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Several Problems in Extremal Combinatorics

A dissertation submitted in partial satisfaction
of the requirements for the degree
Doctor of Philosophy in Mathematics

by

Wenying Gan

2014

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ABSTRACT OF THE DISSERTATION

Several Problems in Extremal Combinatorics

by

Wenying Gan

Doctor of Philosophy in Mathematics

University of California, Los Angeles, 2014

Professor Benjamin Sudakov, Chair

Extremal combinatorics is one of the central branches of discrete mathematics. It focuses on determining or estimating the optimal possible size of a discrete structure (e.g. set systems, graphs) with certain properties. One beauty of problems in this field is that the statements are always easy to understand, while the approaches to solve are difficult and intriguing. The other beauty is the connection with other areas like analysis, number theory, probability and computer science, namely many extremal combinatorics problems have application to these fields and the tools researchers developed in recent decades rely on these fields as well. That is why this branch of mathematics has undergone a period of a spectacular growth in the past half a century and many interesting open problems arose from it. In this dissertation, we discuss several problems in this field. These problems are chosen among the author's work in order to represent various aspects of this area.

In Chapter 2, we study an extremal problem on set systems and partially solve an almost 50 years old problem of Erdős-Katona-Kleitman. In Chapter 3, we focus on saturated bipartite graphs and prove a conjecture of Moshkovitz and Shapira up to a constant. In Chapter 4, we study an extremal problem on graphs and verify a conjecture of Engbers and Galvin. In Chapter 5, we provide some partial results for the generalization of the conjecture in Chapter 4. All these researches

were carried under the supervision of Benny Sudakov.

The dissertation of Wenying Gan is approved.

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2014

*To my parents and my girlfriend
for their endless support*

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Chapter Two is a version of the paper *Sperner's Theorem and a problem of Erdős-Katona-Kleitman* coauthored with Shagnik Das and Benny Sudakov, which has been accepted to *Combinatorics, Probability and Computing*. Chapter Three is a version of the paper *$K_{s,t}$ -saturated bipartite graphs* coauthored with Daniel Korándi and Benny Sudakov, which has been accepted to *European Journal of Combinatorics*. Chapter Four is a version of the paper *Maximizing the number of independent sets of a fixed size* coauthored with Po-Shen Loh and Benny Sudakov, which has been accepted to *Combinatorics, Probability and Computing*. In Chapter Five, we provide some unpublished partial results for the generalization of the main conjecture in Chapter Four.

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S. Das, W. Gan and B. Sudakov, The minimal number of disjoint pairs in set systems and related problems, *Combinatorica*, to appear.

W. Gan, P. Loh and B. Sudakov, Maximizing the number of independent sets of a fixed size, *Combinatorics, Probability and Computing*, to appear.

W. Gan, D. Korándi and B. Sudakov, $K_{s,t}$ -saturated bipartite graphs, *European Journal of Combinatorics*, to appear.

CHAPTER 1

Introduction

Extremal combinatorics is a subfield of combinatorics that studies the maximum or minimum size of certain discrete structures such as set systems, graphs under certain constraints. In this dissertation, we will study several problems in extremal combinatorics.

The first problem that we study is an extremal problem on set systems in Chapter 2. A central result in extremal set theory is the celebrated theorem of Sperner from 1928, which gives the size of the largest family of subsets of $[n]$ not containing a 2-chain $F_1 \subset F_2$. Erdős extended this theorem to determine the largest family without a k -chain $F_1 \subset F_2 \subset \dots \subset F_k$. Erdős and Katona, followed by Kleitman, asked how many chains must appear in families with sizes larger than the corresponding extremal bounds. In 1966, Kleitman resolved this question for 2-chains, showing that the number of such chains is minimized by taking sets as close to the middle level as possible. Moreover, he conjectured the extremal families were the same for k -chains, for all k . In this chapter, making the first progress on this problem, we verify Kleitman's conjecture for the families whose size is at most the size of the $k + 1$ middle levels. We also characterize all extremal configurations.

In Chapter 3, we study the minimum number of edges in saturated bipartite graphs. An n -by- n bipartite graph is H -saturated if the addition of any missing edge between its two parts creates a new copy of H . In 1964, Erdős, Hajnal and Moon made a conjecture on the minimum number of edges in a $K_{s,s}$ -saturated bi-

partite graph. This conjecture was proved independently by Wessel and Bollobás in a more general, but ordered, setting: they showed that the minimum number of edges in a $K_{(s,t)}$ -saturated bipartite graph is $n^2 - (n-s+1)(n-t+1)$, where $K_{(s,t)}$ is the “ordered” complete bipartite graph with s vertices in the first color class and t vertices in the second. However, the very natural question of determining the minimum number of edges in the unordered $K_{s,t}$ -saturated case remained unsolved. This problem was considered recently by Moshkovitz and Shapira and in a special case by Balogh, Bollobás, Morris and Riordan. Moshkovitz and Shapira also conjectured what the answer should be. In this chapter, we give an asymptotically tight bound on the minimum number of edges in a $K_{s,t}$ -saturated bipartite graph, which is only smaller by an additive constant than the conjecture of Moshkovitz and Shapira. We also prove their conjecture for $K_{2,3}$ -saturation, which was the first open case.

In Chapter 4, we study the maximum number of independent sets of a fixed size in graphs with a minimum degree condition. Let $i_t(G)$ be the number of independent sets of size t in a graph G . Engbers and Galvin asked how large $i_t(G)$ could be in graphs with minimum degree at least δ . They further conjectured that when $n \geq 2\delta$ and $t \geq 3$, $i_t(G)$ is maximized by the complete bipartite graph $K_{\delta, n-\delta}$. This conjecture has drawn the attention of many researchers recently. In this chapter, we prove this conjecture.

In Chapter 5, we consider a generalized conjecture mentioned in Chapter 4 and provide some unpublished partial results on it.

