**Title**
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**Journal**
SIAM Journal on Control and Optimization, 52(6)

**ISSN**
0363-0129

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**Publication Date**
2014

**DOI**
10.1137/130938451

# GENERALIZING THE KYP LEMMA TO MULTIPLE FREQUENCY INTERVALS

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**Abstract.** A recent generalization of the Kalman-Yakubovich-Popov (KYP) lemma establishes the equivalence between a semi-infinite inequality on a segment of a circle or straight line in the complex plane and a linear matrix inequality. In this paper we further generalize the KYP lemma to particular curves in the complex plane, described by a polynomial equality and inequality that satisfy certain conditions. The considered set of curves is shown to include the union of segments of a circle or line as a special case.

**Key words.** Kalman-Yakubovich-Popov (KYP) lemma

**AMS subject classifications.** 90C22, 93C05

**1. Introduction.** The Kalman-Yakubovich-Popov (KYP) lemma [11, 16, 20] is a key result in modern system and control theory. The classical version of the lemma states the equivalence between a semi-infinite inequality on an entire circle or straight line in the complex plane and a finite-dimensional linear matrix inequality (LMI). Recently, the KYP lemma has been generalized to allow for semi-infinite inequalities on only a segment of a circle or straight line [9, 18]. Such segments can be described by quadratic equality and inequality:

$$\mathbf{\Lambda}(\Phi,\Psi) = \left\{ \lambda \in \mathbb{C} \; : \; \begin{bmatrix}\lambda \\ 1\end{bmatrix}^* \Phi \begin{bmatrix}\lambda \\ 1\end{bmatrix} = 0 \, , \; \begin{bmatrix}\lambda \\ 1\end{bmatrix}^* \Psi \begin{bmatrix}\lambda \\ 1\end{bmatrix} \geq 0 \right\} ,$$

with given Hermitian $\Phi, \Psi$, $\det(\Phi) < 0$. The generalized KYP lemma of [9, 18] then states that, given matrices $A, B$ and Hermitian matrix $\Theta$, the inequality

$$\begin{bmatrix}(\lambda I - A)^{-1}B \\ I\end{bmatrix}^* \Theta \begin{bmatrix}(\lambda I - A)^{-1}B \\ I\end{bmatrix} \prec 0 \tag{1.1}$$

holds for all $\lambda \in \mathbf{\Lambda}(\Phi,\Psi)$ if and only if there exist Hermitian matrices $P, Q$, such that $Q \succ 0$ and

$$\begin{bmatrix}A & B \\ I & 0\end{bmatrix}^* (\Phi \otimes P + \Psi \otimes Q) \begin{bmatrix}A & B \\ I & 0\end{bmatrix} + \Theta \prec 0 \, .$$

With $\Psi = 0$, $\mathbf{\Lambda}(\Phi,\Psi)$ corresponds to an entire circle or straight line in $\mathbb{C}$ and the original KYP lemma is retrieved.

In this paper we take a first step in extending the KYP lemma to semi-infinite inequalities of the form (1.1) on curves in $\mathbb{C}$ that are characterized by a higher-degree polynomial equality and inequality. Such curves for instance appear when considering multiple segments on a circle or straight line, which correspond to multiple frequency ranges for discrete-time and continuous-time systems, respectively. The more interesting situation that fits into our setting is the frequency domain analysis of linear

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time-invariant systems with a generalized frequency variables [6, 8]. The transfer function of such systems is expressed by $C(\lambda I - A)^{-1}B + D$ where the generalized frequency variable $\lambda = p(s)$ varies on the Nyquist plot of $p(s)$ in the complex plane. This class of systems was proposed to treat a class of linear homogeneous multi-agent systems, where $1/p(s)$ is the common proper transfer function of the agents and $(A, B, C, D)$ captures the interaction or information exchange structure. When $p(s)$ is a rational function, its Nyquist plot coincides with the set of roots of a polynomial equality [6, 8].

We generalize the KYP lemma to a particular subset of such higher-degree polynomial curves, characterized in terms of two assumptions on the describing polynomial equality and inequality. The union of multiple segments of a circle or straight line in $\mathbb{C}$ is shown to be included in the considered set of curves. Although an LMI equivalent to a semi-infinite inequality on a union of segments can also be obtained by application of the results of [9, 18] to each of the segments separately, the LMI obtained by our novel result is generally smaller in both dimension and number of variables. The application of our result to the analysis of systems with a generalized frequency variable is still under investigation. As the polynomial equality describing the Nyquist plot of $p(s)$ does generally not satisfy our two assumptions, only a sufficient LMI condition is currently obtained.

The paper is organized as follows: Section 2 describes our generalization of the KYP lemma and elaborates its proof. In Section 3 we discuss the application of our result to the union of segments of a circle or straight line, while Section 4 concludes the paper. A preliminary version of this paper appeared in [14]. In the current paper we present a more general formulation of our result, allowing for nonstrict inequalities and the state matrix $A$ to have eigenvalues on the curve, and complete some technical proofs.

**Notation.** The set of positive integers is denoted by $\mathbb{I}$, and its subset up to $n \in \mathbb{I}$ is $\mathbb{I}_n = \{1, \ldots, n\}$. $\mathbb{R}_+$ and $\mathbb{R}_{++}$ correspond to the set of nonnegative, respectively positive, real numbers. The sets of $n \times n$ real symmetric and complex Hermitian matrices are indicated by $\mathbb{S}^n$ and $\mathbb{H}^n$, respectively. For a matrix $X \in \mathbb{C}^{n\times m}$, $\bar{X}$ denotes its complex conjugate, $X^\intercal$ its transpose, and $X^*$ its complex conjugate transpose: $X^* = \bar{X}^\intercal$. For a vector $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$ we write $x^{\mathsf{F}} = (x_n, \ldots, x_1)$. $0_{n,m}$ is the $n\times m$ zero matrix, and $I_n$ the $n\times n$ identity matrix. The subscripts are omitted when the dimensions can be inferred from the context. The matrix Kronecker product is indicated by $\otimes$, and the imaginary unit is denoted by $\mathrm{i} = \sqrt{-1}$. For vectors $x, y$, $\operatorname{conv}(x, y)$ is the vector convolution, while $\operatorname{conv}^1(x) = x$ and for $k \geq 2$, $\operatorname{conv}^k(x)$ is defined by the recursion $\operatorname{conv}^k(x) = \operatorname{conv}(x, \operatorname{conv}^{k-1}(x))$.

## 2. Further Generalization of the KYP Lemma.

In this section, we generalize the KYP lemma to a particular subset of curves in $\mathbb{C}$ that are described by a polynomial equality and inequality. The considered subset is specified in Section 2.1, while Section 2.2 presents the corresponding generalized KYP lemma. Section 2.3 elaborates the proof of our result.

### 2.1. Curves in the Complex Plane.

The considered curves are of the form

$$\mathbf{\Lambda}(\Phi, \Psi) = \{\lambda \in \mathbb{C} : l_\ell(\lambda)^*\Phi l_\ell(\lambda) = 0\,,\ l_\ell(\lambda)^*\Psi l_\ell(\lambda) \geq 0\}\ , \tag{2.1}$$

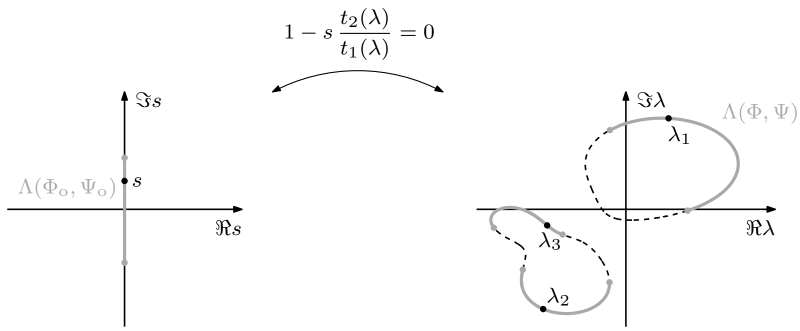


FIG. 2.1. *Decomposition (2.3) defines a mapping between* $\mathbf{\Lambda}(\Phi_o, \Psi_o)$ *and* $\mathbf{\Lambda}(\Phi, \Psi)$, *where* $s \in \mathbf{\Lambda}(\Phi_o, \Psi_o)$ *is mapped onto* $\{\lambda_1, \lambda_2, \lambda_3\} = \mathcal{R}_T(s) \subset \mathbf{\Lambda}(\Phi, \Psi)$. *Assumption 1 requires the roots* $\lambda_1, \lambda_2, \lambda_3$ *to be distinct for every* $s \in \mathbf{\Lambda}(\Phi_o, \Psi_o)$.

where $\ell \in \mathbb{I}$ is a given positive integer, $\Phi, \Psi \in \mathbb{H}^{\ell+1}$ are given Hermitian matrices, and the mapping $l_\ell(\lambda) : \mathbb{C} \to \mathbb{C}^{\ell+1}$ is defined as

$$\begin{aligned} l_\ell(\lambda) &= \begin{bmatrix} \lambda^\ell & \lambda^{\ell-1} & \cdots & \lambda & 1 \end{bmatrix}^\intercal, \quad \text{for } \ell \in \mathbb{I}\,, \\ l_0(\lambda) &= 1\,. \end{aligned} \tag{2.2}$$

In case $\mathbf{\Lambda}(\Phi, \Psi)$ is unbounded, it is extended with $\infty$. We agree that for $\ell \in \mathbb{I}$,

$$l_\ell(\infty) = \begin{bmatrix} 1 & 0_{1,\ell} \end{bmatrix}^\intercal\,.$$

We will enforce the following two assumptions on the matrices $\Phi$ and $\Psi$:

ASSUMPTION 1. *The matrices* $\Phi, \Psi \in \mathbb{H}^{\ell+1}$ *admit a decomposition of the form*

$$\Phi = T^*\Phi_o T\,, \qquad \Psi = T^*\Psi_o T\,, \tag{2.3a}$$

*where*

$$\Phi_o = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \Psi_o = \begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix}, \qquad \begin{aligned} &0 \le \alpha \le \gamma, \quad \textit{or} \\ &\alpha < 0 < \gamma\,, \end{aligned} \tag{2.3b}$$

*for some matrix* $T \in \mathbb{C}^{2\times(\ell+1)}$ *of full row rank, and some* $\alpha, \beta, \gamma \in \mathbb{R}$. *In addition, for each* $s \in \mathbf{\Lambda}(\Phi_o, \Psi_o)$, *the* $\ell$*th degree polynomial equality in* $\lambda$ *given by*

$$\begin{aligned} \begin{bmatrix} 1 & -s \end{bmatrix} T\,l_\ell(\lambda) \;&=\; 0\,, \quad \text{for } s \neq \infty\,, \\ \begin{bmatrix} 0 & 1 \end{bmatrix} T\,l_\ell(\lambda) \;&=\; 0\,, \quad \text{for } s = \infty\,, \end{aligned} \tag{2.4}$$

*has* $\ell$ *distinct roots. These roots are grouped in the set* $\mathcal{R}_T(s)$.

