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Structural Properties of Equivariant Spectra with Incomplete Transfers

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ABSTRACT OF THE DISSERTATION

Structural Properties of Equivariant Spectra with Incomplete Transfers

by

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Blumberg and Hill defined categories of equivariant spectra interpolating between the equivariant stable categories indexed by universes. Indexed by an N_∞ operad $\mathcal{O}$, these $\mathcal{O}$-spectra are characterized by what transfers they admit between their fixed points. We study structural properties of the $\mathcal{O}$-incomplete equvariant stable categories. We first show an analog of a theorem of Guillou and May, giving an equivalence between $\mathcal{O}$-spectra and spectrally enriched presheaves on a spectral enhancement of the incomplete Burnside category. Using this, we define the smash product and geometric fixed points of $\mathcal{O}$-spectra in terms of Kan extensions and show that $\mathcal{O}$-spectra can be recovered from gluing diagrams between their geometric fixed points. Finally, we apply this to give Mayer-Vietoris sequences describing the Picard group of $\mathcal{O}$-spectra. In the presence of an appropriate Segal conjecture, we compare this to the Picard group of invertible abelian Mackey functors, giving a partial generalization of a theorem by Fausk, Lewis, and May.

The dissertation of Andrew Smith is approved.

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CHAPTER 1

Introduction

The explosion of results in equivariant stable homotopy theory in the years since [HHR16] has largely been driven by the study of norm maps in equivariant ring spectra. These multiplicative maps between different fixed points can give precise control over their behavior, when they can be constructed. But not every ring spectrum admits all norms, and it is not always straightforward to classify which norms a given ring spectrum may have. For instance, Carrick showed that even if we know the norms a ring spectrum admits, it may have localizations with more interesting norms [Car19].

Equivariant operads were introduced by Blumberg and Hill to package the homotopy theoretic data of these norm maps [BH15]. Interestingly, these N_∞ operads are a natural generalization of the nonequivariant E_∞ operads whose actions characterize spectra via the recognition principle [May89]. Essentially, where the action of a nonequivariant E_∞ operad provides a homotopy coherent commutative addition, an the action of an equivariant N_∞ operad also provides a "twisted" addition indexed by certain finite G-sets, for G a fixed finite group.

In particular, the little disks operad on any G-universe $\mathcal{U}$ is N_∞, and enjoys a similar recognition principle characterizing G-spectra indexed on $\mathcal{U}$ [GM17]. For a general N_∞ operad $\mathcal{O}$, Blumberg and Hill construct a stable category $\mathbf{Sp}^G_{\mathcal{O}}$ defined by such a recognition principle [BH19]. This provides a finer interpolation between the so-called "naive" and "genuine" notions of equivariant spectra.

These $\mathcal{O}$-spectra enjoy properties that one should expect of a category of G-spectra: for example, they admit a tom Dieck splitting describing the fixed points of suspension spectra. Unfortunately, other constructions are more elusive. In particular, they do not have an accessible construction of geometric fixed points, or even the smash product of $\mathcal{O}$-spectra.

Recently, Guillou and May showed that that genuine G-spectra can be identified, up to a zig-zag of Quillen equivalences, with a suitable spectral enhancement of Mackey functors [GM11]. Our first theorem, Theorem 4.13, is an analog to a slightly stronger version of the Guillou-May theorem in the context of N_∞ spectra. Under this equivalence, we can view spectral Mackey functors as another model for $\mathcal{O}$-spectra. This model admits an easily described smash product, so we view it as a better "category of structured $\mathcal{O}$-spectra".

In Chapter 5, we explore the construction of geometric fixed points of Mackey functors, along with their role in isotropy separation. We show (Proposition 5.7) that this can be written in terms of enriched Kan extensions along certain functors between Burnside categories. Using this, we express isotropy separation as a recollement (Theorem 5.12) decomposing the category of Mackey functors into pieces coming from any downward-closed family of subgroups and its upward-closed complement. This allows us to recover $\mathcal{O}$-spectra from their geometric fixed points and certain gluing data between them.

The gluing data is controlled by functors between geometric fixed points called *Tate functors*. Such functors are, in general, difficult to compute. They are described in the genuine case via completion conjectures which are not known to be true for arbitrary G-spectra [GM95]. In particular, the celebrated Segal conjecture [Car84], whose proof motivated the foundations of genuine equivariant stable homotopy theory in the 1980s [Ada84], can be expressed as giving the value of the Tate functor on spheres as a completion of G-spectra at an ideal in the Burnside ring [AHJ88a]. In Chapter 6, we construct cohomological completions of $\mathcal{O}$-spectra and use it to state a generalization of the Segal conjecture in this context.

In Chapter 7, we show how to use the recollement of Theorem 5.12 to give Mayer-Vietoris sequences computing the Picard groups of invertible $\mathcal{O}$-spectra (Corollary 7.4, Corollary 7.6). Although there is no such recollement for abelian Mackey functors, we give a similar Mayer-

Vietoris sequence for these (Proposition 7.12). Assuming the Segal conjecture, we compare these sequences to show that the subgroup of locally trivial invertible $\mathcal{O}$-spectra embeds into the group of invertible Mackey functors (Theorem 7.15). We view this as a partial generalization of [FLM01, Theorem 0.1].

Finally, we apply these techniques in Chapter 8 to compute the locally trivial Picard group for any operad on a cyclic group of odd prime power order. This has some interesting implications even in the genuine case: it shows that most of the invertible G-spectra corresponding to invertible modules over the Burnside ring have nontrivial Weyl actions on their geometric fixed points (Remark 8.4).

CHAPTER 2

Preliminaries

Throughout the rest of this thesis, we will fix G to be a finite group.

Convention 2.1. In general, we will denote categories with calligraphy (e.g. $\mathcal{A}$,$\mathcal{C}$), 2-categories of any flavor with script ($\mathscr{A}$,$\mathscr{C}$), quasicategories in normal typeface (A, C), and enriched categories with boldface ($\mathbf{A}$, $\mathbf{C}$), at least when the enriching category is not usually though of as "sets with extra structure"; in particular, we will use boldface for spectrally enriched categories.

Let us fix a number of standard named categories:

1. $\mathcal{S}et$ is the category of sets, while $\mathcal{S}et_*$ is pointed sets. Similarly $\mathcal{F}in\mathcal{S}et$ is finite sets and $\mathcal{F}in\mathcal{S}et_*$ is finite pointed sets.

2. $\mathcal{A}b$ is abelian groups.

3. $\mathcal{T}op$ is compactly generated weak Hausdorff topological spaces, and $\mathcal{T}op_*$ is pointed CGWH spaces. These will be used as cofibrantly generated model categories in the standard way [Qui67, II.3]: generating cofibrations are the inclusions

$$S^{n-1} \hookrightarrow D^n$$

boundaries of disks, and generating acyclic cofibrations are the inclusions

$$D^n \times \{0\} \hookrightarrow D^n \times [0, 1].$$

4. $\mathcal{B}G$ is the one-object category with elements of G as automorphisms. $\mathcal{T}op^{BG}$, $\mathcal{S}et^{BG}$ etc. are the categories of G-objects, but to make this line up with our other equivariant

categories, which are categories of presheaves, we'll identify this with e.g. $[\mathcal{B}G^{\mathrm{op}}, \mathcal{T}op]$ along the usual isomorphism $G \cong G^{\mathrm{op}}$. Usually, we will drop the B and refer to these categories as e.g. $\mathcal{S}et^G$. However, when we write $\mathcal{T}op^G$ we will mean the category of coefficient systems of spaces instead, or equivalently $\mathcal{T}op^{BG}$ in a different model structure. This will be expounded upon in Section 3.1.

5. Inside the category $\mathcal{F}in\mathcal{S}et^G$, the orbit category $\mathcal{O}rb^G$ is the full subcategory on the objects G/H for $H < G$. More generally, for any class $\mathcal{H}$ of subgroups of G, $\mathcal{O}rb^{\mathcal{H}}$ is the full subcategory on the "$\mathcal{H}$-free" orbits, G/H with $H \in \mathcal{H}$. $\mathcal{S}et^{\mathcal{H}}$ and $\mathcal{F}in\mathcal{S}et^{\mathcal{H}}$ are the closure of $\mathcal{O}rb^{\mathcal{H}}$ under arbitrary and finite coproducts, respectively: those (finite) G-sets with every stabilizer conjugate to a subgroup in $\mathcal{H}$. (In general, we will require that $\mathcal{H}$ be closed under conjugates for this reason.) Presheaves on $\mathcal{O}rb^G$ are called *coefficient systems.*

Remark 2.2. Most of the results of this thesis would go through without change with $\mathcal{O}rb^G$ replaced by any other small category that has the property of being *orbital*: its free cocompletion under finite coproducts admits binary pullbacks which distribute over coproducts. (The finite coproduct cocompletion of $\mathcal{O}rb^G$ is, of course, $\mathcal{F}in\mathcal{S}et^G$.) In making this generalization, one would pass from the domain of equivariant homotopy theory to parametrized homotopy theory in the sense of [Jam69]. This approach is pursued in the project of Barwick, Dotto, Glasman, Nardin, and Shah laid out in [BDG16], with applications in functor calculus [Gla17] and factorization homology [AMR17]. For the sake of brevity, we will restrict our attention to the orbit category, or full subcategories $\mathcal{O}rb^{\mathcal{H}}$ as needed.

2.1 Categories Enriched in Multicategories

Let us recall the basic constructions of enriched categories, enriched in a monoidal category or more generally a multicategory. A more thorough account can be found in [Lei04], or discussions closer to how we will be using these objects in [EM06] or [BO15].

Definition 2.3 ([Lam72])**.** A *multicategory* $\mathcal{M}$ consists of

1. a collection of objects $\operatorname{ob}\mathcal{M}$,

2. for each finite list of objects $a_1, \ldots, a_k, b$ a collection $\mathcal{M}(a_1, \ldots, a_k; b)$, thought of as k-ary morphisms from the tuple $(a_1, \ldots, a_k)$ to the object b,

3. composition maps

$$\begin{gathered}\mathcal{M}(b_1, \ldots, b_n; c) \times \mathcal{M}(a^1_1, \ldots, a^1_{k_1}; b_1) \times \cdots \times \mathcal{M}(a^n_1, \ldots, a^n_{k_n}; b_n) \\ \downarrow \\ \mathcal{M}(a^1_1, \ldots, a^1_{k_1}, \ldots, a^n_1, \ldots, a^n_{k_n}; c)\end{gathered}$$

   which are usually written as sending $g, f_1, \ldots, f_n$ to $g \circ (f_1, \ldots, f_n)$, and

4. for each object a a distinguished morphism $\mathrm{id}_a \in \mathcal{M}(a; a)$, such that

5. diagrams encoding associativity and unitality of composition commute. Since we will not verify these diagrams, we refer to their specification in [Lei04, 2.1.1].

A *symmetric multicategory* further has, for each permutation $\sigma \in \Sigma_k$ and objects $a_1, \ldots, a_k, b$ a map

$$\sigma^* : \mathcal{M}(a_1, \ldots, a_k; b) \to \mathcal{M}(a_{\sigma(1)}, \ldots, a_{\sigma(k)}; b)$$

which collect into a right action of Σ_k on the collection of all k-ary morphisms of $\mathcal{M}$; the composition maps above then must be $(\Sigma_n \wr (\Sigma_{k_1}, \ldots, \Sigma_{k_n}))$-equivariant [EM09, 2.1].

Example 2.4. Most of the multicategories we encounter come from monoidal categories: if $\mathcal{M}$ is a monoidal category under monoidal product $\otimes$, it can be regarded as a multicategory with k-ary morphisms $\mathcal{M}(a_1, \ldots, a_k; b)$ given by the morphisms $a_1 \otimes \cdots \otimes a_k \to b$. If $\mathcal{M}$ admits a symmetric monoidal structure, then precomposition with the braidings makes it a symmetric multicategory. Outside of Chapter 4 these will be the only multicategories we will deal with.

Example 2.5. The multicategory $\mathcal{P}erm$ has as objects *permutative categories*: symmetric monoidal categories which are strict in the sense that the associators and unitors are identities, though the braidings need not be. k-ary morphisms $(\mathcal{C}_1, \dots, \mathcal{C}_k) \to \mathcal{D}$ are *k-linear functors*. These consist of a functor

$$F : \mathcal{C}_1 \times \cdots \times \mathcal{C}_k \to \mathcal{D}$$

along with natural transformations

$$\delta_i : F(c_1, \dots, c_i, \dots, c_k) \otimes F(c_1, \dots, c_i', \dots, c_k) \to F(c_1, \dots, c_i \otimes c_i', \dots, c_k)$$

expressing the "lax linearity" of F in each variable separately. These δ_i must be identities when c_i or c_i' is the monoidal unit, and satisfy associativity diagrams found in [EM06, 3.2]. $\mathcal{P}erm$ is a symmetric multicategory, with the Σ_k-actions simply permuting the inputs of each k-linear functor.

Definition 2.6. Let $\mathcal{M}$ be a multicategory. An *$\mathcal{M}$-enriched category* $\mathbf{C}$ consists of

1. a collection of objects $\operatorname{ob} \mathbf{C}$,
2. for each pair of objects a, b a morphism object $\mathbf{C}(a, b) \in \operatorname{ob} \mathcal{M}$,
3. for each object a a 0-ary identity $\mathrm{id}_a \in \mathcal{M}(; \mathbf{C}(a, a))$, and
4. composition maps $\mathrm{comp}_{a,b,c} \in \mathcal{M}(\mathbf{C}(b, c), \mathbf{C}(a, b); \mathbf{C}(a, c))$, such that
5. once again, certain diagrams encoding unitality and associativity commute, which can be found in [BO15, 2.5].

Remark 2.7. As one might expect, we can define multifunctors between multicategories as maps $F : \operatorname{ob} \mathcal{M} \to \operatorname{ob} \mathcal{N}$ along with collections of maps

$$\mathcal{M}(a_1, \dots, a_n; b) \to \mathcal{N}(Fa_1, \dots, Fa_n; Fb)$$

which satisfy coherence diagrams with the composition maps [EM06, 2.2]; similarly, we can define $\mathcal{M}$-enriched functors between $\mathcal{M}$-enriched categories (as spelled out in [BO15, 2.9]).

These fit into 2-category structures, both on multicategories themselves and on $\mathcal{M}$-enriched categories for any multicategory $\mathcal{M}$. In particular, we can define standard 2-categorical conventions such as adjunctions between $\mathcal{M}$-enriched categories.

Ultimately, in Chapter 4 we want to construct various categories of spectra as categories enriched in the monoidal category of spectra (specifically, symmetric spectra, following Example 2.14). However, some constructions will provide us instead with categories enriched in a multicategory of symmetric monoidal categories. To fix this, we will need to change the enrichment along multifunctors.

Proposition 2.8 ([BO15, 2.11]). *For any multifunctor $F : \mathcal{M} \to \mathcal{N}$ between multicategories, there is a canonical construction of a $\mathcal{N}$-enriched category $F_\bullet \mathbf{C}$ for any $\mathcal{M}$-enriched category $\mathbf{C}$. $F_\bullet \mathbf{C}$ has the same objects as $\mathbf{C}$, and morphism objects $F(\mathbf{C}(a,b))$. This extends to a 2-functor between the 2-categories of enriched categories.*

2.2 Categories Enriched in Monoidal Model Categories

The enriched categories we will be constructing will be different kinds of categories of equivariant spectra, enriched in the symmetric monoidal category of spectra. Therefore, we really want them to be presenting homotopical information: they should be model categories, in ways that make the enrichment homotopically meaningful. In this section we review what that means.

Definition 2.9 ([Hov98, 1.2]). A *monoidal model category* is a bicomplete closed monoidal category $\mathbf{M}$ (say with monoidal product $\wedge$ and unit 1) which also has the structure of a model category, in the sense of [Hov99, 1.1.3] — in particular, it has a cofibrant replacement functor Q and a fibrant replacement functor R. In addition, the monoidal and model structures must be compatible in the following ways:

1. Whenever $f : A \to B$ and $g : X \to Y$ are cofibrations, the *pushout-product*

$$f \,\Box\, g : (A \wedge Y) \underset{A \wedge X}{\sqcup} (B \wedge X) \to B \wedge Y$$

is also a cofibration. If in addition f or g is a weak equivalence, then so is $f \,\Box\, g$.

2. Whenever A is cofibrant, the maps

$$Q1 \wedge A \to 1 \wedge A \cong A$$

and

$$A \wedge Q1 \to A \wedge 1 \cong A$$

are weak equivalences. (This is automatically true if 1 is cofibrant.)

Remark 2.10. We have chosen to use $\mathbf{M}$ to denote our arbitrary monoidal model category, rather than $\mathcal{M}$, to highlight an important feature: as a closed monoidal category, $\mathbf{M}$ is enriched over itself.

Most model categories arising in nature with monoidal structure are monoidal model categories in the above sense, which is usually verified via the following:

Proposition 2.11 ([Hov99, 4.2.5])**.** *Suppose $\mathcal{M}$ is a bicomplete closed monoidal category with the structure of a cofibrantly generated model category. Then it is a monoidal model category as soon as pushout-products of generating cofibrations are cofibrations, and pushout-products of generating cofibrations with generating acyclic cofibrations (in either order) are weak equivalences.*

Definition 2.12. Let $\mathbf{M}$ be a monoidal model category as in Definition 2.9. Further suppose that $\mathbf{M}$ is concrete in the sense that it is equipped with a faithful, conservative, lax monoidal functor $U : \mathbf{M} \to \mathcal{S}et$. An $\mathbf{M}$-*enriched model category* consists of

1. a category $\mathbf{C}$ enriched in the monoidal category $\mathbf{M}$ as a multicategory, and

2. a model structure on the "underlying category" $U_\bullet \mathbf{A}$, using the change-of-enrichment along U from Proposition 2.8.

3. $\mathbf{C}$ must be bitensored over $\mathbf{M}$, meaning $\mathbf{C}(-,-)$ fits into a two-variable adjunction of enriched categories, i.e. there are enriched functors

$$- \wedge - : \mathbf{M} \times \mathbf{C} \to \mathbf{C}$$

and

$$(-)^{(-)} : \mathbf{C} \times \mathbf{M}^{\mathrm{op}} \to \mathbf{C}$$

such that

$$\mathbf{C}(m \wedge a, b) \cong \mathbf{M}(m, \mathbf{C}(a,b)) \cong \mathbf{C}(a, b^m).$$

4. The bitensoring adjunction of two variables must satisfy the same property as the monoidal product in $\mathbf{M}$. That is, if $f : m \to n$ is a cofibration in $U\mathbf{M}(m,n)$ and $g : a \to b$ is a cofibration in $U\mathbf{C}(c,d)$, then

$$f \,\Box\, g \in U\mathbf{C}(m \wedge b \coprod_{m \wedge a} n \wedge a, n \wedge b)$$

is a cofibration, and if either f or g is acyclic, then so is $f \wedge g$. (In forming this pushout-product, we are implicitly using the fact that pushouts in $U_\bullet \mathbf{C}$ agree with $\mathbf{M}$-enriched pushouts in $\mathbf{C}$. It is here that we use the assumption that U was faithful and conservative — note that we really do need these properties on-the-nose, not on the homotopy category.)

Definition 2.12 is good enough to ensure that $\mathbf{C}(-,-)$ is homotopically meaningful when mapping from a cofibrant object to a fibrant one; in particular, the homotopy category of $\mathbf{C}$ is enriched over the homotopy category of $\mathbf{M}$. Yet it is sufficiently easy to verify to allow construction of presheaf categories as enriched model categories in the next section.

Example 2.13. The model category $\mathcal{T}op$ is a concrete category, simply by taking the underlying set of points of a space. Thus we have its enriched model categories, or *topological model categories*.

Example 2.14. We must be slightly more careful in choosing a model of spectra which allows us to form spectrally enriched model categories. We will define **Sp** to be the category of symmetric spectra, under the smash product of symmetric spectra and the stable model structure [HSS00, 6.3.2]. This is concrete, taking U to be the functor sending a spectrum $\mathbb{E}$ to the set of points of the product of all the level spaces $\prod_n E_n$.

2.3 Enriched Model Categories of Presheaves

Let $\mathbf{M}$ be a fixed concrete monoidal model category, and $\mathbf{D}$ any small $\mathbf{M}$-enriched category (but we assume no model structure on $U\mathbf{D}$). The construction of the enriched presheaf category $\mathbf{Pre}(\mathbf{D})$ is standard: its objects are enriched functors $\mathbf{D}^{\mathrm{op}} \to \mathbf{M}$ (cf. Remark 2.7), and the morphism object $\mathbf{Pre}(\mathbf{D})(F, G)$ between two such enriched presheaves is the natural transformation object, given by the equalizer of the usual parallel pair defining natural transformations, written out in [GM20, 5.1].

We would like, using the model structure on $U\mathbf{M}$, to give $\mathbf{Pre}(\mathbf{D})$ a model structure making it an enriched model category as in the previous section, such that the weak equivalences are defined levelwise. Assuming the model structure on $\mathbf{M}$ is cofibrantly generated, the *projective model structure* [Pia91, 5.1] has fibrations also defined levelwise.

Proposition 2.15 ([Hir09, 11.6.1])**.** *The projective model structure makes* $\mathbf{Pre}(\mathbf{D})$ *a cofibrantly generated* $\mathbf{M}$*-model category. It has generating cofibrations, and generating acyclic cofibrations respectively, given by the images of the generating cofibrations and generating acyclic cofibrations, respectively, of* $\mathbf{M}$ *under the adjoints* F_d *to each evaluation-at-d map* $\mathbf{Pre}(\mathbf{D}) \to \mathbf{M}$ *for* $d \in \operatorname{ob} \mathbf{D}$*, explicitly given by*

$$F_d(m)(e) = \mathbf{D}(d, e) \wedge m.$$

The usual Yoneda viewpoint is to consider $\mathbf{Pre}(\mathbf{D})$ as the free cocompletion of $\mathbf{D}$ under weighted colimits by small $\mathbf{M}$-enriched categories [DL07]. Thus, if $\mathbf{D}$ is a *monoidal* enriched category, say with monoidal product $\otimes$, this should extend to a monoidal structure on $\mathbf{Pre}(\mathbf{D})$ by universal property; this is the enriched Day convolution [Day70]. Explicitly, the Day convolution of two presheaves $F, G : \mathbf{D}^{\mathrm{op}} \to \mathbf{M}$ is computed by the coend

$$F \boxtimes G(d) = \int^{a,b\in\mathbf{D}} \mathbf{D}(d, a\otimes b) \wedge F(a) \wedge G(b). \tag{2.1}$$

The monoidal unit is the represented presheaf $d \mapsto \mathbf{D}(d, 1_{\mathbf{D}})$.

Proposition 2.16 ([HHR21, 5.6.35])**.** *If* $\mathbf{M}$ *is a concrete, cofibrantly generated symmetric monoidal model category and* $\mathbf{D}$ *is a (symmetric) monoidal* $\mathbf{M}$*-enriched category in which all morphism objects are cofibrant, then the cofibrantly generated* $\mathbf{M}$*-enriched model category* $\mathbf{Pre}(\mathbf{D})$ *is a monoidal model category under Day convolution.*

Unfortunately, the categories we will be considering presheaves over do not come with natural monoidal structures until all finite coproducts are considered. For example, $\mathcal{F}in\mathcal{S}et^G$ is monoidal under the Cartesian product, but $\mathcal{O}rb^G$ is not since nontrivial products of orbits are not single orbits themselves. Moreover, if a class $\mathcal{H}$ of subgroups does not include G, the Cartesian monoidal structure on $\mathcal{F}in\mathcal{S}et^H$ lacks a unit. Therefore, we will need to consider monoidal structures on $\mathbf{Pre}(\mathbf{D})$ induced from promonoidal structures on $\mathbf{D}$.

Definition 2.17 ([Bé73])**.** A *profunctor* $\mathbf{C} \to \mathbf{D}$ between $\mathbf{M}$-enriched categories is an enriched functor $\mathbf{C} \to \mathbf{Pre}(\mathbf{D})$. Enriched categories, profunctors, and natural transformations assemble into a 2-category, monoidal under products of categories; we can write out the composition of $F : \mathbf{A} \to \mathbf{Pre}(\mathbf{B})$ and $G : \mathbf{B} \to \mathbf{Pre}(\mathbf{C})$ as

$$(G \circ F)(a)(c) = \int^{b\in\mathbf{B}} F(a)(b) \wedge G(b)(c).$$

A (symmetric) promonoidal category is a (symmetric) pseudomonoid in this 2-category. Explicitly, this means it has a unit presheaf

$$J : \mathbf{D}^{\mathrm{op}} \to \mathbf{M}$$

and a multiplication pairing

$$P : \mathbf{D}^{\mathrm{op}} \times \mathbf{D} \times \mathbf{D} \to \mathbf{M}.$$

Unitors are of the form

$$\int^{x \in \mathbf{D}} J(x) \wedge P(b; a, x) \to \mathbf{D}(b, a)$$

and

$$\int^{x \in \mathbf{D}} J(x) \wedge P(b; x, a) \to \mathbf{D}(b, a)$$

while the associators are of the form

$$\int^{x \in \mathbf{D}^{\mathrm{op}}} P(x; a, b) \wedge P(d; x, c) \to \int^{x \in \mathbf{D}^{\mathrm{op}}} P(x; b, c) \wedge P(d; a, x).$$

Day convolution can be naturally extended to promonoidal categories. In formula 2.1, we need only replace $\mathbf{D}(d, a \otimes b)$ with $P(d; a, b)$:

$$F \boxtimes G(d) = \int^{a, b \in \mathbf{D}} P(d; a, b) \wedge F(a) \wedge G(b).$$

This is the generality in which Day originally constructed convolution in [Day70]. However, promonoidal indexing categories are not considered in [HHR21, 5.6], so we will need a slight generalization of Proposition 2.16.