CHAPTER 2

A Problem of Erdős, Katona and Kleitman

2.1 Introduction

Sperner's Theorem is a central result in extremal set theory, giving the size of the largest family of sets not containing a 2-chain $F_1 \subset F_2$. Erdős later extended this theorem to determine the largest family without a k -chain $F_1 \subset F_2 \subset \dots \subset F_k$. A natural question is to ask how many k -chains must appear in a family larger than this extremal bound.

More precisely, we consider the following problem, first posed by Erdős and Katona and then extended by Kleitman some fifty years ago. Given a family \mathcal{F} of s subsets of $[n]$, how many k -chains must \mathcal{F} contain? We denote this minimum by $c_k(n, s)$, and determine it for a wide range of values of s . This provides a quantitative strengthening of the Erdős result on the size of k -chain-free families.

We shall now discuss the background of Sperner's Theorem and this problem further, before presenting our new results.

2.1.1 Background

Extremal set theory is one of the most rapidly developing areas in combinatorics, having applications to other branches of mathematics and computer science including discrete geometry, functional analysis, number theory and complexity. The typical extremal problem has the following form: how large can a structure be without containing some forbidden configuration? A classical example, con-

sidered by many to be the starting point of extremal set theory, is a theorem of Sperner [33]. An *antichain* is a family of subsets of $[n]$ that does not contain sets $F_1 \subset F_2$. Sperner's Theorem states that the largest antichain has $\binom{n}{\lfloor n/2 \rfloor}$ sets, a bound that is easily seen to be tight by considering the family of sets of size $\lfloor \frac{n}{2} \rfloor$. This celebrated result enjoys numerous applications and has many extensions, many of which are discussed in Engel's book [12]. One particular extension, due to Erdős [13], shows that the size of the largest set family without a k -chain, that is, k -sets $F_1 \subset F_2 \subset \dots \subset F_k$, is the sum of the $k - 1$ largest binomial coefficients, $M_{k-1} = \sum_{i=\lfloor \frac{n-k+2}{2} \rfloor}^{\lfloor \frac{n+k-2}{2} \rfloor} \binom{n}{i}$. When $k = 2$, we recover Sperner's Theorem.

Our problem is what we refer to as an Erdős–Rademacher-type extension of Erdős' theorem, a name we now explain. Arguably the most well-known result in extremal combinatorics is a theorem of Mantel [30] from 1907, which states that an n -vertex triangle-free graph can have at most $\lfloor \frac{n^2}{4} \rfloor$ edges. In an unpublished result, Rademacher strengthened this theorem by showing that any graph with $\lfloor \frac{n^2}{4} \rfloor + 1$ edges must contain at least $\lfloor \frac{n}{2} \rfloor$ triangles. Erdős [14] then extended this to graphs with a linear number of extra edges, and in [15] studied the problem for larger cliques. More generally, for any extremal problem, the corresponding Erdős–Rademacher problem asks how many copies of the forbidden configuration must appear in a structure larger than the extremal bound.

In the context of Sperner's Theorem, this problem was first considered by Erdős and Katona, who conjectured that a family with $\binom{n}{\lfloor n/2 \rfloor} + t$ sets must contain at least $t \lfloor \frac{n+1}{2} \rfloor$ 2-chains. Kleitman [25] confirmed the conjecture, and, in a far-reaching generalization, showed the minimum number of 2-chains in a family of any fixed size is obtained by choosing sets of size as close to $\frac{n}{2}$ as possible. He then conjectured (see [18, 25]) that the same families minimize the number of k -chains, a problem that has remained open for nearly fifty years.

Conjecture 2.1.1. *The number of k -chains in a family is minimized by choosing sets of sizes as close to $\frac{n}{2}$ as possible.*

2.1.2 Our results

In this chapter we study these Erdős–Rademacher-type extensions of the theorems of Sperner and Erdős. We began by considering the case of 2-chains, and determined the minimum number of 2-chains in a family of any number of sets. Later, we discovered Kleitman had earlier obtained the same result. However, through slightly more careful calculations, and by introducing an additional argument, we are able to characterize all extremal families, as given below.

Theorem 2.1.2. *Let \mathcal{F} be a family of subsets of $[n]$, with $|\mathcal{F}| = s \geq \binom{n}{\lfloor n/2 \rfloor}$. Let $r \in \frac{1}{2}\mathbb{N}$ be the unique half-integer such that $\sum_{i=\frac{n}{2}-r+1}^{\frac{n}{2}+r-1} \binom{n}{i} < s \leq \sum_{i=\frac{n}{2}-r}^{\frac{n}{2}+r} \binom{n}{i}$. Then \mathcal{F} minimizes the number of 2-chains if and only if the following conditions are satisfied:*

1. For every $F \in \mathcal{F}$, $\frac{n}{2} - r \leq |F| \leq \frac{n}{2} + r$.
2. For any $A \subset [n]$ with $\frac{n}{2} - r + 1 \leq |A| \leq \frac{n}{2} + r - 1$, we have $A \in \mathcal{F}$.
3. If $s \leq \sum_{i=\frac{n}{2}-r}^{\frac{n}{2}+r-1} \binom{n}{i}$, then $\{F \in \mathcal{F} : |F| = \frac{n}{2} \pm r\}$ forms an antichain.
4. If $s \geq \sum_{i=\frac{n}{2}-r}^{\frac{n}{2}+r-1} \binom{n}{i}$, then $\{F \notin \mathcal{F} : |F| = \frac{n}{2} \pm r\}$ forms an antichain.

Our main results verify Conjecture 2.1.1 for families of certain sizes. To begin with, recall that Erdős showed the largest family without k -chains consists of the $k - 1$ middle levels of the hypercube, whose size we denote by M_{k-1} . If we were to add one set to this family, the best we could do would be to add it to the k th level, in which case we would create $\binom{\lfloor (n+k)/2 \rfloor}{k-1} (k - 1)!$ k -chains. Indeed, we show that every additional set must contribute at least this many new k -chains, and the above construction shows this is tight when our extremal family is contained within the k middle levels.