ASSUMPTION 2. *When* $\ell \ge 2$, *there exists a Hermitian matrix* $R \in \mathbb{H}^\ell$ *such that*

$$l_{\ell-1}(\lambda)^* R\, l_{\ell-1}(\lambda) > 0\,, \qquad \forall \lambda \in \mathbf{\Lambda}(\Phi, \Psi)\,, \tag{2.5a}$$

$$l_{\ell-1}(\lambda_i)^* R\, l_{\ell-1}(\lambda_j) = 0\,, \qquad \forall \lambda_i, \lambda_j \in \mathcal{R}_T(s)\,,\ i \neq j\,,\ \forall s \in \mathbf{\Lambda}(\Phi_o, \Psi_o)\,. \tag{2.5b}$$

As illustrated in Figure 2.1, decomposition (2.3) defines a mapping between the curve $\mathbf{\Lambda}(\Phi, \Psi)$ and $\mathbf{\Lambda}(\Phi_o, \Psi_o)$, which corresponds to (a segment of) the imaginary axis.

Each $s \in \mathbf{\Lambda}(\Phi_o, \Psi_o)$ is mapped onto the $\ell$ roots of (2.4), grouped in $\mathcal{R}_T(s) \subset \mathbf{\Lambda}(\Phi, \Psi)$, while all $\lambda \in \mathcal{R}_T(s)$ are mapped onto the same $s$:

$$
\begin{aligned}
s &= \frac{\begin{bmatrix}1 & 0\end{bmatrix} T\, l_\ell(\lambda)}{\begin{bmatrix}0 & 1\end{bmatrix} T\, l_\ell(\lambda)} = \frac{t_1(\lambda)}{t_2(\lambda)}\,, && \text{if } t_2(\lambda) \neq 0\,, \\
&= \infty\,, && \text{otherwise}\,.
\end{aligned}
\tag{2.6}
$$

Using the polynomials $t_1(\lambda)$ and $t_2(\lambda)$ as defined above, equation (2.4) can be reformulated as

$$
1 - s\,\frac{t_2(\lambda)}{t_1(\lambda)} = 0\,.
$$

Hence, the curve $\Lambda(\Phi, \Psi)$ corresponds to the set of complex numbers $\lambda$ that solve this root locus equation for some $s \in \Lambda(\Phi_o, \Psi_o)$. As for every $s \in \Lambda(\Phi_o, \Psi_o)$ the roots in $\mathcal{R}_T(s)$ must be distinct, only root loci without branching points are allowed.

Assumption 2 implies that $R \succ 0$ as it states that every congruence transformation with a matrix of the form

$$
\begin{bmatrix} l_{\ell-1}(\lambda_1) & \cdots & l_{\ell-1}(\lambda_\ell) \end{bmatrix}\,,
$$

with $\{\lambda_1, \ldots, \lambda_\ell\} = \mathcal{R}_T(s)$ for some $s \in \mathbf{\Lambda}(\Phi_o, \Psi_o)$, yields a positive definite diagonal matrix. Whereas (2.5a) can be satisfied by an arbitrary positive definite matrix $R$, requirement (2.5b) constrains the considered set of curves to great extent. While Assumption 1 still allows for a large variety of curves in $\mathbb{C}$, only the union of segments of a circle or straight line has been found to comply with Assumption 2 as well.

For $\ell = 1$, Assumption 2 is irrelevant, and the curves considered in the generalized KYP lemma of [9, 18] are retrieved: $\mathbf{\Lambda}(\Phi, \Psi)$ corresponds to a nonempty and nonsingleton segment of a circle or straight line in $\mathbb{C}$. If in addition $\Psi = 0$ is considered, the curve corresponds to the entire circle or line, retrieving the original KYP lemma [11, 16, 20].

**2.2. The Corresponding Generalized KYP Lemma.** To formulate our generalized KYP lemma, we define, for given $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times m}$ and $\ell \in \mathbb{I}$, the matrix $F_\ell(A, B) \in \mathbb{C}^{(\ell+1)n\times(n+m\ell)}$ as follows:

$$
F_\ell(A,B) = \begin{bmatrix}
A^\ell & A^{\ell-1}B & A^{\ell-2}B & \cdots & B \\
A^{\ell-1} & A^{\ell-2}B & \cdots & B & 0 \\
\vdots & \vdots & \iddots & \iddots & \vdots \\
A & B & 0 & & 0 \\
I & 0 & \cdots & \cdots & 0
\end{bmatrix},
\tag{2.7}
$$

and the matrix $G_\ell(A, B) \in \mathbb{C}^{\ell(n+m)\times(n+m\ell)}$ as

$$
\begin{aligned}
G_\ell(A,B) = {} & \left(I_\ell \otimes \begin{bmatrix} I_n \\ 0_{m,n} \end{bmatrix}\right) \begin{bmatrix} F_{\ell-1}(A,B) & 0_{n\ell,m} \end{bmatrix} + \\
& \left(I_\ell \otimes \begin{bmatrix} 0_{n,m} \\ I_m \end{bmatrix}\right) \begin{bmatrix} 0_{m\ell,n} & F_\ell(0_{m,m}, I_m) \end{bmatrix}\,.
\end{aligned}
\tag{2.8}
$$

For instance for $\ell = 3$:

$$G_3(A,B) = \begin{bmatrix} A^2 & AB & B & 0 \\ 0 & 0 & 0 & I \\ A & B & 0 & 0 \\ 0 & 0 & I & 0 \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{bmatrix}.$$

In addition, for $\lambda \in \mathbb{C}$ we define the set $\mathcal{N}_{A,B}(\lambda)$ as follows:

$$\begin{aligned} \mathcal{N}_{A,B}(\lambda) &= \{(x,u) \in \mathbb{C}^n \times \mathbb{C}^m \ : \ (\lambda I - A)x = Bu\}\,, && \text{for } \lambda \neq \infty\,, \\ &= \{0\} \times \mathbb{C}^m\,, && \text{for } \lambda = \infty\,. \end{aligned}$$

Note that if $\det(\lambda I - A) \neq 0$, every element of $\mathcal{N}_{A,B}(\lambda)$ is of the form

$$\begin{bmatrix} (\lambda I - A)^{-1}B \\ I \end{bmatrix} u\,,$$

for some $u \in \mathbb{C}^m$.

We are now ready to state our generalized KYP lemma:

THEOREM 2.1 (Generalized KYP Lemma, strict inequalities). *Let Hermitian matrices $\Phi, \Psi \in \mathbb{H}^{\ell+1}$ and $\Theta \in \mathbb{H}^{m+n}$, and matrices $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}^{n\times m}$ be given, with $B$ of full column rank. Suppose $\Phi$ and $\Psi$ satisfy Assumptions 1 and 2, and let $R \in \mathbb{H}^\ell$ be a matrix satisfying (2.5). Then, the following statements are equivalent:*

*(i) The inequality*

$$\begin{bmatrix} x \\ u \end{bmatrix}^* \Theta \begin{bmatrix} x \\ u \end{bmatrix} < 0\,, \tag{2.9}$$

*holds for all nonzero $(x,u) \in \mathcal{N}_{A,B}(\lambda)$ and all $\lambda \in \mathbf{\Lambda}(\Phi,\Psi)$.*

*(ii) There exist $P, Q \in \mathbb{H}^n$ that satisfy $Q \succ 0$ and*

$$\begin{aligned} F_\ell(A,B)^*(\Phi \otimes P + \Psi \otimes Q)\, F_\ell(A,B) &+ \\ G_\ell(A,B)^*(R \otimes \Theta)\, G_\ell(A,B) &\prec 0\,. \end{aligned} \tag{2.10}$$

Under the additional assumption that $(A,B)$ is controllable, the nonstrict version of this theorem also holds:

THEOREM 2.2 (Generalized KYP Lemma, nonstrict inequalities). *Let Hermitian matrices $\Phi, \Psi \in \mathbb{H}^{\ell+1}$ and $\Theta \in \mathbb{H}^{m+n}$, and matrices $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}^{n\times m}$ be given, with $B$ of full column rank and $(A,B)$ controllable. Suppose $\Phi$ and $\Psi$ satisfy Assumptions 1 and 2, and let $R \in \mathbb{H}^\ell$ be a matrix satisfying (2.5). Then, the following statements are equivalent:*

*(i) The inequality*

$$\begin{bmatrix} x \\ u \end{bmatrix}^* \Theta \begin{bmatrix} x \\ u \end{bmatrix} \leq 0\,, \tag{2.11}$$

*holds for all $(x,u) \in \mathcal{N}_{A,B}(\lambda)$ and all $\lambda \in \mathbf{\Lambda}(\Phi,\Psi)$.*

*(ii) There exist $P, Q \in \mathbb{H}^n$ that satisfy $Q \succeq 0$ and*

$$\begin{aligned} F_\ell(A,B)^*(\Phi \otimes P + \Psi \otimes Q)\, F_\ell(A,B) &+ \\ G_\ell(A,B)^*(R \otimes \Theta)\, G_\ell(A,B) &\preceq 0\,. \end{aligned} \tag{2.12}$$

Irrespective of Assumptions 1 and 2, and with an arbitrary $R \succ 0$, the LMI (2.10), respectively (2.12), provide a sufficient condition for the semi-infinite inequality (2.9), respectively (2.11), to hold. To reduce conservatism, the matrix $R \succ 0$ can be considered an additional optimization variable.

As further elaborated below, Assumptions 1 and 2 are needed to prove the necessity of the LMI conditions, which is done by contradiction. Herein, Assumption 1 plays a major role, as it allows constructing vectors $(x, u)$ of the form considered in the semi-infinite inequality (2.9,2.11) from the LMI's infeasibility certificate. In the last step of the necessity proof Assumption 2 is used to show that at least one of these vectors must violate the inequality (2.9,2.11). Although this violation is no longer guaranteed in case Assumption 2 doesn't hold, it may still occur. Hence, in case the LMI condition (2.10,2.12) is found infeasible, one can assess the conservatism of the LMI by constructing vectors $(x, u)$ of the appropriate form as outlined in the proof below. For all these vectors the inequality (2.9,2.11) is checked and the smallest margin to constraint violation provides an indication of the conservatism involved.