Proposition 2.18. *The results of Proposition 2.16 still hold if* $\mathbf{D}$ *is instead a promonoidal category such that each monoidal product presheaf* $P(-; a, b)$ *takes values in cofibrant objects of* $\mathbf{M}$*. In particular, this is true if morphism objects of* $\mathbf{D}$ *are all cofibrant and* $P(-; a, b)$ *is a direct sum of representables.*

Proof. The proof of [HHR21, 5.6.35(iii)] still goes through, where our $\mathbf{D}$ is the opposite of their $\mathscr{J}$ (note that their theorem is about $[\mathscr{J}, \mathcal{M}]$, not the presheaf category). We need only replace their $\mathscr{J}(j' + j'', k)$, which is our term $\mathbf{D}(k, j' + j'')$ in formula 2.1, with the promonoidal product $P(k; j', j'')$. By assumption, this is still cofibrant. $\square$

Remark 2.19. A promonoidal structure is *Cartesian* when $P(d; a, b)$ is simply the product $\mathbf{D}(d, a) \wedge \mathbf{D}(d, b)$ and $J(a)$ is the constant presheaf on $1_{\mathbf{M}}$. When this is the case, the coend in equation 2.1 collapses to $F \boxtimes G(d) = F(d) \wedge G(d)$, the pointwise monoidal product.

CHAPTER 3

A Revisionist History of Equivariant Stable Homotopy Theory

In this chapter, we will briefly review the construction of the classical categories of equivariant spectra (Borel, trivial, and genuine) as well as the operadic spectra of [BH19]. We emphasize that the key distinction between these categories is which fixed-point functors they admit, and which natural transformations between these can be defined.

3.1 Borel and Trivial Equivariant Spectra

We begin with the most obvious notion of equivariant spaces and spectra: G-objects in these categories.

Definition 3.1. A *Borel G-spectrum* (respectively, *Borel G-space*) is a G-object in spectra (resp. spaces); that is, a functor $\mathcal{B}G^{\mathrm{op}} \to \mathbf{Sp}$ (resp. $\mathcal{B}G^{\mathrm{op}} \to \mathcal{T}op$), or a spectrum (resp. space) equipped with an action of G. The category of such will be denoted $\mathbf{Sp}^{BG}$ (resp. $\mathcal{T}op^{BG}$). Regarding $\mathcal{B}G$ as a discrete spectrally topologically enriched category, or (via the suspension spectrum of the discrete space G) a spectrally enriched category, we may view either Borel category as an enriched presheaf category. In particular, it comes equipped with the structure of an enriched model category under the projective model structure, whose weak equivalences are equivariant maps whose underlying map of spectra (resp. spaces) is a weak equivalence.

While this is a suitable *category* of equivariant objects, the Borel model structure is extremely coarse. In particular, since weak equivalences only care about the map on underlying spectra, it is easy to come up with "weak equivalences" which don't look much like equivalences at all.

Example 3.2. Let $G = C_n$ be a cyclic group, and let λ be the two-dimensional irreducible representation given by the action on $\mathbb{C}$ by a choice of nth root of unity ζ. Let S^λ denote its one-point compactification, and note that this has fixed points $(S^\lambda)^{C_n} = S^0$.

The nth power map $\mathbb{C} \to \mathbb{C}$ gives a G-equivariant map $S^\lambda \to S^2$. This is not an equivalence: it acts on π_* by multiplication by n. But the map $S^\lambda[\frac{1}{n}] \to S^2[\frac{1}{n}]$ is an equivalence on underlying spaces, even though the induced map on fixed points $S^0[\frac{1}{n}] \to S^2[\frac{1}{n}]$ is not.

As Example 3.2 shows, we want to consider equivariant objects *along with their fixed points*, whereas in the Borel model structure, fixed points aren't homotopically meaningful. We thus turn to a finer model structure on the categories $\mathbf{Sp}^{BG}$ and $\mathcal{T}op^G$, the universal one which makes fixed points meaningful:

Definition 3.3. The *fixed-points model structure* is another model structure on $\mathcal{T}op^{BG}$ or $\mathbf{Sp}^{BG}$. Its weak equivalences and fibrations are the maps $X \to Y$ such that each map $X^H \to Y^H$ on categorical fixed points is a weak equivalence (respectively, fibration) of spaces or spectra, for any subgroup $H \leq G$.

In making this definition, we have lost a useful tool: a nice presentation of our model category as an enriched presheaf category. We recover this via Elmendorf's theorem:

Theorem 3.4 ([Elm83])**.** *The model category $\mathcal{T}op^{BG}$, with its fixed-points model structure, is Quillen equivalent to the presheaf category $\mathcal{T}op^{(\mathcal{O}rb^G)^{\mathrm{op}}}$ with its projective model structure. Similarly,* $\mathbf{Sp}^{BG}$ *with its fixed-points model structure is Quillen equivalent to* $\mathbf{Sp}^{(\mathcal{O}rb^G)^{\mathrm{op}}}$ *with its projective model structure.*

Definition 3.5. The topologically enriched model category $\mathcal{T}op^G$ is $\mathcal{T}op^{(\mathcal{O}rb^G)^{\mathrm{op}}}$ with its projective model structure. Similarly, the spectrally enriched model category $\mathbf{Sp}^G_{\mathrm{triv}}$ of *trivial G-spectra* is $\mathbf{Sp}^{(\mathcal{O}rb^G)^{\mathrm{op}}}$ with its projective model structure.

Remark 3.6. The left adjoint in each Quillen equivalence in Lemma 3.4 takes a spectrum or space X with a G-action, and forms the coefficient system $G/H \mapsto X^H$. We will implicitly follow this equivalence to refer to G-spaces as objects in $\mathcal{T}op^G$, or G-spectra as objects in $\mathbf{Sp}^G_{\text{triv}}$.

Remark 3.7. The category $\mathbf{Sp}^G_{\text{triv}}$ might be more traditionally called something like "naive G-spectra". Unfortunately this term is also frequently used for $\mathbf{Sp}^{BG}$, so we will avoid this terminology to disambiguate between the two categories (or, equivalently, the two model structures on $\mathbf{Sp}^{BG}$). As with "naive", "trivial" contrasts $\mathbf{Sp}^G_{\text{triv}}$ with genuine G-spectra; in Proposition 4.11, we will identify this with spectra indexed on a trivial N_∞ operad for G. We note that this terminology still may be slightly counterintuitive: trivial G-spectra are, in some sense, *less* trivial than Borel G-spectra.

Remark 3.8. By design, the main feature that distinguishes trivial G-spectra from Borel G-spectra is the existence of homotopically meaningful fixed-point functors

$$(-)^H : \mathbf{Sp}^G_{\text{triv}} \to \mathbf{Sp}.$$

Each of these is formed by evaluating an $\mathcal{O}rb^G$-presheaf at the object G/H, and as such it naturally carries an action of the automorphism group $W_G H = \mathrm{Aut}_{\mathcal{O}rb^G}(G/H)$, i.e. it factors through

$$(-)^H : \mathbf{Sp}^G_{\text{triv}} \to \mathbf{Sp}^{BW_G H}.$$

Indeed, this functor factors through a right Quillen functor

$$(-)^H : \mathbf{Sp}^G_{\text{triv}} \to \mathbf{Sp}^{W_G H}_{\text{triv}}.$$

This is the pullback of presheaves along the map

$$G \times_{N_G H} (-) : \mathcal{O}rb^{W_G H} \to \mathcal{O}rb^G$$

sending $W_G H/(K/H) = N_G H/K$ to G/K.

Definition 3.9. Trivial G-spectra are a symmetric monoidal category under the objectwise smash products (i.e. $(X \wedge Y)^H = X^H \wedge Y^H$).

By Remark 2.19, we can also think of this as the Day monoidal structure induced by the Cartesian promonoidal structure on $\mathcal{O}rb^G$.

3.2 Genuine G-Spectra

In Section 3.1, we defined trivial G-spectra to address a deficiency with Borel G-spectra, namely the lack of fixed points, Let us now consider deficiencies that remain (or were created) in $\mathbf{Sp}^G_{\mathrm{triv}}$:

1. In the objectwise monoidal structure, finite spectra are not guaranteed to be dualizable; in fact, *no* proper orbit spectrum $\Sigma^\infty_G(G/H)_+$ is dualizable [Lew00, 7.1]. This breaks any number of nonequivariant constructions. In particular, tools such as the Balmer spectrum [Bal10] are far less useful for non-rigid homotopy categories, and any result for "stable homotopy categories" in the sense of [HPS97, 1.1.4] does not apply to the homotopy category of trivial G-spectra.

2. Many equivariant spectra arising in nature carry more structure that is not homotopy-invariant in $\mathbf{Sp}^G_{\mathrm{triv}}$. Namely, in addition to the restriction maps
$$X^H \to X^K$$
for $K \leq H$ that are part of the coefficient system structur, they also have *transfer maps*
$$X^K \to X^H$$
in the opposite direction. These tend to interact well with the restrictions, in particular making homotopy groups assemble into Mackey functors (see Definition 4.1).

Example 3.10. As a presheaf, the equivariant K-theory spectrum K_G is defined on each G/H to be the K-theory spectrum of H-equivariant complex vector bundles, with restriction maps given by restriction of G-equivariant bundles to H-equivariant bundles. Then the induction functors give transfer maps.

Remark 3.11. These problems are not unrelated: any solution to the second also solves the first. That is, imagine a hypothetical monoidal model category of spectra which still has H-fixed point functors represented by spectra $\Sigma^\infty (G/H)_+$, and extend this to $\Sigma^\infty T_+$ of any finite G-set by direct sum. If this hypothetical category of spectra admits transfers, then by Yoneda it has extra maps between the representing spectra. In particular, we get a map

$$\Sigma^\infty \left((G/H) \times (G/H)_+\right) \to \Sigma^\infty (G/H)_+$$

by breaking the left hand side down into its orbits. Now let's assume that smash products of these suspension spectra agree with products of finite G-sets. One then verifies that any reasonable axioms on the relationships between the transfers and the orbits will ensure that this map is actually a self-duality on $\Sigma^\infty (G/H)_+$. If our category of spectra is also *generated* by these representables, which simply means that weak equivalences are still detected on fixed points, then this implies that all finite objects are dualizable.

Conversely, of course if the spectra $\Sigma^\infty (G/H)_+$ are self-dual, the transfers are dual to the restrictions.

Thus, any category of equivariant spectra that is still a good monoidal model category with fixed-point functors that detect weak equivalences, but in addition admits transfer maps between those fixed points, could be considered a solution to these deficiencies in trivial equivariant spectra.

Example 3.12. The usual way of obtaining these transfers is by inverting representation spheres.

Let $\mathcal{U}$ be a G-universe — an infinite-dimensional real G-representation that, for each irreducible representation V, either does not contain V as a subrepresentation or contains an infinite direct sum $V^{\oplus\infty}$, and at least contains the trivial representations.

Let $\mathbf{Sp}^G_{\mathcal{U}}$ denote the model category of orthogonal G-spectra indexed on $\mathcal{U}$ as in [MM02].

If $\mathcal{U}$ is complete (i.e. it contains every irreducible representation), then $\mathbf{Sp}^G_{\mathcal{U}}$ has all transfers, and thus has duals for finite spectra. More generally, $\mathcal{U}$-spectra admit transfers $X^K \to X^H$ for $K < H$ whenever H/K embeds H-equivariantly into $\mathcal{U}$.

3.3 Universal Spaces and Geometric Fixed Points

Like trivial G-spectra, orthogonal G-spectra admit fixed-point functors

$$(-)^H : \mathbf{Sp}^G_{\mathcal{U}} \to \mathbf{Sp}^{W_G H}_{\mathcal{U}^H}$$

which are right Quillen. However, these fixed points exhibit some undesirable properties. For example, they fail to be strong monoidal; in fact, they do not even send the monoidal unit (the sphere $\mathbb{S}_{\mathcal{U}} = \Sigma^\infty_{\mathcal{U}}(S^0)$) to the sphere spectrum. Instead, the fixed points of suspension spectra are given by the tom Dieck splitting.

Proposition 3.13 (tom Dieck splitting; [Die72][LMS84])**.** *Let X be a pointed G-space. The suspension spectrum $\Sigma^\infty_{\mathcal{U}}(X) \in \mathbf{Sp}^G_{\mathcal{U}}$ has H-fixed points with underlying spectrum given by*

$$\Sigma^\infty_{\mathcal{U}}(X)^H \simeq \bigvee_{(K) \leq H} \Sigma^\infty \left(EW_H K \underset{W_H K}{\wedge} X^K \right).$$

That is, the H-fixed points of an orthogonal spectrum contain information about isotropy from all subgroups contained in H, not only H. We must further split these apart to get information resembling the fixed points of a space.

Let $\mathcal{F}$ be a *family* of subgroups of G — that is, a collection of subgroups closed under subconjugates. A G-space X is *$\mathcal{F}$-free* when the stabilizer of any point $x \in X$ lies in $\mathcal{F}$. Extending this notion to G-spectra gives us a Bousfield localization:

Definition 3.14. 1. The *$\mathcal{F}$-free* G-spectra are the triangulated localizing subcategory generated by the objects $\Sigma^\infty_{\mathcal{U}}(G/H)_+$ for $H \in \mathcal{F}$. This is also the triangulated localizing subcategory generated by suspension spectra of all $\mathcal{F}$-free spaces.

2. A G-spectrum X is *$\mathcal{F}$-contractible* if $E \wedge X \simeq 0$ for any $\mathcal{F}$-free G-spectrum E. Equivalently, the fixed points X^H for $H \in \mathcal{F}$ are all contractible.

3. A map $f : X \to Y$ of G-spectra is an *$\mathcal{F}$-equivalence* if its cofiber is $\mathcal{F}$-contractible. Equivalently, $E \wedge f : E \wedge X \to E \wedge Y$ is an equivalence for any $\mathcal{F}$-free E, or each map on fixed points $f^H : X^H \to Y^H$ is an equivalence for $H \in \mathcal{F}$.

This localization is best understood in terms of universal spaces:

Definition 3.15. The *universal space* $E\mathcal{F}$ is the G-space defined up to fixed-points equivalence by

$$(E\mathcal{F})^H \simeq \begin{cases} *, & H \in \mathcal{F} \\ \emptyset, & H \notin \mathcal{F}. \end{cases}$$

The cofiber of the canonical map $E\mathcal{F}_+ \to S^0$ is denoted $\widetilde{E}\mathcal{F}$, so that

$$(\widetilde{E}\mathcal{F})^H \simeq \begin{cases} *, & H \in \mathcal{F} \\ S^0, & H \notin \mathcal{F}. \end{cases}$$

Proposition 3.16. *For a G-spectrum X, the map $E\mathcal{F}_+ \wedge X \to X$ is the homotopy terminal map from an $\mathcal{F}$-free G-spectrum to X — i.e., $E\mathcal{F}_+ \wedge -$ is the colocalization into $\mathcal{F}$-free spectra. Thus $\widetilde{E}\mathcal{F} \wedge -$ is the localization into $\mathcal{F}$-contractible spectra.*

The exact triangle associated to this localization, $E\mathcal{F}_+ \wedge X \to X \to \widetilde{E}\mathcal{F} \wedge X$, is called the *isotropy separation sequence for X at $\mathcal{F}$.*

Definition 3.17. The *geometric fixed points* functor Φ^H for $H < G$ is defined by

$$\Phi^H(X) := \left(\widetilde{E}\mathcal{F}_{<H} \wedge X\right)^H$$

where $\mathcal{F}_{<H}$ is the family of all proper subgroups of H, or any family containing all proper subgroups of H but not H itself.

This is in particular constructed to kill off all the terms in the tom Dieck splitting except for the one we want.

Proposition 3.18 ([MM02, 4.6])**.** *The geometric fixed points $\Phi^H(\Sigma^\infty_{\mathcal{U}} X)$ of a suspension G-spectrum have underlying spectrum $\Sigma^\infty(X^H)$.*

Proposition 3.19 ([MM02, 4.7])**.** *Each of the geometric fixed points functors Φ^H is strong monoidal.*

3.4 $\mathcal{O}$-Spectra

Motivated by the discussion in Section 3.2, we consider a category of spectra which is characterized by the existence of transfers. More precisely, we will define a *family* of categories of spectra, depending on which transfers are present.

Definition 3.20 ([BH18, Section 3][Rub19, 3.1])**.** An *indexing category* in $\mathcal{O}rb^G$ is a wide subcategory $\mathcal{T} \subseteq \mathcal{O}rb^G$ whose finite-coproduct completion $\mathcal{T}^{\sqcup} \subseteq \mathcal{F}in\mathcal{S}et^G$ (those maps such that the restriction to each orbit falls in $\mathcal{T}$) is closed under pullbacks along any map in $\mathcal{F}in\mathcal{S}et^G$.

It can be helpful to understand these wide subcategories as instead giving full subcategories of each overcategory $\mathcal{F}in\mathcal{S}et^G{}_{/(G/H)}$, which is equivalent to $\mathcal{F}in\mathcal{S}et^H$.

Definition 3.21 ([BH18, 1.2][Rub19, 2.12])**.** An *indexing system* $\mathcal{I}$ is a collection of full replete subcategories $\mathcal{I}(H) \subseteq \mathcal{F}in\mathcal{S}et^H$ for each $H \leq G$, satisfying the following conditions:

1. (restriction) If $T \in \mathcal{I}(H)$ and $K \leq H$, then $T|_K \in \mathcal{I}(K)$.
2. (conjugation) If $T \in \mathcal{I}(H)$ and $g \in G$, then $gT \in \mathcal{I}(gHg^{-1})$. (Recall that gT has elements gt for $t \in T$, with the gHg^{-1}-action $(ghg^{-1})(gt) = g(ht)$. Taken together with the previous condition, this says that the collection of $I(H)$ form a coefficient system of categories.)

3. (trivial sets) Any set with a trivial action is in $\mathcal{I}(G)$, hence in every $\mathcal{I}(H)$.

4. (subobjects) If $T \in \mathcal{I}(H)$ and $S \subseteq T$, then $S \in \mathcal{I}(H)$.

5. (coproducts) If $S, T \in \mathcal{I}(H)$, then $S \sqcup T \in \mathcal{I}(H)$. (With the previous condition, this means that $\mathcal{I}(H)$ is determined by which orbits H/K it contains.)

6. (self-induction) If $K \leq H$ is such that $H/K \in \mathcal{I}(H)$ and $T \in \mathcal{I}(K)$, then

$$H \times_K T \in \mathcal{I}(H).$$

Proposition 3.22 ([BH18, 1.4]). *There is an isomorphism between the poset of indexing categories and the poset of indexing systems. In one direction, an indexing system $\mathcal{I}$ gives the category $\mathcal{T}_{\mathcal{I}}$ consisting of those maps $f : G/H \to G/K$ with fibers $f^{-1}(eK) \in \mathcal{I}(K)$. In the other, $\mathcal{T}$ gives the indexing system of $\mathcal{T}$-admissible H-sets:*

$$\mathcal{I}_{\mathcal{T}}(H) = \{T \in \mathcal{F}in\mathcal{S}et^H \mid (G \times_H T \to G/H) \in \mathcal{T}\}.$$

Using the latter poset, we shift from looking at transfers $X^H \to X^K$ for $G/H \to G/K \in \mathcal{T}$ to looking at H-equivariant norms $X^T \to X$ for $T \in \mathcal{I}_{\mathcal{T}}(H)$; the transfer is then the H-fixed points of the norm for $T = H/K$.

Blumberg and Hill define operads whose action codifies the existence of all norms from an indexing system, commuting up to coherent homotopy.

Definition 3.23 ([BH19, 3.7]). An N_∞ *operad* $\mathcal{O}$ is a symmetric operad in $\mathbf{Top}^G$ with the following properties:

1. Each action of Σ_n on $\mathcal{O}(n)$ is free,

2. Each subgroup $\Gamma \subseteq G \times \Sigma_n$ has fixed points $\mathcal{O}(n)^\Gamma$ either empty or contractible, and

3. $\mathcal{O}(0)$ and $\mathcal{O}(2)$ are nonempty (hence contractible).

By freeness, subgroups $\Gamma \subseteq G \times \Sigma_n$ can have no fixed points in $\mathcal{O}(n)$ unless they meet $1 \times \Sigma_n$ trivially. Such Γ are called *graph subgroups*, because they are necessarily the graph of a homomorphism $H \to \Sigma_n$ for $H \leq G$, i.e. an action of H on $[n] = \{1, \ldots, n\}$. Any H-set T of cardinality n, along with a choice of enumeration $[n] \cong T$, yields such a subgroup Γ_T; this is well-defined up to conjugation by an element of Σ_n. Any fixed point $f \in \mathcal{O}(n)^{\Gamma_T}$ acts on each $\mathcal{O}$-algebra X via an H-equivariant map $X^T \to X$.

Definition 3.24. Let $\mathcal{O}$ be an N_∞ G-operad, and $H \leq G$ a subgroup. A finite H-set T of cardinality n is $\mathcal{O}$-*admissible* if $\mathcal{O}(n)^{\Gamma_T}$ is nonempty (hence contractible).

The $\mathcal{O}$-admissible H-sets form an indexing system, which we will denote $\mathrm{adm}(\mathcal{O})$. This gives a fully faithful embedding of the homotopy category of N_∞ operads into the poset indexing systems [BH19, 3.24]. N_∞ operads realizing any indexing system were independently constructed by Bonventre and Pereira [BP17, Corollary IV], Gutiérrez and White [GW18, 4.7], and Rubin [Rub19, 2.16], proving that the homotopy category of N_∞ operads is equivalent to the poset of indexing systems or indexing categories. Because of this, we will often use $\mathcal{O}$ to refer to the operad, the indexing system, and the indexing category interchangeably.

Remark 3.25. If $\mathcal{O}$ is an N_∞ operad for G and $H \leq G$, then by restriction we can also view $\mathcal{O}$ as an N_∞ operad for H. Fortunately, there is no confusion here: the K-sets $\mathrm{adm}(\mathcal{O})(K)$ for $K \leq H$ are the same whether we view $\mathcal{O}$ as an operad in G-spaces or in H-spaces. In particular, when H is the trivial group, we can take the underlying nonequivariant spectrum of $\mathcal{O}$, which is an E_∞ operad. We can also restrict $\mathcal{O}$-algebras to $\mathcal{O}|_H$-algebras in H-spaces, and in particular the underlying space of any $\mathcal{O}$-algebra is an E_∞-algebra.

Example 3.26. If G acts trivially on a nonequivariant E_∞ operad, the result is an N_∞ operad $\mathcal{O}^{\mathrm{triv}}$ for which only trivial H-sets are admissible. In the other extreme, there is a terminal N_∞ operad $\mathcal{O}^{\mathrm{gen}}$ for which all H-sets are admissible; then $\mathcal{O}^{\mathrm{gen}}$ is genuinely G-E_∞.

Example 3.27. Let $\mathcal{U}$ be a G-universe. Then the little disks operad $\mathcal{D}_\mathcal{U}$ and the Steiner operad $\mathcal{K}_\mathcal{U}$ are both N_∞ operads, and are equivalent; the corresponding indexing system

$\mathrm{adm}(\mathcal{D}_{\mathcal{U}})(H)$ consists of those H-sets which embed H-equivariantly into $\mathcal{U}$ [BH15, 4.19]. By [GM17, 1.14], these admit a recognition principle: grouplike $\mathcal{K}_{\mathcal{U}}$-algebras admit deloopings by the representation spheres of all finite representations in the universe $\mathcal{U}$, and thus are equivalent to connective orthogonal spectra indexed on $\mathcal{U}$ [BH19, 6.1].

Example 3.27 gives the motivating examples of N_∞ operads, but this example is far from exhaustive: being a Steiner operad is a fairly strong property among general N_∞ operads [Rub20, 4.11]. For all other $\mathcal{O}$, we can define $\mathcal{O}$-spectra formally to satisfy the same recognition principle:

Definition 3.28 ([BH19, 1.5])**.** Let $\mathcal{O}$ be an N_∞ operad for G. The spectrally enriched model category $\mathbf{Sp}^G_{\mathcal{O}}$ of $\mathcal{O}$-spectra is the spectrum objects in the category $\mathbf{Top}^G_*[\mathcal{O}]$ of $\mathcal{O}$-algebras in pointed G-spaces.

$\mathcal{O}$-spectra form a stable, spectrally enriched model category. They admit a topologically enriched Quillen adjunction $\Sigma^\infty_{\mathcal{O}} \dashv \Omega^\infty_{\mathcal{O}}$ from $\mathcal{T}op^G_*$, where $\Sigma^\infty_{\mathcal{O}}$ is the composition of the free $\mathcal{O}$-algebra functor $\widetilde{\mathbb{P}}_{\mathcal{O}}$ and the suspension spectrum object functor Σ^∞. By [BH19, 3.6], the composition $\Omega\Sigma$ (and thus also $\Omega^\infty\Sigma^\infty$) on $\mathcal{O}$-algebras is group-completion on each fixed-points space.

Proposition 3.29 (tom Dieck splitting for $\mathcal{O}$-spectra [BH19, 3.32])**.** *The fixed points of suspension $\mathcal{O}$-spectra are given by*

$$(\Sigma^\infty_{\mathcal{O}} X_+)^G = \bigvee_{G/H \in \pi_0 \mathrm{adm}(\mathcal{O})} EW_G H_+ \underset{W_G H}{\wedge} \Sigma^\infty_+(X^H).$$

CHAPTER 4

Spectral Mackey Functors

The main result of this chapter will be Theorem 4.13, establishing the Quillen equivalence between $\mathcal{O}$-spectra and spectrally enriched Mackey functors of spectra, with restricted transfers determined by $\mathcal{O}$. This is, essentially, the analog for $\mathcal{O}$-spectra of Elmendorf's theorem for trivial G-spectra: it describes $\mathcal{O}$-spectra via their fixed points, as the free homotopy cocompletion of a combinatorially defined category of G-orbits.

First, we will establish several variants of the Burnside category. In each of these constructions, we will need to fix an indexing category, which will determine which maps will be considered in spans. Recall that the choice of such a category is equivalent to a choice of N_∞ operad $\mathcal{O}$; where it is needed, we will refer to the corresponding indexing system as $\mathcal{T}_{\mathcal{O}}$.