Theorem 2.1.3. *If \mathcal{F} is a set family over $[n]$ of size $s = M_{k-1} + t$, then \mathcal{F} contains at least $t \binom{\lfloor (n+k)/2 \rfloor}{k-1} (k - 1)!$ k -chains.*

We are then able to extend our argument to work for larger set families, obtaining a result that is tight when the extremal family is contained within the $k + 1$ middle levels.

Theorem 2.1.4. *Provided $n \geq 15$ and $k \leq n - 6$, if \mathcal{F} is a set family over $[n]$ of size $s = M_k + t$, then the number of k -chains in \mathcal{F} is at least*

$$\binom{n}{\lceil (n-k)/2 \rceil} \binom{\lfloor (n+k)/2 \rfloor}{k-1} (k-1)! + t \left(\binom{\lceil (n+k)/2 \rceil}{k-1} + \binom{\lceil (n+k)/2 \rceil}{k} \binom{k}{2} \right) (k-1)!$$

In both cases, we actually obtain stronger results (see Theorems 2.3.1 and 2.4.2 respectively), providing stability versions of the above theorems, showing that if a family has close to the minimum number of k -chains, it must be close in structure to the extremal example. These stability results are of interest even in the case $k = 2$, as one does not obtain any stability from the Kleitman proof for 2-chains. We then use the stability results to show that when the above bounds are tight, the extremal families are exactly as in Theorem 2.1.2.

2.1.3 Outline and notation

The remainder of this chapter is organized as follows. Section 2.2 contains a proof of Theorem 2.1.2. In Section 2.3, we prove Theorem 2.1.3, and then in Section 2.4 prove Theorem 2.1.4. In the final section we present some concluding remarks and open problems.

We let $[n]$ denote the set of the first n integers. For a ground set X and an integer i , we denote the family of i -subsets of X by $\binom{X}{i} = \{Y \subseteq X : |Y| = i\}$. We let $M_k = \sum_{i=\lceil n-k+1/2 \rceil}^{\lceil n+k-1/2 \rceil} \binom{n}{i}$ be the size of the k middle, and thus largest, levels. Given a family \mathcal{F} of subsets of $[n]$, we let $\mathcal{F}_i = \mathcal{F} \cap \binom{[n]}{i}$ denote those sets in \mathcal{F} of size i . The ℓ -shadow of a family is given by $\partial^\ell \mathcal{F} = \{G : \exists F \in \mathcal{F}, G \subset F, |G| = |F| - \ell\}$. For a subset $F \subset [n]$, we define $m(F) = \max\{|F|, n - |F|\}$.

Given a set family \mathcal{F} , $c_k(\mathcal{F})$ denotes the number of k -chains in \mathcal{F} . For any

$n \in \mathbb{N}$ and $0 \leq s \leq 2^n$, we let $c_k(n, s)$ denote the minimum of $c_k(\mathcal{F})$ over all families \mathcal{F} of s subsets of $[n]$. When $k = 2$, if we have two families \mathcal{F} and \mathcal{G} , then we let $c_2(\mathcal{F}, \mathcal{G})$ denote the number of 2-chains with one set from \mathcal{F} and one set from \mathcal{G} .

2.2 Counting 2-chains

In this section we will prove Theorem 2.1.2, characterizing those families that minimize the number of 2-chains. We essentially show that it is optimal to take sets of sizes as close to $\frac{n}{2}$ as possible. The theorem then prescribes how the boundary sets can be distributed.

Sperner's Theorem shows that the largest antichain is given by one of the middle levels, that is either all sets of size $\lfloor \frac{n}{2} \rfloor$ or all sets of size $\lceil \frac{n}{2} \rceil$. Obviously, an antichain minimizes the number of 2-chains, as it has none. This theorem is then a natural extension of Sperner's Theorem, as it shows that to construct a family of any size that minimizes the number of 2-chains, one should start by taking sets of size $\lfloor \frac{n}{2} \rfloor$, then sets of size $\lfloor \frac{n}{2} \rfloor + 1$, then $\lfloor \frac{n}{2} \rfloor - 1$, and so on until one has a family of the desired size. As we shall show, these families are optimal, and so we may denote the number of 2-chains in the first s such sets by $c_2(n, s)$.

The idea behind the proof is as follows. If our family \mathcal{F} contains a set F that is too far away from the middle (i.e. $||F| - \frac{n}{2}| > r$), then we will show that we can shift F closer to the middle and decrease the number of 2-chains. Once we have our family contained in the $2r + 1$ middle layers, a simple counting argument will give the characterization of extremal families.

Proposition 2.2.1. *Let \mathcal{F} be a family of $s > \binom{n}{\lfloor n/2 \rfloor}$ subsets of $[n]$ minimizing the number of 2-chains. If $A \in \mathcal{F}$ is of maximal cardinality, with $|A| = \frac{n}{2} + m$, then for any $B \subset A$, $|B| \geq \frac{n}{2} - m + 1$, we have $B \in \mathcal{F}$.*

Assuming this proposition, we shall proceed to prove Theorem 2.1.2.

Proof of Theorem 2.1.2. We prove the theorem by induction on s .

For the base case, we take $s = \binom{n}{\lfloor n/2 \rfloor}$. By Sperner's Theorem, it follows that any family \mathcal{F} of this size that minimizes the number of 2-chains must be an antichain. It is well known that the only antichains of this size are the family of all sets of size $\lfloor \frac{n}{2} \rfloor$, or the family of sets of size $\lceil \frac{n}{2} \rceil$. It is easy to see that these families are the only ones satisfying Properties 1 through 4, with $r = 0$ or $\frac{1}{2}$ depending on whether n is even or odd respectively.