**2.3. Proof of the Generalized KYP Lemma.** To lighten the notation, we will omit the arguments $(A, B)$ in $F_\ell(A, B)$ and $G_\ell(A, B)$ below. In the proof of our generalized KYP lemma we rely on the following auxiliary results:

LEMMA 2.3. *Let matrices $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times m}$ and $T \in \mathbb{C}^{2\times(\ell+1)}$ be given, and assume $B$ has full column rank and $T$ full row rank. In addition, let $s \in \mathbb{C} \cup \{\infty\}$ be given and assume that the $\ell$ roots $\lambda_i$, $i \in \mathbb{I}_\ell$, in $\mathcal{R}_T(s)$ are all distinct. Then, a vector $z \in \mathbb{C}^{n+m\ell}$ satisfies*

$$\begin{aligned} \left(\begin{bmatrix}1 & -s\end{bmatrix} \otimes I_n\right)(T \otimes I_n)\, F_\ell\, z &= 0\,, \qquad \text{for } s \neq \infty \\ \left(\begin{bmatrix}0 & 1\end{bmatrix} \otimes I_n\right)(T \otimes I_n)\, F_\ell\, z &= 0\,, \qquad \text{for } s = \infty \end{aligned} \tag{2.13}$$

*if and only if it can be decomposed as*

$$z = \sum_{i=1}^{\ell} \begin{bmatrix} x_i \\ l_{\ell-1}(\lambda_i)^{\mathsf{F}} \otimes u_i \end{bmatrix}, \tag{2.14}$$

*with $(x_i, u_i) \in \mathcal{N}_{A,B}(\lambda_i)$, for all $i \in \mathbb{I}_\ell$.*

The proof of this lemma is presented in Appendix A.

LEMMA 2.4. *Let $\Phi_\mathrm{o}, \Psi_\mathrm{o} \in \mathbb{H}^2$ of the form (2.3b) be given, as well as $X, Y \in \mathbb{C}^{n\times m}$. Then*

$$\begin{bmatrix} X & Y \end{bmatrix} (\Phi_\mathrm{o} \otimes I) \begin{bmatrix} X^* \\ Y^* \end{bmatrix} = 0\,, \quad \text{and} \quad \begin{bmatrix} X & Y \end{bmatrix} (\Psi_\mathrm{o} \otimes I) \begin{bmatrix} X^* \\ Y^* \end{bmatrix} \succeq 0\,, \tag{2.15}$$

*hold if and only if $X$ and $Y$ can be factored as*

$$X = W \operatorname{diag}(s_1, \ldots, s_m) V^*\,, \qquad Y = W V^*\,, \tag{2.16}$$

*with some $W \in \mathbb{C}^{n\times m}$, unitary $V \in \mathbb{C}^{m\times m}$, and $s_i \in \mathbf{\Lambda}(\Phi_\mathrm{o}, \Psi_\mathrm{o})$ for all $i \in \mathbb{I}_m$.*

The proof of this lemma is presented in Appendix B.

*Proof.* [Proof of Theorem 2.1] To prove that (ii) implies (i) we multiply (2.10) by $z^*$ on the left and $z$ on the right, where $z$ is a vector of the form

$$z = \begin{bmatrix} x \\ l_{\ell-1}(\lambda)^{\mathsf{F}} \otimes u \end{bmatrix},$$

with arbitrary $\lambda \in \mathbf{\Lambda}(\Phi,\Psi)$ and arbitrary $(x,u) \in \mathcal{N}_{A,B}(\lambda)$. Straightforward calculations reveal that for such $z$

$$F_\ell\, z = l_\ell(\lambda) \otimes \chi(x,u)\,, \quad \text{with} \quad \chi(x,u) = \begin{cases} x\,, & \text{if } \lambda \neq \infty\,, \\ Bu\,, & \text{if } \lambda = \infty\,, \end{cases} \tag{2.17a}$$

$$G_\ell\, z = l_{\ell-1}(\lambda) \otimes \begin{bmatrix} x \\ u \end{bmatrix}\,. \tag{2.17b}$$

Combining these equalities with the mixed-product property of the matrix Kronecker product, the multiplication of (2.10) by $z^*$ and $z$ yields

$$l_\ell(\lambda)^*\Phi l_\ell(\lambda) \cdot \chi(x,u)^* P \chi(x,u) + l_\ell(\lambda)^*\Psi l_\ell(\lambda) \cdot \chi(x,u)^* Q \chi(x,u) + \\ l_{\ell-1}(\lambda)^* R l_{\ell-1}(\lambda) \cdot \begin{bmatrix} x \\ u \end{bmatrix}^* \Theta \begin{bmatrix} x \\ u \end{bmatrix} < 0\,.$$

By definition (2.1) of $\mathbf{\Lambda}(\Phi,\Psi)$ and the constraint $Q \succ 0$, the first term equals zero for all $\lambda \in \mathbf{\Lambda}(\Phi,\Psi)$, while the second term is nonnegative. In addition, $l_{\ell-1}(\lambda)^* R l_{\ell-1}(\lambda) > 0$ for all $\lambda \in \mathbf{\Lambda}(\Phi,\Psi)$ on account of Assumption 2, and hence, (2.9) holds for all $\lambda \in \mathbf{\Lambda}(\Phi,\Psi)$ and $(x,u) \in \mathcal{N}_{A,B}(\lambda)$.

We prove that (i) implies (ii) by contradiction, using a theorem of alternatives [1, Theorem 1] [2, Theorem 1.3]: from the infeasibility certificate of (2.10) we construct a $\lambda \in \mathbf{\Lambda}(\Phi,\Psi)$ and $(x,u) \in \mathcal{N}_{A,B}(\lambda)$ that violates (2.9). Suppose (2.10) is infeasible. Then there exists a nonzero positive semidefinite matrix $Z \in \mathbb{H}^{n+m\ell}$ such that

$$\mathbf{tr}(G_\ell^*(R\otimes\Theta)G_\ell\, Z) \geq 0\,, \tag{2.18a}$$

$$\tilde{F}_\ell(\bar{\Phi}\otimes Z)\tilde{F}_\ell^* = 0\,, \tag{2.18b}$$

$$\tilde{F}_\ell(\bar{\Psi}\otimes Z)\tilde{F}_\ell^* \succeq 0\,. \tag{2.18c}$$

The matrix $\tilde{F}_\ell \in \mathbb{H}^{n\times(\ell+1)(n+m\ell)}$ is defined as

$$\tilde{F}_\ell = \begin{bmatrix} F_{\ell,1} & \cdots & F_{\ell,\ell+1} \end{bmatrix}\,,$$

where $F_{\ell,i} \in \mathbb{H}^{n\times(n+m\ell)}$, $i \in \mathbb{I}_{\ell+1}$, denote the block rows of $F_\ell$:

$$F_\ell = \begin{bmatrix} F_{\ell,1} \\ \vdots \\ F_{\ell,\ell+1} \end{bmatrix}\,.$$

Let $Z = \Upsilon\Upsilon^*$ with $\Upsilon \in \mathbb{C}^{(n+m\ell)\times r}$ be a full-rank factorization of $Z$, where $r$ is the rank. Equality (2.18b) and inequality (2.18c) can then be written as

$$\tilde{E}_\ell(\bar{\Phi}\otimes I_r)\tilde{E}_\ell^* = 0\,, \qquad \tilde{E}_\ell(\bar{\Psi}\otimes I_r)\tilde{E}_\ell^* \succeq 0\,, \tag{2.19}$$

where

$$\tilde{E}_\ell = \begin{bmatrix} F_{\ell,1}\Upsilon & \cdots & F_{\ell,\ell+1}\Upsilon \end{bmatrix}\,.$$

Substituting decomposition (2.3) of $\Phi$ and $\Psi$ in (2.19) transforms the equality and inequality to the form (2.15) with

$$\begin{bmatrix} X & Y \end{bmatrix} = \tilde{E}_\ell(T^\intercal \otimes I)\,, \quad \to \quad \begin{bmatrix} X \\ Y \end{bmatrix} = (T\otimes I)F_\ell\Upsilon\,.$$

By Lemma 2.4, matrices $X, Y$ admit a decomposition of the form (2.16), which yields

$$(T \otimes I)F_\ell \Upsilon = \begin{bmatrix} W \operatorname{diag}(s_k) \\ W \end{bmatrix} V^*,$$

for some $W \in \mathbb{C}^{n\times r}$, unitary $V \in \mathbb{C}^{r\times r}$, and $s_k \in \mathbf{\Lambda}(\Phi_\mathrm{o}, \Psi_\mathrm{o})$ with $k \in \mathbb{I}_r$. Let $w_k$ and $v_k$ denote the $k$th column of $W$ and $V$, respectively, then this equality corresponds to

$$\begin{bmatrix} s_k \\ 1 \end{bmatrix} \otimes w_k = (T \otimes I)F_\ell \Upsilon v_k, \quad \forall k \in \mathbb{I}_r.$$

Let $\lambda_{k,i}$ with $i \in \mathbb{I}_\ell$ be the distinct roots in $\mathcal{R}_T(s_k)$. Then by Lemma 2.3, there exist $(x_{k,i}, u_{k,i}) \in \mathcal{N}_{A,B}(\lambda_{k,i})$ such that

$$\Upsilon v_k = \sum_{i=1}^{\ell} \begin{bmatrix} x_{k,i} \\ l_{\ell-1}(\lambda_{k,i})^{\mathsf{F}} \otimes u_{k,i} \end{bmatrix} = \sum_{i=1}^{\ell} z_{k,i}.$$

Because $V$ is unitary, we have

$$Z = \Upsilon V V^* \Upsilon^* = \sum_{k=1}^{r} \left( \sum_{i=1}^{\ell} z_{k,i} \right) \left( \sum_{i=1}^{\ell} z_{k,i} \right)^*.$$

Substituting this decomposition into (2.18a) yields

$$\sum_{k=1}^{r} \sum_{i=1}^{\ell} \sum_{j=1}^{\ell} z_{k,i}^* \, G_\ell^* (R \otimes \Theta) G_\ell \, z_{k,j} \;\geq\; 0, \tag{2.20}$$

Elaborating $z_{k,i}^* G_\ell^* (R \otimes \Theta) G_\ell z_{k,i}$ using (2.17b) yields

$$(z_{k,i})^* G_\ell^* (R \otimes \Theta) G_\ell z_{k,j} = l_{\ell-1}(\lambda_{k,i})^* R l_{\ell-1}(\lambda_{k,j}) \cdot \begin{bmatrix} x_{k,i} \\ u_{k,i} \end{bmatrix}^* \Theta \begin{bmatrix} x_{k,j} \\ u_{k,j} \end{bmatrix},$$

On account of (2.5b), this contribution vanishes for $i \neq j$ such that (2.20) amounts to

$$\sum_{k=1}^{r} \sum_{i=1}^{\ell} l_{\ell-1}(\lambda_{k,i})^* R l_{\ell-1}(\lambda_{k,i}) \cdot \begin{bmatrix} x_{k,i} \\ u_{k,i} \end{bmatrix}^* \Theta \begin{bmatrix} x_{k,i} \\ u_{k,i} \end{bmatrix} \geq 0.$$

Combining this with (2.5a) we obtain that at least for one $(x_{k,i}, u_{k,i})$, (2.9) cannot hold and we reach a contradiction with statement (i). ◻

The extension of this proof to Theorem 2.2 is discussed in Appendix C.