For the sake of the decomposition in Chapter 5, we will also need to fix a class $\mathcal{H}$ of subgroups which will be allowed as stabilizers of G-sets in our Burnside category. We do not require that $\mathcal{H}$ be a family, but it should be at least closed under subgroup conjugation, and it should be an interval: if $H < J < K$ with $H, K \in \mathcal{H}$ then $J \in \mathcal{H}$. Our Burnside categories will be indexed on the pair $(\mathcal{H}; \mathcal{O})$. When $\mathcal{H}$ is the maximal class of all subgroups (including G itself), we will simply write this as $(G; \mathcal{O})$.

We will begin by briefly reviewing the definition of the Burnside category and Mackey functors in Section 4.1. In particular, the homotopy groups of the H-fixed points of an $\mathcal{O}$-spectrum for all $H < G$ assemble together into a Mackey functor for $\mathcal{O}$ [BH19, 1.5]. By Yoneda, this should tell us about the existence of maps

$$\Sigma^\infty_{\mathcal{O}}(A)_+ \to \Sigma^\infty_{\mathcal{O}}(B)_+$$

coming from spans $A \leftarrow T \rightarrow B$. Indeed, we can construct this directly from the $\mathcal{O}$-action: when $A = G/H$, T is $G \times_H S$ for S an $\mathcal{O}$-admissible H-set, so there is a contractible choice of $f \in \mathcal{O}(|S|)^{\Gamma_S}$, whose action on $\Sigma^\infty_{\mathcal{O}}(B)_+$ yields our map.

While this defines our map uniquely up to homotopy, the space of maps it parametrizes is not contractible: instead, it is the classifying space of the group of span automorphisms. Thus we need to consider the full 2-category of spans, of which the Burnside category is the homotopy 1-category. Iweak n Section 4.2 we construct this as a category enriched in permutative categories, and use change of enrichment (Proposition 2.8) to produce a spectrally enriched Burnside category. With this we will be ready to prove Theorem 4.13 in Section 4.3. We discuss some consequences of this in Section 4.4.

Finally, in Section 4.5 we will discuss how to present the Burnside ∞-category as a quasicategory instead; in Chapter 5, we will use this presentation to combinatorially describe some homotopy fibers.

4.1 The Burnside Category and Mackey Functors

Definition 4.1. The *effective Burnside category supported on $\mathcal{H}$ with transfers in $\mathcal{O}$*, written $\mathcal{A}^{\text{eff}}_{\mathcal{H};\mathcal{O}}$, has the same objects as those of $\mathcal{F}in\mathcal{S}et^{\mathcal{H}}$. Morphisms $\mathcal{A}^{\text{eff}}_{\mathcal{H};\mathcal{O}}(A, B)$ are isomorphism classes of spans $A \leftarrow T \rightarrow B$ in $\mathcal{F}in\mathcal{S}et^{\mathcal{H}}$, such that the map $T \rightarrow A$ is in $\mathcal{T}_{\mathcal{O}}$. Composition is by pullback, so that

$$(B \leftarrow S \rightarrow C) \circ (A \leftarrow T \rightarrow B) = (A \leftarrow (S \times_B T) \rightarrow C).$$

(Here the pullback is in $\mathcal{F}in\mathcal{S}et^{\mathcal{H}}$, so contains only those elements of the pullback in $\mathcal{F}in\mathcal{S}et^G$ which have stabilizer in $\mathcal{H}$).

This is a semiadditive category, whose biproduct is given by disjoint unions of G-sets. For any other semiadditive category $\mathcal{C}$, a *Mackey functor for ($\mathcal{H}$,$\mathcal{O}$) valued in $\mathcal{C}$* is an additive presheaf $\underline{M} : (\mathcal{A}^{\text{eff}}_{\mathcal{H};\mathcal{O}})^{\text{op}} \rightarrow \mathcal{C}$. The category of such is denoted $\mathcal{M}\text{ack}_{\mathcal{H};\mathcal{O}}(\mathcal{C})$. In particular, we will refer to $\mathcal{M}\text{ack}_{\mathcal{H};\mathcal{O}}(\mathcal{A}b)$ as simply $\mathcal{M}\text{ack}_{\mathcal{H};\mathcal{O}}$.

Mackey functors are determined by their values on orbits G/H, and we will usually write $\underline{M}(G/H)$ as simply $M(H)$. Explicitly, Mackey functors have restriction maps

$$r^H_K : M(H) \to M(K)$$

for $K < H$, conjugation maps

$$c_g : M(H) \to M(g^{-1}Hg)$$

for $g \in G$, and transfers

$$M(K) \to M(H)$$

for $K < H$ provided $G/K \to G/H \in \mathcal{T}_{\mathcal{O}}$. These are related by the formula

$$r^H_K \circ t^H_L = \sum_{KgL \in K\backslash H/L} t^K_{L\cap g^{-1}Kg} \circ c_g \circ r^K_{gLg^{-1}\cap K}.$$

Remark 4.2. We have made a few departures from the standard terminology here. First of all, our indexing category restricts the "backwards" map $T \to A$ in a span $A \leftarrow T \to B$; it is usually written restricting the "forwards" map $T \to B$. This is because we define Mackey functors as *presheaves* on the Burnside category, whereas they are covariant functors in the original definition [Dre71]. (Of course, Dress's Burnside category had no restrictions on its transfers, and thus was self-dual).

Remark 4.3. The "effective" here denotes that we are dealing with literal spans of G-sets, rather than virtual spans. To form the full Burnside category, one group-completes each commutative monoid of spans; that is, if K is the group-completion functor from commutative monoids to abelian groups, $\mathcal{A}_{\mathcal{H};\mathcal{O}} = K_\bullet \mathcal{A}^{\text{eff}}_{\mathcal{H};\mathcal{O}}$ using the change-of-enrichment functor from 2.8. When forming Mackey functors valued in additive categories, additive presheaves on the effective and full Burnside categories are equivalent.

Moreover, additive presheaves on either are equivalent to $\mathcal{A}b$-enriched presheaves on $\mathcal{A}^{-}_{\mathcal{H};\mathcal{O}}$, the full subcategory of $\mathcal{A}_{\mathcal{H};\mathcal{O}}$ on orbits G/H, since all objects are direct sums of these.

Definition 4.4. As long as $G \in \mathcal{H}$, the Burnside category $\mathcal{A}_{\mathcal{H};\mathcal{O}}$ is symmetric monoidal. The monoidal product is the extension by linearity of the product of $\mathcal{H}$-free G-sets and spans — again, this consists only of the points of the product in $\mathcal{F}in\mathcal{S}et^G$ that have stabilizer in $\mathcal{H}$. (Note that this is *not* the categorical product of spans; that is given by the disjoint union instead.) The monoidal unit is $1 = G/G$.

If G is not in $\mathcal{H}$, then $\mathcal{A}_{\mathcal{H};\mathcal{O}}$ is no longer monoidal, because it lacks a monoidal unit. However, it is still promonoidal, with the unit given by the *Burnside ring Mackey functor* $\underline{A}_{\mathcal{H};\mathcal{O}}$, which sends H to the free abelian group $A_{\mathcal{H};\mathcal{O}}(H)$ on the subgroups $K < H$ in $\mathcal{H}$ such that H/K is $\mathcal{O}$-admissible. If $G \in \mathcal{H}$, $\underline{A}_{\mathcal{H};\mathcal{O}}$ is equivalent to the representable presheaf on G/G.

Either way, this descends to a promonoidal structure on $\mathcal{A}^{-}_{\mathcal{H};\mathcal{O}}$. Therefore abelian Mackey functors are symmetric monoidal under Day convolution. Explicitly, the box product of Mackey functors is given by adding formal transfers into the objectwise tensor product

$$(M \boxtimes N)(H) = \left(\bigoplus_{(K)<H} M(K) \otimes N(K) \right) / \sim$$

where the direct sum is under conjugacy classes of subgroups $K < H$ in $\mathcal{H}$ such that $G/H \to G/K \in \mathcal{T}_{\mathcal{O}}$, and the Frobenius relation $\sim$ is generated by

$$t^H_K(a \otimes r^H_K b) \sim (t^H_K a) \otimes b.$$

and the same relations with the tensor products in the other order. The monoidal unit is, once again, the Burnside ring Mackey functor.

4.2 The Burnside 2-Category and Spectral Category

In forming $\mathcal{A}^{\text{eff}}_{\mathcal{H};\mathcal{O}}$, we had to take isomorphism classes of spans to get the right notion of Mackey functors. By doing so, we are really truncating from a 2-category of spans down to its homotopy category.

Definition 4.5. The *effective Burnside 2-category supported on $\mathcal{H}$ with transfers from $\mathcal{O}$*, $\mathscr{A}^{\text{eff}}_{\mathcal{H};\mathcal{O}}$, is the (2,1)-category defined as follows.

1. Objects are finite $\mathcal{H}$-sets T equipped with an enumeration $T \cong \{1, \ldots, n\}$.

2. The morphism category $\mathscr{A}^{\text{eff}}_{\mathcal{H};\mathcal{O}}(A, B)$ is the groupoid of spans $A \leftarrow T \to B$ with $T \to A$ in $\mathcal{T}_{\mathcal{O}}$, and isomorphisms between spans.

3. Composition of spans is again given by pullback, specifically the choice of pullback specified in [BO15, 7.2]. That is, pullback along any identity is given by the identity, and otherwise pullback is given as a subset of the product with the lexicographic ordering. This ensures that composition is strictly unital and strictly associative.

Just as $\mathcal{A}^{\text{eff}}_{\mathcal{H};\mathcal{O}}$ is naturally enriched in the monoidal category of commutative monoids, $\mathscr{A}^{\text{eff}}_{\mathcal{H};\mathcal{O}}$ is enriched in the multicategory $\mathcal{P}erm$ of permutative categories (Example 2.5). The bilinear functor structure on composition is given by the distributivity of pullbacks over disjoint union, and the coherence conditions are equivalent to $\mathcal{F}in\mathcal{S}et^{\mathcal{H}}$ being a disjunctive category [BO15, Remark 4.2].

Remark 4.6. Once again, we have chosen to call this the "effective" Burnside 2-category, because its morphism categories are not group-complete under disjoint union. In the 1-categorical case, we corrected this by changing enrichment along the group-completion functor. Of course, there is no such group-completion functor for permutative categories. However, embedding them into spaces via the classifying space, we can view the K-theory spectrum of a permutative category as its derived group-completion.

Proposition 4.7 ([EM06, 1.1])**.** *The construction of K-theory spectra extends to a symmetric multifunctor $\mathbb{K} : \mathcal{P}erm \to$ **Sp** from the symmetric multilinear category of permutative categories to the symmetric monoidal category of symmetric spectra. This multifunctor takes values in cofibrant spectra.*

Consequently, there is a change-of-enrichment functor $\mathbb{K}_\bullet$ from categories enriched in permutative categories to spectrally enriched categories.

Definition 4.8. The *spectrally enriched Burnside category* with respect to $(\mathcal{H};\mathcal{O})$ is given by $\mathbf{A}_{\mathcal{H};\mathcal{O}} := \mathbb{K}_\bullet \mathscr{A}^{\text{eff}}_{\mathcal{H};\mathcal{O}}$. Inside this, let $\mathbf{A}^{-}_{\mathcal{H};\mathcal{O}}$ denote the full enriched subcategory on the orbits G/H. For $\mathbf{C}$ any spectrally enriched category, a *spectral Mackey functor* for $(\mathcal{H},\mathcal{O})$ in $\mathbf{C}$ is a spectrally enriched presheaf $(\mathbf{A}^{-}_{\mathcal{H};\mathcal{O}})^{\text{op}} \to \mathbf{C}$, which determines a biproduct-preserving presheaf $(\mathbf{A}_{\mathcal{H};\mathcal{O}})^{\text{op}} \to \mathbf{C}$. These form the spectrally enriched category $\mathbf{Mack}_{\mathcal{H};\mathcal{O}}(\mathbf{C})$. In particular, we will denote $\mathbf{Mack}_{\mathcal{H};\mathcal{O}}(\mathbf{Sp}) = \mathbf{Pre}(\mathbf{A}^{-}_{\mathcal{H};\mathcal{O}})$ simply as $\mathbf{Mack}_{\mathcal{H};\mathcal{O}}$.

Definition 4.9. Like $\mathcal{A}_{\mathcal{H};\mathcal{O}}$, $\mathbf{A}_{\mathcal{H};\mathcal{O}}$ has a symmetric monoidal structure (or promonoidal if $G \notin \mathcal{H}$). On objects this is given the product of $\mathcal{H}$-free G-sets, and on morphism spectra it is the map induced by $\mathbb{K}_\bullet$ on the product-of-spans functor (which, again, is a bilinear functor since it distributes over disjoint union).

The *smash product of spectral Mackey functors* is the symmetric monoidal structure on $\mathbf{Mack}_{\mathcal{H};\mathcal{O}}$ given by Day convolution over the induced promonoidal structure on $\mathbf{A}^{-}_{\mathcal{H};\mathcal{O}}$.

As promised, we can recover Borel and trivial G-spectra as special cases of this definition.

Proposition 4.10. *When $\mathcal{H}$ is a single conjugacy class (H), there is an enriched Quillen equivalence*

$$\mathbf{Mack}_{(H);\mathcal{O}} \simeq_Q \mathbf{Sp}^{BW_GH}.$$

for any operad $\mathcal{O}$. The left adjoint

$$\mathbf{Mack}_{(H);\mathcal{O}} \to \mathbf{Sp}^{BW_GH}.$$

in this equivalence is strong monoidal.

Moreover, this factors through a monoidal equivalence *(not just Quillen)*

$$\mathbf{Mack}_{(H);\mathcal{O}} \simeq \mathbf{Mack}_{(e);\mathcal{O}'}$$

where on the right we have the family containing only the trivial subgroup of W_GH, and any N_∞ operad $\mathcal{O}'$ for W_GH.

Proof. $\mathbf{A}^{-}_{(H);\mathcal{O}}$ has a single isomorphism class of object given by G/H. Its endomorphism spectrum at this object is $\mathbb{K}_{\bullet}$ applied to the permutative category of spans $\mathscr{A}^{\mathrm{eff}}_{(H);\mathcal{O}}(G/H, G/H)$ all of whose objects are sums of spans of the form

$$G/H \xleftarrow{g} G/H \xrightarrow{h} G/H$$

in which g and h are isomorphisms given by elements of the Weyl group $W_G H$ (which are automatically transfers for any $\mathcal{O}$). Each such span is isomorphic to

$$G/H = G/H \xrightarrow{hg^{-1}} G/H$$

and has trivial automorphism group in the category of spans. So $\mathscr{A}_{G;\mathcal{O}}(G/H, G/H)^{\mathrm{eff}}$ is the free permutative category on the discrete category with objects $W_G H$.

In particular, the categories $\mathscr{A}_{G;\mathcal{O}}(G/H, G/H)^{\mathrm{eff}}$ are isomorphic for any choice of category G and subgroup H with isomorphic Weyl group, and any choice of operad $\mathcal{O}$. This gives the second equivalence of the proposition.

For the first Quillen equivalence, our description of this hom-spectrum as $\mathbb{K}$ applied to a free permutative category on a discrete category gives an equivalence of ring spectra

$$\bigvee_{g \in W_G H} \mathbb{S} =: \mathbb{S}[W_G H] \simeq \mathbb{K}\mathscr{A}^{\mathrm{eff}}_{(H);\mathcal{O}}(G/H, G/H).$$

This defines an enriched functor from the discrete spectrally enriched category on $\mathcal{B}W_G H$ to $\mathbf{A}^{-}_{(H);\mathcal{O}}$, by sending the unique object to G/H. This functor is essentially surjective, and on hom-spectra induces a weak equivalence between cofibrant spectra. We conclude that restriction and left Kan extension along it gives a Quillen equivalence by [GM20, 2.4].

Finally, we observe that the product $(G/H) \times (G/H)$ decomposes as a disjoint union of subgroups $G/H \cap gHg^{-1}$ where the conjugates gHg^{-1} range over double cosets HgH. The components admitting maps from G/H are thus only those with $g \in N_G H$. Thus our spectrally enriched weak equivalence pulls back the promonoidal product presheaf represented by $G/H \times G/H$ on $\mathbf{A}^{-}_{(H);\mathcal{O}}$ to the one represented by $W_G H \times W_G H \in \mathcal{F}in\mathcal{S}et^{W_G H}$ as a presheaf on $\mathcal{B}W_G H$, so it is strong promonoidal. $\square$

Proposition 4.11. *When $\mathcal{O}$ is the trivial operad $\mathcal{O}^{\mathrm{triv}}$, there is an enriched Quillen equivalence*

$$\mathbf{Mack}_{G;\mathcal{O}^{\mathrm{triv}}} \simeq_Q \mathbf{Sp}^G_{\mathrm{triv}}.$$

Proof. This is similar to Proposition 4.10. If $\mathcal{O}$ has no transfers, then

$$\mathscr{A}^{\mathrm{eff}}_{G;\mathcal{O}^{\mathrm{triv}}}(G/H, G/K)$$

has as objects only the spans

$$G/H = G/H \to G/K$$

so it is the free permutative category on the discrete category with objects $\mathcal{A}_{G;\mathcal{O}^{\mathrm{triv}}}(G/H, G/K)$. Once again, this defines a spectrally enriched weak equivalence from the discrete spectral category on $\mathcal{O}rb^G$ to $\mathbf{A}^{-}_{G;\mathcal{O}^{\mathrm{triv}}}$, and we conclude that it induces a Quillen equivalence on presheaves by [GM20, 2.4]. □

4.3 The Guillou-May Theorem

Lemma 4.12. *There is a spectrally enriched functor $\sigma : \mathbf{A}^{-}_{G;\mathcal{O}} \to \mathbf{Sp}^G_{\mathcal{O}}$, which sends each orbit G/H to a fibrant replacement $Q\Sigma^\infty_{\mathcal{O}}(G/H)_+$ of its suspension spectrum. Furthermore, this σ is weakly full and faithful, in the sense that the induced map on each mapping spectrum is an equivalence of spectra.*

Proof. Consider, for $H \leq G$ and finite G-set T, the mapping spectrum

$$\begin{aligned}\mathbf{Sp}^G_{\mathcal{O}}(Q\Sigma^\infty_{\mathcal{O}}(G/H)_+, Q\Sigma^\infty_{\mathcal{O}}T_+) &\simeq \mathbf{Sp}^G_{\mathcal{O}}(\Sigma^\infty_{\mathcal{O}}(G/H)_+, Q\Sigma^\infty_{\mathcal{O}}T_+)\\ &\simeq \mathcal{T}op^G((G/H)_+, \Omega^\infty_{\mathcal{O}}Q\Sigma^\infty_{\mathcal{O}}T_+)\\ &\simeq (\Omega^\infty_{\mathcal{O}}Q\Sigma^\infty_{\mathcal{O}}T_+)^H\end{aligned}$$

which deloops the mapping space

$$\begin{aligned}\mathcal{S}p^G_{\mathcal{O}}(\Sigma^\infty_{\mathcal{O}}(G/H)_+, Q\Sigma^\infty_{\mathcal{O}}T_+) &\simeq \mathcal{T}op^G((G/H)_+, \Omega^\infty_{\mathcal{O}}Q\Sigma^\infty_{\mathcal{O}}T_+)\\ &\simeq (\Omega^\infty_{\mathcal{O}}Q\Sigma^\infty_{\mathcal{O}}T_+)^H\\ &\simeq (\Omega^\infty Q\Sigma^\infty \mathbb{P}_{\mathcal{O}}(T))^H.\end{aligned}$$

Since Q can be taken to be $\Omega^\infty\Sigma^\infty$, and $\Omega\Sigma$ on $\mathcal{O}$-algebras gives group-completion on each fixed points [BH19, 3.6], this mapping spectrum is a group-completed delooping of the fixed points of the free $\mathcal{O}$-algebra $\mathbb{P}_{\mathcal{O}}(T)^H$. One variant of the tom Dieck splitting [BH19, 4.2] for the restriction $T|_H$ as an H-space expresses these fixed points as

$$\bigvee_{X\in\pi_0\mathrm{adm}(\mathcal{O})(H)} E\mathrm{Aut}(X)_+ \underset{\mathrm{Aut}(X)}{\wedge} \mathrm{Maps}^H(X,T).$$

This is the classifying space of the groupoid $\mathrm{adm}(\mathcal{O})(H)_{/T}$, whose objects are given by an admissible H-set $X \in \mathrm{adm}(\mathcal{O})(H)$ along with an H-equivariant map $X \to T$ — or equivalently a G-equivariant map $G \times_H X \to T$. Now $G\times_H -$ gives an equivalence between $\mathcal{F}in\mathcal{S}et^H$ and $(\mathcal{F}in\mathcal{S}et^G)_{/(G/H)}$ which carries $\mathrm{adm}(\mathcal{O})(H)$ to maps in $\mathcal{T}_{\mathcal{O}}$, so $\mathrm{adm}(\mathcal{O})(H)_{/T}$ is equivalent to the groupoid of spans $\mathscr{A}^{\mathrm{eff}}_{G;\mathcal{O}}(G/H,T)$.

Hence the delooping $\mathbf{Sp}^G_{\mathcal{O}}(\Sigma^\infty_{\mathcal{O}}(G/H)_+, \Sigma^\infty_{\mathcal{O}}T_+)$ is equivalent to $\mathbb{K}\mathscr{A}^{\mathrm{eff}}_{G;\mathcal{O}}(G/H,T)$, which is by definition $\mathbf{A}_{G;\mathcal{O}}(G/H,T)$.

What remains is to verify that all these equivalences on mapping spectra collect together to an associative spectrally-enriched functor. This is equivalent to the claim that our identification

$$\mathbb{P}_{\mathcal{O}}(T)^H \simeq B(\mathrm{adm}(\mathcal{O})(H)_{/T})$$

is natural, both with respect to maps $T \to T'$ and to restrictions $K \le H$. The former is part of the statement of our tom Dieck splitting [BH19, 4.2]. For the latter, examining the proof of [BH19, 4.2] we reduce to the naturality of the identification

$$(\mathcal{O}(n) \underset{\Sigma_n}{\times} T)^H \xrightarrow{\sim} |B(*, \mathcal{F}_{n,H}, J_H \underset{\Sigma_n}{\times} T)|^H$$

with respect to restriction to $K \leq H$. Here the target is a bar construction over $\mathcal{F}_{n,H}$, defined as the full subcategory of $\mathcal{O}rb^{H\times\Sigma_n}$ on orbits of the form $(H \times \Sigma_n)/\Gamma_T$ for $L \leq K$ and T an admissible L-set of cardinality n. J_H is simply the inclusion of such orbits into H-spaces.

To show that this is natural with respect to restriction to $K \leq H$, consider the diagram

$$\begin{tikzcd}
(\mathcal{O}(n) \underset{\Sigma_n}{\times} T)^K \arrow[r] & \left|B(*, \mathcal{F}_{n,K}, J_K \underset{\Sigma_n}{\times} T)\right|^K \\
 & \left|B(*, \mathcal{F}_{n,K}, J_K \underset{\Sigma_n}{\times} T)\right|^H \arrow[u] \\
(\mathcal{O}(n) \underset{\Sigma_n}{\times} T)^H \arrow[r] \arrow[uu] & \left|B(*, \mathcal{F}_{n,H}, J_H \underset{\Sigma_n}{\times} T)\right|^H \arrow[u, "\alpha"']
\end{tikzcd}$$

The bottom right map α is the H-fixed points of the map on bar constructions resulting from the functor $H \times_K - : \mathcal{F}_{n,K} \to \mathcal{F}_{n,H}$ and the natural map $J_H \circ (H \times_K -) \to J_K$. Thus its composition with the bottom rightwards map is the H-fixed points of the same map

$$\mathcal{O}(n) \underset{\Sigma_n}{\times} T \to |B(*, \mathcal{F}_{n,K}, J_K)| \underset{\Sigma_n}{\times} T$$

whose K-fixed points give the top rightwards map. The result then follows from naturality of the restriction map between fixed points. $\square$

Theorem 4.13. *Let $\mathcal{O}$ be any N_∞ operad for G. There is a spectrally enriched Quillen equivalence*

$$\mathbf{Sp}^G_{\mathcal{O}} \simeq_Q \mathbf{Mack}_{G;\mathcal{O}}$$

between the spectral model categories of G-spectra indexed by $\mathcal{O}$ and spectral Mackey functors for $(G;\mathcal{O})$.

Proof. The suspension spectra $\Sigma^\infty_{\mathcal{O}}(G/H)_+$ form a set of compact generators [BH19, 3.25], so their fibrant replacements are bifibrant compact generators. This is sufficient to ensure the spectrally enriched restricted Yoneda functor $Y \mapsto \mathbf{Sp}^G_{\mathcal{O}}(-, Y)$ is the right adjoint in a Quillen equivalence [SS03, 3.9.3] between $\mathbf{Sp}^G$ and spectral presheaves on the full subcategory at these $Q\Sigma^\infty_{\mathcal{O}}(G/H)_+$. Then by Lemma 4.12, σ forms a mapping-spectrum-wise weak equivalence of spectrally enriched categories from $\mathbf{A}^-_{G;\mathcal{O}}$ to this full subcategory, so the extension $\sigma_! \dashv \sigma^*$ is also a Quillen equivalence [GM20, 2.4]. □

Remark 4.14. Our statement of Theorem 4.13 sounds a bit stronger than [GM11, 1.1]: rather than a zig-zag of Quillen equivalences, we have produced a single Quillen equivalence given by restricted Yoneda. Ultimately, this is possible because our starting point of $\mathcal{O}$-spectra is much closer to Mackey functors than orthogonal G-spectra are. Indeed, the existence of transfers is precisely what an $\mathcal{O}$-action encodes. Because of this, we have a much easier time constructing the Barratt-Priddy-Quillen map of Lemma 4.12 and ensuring that it is on-the-nose associative and takes values in bifibrant spectra. In contrast, note that our $\mathcal{O}$-spectra themselves, for $\mathcal{O}$ a Steiner operad, only compare to orthogonal spectra via a zig-zag in [BH19, 6.2].