For the induction step, assume $s > \binom{n}{\lfloor n/2 \rfloor}$, and let \mathcal{F} be an optimal family of size s . Suppose Property 1 were not satisfied. Since \mathcal{F} and $\mathcal{F}' = \{[n] \setminus F : F \in \mathcal{F}\}$ have the same number of 2-chains, we may assume there is a largest set $F \in \mathcal{F}$ with $|F| = \frac{n}{2} + t$ for some $t > r$. By Proposition 2.2.1, it follows that for every $G \subset F$, $|G| \geq \frac{n}{2} - t + 1$, we have $G \in \mathcal{F}$. Hence F is in at least $\sum_{i=1}^{2t-1} \binom{\frac{n}{2}+t}{i}$ 2-chains in \mathcal{F} . Since $\mathcal{F} \setminus \{F\}$ is a family of $s - 1$ sets, there are at least $c_2(n, s - 1)$ 2-chains in \mathcal{F} not involving F . Thus the number of 2-chains in \mathcal{F} is at least $c_2(n, s - 1) + \sum_{i=1}^{2t-1} \binom{\frac{n}{2}+t}{i} > c_2(n, s - 1) + \sum_{i=1}^{2r} \binom{\frac{n}{2}+r}{i} \geq c_2(n, s)$, and so \mathcal{F} cannot be optimal, giving a contradiction. Hence if \mathcal{F} is optimal, each $F \in \mathcal{F}$ has $\frac{n}{2} - r \leq |F| \leq \frac{n}{2} + r$, and so Property 1 is established.

Now consider the case $\sum_{i=\frac{n}{2}-r+1}^{\frac{n}{2}+r-1} \binom{n}{i} < s \leq \sum_{i=\frac{n}{2}-r}^{\frac{n}{2}+r-1} \binom{n}{i}$. Since $s > \sum_{i=\frac{n}{2}-r+1}^{\frac{n}{2}+r-1} \binom{n}{i}$, and in light of Property 1, it follows that there exists some $F \in \mathcal{F}$ with $|F| = \frac{n}{2} \pm r$; by symmetry, we may assume $|F| = \frac{n}{2} + r$. By Proposition 2.2.1, F must be contained in $\sum_{i=1}^{2r-1} \binom{\frac{n}{2}+r}{i} = c_2(n, s) - c_2(n, s - 1)$ 2-chains with sets in \mathcal{F} of sizes between $\frac{n}{2} - r + 1$ and $\frac{n}{2} + r - 1$. If F is contained in any 2-chains with sets of size $\frac{n}{2} - r$, then by induction it follows that \mathcal{F} has more than $c_2(n, s)$ 2-chains, contradicting the optimality of \mathcal{F} . Thus F is incomparable to the other sets in \mathcal{F} of sizes $\frac{n}{2} \pm r$. Removing F , we find that $\mathcal{F} \setminus \{F\}$ must also be optimal, and thus Properties 2 and 3 follow.

Finally, suppose $\sum_{i=\frac{n}{2}-r}^{\frac{n}{2}+r-1} \binom{n}{i} \leq s \leq \sum_{i=\frac{n}{2}-r}^{\frac{n}{2}+r} \binom{n}{i}$. By Property 1, we know all sets in \mathcal{F} have sizes between $\frac{n}{2} - r$ and $\frac{n}{2} + r$. Let $\mathcal{H} = \cup_{i=\frac{n}{2}-r}^{\frac{n}{2}+r} \binom{[n]}{i}$ be the family of all subsets of $[n]$ of sizes between $\frac{n}{2} - r$ and $\frac{n}{2} + r$, and let $\mathcal{G} = \mathcal{H} \setminus \mathcal{F}$ be those sets not in \mathcal{F} . We have $c_2(\mathcal{F}) = c_2(\mathcal{H}) - c_2(\mathcal{G}, \mathcal{H}) + c_2(\mathcal{G})$. $c_2(\mathcal{H})$ depends only on r , and hence on s , and is independent of the structure of \mathcal{F} . We have

$$c_2(\mathcal{G}, \mathcal{H}) = \sum_{G \in \mathcal{G}} c_2(\{G\}, \mathcal{H}) = \sum_{G \in \mathcal{G}} \left(\sum_{i=\frac{n}{2}-r}^{|G|-1} \binom{|G|}{i} + \sum_{i=|G|+1}^{\frac{n}{2}+r} \binom{n-|G|}{i-|G|} \right).$$

The parenthetical term is maximized when $|G| = \frac{n}{2} \pm r$, and so $c_2(\mathcal{G}, \mathcal{H})$ is maximized when for every $G \in \mathcal{G}$ we have $|G| = \frac{n}{2} \pm r$. Finally, $c_2(\mathcal{G})$ is minimized when \mathcal{G} is an antichain, in which case $c_2(\mathcal{G}) = 0$. Both of these conditions are satisfied by the construction outlined at the beginning of this section, and hence must also be true of any other extremal family. Thus to minimize $c_2(\mathcal{F})$, \mathcal{F} must contain all sets of sizes between $\frac{n}{2} - r + 1$ and $\frac{n}{2} + r - 1$, and $\mathcal{G} = \left(\binom{[n]}{\frac{n}{2}+r} \cup \binom{[n]}{\frac{n}{2}-r} \right) \setminus \mathcal{F}$ should be an antichain, establishing Properties 2 and 4. This completes the induction step, and with it the proof of Theorem 2.1.2.

□

Now we prove Proposition 2.2.1, which enables us to perform the shifting necessary for Theorem 2.1.2. This is essentially the same shifting argument used in the original proof of Kleitman in [25]. We provide the proof here as the details of the calculations are not included in Kleitman's paper.