**3. Union of Segments of a Circle or Straight Line.** As a particular application of our result, the union of $\ell$ nonintersecting, nonempty and nonsingleton segments of a circle or straight line in $\mathbb{C}$ admits a description of the form (2.1) with Hermitian matrices $\Phi, \Psi \in \mathbb{H}^{\ell+1}$ that satisfy Assumptions 1 and 2. In Section 3.1 we elaborate the construction of these $\Phi, \Psi$, and in Section 3.2 we apply our result to obtain a sum-of-squares certificate for the positivity of a univariate polynomial on a union of intervals. Section 3.3 compares our result to existing results from the literature.

**3.1. Description as $\mathbf{\Lambda}(\Phi,\Psi)$ with $\Phi,\Psi$ Satisfying Assumptions 1 and 2.** We first elaborate the construction of $\Phi,\Psi$ for the union of segments of the real axis. The extension to other lines and circles in $\mathbb{C}$ relies on a Möbius transform that maps the circle or line to the real axis [9], and is discussed at the end of this section. To clarify our elaborations, we define the matrices $\Phi_\mathrm{r},\Psi_\mathrm{r}$ as follows:

$$\Phi_\mathrm{r} = \begin{bmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{bmatrix}, \qquad \Psi_\mathrm{r} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \tag{3.1}$$

such that $\mathbf{\Lambda}(\Phi_\mathrm{r},\Psi_\mathrm{r}) = \mathbb{R}_+ \cup \{\infty\}$. In addition, for $\ell \in \mathbb{I}$, the matrix $J_\ell \in \mathbb{R}^{2\ell\times(\ell+1)}$ is defined as

$$J_\ell = \left[\begin{array}{cc} I_\ell & 0_{\ell,1} \\ \hline 0_{\ell,1} & I_\ell \end{array}\right]. \tag{3.2}$$

Lemma 3.1. *Let $2\ell$ scalars $\alpha_i,\beta_i \in \mathbb{R}\cup\{\infty\}$, $i \in \mathbb{I}_\ell$, be given that satisfy*

$$\alpha_1 < \beta_1 < \alpha_2 < \ldots < \beta_{\ell-1} < \alpha_\ell < \beta_\ell\,. \tag{3.3}$$

*Let the vectors $a,b \in \mathbb{R}^{\ell+1}$ be defined by*

$$\prod_{i=1}^{\ell}(\lambda-\alpha_i) = a^\intercal l_\ell(\lambda)\,, \qquad \prod_{i=1}^{\ell}(\lambda-\beta_i) = b^\intercal l_\ell(\lambda)\,, \tag{3.4}$$

*where $(\lambda-\alpha_1) = 1$ for $\alpha_1 = -\infty$ and $(\lambda-\beta_\ell) = -1$ for $\beta_\ell = \infty$, and set $\tilde{T} = \begin{bmatrix} -a & b \end{bmatrix}^\intercal$. Then, the matrices $\Phi,\Psi \in \mathbb{H}^{\ell+1}$ given by*

$$\Phi = \tilde{T}^*\Phi_\mathrm{r}\tilde{T}\,, \qquad \Psi = \tilde{T}^*\Psi_\mathrm{r}\tilde{T}\,, \tag{3.5}$$

*satisfy Assumptions 1 and 2, and*

$$\mathbf{\Lambda}(\Phi,\Psi) = \bigcup_{i=1}^{\ell}[\alpha_i,\beta_i]\,. \tag{3.6}$$

*In particular, a matrix $R \in \mathbb{S}^\ell$ satisfying (2.5) is given by the unique solution of*

$$\Phi = J_\ell^*(\Phi_\mathrm{r}\otimes R)J_\ell\,. \tag{3.7}$$

*Proof.* Decomposition (3.5) defines a mapping between $\kappa \in \mathbf{\Lambda}(\Phi_\mathrm{r},\Psi_\mathrm{r}) = \mathbb{R}_+\cup\{\infty\}$ and $\lambda \in \mathbf{\Lambda}(\Phi,\Psi)$, similar to (2.6), where every $\kappa$ is mapped onto the $\ell$ roots of

$$\begin{bmatrix} 1 & -\kappa \end{bmatrix} \tilde{T}\, l_\ell(\lambda) = -a^\intercal l_\ell(\lambda) - \kappa b^\intercal l_\ell(\lambda) = 0\,,$$

while for $\kappa = \infty$, this equation reads as $b^\intercal l_\ell(\lambda) = 0$. Hence, $\mathbf{\Lambda}(\Phi,\Psi)$ corresponds to the 180° root locus of

$$\frac{b^\intercal l_\ell(\lambda)}{a^\intercal l_\ell(\lambda)} = \frac{\prod_{i=1}^{\ell}(\lambda-\beta_i)}{\prod_{i=1}^{\ell}(\lambda-\alpha_i)}\,,$$

which indeed corresponds to the union of the intervals $[\alpha_i,\beta_i]$, $i \in \mathbb{I}_\ell$. Consequently, (3.6) holds. To show that the matrices $\Phi,\Psi$ given by (3.5) satisfy Assumption 1 we substitute the following common congruence transformation of $\Phi_\mathrm{r},\Psi_\mathrm{r}$:

$$\Phi_\mathrm{r} = T_\mathrm{r}^*\Phi_\mathrm{o}T_\mathrm{r}\,, \qquad \Psi_\mathrm{r} = T_\mathrm{r}^*\Psi_\mathrm{o}T_\mathrm{r}\,, \tag{3.8a}$$

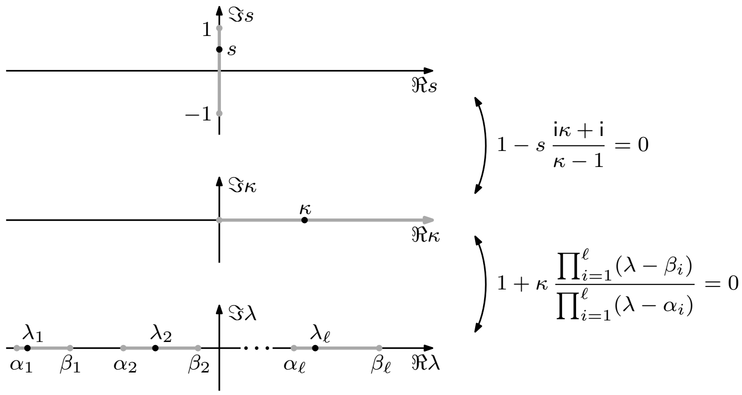


FIG. 3.1. *To compute the mapping between $s \in \mathbf{\Lambda}(\Phi_{\mathrm{o}}\Psi_{\mathrm{o}})$ and $\lambda \in \mathbf{\Lambda}(\Phi, \Psi)$, $s$ is first mapped onto $\kappa \in \mathbb{R}_+ \cup \{\infty\}$, which is subsequently mapped onto $\ell$ distinct roots $\lambda_i \in [\alpha_i, \beta_i]$, $i \in \mathbb{I}_\ell$.*

with

$$T_{\mathrm{r}} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ \mathrm{i} & \mathrm{i} \end{bmatrix}, \qquad \Psi_{\mathrm{o}} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{3.8b}$$

in (3.5) to obtain $\Phi, \Psi$ in the form (2.3) with $\Psi_{\mathrm{o}}$ given above and $T = T_{\mathrm{r}}\tilde{T}$. $T_{\mathrm{r}}$ defines a bijective mapping between $s \in \mathbf{\Lambda}(\Phi_{\mathrm{o}}, \Psi_{\mathrm{o}})$ and $\kappa \in \mathbf{\Lambda}(\Phi_{\mathrm{r}}, \Psi_{\mathrm{r}})$, and by the root locus argument above, every $\kappa \in \mathbf{\Lambda}(\Phi_{\mathrm{r}}, \Psi_{\mathrm{r}})$ is mapped onto $\ell$ roots $\lambda_i \in [\alpha_i, \beta_i]$, $i \in \mathbb{I}_\ell$. This composition of mappings is illustrated in Figure 3.1. As the intervals $[\alpha_i, \beta_i]$ are nonintersecting, the roots $\lambda_i$ are distinct.

Solving (3.7) for $R \in \mathbb{S}^\ell$ amounts to solving a set of linear equations in the $0.5\ell(\ell+1)$ entries of $R$. Since both $\Phi_{\mathrm{r}}$ and $\Phi$ are pure imaginary, the only nontrivial constraints relate to the off-diagonal entries of the imaginary part of $\Phi$. The corresponding $0.5\ell(\ell+1)$ linear constraints are readily verified to be linearly independent, and hence, the corresponding solution $R$ is unique. The proof that this $R$ satisfies (2.5) is presented in Appendix D $\square$

Next, we consider the union of $\ell$ nonempty, nonsingleton, nonintersecting segments $\mathbf{\Lambda}(\phi, \psi_i)$, $i \in \mathbb{I}_\ell$, on an arbitrary circle or line $\mathbf{\Lambda}(\phi, 0)$. To transfer the results of Lemma 3.1 to $\bigcup_{i=1}^{\ell} \mathbf{\Lambda}(\phi, \psi_i)$ we rely on a Möbius transform that maps $\mathbf{\Lambda}(\phi, 0)$ onto the extended real axis $\mathbf{\Lambda}(\Phi_{\mathrm{r}}, 0)$, and every segment $\mathbf{\Lambda}(\phi, \psi_i)$ onto an interval of the form $[\alpha_i\,,\,\beta_i]$ with $\beta_i > \alpha_i$. One way to compute such a Möbius transform is illustrated in Figure 3.2. For given distinct points $z_1, z_2, z_3 \in \mathbb{C} \cup \{\infty\}$, the Möbius transform

$$\mu(\lambda) = \frac{(\lambda - z_1)(z_2 - z_3)}{(\lambda - z_3)(z_2 - z_1)} \tag{3.9}$$

maps $\{z_1, z_2, z_3\}$ onto $\{0, 1, \infty\}$ and the circle or line through $z_1$, $z_2$ and $z_3$ onto the real axis [12]. Hence, with any set of distinct points $\{z_1, z_2, z_3\} \subset \mathbf{\Lambda}(\phi, 0)$, $z_3 \notin \bigcup_{i=1}^{\ell} \mathbf{\Lambda}(\phi, \psi_i)$, this transformation satisfies the requirements. Let $\hat{\Phi}, \hat{\Psi} \in \mathbb{H}^{\ell+1}$ and $\hat{R} \in \mathbb{H}^\ell$ be the result of Lemma 3.1 applied to the image of $\bigcup_{i=1}^{\ell} \mathbf{\Lambda}(\phi, \psi_i)$ under the Möbius transform (3.9). In addition, set

$$M = \begin{bmatrix} z_2 - z_3 & -z_1(z_2 - z_3) \\ z_2 - z_1 & -z_3(z_2 - z_1) \end{bmatrix} = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix},$$