Remark 4.15. The spectrally enriched category of Mackey functors has some advantages over $\mathbf{Sp}^G_{\mathcal{O}}$. They are often easier to construct explicitly; for example, [BO15] shows how to construct spectral Mackey functors out of combinatorial data, and in particular the application of [BO15, 8.5] in our setting gives an explicit construction of Eilenberg-MacLane $\mathcal{O}$-spectra for abelian $\mathcal{O}$-Mackey functors. While we could pass these constructions into $\mathcal{O}$-spectra via the right Quillen adjoint, given by a coend, it would become much less explicit.

Similarly, spectral Mackey functors admit a direct construction of the geometric fixed points as a Kan extension along a map between spectral Burnside categories, which we will discuss in Section 5.2. We contrast this with the construction of [BH19, 3.33], which requires taking a fibrant replacement after smashing with $\widetilde{E}\mathcal{P}$.

Furthermore, per Definition 4.9, $\mathbf{Mack}_{G;\mathcal{O}}$ is automatically an enriched monoidal model category under Day convolution. Presumably, one could define the operadic smash product of $\mathcal{O}$-spectra to get a symmetric monoidal model category; one would then have to verify the monoid axiom by hand, as well as other properties such as monoidality of the geometric fixed points. No such construction exists in the literature.

For this reason, we will largely work with $\mathbf{Mack}_{G;\mathcal{O}}$ as our category of $\mathcal{O}$-spectra for the rest of this thesis. In this light, we may view Theorem 4.13 as a recognition principle for $\mathcal{O}$-algebras.

4.4 Applications of Eilenberg-MacLane $\mathcal{O}$-Spectra

Let us now zero in on one particular benefit of working in spectral Mackey functors for $\mathcal{O}$: the existence of an Eilenberg-MacLane spectrum $H\underline{M}$ for any abelian Mackey functor $\underline{M}$ for $(\mathcal{H}, \mathcal{O})$, such that each $(H\underline{M})^H$ is an Eilenberg-MacLane spectrum on $\underline{M}(H)$.

First of all, we can use these as the basis of induction up the Postnikov tower. That is, any map which commutes with limits and is an equivalence on Eilenberg-MacLane spectra must be an equivalence on any spectral Mackey functor X which is *bounded below*, in the sense that $\underline{\pi}_i X = 0$ for small enough i. This is formal once we construct Postnikov towers:

Proposition 4.16. *Let X be a bounded-below spectral Mackey functor for $(\mathcal{H},\mathcal{O})$. Then X is the homotopy limit of a Postnikov tower*

$$X \to \cdots \to X_n \to X_{n-1} \to \dots$$

such that

1. *for $i > n$, $\underline{\pi}_i X_n = 0$, i.e. X_n is n-coconnective, and*
2. *for $i \leq n$, $\underline{\pi}_i X \to \underline{\pi}_i X_n$ is an isomorphism.*

In particular, each fiber

$$F_n \to X_n \to X_{n-1}$$

is $\Sigma^n H\underline{\pi}_n X$.

Proof. $\mathbf{Mack}_{\mathcal{H};\mathcal{O}}$ is a cofibrantly generated model category, so we can form the homotopy initial map $X \to X_n$ among spectra Y with $\underline{\pi}_i Y = 0$ for $i > n$ by the small object argument. Explicitly, set $X^{(0)} = X$ and form $X^{(k+1)}$ from $X^{(k)}$ as the homotopy cofiber of the sum of all the maps defining homotopy groups above i:

$$\bigvee_{i>n} \bigvee_{f \in \pi_i(X)(H)} \Sigma^\infty_{\mathcal{O}} \left(S^i \wedge (G/H)_+\right) \to X^{(k)}$$

and take the homotopy colimit $X_n = X^{(\omega)}$.

To verify that $X \to X_n$ is an isomorphism on $\underline{\pi}_i$ for $i \leq n$, it suffices to know that

$$\underline{\pi}_i \left(\Sigma^\infty_{\mathcal{O}} (S^j \wedge (G/K)_+)^H\right)$$

vanishes for $j > n \geq i$, i.e. that $\left(\Sigma^\infty_{\mathcal{O}} (G/K)_+\right)^H$ is connective. This follows from the tom Dieck splitting. □

As a second application, if we can describe an $\mathcal{O}$-spectrum as a module over an Eilenberg-MacLane spectrum, we can reduce to algebraic computations in the derived category of abelian Mackey functors.

Proposition 4.17. *The category of $H\underline{A}_{\mathcal{H};\mathcal{O}}$-modules is Quillen equivalent to the category of chain complexes of abelian Mackey functors for ($\mathcal{H}$,$\mathcal{O}$).*

Proof. Tracing out the construction of Eilenberg-MacLane spectra from [BO15, 7.5], an $H\underline{A}_{\mathcal{H};\mathcal{O}}$-module structure on a spectral Mackey functor X amounts to a factorization of the presheaf

$$X : \mathbf{A}^-_{\mathcal{H};\mathcal{O}} \to \mathbf{Sp}$$

through the spectral category which is $\mathbb{K}_\bullet$ applied to the discrete permutative category on $\mathcal{A}^-_{\mathcal{H};\mathcal{O}}$. But presheaves on this spectral category are Quillen equivalent to chain complexes of abelian presheaves on $\mathcal{A}^-_{\mathcal{H};\mathcal{O}}$ by [SS03, 5.1.6]. □

Finally, the existence of Eilenberg-MacLane spectral Mackey functors for $\mathcal{O}$, along with the closed monoidal structure of $\mathbf{Mack}_{G;\mathcal{O}}$, allows us to define $\mathcal{O}$-Bredon cohomology.

Definition 4.18. For X a spectral $\mathcal{O}$-Mackey functor and $\underline{M}$ an abelian $\mathcal{O}$-Mackey functor, the Bredon cohomology $\mathcal{O}$-Mackey functors $\underline{H}^\bullet(X; \underline{M})$ are the homotopy Mackey functors of the internal hom spectral Mackey functor $F(X, H\underline{M})$.

4.5 The Burnside Quasicategory

We close this section by considering another way of modeling Mackey functors of spectra.

Recall that if Y is any spectrum, and $\mathcal{G}$ is some small permutative groupoid, giving a map of spectra $\mathbb{K}\mathcal{G} \to Y$ is equivalent to giving a map of E_∞-spaces $B\mathcal{G} \to \Omega^\infty Y$. In particular, the data of a spectrally enriched Mackey functor $\mathbb{K}_\bullet(\mathscr{A}^{\text{eff}}_{\mathcal{H};\mathcal{O}})^{\text{op}} \to \mathbf{Sp}$ is the same as the data of a E_∞-space-enriched functor $B_\bullet(\mathscr{A}^{\text{eff}}_{G;\mathcal{O}})^{\text{op}} \to \mathcal{S}p$.

Next we consider the equivalence between topologically enriched categories and quasicategories; E_∞-space-enriched functors are sent to additive functors between quasicategories. Thus in particular, as triangulated categories, the homotopy category of $\mathbf{Mack}_{\mathcal{H};\mathcal{O}}(\mathbf{Sp})$ is equivalent to that of the stable quasicategory $\text{Fun}_\oplus(N(\mathscr{A}^{\text{eff}}_{\mathcal{H};\mathcal{O}})^{\text{op}}, Sp)$ of additive presheaves from the nerve of the $(2,1)$-category $\mathscr{A}^{\text{eff}}_{\mathcal{H};\mathcal{O}}$ (with its semiadditive structure) to the stable quasicategory of spectra.

Fortunately, Barwick has given us a succinct description of this nerve in [Bar17]:

Definition 4.19 ([Bar17, 5.7])**.** The *effective Burnside quasicategory for ($\mathcal{H}$;$\mathcal{O}$)* is the quasicategory $A^{\text{eff}}_{\mathcal{H};\mathcal{O}}$ whose n-cells are diagrams in $\mathcal{F}in\mathcal{S}et^{\mathcal{H}}$ as in Figure 4.1, such that each square is Cartesian and each leftwards map lies in $\mathcal{T}_\mathcal{O}$. This is a 2-category in the sense of [Lur17, 2.3.4.1], and semiadditive in the sense of [Lur17, 6.1.6.13]. For another additive quasicategory C, the quasicategory $\text{Mack}_{\mathcal{H};\mathcal{O}}(C)$ of *Mackey ∞-functors for ($\mathcal{H}$;$\mathcal{O}$) in C* is the additive presheaves $\text{Fun}_\oplus((A^{\text{eff}}_{G;\mathcal{O}})^{\text{op}}, C)$.

$$
\begin{tikzcd}[column sep=small]
 & & & & A_{0n} \arrow[dl] \arrow[dr] & & & & \\
 & & & \dots \arrow[dl] & & A_{1n} \arrow[dl] \arrow[dr] & & & \\
 & & A_{02} \arrow[dl] \arrow[dr] & & \dots \arrow[dl] & & A_{2n} \arrow[dl] \arrow[dr] & & \\
 & A_{01} \arrow[dl] \arrow[dr] & & A_{12} \arrow[dl] \arrow[dr] & & \dots \arrow[dl] & & \dots \arrow[dr] & \\
A_{00} & & A_{11} & & A_{22} & & \dots & & A_{nn}
\end{tikzcd}
$$

Figure 4.1: An n-cell in $A^{\mathrm{eff}}_{\mathcal{O}}(G)$

CHAPTER 5

Recollements and Geometric Fixed Points of $\mathcal{O}$-Spectra

A major disadvantage in working with $\mathcal{O}$-spectra compared to orthogonal G-spectra is that they lack a good point-set model of geometric fixed points, to separate spectra into information coming from different isotropy subgroups. Our goal in this chapter is to address this by using spectral Mackey functors.

Recall from Section 3.3 the classical isotropy separation sequence for genuine G-spectra. This becomes particularly useful when we extend it by considering the isotropy separation sequence at the "cofree" spectrum $F(E\mathcal{F}_+, X)$ as in Figure 5.1, which is [GM95, Diagram C].

$$\begin{tikzcd}
E\mathcal{F}_+ \wedge X \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \widetilde{E}\mathcal{F} \wedge X \arrow[d] \\
E\mathcal{F}_+ \wedge F(E\mathcal{F}_+, X) \arrow[r] & F(E\mathcal{F}_+, X) \arrow[r] & \widetilde{E}\mathcal{F} \wedge F(E\mathcal{F}_+, X)
\end{tikzcd}$$

Figure 5.1: Isotropy separation diagram for G-spectrum X at family of subgroups $\mathcal{F}$

The map $X \to F(E\mathcal{F}_+, X)$ is an $\mathcal{F}$-equivalence [GM95, 1.2], so the right-hand square (the *fracture square*) is homotopy Cartesian [Gla17, 3.16]. Thus we can recover a G-spectrum X by knowing only the span in the bottom right corner

$$F(E\mathcal{F}_+, X) \to \widetilde{E}\mathcal{F} \wedge F(E\mathcal{F}_+, X) \leftarrow \widetilde{E}\mathcal{F} \wedge X.$$

Since the horizontal morphisms in the bottom row are determined by $F(E\mathcal{F}_+, X)$, we only need three pieces of data [Gla17, 3.26]: the $\mathcal{F}$-contractible part $\widetilde{E}\mathcal{F} \wedge X$, the cofree part

$F(E\mathcal{F}_+, X)$, and the map

$$\widetilde{E}\mathcal{F} \wedge X \to \widetilde{E}\mathcal{F} \wedge F(E\mathcal{F}_+, X).$$

In other words, this specifies a recollement of the ∞-category of G-spectra into those of $\mathcal{F}$-contractible G-spectra and $\mathcal{F}$-cofree G-spectra. We will review what this means in Section 5.1, and then bring these constructions to spectral Mackey functors. In Section 5.2, we will show that the analogous isotropy separation functors for spectral Mackey functors can be recovered as Kan extensions along maps between Burnside categories.

In order to show that the maps give a recollement, we will make use of Glasman's computation of deformation retractions in homotopy fibers of the Burnside quasicategory [Gla17]. Therefore, we will need to work with the *quasicategory* of spectral Mackey functors from Section 4.5. Since we are most interested in the *spectral model category* of Section 4.2 in the later chapters, we will keep a close eye on the behavior of the homotopy categories.

Along the way, we will also verify that the decomposition along $\mathcal{F}$ also respects the monoidal structure. In doing so, we are presented with an obstacle: very little exists in the literature about Day convolution over monoidal quasicategories [Gla16] and even less over promonoidal quasicategories [BGS20]. We will only show that the recollement is "homotopy monoidal", i.e. it respects the monoidal structure on the homotopy category, which is equivalent to the homotopy category of the spectrally enriched category $\mathbf{Mack}_{G;\mathcal{O}}$ (and thus that of $\mathbf{Sp}^G_{\mathcal{O}}$, though the latter has no monoidal structure defined). This will be sufficient for our Picard group computations.

5.1 Recollements

Definition 5.1 ([BBD82, 1.4.4][Lur17, A.8.1])**.** A diagram

$$X \underset{\xrightarrow[i_*]{}}{\overset{\xrightarrow{i_!}}{\xleftarrow{i^*}}} Y \underset{\xrightarrow[p_*]{}}{\overset{\xrightarrow{p_!}}{\xleftarrow{p^*}}} Z \tag{5.1}$$

of stable quasicategories (resp. triangulated categories), along with adjunctions $i_! \dashv i^* \dashv i_*$ and $p_! \dashv p^* \dashv p_*$ forms a *recollement of Y into X and Z* when

1. All six functors are exact (which is guaranteed from the adjunction in the quasicategory setting),
2. i_*, $i_!$, and p^* are fully faithful,
3. $p_! i_!$ is zero, and
4. $p_!$ and i^* are jointly conservative, i.e. if $p_! f$ and $i^* f$ are both equivalences (resp. isomorphisms), so is f.

In particular, a recollement of stable quasicategories induces a recollement of their triangulated homotopy categories.

Proposition 5.2 ([Lur17, A.8.11])**.** *Suppose X, Y, Z are stable quasicategories, and we have a recollement of Y into X and Z. Then there is a Cartesian fibration $t : M \to \Delta^1$ with fibers $t^{-1}(0) \simeq X$, $t^{-1}(0) \simeq Z$ determining an exact functor $T : X \to Z$, such that $Y \simeq M$. In particular, the homotopy category $\pi_1 Y$ is equivalent to the triangulated category whose objects are triples*

$$(x \in X, z \in Z, f \in \pi_1 Z(z, Tx)).$$

Definition 5.3. The functor T in Proposition 5.2 is called the *Tate functor* of the recollement. It can be computed from Diagram 5.1 as the composite $p_! i_*$.

Remark 5.4. The equivalence of Proposition 5.2 sends $y \in Y$ to the triple

$$(i^* y \ , \ p_! y \ , \ p_! \eta_y : p_! y \to p_! i_* i^* y)$$

if η is the unit of the $i^* \dashv i_*$ adjunction. Its inverse sense (x, z, f) to the homotopy pullback of the span

$$i_* x \xrightarrow{\xi_{i_* x}} p^* p_! i_* x \xleftarrow{p^* f} p^* z$$

if ξ is the counit of the $p_! \dashv p^*$ adjunction.

Definition 5.5. Suppose X, Y, Z are monoidal stable quasicategories (resp. tensor triangulated categories). A *monoidal recollement* is a recollement as in Definition 5.1, along with the structure of a strong monoidal functor on $p_!$ and i^*.

A *homotopy monoidal* recollement is a recollement of stable quasicategories which induces a monoidal recollement on their homotopy categories.

In such a recollement, the triangulated category of triples (x, z, f) from Proposition 5.2 has a natural monoidal structure:

$$(x, z, f) \otimes (x', z', f') = (x \otimes x', z \otimes z', z \otimes z' \to Tx \otimes Tx' \to T(x \otimes x'))$$

using the lax monoidal structure on $T = p_! i_*$, since $p_!$ is strong monoidal and i_* is the right adjoint to strong monoidal i^*. Expanding out the induced lax monoidal structure, the following is immediate.

Proposition 5.6. *If Y has the structure of a homotopy monoidal recollement into X and Z, then the equivalence on homotopy categories of Proposition 5.2 is strong monoidal.*

5.2 Isotropy Separation via Kan Extensions

The inclusion

$$i : \mathcal{F}in\mathcal{S}et^{\mathcal{F}} \hookrightarrow \mathcal{F}in\mathcal{S}et^{G}$$

preserves pullbacks, and so it respects compositions of spans and gives a 2-functor, which we will also denote i,

$$i : \mathscr{A}^{\text{eff}}_{\mathcal{F};\mathcal{O}} \hookrightarrow \mathscr{A}^{\text{eff}}_{G;\mathcal{O}}.$$

This is moreover a $\mathcal{P}erm$-enriched functor, since the inclusion also preserves coproducts. Thus it induces a spectrally-enriched functor

$$\mathbb{K}_\bullet(i) : \mathbf{A}^{\text{eff}}_{\mathcal{F};\mathcal{O}} \hookrightarrow \mathbf{A}^{\text{eff}}_{G;\mathcal{O}}.$$

By an abuse of notation, we will also call this functor i for brevity. Meanwhile, diagrams as in Figure 4.1 remain n-cells since the squares are still Cartesian, so we get an ∞-functor

$$i : A^{\mathrm{eff}}_{\mathcal{F};\mathcal{O}} \hookrightarrow A^{\mathrm{eff}}_{G;\mathcal{O}}.$$

Since heads of spans between $\mathcal{F}$-free G-sets are again $\mathcal{F}$-free as $\mathcal{F}in\mathcal{S}et^{\mathcal{F}}$ is a sieve, each such i is full and faithful.

In particular, we can form the restriction

$$i^* : \mathbf{Mack}_{G;\mathcal{O}} \to \mathbf{Mack}_{\mathcal{F};\mathcal{O}}$$

along the spectrally enriched functor i. This has both left and right adjoints

$$i_*, i_! : \mathbf{Mack}_{\mathcal{F};\mathcal{O}} \to \mathbf{Mack}_{G;\mathcal{O}}$$

given by left and right Kan extension along i. Both adjunctions descend to the homotopy category: $i_! \dashv i^*$ is Quillen in our projective model structure [HHR21, 5.4.18], while $i^* \dashv i_*$ is Quillen on the injective model structure. Both i_* and $i_!$ are fully faithful because i is.

Working instead with the ∞-functor i lets us prove some useful things on the homotopy category:

Proposition 5.7. *The composition $i_! i^*$ of restriction to $A_{\mathcal{F};\mathcal{O}}$ followed by left Kan extension is identified, under the equivalence of the homotopy category of $\mathbf{Mack}_{G;\mathcal{O}}$ with that of* $\mathrm{Mack}_{G;\mathcal{O}}$*, with $E\mathcal{F}_+ \wedge -$. Therefore the right adjoint $i_* i^*$ is $F(E\mathcal{F}_+, -)$.*

Proof. First of all, we must know this composition preserves Mackey functors, i.e. *additive* presheaves. This follows from [Gla17, 2.20].

Construct $E\mathcal{F}_+$ as the bar construction

$$E\mathcal{F}_+ = |B(*, \mathcal{O}rb^{\mathcal{F}}, J)|$$

of [Elm83] as simplified in [BH19, 4.6], where $J(G/H) = (G/H)_+$. Then its fixed points, $(E\mathcal{F}_+ \wedge X)^G$ are given by the bar construction

$$(E\mathcal{F}_+ \wedge X)^G = |B(*, \mathcal{O}rb^{\mathcal{F}}, (J \wedge X)^G)|$$

because the finite limit $(-)^G$ commutes with geometric realizations. This expresses these fixed points as the homotopy colimit of X^H over $G/H \in \mathcal{F}in\mathcal{S}et^{\mathcal{F}}$. That this also computes the left Kan extension at G/G is [Gla17, 2.28]: spans $i(S) \leftarrow X \rightarrow T$ with $X \rightarrow i(S)$ an identity are homotopy cofinal in all such spans, or in all such spans with $X \rightarrow i(S)$ lying in $\mathcal{T}_{\mathcal{O}}$. By restricting to $K < G$ and applying the same argument, we see $E\mathcal{F}_+ \wedge X$ agrees with $i_! i^* X$ on each fixed-point spectrum. $\square$

On the other hand, set $\mathcal{N}$ to be the class of subgroups not in $\mathcal{F}$. The inclusion

$$j : \mathcal{F}in\mathcal{S}et^{\mathcal{N}} \rightarrow \mathcal{F}in\mathcal{S}et^{G}$$

does *not*, in general, preserve pullbacks: intersections of subgroups in $\mathcal{N}$ could be in $\mathcal{F}$. So j, unlike i, does not define a functor on any of our Burnside categories. However, it has a right adjoint: the functor

$$p : \mathcal{F}in\mathcal{S}et^{G} \rightarrow \mathcal{F}in\mathcal{S}et^{\mathcal{N}}$$

defined by

$$p(T) = \{t \in T \mid \mathrm{stab}(t) \in \mathcal{N}\} = \bigcup_{K \in \mathcal{N}} T^K.$$

As a right adjoint, this *does* preserve pullbacks. Again, we will write each resulting functor between each variant of Burnside categories as p as well.

Kan extension along i and p thus gives us a diagram

$$\begin{tikzcd} \mathrm{Mack}^{\mathrm{eff}}_{\mathcal{F};\mathcal{O}} \arrow[r, shift left=2, "i_!"] \arrow[r, shift right=2, "i_*"'] & \mathrm{Mack}^{\mathrm{eff}}_{G;\mathcal{O}} \arrow[l, "i^*"'] \arrow[r, shift left=2, "p_!"] \arrow[r, shift right=2, "p_*"'] & \mathrm{Mack}^{\mathrm{eff}}_{\mathcal{N};\mathcal{O}} \arrow[l, "p^*"'] \end{tikzcd}$$

like Diagram 5.1, which we will show induces a recollement on their homotopy categories. As a corollary, the cofiber $X \rightarrow \widetilde{E}\mathcal{F} \wedge X$ of $E\mathcal{F}_+ \wedge X \rightarrow X$ must be equivalent to the cofiber $X \rightarrow p_! p^* X$ of $i_! i^* X \rightarrow X$.

More generally, if $\mathcal{H}$ is any class of subgroups, a family $\mathcal{F} \subseteq \mathcal{H}$ defines a full subcategory $i : A^{\text{eff}}_{\mathcal{F};\mathcal{O}} \hookrightarrow A^{\text{eff}}_{\mathcal{H};\mathcal{O}}$ as long as $\mathcal{F}$ is closed under taking smaller subgroups in $\mathcal{H}$. As long as this is the case, we can still define $p : A^{\text{eff}}_{\mathcal{H};\mathcal{O}} \to A^{\text{eff}}_{\mathcal{N};\mathcal{O}}$ as above, where $\mathcal{N} = \mathcal{H} - \mathcal{F}$.

5.3 Short Exact Sequences of Additive Quasicategories

Again let $\mathcal{F}$ be downward-closed in $\mathcal{H}$ and $\mathcal{N}$ its upward-closed complement. The sequence

$$A^{\text{eff}}_{\mathcal{F};\mathcal{O}} \xrightarrow{i} A^{\text{eff}}_{\mathcal{H};\mathcal{O}} \xrightarrow{p} A^{\text{eff}}_{\mathcal{N};\mathcal{O}} \tag{5.2}$$

of quasicategories induces a short exact sequence of commutative monoids of homotopy classes of objects. We will now show that it should be thought of as a short exact sequence in a much stronger sense.

Consider the idea of a diagram

$$A \xrightarrow{i} B \xrightarrow{p} C$$

being a "short exact sequence on morphisms". We begin by asking that for each $b, b' \in B$ the map of commutative monoid spaces

$$B(b, b') \to C(pb, pb')$$

be surjective on π_0. Then we should ask that its kernel be identified with maps "coming from A". We do this by Yoneda, identifying b, b' with their represented functors $B(b, -)$ and $B(b', -)$ and considering maps from $i^*B(b, -)$ to $i^*B(b', -)$. By adjunction, this should be the same as evaluating $i_!i^*B(b, -)$ on b'. Thus we want a cofiber sequence

$$i_!i^*B(b, -) \to B(b, -) \to C(pb, p(-))$$

of maps from B to the quasicategory of commutative monoid spaces. Equivalently, the counit map

$$i_!i^*B(b, -) \to B(b, -)$$

should give a homotopy equivalence from its source to the full subcategory of $B(b, b')$ on those $f : b \to b'$ with $pf \simeq 0$.

Proposition 5.8 ([Gla17, B.3])**.** *If $i : A \to B$ is a full and faithful map of semiadditive quasicategories, the homotopy fiber of $i_! i^* B(b, -) \to B(b, -)$ over $f : b \to b'$ is equivalent to a Kan complex $\mho_f$ whose n-cells are given by $(n+2)$-cells*

$$\sigma : \Delta^{n+2} \to B$$

such that

1. *The long edge $\sigma(\Delta^{\{0,n+2\}})$ is f, and*
2. *Each internal vertex $\sigma(i)$ for $0 < i < n+2$ is sent into the image of i.*

We think of this as a "space of factorizations of f through i".

Definition 5.9. A sequence

$$A \xrightarrow{i} B \xrightarrow{p} C \tag{5.3}$$

of semiadditive quasicategories is a *short exact sequence* if it satisfies the following conditions:

1. (Injectivity on objects) i is full and faithful.
2. (Injectivity on morphisms) For each $b \in B$, the homotopy fiber $\mho_f$ of Proposition 5.8 is either empty or contractible.
3. (Surjectivity on objects) p is essentially surjective.
4. (Surjectivity on morphisms) For each $b, b' \in B$, the map $B(b, b') \to C(pb, pb')$ is surjective on π_0.
5. (Exactness on objects) The essential image of i is those $b \in B$ such that $pb \simeq 0$.
6. (Exactness on morphisms) Each edge $f : b \to b'$ in B with $pf \simeq 0$ admits a factorization $b \to ia \to b'$ for $a \in A$.

