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Author

Large, David B.

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University of California

Ernest O. Lawrence Radiation Laboratory

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David B. Large

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David B. Large

Lawrence Radiation Laboratory
University of California
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The proposed problem is to design a resonant section of TEM transmission line such that a high-voltage point corresponds to an open-circuit end, and such that resonance occurs at all integer multiples of some fundamental frequency. This communication is a discussion of the theoretical solution to the problem.

Let the series impedance per unit length and the shunt admittance per unit length of a lossless TEM transmission system be given by $Z(x) = j\omega L(x)$ and $Y(x) = j\omega C(x)$ respectively. The independent variable, x , refers to the distance measured along the line from some reference $x = 0$, and $L(x)$ and $C(x)$ are the distributed inductance and capacitance of the system and are allowed to change with distance.

The voltage along the line, $V(x)$, is assumed to satisfy the wave equation for nonuniform transmission lines,

$$V''(x) - \frac{Z'(x)}{Z(x)} V'(x) - Y(x) Z(x) V(x) = 0,$$

where the "primed" quantities are differentiated with respect to x . [In most cases, the wave will not be purely TEM (as in a coaxial line with changing inner conductor), and ambiguity can arise in the definition of "V" and "x."] Putting $Z(x) = j\omega L(x)$ and $Y(x) = j\omega C(x)$ into (1) yields

$$V''(x) - \frac{L'(x)}{L(x)} V'(x) + \omega^2 L(x) C(x) V(x) = 0. \quad (1)$$

Stated mathematically, our problem is: Determine two piecewise continuous functions $L(x)$ and $C(x)$ such that Eq. (1) has an infinite, discrete set of solutions $V_n(x)$ corresponding to an infinite, discrete set of "eigenfrequencies"

$\omega_n = n\omega_0$; $n = 1, 2, 3, \dots$; ω_0 a constant, with the boundary conditions

$$V_n(l) = 0; \quad V_n'(0) = 0.$$

Physically, the above statement asks for a length of transmission line closed by an electrical short circuit at $x = l$ and open-circuited at $x = 0$ such that the section resonates at the radian frequencies $\omega_0, 2\omega_0, 3\omega_0$, etc. (Note that a uniform line "shorted" at one end and "open" at the other resonates at some fundamental and all odd multiples thereof.)

Most of the elementary mathematical literature that discusses the "spectral distribution" of solutions to linear second-order differential equations considers equations of the simplified "Sturm-Liouville type", equations of the form

$$y''(x) + [\lambda - q(x)] y(x) = 0, \quad (2)$$

with some specified boundary conditions on y .

In order to use the large amount of theory already applied to equations of the form (2), we began our analysis by transforming Eq. (1) into one of the form (2). The required transformation is a standard one:

$$V(x) \equiv z(x) \exp \left[\frac{1}{2} \int_0^x \frac{L'(\xi)}{L(\xi)} d\xi \right]. \quad (3)$$

Then Eq. (1) becomes

$$z''(x) + z(x) \left\{ \omega^2 L(x) C(x) + \frac{1}{2} \left[\frac{L'(x)}{L(x)} \right]^2 - \frac{1}{4} \left[\frac{L'(x)}{L(x)} \right]^2 \right\} = 0. \quad (4)$$

We now wish to consider solutions of Eq. (4) that generate a voltage function of the desired character. Obviously, Eq. (4) is difficult to solve for general forms of

$L(x)$ and $C(x)$. Some physical intuition can be employed, however, which greatly simplifies the problem. It is known that for most geometrical configurations of metal and dielectric materials, the inductance and capacitance values vary in opposite manner with the geometrical parameters of the system. For example, in all uniform TEM transmission systems the product LC is a constant dependent only upon the μ and ϵ of the medium. Therefore we make the tentative assumption that a solution exists for which $L(x) C(x) = L_0 C_0$, a constant independent of x . Equation (4) then becomes

$$z''(x) + z(x) \left\{ \omega^2 L_0 C_0 + \frac{1}{2} \left[\frac{L'(x)}{L(x)} \right]^2 - \frac{1}{4} \left[\frac{L''(x)}{L(x)} \right]^2 \right\} = 0.$$