Proof of Proposition 2.2.1. Suppose not. Note that we must have $m \geq 1$, otherwise there is nothing to prove. Let $\ell \leq 2m - 1$ be the minimal integer such that there exists a largest set of size $\frac{n}{2} + m$ with a subset of size $\frac{n}{2} + m - \ell$ that is not in the family. Let $\mathcal{A} = \{A \in \mathcal{F} : |A| = \frac{n}{2} + m, \partial^\ell A \notin \mathcal{F}\}$, and let $\mathcal{B} = \partial^\ell \mathcal{A} \setminus \mathcal{F}$. We can construct an auxiliary bipartite inclusion graph on $\mathcal{A} \cup \mathcal{B}$, with an edge (A, B) iff $B \subset A$.

Consider first the case where we have a matching $M : \mathcal{A} \rightarrow \mathcal{B}$, so that for every set $A \in \mathcal{A}$ there exists a set $M(A) \subset A$, $M(A) \notin \mathcal{F}$. We shift the family from \mathcal{F} to $\tilde{\mathcal{F}}$ by replacing each set $A \in \mathcal{A}$ by $M(A) \in \mathcal{B}$, and claim that this reduces the number of 2-chains. Note that if $B = M(A)$ is a newly-introduced set, and $C \in \mathcal{F}$ is a set with $C \subset B$, then we must have had $C \subset A$ as well. Thus the only 2-chains that we need to consider are those between the levels $\frac{n}{2} + m$ and $\frac{n}{2} + m - \ell$; we call these *intermediate chains*.

Suppose $\ell > 1$. By the minimality of our choice of ℓ , we must have $\partial^i \mathcal{A} \subset \mathcal{F}$ for every $1 \leq i \leq \ell - 1$. Thus the number of intermediate chains in \mathcal{F} that we lose is at least $|\mathcal{A}| \sum_{i=1}^{\ell-1} \binom{\frac{n}{2}+m}{i}$. On the other hand, all sets of size $\frac{n}{2} + m$ in $\tilde{\mathcal{F}}$ are the sets from \mathcal{F} with i -shadow completely in \mathcal{F} for all $i \leq \ell$. These sets cannot be involved in any 2-chains with sets in \mathcal{B} , and therefore we only gain intermediate chains between the levels $\frac{n}{2} + m - 1$ and $\frac{n}{2} + m - \ell$. The number of such chains that we gain is at most $|\mathcal{A}| \sum_{i=1}^{\ell-1} \binom{\frac{n}{2}-m+\ell}{i}$. Since $\ell \leq 2m - 1$, it follows that $\binom{\frac{n}{2}-m+\ell}{i} < \binom{\frac{n}{2}+m}{i}$ for every $1 \leq i \leq \ell - 1$, and hence the number of 2-chains decreases.

Thus we may assume $\ell = 1$. If we had $A \in \mathcal{A}$ and $B \in \mathcal{F}$ with $B \subset A$, $|B| = \frac{n}{2} + m - 1$, then upon shifting to $\tilde{\mathcal{F}}$, we lose the 2-chain $B \subset A$ and gain no pairs. Hence we may assume $\partial \mathcal{A} \cap \mathcal{F} = \emptyset$, so $\mathcal{B} = \partial \mathcal{A}$. We now claim that the sets $A \in \mathcal{A}$ cannot be involved in any 2-chains $C \subset A$ in \mathcal{F} . Suppose to the contrary we had such a 2-chain. Let $x \in C$ be an arbitrary element of C , and shift A to $A \setminus \{x\}$ (recall that $A \setminus \{x\} \notin \mathcal{F}$). Shift the remaining sets in \mathcal{A} by an arbitrary matching from $\mathcal{A}' = \mathcal{A} \setminus \{A\}$ to $\mathcal{B}' = \partial \mathcal{A} \setminus \{A \setminus \{x\}\}$; we can do this by Hall's Theorem, since every set in \mathcal{A}' has at least $\frac{n}{2} + m - 1$ neighbors in \mathcal{B}' , while each set in \mathcal{B}' has at most $\frac{n}{2} - m + 1$ neighbors. In this shifted set we have lost the 2-chain $C \subset A$, and hence $\tilde{\mathcal{F}}$ has fewer 2-chains.

Hence we may assume that there are no 2-chains in \mathcal{F} involving sets in \mathcal{A} . Thus in the shifted family $\tilde{\mathcal{F}}$, sets in $\partial \mathcal{A}$ will also not be in any 2-chains. Now,

since $|\mathcal{F}| > \binom{n}{\lfloor n/2 \rfloor}$, it follows from Sperner's Theorem that there is some 2-chain $C \subset D$ in \mathcal{F} . In $\tilde{\mathcal{F}}$, we may also shift D to some set in $\partial\mathcal{A}$, since $m \geq 1$ implies $|\partial\mathcal{A}| > |\mathcal{A}|$. As no set in $\partial\mathcal{A}$ is involved in any 2-chain, this reduces the number of 2-chains, which contradicts the minimality of \mathcal{F} .

Therefore we conclude that there cannot be a matching from \mathcal{A} to \mathcal{B} in the auxiliary bipartite inclusion graph, and so we will not shift all sets in \mathcal{A} . Instead, we use the following lemma, to be proven shortly, to find a collection of sets to shift.

Lemma 2.2.2. *Let G be a bipartite graph on $U \cup V$ with minimum degree $\delta_U \geq 1$ in U and maximum degree Δ_V in V . Suppose there is no matching from U to V . Then there exist nonempty subsets $U_1 \subset U$ and $V_1 \subset V$ with a perfect matching $M : U_1 \rightarrow V_1$ and $e(U_1, V) + e(U \setminus U_1, V_1) \leq |U_1|\Delta_V$.*

Our auxiliary graph satisfies the conditions of the lemma, with $U = \mathcal{A}, V = \mathcal{B}, \Delta_V = \binom{\frac{n}{2}-m+\ell}{\ell} < \binom{\frac{n}{2}+m}{\ell}$, and so we can find a collection of sets $\mathcal{A}_1 \subset \mathcal{A}$ and a matching $M : \mathcal{A}_1 \rightarrow \mathcal{B}_1 \subset \mathcal{B}$ as given by the lemma. Consider the shifted family $\tilde{\mathcal{F}}$ where we replace the sets in \mathcal{A}_1 by the corresponding sets in \mathcal{B}_1 . As before, since for every $A \in \mathcal{A}_1$ we have $M(A) \subset A$, we need only consider the intermediate chains.

