle, \langle 4 \rangle\}$, $\Sigma_0 \neq \{\langle 2 \rangle, \langle 3 \rangle\}$, and $\Sigma_0 \neq \{\langle 2 \rangle, \langle 4 \rangle\}$. However, if $\Sigma_0 = \{\langle 1 \rangle, \langle 2 \rangle\}$ we see that

$$\Delta = \{\langle 1 \rangle, \langle 2 \rangle\} \cdot \{\langle 1,2 \rangle, \langle 1,3 \rangle, \langle 1,4 \rangle, \langle 2,3 \rangle\}.$$

Example 4.1.8. We can observe that

$$\Sigma = \{\langle 1,1,2 \rangle, \langle 1,1,3 \rangle, \langle 1,1,4 \rangle, \langle 1,2,2 \rangle, \langle 1,2,3 \rangle, \langle 2,2,3 \rangle\}$$

is not a product because if $\Sigma_0 = \{\langle 1\rangle, \langle 2\rangle\}$ then since $\langle 1, 1, 4\rangle \in \Sigma$ we should have $\langle 1, 2, 4\rangle \in \Sigma$ which is false. We can similarly eliminate the possibility of any other 0-complex factor.

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