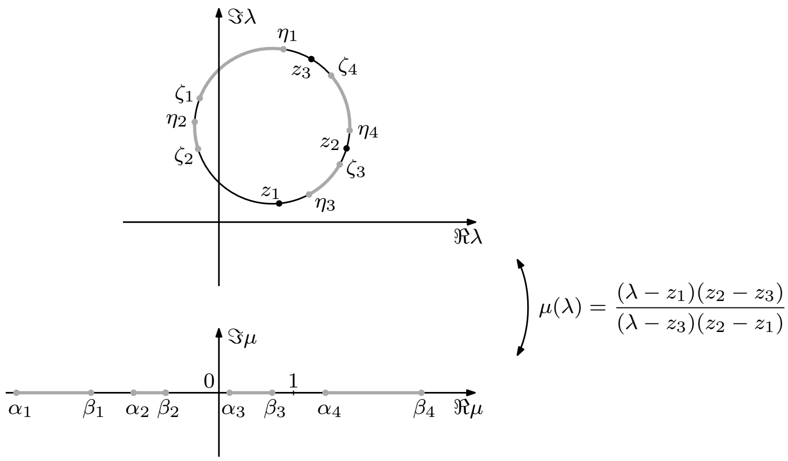


FIG. 3.2. *For $\{z_1, z_2, z_3\} \subset \mathbf{\Lambda}(\phi, 0)$, $z_3 \notin \bigcup_{i=1}^{\ell} \mathbf{\Lambda}(\phi, \psi_i)$, the Möbius transform $\mu(\lambda)$ maps $\mathbf{\Lambda}(\phi, 0)$ onto $\mathbf{\Lambda}(\Phi_{\mathrm{r}}, 0)$, and every segment $\mathbf{\Lambda}(\phi, \psi_i)$ onto an interval of the form $[\alpha_i\,,\, \beta_i]$ with $\alpha_i = \mu(\eta_i) < \beta_i = \mu(\zeta_i)$.*

and for $k \in \mathbb{I}$ we define the matrix $\mathbf{M}_k \in \mathbb{R}^{(k+1)\times(k+1)}$ as follows:

$$\mathbf{M}_k = \begin{bmatrix} \mathrm{conv}^k(M_1) \\ \mathrm{conv}(\mathrm{conv}^{k-1}(M_1), M_2) \\ \vdots \\ \mathrm{conv}^k(M_2) \end{bmatrix}.$$

Then, it is readily verified that $\Phi, \Psi \in \mathbb{H}^{\ell+1}$ and $R \in \mathbb{H}^{\ell}$ defined by

$$\Phi = \mathbf{M}_\ell^* \hat{\Phi} \mathbf{M}_\ell\,, \quad \Psi = \mathbf{M}_\ell^* \hat{\Psi} \mathbf{M}_\ell\,, \quad R = \mathbf{M}_{\ell-1}^* \hat{R} \mathbf{M}_{\ell-1}\,,$$

satisfy $\mathbf{\Lambda}(\Phi, \Psi) = \bigcup_{i=1}^{\ell} \mathbf{\Lambda}(\phi, \psi_i)$ and $\Phi = J_\ell^*(\phi \otimes R) J_\ell$. In addition, $R$ satisfies (2.5) and $\Phi, \Psi$ admit a decomposition of the form (2.3)

EXAMPLE 1 (Union of two continuous-time frequency intervals). *To clarify the elaborations above let us derive the matrices $\Phi$, $\Psi$ and $R$ corresponding to the union of two continuous-time frequency intervals:*

$$\mathbf{\Lambda}(\Phi, \Psi) = \{\lambda = i\omega \,:\, \omega \in [\alpha_1, \beta_1] \cup [\alpha_2, \beta_2]\}\,,$$

*with $-\infty < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \infty$. A Möbius transform $\mu(\lambda)$ that maps the imaginary axis to the real axis is given by*

$$\mu(\lambda) = -i\lambda \qquad \rightarrow \qquad M = \begin{bmatrix} -i & 0 \\ 0 & 1 \end{bmatrix},$$

*which is of the form (3.9) with $z_1 = 0$, $z_2 = i$ and $z_3 = \infty$. Combination of the results above yields*

$$\Phi = \tilde{T}^* \Phi_{\mathrm{r}} \tilde{T}\,, \quad \Psi = \tilde{T}^* \Psi_{\mathrm{r}} \tilde{T}\,, \quad \textit{with} \quad \tilde{T} = \begin{bmatrix} -1 & i\alpha_1 + i\alpha_2 & \alpha_1\alpha_2 \\ 1 & -i\beta_1 - i\beta_2 & -\beta_1\beta_2 \end{bmatrix}.$$

*The corresponding matrix $R$ equals*

$$R = M^* \begin{bmatrix} \beta_1 + \beta_2 - \alpha_1 - \alpha_2 & \alpha_1\alpha_2 - \beta_1\beta_2 \\ \alpha_1\alpha_2 - \beta_1\beta_2 & \beta_1\beta_2(\alpha_1 + \alpha_2) - \alpha_1\alpha_2(\beta_1 + \beta_2) \end{bmatrix} M\,,$$

*and can also be solved from*

$$\Phi = J_\ell^* \big( (M^* \Phi_{\mathrm{r}} M) \otimes R \big) J_\ell \,.$$

Example 2 (Union of two discrete-time frequency intervals). *Next, we consider the union of two discrete-time frequency intervals:*

$$\mathbf{\Lambda}(\Phi, \Psi) = \left\{ \lambda = e^{i\theta} \,:\, \theta \in [\eta_1, \zeta_1] \cup [\eta_2, \zeta_2] \right\} \,,$$

*with* $-\pi < \eta_1 < \zeta_1 < \eta_2 < \zeta_2 < \pi$*. A Möbius transform that maps the unit circle to the real axis is given by*

$$\mu(\lambda) = -i \frac{\lambda - 1}{\lambda + 1} \qquad \rightarrow \qquad M = \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix} \,, \tag{3.10}$$

*which is of the form (3.9) with* $z_1 = 1$*,* $z_2 = i$ *and* $z_3 = -1$*. The two frequency intervals are mapped into the real intervals* $[\alpha_i, \beta_i]$*, with* $\alpha_i = \mu(e^{i\eta_i})$ *and* $\beta_i = \mu(e^{i\zeta_i})$*,* $i \in \mathbb{I}_2$*. The corresponding matrices* $\Phi$ *and* $\Psi$ *can be reformulated as*

$$\Phi = \tilde{T}^* \begin{bmatrix} 0 & ic \\ -i\bar{c} & 0 \end{bmatrix} \tilde{T} \,, \qquad \Psi = \tilde{T}^* \begin{bmatrix} 0 & c \\ \bar{c} & 0 \end{bmatrix} \tilde{T} \,,$$

*with*

$$\tilde{T} = \begin{bmatrix} -1 & e^{i\eta_1} + e^{i\eta_2} & -e^{i\eta_1} e^{i\eta_2} \\ 1 & -e^{i\zeta_1} - e^{i\zeta_2} & e^{i\zeta_1} e^{i\zeta_2} \end{bmatrix} \,, \quad c = (1 + i\alpha_1)(1 + i\alpha_2)(1 - i\beta_1)(1 - i\beta_2) \,.$$

*The computation of the corresponding matrix* $R$ *is similar to the continuous-time case, yet with the matrix* $M$ *from (3.10) filled in.*

**3.2. Application to Univariate Polynomials.** To provide more insight in the LMI condition obtained from our generalized KYP lemma, we apply Theorem 2.2 to obtain a sum-of-squares certificate for the nonnegativity of a real univariate polynomial on a union of intervals. The ring of real polynomials in $\lambda \in \mathbb{R}$ is denoted by $\mathbb{R}[\lambda]$ and $\Sigma$ is its subset of sum-of-squares polynomials:

$$\Sigma = \left\{ \sum_i p_i^2 \,:\, p_i \in \mathbb{R}[\lambda] \right\} .$$

As exposed in e.g. [3, 17], $\Sigma_n$, the set of polynomials in $\Sigma$ of degree less than or equal to $2n$, is related to $\mathbb{S}^{n+1}$:

$$p \in \Sigma_n \quad \Leftrightarrow \quad \exists P \in \mathbb{S}^{n+1} \,:\, P \succeq 0 \,, \;\; p(\lambda) = l_n(\lambda)^{\intercal} P l_n(\lambda) \,. \tag{3.11}$$

For statement (i) of Theorem 2.2 to correspond to a polynomial inequality, we select $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times 1}$ as follows [7]:

$$\begin{bmatrix} A & B \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 & 0 \\ 0 & \ldots & \cdots & 0 & 1 \end{bmatrix} \,, \tag{3.12}$$

such that

$$\begin{bmatrix} (\lambda I - A)^{-1}B \\ I \end{bmatrix} = \frac{1}{\lambda^n}\, l_n(\lambda)^{\mathsf{F}} .$$

Let $\Phi, \Psi$ be constructed according to Lemma 3.1, then statement (i) of Theorem 2.2 is equivalent to

$$(l_n(\lambda)^{\mathsf{F}})^{\mathsf{T}}\, \Theta\, l_n(\lambda)^{\mathsf{F}} \leq 0\,, \qquad \forall \lambda \in \bigcup_{i=1}^{\ell} [\alpha_i\,,\, \beta_i]\,.$$

Hence, to relate this condition to the nonnegativity of a given polynomial $\theta \in \mathbb{R}[\lambda]$ of degree $2n$, we compute a matrix $\Theta \in \mathbb{S}^{n+1}$ such that

$$\theta(\lambda) = -\,(l_n(\lambda)^{\mathsf{F}})^{\mathsf{T}}\, \Theta\, l_n(\lambda)^{\mathsf{F}}\,. \tag{3.13}$$

The corresponding sum-of-squares certificate is obtained from statement (ii) of Theorem 2.2. Let us denote the left-hand side of (2.12) by $Z$. Then, with the help of (2.17), one readily verifies that

$$\begin{aligned}(l_{n+\ell-1}(\lambda)^{\mathsf{F}})^{\mathsf{T}} Z\, l_{n+\ell-1}(\lambda)^{\mathsf{F}} = l_\ell(\lambda)^{\mathsf{T}} \Psi\, l_\ell(\lambda)\, \cdot\, (l_{n-1}(\lambda)^{\mathsf{F}})^{\mathsf{T}}\, Q\, l_{n-1}(\lambda)^{\mathsf{F}} - \\ l_{\ell-1}(\lambda)^{\mathsf{T}} R\, l_{\ell-1}(\lambda)\, \cdot\, \theta(\lambda)\,.\end{aligned}$$