Again, by the minimality of ℓ , we know that \mathcal{A}_1 has full shadow in \mathcal{F} up until the ℓ th shadow, and so the same calculation as before implies that we remove more chains than we gain, and thus have fewer intermediate chains in $\tilde{\mathcal{F}}$. Hence it suffices to consider only the new chains formed between levels $\frac{n}{2} + m$ and $\frac{n}{2} + m - \ell$.

The number of new chains between these levels we gain is exactly $e(\mathcal{A} \setminus \mathcal{A}_1, \mathcal{B}_1)$. On the other hand, we lose all chains between \mathcal{A}_1 and $\partial^\ell \mathcal{A}_1 \cap \mathcal{F} = \partial^\ell \mathcal{A}_1 \setminus \mathcal{B}$. Thus the number of chains we are losing is $|\mathcal{A}_1| \binom{\frac{n}{2}+m}{\ell} - e(\mathcal{A}_1, \mathcal{B})$. By the lemma, we have $|\mathcal{A}_1| \binom{\frac{n}{2}+m}{\ell} - e(\mathcal{A}_1, \mathcal{B}) > |\mathcal{A}_1| \binom{\frac{n}{2}-m+\ell}{\ell} - e(\mathcal{A}_1, \mathcal{B}) \geq e(\mathcal{A} \setminus \mathcal{A}_1, \mathcal{B}_1)$, and hence $\tilde{\mathcal{F}}$ has fewer 2-chains than \mathcal{F} , contradicting the optimality of \mathcal{F} .

Thus if \mathcal{F} minimizes the number of 2-chains, and A is the largest set in \mathcal{F} with $|A| = \frac{n}{2} + m$, then whenever $B \subset A$ with $|B| \geq \frac{n}{2} - m + 1$, we must have $B \in \mathcal{F}$ as well.

□

It remains to furnish a proof of Lemma 2.2.2, which we now provide.

Proof of Lemma 2.2.2. As there is no matching from U to V , by Hall's Theorem there exists a minimal subset $U_0 \subseteq U$ with $|N(U_0)| < |U_0|$. Since $\delta_U \geq 1$, we must have $|U_0| \geq 2$. Let $u \in U_0$ be an arbitrary element, and take $U_1 = U_0 \setminus \{u\}$. By the minimality of U_0 , it follows that $|N(U_1)| \geq |U_1|$, and so we must have $|N(U_1)| = |U_1|$. Set $V_1 = N(U_1)$. Again by the minimality of U_0 , for any subset $X \subset U_1$, $|N(X)| \geq |X|$, and so by Hall's Theorem there exists a perfect matching $M : U_1 \rightarrow V_1$.

Now $e(U_1, V) + e(U \setminus U_1, V_1) = e(U_1, V) + e(U, V_1) - e(U_1, V_1) = e(U, V_1) \leq |V_1|\Delta_V = |U_1|\Delta_V$, where the second equality follows from the fact that $N(U_1) = V_1$, and thus we have the desired inequality. □

2.3 Counting k -chains

We now seek a similar result for k -chains, and thus to make some progress on Conjecture 2.1.1. In this section we verify the conjecture when the number of sets is at most that in the k middle levels, and in the next section we shall extend the result to the $k + 1$ middle levels.

Note that if we take all M_{k-1} sets in the $k - 1$ middle levels, that is of sizes between $\lfloor \frac{n-k}{2} \rfloor + 1$ and $\lfloor \frac{n+k}{2} \rfloor - 1$, and then add t sets of size $\lfloor \frac{n+k}{2} \rfloor$, we would create precisely $t \binom{\lfloor (n+k)/2 \rfloor}{k-1} (k-1)!$ k -chains. Hence Theorem 2.1.3 is tight when $0 \leq t \leq \binom{n}{\lfloor (n+k)/2 \rfloor}$. We shall in fact prove the following stronger theorem, which provides a stability result. In the following notation, we let $r \in \{\frac{k-1}{2}, \frac{k}{2}\}$ be such

that the sets in the k middle levels have sizes between $\frac{n}{2} - r$ and $\frac{n}{2} + r$, and for a set F , we define $m(F) = \max\{|F|, n - |F|\}$.

Theorem 2.3.1. *Let \mathcal{F} be a family of subsets of $[n]$. Then the number of k -chains in \mathcal{F} is bounded by*

$$c_k(\mathcal{F}) \geq \left(\sum_{i=n/2-r}^{n/2+r} \frac{|\mathcal{F}_i|}{\binom{n}{i}} - (k-1) \right) \binom{n}{n/2+r} \binom{n/2+r}{k-1} (k-1)! \\ + \sum_{\substack{F \in \mathcal{F}: \\ \left| |F| - n/2 \right| \geq r+1}} \binom{m(F)}{k-1} (k-1)!.$$

Proof. We prove the theorem by induction on $|\mathcal{F}|$. If $|\mathcal{F}| = 0$, then there is nothing to show, as the desired lower bound is negative.

For the induction step, we begin by noting that for every set $F \in \mathcal{F}$ with $\left| |F| - \frac{n}{2} \right| \geq r$, F can be in at most $\binom{m(F)}{k-1} (k-1)!$ k -chains. If not, then we could remove F , and applying the inductive hypothesis to $\mathcal{F} \setminus \{F\}$, we would have the desired inequality.