Lemma 5.10. *The sequence of semiadditive quasicategories (5.2) is a short exact sequence, as is its opposite.*

Proof. Criteria 1, 3, 4, and 5 are clear from the definitions. So is 6: a map in $A^{\mathrm{eff}}_{G;\mathcal{O}}$ projecting to 0 is given by $S \leftarrow X \rightarrow T$ with X $\mathcal{F}$-free, in which case it factors as $S \leftarrow X = X$ and $X = X \rightarrow T$, where $X \in A^{\mathrm{eff}}_{\mathcal{F};\mathcal{O}}$. So we just need to show the homotopy fibers of Proposition 5.8 are contractible. We follow Glasman [Gla17, B.4] in providing a deformation retract onto the choice of factorization we have just described.

Explicitly, an $(n-2)$-cell in the homotopy fiber $\mho$ of

$$i_! i^* A^{\mathrm{eff}}_{\mathcal{H};\mathcal{O}}(S,-) \rightarrow A^{\mathrm{eff}}_{G;\mathcal{O}}(S,-)$$

over $S \leftarrow X \rightarrow T$ is given by an n-cell in $A^{\mathrm{eff}}_{\mathcal{H};\mathcal{O}}$, i.e. a diagram as in Figure 4.1, such that

1. each vertex in the diagram except A_{00} and A_{nn} is $\mathcal{F}$-free, and

2. the composition $A_{00} \leftarrow A_{0n} \rightarrow A_{nn}$ is our $S \leftarrow X \rightarrow T$.

Using this description, the chosen factorization in the previous paragraph gives the 0-cell

$$\begin{tikzcd}
 & & X \arrow[dl, equal] \arrow[dr, equal] & & \\
 & X \arrow[dl] \arrow[dr, equal] & & X \arrow[dl, equal] \arrow[dr] & \\
S & & X & & T
\end{tikzcd}$$

We give a homotopy

$$h : \mho \times \Delta^1 \rightarrow \mho$$

which restricts to the identity on $0 \in \Delta^1$ and the constant map at this factorization on $1 \in \Delta^1$. Our h takes any $(n-2)$-cell and each of the $(n-1)$ maps

$$\sigma : [n-2] \rightarrow [1]$$

and outputs the $(n-2)$-cell in which the top square of A_{ij} such that $\sigma(n+2-i) = \sigma(j) = 0$ is replaced with the identity square on A_{0n}. $\square$

5.4 Recollements for Functor Categories

Our use for Lemma 5.10 is that it is exactly the condition necessary to get a recollement on the homotopy category of Mackey functors.

Lemma 5.11. *Suppose we have a short exact sequence of quasicategories as in Definition 5.9. Let S be any stable quasicategory. Then restriction and Kan extensions along i and p yield a recollement (Definition 5.1) of* $\mathrm{Fun}_{\oplus}(B, S)$ *into* $\mathrm{Fun}_{\oplus}(A, S)$ *and* $\mathrm{Fun}_{\oplus}(C, S)$*.*

Proof. i_* and $i_!$ are automatically fully faithful since i is, and $p_! i_!$ is zero since pi is. Since the quasicategories in the diagram are stable, adjoints between them must be exact. So we just need p^* to be fully faithful and $p_!, i^*$ jointly conservative.

Suppose the sequence

$$i_! i^* F \to F \to p^* p_! F$$

is a cofiber sequence for every $F : B \to S$. Then $p_!, i^*$ are jointly conservative since if they both vanish on F, both sides of this cofiber sequence vanish so F does too. On the other hand, for any $G : C \to S$, $i^* p^* G = 0$, so the cofiber sequence tells us $p^* G \simeq p^* p_! p^* G$. Since p is surjective, p^* is conservative, so $p \simeq p_! p^*$, and p^* is fully faithful.

We thus reduce the recollement to verifying that $i_! i^* \to 1 \to p^* p_!$ is a cofiber sequence of functors in $\mathrm{Fun}_{\oplus}(B, S)$, for any stable quasicategory S. (This condition is necessary as well, cf. [BG16]).

The proof of [Gla17, 2.32] shows that our sequence is a cofiber sequence for maps into any S as soon as it is a cofiber sequence for maps into the additive category of commutative monoid spaces (though the latter is not stable). But the exactness of this sequence on representables was part of the assumption that $A \to B \to C$ is a short exact sequence of semiadditive quasicategories, and of course these representables generate the entire functor quasicategory. $\square$

By combining Lemma 5.11 with Lemma 5.10, we immediately get the recollement portion of the following theorem.

Theorem 5.12. *Restrictions and left Kan extensions along the sequence (5.2) of Mackey functors gives a homotopy monoidal recollement of* $\mathrm{Mack}_{\mathcal{H};\mathcal{O}}$ *into* $\mathrm{Mack}_{\mathcal{F};\mathcal{O}}$ *and* $\mathrm{Mack}_{\mathcal{N};\mathcal{O}}$*.*

We finish the proof by showing the recollement is homotopy monoidal.

Lemma 5.13. *Using Day convolution [Gla16] over the promonoidal structure on the effective Burnside quasicategories constructed in [BGS20, 2.6], the recollement of Theorem 5.12 is homotopy monoidal.*

Proof. Let us separately show that $p_!$ and i^* are strong monoidal on the homotopy categories.

1. The functor p is strong promonoidal, because $p : \mathcal{F}in\mathcal{S}et^{\mathcal{H}} \to \mathcal{F}in\mathcal{S}et^{\mathcal{N}}$ commutes with products. So we reduce to the fact that left Kan extension along *any* strong promonoidal $p : B \to C$ is strong monoidal, at least on the homotopy categories. We can verify this in the standard way [DS95], replacing colimits with homotopy colimits. That is, for $F, G : B^{\mathrm{op}} \to S$, we can compute $p_!(F \boxtimes G)$ on an object c as a homotopy colimit

$$\operatorname*{colim}_{pb \to c} (F \boxtimes G)(b).$$

   In turn, we can write $(F \boxtimes G)(b)$ as a homotopy colimit

$$\operatorname*{colim}_{b_1 \otimes b_2 \to b} Fb_1 \otimes Gb_2$$

   where, to be precise, the colimit is over the overcategory $(B^2)_{/b}$ using the embedding of B^2 and b in the flat inner fibration over $N(\mathcal{F}in\mathcal{S}et_*)$ defining the promonoidal structure. Combining these homotopy colimits, we have

$$p_!(F \boxtimes G)(c) \simeq \operatorname*{colim}_{p(b_1 \otimes b_2) \to c} Fb_1 \otimes Fb_2.$$

Using the strong promonoidality of p, we can rewrite this as

$$p_!(F \boxtimes G)(c) \simeq \operatorname*{colim}_{pb_1 \otimes pb_2 \to c} Fb_1 \otimes Fb_2 \simeq \operatorname*{colim}_{c_1 \otimes c_2 \to c} \operatorname*{colim}_{\substack{pb_1 \to c_1 \\ pb_2 \to c_2}} Fb_1 \otimes Fb_2.$$

Using the distributivity of the promonoidal structure, this separates as

$$\operatorname*{colim}_{c_1 \otimes c_2 \to c} \left(\operatorname*{colim}_{pb_1 \to c_1} Fb_1 \right) \otimes \left(\operatorname*{colim}_{pb_2 \to c_2} Fb_2 \right) \simeq \operatorname*{colim}_{c_1 \otimes c_2 \to c} p_! F(c_1) \otimes p_! F(c_2).$$

This, finally, gives the Day convolution $(p_! F \boxtimes p_! F)(c)$. Thus $p_!$ induces a strong monoidal functor on the homotopy categories.

2. We have no such general reason for i^* to be monoidal; restrictions are, in general, not monoidal even along (pro)monoidal functors.

   But let us directly verify: for F, G Mackey functors on the whole $A^{\mathrm{eff}}_{\mathcal{H};\mathcal{O}}$, $i^*(F \boxtimes G)$ takes value on a given by the homotopy colimit

$$(F \boxtimes G)(ia) = \operatorname*{colim}_{a \to b_1 \times b_2} Fb_1 \wedge Gb_2.$$

   This is indexed by the opposite of the quasicategory $\left(A^{\mathrm{eff}}_{\mathcal{H};\mathcal{O}} \times A^{\mathrm{eff}}_{\mathcal{H};\mathcal{O}}\right)_{ia/}$. On the other hand, $i^*F \boxtimes i^*G$ takes value

$$(i^*F \boxtimes i^*G)(a) = \operatorname*{colim}_{a \to a_1 \times a_2} F(ia_1) \wedge G(ia_2).$$

   This is indexed by the opposite of $\left(A^{\mathrm{eff}}_{\mathcal{F};\mathcal{O}} \times A^{\mathrm{eff}}_{\mathcal{F};\mathcal{O}}\right)_{a/}$. Thus we must show that the map $i \times i$ from the latter to the former is homotopy cofinal, i.e. the undercategory under each span $b_1 \times b_2 \leftarrow s \rightarrow ia$ must be contractible. Indeed, it has an initial object: since s has a map to ia, it must be $\mathcal{F}$-free; for clarity, write $s = it$. Then the factorization of our span given by the 2-cell

$$\begin{tikzcd}
 & & s \arrow[dl, "\Delta"'] \arrow[dr, equal] & & \\
 & s \times s \arrow[dl] \arrow[dr, equal] & & it \arrow[dl, "\Delta"] \arrow[dr] & \\
b_1 \times b_2 & & it \times it & & ia
\end{tikzcd}$$

   is an initial object in the undercategory.

$\square$

5.5 Geometric Fixed Points

Now that we have defined the maps p and i coming from any family of subgroups, we can use these to construct a better model of geometric fixed points on $\mathbf{Mack}_{G;\mathcal{O}}$. The following definition agrees on the homotopy category with geometric fixed points of [BH19, 3.3] on $\mathbf{Sp}^G_{\mathcal{O}}$ by Proposition 5.7.

Definition 5.14. For $H \leq G$, the *geometric fixed points* functor

$$\Phi^H : \mathbf{Mack}_{G;\mathcal{O}} \to \mathbf{Sp}^{W_G H}$$

is the composition of restriction

$$i^* : \mathbf{Mack}_{G;\mathcal{O}} \to \mathbf{Mack}_{\mathcal{F};\mathcal{O}}$$

to the family $\mathcal{F}_{\leq H}$ of all subconjugates of H, followed by the left Kan extension

$$p_! : \mathbf{Mack}_{\mathcal{F};\mathcal{O}} \to \mathbf{Mack}_{(H);\mathcal{O}}$$

away from the family $\mathcal{F}_{<H} := \mathcal{F}_{\leq H} - (H)$ of proper subconjugates, and finally the Quillen equivalence

$$\mathbf{Mack}_{(H);\mathcal{O}} \simeq_Q \mathbf{Sp}^{W_G H}$$

of Proposition 4.10.

Remark 5.15. This geometric fixed points functor is a composition of the right Quillen adjoint i_*, the left Quillen adjoint $p_!$, and the right Quillen adjoint of Proposition 4.10. As such is not very pleasant to compute on the homotopy category in general. However, when $H = G$, our family $\mathcal{F}_{\leq H}$ is all subgroups, the Borel group $W_G H$ is trivial, and both of these right Quillen equivalences are actually equivalences.

Lemma 5.16. *For a map of operads $\mathcal{O} \to \mathcal{O}'$, let $\iota = \iota_{\mathcal{O}}^{\mathcal{O}'}$ denote the extension-of-operad functor, given by left Kan extension along the inclusion*

$$\mathbf{A}^{-}_{G;\mathcal{O}} \to \mathbf{A}^{-}_{G;\mathcal{O}'}.$$

There is a natural equivalence

$$\Phi^H(X) \simeq \Phi^H(\iota_{\mathcal{O}}^{\mathcal{O}'}(X))$$

where the left hand side is the geometric fixed points of $\mathcal{O}$-spectra, and the right hand side is the geometric fixed points of $\mathcal{O}'$-spectra. That is, "adding transfers preserves geometric fixed points."

Proof. Assume without loss of generality that H is G. (We may do this because restriction from G-spectra for $\mathcal{O}$ to H-spectra for $\mathcal{O}|_H$ is left Kan extension along the map on Burnside categories induced by restriction from $\mathcal{F}in\mathcal{S}et^G$ to $\mathcal{F}in\mathcal{S}et^H$, and so commutes with the extension-of-operad functors).

Now both geometric fixed point functors are left Kan extensions, and the result follows because the diagram along which they are extensions

$$\begin{tikzcd}
\mathbf{A}_{G;\mathcal{O}} \arrow[r, "p"] \arrow[d] & \mathbf{A}_{(G);\mathcal{O}} \arrow[d, equal] \\
\mathbf{A}_{G;\mathcal{O}'} \arrow[r, "p"] & \mathbf{A}_{(G);\mathcal{O}'}
\end{tikzcd}$$

commutes. □

Corollary 5.17. *For a G-space X, the geometric fixed points of the suspension spectrum are given by*

$$\Phi^H(\Sigma^\infty_{\mathcal{O}} X_+) \simeq \Sigma^\infty_+(X^H).$$

Proof. When $\mathcal{O}$ is $\mathcal{O}^{\mathrm{triv}}$, this is the tom Dieck splitting (Proposition 3.29). It remains true after extension-of-operad along $\mathcal{O}^{\mathrm{triv}} \to \mathcal{O}$ by the previous lemma. □

Proposition 5.18. *Each geometric fixed point functor is strong monoidal.*

Proof. Φ^H is composition of a restriction to a family followed by a left Kan extension $p_!$. Both are strong monoidal by Lemma 5.13. $\square$

CHAPTER 6

Completions of $\mathcal{O}$-Spectra and the Segal Conjecture

The recollement of Theorem 5.12 is only as useful as our understanding of the components $\mathbf{Mack}_{\mathcal{F};\mathcal{O}}$ and $\mathbf{Mack}_{\mathcal{N};\mathcal{O}}$, and the Tate functor

$$\widetilde{E}\mathcal{F} \wedge F(E\mathcal{F}_+, -).$$

Unfortunately, the latter map is not well-understood in general.

To understand it, one might compare it to other recollements. The most well-understood example, and the one motivating the definition [BBD82], is the recollement of the derived category of a scheme into that of a closed subscheme and its closed complement. Indeed, in a broad enough sense all recollements are generalizations of this: a monoidal recollement of any (rigid, compactly generated) tensor triangulated category is equivalent to a Thomason closed subset of its Balmer spectrum [BF11, 5.9].

In the affine case, the closed subscheme is cut out by an ideal I and the functor playing the role of our $E\mathcal{F}_+ \wedge -$ is the inclusion of the I-torsion complexes, while the analog to our $F(E\mathcal{F}_+, -)$ is derived completion at I.

The completion conjecture therefore relates $F(E\mathcal{F}_+, -)$ to completion at an ideal corresponding to $\mathcal{F}$. Viewing $\operatorname{Spec} A(G)$ as a union of copies of $\operatorname{Spec}(\mathbb{Z})$ coming from the H-fixed-points maps for each $H \leq G$, the ideal $I\mathcal{F}$ is the ideal cutting out the image of those copies of $\operatorname{Spec}(\mathbb{Z})$ coming from $H \in \mathcal{F}$.

Conjecture 6.1 (Genuine Completion Conjecture for X, $\mathcal{F}$)**.** *Fix a genuine G-spectrum X and a family $\mathcal{F}$. The G-spectrum $F(E\mathcal{F}_+, X)$ is equivalent to the $I\mathcal{F}$-adic completion $X^{\wedge}_{I\mathcal{F}}$,* *at the ideal*

$$I\mathcal{F} := \bigcap_{H \in \mathcal{F}} \ker\left(r^G_H : A(G) \to A(H)\right).$$

As functors, $F(E\mathcal{F}_+, -)$ cannot be the same as $(-)^\wedge_{I\mathcal{F}}$, because the classes of $\mathcal{F}$-equivalences and $I\mathcal{F}$-adic equivalences are not the same. However, Conjecture 6.1 holds when X is equivariant K-theory by the Atiyah-Segal completion theorem [Ati61][AHJ88b] or the genuine equivariant sphere spectrum by the solved Segal conjecture [Car84][AHJ88a]. The latter implies it is also true for the dual of any G-spectrum (and thus any finite G-spectra, since in genuine G-spectra these are dualizable).

In Section 6.3, we will discuss the appropriate generalization of the Segal conjecture to $\mathcal{O}$-spectra.

First, though, we must define the notion of completions to which we hope to compare our cofree functor $F(E\mathcal{F}_+, -)$. This generalizes the completion $X^\wedge_I$ of a genuine G-spectrum X at an ideal $I \leq A(G)$ in the Burnside ring [GM92] in two important ways:

1. Of course, genuine G-spectra are replaced with $\mathcal{O}$-spectra for any N_∞ operad $\mathcal{O}$, and
2. The ideal $I \leq A(G)$ in the Burnside ring is replaced with a Mackey functor $\underline{I}$ of ideals $I(H) \leq A_{\mathcal{O}}(H)$ in the Burnside ring $\mathcal{O}$-Mackey functor $\underline{A}_{\mathcal{O}}$ of $\mathcal{O}$-admissible H-sets.

Remark 6.2. The second generalization is necessary, because there exist N_∞ operads $\mathcal{O}$ for which the ring $A_{\mathcal{O}}(G)$ is too small to capture meaningful information about $\mathcal{O}$. For example, if $G = C_{p^2}$, there is an indexing system in which C_p is admissible as a C_p-set, but there are only trivial admissible C_{p^2}-sets and so $A_{\mathcal{O}}(C_{p^2}) = \mathbb{Z}$, and any ideal defined by kernels of restriction maps must be zero.

6.1 Completions of Mackey Functors

We begin with the construction of completion in the monoidal abelian category of $\mathcal{O}$-Mackey functors.

Definition 6.3. 1. A *Mackey ideal* $\underline{I}$ in $\underline{A}_{\mathcal{O}}$ is any sub-Mackey functor. Notice that, by having restrictions and transfers, each $I(H)$ automatically has an $A_{\mathcal{O}}(H)$-action, i.e. it is indeed an ideal in $A_{\mathcal{O}}(H)$.

2. The *Mackey powers* $\underline{I^r}$ of $\underline{I}$ are the image of the r-fold box product of $\underline{I}$ with itself under the multiplication map

$$\underline{I}^{\boxtimes r} \hookrightarrow \underline{A}_{\mathcal{O}}^{\boxtimes r} \to \underline{A}_{\mathcal{O}}.$$

In general, $I^r(H)$ is larger than $I(H)^r$; the latter form a coefficient system, but may fail to assemble into a Mackey functor, since transfers are not multiplicative and need not preserve powers of $\underline{I}(-)$. Instead, $\underline{I^r}$ is the Mackey functor generated by this coefficient system:

$$I^r(H) := \sum_{\substack{K \leq H \\ H/K \in \mathrm{adm}(\mathcal{O})(H)}} t_K^H \left(I(K)^r\right).$$

Similarly, for any $\mathcal{O}$-Mackey functor $\underline{M}$, the product $\underline{IM}$ is the image of

$$\underline{I} \boxtimes \underline{M} \hookrightarrow \underline{A}_{\mathcal{O}} \boxtimes \underline{M} \to \underline{M}.$$

Again, this comprises all transfers of elements of the form im for $i \in I(K)$, $m \in M(K)$, though the transfers may not have this form themselves.

3. For any $\mathcal{O}$-Mackey functor $\underline{M}$, the completion $\underline{M}_I^\wedge$ at $\underline{I}$ is the limit of the sequence $\{\underline{M}/\underline{I^r M}\}$. Note that the evalutation $M_I^\wedge(H)$ may not agree with the $A_{\mathcal{O}}(H)$-module completion $M(H)_{I(H)}^\wedge$ or the $A_{\mathcal{O}}(G)$-module completion $M(H)_{I(G)}^\wedge$, which may also not agree with each other.

The above cautions are most apparent when considering $M(H)^{\wedge}_{I(G)}$. Indeed, if $\mathcal{O}$ is any transfer system as in Remark 6.2 where $A_{\mathcal{O}}(G) \cong \mathbb{Z}$ but $A(H)$ is larger, the augmentation ideal $\mathcal{O}$-Mackey functor

$$I_{\mathcal{O}}(H) = I(H) \cap A_{\mathcal{O}}(H)$$

has $I_{\mathcal{O}}(G) = 0$ but $I_{\mathcal{O}}(H)$ nontrivial.

Fortunately, for the ideals we most care about, we can often leverage multiplicative as well as additive norms to ensure the other two notions of completion are the same.

Lemma 6.4. *Suppose G is abelian, and $\mathcal{O}$-Mackey ideal $\underline{I}$ is closed under the multiplicative norms*

$$N_H^K : A(K) \to A(H)$$

whenever H/K is $\mathcal{O}$-admissible. Then for any $\mathcal{O}$-Mackey functor M, the completions $M_I^{\wedge}(H)$ and $M(H)^{\wedge}_{I(H)}$ agree. Furthermore, if L/H is $\mathcal{O}$-admissible for $H \leq L \leq G$, the $A_{\mathcal{O}}(L)$-module completion $M(H)^{\wedge}_{I(L)}$ also agrees with these.

Proof. For any $i \in I(K)$ with H/K admissible, since $K \lhd H$ is normal, $i^{[H:K]} = r_K^H N_K^H i$. (Without the abelianness assumption, we would have a double-coset formula here instead). So for $m \in M(K)$, we have

$$t_K^H(i^{[H:K]}m) = t_K^H(r_K^H(N_K^H i)m) = (N_K^H i)t_K^H(m) \in I(H)M(H).$$

Pulling out such powers by pigeonhole,

$$t_K^H(I(K)^{(n+r)[H:K]}M(K)) \leq I(H)^r M(H)$$

if n is larger than the maximum number of generators necessary to generate any $I(K)$ for $K \leq H$. Hence the filtrations defining the two completions are cofinal.

The second statement is similar: if $i \in I(H)$, $i^{[L:H]} = r_H^L N_H^L i$ is in the image of $I(L)$ if we know this norm lies in $I(L)$. □

In particular, if $\mathcal{O}$ is genuine, Lemma 6.4 tells us that to complete at an ideal Mackey functor closed under all norms (a *Tambara ideal*), we can simply complete each $M(H)$ at the ideal $I(G) \leq A(G)$. So completing at such a Mackey ideal does indeed generalize the completion of [GM92].

6.2 Completions of $\mathcal{O}$-Spectra

We can now define completions of $\mathcal{O}$-spectra, analogous to [GM92]. We will construct only cohomological $\underline{I}$-completions, which are only well-behaved for bounded-below $\mathcal{O}$-spectra of finite type (i.e. their homotopy Mackey functors are made up of finitely generated groups). To construct completions of arbitrary $\mathcal{O}$-spectra, the standard construction would involve something like a Koszul complex for $\underline{I}$ that we do not know how to define. Since our aim is the Segal conjecture, which involves only the sphere spectrum, cohomological completion will be sufficient.

Definition 6.5. A spectral Mackey functor X is *cohomologically $\underline{I}$-acyclic* if $\underline{H}^\bullet(X;\underline{M}) = 0$ whenever $\underline{IM} = 0$. A spectral Mackey functor Y is *cohomologically $\underline{I}$-complete* if $F(X,Y)$ vanishes for $\underline{I}$-acyclic X (so in particular, $H\underline{M}$ is $\underline{I}$-complete whenever $\underline{IM} = 0$). A map $f : Y \to Z$ is a *cohomological $\underline{I}$-adic equivalence* if its cofiber is cohomologically $\underline{I}$-acyclic. A map $X \to X^\wedge_I$ is a *cohomological $\underline{I}$-completion* if $X^\wedge_I$ is cohomologically $\underline{I}$-complete and the map is a cohomological $\underline{I}$-adic equivalence.

Since this cohomological completion is the only notion of completion we will be working with, we will omit the "cohomological" qualifier.

Lemma 6.6. *If $\underline{M}$ is an $\underline{I}$-complete Mackey functor, the Eilenberg-MacLane $\mathcal{O}$-spectrum $H\underline{M}$ is $\underline{I}$-complete.*

Proof. $H(\underline{M}/\underline{IM})$ is $\underline{I}$-complete by definition. Inductively, each $H(\underline{M}/\underline{I^r M})$ is $\underline{I}$-complete by the cofiber sequence resulting from the Bockstein short exact sequence

$$0 \to \underline{I^{r-1}M}/\underline{I^rM} \to \underline{M}/\underline{I^rM} \to \underline{M}/\underline{I^{r-1}M} \to 0.$$

The completion of $\underline{M}$ is then computed with the short exact sequence

$$0 \to \underline{M}_I^\wedge \to \prod \underline{M}/\underline{I^rM} \to \prod \underline{M}/\underline{I^rM} \to 0.$$

The corresponding sequence of Eilenberg-MacLane spectra is therefore a fiber sequence expressing $H\left(\underline{M}_I^\wedge\right)$ as a fiber of a map between $\underline{I}$-complete spectra, so it is complete itself. In particular, if $\underline{M}$ was already complete, so is $H\underline{M}$. □

Corollary 6.7. *If X is a bounded-below $\mathcal{O}$-Mackey functor such that each $\underline{\pi}_nX$ is an $\underline{I}$-complete Mackey functor, then X is $\underline{I}$-complete.*

Proof. Since the property of being $\underline{I}$-complete is preserved by homotopy limits, this follows by induction up the Postnikov tower. □

Lemma 6.8. *Completion $(-)_I^\wedge$ is lax monoidal.*

Proof. This is formal as soon as the class of acyclics is closed under smash products. Indeed, these are a $\otimes$-ideal: if $F(Y, H\underline{M}) \simeq 0$ for $\underline{M}$ annihilated by $\underline{I}$, then for any other X,

$$F(X \wedge Y, H\underline{M}) \simeq F(X, F(Y, H\underline{M})) \simeq 0$$

for such $\underline{M}$ as well. □

Lemma 6.9. *For any $\mathcal{O}$-Mackey functor $\underline{M}$ such that each $M(H)$ is a finitely generated abelian group, the $\underline{I}$-completion of $H\underline{M}$ is given by the map $H\underline{M} \to H(\underline{M}_I^\wedge)$.*

Proof. By Lemma 6.6, $H(\underline{M}_I^\wedge)$ is $\underline{I}$-complete. Thus we reduce to showing the map

$$H\underline{M} \to H(\underline{M}_I^\wedge)$$

is an $\underline{I}$-adic equivalence.