Using decomposition (3.5) of $\Psi$ and the equivalency (3.11), the result is interpreted as follows:

Corollary 3.2. *Let a polynomial $\theta \in \mathbb{R}[\lambda]$ of degree $2n$ be given, as well as $2\ell$ scalars $\alpha_i, \beta_i \in \mathbb{R}$, $i \in \mathbb{I}_\ell$, that satisfy (3.3). Let $R$ be the solution of (3.7). Then, the following statements are equivalent:*

*(i) $\theta(\lambda) \geq 0$ for all $\lambda \in \bigcup_{i=1}^{\ell}[\alpha_i, \beta_i]$.*

*(ii) There exist $q \in \Sigma_{n-1}$ and $z \in \Sigma_{n+\ell-1}$ such that*

$$r(\lambda)\theta(\lambda) = z(\lambda) + q(\lambda)\psi(\lambda)\,, \tag{3.14}$$

*where*

$$\begin{aligned}\psi(\lambda) &= l_\ell(\lambda)^{\mathsf{T}} \Psi\, l_\ell(\lambda) = -\prod_{i=1}^{\ell}(\lambda - \alpha_i)(\lambda - \beta_i)\,, \\ r(\lambda) &= l_{\ell-1}(\lambda)^{\mathsf{T}} R\, l_{\ell-1}(\lambda) = \sum_{i=1}^{\ell}(\beta_i - \alpha_i)\prod_{j\neq i}(\lambda - \beta_j)(\lambda - \alpha_j)\,.\end{aligned}$$

The elaboration of $r(\lambda)$ is derived in Appendix D. As Assumption 2 implies $R \succ 0$, $r$ is a positive sum-of-squares polynomial. Since $\Phi$, $A$, $B$, and $\Theta$ are real and $\Psi$ is pure imaginary, we may select $Q$ real and $P$ imaginary in (2.12) without loss of generality. To expose the role of the imaginary matrix $P$, we use (3.7) and definitions (2.7,2.8) to elaborate its contribution:

$$F_\ell^{\mathsf{T}}(\Phi \otimes P)F_\ell = G_\ell^{\mathsf{T}}\big(R \otimes (F_1^{\mathsf{T}}(\Phi_{\mathrm{r}} \otimes P)F_1)\big)G_\ell\,.$$

As noted in [7], with $A, B$ given by (3.12), $\Theta(P) = \Theta + F_1^{\mathsf{T}}(\Phi_{\mathrm{r}} \otimes P)F_1$ corresponds to an explicit parametrization of all matrices $\Theta$ that satisfy (3.13)

**3.3. Comparison of Computational Aspects to Existing Results.** The generalization of the KYP lemma presented in [9, 18] provides an LMI equivalent for an inequality of the form (2.9) on a segment of a circle or straight line in $\mathbb{C}$, i.e. a curve of the form $\mathbf{\Lambda}(\Phi, \Psi)$ for $\Phi, \Psi \in \mathbb{H}_2$ that satisfy Assumption 1. In case the inequality must hold on a union of $\ell$ segments, an LMI reformulation can be obtained by applying the result of [9, 18] for each of the intervals separately. This LMI involves $2\ell$ matrix variables in $\mathbb{H}_n$, and comprises $\ell$ inequalities in $\mathbb{H}_n$ and $\ell$ inequalities in $\mathbb{H}_{n+m}$. Theorem 2.1 provides an alternative LMI condition, involving only 2 matrix variables in $\mathbb{H}_n$ and comprising one inequality in $\mathbb{H}_n$, and one in $\mathbb{H}_{n+m\ell}$. Consequently, this LMI is generally smaller in both dimension and number of variables.

Condition (3.14) differs from current sum-of-squares certificates for the nonnegativity of a univariate polynomial on a union of intervals. To compare, let us recall the existing result. Let $V \subset \mathbb{R}[\lambda]$ be given as:

$$V = \{(\beta_\ell - \lambda)(\lambda - \alpha_1), (\lambda - \beta_1)(\lambda - \alpha_2), \ldots, \\ (\lambda - \beta_{\ell-1})(\lambda - \alpha_\ell)\},$$

and the corresponding preordering $T_V$:

$$T_V = \left\{ \sum_{e \in \{0,1\}^\ell} s_e(\lambda) \prod_{i=1}^{\ell} v_i(\lambda)^{e_i} \; : \; s_e(\lambda) \in \Sigma \right\},$$

where $v_i(\lambda)$ are the elements of $V$. Let $T_{V,n}$ be the subset of $T_V$ obtained by limiting the degree of the terms in the summation to $2n$. Then, the following result is proven in [4]:

LEMMA 3.3. *Let $V$ and $T_{V,n}$ be as defined above. Then, a polynomial $\theta(\lambda) \in \mathbb{R}[\lambda]$ of degree $2n$ is nonnegative on $\bigcup_{i=1}^{\ell} [\alpha_i, \beta_i]$ if and only if $\theta(\lambda) \in T_{V,n}$.*

For instance, for $\ell = 3$ and $n \geq 3$, this condition involves $s_{0,0,0} \in \Sigma_n$, $s_{1,0,0}$, $s_{0,1,0}$, $s_{0,0,1} \in \Sigma_{n-1}$, $s_{1,1,0}, s_{0,1,1}, s_{1,0,1} \in \Sigma_{n-2}$, and $s_{1,1,1} \in \Sigma_{n-3}$. On the other hand, Corollary 3.2 yields a sum-of-squares certificate that involves $z \in \Sigma_{n+\ell-1}$ and $q \in \Sigma_{n-1}$. Converting the cones of sum-of-squares polynomials to positive semidefinite cones according to (3.11), the LMI corresponding to (3.14) is generally smaller in both dimension and number of variables compared to the LMI representation of the result of Lemma 3.3. In [5], a counterpart of Lemma 3.3 for trigonometric polynomials nonnegative on nonintersecting segments of the unit circle is presented.

To illustrate the computational advantage of our result to the LMIs obtained from [9, 18] and [4, 5], we apply all three approaches to a finite impulse response filter design example adopted from [9]. Herein, the coefficients $h_i$, $i = 0, \ldots, n$ of the filter $H(z) = \sum_{i=0}^{n} h_i z^{-i}$ are computed by solving the following problem:

$$\text{(3.15a)} \qquad \text{minimize} \quad \varepsilon_\mathrm{p}$$

$$\text{(3.15b)} \qquad \text{subject to} \quad |H(e^{\mathrm{i}\theta})| \leq \varepsilon_\mathrm{s}, \qquad \forall \theta \in [-\theta_\mathrm{sl}, \theta_\mathrm{sl}] \cup [\theta_\mathrm{sh}, 2\pi - \theta_\mathrm{sh}]$$

$$\text{(3.15c)} \qquad |H(e^{\mathrm{i}\theta}) - e^{-\mathrm{i}d\theta}| \leq \varepsilon_\mathrm{p}, \qquad \forall \theta \in [\theta_\mathrm{pl}, \theta_\mathrm{ph}]$$

where we use $\varepsilon_\mathrm{s} = 5 \cdot 10^{-3}$, $\theta_\mathrm{sl} = 0.1\pi$, $\theta_\mathrm{sh} = 0.7\pi$, $\theta_\mathrm{pl} = 0.3\pi$, $\theta_\mathrm{ph} = 0.5\pi$, and delay $d = 15$. A filter of length $n = 30$ is designed. While all three approaches yield the same LMI equivalent to (3.15c), they differ on (3.15b). In all cases, the LMI variables can be selected real symmetric instead of complex Hermitian without introducing conservatism [15]. The resulting SDPs are parsed with YALMIP [13] and solved with

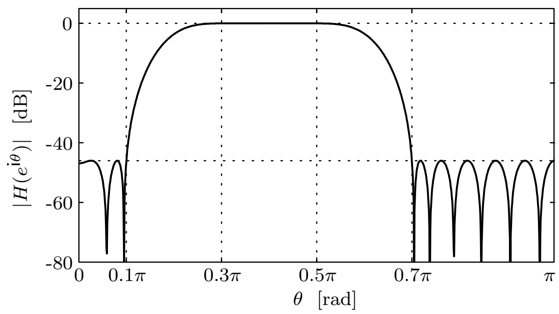


FIG. 3.3. *Solution of the optimal filter design example.*

SDPT3 [19] on a general purpose laptop (Intel® Core™ i7, 2.8 GHz, 8 GB of RAM). The optimal value is $\varepsilon_{\mathrm{p}} = 2.5 \cdot 10^{-4}$ and Figure 3.3 shows the amplitude response of the optimal filter. When applying [9, 18] to each of the intervals considered in (3.15b) separately, SDPT3 requires 22 CPU seconds to solve the resulting SDP. If [5] is used to convert (3.15b) to an LMI, the SDP is solved in 35 CPU seconds, while the SDP obtained from Theorem 2.2 is solved in 15 CPU seconds. Note that the three LMI equivalents to (3.15b) have very different structures, and we did not explore the computational benefits that can be obtained by exploiting these structures.

**4. Conclusion.** This paper generalizes the KYP lemma to particular curves in the complex plane that are characterized by a polynomial equality and inequality of degree largen than two. As a first application of our result we show that the considered set of curves includes the union of segments on a circle or line as a special case. When dealing with these special cases, the novel LMI condition is smaller in dimension and number of variables compared to existing results from the literature. Currently, we are investigating the potential of our result in cases where Assumption 2 does not hold, as well as its application to the frequency domain analysis of linear time-invariant systems with a generalized frequency variable. In the latter application, both assumptions are generally violated.

**Acknowledgement.** This work was supported in part by the Ministry of Education, Culture, Sports, Science and Technology in Japan through Grant-in-Aid for Scientific Research (A) No. 21246067. Tetsuya Iwasaki was supported in part by the National Science Foundation, No. 1068997. Goele Pipeleers is a Postdoctoral Fellow of the Research Foundation – Flanders (FWO–Vlaanderen). Her work benefits from KU Leuven–BOF PFV/10/002 Center-of-Excellence Optimization in Engineering (OPTEC), and the Belgian Network Dynamical Systems, Control and Optimization (DYSCO) initiated by the Belgian Science Policy Office.