We now use an LYM-type inequality, counting the number of k -chains in our family by considering permutations. We say that a permutation $\sigma \in S_n$ contains a set $F \subset [n]$, denoted $F \in \sigma$, if $\{\sigma(1), \sigma(2), \dots, \sigma(|F|)\} = F$; that is, F is an initial segment of σ . Note that if σ contains k sets F_1, F_2, \dots, F_k , then those k sets must form a k -chain. For any set $F \subset [n]$, we let $S_n[F] = \{\sigma \in S_n : F \in \sigma\}$. Since every permutation containing m sets contributes $m - \binom{m}{k} \leq k-1$ to the right-hand side of the sum below, it follows that

$$(k-1)n! \geq \sum_{F \in \mathcal{F}} |S_n[F]| - \sum_{F_1 \subset F_2 \subset \dots \subset F_k \in \mathcal{F}} \left| \bigcap_{i=1}^k S_n[F_i] \right|. \quad (2.1)$$

As the second sum is over all k -chains in our family \mathcal{F} , this inequality will allow us to bound the number of k -chains. Note that for any $F \in \mathcal{F}$, we have $|S_n[F]| = |F|! |[n] \setminus F|!$, and for a k -chain $F_1 \subset F_2 \subset \dots \subset F_k$, $\left| \bigcap_{i=1}^k S_n[F_i] \right| =$

$|F_1|! \prod_{i=1}^{k-1} |F_{i+1} \setminus F_i|! |[n] \setminus F_k|!$ gives the number of permutations containing the k -chain.

We shall associate every k -chain in \mathcal{F} with either its minimum or maximum set, depending on which is further away from the middle level. For $F \in \mathcal{F}$ with $|F| < \frac{n}{2}$, let $\mathcal{C}(F) = \{F \subset F_2 \subset \dots \subset F_k : |F| + |F_k| < n\}$, and if $F \in \mathcal{F}$ with $|F| \geq \frac{n}{2}$, let $\mathcal{C}(F) = \{F_1 \subset \dots \subset F_{k-1} \subset F : |F_1| + |F| \geq n\}$, and, for convenience, define $C(F) = |\mathcal{C}(F)|$. Note that we have partitioned the set of k -chains $\{F_1 \subset F_2 \subset \dots \subset F_k : F_i \in \mathcal{F}\}$ into the disjoint sets $\{\mathcal{C}(F)\}_{F \in \mathcal{F}}$. We can thus rewrite inequality (2.1) as follows:

$$(k-1)n! \geq \sum_{F \in \mathcal{F}} \left[|S_n[F]| - \sum_{F_1 \subsetneq F_2 \subsetneq \dots \subsetneq F_k \in \mathcal{C}(F)} |\cap_{i=1}^k S_n[F_i]| \right]. \quad (2.2)$$

To bound $|\cap_{i=1}^k S_n[F_i]|$ appropriately, we require that \emptyset and $[n]$ not be members of our family. This is given by the following lemma.

Lemma 2.3.2. *If $|\mathcal{F}| \leq M_{n-1}$, and \mathcal{F} minimizes the number of k -chains, then $\emptyset \notin \mathcal{F}$ and $[n] \notin \mathcal{F}$.*

Proof. Suppose we had $[n] \in \mathcal{F}$. Since $|F| \leq 2^n - 2$, there must be some $\emptyset \neq F \notin \mathcal{F}$. We decrease the number of k -chains in \mathcal{F} by replacing $[n]$ with F , since any new k -chain involving F was a k -chain with $[n]$ before.

Similarly, if $\emptyset \in \mathcal{F}$, we can replace it with any set $[n] \neq F \notin \mathcal{F}$. □

Note that $M_{n-1} = 2^n - 2$. Thus, if $|\mathcal{F}| > M_{n-1}$, then either we have all subsets of $[n]$, or our family is missing just one set, in which case (as we explained above) it is best to remove either \emptyset or $[n]$. In either case, the bound in Theorem 2.3.1 remains true.

We now assume $1 \leq |F| \leq n-1$ for all $F \in \mathcal{F}$. If we fix $F_1, |F_1| < n/2$, then we maximize $|\cap_{i=1}^k S_n[F_i]| = |F_1|! \prod_{i=1}^{k-1} |F_{i+1} \setminus F_i|! |[n] \setminus F_k|!$ by taking

$|F_{i+1} \setminus F_i| = 1$ for $1 \leq i \leq k-1$, and so $|\cap_{i=1}^k S_n[F_i]| \leq |F_1|! (n - |F_1| - (k-1))!$. The same holds true if we instead fix F_k , $|F_k| \geq n/2$, and thus $|\cap_{i=1}^k S_n[F_i]| \leq (|F_k| - (k-1))! (n - |F_k|)!$. We can unify both bounds in the form $\frac{|F|!(n-|F|)!}{\binom{m(F)}{k-1}(k-1)!}$.

Moreover, by definition we must have $\mathcal{C}(F) = \emptyset$ for any F with $||F| - \frac{n}{2}| \leq r-1$. Hence we split our sum based on how $||F| - \frac{n}{2}|$ compares to r . Dividing through by $n!$, inequality (2.2) leads to

$$k-1 \geq \sum_{F \in \mathcal{F}} \frac{1}{\binom{n}{|F|}} \left(1 - \frac{C(F)}{\binom{m(F)}{k-1}(k-1)!} \right) = \Sigma_1 + \Sigma_2 + \Sigma_3,$$

where

$$\begin{aligned} \Sigma_1 &= \sum_{\substack{F \in \mathcal{F} \\ ||F| - n/2| \leq r-1}} \frac{1}{\binom{n}{|F|}} = \sum_{i=n/2-r+1}^{n/2+r-1} \frac{|\mathcal{F}_i|}{\binom{n}{i}}, \\ \Sigma_2 &= \sum_{\substack{F \in \mathcal{F} \\ ||F| - n/2| = r}} \frac{1}{\binom{n}{n/2+r}} \left(1 - \frac{C(F)}{\binom{n/2+r}{k-1}(k-1)!} \right) \\ &= \frac{|\mathcal{F}_{n/2-r}| + |\mathcal{F}_{n/2+r}|}{\binom{n}{n/2+r}} - \frac{\sum_{||F| - n/2| = r} C(F)}{\binom{n}{n/2+r} \binom{n/2+r}{k-1} (k-1)!}, \end{aligned}$$

and

$$\Sigma_3 = \sum_{\substack{F \in \mathcal{F} \\ ||F| - n/2| \geq r+1}} \frac{1}{\binom{n}{|F|}} \left(1 - \frac{C(F)}{\binom{m(F)}{k-1}} \right).$$