Now note that if $L(x)$ is chosen so that $\frac{1}{2} \left[\frac{L'(x)}{L(x)} \right]^2 - \frac{1}{4} \left[\frac{L''(x)}{L(x)} \right]^2 = 0$, a simple solution for $z(x)$ is obvious:

$$z(x) = A \sin(\omega \sqrt{L_0 C_0} x) + B \cos(\omega \sqrt{L_0 C_0} x),$$

where A and B are arbitrary constants. Also, we see that if $B = 0$, then forcing $z(x)$ to satisfy the condition $z(l) = 0$ implies $V(l) = 0$ (assuming

$\int_0^l \frac{L'(\xi)}{L(\xi)} d\xi$ is well defined) and further yields the correct form for the spectrum,

$$\sin(\omega \sqrt{L_0 C_0} l) = 0 \Rightarrow \omega = 0, \frac{\pi}{l \sqrt{L_0 C_0}}, \frac{2\pi}{l \sqrt{L_0 C_0}}, \dots$$

The critical question at this point is: Will this particular $L(x)$ function exist, and if so, will it cause the voltage function $V(x)$ to satisfy $V'(0) = 0$?

To answer this question, we need to solve the following equation for $L(x)$:

$$\frac{1}{2} \left[\frac{L'(x)}{L(x)} \right]^2 - \frac{1}{4} \left[\frac{L''(x)}{L(x)} \right]^2 = 0, \quad (5)$$

A trial-and-error method applied to (5) shows that all solutions $L(x)$ are of the form

$$L(x) = \frac{L_0}{(x+c)^2}; \quad L_0 \text{ and } c \text{ are constants.}$$

[Since Eq. (5) is nonlinear, there exists an infinity of linearly independent solutions.]

It now remains to determine what value, if any, can be found for the parameter c such that $V(x)$ will satisfy the boundary condition at the origin.

Setting $L(\xi) = L_0 (\xi + c)^{-2}$ and $\omega = \frac{n\pi}{l\sqrt{L_0 C_0}}$ yields $z(x) = \sin(\frac{n\pi x}{l})$, and Eq. (3) gives

$$V(x) = \sin(\frac{n\pi x}{l}) \exp \left[- \int_0^x \frac{d\xi}{(\xi+c)} \right] = \frac{\sin(\frac{n\pi x}{l})}{(x+c)}.$$

Hence

$$V'(x) = \frac{\frac{n\pi}{l}(x+c) \cos(\frac{n\pi x}{l}) - \sin(\frac{n\pi x}{l})}{(x+c)^2}.$$

Now, for $c \neq 0$, we have $V'(0) = \frac{n\pi}{cl} \neq 0$. Therefore, the only possible c that may work is $c = 0$:

$$V(x) \Big|_{c=0} = \frac{\sin(\frac{n\pi x}{l})}{x};$$

and application of the usual limit theorems shows that

$$\lim_{x \rightarrow 0} V'(x) = 0 \quad \text{for} \quad V(x) = \frac{\sin(\frac{n\pi x}{l})}{x}.$$

Hence the solution has come quite easily, and no real synthesis was needed from the mathematical viewpoint. The results are summarized in the following paragraph.

The wave equation for general TEM systems, with $L(x) = \frac{L_0}{x^2}$ and $C(x) = C_0 x^2$, becomes

$$V''(x) + (2/x) V'(x) + \omega^2 L_0 C_0 V(x) = 0 .$$

The solutions of this equation subject to the boundary conditions $V(l) = 0$ and $V(0) = 0$ are of familiar form,

$$V_n(x) = \frac{\sin(\frac{n\pi x}{l})}{x} .$$

and the corresponding resonant frequencies are

$$f_n = \frac{\omega n}{2\pi} = \frac{n}{2l \sqrt{L_0 C_0}} .$$

We therefore may design a section of resonant line whose frequency spectrum corresponds to that of a half-wavelength resonator but whose physical appearance is that of a quarter-wavelength resonator. Applications of this result will be discussed in later correspondence.

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