Note that $(H\underline{M})^\wedge_I$ is canonically an $(H\underline{A}_{\mathcal{O}})^\wedge_I$-module, and in particular an $H\underline{A}_{\mathcal{O}}$-module. Thus it must be $H\underline{N}$ for some chain complex $\underline{N}$ of Mackey functors, which must be homotopy universal among maps from $\underline{M}$ to a chain complex of $\underline{I}$-complete Mackey functors. That is, $\underline{N}$ is the derived completion of $\underline{M}$.

But since each $M(H)$ is finitely generated, the derived completion is simply $M^\wedge_I(H)$. $\square$

Corollary 6.10. *Suppose X is a bounded-below spectral Mackey functor for $\mathcal{O}$ such that each homotopy group $\pi_*(X)(H)$ is finitely generated. Then a map $X \to Y$ is an $\underline{I}$-completion of X iff $\underline{\pi}_* X \to \underline{\pi}_* Y$ is an $\underline{I}$-completion of $\mathcal{O}$-Mackey functors. Moreover, such a completion exists.*

Proof. This is a standard argument, and exists in the genuine case as [GM92, 1.6], but we reproduce it here for completeness. As in Corollary 6.7, we can use the existence of Postnikov towers to enable induction.

We will construct an $\underline{I}$-completion of X and show that it induces $\underline{I}$-completion on homotopy Mackey functors, giving the existence and the forward implication by uniqueness.

Form a Postnikov tower $\{X_n\}$ for X. We will inductively construct a completion $X_n \to (X_n)^\wedge_I$ for each n which also induces $\underline{I}$-completion on $\underline{\pi}_*$. Suppose we have already constructed such a completion $X_{n-1} \to (X_{n-1})^\wedge_I$.

Consider the k-invariant of X_{n-1}; that is, the cofiber

$$X_n \to X_{n-1} \to \Sigma^{n+1} H\underline{\pi}_n X.$$

By the previous lemma, the rightmost arrow in the following diagram is an $\underline{I}$-completion:

$$\begin{tikzcd}
X_n \arrow[r] \arrow[d, dotted] & X_{n-1} \arrow[r] \arrow[d] & \Sigma^{n+1} H\underline{\pi}_n X \arrow[d] \\
(X_n)^\wedge_I \arrow[r, dotted] & (X_{n-1})^\wedge_I \arrow[r, dashed] & \Sigma^{n+1} H\left((\underline{\pi}_n X)^\wedge_I\right)
\end{tikzcd}$$

In particular the bottom right corner is $\underline{I}$-complete, so there is a unique dashed arrow making the right square commute up to homotopy. Defining $(X_n)^\wedge_I$ to be the fiber of this map between $\underline{I}$-complete $\mathcal{O}$-spectra, it must itself be $\underline{I}$-complete. Yet since both solid downwards arrows are $\underline{I}$-adic equivalences, so is the map on fibers $X_n \to (X_n)^\wedge_I$. Thus this map is the $\underline{I}$-completion map of X. But by construction, its homotopy Mackey functors are all either those of $(X_{n-1})^\wedge_I$ or $H(\underline{\pi}_n X)^\wedge_I$, so the completion map is indeed a completion on $\underline{\pi}_*$.

In the other direction, write any map $X \to Y$ as a limit of maps $X_n \to Y_n$ between their Postnikov truncations. If each $\underline{\pi}_n X \to \underline{\pi}_n Y$ is an $\underline{I}$-completion, then the fibers

$$\Sigma^n H\underline{\pi}_n X \to \Sigma^n H\underline{\pi}_n Y$$

are $\underline{I}$-completions, so by induction, each $X_n \to Y_n$ is too and thus so is the limit $X \to Y$. $\square$

Corollary 6.11. *If bounded-below spectral $\mathcal{O}$-Mackey functor Y is $\underline{I}$-complete and of finite type over $\underline{A}_{\mathcal{O}}{}^\wedge_I$, and $H < G$ is any subgroup, the restriction $Y|_H$ to a spectral $\mathcal{O}|_H$-Mackey functor for the group H is $\underline{I}|_H$-complete.*

Lemma 6.12. *If $X \to X^\wedge_I$ is an $\underline{I}$-completion of a bounded below spectral Mackey functor of finite type, then the induced map on geometric fixed points*

$$\Phi^H X \to \Phi^H(X^\wedge_I)$$

is a completion of spectra at the ideal $f_H(I(H)) \subseteq \mathbb{Z}$, where

$$f_H : A_{\mathcal{O}}(H) \to \mathbb{Z}$$

is the H-fixed points map sending $[H/H]$ to 1 and any other $[H/K]$ to 0.

Proof. Taking the restriction $X|_H$ if necessary, we may assume without loss of generality that $G = H$.

Now let us show that Φ^G carries $\underline{I}$-adic equivalences to cohomological $f_G(I(G))$-adic equivalences of spectra. Since it is exact, this is the same as carrying $\underline{I}$-acyclic spectral Mackey functors to cohomologically $f_G(I(G))$-acyclic spectra.

Φ^G is nothing more than the projection $p_!$ corresponding to the family of all proper subgroups; that is, the map of Burnside categories induced by taking the fixed points of any G-set. Thus it is an enriched left adjoint to p^*; explicitly, this sends a spectrum E to the spectral Mackey functor defined by $(p^*E)^G = E$ and

$$(p^*E)^H = \begin{cases} E, & H = G \\ 0, & H < G. \end{cases}$$

The action of $A_{\mathcal{O}}(G)$ on $(p^*E)^G$ is via the fixed points map f_G since $[G/H]$ acts by restriction to $(p^*E)^H = 0$ followed by transfer.

Thus for any abelian group N,

$$[\Phi^G(X), HN] \simeq [X, i^*HN]$$

where i^*HN is Eilenberg-MacLane on the abelian Mackey functor $\underline{i^*N}$ defined the same way. Now the box multiplication $\underline{I i^*N}$ vanishes iff $f_G(I(G))N = 0$, since there are no transfers from subgroups. So if X is $\underline{I}$-acyclic, this cohomology group vanishes for such N, and $\Phi^G X$ is cohomologically $f_G(I(G))$-acyclic.

Next we must verify that Φ^G carries $\underline{I}$-complete spectral Mackey functors to $f_G(I(G))$-complete spectra, at least when they are bounded below and finite type over $\underline{A}^\wedge_I$. Once again, we can prove this by induction up the Postnikov tower, so it suffices to show that if $\underline{M}$ has $\underline{IM} = 0$, then $\Phi^G(H\underline{M})$ is $f_G(I(G))$-complete.

For $H < G$ with G/H $\mathcal{O}$-admissible,

$$(G/H)_+ \wedge \widetilde{E}\mathcal{P}$$

is contractible for $\mathcal{P}$ the family of proper subgroups, so $[G/H] \in A_{\mathcal{O}}(G)$ acts by 0 on the homotopy groups of $\widetilde{E}\mathcal{P} \wedge X$. Thus the action of $A_{\mathcal{O}}(G)$ on the homotopy groups of

$$(\widetilde{E}\mathcal{P} \wedge X)^G = \Phi^G X$$

factors through f_G, and so if $I(G)$ acts by 0 on $\underline{\pi}_* H\underline{M}$, $f_G(I(G))$ acts by 0 on $\pi_*(\Phi^G H\underline{M})$. $\square$

Corollary 6.13. *Let $\underline{I}$ be a Mackey ideal for $\mathcal{O}$.*

If X is a bounded-below spectral Mackey functor for $\mathcal{O}$ of finite type, X is $\underline{I}$-complete iff every $\Phi^H X$ is an $f_H(I(H))$-complete spectrum.

A map $f : X \to Y$ between bounded-below spectral Mackey functors is an $\underline{I}$-adic equivalence iff every

$$\Phi^H(f) : \Phi^H X \to \Phi^H Y$$

is an $f_H(I(H))$-adic equivalence of spectra.

Proof. The forward implications follow from Lemma 6.12.

In the other direction, for any X meeting our hypotheses, we can form the completion $X \to X_I^\wedge$, and by Lemma 6.12 this map induces completion on each geometric fixed points. In particular, X is $\underline{I}$-complete iff this map is an equivalence iff each map

$$\Phi^H X \to \Phi^H(X_I^\wedge) \simeq \Phi^H(X)^\wedge_{f_H(I(H))}$$

is an equivalence iff each $\Phi^H(X)$ is $f_H(I(H))$-complete.

Similarly, any map $f : X \to Y$ is an $\underline{I}$-adic equivalence iff $X_I^\wedge \to Y_I^\wedge$ is an equivalence iff each map

$$\Phi^H(X)^\wedge_{f_H(I(H))} \simeq \Phi^H(X_I^\wedge) \to \Phi^H(Y_I^\wedge) \simeq \Phi^H(Y)^\wedge_{f_H(I(H))}$$

is an equivalence iff each $\Phi^H(f)$ is an $f_H(I(H))$-adic equivalence. □

6.3 The Segal Conjecture

With the language of completion of $\mathcal{O}$-spectra, we can give the incomplete analog to the completion conjecture for a fixed spectral Mackey functor X for $\mathcal{O}$ and fixed family $\mathcal{F}$.

This should compare $F(E\mathcal{F}_+, X)$ to a completion $X_I^\wedge$ at some Mackey ideal $\underline{I}$ for $\mathcal{O}$. In particular, $\mathcal{F}$-cofree Mackey functors will be automatically $\underline{I}$-complete if $\underline{I}$-adic equivalences are all $\mathcal{F}$-equivalences, but that can only hold if $\underline{I}$ restricts to 0 on each $H \in \mathcal{F}$. The maximal such $\underline{I}$ is therefore, in some sense, the closest a completion at a Mackey ideal can be to the $\mathcal{F}$-cofree functor.

Conjecture 6.14 (The Strong Conjecture for X at $\mathcal{F}$)**.** *The map*

$$X \to F(E\mathcal{F}_+, X)$$

is an $\underline{I\mathcal{F}}$-adic equivalence and so exhibits $F(E\mathcal{F}_+, X)$ as the $\underline{I\mathcal{F}}$-adic completion of X, where $\underline{I\mathcal{F}} = \underline{I(\mathcal{F}, \mathcal{O})}$ is the $\mathcal{O}$-Mackey ideal

$$I\mathcal{F}(H) = \bigcap_{\substack{K \le H \\ K \in \mathcal{F}}} \ker\left(r^H_K : A_{\mathcal{O}}(H) \to A_{\mathcal{O}}(K)\right).$$

Remark 6.15. As a right homotopy Kan extension up from $\mathbf{A}^-_{\mathcal{F};\mathcal{O}}$, the behavior of $F(E\mathcal{F}_+, -)$ cannot depend on the transfers in $\mathcal{O}$ ending in subgroups outside $\mathcal{F}$.

Indeed, for any $\mathcal{O}$-algebra X, the space $F(E\mathcal{F}_+, X)$ is canonically an $F(E\mathcal{F}_+, \mathcal{O})$-algebra. This operad is also N_∞; an H-set T is $F(E\mathcal{F}, \mathcal{O})$-admissible iff each restriction $T|_K$ to $K \in \mathcal{F}$ is $\mathcal{O}$-admissible. That is, $F(E\mathcal{F}_{\mathcal{O}})$ is the maximal N_∞ operad which restricts to $\mathcal{O}|_H$ on each $H \in \mathcal{F}$.

Therefore the completion map $X \to F(E\mathcal{F}_+, X)$ must factor through the unit

$$X \to \iota_{\mathcal{O}}^{F(E\mathcal{F},\mathcal{O})} X$$

of the extension-of-operad and restriction-of-operad adjunction from Lemma 5.16. If $\mathcal{O}$ is not equivalent to $F(E\mathcal{F}, \mathcal{O})$, this is rarely an $\underline{I}$-adic equivalence. In particular, if any H-set H/K is admissible for the latter but not the former, then

$$\pi_0^H(\mathbb{S}_{\mathcal{O}}) = A_{\mathcal{O}}(H) \hookrightarrow A_{F(E\mathcal{F},\mathcal{O})}(H) = \pi_0^H(\mathbb{S}_{F(E\mathcal{F},\mathcal{O})}) = \pi_0^H(\iota_{\mathcal{O}}^{F(E\mathcal{F},\mathcal{O})} \mathbb{S}_{\mathcal{O}})$$

is an inclusion of free abelian groups strictly increasing the rank, so it cannot ever be an I-adic equivalence.

Thus we should never expect Conjecture 6.14 to be true unless $\mathcal{O} \simeq F(E\mathcal{F}, \mathcal{O})$.

Conjecture 6.16 (Incomplete Segal Conjecture)**.** *For any family $\mathcal{F}$ and any operad $\mathcal{O}$ such that $\mathcal{O} \simeq F(E\mathcal{F}, \mathcal{O})$, the completion conjecture 6.14 holds for the $\mathcal{O}$-sphere $\mathbb{S}_{\mathcal{O}}$ at $\mathcal{F}$.*

Proposition 6.17. *Suppose the Segal conjecture 6.16 is true. Let $\mathcal{F}$ be any family of subgroups and $\mathcal{O}$ any N_∞ operad (not necessarily satisfying $\mathcal{O} \simeq F(E\mathcal{F}, \mathcal{O})$).*

1. *The automorphisms of the monoidal unit $\mathbb{S}_{\mathcal{F};\mathcal{O}}$ of $\mathbf{Mack}_{\mathcal{F};\mathcal{O}}$ in the homotopy category are given by*

$$\mathrm{Aut}_{\mathbf{Mack}_{\mathcal{F};\mathcal{O}}}(\mathbb{S}_{\mathcal{F};\mathcal{O}}) = (A_{F(E\mathcal{F},\mathcal{O})})^\wedge_I(G).$$

2. *If H is any subgroup such that each proper subgroup $K < H$ lies in $\mathcal{F}$, then*

$$\pi_0 \Phi^H(F(E\mathcal{F}_+, \mathbb{S}_{\mathcal{O}}))$$

is the quotient of the ring

$$(A_{F(E\mathcal{F},\mathcal{O})})^\wedge_I(H)$$

by the ideal generated by elements of the form $[G/H]$ where G/H is $\mathcal{O}$-admissible.

Proof. Since $F(E\mathcal{F}_+, -)$ is a full and faithful inclusion of $\mathbf{Mack}_{\mathcal{F};\mathcal{O}}$ into $\mathbf{Mack}_{G;\mathcal{O}}$, the endomorphism spectrum of $\mathbb{S}_{\mathcal{F};\mathcal{O}}$ is equivalent to that of $F(E\mathcal{F}_+, \mathbb{S}_{\mathcal{O}})$. Then since

$$\mathbb{S}_{\mathcal{O}} \to F(E\mathcal{F}_+, \mathbb{S}_{\mathcal{O}})$$

is an $\mathcal{F}$-equivalence, precomposition induces an equivalence

$$F\left(F(E\mathcal{F}_+, \mathbb{S}_{\mathcal{O}}), F(E\mathcal{F}_+, \mathbb{S}_{\mathcal{O}})\right) \to F\left(\mathbb{S}_{\mathcal{O}}, F(E\mathcal{F}_+, \mathbb{S}_{\mathcal{O}})\right).$$

The units in π_0 of this spectrum therefore give the desired automorphism group. Next, writing $E\mathcal{F}$ as a bar construction and thus evaluating $F(E\mathcal{F}_+, \mathbb{S}_{\mathcal{O}})^G$ as a homotopy limit in which only fixed points in $\mathcal{F}$ arise, we see that

$$F(E\mathcal{F}_+, \mathbb{S}_{\mathcal{O}})^G \simeq F(E\mathcal{F}_+, \mathbb{S}_{F(E\mathcal{F},\mathcal{O})})^G$$

because their fixed points agree on subgroups in $\mathcal{F}$. Finally, the homotopy groups of the latter are given by the Segal conjecture 6.16 and Corollary 6.10.

For the second claim, we are looking at the geometric fixed points of the same $F(E\mathcal{F}_+, \mathbb{S}_{\mathcal{O}})$ which we have just verified is equivalent to the restriction to $\mathcal{O}$ of the $F(E\mathcal{F}, \mathcal{O})$-spectrum

$$(\mathbb{S}_{F(E\mathcal{F},\mathcal{O})})^{\wedge}_{I(\mathcal{F},F(E\mathcal{F},\mathcal{O}))}.$$

Unfortunately, restriction of operads does *not* preserve completions, so we cannot apply Lemma 6.12. Instead, we observe that the geometric fixed points are the cofiber of the map

$$E\mathcal{P}_+ \wedge (\mathbb{S}_{F(E\mathcal{F},\mathcal{O})}) \to (\mathbb{S}_{F(E\mathcal{F},\mathcal{O})})^{\wedge}_{I(\mathcal{F},F(E\mathcal{F},\mathcal{O}))}.$$

where $\mathcal{P}$ is the family of all proper subgroups of H. The fixed points of the source at any H must agree with those of $E\mathcal{P}_+ \wedge \mathbb{S}_{\mathcal{O}}$, again by writing it as a homotopy colimit via the bar construction giving $E\mathcal{P}$, and noting that $\mathcal{P} \subseteq \mathcal{F}$. Since both terms are connective, we can compute π_0 of the map

$$E\mathcal{P}_+ \wedge \mathbb{S}_{\mathcal{O}} \to (\mathbb{S}_{F(E\mathcal{F},\mathcal{O})})^{\wedge}_{I(\mathcal{F},F(E\mathcal{F},\mathcal{O}))}$$

as the cokernel of the induced map on π_0. $\square$

CHAPTER 7

Picard Groups

As a consequence of the monoidal recollement of Theorem 5.12, we can understand a spectral Mackey functor X, with its behavior under smash product, by understanding each geometric fixed points spectrum

$$\Phi^H X := (\widetilde{E}\mathcal{F}_{<H} \wedge X)^H$$

where $\mathcal{F}_{<H}$ is any family that contains all proper subgroups of H but not H itself. Collecting all of these fixed-points gives a strong monoidal functor

$$\Phi : \mathbf{Mack}_{G;\mathcal{O}} \to \prod_{(H)} \mathbf{Sp}^{BW_G H}$$

which categorifies the "ghost map" $A(G) \to \prod_{(H)} \mathbb{Z}$ [Dre71].

More generally, for any class $\mathcal{H}$ of subgroups we can consider its ghost functor

$$\Phi_{\mathcal{H}} : \mathbf{Mack}_{\mathcal{H};\mathcal{O}} \to \prod_{(H)\in\mathcal{H}} \mathbf{Sp}^{BW_G H}$$

Definition 7.1. The group $\mathrm{Pic}_0(\mathbf{Mack}_{\mathcal{H};\mathcal{O}})$ of *locally trivial* spectral Mackey functors for $(\mathcal{H};\mathcal{O})$ is the kernel of the induced map

$$\Phi_{\mathcal{H}} : \mathrm{Pic}(\mathbf{Mack}_{\mathcal{H};\mathcal{O}}) \to \prod_{(H)\in\mathcal{H}} \mathrm{Pic}(\mathbf{Sp}^{BW_G H}).$$

This is those invertible Mackey functors such that each geometric fixed points spectrum is the sphere spectrum with trivial Weyl action.

Remark 7.2. This group Pic_0 is, in general, smaller than the kernel of the composed map

$$\dim \Phi_{\mathcal{H}} : \mathrm{Pic}(\mathbf{Mack}_{\mathcal{H};\mathcal{O}}) \to C(\mathcal{H}) := \prod_{(H)\in\mathcal{H}} \mathbb{Z}$$

which we might call the "locally zero-dimensional" spectral Mackey functors. In the genuine case, the group of locally zero dimensional G-spectra is equivalent to the Picard group of the Burnside ring [FLM01, 0.1]. It also agrees with the group of invertible "Künneth objects": those X such that smashing with X commutes with taking π_0 [FLM01, 3.2].

7.1 Mayer-Vietoris Sequences for Locally Trivial Mackey Functors

The situation of a monoidal recollement allows us to recover invertible objects in the glued category from invertible objects in each component. We collect this information together into a Mayer-Vietoris sequence relating Picard groups and groups of units. Similar Mayer-Vietoris sequences for tensor triangulated categories exist in other circumstances [BF07, 6.7].

Lemma 7.3 (Mayer-Vietoris for Monoidal Recollements)**.** *Suppose we have a monoidal recollement*

$$\begin{tikzcd} \mathcal{X} \arrow[r, shift left=2, "i_!"] \arrow[r, leftarrow, "i^*"] \arrow[r, shift right=2, "i_*"'] & \mathcal{Y} \arrow[r, shift left=2, "p_!"] \arrow[r, leftarrow, "p^*"] \arrow[r, shift right=2, "p_*"'] & \mathcal{Z} \end{tikzcd}$$

of tensor triangulated categories. There is an exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Aut}(1_{\mathcal{Y}}) \arrow[r] & \mathrm{Aut}(1_{\mathcal{X}}) \oplus \mathrm{Aut}(1_{\mathcal{Z}}) \arrow[r] & \pi_0 p_! i_*(1_{\mathcal{X}})^{\times} \arrow[dll, out=-10, in=170] \\ & \mathrm{Pic}(\mathcal{Y}) \arrow[r] & \mathrm{Pic}(\mathcal{X}) \oplus \mathrm{Pic}(\mathcal{Z}) & \end{tikzcd}$$

where $1_{\mathcal{X}}$, $1_{\mathcal{Y}}$, $1_{\mathcal{Z}}$ denote the monoidal units in each category of Mackey functors, and $\pi_0 p_! i_*(1_{\mathcal{X}})$ is the ring of maps from $1_{\mathcal{Z}}$ to the Tate object on $1_{\mathcal{X}}$.

Proof. The final map in the sequence is the induced map on Picard groups from the strong monoidal (and conservative) functor

$$(i^*, p_!) : \mathcal{Y} \to \mathcal{X} \times \mathcal{Z}.$$

An element in the kernel of this is $Y \in \mathcal{Y}$ with $i^*Y \cong 1_\mathcal{X}$ and $p_!Y \cong 1_\mathcal{Z}$. By the recollement, such objects are determined by a gluing map

$$g : 1_\mathcal{Z} \to p_! i_*(1_\mathcal{X})$$

i.e. an element $g \in \pi_0 p_! i_*(1_\mathcal{X})$. The monoidal product of such objects induces the product in this ring, so invertible objects are those with invertible gluing map. Thus exactness at the bottom left.

Next, two gluing maps g_1, g_2 determine isomorphic objects precisely when there are isomorphisms $\alpha : 1_\mathcal{X} \to 1_\mathcal{X}$ and $\beta : 1_\mathcal{Z} \to 1_\mathcal{Z}$ making the diagram

$$\begin{tikzcd} 1_\mathcal{Z} \arrow[r, "g_1"] \arrow[d, "\beta"] & p_! i_*(1_\mathcal{X}) \arrow[d, "p_! i_*(\alpha)"] \\ 1_\mathcal{Z} \arrow[r, "g_2"] & p_! i_!(1_\mathcal{X}) \end{tikzcd}$$

commute. Thus exactness at the top right.

When $g_1 = g_2 = 1$, such an α, β determine an automorphism of the Mackey functor determined by g, giving exactness at the top middle. Finally, exactness at the top left follows from conservativity of $(i^*, p_!)$. $\square$

Corollary 7.4. *Let $\mathcal{F}_1$,$\mathcal{F}_2$ be any two families of subgroups of G which are H-adjacent for some conjugacy class (H), in the sense that $\mathcal{F}_1 = \mathcal{F}_2 - (H)$. For any N_∞ operad $\mathcal{O}$, there is an exact sequence*

$$\begin{tikzcd} 0 \arrow[r] & \pi_0 F(E\mathcal{F}_{2+}, \mathbb{S}_\mathcal{O})^\times \arrow[r] & \pi_0 F(E\mathcal{F}_{1+}, \mathbb{S}_\mathcal{O})^\times \oplus (A(W_G H)^\wedge_I)^\times \arrow[r] & \pi_0(\Phi^H F(E\mathcal{F}_{1+}, \mathbb{S}_\mathcal{O}))^\times \arrow[dll, out=-10, in=170] \\ & \mathrm{Pic}_0(\mathbf{Mack}_{\mathcal{F}_2;\mathcal{O}}) \arrow[r] & \mathrm{Pic}_0(\mathbf{Mack}_{\mathcal{F}_1;\mathcal{O}}). \end{tikzcd}$$

Proof. If the bottom row were

$$\mathrm{Pic}(\mathbf{Mack}_{\mathcal{F}_2;\mathcal{O}}) \to \mathrm{Pic}(\mathbf{Mack}_{\mathcal{F}_1;\mathcal{O}}) \oplus \mathrm{Pic}(\mathbf{Sp}^{BW_G H})$$

then this would simply be an application of Lemma 7.3 to the monoidal recollement of $\mathbf{Mack}_{\mathcal{F}_2;\mathcal{O}}$ into $\mathbf{Mack}_{\mathcal{F}_1;\mathcal{O}}$ and $\mathbf{Mack}_{(H);\mathcal{O}} \cong \mathbf{Sp}^{BW_GH}$ from Theorem 5.12. $F(E\mathcal{F}_{i+}, \mathbb{S}_{\mathcal{O}})$ is i_* applied to the monoidal unit $\mathbb{S}_{\mathcal{O}}$, and Φ^H is $p_!$, while the endomorphisms of the Borel W_GH-sphere are the completion of $A(W_G^H)$ at its augmentation ideal by classical Segal conjecture.