**Appendix A. Proof of Lemma 2.3.**

Based on the mixed-product property of the matrix Kronecker product, Eq. (2.13) amounts to

$$\begin{aligned} \left(\begin{bmatrix}1 & -s\end{bmatrix} T \otimes I_n\right) F_\ell z = 0\,, \qquad & \text{for } s \neq \infty \\ \left(\begin{bmatrix}0 & 1\end{bmatrix} T \otimes I_n\right) F_\ell z = 0\,, \qquad & \text{for } s = \infty \end{aligned} \tag{A.1}$$

To verify that a vector $z$ of the form (2.14) satisfies (A.1) we elaborate $F_\ell z_i$, where $z_i$ denotes the $i$th term in (2.14), from bottom to top, which yields

$$F_\ell \begin{bmatrix} x_i \\ l_{\ell-1}(\lambda_i)^{\mathsf{F}} \otimes u_i \end{bmatrix} = l_\ell(\lambda_i) \otimes \chi(x_i, u_i)\,,$$

where

$$\chi(x_i, u_i) = \begin{cases} x_i\,, & \text{if } \lambda_i \neq \infty\,, \\ Bu_i\,, & \text{if } \lambda_i = \infty\,, \end{cases}\,.$$

As $\lambda_i$ are the $\ell$ roots of (2.4), each $z_i$ satisfies (A.1), and consequently, so does $z = \sum_{i=1}^{\ell} z_i$.

To show that (A.1) implies (2.14) we observe that the null space of $\begin{bmatrix}1 & -s\end{bmatrix} T$ for $s \neq \infty$ and $\begin{bmatrix}0 & 1\end{bmatrix} T$ for $s = \infty$ is spanned by $\{l_\ell(\lambda_1), \dots, l_\ell(\lambda_\ell)\}$ with $\lambda_i$, $i \in \mathbb{I}_\ell$, the $\ell$ distinct roots in $\mathcal{R}_T(s)$. Hence, (A.1) holds if and only if there exist $x_i \in \mathbb{C}^n$ such that

$$F_\ell z = \sum_{i=1}^{\ell} l_\ell(\lambda_i)\chi_i\,. \tag{A.2}$$

The vector $z$ is partitioned as follows

$$z = \begin{bmatrix} x^\intercal & u^\intercal & v_1^\intercal & \cdots & v_{\ell-1}^\intercal \end{bmatrix}^\intercal\,,$$

with $x \in \mathbb{C}^n$ and $u, v_1, \dots, v_{\ell-1} \in \mathbb{C}^m$. Let us first assume that $\infty \notin \mathcal{R}_T(s)$. Then, by elaborating (A.2) from bottom to top we obtain the following equalities

$$x = \sum_{i=1}^{\ell} \chi_i\,, \qquad \text{and}$$

$$\left(\begin{bmatrix} l_{\ell-1}(\lambda_1)^\mathsf{F} & \cdots & l_{\ell-1}(\lambda_\ell)^\mathsf{F} \end{bmatrix} \otimes I_n\right) \begin{bmatrix} (\lambda_1 I - A)\chi_1 \\ \vdots \\ (\lambda_\ell I - A)\chi_\ell \end{bmatrix} = \begin{bmatrix} Bu \\ Bv_1 \\ \vdots \\ Bv_{\ell-1} \end{bmatrix}\,.$$

As the Vandermonde matrix $\begin{bmatrix} l_{\ell-1}(\lambda_1)^\mathsf{F} & \cdots & l_{\ell-1}(\lambda_\ell)^\mathsf{F} \end{bmatrix}$ is invertible, the last equation implies that all $(\lambda_i I - A)\chi_i$ lie in the range of $B$. Consequently, one can decompose $u$ as follows

$$u = \sum_{i=1}^{\ell} u_i\,, \quad \text{where} \quad (\lambda_i I - A)\chi_i = Bu_i\,,$$

and hence, $(\chi_i, u_i) \in \mathcal{N}_{A,B}(\lambda_i)$. Similarly, $v_1$ can be written as

$$v_1 = \sum_{i=1}^{\ell} v_{1,i}\,, \quad \text{where} \quad \lambda_i B u_i = B v_{1,i}\,.$$

Since $B$ is assumed to have full column rank, this implies $v_{1,i} = \lambda_i u_i$. Continuing this way up to $v_{\ell-1}$ yields decomposition (2.14). Let us now consider the case $\infty \in \mathcal{R}_T(s)$, where we take $\lambda_\ell = \infty$. Then, the elaboration of (A.2) yields

$$x = \sum_{i=1}^{\ell-1} \chi_i\,, \qquad \text{and}$$

$$\left(\begin{bmatrix} l_{\ell-1}(\lambda_1)^\mathsf{F} & \cdots & l_{\ell-1}(\lambda_\ell)^\mathsf{F} \end{bmatrix} \otimes I_n\right) \begin{bmatrix} (\lambda_1 I - A)\chi_1 \\ \vdots \\ (\lambda_{\ell-1} I - A)\chi_{\ell-1} \\ \chi_\ell \end{bmatrix} = \begin{bmatrix} Bu \\ Bv_1 \\ \vdots \\ Bv_{\ell-1} \end{bmatrix}\,,$$

and similarly as above, we obtain

$$u = \sum_{i=1}^{\ell-1} u_i\,, \quad \text{where} \quad (\lambda_i I - A)\chi_i = B u_i\,,$$
$$v_1 = \sum_{i=1}^{\ell-1} \lambda_i u_i\,, \quad \dots \quad v_{\ell-2} = \sum_{i=1}^{\ell-1} \lambda_i^{\ell-2} u_i\,,$$
$$v_{\ell-1} = \sum_{i=1}^{\ell-1} \lambda_i^{\ell-1} u_i + u_\ell\,, \quad \text{where} \quad \chi_\ell = B u_\ell\,.$$

This way, decomposition (2.14) is retrieved.

**Appendix B. Proof of Lemma 2.4.**
Expanding (2.15) with $\Phi_\mathrm{o}, \Psi_\mathrm{o}$ of the form (2.3b) yields

$$XY^* + YX^* = 0\,, \qquad \alpha XX^* + \gamma YY^* \succeq 0\,,$$

and $s_i \in \mathbf{\Lambda}(\Phi_\mathrm{o}, \Psi_\mathrm{o})$ means $s_i + \bar{s}_i = 0$ and $\alpha|s_i|^2 + \gamma \geq 0$. We distinguish two cases.

1. $0 \leq \alpha \leq \gamma$. In this case the inequality in (2.15) is redundant. The equality is equivalent to $(X+Y)(X+Y)^* = (X-Y)(X-Y)^*$. Therefore $X+Y$ and $X-Y$ have the same left singular vectors and singular values, so they can be written as

$$X + Y = P\Sigma Q_1^*, \qquad X - Y = P\Sigma Q_2^*.$$

with a unitary matrix $P \in \mathbb{C}_{n\times n}$, diagonal $\Sigma \in \mathbb{R}_{n\times m}$, and unitary $Q_1, Q_2 \in \mathbb{C}_{m\times m}$. The matrix $Q_1 Q_2^*$ is unitary, and therefore has a Schur factorization $V \operatorname{diag}(\gamma) V^*$ with $|\gamma_i| = 1$, $i = 1, \dots, m$. Defining $W = YV$ and $s_i = (1+\gamma_i)/(1-\gamma_i)$ provides the factorization (2.16).
2. $\alpha < 0 < \gamma$. It follows from lemma 5 in [10] that $X$ can be written as $X = Y\Delta$, where $\Delta + \Delta^* = 0$, and $\alpha\Delta\Delta^* + \gamma I \preceq 0$. The equality $\Delta + \Delta^* = 0$ implies that $\Delta$ has a Schur factorization $\Delta = V \operatorname{diag}(s) V^*$ with imaginary eigenvalues $s_i$ (i.e., $s_i + \bar{s}_i = 0$). The inequality $\alpha\Delta\Delta^* + \gamma I \preceq 0$ implies that $\alpha|s_i|^2 + \gamma \leq 0$. Defining $W = YV$ gives (2.16).

**Appendix C. Proof of Theorem 2.2.**
To extend the proof of Theorem 2.1 presented in Section 2.3 to Theorem 2.2, we show that following LMIs are strong alternatives in case $(A, B)$ is controllable:

(i) There exist $P, Q \in \mathbb{H}^n$ that satisfy $Q \succeq 0$ and

$$F_\ell^* (\Phi \otimes P + \Psi \otimes Q)\, F_\ell + G_\ell^* (R \otimes \Theta)\, G_\ell \preceq 0\,.$$

(ii) There exists $Z \in \mathbb{H}^{n+m\ell}$ that satisfies $Z \succeq 0$ and

$$\begin{aligned} \mathbf{tr}(G_\ell^* (R \otimes \Theta) G_\ell\, Z) > 0\,, \\ \tilde{F}_\ell (\bar{\Phi} \otimes Z) \tilde{F}_\ell^* = 0\,, \\ \tilde{F}_\ell (\bar{\Psi} \otimes Z) \tilde{F}_\ell^* \succeq 0\,. \end{aligned}$$

We focus on the case where $\Psi$ is of the form (2.3) with $\alpha < 0 < \gamma$. In case $0 \leq \alpha \leq \gamma$, without loss of generality $Q = 0$ may be enforced in (i) and the last matrix inequality in (ii) may be omitted. After these modifications, the proof proceeds along the same lines as outlined below.

To meet Slater's constraint qualification, we show that the controllability of $(A, B)$ implies that there exists $Z \succ 0$ such that

$$\tilde{F}_\ell(\bar{\Phi} \otimes Z)\tilde{F}_\ell^* = 0\,, \qquad \tilde{F}_\ell(\bar{\Psi} \otimes Z)\tilde{F}_\ell^* \succ 0\,. \tag{C.1}$$

To this end, we construct $n+m\ell$ linearly independent vectors $z_i \in \mathbb{C}^{n+m\ell}$ of the form

$$z_i = \begin{bmatrix} x_i \\ l_{\ell-1}(\lambda_i)^{\mathsf{F}} \otimes u_i \end{bmatrix}, \quad \text{with} \quad \lambda_i \in \mathring{\mathbf{\Lambda}}(\Phi, \Psi)\,,\ (x_i, u_i) \in \mathcal{N}_{A,B}(\lambda_i)\,,$$

where

$$\mathring{\mathbf{\Lambda}}(\Phi, \Psi) = \{\lambda \in \mathbb{C} \,:\, l_\ell(\lambda)^* \Phi l_\ell(\lambda) = 0\,,\ l_\ell(\lambda)^* \Psi l_\ell(\lambda) > 0\}\ .$$