For the given sequence, we must make an additional verification. $\mathrm{Pic}_0(\mathbf{Mack}_{\mathcal{F}_2;\mathcal{O}})$ is by definition precisely those elements of the Picard group which are sent to an element of $\mathrm{Pic}_0(\mathbf{Mack}_{\mathcal{F}_1;\mathcal{O}})$ and $0 \in \mathbb{Z}$. Thus the sequence is exact at the bottom left. $\square$

This Mayer-Vietoris sequence is unfortunately "non-local": although it tells us about the ways the geometric fixed points can be glued into an $\mathcal{F}_2$-spectrum, its first two terms depend on information about the entire restriction to $\mathcal{F}_1$, not only those subgroups contained in H. The next proposition shows that we can glue in the H-fixed points after pulling back to H, essentially reducing any Mayer-Vietoris calculation to one in which the two families $\mathcal{F}_1, \mathcal{F}_2$ are all proper subgroups and all subgroups, respectively.

Proposition 7.5. *Again suppose $\mathcal{F}_1 = \mathcal{F}_2 - (H)$ are families of subgroups of G and $\mathcal{O}$ is any N_∞ operad for G. Let $\mathcal{P}$ denote the family of all proper subgroups of H.*

There is an exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \pi_0(\mathbb{S}_{(\mathcal{O}|_H)})^\times \arrow[r] & \pi_0 F(E\mathcal{P}_+, \mathbb{S}_{(\mathcal{O}|_H)})^\times \oplus \{\pm 1\} \arrow[r] & \pi_0 \Phi^H F(E\mathcal{P}_+, \mathbb{S}_{(\mathcal{O}|_H)})^\times \arrow[dll, out=-10, in=170] \\ & \mathrm{Pic}_0(\mathbf{Mack}_{\mathcal{F}_2;\mathcal{O}}) \arrow[r] & \mathrm{Pic}_0(\mathbf{Mack}_{\mathcal{F}_1;\mathcal{O}}). & \end{tikzcd}$$

Proof. The top row of this Mayer-Vietoris sequence is just the top row for the Mayer-Vietoris sequence of Corollary 7.4, taken on the group H, the operad $\mathcal{O}|_H$, and the families $\mathcal{P}$ and all subgroups of H. As families of subgroups of H, these are H-adjacent.

It therefore suffices to show that the square

$$\begin{tikzcd}
\mathrm{Pic}_0(\mathbf{Mack}_{\mathcal{F}_2;\mathcal{O}}) \arrow[r] \arrow[d] & \mathrm{Pic}_0(\mathbf{Mack}_{\mathcal{F}_1;\mathcal{O}}) \arrow[d] \\
\mathrm{Pic}_0(\mathbf{Mack}_{H;(\mathcal{O}|_H)}) \arrow[r] & \mathrm{Pic}_0(\mathbf{Mack}_{\mathcal{P};(\mathcal{O}|_H)})
\end{tikzcd}$$

is a pullback square. Indeed, this is a pullback square of enriched categories, because the overcategories

$$(\mathbf{A}_{\mathcal{F}_1;\mathcal{O}})_{/(G/H)}, (\mathbf{A}_{\mathcal{P};(\mathcal{O}|_H)})_{/(H/H)}$$

both have (the discrete spectral category on) $\mathcal{O}rb^H$ as a cofinal subcategory, and the undercategories

$$(\mathbf{A}_{\mathcal{F}_1;\mathcal{O}})_{\backslash(G/H)}, (\mathbf{A}_{\mathcal{P};(\mathcal{O}|_H)})_{\backslash(H/H)}$$

both have $\mathrm{adm}(\mathcal{O})(H)$ as a cofinal subcategory.

$\square$

If we assume the Segal conjecture, then we can write the homotopy groups in the Mayer-Vietoris sequence explicitly.

Corollary 7.6. *Again suppose $\mathcal{F}_1, \mathcal{F}_2$ are H-adjacent as in Corollary 7.4. Assume the Segal conjecture 6.16 is true, or at least the conclusions of Proposition 6.17 hold. Then we have an exact sequence*

$$(A_{F(E\mathcal{P},\mathcal{O}|_H)})^{\wedge}_{I\mathcal{P}}(H)^{\times} \to (\Phi^H(\underline{A}_{F(E\mathcal{P},\mathcal{O}|_H)})^{\wedge}_{I\mathcal{P}})^{\times} \to \mathrm{Pic}_0(\mathbf{Mack}_{\mathcal{F}_2;\mathcal{O}}) \to \mathrm{Pic}_0(\mathbf{Mack}_{\mathcal{F}_1;\mathcal{O}}).$$

Here $\mathcal{P}$ is the family of all proper subgroups of H, and

$$\Phi^H(\underline{A}_{F(E\mathcal{P},\mathcal{O}|_H)})^{\wedge}_{I\mathcal{P}}$$

is the Tate ring

$$\pi_0\Phi^H(F(E\mathcal{P}_+, \mathbb{S}_{\mathcal{O}|_H}))$$

described in Proposition 6.17 as the quotient of

$$(A_{F(E\mathcal{F},\mathcal{O}|_H)})^{\wedge}_{I\mathcal{P}}(H)$$

by the ideal generated by the elements representing nontrivial $\mathcal{O}$-admissible H-sets.

Corollary 7.7. *If $\mathcal{F}_1$ and $\mathcal{F}_2$ are H-adjacent, and $\mathcal{O}$ has no transfers ending in H, then the restriction*

$$\mathrm{Pic}_0(\mathbf{Mack}_{\mathcal{F}_2;\mathcal{O}}) \to \mathrm{Pic}_0(\mathbf{Mack}_{\mathcal{F}_1;\mathcal{O}})$$

is injective.

Proof. The first two terms of the Mayer-Vietoris sequence of Corollary 7.6 are isomorphic in this case. □

Corollary 7.8. *The trivial N_∞ operad has*

$$\mathrm{Pic}_0(\mathbf{Mack}_{G;\mathrm{triv}}) = 0.$$

Proof. We show by induction that any $\mathrm{Pic}_0(\mathbf{Mack}_{\mathcal{F};\mathrm{triv}})$ vanishes for any family $\mathcal{F}$. If $\mathcal{F}$ contains only the trivial subgroup $\{e\}$, then the geometric fixed points are a Quillen equivalence

$$\mathbf{Mack}_{\mathcal{F};\mathrm{triv}} \simeq_Q \mathbf{Sp}^{BW_GH}$$

by Proposition 4.10 and we are done. Otherwise, choose a maximal conjugacy class $H \in \mathcal{F}$ and set $\mathcal{F}_1 = \mathcal{F} - (H)$. Then by Corollary 7.7,

$$\mathrm{Pic}_0(\mathbf{Mack}_{\mathcal{F};\mathrm{triv}}) \to \mathrm{Pic}_0(\mathbf{Mack}_{\mathcal{F}_1;\mathrm{triv}})$$

is injective. Since the latter Picard group vanishes by inductive hypothesis, so does the former. □

7.2 Mayer-Vietoris for Abelian Mackey Functors

Continuing to let $\mathcal{F}_1 = \mathcal{F}_2 - \mathcal{H}$, we now want to understand the group of invertible abelian Mackey functors for $(\mathcal{F}_2; \mathcal{O})$ by comparing to those for $(\mathcal{F}_1; \mathcal{O})$. As in the spectrally enriched case, the restriction i^* of Mackey functors supported on $\mathcal{F}_2$ to $\mathcal{F}_1$ has both adjoints. The left adjoint $i_!$ extends $\underline{N}$ from $\mathcal{F}_1$ to $\mathcal{F}_2$ by

$$i_!N(H) = \operatorname*{colim}_{H/K \in \mathrm{adm}(\mathcal{O})} N(K)$$

where the colimit is over the transfers, and the right adjoint i_* is the dual

$$i_*N(H) = \lim_{J<H} N(J)$$

where the limit is over the restriction maps. The restriction maps $i_!N(H) \to N(J)$, and the transfers $N(K) \to i_*N(H)$, are both defined by universal property from the same collection of maps: the maps $N(K) \to N(J)$ defining what $r^H_J t^H_K$ should be, namely

$$\sum_{KhJ \in K\backslash H/J} t^J_{J\cap h^{-1}Kh} \circ c_h \circ r^K_{hJh^{-1}\cap K}.$$

Setting $\Phi^H(\underline{M})$ to be the cokernel of $i_!i^*M(H) \to M(H)$, which is the quotient of $M(H)$ by all transfers from proper subgroups, we can form the diagram

$$\begin{tikzcd}
i_!i^*M(H) \arrow[r] \arrow[d, equal] & M(H) \arrow[r] \arrow[d] & \Phi^H \underline{M} \arrow[d] \\
i_!i^*i_*i^*M(H) \arrow[r] & i_*i^*M(H) \arrow[r] & \Phi^H(i_*i^*\underline{M})
\end{tikzcd}$$

as we did for spectral Mackey functors. However, this is *not* a recollement even in the abelian sense, and the right square is not in general Cartesian. Nor are the rows short exact sequences: $i_!i^*M(H) \to M(H)$ need not be injective.

Example 7.9. Set $G = H = C_p$, and let $\mathcal{F}_1 = \{e\}$ while $\mathcal{F}_2$ is both subgroups $\{e, C_p\}$. Let $\underline{M}$ be the Mackey functor where $M(C_p)$ and $M(e)$ are both $\mathbb{Z}/p$, with the trivial C_p-action

$$\begin{tikzcd}
\mathbb{Z}/p \arrow[d, bend left, "0"] \\
\mathbb{Z}/p \arrow[u, bend left, "1"] \arrow[loop left, "1"]
\end{tikzcd}$$

Figure 7.1: A Lewis diagram for the Mackey functor of Example 7.9

on $M(e)$. The transfer $M(e) \to M(C_p)$ is the identity, while the restriction $M(e) \to M(C_p)$ is zero.

Then $i_! i^* M(C_p)$ is the quotient $M(e)/C_p$, which is still $\mathbb{Z}/p$, while $i_* i^* M(C_p)$ is the fixed points $M(e)^{C_p}$, which is also $\mathbb{Z}/p$. However, the map $i_! i^* M(C_p) \to M(C_p)$ is an identity, while the map $i_! i^* M(C_p) \to i_* i^* M(C_p)$ is zero. Thus our would-be fracture diagram becomes

$$\begin{tikzcd}
\mathbb{Z}/p \arrow[r, "1"] \arrow[d, equal] & \mathbb{Z}/p \arrow[r] \arrow[d, "0"] & 0 \arrow[d] \\
\mathbb{Z}/p \arrow[r, "0"] & \mathbb{Z}/p \arrow[r, equal] & \mathbb{Z}/p
\end{tikzcd}$$

and obviously the right square is not Cartesian.

The key fact in Example 7.9 is that the restriction map $M(C_p) \to M(e)$ is 0, but it is still a Mackey functor because the orbit sum map on $M(e)$ is also 0 — in this case because $M(e)$ is p-torsion. Such torsion cannot exist in invertible Mackey functors.

Lemma 7.10. *If $\underline{M}$ is projective in the abelian category of Mackey functors for $(\mathcal{H};\mathcal{O})$, then each $M(H)$ is a free abelian group.*

Proof. Consider the *category algebra* of the Burnside category, the noncommutative ring

$$\mathbb{Z}[\mathcal{A}^{-}_{\mathcal{H};\mathcal{O}}] := \bigoplus_{H,K \in \mathcal{H}} \mathcal{A}_{\mathcal{H};\mathcal{O}}(G/H, G/K)$$

where multiplication of two spans is given by composition if they are composable and 0 otherwise:

$$(G/H \leftarrow S \to G/K) * (G/L \leftarrow T \to G/J) = \begin{cases} G/L \leftarrow (S \underset{G/H}{\times} T) \to G/K, & J = H \\ 0, & \text{else.} \end{cases}$$

Then Mackey functors are equivalent to left $\mathbb{Z}[\mathcal{A}^{-}_{\mathcal{H};\mathcal{O}}]$-modules. The equivalence sends $\underline{M}$ to to the sum of all its values

$$\bigoplus_{H\in\mathcal{H}} M(H)$$

and its inverse sends a module N to the Mackey functor $N(H) = 1_{G/H}N$, using that each identity morphism is idempotent in the category algebra.

To verify this equivalence, note that there is a projective generator of $\mathcal{M}\mathrm{ack}_{\mathcal{H};\mathcal{O}}$ given by the direct sum of all the representable functors on the objects G/H. $\mathbb{Z}[\mathcal{A}^{-}_{\mathcal{H};\mathcal{O}}]$ is its endomorphism ring.

In particular, for any projective Mackey functor $\underline{M}$, $\bigoplus_H M(H)$ is a retract of a direct sum of copies of $\mathbb{Z}[\mathcal{A}^{-}_{\mathcal{H};\mathcal{O}}]$. Since the category algebra is a free $\mathbb{Z}$-module, each $M(H)$ must also be free. $\square$

Lemma 7.11. *Suppose $\underline{M}$ is an abelian $\mathcal{O}$-Mackey functor such that the following conditions are true.*

1. *Each $M(H)$ is a torsion-free abelian group. This holds in particular if $\underline{M}$ is projective, by Lemma 7.10.*
2. *The map*

$$\underline{M} \to \underline{M} \boxtimes \underline{A}_{\mathcal{O}^{gen}}$$

is a monomorphism. This holds in particular if $\underline{M}$ is flat.

Then the square

$$\begin{array}{ccc} M(H) & \longrightarrow & \Phi^H\underline{M} \\ \downarrow & & \downarrow \\ i_*i^*M(H) & \longrightarrow & \Phi^H(i_*i^*\underline{M}) \end{array}$$

is both Cartesian and coCartesian.

Proof. Both rows are surjective, so we need their kernels to be isomorphic. The kernel of the top row is the image of

$$i_! i^* M(H) \to M(H),$$

generated by the elements $t_K^H(x)$ for $x \in M(K)$. Similarly, the kernel of the bottom row is is the image of

$$i_! i^* M(H) \to i_* i^* M(H).$$

Thus to show they are isomorphic, we must show that the map

$$M(H) \to i_* i^* M(H)$$

is injective on the subgroup generated by transfers.

Now consider the free $\mathcal{O}^{\text{gen}}$-Mackey functor on $\underline{M}$; since these are equivalently $\underline{A}_{\mathcal{O}^{\text{gen}}}$-modules, this is

$$\underline{M}_{\text{gen}} := \underline{A}_{\mathcal{O}^{\text{gen}}} \boxtimes \underline{M}.$$

By assumption, the map $\underline{M} \to \underline{M}_{\text{gen}}$ is injective on each $M(H)$. Therefore the image of the transfers (from subgroups K with $\mathcal{O}$-admissible quotients H/K) in $\underline{M}(H)$ sits inside the image of the transfers (from potentially more subgroups) in $\underline{M}_{\text{gen}}(H)$. So it suffices to show that the restriction maps in $\underline{M}_{\text{gen}}$ are jointly injective on this larger group. That is, without loss of generality we may assume $\mathcal{O}$ was $\mathcal{O}^{\text{gen}}$.

Now form the rationalization $\underline{M}_{\mathbb{Q}}$, defined in the obvious way:

$$M_{\mathbb{Q}}(H) := M(H) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Since each $M(H)$ is a free $\mathbb{Z}$-module, the transfers and restrictions extend to $\underline{M}_{\mathbb{Q}}$ uniquely, making $\underline{M}_{\mathbb{Q}}$ a Mackey functor. For the same reason,

$$\underline{M} \to \underline{M}_{\mathbb{Q}}$$

is an injection. Moreover, this exact localization commutes with the finite colimit defining $i_!$ and the finite limit defining i_*. Thus it suffices to show that the restriction cone

$$M_{\mathbb{Q}}(H) \to i_* i^* M_{\mathbb{Q}}(H)$$

is injective on the image of $i_! i^* M_{\mathbb{Q}}(H)$.

Now finally, we have reduced to genuine rational Mackey functors, which are completely decomposable. The particular injection we need follows from [TW90, 9.4]. □

We use this to reconstruct invertible Mackey functors by adding geometric fixed points one at a time, as in the spectral case.

Proposition 7.12. *Let $\mathcal{F}_1 = \mathcal{F}_2 - (H)$ be two H-adjacent families. Let i denote the inclusion of $\mathcal{F}_1$ into $\mathcal{F}_2$ and let j denote the inclusion of $\mathcal{F}_1$ into the family of all subgroups of G. There is an exact sequence*

$$j_* A_{\mathcal{F}_1;\mathcal{O}}(G)^\times \to \Phi^H(i_* \underline{A}_{\mathcal{F}_1;\mathcal{O}})^\times \to \mathrm{Pic}(\mathcal{M}\mathrm{ack}_{\mathcal{F}_2;\mathcal{O}}) \to \mathrm{Pic}(\mathcal{M}\mathrm{ack}_{\mathcal{F}_1;\mathcal{O}}).$$

Proof. The kernel of the rightmost map is all the ways of taking the monoidal unit Mackey functor for $(\mathcal{F}_1;\mathcal{O})$ and extending it to an invertible Mackey functor $\underline{M}$ for $(\mathcal{F}_2;\mathcal{O})$.

Before worrying about invertibility, consider the problem of defining any extension $\underline{M}$. This means defining a $W_G H$-module $M(H)$ along with all the restriction and transfer maps. The former define a cone giving a map

$$r : M(H) \to i_* A_{\mathcal{F}_1;\mathcal{O}}(H)$$

and the latter define a cocone giving a map

$$t : i_! A_{\mathcal{F}_1;\mathcal{O}}(H) \to M(H)$$

The Mackey relations on transfer followed by restriction now simply say that the composite

$$rt : i_! A_{\mathcal{F}_1;\mathcal{O}}(H) \to i_* A_{\mathcal{F}_1;\mathcal{O}}(H)$$

must be the canonical map whose cokernel is $\Phi^H(i_* \underline{A}_{\mathcal{F}_1;\mathcal{O}})$.

Thus assigning to $M(H)$ the pullback of

$$i_* A_{\mathcal{F}_1;\mathcal{O}}(H) \to \Phi^H(i_* \underline{A}_{\mathcal{F}_1;\mathcal{O}})$$

along any $W_G H$-module map

$$g : P \to \Phi^H(i_* \underline{A}_{\mathcal{F}_1;\mathcal{O}})$$

defines a Mackey functor $\underline{M}_g \in \mathcal{M}\mathrm{ack}_{\mathcal{F}_2;\mathcal{O}}$ with $\Phi^H(\underline{M}_g) = P$.

Of course, the result cannot be invertible unless P is invertible as a $W_G H$-module. Moreover, if P is a nontrivial invertible $W_G H$-module, it cannot have any nonzero maps to $\Phi^H(i_* \underline{A}_{\mathcal{F}_1;\mathcal{O}})$, on which the Weyl group acts trivially. So let us now specialize to extensions by $P = \mathbb{Z}$, so that the map g simply picks out an element of $\Phi^H(i_* \underline{A}_{\mathcal{F}_1;\mathcal{O}})$. Once again, the box product of two such Mackey functors induces the product of these elements, so $\underline{M}_g$ is invertible iff g is a unit.

Thus we have our map

$$\Phi^H(i_* \underline{A}_{\mathcal{F}_1;\mathcal{O}})^\times \to \mathrm{Pic}(\mathcal{M}\mathrm{ack}_{\mathcal{F}_2;\mathcal{O}})$$

with image in the kernel of the restriction to $\mathrm{Pic}(\mathcal{M}\mathrm{ack}_{\mathcal{F}_1;\mathcal{O}})$. Indeed, its image is the entire kernel, because any extension of $\underline{A}_{\mathcal{F}_1;\mathcal{O}}$ to an invertible Mackey functor on $\mathcal{F}_2$ must be a such pullback $\underline{M}_g$ by Lemma 7.11.

Now we consider the kernel of this map: when are $\underline{M}_g$ and $\underline{M}_{g'}$ isomorphic? A map

$$\underline{M}_g \to \underline{M}_{g'}$$

must induce a map on their restrictions, which is an endomorphism of $\underline{A}_{\mathcal{F}_1;\mathcal{O}}$, and a map

$$\mathbb{Z} = \Phi^H(\underline{M}_g) \to \Phi^H(\underline{M}_{g'}) = \mathbb{Z}$$

of which the only invertible ones are ± 1. Therefore g describes the monoidal unit iff it is the image of $1 \in \Phi^H(i_* \underline{A}_{\mathcal{F}_1;\mathcal{O}})$ under some automorphism of $\underline{A}_{\mathcal{F}_1;\mathcal{O}}$.

We therefore conclude by examining this automorphism group $\mathrm{Aut}(\underline{A}_{\mathcal{F}_1;\mathcal{O}})$. Any endomorphism

$$\phi : \underline{A}_{\mathcal{F}_1;\mathcal{O}} \to \underline{A}_{\mathcal{F}_1;\mathcal{O}}$$

is determined by an $A_{\mathcal{O}}(H)$-linear map

$$\phi_H : A_{\mathcal{O}}(H) \to A_{\mathcal{O}}(H)$$

for each $H \in \mathcal{F}_1$, which is multiplication by some $x_H \in A_{\mathcal{O}}(H)$. Thus we have an injection

$$\mathrm{End}(\underline{A}_{\mathcal{F}_1;\mathcal{O}}) \to \lim_{G/H \in \mathcal{O}rb^{\mathcal{F}_1}} A_{\mathcal{O}}(H) = j_* A_{\mathcal{F}_1;\mathcal{O}}(G).$$

The only other condition is that the ϕ_H commute with transfers in $\mathcal{O}$, but this is automatic from the Frobenius relation

$$t_H^K(\phi_H(y)) = t_H^K(x_H y) = t_H^K(r_H^K(x_K)y) = x_K t_H^K(y) = \phi_K(t_H^K(y)).$$

So the endomorphism ring is $j_* A_{\mathcal{F}_1;\mathcal{O}}(G)$, and the automorphisms are its units.

□

Corollary 7.13. *In the setting of Proposition 7.12, we also have an exact sequence*

$$i_* A_{\mathcal{P};\mathcal{O}|_H}(H)^\times \to \Phi^H(i_* \underline{A}_{\mathcal{P};\mathcal{O}|_H})^\times \to \mathrm{Pic}(\mathcal{M}\mathrm{ack}_{\mathcal{F}_2;\mathcal{O}}) \to \mathrm{Pic}(\mathcal{M}\mathrm{ack}_{\mathcal{F}_1;\mathcal{O}})$$

where $\mathcal{P}$ is the family of all proper subgroups of H.

Proof. We use the same fact that we used in Proposition 7.5. Namely, the square of abelian categories

$$\begin{array}{ccc} \mathcal{M}\mathrm{ack}_{\mathcal{F}_2;\mathcal{O}} & \longrightarrow & \mathcal{M}\mathrm{ack}_{\mathcal{F}_1;\mathcal{O}} \\ \downarrow & & \downarrow \\ \mathcal{M}\mathrm{ack}_{H;(\mathcal{O}|_H)} & \longrightarrow & \mathcal{M}\mathrm{ack}_{\mathcal{P};(\mathcal{O}|_H)} \end{array}$$

is a pullback square, by inspecting the overcategories. □

7.3 Comparing the Exact Sequences

Consider the functor

$$\underline{H}_0 = \underline{\pi}_0(- \wedge H\underline{A}_{\mathcal{O}}) : \mathbf{Mack}_{G;\mathcal{O}} \to \mathcal{M}\mathrm{ack}_{G;\mathcal{O}}.$$

Invertible spectral Mackey functors are sent to invertible $H\underline{A}_{\mathcal{O}}$-modules, and then to invertible abelian Mackey functors by Proposition 4.17. This gives us, for each H-adjacent pair of families, a map of Mayer-Vietoris sequences:

$$\begin{array}{ccccccc}
(A_{F(E\mathcal{P},\mathcal{O}|_H)})^{\wedge}_{I\mathcal{P}}(H)^{\times} & \longrightarrow & \Phi^H((\underline{A}_{F(E\mathcal{P},\mathcal{O}|_H)})^{\wedge}_{I\mathcal{P}})^{\times} & \longrightarrow & \mathrm{Pic}_0(\mathbf{Mack}_{\mathcal{F}_2;\mathcal{O}}) & \longrightarrow & \mathrm{Pic}_0(\mathbf{Mack}_{\mathcal{F}_1;\mathcal{O}}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
i_* A_{\mathcal{P};\mathcal{O}|_H}(H)^{\times} & \longrightarrow & \Phi^H(i_* \underline{A}_{\mathcal{P};\mathcal{O}|_H})^{\times} & \longrightarrow & \mathrm{Pic}(\mathcal{M}\mathrm{ack}_{\mathcal{F}_2;\mathcal{O}}) & \longrightarrow & \mathrm{Pic}(\mathcal{M}\mathrm{ack}_{\mathcal{F}_1;\mathcal{O}})
\end{array}$$

Figure 7.2: The comparison map between the spectral and abelian Mayer-Vietoris sequences

Lemma 7.14. *For any group H and operad $\mathcal{O}$ on H, the square*

$$\begin{array}{ccc}
(A_{\mathcal{O}})^{\wedge}_{I\mathcal{P}}(H)^{\times} & \longrightarrow & \Phi^H((\underline{A}_{\mathcal{O}})^{\wedge}_{I\mathcal{P}})^{\times} \\
\downarrow & & \downarrow \\
i_* A_{\mathcal{P};\mathcal{O}}(H)^{\times} & \longrightarrow & \Phi^H(i_* \underline{A}_{\mathcal{P};\mathcal{O}})^{\times}
\end{array}$$

has the property that the map from the pushout to $(\Phi^H(i_* \underline{A}_{\mathcal{P};\mathcal{O}}))^{\times}$ is an injection.

Proof. Note that the ring in the top left corner $(A_{\mathcal{O}})^{\wedge}_{I\mathcal{P}}(H)$ is actually $A_{\mathcal{O}}(H)^{\wedge}_{I\mathcal{P}(H)}$, because for $K < H$, $I\mathcal{P}(K) = 0$, so we have no additional transfers to worry about. By the same vanishing, $(A_{\mathcal{O}})^{\wedge}_{I\mathcal{P}}$ restricts to $A_{\mathcal{P};\mathcal{O}}$ on $\mathcal{P}$. Thus this square is actually the square of Lemma 7.11 on this Mackey functor $(\underline{A}_{\mathcal{P};\mathcal{O}})^{\wedge}_{I\mathcal{P}}$.