It is then readily verified that

$$Z = \sum_{i=1}^{n+m\ell} z_i z_i^*$$

satisfies $Z \succ 0$ and (C.1). To construct the first $n$ vectors $z_i$, $i \in \mathbb{I}_n$, we pick $n$ distinct values $\lambda_i \in \mathring{\mathbf{\Lambda}}(\Phi, \Psi)$ that correspond to $n$ distinct values $s_i \in \mathring{\mathbf{\Lambda}}(\Phi_{\mathrm{o}}, \Psi_{\mathrm{o}})$. Next, the matrix $K \in \mathbb{C}^{m\times n}$ is computed such that $A + BK$ has $\lambda_1, \dots, \lambda_n$ as eigenvalues. The existence of $K$ is guaranteed by the controllability of $(A, B)$. The vector $x_i$, $i \in \mathbb{I}_n$, is set equal to the eigenvector of $A + BK$ corresponding to $\lambda_i$, and $u_i = Kx_i$. This way, $(x_i, u_i) \in \mathcal{N}_{A,B}(\lambda_i)$ since

$$Ax_i + Bu_i = (A + BK)x_i = \lambda_i x_i\,,$$

and the corresponding vectors $z_i$, $i \in \mathbb{I}_n$, are linearly independent since $x_i$ are eigenvectors of $A+BK$ for distinct eigenvalues and hence, linearly independent. To construct the remaining $m\ell$ vectors $z_i$, $i \in n+\mathbb{I}_{m\ell}$, we pick $s_0 \in \mathring{\mathbf{\Lambda}}(\Phi_{\mathrm{o}}, \Psi_{\mathrm{o}})$ and $s_0 \notin \{s_1, \dots, s_n\}$. The corresponding roots in $\mathcal{R}_T(s_0)$ be denoted by $\lambda_{0,k}$, $k \in \mathbb{I}_\ell$. For ease of exposition, we assume $s_0 \neq \infty$ and $\infty \notin \mathcal{R}_T(s_0)$. Next, we compute the matrices $X_{0,k} \in \mathbb{C}^{n\times m}$ and $U_{0,k} \in \mathbb{C}^{m\times m}$ such that

$$\begin{bmatrix} A - \lambda_{0,k} I & B \end{bmatrix} \begin{bmatrix} X_{0,k} \\ U_{0,k} \end{bmatrix} = 0\,, \quad \text{and} \quad \operatorname{rank} U_{0,k} = m\,.$$

By the controllability of $(A, B)$, such $X_{0,k}, U_{0,k}$ exist and the columns of $\begin{bmatrix} X_{0,k}^\mathsf{T} & U_{0,k}^\mathsf{T} \end{bmatrix}^\mathsf{T}$ constitute a basis for $\mathcal{N}_{A,B}(\lambda_{0,k})$. The vectors $z_i$, $i \in n + \mathbb{I}_{m\ell}$ are then set as follows:

$$\begin{bmatrix} z_{n+1} & z_{n+2} & \cdots & z_{n+m\ell} \end{bmatrix} = \begin{bmatrix} X_{0,1} & X_{0,2} & \cdots & X_{0,\ell} \\ U_{0,1} & U_{0,2} & \cdots & U_{0,\ell} \\ \vdots & \vdots & & \vdots \\ \lambda_{0,1}^{\ell-1} U_{0,1} & \lambda_{0,2}^{\ell-1} U_{0,2} & \cdots & \lambda_{0,\ell}^{\ell-1} U_{0,\ell} \end{bmatrix}.$$

As the roots $\lambda_{0,k}$, $k \in \mathbb{I}_\ell$, are all distinct and matrices $U_{0,k}$ have full rank, the $m\ell\times m\ell$ bottom part of the right-hand side matrix has full rank, and consequently, the vectors $z_i$, $i \in n + \mathbb{I}_{m\ell}$ are linearly independent.

To complete the proof, we show that $S_1 \cap S_2 = \{0\}$, where

$$S_1 = \operatorname{span}\{z_1, \ldots, z_n\}\,, \qquad S_2 = \operatorname{span}\{z_{n+1}, \ldots, z_{n+m\ell}\}\,.$$

Suppose $v \in S_1 \cap S_2$. The containment $v \in S_1$ is equivalent to the existence of scalars $\alpha_i$ such that

$$v = \sum_{i=1}^{n} \alpha_i z_i\,.$$

Combining Lemma 2.3 with the property that the vectors $z_{n+(k-1)m+1}, \ldots, z_{n+km}$ constitute a basis for $\mathcal{N}_{A,B}(\lambda_{0,k})$ reveals that the vectors $z_i$, $i \in n + \mathbb{I}_{m\ell}$, constitute a basis for the nullspace of

$$\left(\begin{bmatrix} 1 & -s_0 \end{bmatrix} \otimes I_n\right)(T \otimes I_n)\, F_\ell\,.$$

Consequently, $v \in S_2$ is equivalent to

$$\left(\begin{bmatrix} 1 & -s_0 \end{bmatrix} \otimes I_n\right)(T \otimes I_n)\, F_\ell\, v = 0\,.$$

Elaborating the left-hand side using relations (2.17a) and (2.6) yields

$$\sum_{i=1}^{n} \alpha_i\, \frac{(s_i - s_0)}{t_2(\lambda_i)}\, x_i = 0\,.$$

Since $s_0 \neq s_i$ for all $i \in \mathbb{I}_n$ and the vectors $x_1, \ldots, x_n$ are linearly independent, this implies $\alpha_i = 0$ for all $i \in \mathbb{I}_n$, and consequently, $v = 0$.

**Appendix D. Proof of Lemma 3.1.**

To complete the proof of Lemma 3.1, we show that the matrix $R \in \mathbb{S}^\ell$ defined by (3.7) satisfies the conditions (2.5) of Assumption 2. First we prove (2.5a) by showing that

$$l_{\ell-1}(\lambda)^* R\, l_{\ell-1}(\lambda) > 0\,, \quad \forall \lambda \in \mathbb{R}\,. \tag{D.1}$$

From (3.7) and definition (3.2), we obtain

$$l_\ell(\lambda)^* \Phi l_\ell(\lambda) = l_1(\lambda)^* \Phi_{\mathrm{r}} l_1(\lambda) \cdot l_{\ell-1}(\lambda)^* R l_{\ell-1}(\lambda)\,.$$

Solving this equation for the second term of the right-hand side yields

$$l_{\ell-1}(\lambda)^* R l_{\ell-1}(\lambda) = \sum_{i=1}^{\ell} (\beta_i - \alpha_i) \prod_{j \neq i} (\lambda - \beta_j)(\lambda - \alpha_j)\,. \tag{D.2}$$

To prove (D.1) we show that for given $\alpha_i, \beta_i$, $i \in \mathbb{I}_\ell$, that satisfy (3.3)

$$f_k(\lambda) = \sum_{i=1}^{k} (\beta_i - \alpha_i) \prod_{j \neq i} (\lambda - \beta_j)(\lambda - \alpha_j) \;>\; 0\,,\ \forall \lambda \in \mathbb{R}. \tag{D.3}$$

holds for all $k \in \mathbb{I}_\ell$. We do this by induction on $k$. For $k = 1$, $f_1(\lambda) = (\beta_1 - \alpha_1) > 0$ on account of the relations (3.3). Next, we assume that (D.3) holds for a given $k < \ell$ and show that it then also holds for $k+1$. To this end, we elaborate $f_{k+1}(\lambda)$ as follows:

$$\begin{aligned} f_{\ell+1}(\lambda) &= (\lambda-\alpha_{\ell+1})(\lambda-\beta_{\ell+1})f_\ell(\lambda) \\ &\qquad + (\beta_{\ell+1}-\alpha_{\ell+1})\prod_{j=1}^{\ell}(\lambda-\beta_j)(\lambda-\alpha_j)\,, \qquad \text{(D.4a)} \\ &= -(\lambda-\alpha_1)(\lambda-\beta_{\ell+1})\tilde f_\ell(\lambda) \\ &\qquad + (\beta_{\ell+1}-\alpha_1)\prod_{j=1}^{\ell}(\lambda-\tilde\beta_j)(\lambda-\tilde\alpha_j)\,, \qquad \text{(D.4b)} \end{aligned}$$

where $\tilde\alpha_j = \beta_j$ and $\tilde\beta_j = \alpha_{j+1}$, $j \in \mathbb{I}_\ell$, and

$$\tilde f_k(\lambda) = \sum_{i=1}^{k}(\tilde\beta_i - \tilde\alpha_i)\prod_{j\neq i}(\lambda-\tilde\beta_j)(\lambda-\tilde\alpha_j)\,.$$

On account of the induction hypothesis, both $f_k(\lambda)$ and $\tilde f_k(\lambda)$ are positive for all $\lambda \in \mathbb{R}$. From decomposition (D.4b) we readily obtain that $f_{k+1}(\lambda) > 0$ for $\lambda \in \bigcup_{i=1}^{k+1}[\alpha_i,\beta_i]$, as for these $\lambda$ both terms in (D.4b) are nonnegative and at least one of them is positive. On the other hand, $f_{\ell+1}(\lambda) > 0$ also holds for $\lambda \in \mathbb{R}\setminus\bigcup_{i=1}^{k+1}[\alpha_i,\beta_i]$ as for these $\lambda$ both terms in (D.4a) are positive. This concludes the inductive step. Hence, (D.1) holds and condition (2.5a) of Assumption 2 is satisfied.

To show (2.5b) we consider arbitrary $\lambda_i, \lambda_j \in \mathcal{R}_T(s)$, $i, j \in \mathbb{I}_\ell$, $i \neq j$, for some arbitrary $s \in \mathbf{\Lambda}(\Phi_\mathrm{o}, \Psi_\mathrm{o})$. From the root-locus argument in the proof of Lemma 3.1 we note that $\lambda_i \in [\alpha_i, \beta_i]$ and $\lambda_j \in [\alpha_j, \beta_j]$, and hence $\lambda_i \neq \lambda_j$. Using the decomposition $\Phi = T^*\Phi_\mathrm{o}T$ and result (2.6), we obtain

$$l_\ell(\lambda_i)^*\Phi\, l_\ell(\lambda_j) = t_2(\lambda_i)^* t_2(\lambda_j)\cdot l_1(s)^*\Phi_\mathrm{o} l_1(s) = 0\,,$$

for $s \neq \infty$, while for $s = \infty$

$$l_\ell(\lambda_i)^*\Phi\, l_\ell(\lambda_j) = t_1(\lambda_i)^* t_1(\lambda_i)\cdot l_1(s)^*\Phi_\mathrm{o} l_1(s) = 0\,.$$

Elaborating the left-hand side using (3.7) yields

$$\begin{aligned} l_\ell(\lambda_i)^*\Phi\, l_\ell(\lambda_j) &= l_1(\lambda_i)^*\Phi_\mathrm{r} l_1(\lambda_j)\cdot l_{\ell-1}(\lambda_i)^* R\, l_{\ell-1}(\lambda_j)\,, \\ &= \mathrm{i}(\lambda_i-\lambda_j)\cdot l_{\ell-1}(\lambda_i)^* R\, l_{\ell-1}(\lambda_j)\,. \end{aligned}$$

As this must equal 0 while $\lambda_i \neq \lambda_j$, (2.5b) holds.

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