Now we verify that this Mackey functor actually meets the conditions of that lemma. $A_{\mathcal{O}}(H)^{\wedge}_{I\mathcal{P}(H)}$ is torsion-free, because $I\mathcal{P}(H)$ does not contain any integers. And the extension of operad

$$\left((\underline{A}_{\mathcal{O}})^{\wedge}_{I\mathcal{P}} \boxtimes \underline{A}_{\mathcal{O}^{\mathrm{gen}}}\right)(H)$$

is simply

$$A_{\mathcal{O}^{\mathrm{gen}}}(H)^{\wedge}_{I\mathcal{P}(H)A_{\mathcal{O}^{\mathrm{gen}}}(H)}$$

so the map to it is injective. This map is therefore a pullback of rings, and thus the sequence

$$0 \to (A_{\mathcal{O}})^{\wedge}_{I\mathcal{P}}(H)^{\times} \to \Phi^H((\underline{A}_{\mathcal{O}})^{\wedge}_{I\mathcal{P}})^{\times} \oplus i_* A_{\mathcal{P};\mathcal{O}}(H)^{\times} \to \Phi^H(i_* \underline{A}_{\mathcal{P};\mathcal{O}})^{\times}$$

is left exact. ∎

Theorem 7.15. *Again assume the Segal conjecture, or at least the conclusions of Proposition 6.17 hold for some N_∞ operad $\mathcal{O}$. For any family $\mathcal{F}$, the map*

$$\mathrm{Pic}_0(\mathbf{Mack}_{\mathcal{F};\mathcal{O}}) \to \mathrm{Pic}(\mathcal{M}\mathrm{ack}_{\mathcal{F};\mathcal{O}})$$

is injective.

Proof. Proceed by induction on $\mathcal{F}$. In the base case, $\mathcal{F}$ contains only the trivial subgroup, and $\mathbf{Mack}_{\mathcal{F};\mathcal{O}}$ is equivalent to the Borel G-spectra. By definition, the locally trivial Picard group is 0.

In the inductive case, choose a maximal conjugacy class $H \in \mathcal{F}$, and set $\mathcal{F}_1 = \mathcal{F} - (H)$. Then we can draw the pushout of Lemma 7.14 (applied to the operad $F(E\mathcal{P}, \mathcal{O}|_H)$, which restricts to $\mathcal{O}|_H$ on $\mathcal{P}$) into the map on Mayer-Vietoris sequences of Diagram 7.2:

$$\begin{tikzcd}
(A_{F(E\mathcal{P},\mathcal{O}|_H)})^{\wedge}_{I\mathcal{P}}(H)^\times \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \Phi^H((\underline{A}_{F(E\mathcal{P},\mathcal{O}|_H)})^{\wedge}_{I\mathcal{P}})^\times \arrow[r] \arrow[d] & \mathrm{Pic}_0(\mathbf{Mack}_{\mathcal{F}_2;\mathcal{O}}) \arrow[r] \arrow[d, equal] & \mathrm{Pic}_0(\mathbf{Mack}_{\mathcal{F}_1;\mathcal{O}}) \arrow[d, equal] \\
i_*A_{\mathcal{P};\mathcal{O}|_H}(H)^\times \arrow[r] \arrow[d, equal] & P \arrow[r] \arrow[d, hook, "\pi"] & \mathrm{Pic}_0(\mathbf{Mack}_{\mathcal{F};\mathcal{O}}) \arrow[r] \arrow[d, "f"] & \mathrm{Pic}_0(\mathbf{Mack}_{\mathcal{F}_1;\mathcal{O}}) \arrow[d, hook, "f_1"] \\
i_*A_{\mathcal{P};\mathcal{O}|_H}(H)^\times \arrow[r] & \Phi^H(i_*\underline{A}_{\mathcal{P};\mathcal{O}|_H})^\times \arrow[r] & \mathrm{Pic}(\mathcal{M}\mathrm{ack}_{\mathcal{F};\mathcal{O}}) \arrow[r] & \mathrm{Pic}(\mathcal{M}\mathrm{ack}_{\mathcal{F}_1;\mathcal{O}})
\end{tikzcd}$$

The map f_1 is injective by inductive hypothesis, and the map π is injective by Lemma 7.14. We conclude that f is injective by the four lemma. ∎

7.4 Representation Spheres

Although we have primarily been concerned with invertible G-spectra corresponding to invertible Mackey functors, studying G-spectra via orthogonal spectra $\mathbf{Sp}^G_{\mathcal{U}}$ naturally emphasizes another comparison map: the representation sphere map $S : RO(G) \to \mathrm{Pic}(\mathbf{Sp}^G_{\mathcal{U}})$ sending V to the suspension spectrum on its one-point compactification $\Sigma^\infty_{\mathcal{U}} S^V$.

This comparison is particularly useful in understanding the image of the map

$$\dim \Phi : \mathrm{Pic}(\mathbf{Sp}^G_{\mathcal{U}}) \to C(G)$$

because the image of the composition $\dim : RO(G) \to C(G)$ is characterized by the Borel-Smith conditions [Die11, 5.1]. In [Bau89], Bauer used this to characterized the image of the genuine $\dim \Phi$ for any finite group in terms of $RO(G)$.

In particular, when G is a p-group, Bauer's result implies that the dimension function of any invertible G-spectrum is achieved by a representation. Along with the description of the kernel of $\dim \Phi$, this means that invertible G-spectra for G a p-group are all smash products of a representation sphere with the spectrum corresponding to an invertible Mackey functor.

Let us now consider which representation spheres $\Sigma^\infty_{\mathcal{O}}(S^V)$ are invertible in $\mathbf{Mack}_{G;\mathcal{O}}$.

Proposition 7.16. *For any N_∞ operad $\mathcal{O}$, there is a maximal $\mathcal{O}' < \mathcal{O}$ in the poset of N_∞ operads such that $\mathcal{O}'$ is equivalent to a Steiner operad on some universe $\mathcal{U}$.*

A representation sphere $\Sigma^\infty_{\mathcal{O}}(S^V)$ is invertible in $\mathbf{Mack}_{G;\mathcal{O}}$ precisely when V embeds into this universe $\mathcal{U}$.

Proof. By [Rub20, 2.11], the join of two Steiner operads in the poset of N_∞ operads is

$$\mathcal{K}_{\mathcal{U}} \vee \mathcal{K}_{\mathcal{U}'} \simeq \mathcal{K}_{\mathcal{U} \oplus \mathcal{U}'}.$$

Thus since the poset of N_∞ operads for fixed G is finite, we can form our $\mathcal{O}'$ as the join of the finitely many operads equivalent to some $\mathcal{K}_{\mathcal{U}}$ with $\mathcal{K}_{\mathcal{U}} < \mathcal{O}$.

If a representation V embeds into $\mathcal{U}$, then $\Sigma^\infty_{\mathcal{U}}(S^V)$ is inverted in $\mathbf{Sp}^G_{\mathcal{U}}$. Thus the functor $S^V \wedge -$ is an equivalence on the homotopy category, so it is also an equivalence on the homotopy category of $\mathbf{Sp}^G_{\mathcal{K}_{\mathcal{U}}}$ since the equivalence of [BH19, 6.1] is enriched over G-spaces. Now the suspension spectrum $\Sigma^\infty_{\mathcal{K}_{\mathcal{U}}}(S^V)$ has an inverse $\mathbb{S}^{-V}$ in $\mathbf{Mack}_{G;\mathcal{K}_{\mathcal{U}}}$, so we can apply the strong monoidal change-of-operad functor to produce an inverse to $\Sigma^\infty_{\mathcal{O}} S^V$ in $\mathbf{Mack}_{G;\mathcal{O}}$.

Conversely, suppose $\Sigma^\infty_\mathcal{O} S^V$ is invertible in $\mathbf{Mack}_{G;\mathcal{O}}$. The action of the Steiner operad $\mathcal{K}_{\infty(1+V)}$ on $\operatorname{colim}_n \Omega^{nV}\Sigma^{nV}$ gives a natural action on each $\mathcal{O}$-Mackey functor, and thus each $\mathcal{O}$-spectrum. Hence there is a map $\mathcal{K}_{\infty(1+V)} \to \mathcal{O}$. So $\mathcal{K}_{\infty(1+V)}$ must map to the maximal $\mathcal{K}_\mathcal{U}$ mapping to $\mathcal{O}$, and thus V embeds into this $\mathcal{U}$. □

Remark 7.17. Proposition 7.16 tells us something unfortunate: as far as representation spheres can detect, all N_∞ operads look like those coming from universes. By [Rub20, 4.1], we can view this as a failure to capture information from subgroups smaller than G.

This is remedied by considering the Picard *coefficient system* instead: the collection of Picard groups

$$\underline{\operatorname{Pic}}(\mathcal{M}\mathrm{ack}_{G;\mathcal{O}})(H) = \operatorname{Pic}(\mathcal{M}\mathrm{ack}_{H;(\mathcal{O}|_H)}).$$

with restriction maps because restriction of groups is strong monoidal.

CHAPTER 8

Calculations for Cyclic p-Groups

We now apply our Mayer-Vietoris sequences to compute the locally trivial Picard group of any N_∞ operad on a cyclic p-group when p is odd. We begin by first examining the genuine case, where the locally zero-dimensional Picard group is known, before moving on to incomplete operads in Section 8.2.

8.1 The Genuine Operad

Let p be any odd prime.

The subgroup lattice of C_{p^n} is a total order consisting of the $n+1$ subgroups C_{p^r} for $0 \leq r \leq n$. Each C_{p^r} has Burnside ring of the form

$$A(C_{p^r}) = \mathbb{Z}[x_1, \ldots, x_r]/(x_i x_j - p^i x_j,\ j \geq i)$$

where each generator x_i represents the C_{p^r}-orbit

$$x_i = \left[\frac{C_{p^r}}{C_{p^{r-i}}}\right]$$

of cardinality p^i. As an abelian group, $A(C_{p^r})$ is free on the x_i and 1 (which we can think of as x_0, since it represents $[C_{p^r}/C_{p^r}]$).

We can also choose the generators

$$t_i := p^i - x_i$$

which satisfy similar relations

$$\begin{aligned} t_i t_j &= (p^i - x_i)(p^j - x_j) \\ &= p^{i+j} - p^j x_i - p^i x_j - x_i x_j \\ &= p^{i+j} - p^j x_i - p^i x_j - p^i i x_j \\ &= p^{i+j} - p^j x_i \\ &= p^j t_i \end{aligned}$$

for $i \leq j$. (Note that the i and j have switched places compared to the relation for $x_i x_j$.)

The restriction map $A(C_{p^r}) \to A(C_{p^{r-1}})$ is given by

$$r^{C_{p^r}}_{C_{p^{r-1}}}(x_i) = \begin{cases} p, & i = 1 \\ p x_{i-1}, & i > 1 \end{cases} \qquad \text{so} \qquad r^{C_{p^r}}_{C_{p^{r-1}}}(t_i) = \begin{cases} 0, & i = 1 \\ p t_{i-1}, & i > 1. \end{cases}$$

The transfer $A(C_{p^{r-1}}) \to A(C_{p^r})$ sends x_i to x_{i+1} and 1 to x_1.

The units of $A(C_{p^r})$ are just ± 1, by a standard observation:

Lemma 8.1 ([Bou07, 5.5])**.** *If G is any p-group for p odd, $A(G)^\times = \{\pm 1\}$.*

Proposition 8.2. *The contribution to* $\mathrm{Pic}_0(\mathbf{Mack}_{C_{p^n};\mathrm{gen}})$ *from Corollary 7.6 at $H = C_{p^r}$ is*

$$(\mathbb{Z}/p)^\times / \{\pm 1\}$$

Proof. Since each restriction factors through $C_{p^{r-1}}$, we have

$$I\mathcal{P}(C_{p^r}) = \ker(r^{C_{p^r}}_{C_{p^{r-1}}}) = (t_1).$$

Therefore

$$I\mathcal{P}^n(C_{p^r}) = (t_1^n) = (p^{n-1} t_1)$$

which is spanned as a group by this and the elements $p^{n-1} t_1 t_i = p^{n-1+i} t_1$ for $i > 1$. Therefore the completion is given as a group by

$$A^\wedge_{I\mathcal{P}}(C_{p^r}) \cong \mathbb{Z}^r \oplus \mathbb{Z}_p$$

where the first $\mathbb{Z}^r$ is spanned by $1, t_r, \ldots, t_2$ and the $\mathbb{Z}_p$ summand is spanned as a $\mathbb{Z}_p$-module by t_1. As a ring, the summand $\mathbb{Z}_p$ is an ideal, and its quotient is the first $\mathbb{Z}^r$, which is a subring of $A(C_{p^r})$ and therefore has units $\{\pm 1\}$.

The units of the completion must therefore all be either of the form $\pm 1 + \zeta t_1$ for $\zeta \in \mathbb{Z}_p$. All elements of this form are invertible: the inverse of $1 + \zeta t_1$ is

$$1 - \frac{\zeta}{1+p\zeta} t_1$$

because

$$\begin{aligned}(1-\zeta)\left(1 - \frac{\zeta}{1+p\zeta}\right) &= 1 - t_1\left(\zeta - \frac{\zeta}{1+p\zeta} - p\zeta\frac{\zeta}{1+p\zeta}\right)\\ &= 1 - t_1\left(\frac{(1+p\zeta)\zeta - \zeta - p\zeta^2}{1+p\zeta}\right)\end{aligned}$$

and the top of this fraction cancels out.

Each $x_i = p^i - t_i$ is in the image of the transfers, so we have

$$\Phi^{C_{p^r}}(\underline{A}^{\wedge}_{I\mathcal{P}}) \cong \mathbb{Z}_p$$

and the map between these sends t_i to p^i, and in particular t_1 to p.

The induced map on units

$$A^{\wedge}_{I\mathcal{P}}(C_{p^r})^{\times} \to \mathbb{Z}_p^{\times}$$

therefore has image $\{\pm 1\} + p\mathbb{Z}_p$ (where $1 + p\zeta$ is hit by $1 + t_1\zeta$). The cokernel is therefore the quotient of

$$\frac{\mathbb{Z}_p^{\times}}{1+p\mathbb{Z}_p} \cong (\mathbb{Z}_p/p\mathbb{Z}_p)^{\times} \cong (\mathbb{Z}/p)^{\times}$$

by $\{\pm 1\}$. $\square$

Next we turn to the abelian Mayer-Vietoris sequence.

Proposition 8.3. *The contribution to* $\mathrm{Pic}(\mathcal{M}\mathrm{ack}_{C_{p^n};\mathrm{gen}})$ *from Proposition 7.12 at* C_{p^r} *is*

$$(\mathbb{Z}/p)^{\times}/\{\pm 1\} \times (\mathbb{Z}/p)^{r-1}.$$

Proof. Again, we use that $C_{p^{r-1}}$ is maximal among subgroups of C_{p^r} to write the limit

$$i_* A(C_{p^r}) = \lim_{i<r} A(C_{p^i}) = A(C_{p^{r-1}})^{C_p}$$

where the fixed points are by the trivial action, and similarly

$$i_! A(C_{p^r}) = \operatorname*{colim}_{i<r} A(C_{p^i}) = A(C_{p^{r-1}})_{C_p}$$

so both are just $A(C_{p^{r-1}})$, with units ± 1.

Finally, the norm map $i_! A(C_{p^r}) \to i_* A(C_{p^r})$ sends $x_i \in A(C_{p^{r-1}})$ to $x_{i+1} \in A(C_{p^r})$ and then to $px_i \in A(C_{p^{r-1}})$, and similarly 1 to x_1 and then to p — that is, it is multiplication by p. So the geometric fixed points are the square-zero extension

$$\Phi^{C_{p^r}}(i_* \underline{A}) = (\mathbb{Z}/p)[x_1, \ldots, x_{r-1}]/(x_i x_j)$$

which has units

$$\Phi^{C_{p^r}}(i_* \underline{A}) \cong (\mathbb{Z}/p)^\times \times (\mathbb{Z}/p)^{r-1}.$$

$\square$

Remark 8.4. It is well-known that the Picard group of the Burnside ring $\mathrm{Pic}(A(C_{p^n}))$ is

$$\prod_{s=1}^{n} (\mathbb{Z}/p^s)^\times / \{\pm 1\}.$$

This follows from the Mayer-Vietoris sequence of [Die79, 10.3.8] along with the description of the image of $A(C_{p^n})$ in the ghost ring $C(C_{p^n})$ of [Die79, 1.3.5].

By [FLM01, 0.1], this is isomorphic to the locally zero-dimensional Picard group of genuine G-spectra, with an isomorphism given by taking π_0. Of course, the functor π_0 factors as

$$\underline{\pi}_0 : \mathbf{Sp}_{\mathcal{U}}^{C_{p^n}} \to \mathcal{M}\mathrm{ack}_{C_{p^n};\mathrm{gen}}$$

followed by

$$(-)(G) : \mathcal{M}\mathrm{ack}_{C_{p^n};\mathrm{gen}} \to \mathrm{Mod}(A(C_{p^n}))$$

so the latter is surjective. Since Proposition 8.2 gives the cardinality of the target as an upper bound on the cardinality of the source, evaluation at G must give an isomorphism of Picard groups

$$\mathrm{Pic}(\mathcal{M}\mathrm{ack}_{C_{p^n};\mathrm{gen}}) \cong \mathrm{Pic}(A(C_{p^n})).$$

As a consequence, we know all the restriction functors

$$\mathcal{M}\mathrm{ack}_{\mathcal{F}_2;\mathrm{gen}} \to \mathcal{M}\mathrm{ack}_{\mathcal{F}_1;\mathrm{gen}}$$

must be surjective.

Our abelian Mayer-Vietoris sequences filter each copy of $(\mathbb{Z}/p^s)^\times/\{\pm 1\}$ by the short exact sequences

$$0 \to \mathbb{Z}/p \hookrightarrow (\mathbb{Z}/p^j)^\times/\{\pm 1\} \twoheadrightarrow (\mathbb{Z}/p^{j-1})^\times/\{\pm 1\} \to 0.$$

The inclusion

$$\mathrm{Pic}_0(\mathbf{Mack}_{C_{p^r};\mathrm{gen}}) \hookrightarrow \mathrm{Pic}(\mathcal{M}\mathrm{ack}_{C_{p^r};\mathrm{gen}})$$

from Theorem 7.15 therefore describes the locally trivial invertible spectra inside the locally zero-dimensional ones: they are the elements corresponding to

$$\prod_{s=1}^{n} (\mathbb{Z}/p)^\times/\{\pm 1\} \subseteq \prod_{s=1}^{n} (\mathbb{Z}/p^s)^\times/\{\pm 1\}.$$

Interestingly, this means that the invertible G-spectra outside of this subgroup are *not* locally trivial: their geometric fixed points admit nontrivial Weyl actions that are not seen at the level of π_0.

8.2 Incomplete Operads

Now we consider how the results of Section 8.1 change when we restrict the transfers to some operad $\mathcal{O}$. Continue to assume p is odd.

Each group $A_{\mathcal{O}}(C_{p^r})$ is the subalgebra (and subgroup) of $A(G)$ generated by the x_i or t_i such that $C_{p^r}/C_{p^{r-i}}$ is $\mathcal{O}$-admissible. We will say such a generator is an *$\mathcal{O}$-admissible element* of $A(C_{p^r})$.

Meanwhile, $A_{F(E\mathcal{P},\mathcal{O})}(C_{p^r})$ includes all those x_i or t_i such that x_{i-1} is $\mathcal{O}$-admissible in $A(C_{p^{r-1}})$. (Recall that from Definition 3.21, if x_i is admissible in $A(C_{p^r})$ then x_{i-1} is admissible in $A(C_{p^{r-1}})$.)

Proposition 8.5. *There is no contribution to* $\mathrm{Pic}_0(\mathbf{Mack}_{C_{p^n};\mathrm{gen}})$ *from Corollary 7.6 at* C_{p^r} *if* $C_{p^r}/C_{p^{r-1}}$ *is inadmissible. Otherwise, the contribution is* $(\mathbb{Z}/p)^{\times}/\{\pm 1\}$ *as in the genuine case.*

Proof. 1. First suppose $C_{p^r}/C_{p^{r-1}}$ is not $\mathcal{O}$-admissible.

Then $A_{\mathcal{O}}(C_{p^r})$, which is spanned by 1 and the $\mathcal{O}$-admissible t_i, does not meet

$$I\mathcal{P}(C_{p^r}) = \langle t_1 \rangle$$

in $A(C_{p^r})$. Thus the completion $A_{\mathcal{O}}(\mathcal{P})^{\wedge}_{I\mathcal{P}}$ is just $A_{\mathcal{O}}(\mathcal{P})$, with units ± 1. In this case, the Tate ring is

$$\Phi^{C_{p^r}}(\underline{A}_{\mathcal{O}}) = \mathbb{Z}$$

which also has units ± 1.

2. Now suppose $C_{p^r}/C_{p^{r-1}}$ is $\mathcal{O}$-admissible.

Then if any $C_{p^{r-1}}/C_{p^{r-1-i}}$ is $\mathcal{O}$-admissible, so is $C_{p^r}/C_{p^{r-i}}$ by self-induction. In this case, $\mathcal{O} = F(E\mathcal{P},\mathcal{O})$.

Now the completion

$$(A_{F(E\mathcal{P},\mathcal{O})})^{\wedge}_{I\mathcal{P}}(C_{p^r}) = (A_{\mathcal{O}})^{\wedge}_{I\mathcal{P}}(C_{p^r})$$

is, as a group, $\mathbb{Z}^k \oplus \mathbb{Z}_p$ for some k; the former summand is generated as a $\mathbb{Z}_p$-module by all the $\mathcal{O}$-admissible t_i for $i > 1$.

We can now trace out the rest of the proof of Proposition 8.2. $(A_{\mathcal{O}})^{\wedge}_{I\mathcal{P}}(C_{p^r})$ again has an ideal spanned as a $\mathbb{Z}_p$-module by t_1, whose quotient is a subring of $A(C_{p^r})$. The units of this quotient are therefore still ± 1, so the units of the completion are the elements of the form $1 + \zeta t_1$. The geometric fixed points $\Phi^{C_{p^r}}(\underline{A}_{\mathcal{O}})^{\wedge}_{I\mathcal{P}}$ is again $\mathbb{Z}_p$ with $t_i \mapsto p^i$, and the units of the form $1 + p\zeta$ are hit by $1 + t_1\zeta$.

$\square$

Proposition 8.6. *Writing the contribution of Proposition 8.3 as a product of r terms*

$$(\mathbb{Z}/p)^{\times} \oplus \mathbb{Z}/p \oplus \cdots \oplus \mathbb{Z}/p,$$

the contribution to $\mathrm{Pic}(\mathcal{M}\mathrm{ack}_{C_{p^n};\mathcal{O}})$ *from Proposition 7.12 at C_{p^r} is isomorphic to the subgroup containing the ith factor iff $C_{p^r}/C_{p^{r-i}}$ is admissible.*

Proof. Once again, the limit $i_* A_{F(E\mathcal{P},\mathcal{O})}(C_{p^r})$ is the fixed points of the trivial C_p-action on the value at the maximal proper subgroup, $A_{\mathcal{O}}(C_{p^r})$. The units of this are still ± 1 (as are the units of any $A_{\mathcal{O}}(G)$ for G an odd p-group, since $A_{\mathcal{O}}(G)$ is a subring of $A(G)$).

On the other hand, the colimit $i_! A_{F(E\mathcal{P},\mathcal{O})}(C_{p^r})$ is the quotient of the trivial action on the maximal $A_{\mathcal{O}}(C_{p^j})$ such that C_{p^r}/C_{p^j} is admissible, but this j need not be $r-1$. Instead, let us point out that the image of the transfers in $A(C_{p^{r-1}})$ is spanned by those $px_{i-1} \in A_{\mathcal{O}}(C_{p^{r-1}})$ such that $x_i \in A(C_{p^r})$ is $\mathcal{O}$-admissible.

We now break into cases again on whether $C_{p^r}/C_{p^{r-1}}$ is admissible.

1. If $C_{p^r}/C_{p^{r-1}}$ is not admissible, then p is not a transfer. In this case, $\Phi^{C_{p^r}}(i_* \underline{A}_{\mathcal{O}})$ is a square-zero extension of a torsion-free ring R by a $(\mathbb{Z}/p)$-module M.

   The ring R is generated as a ring by the $x_i \in A(C_{p^{r-1}})$ such that x_i is $\mathcal{O}$-admissible but $x_{i+1} \in A_{C_{p^r}}$ is not; as a group, R is freely spanned by these generators and 1. Note that R is a subring of $A_{\mathcal{O}}(C_{p^{r-1}})$, and so has units ± 1.

   The $(\mathbb{Z}/p)$-module M is generated by those x_i such that x_{i+1} is $\mathcal{O}$-admissible in $A(C_{p^r})$.

Thus the units are $\pm 1 \times (\mathbb{Z}/p)^k$, where k is the number of such admissible x_{i+1} in $A(C_{p^r})$.

2. If $C_{p^r}/C_{p^{r-1}}$ is admissible, then once again $\mathcal{O} = F(E\mathcal{P}, \mathcal{O})$. In this case the image of the transfers is again (p), and so $\Phi^{C_{p^r}}(i_* \underline{A}_{\mathcal{O}})$ is a square-zero extension of $\mathbb{Z}/p$ by the free $\mathbb{Z}/p$-module on all the x_i such that x_{i+1} is $\mathcal{O}$-admissible.

$\square$

Remark 8.7. Recall the filtration of $\operatorname{Pic}(A(C_{p^n}))$ from Remark 8.4. We can view Proposition 8.6 as picking out the subgroup

$$\prod_{s=1}^{n} (\mathbb{Z}/p^{k_s})^{\times}/\{\pm 1\}$$

where k_s is the number of j such that C_{p^j}/C_{p^s} is admissible. Choosing the maximal such j, the restriction $p^{j-i}C_{p^i}/C_{p^s}$ must be admissible for any $s < i < j$, so k_s is just $j - s$.

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