Note that $\sum_{||F| - n/2| = r} C(F) = c_k(\mathcal{F}) - \sum_{||F| - n/2| \geq r+1} C(F)$, and so, substituting in Σ_2 , we obtain

$$\begin{aligned} &\frac{c_k(\mathcal{F})}{\binom{n}{n/2+r} \binom{n/2+r}{k-1} (k-1)!} + k-1 \geq \sum_{i=n/2-r}^{n/2+r} \frac{|\mathcal{F}_i|}{\binom{n}{i}} \\ &+ \sum_{\substack{F \in \mathcal{F}: \\ ||F| - n/2| \geq r+1}} \left(\frac{1}{\binom{n}{|F|}} - \frac{C(F)}{(k-1)!} \left(\frac{1}{\binom{n}{|F|} \binom{m(F)}{k-1}} - \frac{1}{\binom{n}{n/2+r} \binom{n/2+r}{k-1}} \right) \right). \end{aligned}$$

Now, since $\binom{n}{a} \binom{a}{k-1} = \binom{n}{k-1} \binom{n-k+1}{a-k+1}$, it follows that when $m(F) \geq \frac{n}{2} + r$, $\binom{n}{|F|} \binom{m(F)}{k-1} \leq \binom{n}{n/2+r} \binom{n/2+r}{k-1}$, as $\frac{1}{2}(n-k+1) \leq \frac{n}{2} + r - k + 1 \leq m(F) - k + 1$.

Hence the summand is minimized when $C(F)$ is as large as possible, which, by our inductive hypothesis, is $\binom{m(F)}{k-1}(k-1)!$. Substituting this into the inequality above gives

$$\frac{c_k(\mathcal{F})}{\binom{n}{n/2+r} \binom{n/2+r}{k-1} (k-1)!} + k - 1 \geq \sum_{i=n/2-r}^{n/2+r} \frac{|\mathcal{F}_i|}{\binom{n}{i}} + \sum_{\substack{F \in \mathcal{F}: \\ \left| |F| - n/2 \right| \geq r+1}} \frac{\binom{m(F)}{k-1}}{\binom{n}{n/2+r} \binom{n/2+r}{k-1}}.$$

Rearranging gives the desired bound

$$\begin{aligned} c_k(\mathcal{F}) &\geq \left(\sum_{i=n/2-r}^{n/2+r} \frac{|\mathcal{F}_i|}{\binom{n}{i}} - (k-1) \right) \binom{n}{n/2+r} \binom{n/2+r}{k-1} (k-1)! \\ &\quad + \sum_{\substack{F \in \mathcal{F}: \\ \left| |F| - n/2 \right| \geq r+1}} \binom{m(F)}{k-1} (k-1)!. \end{aligned}$$

□

Given Theorem 2.3.1, it is easy to deduce Theorem 2.1.3. In fact, we are able to characterize all extremal families.

Proof of Theorem 2.1.3. Suppose we have a family of sets \mathcal{F} , with $|\mathcal{F}| = M_{k-1} + t$. Note that the contribution each set $F \in \mathcal{F}$ makes to the right-hand side above is $\frac{\binom{n}{n/2+r}}{\binom{n}{|F|}} \binom{n/2+r}{k-1} (k-1)!$ if $\frac{n}{2} - r \leq |F| \leq \frac{n}{2} + r$, and $\binom{m(F)}{k-1} (k-1)!$ otherwise. This contribution increases with $\left| |F| - \frac{n}{2} \right|$, and so to minimize the right-hand side we need all sets to satisfy $\left| |F| - \frac{n}{2} \right| \leq r$. Moreover, since the binomial coefficients $\binom{n}{i}$ are minimized over $n/2 - r \leq i \leq n/2 + r$ when $i = n/2 \pm r$, any extremal family must contain all sets of sizes between $\frac{n}{2} - r + 1$ and $\frac{n}{2} + r - 1$, with the remaining sets having size $\frac{n}{2} \pm r$. It is easy to see that such a collection of sets gives $c_k(\mathcal{F}) \geq t \binom{n/2+r}{k-1} (k-1)!$ above, as required.

To classify the extremal families, note that we already know we must have all sets of sizes between $\frac{n}{2} - r + 1$ and $\frac{n}{2} + r - 1$. If $r = \frac{k}{2}$, this consists of the middle $k-1$ levels, giving M_{k-1} sets. Hence we have t sets of size $\frac{n}{2} \pm r$. To obtain equality in (2.1), we must have every chain passing through either $k-1$ or k sets of \mathcal{F} . As

all the sets in the $k - 1$ middle levels are in \mathcal{F} , the chain can contain at most one set from \mathcal{F} of size $\frac{n}{2} \pm r$. From this, we deduce that $\{F \in \mathcal{F} : |F| = \frac{n}{2} \pm r\}$ must be an antichain.

If $r = \frac{k-1}{2}$, we know the middle $k - 2$ levels are full, and we have $\binom{n}{n/2+r} + t$ sets of size $\frac{n}{2} \pm r$. In order to obtain equality in (2.1), we must therefore have every chain pass through at least one set in \mathcal{F} of size $\frac{n}{2} \pm r$. Hence $\{G \notin \mathcal{F} : |G| = \frac{n}{2} \pm r\}$ must form an antichain.

In particular, we note that the extremal families are exactly the same as for Theorem 2.1.2. □

We remark that Theorem 2.3.1 is a stability result for Theorem 2.1.3, as our bound on $c_k(\mathcal{F})$ increases if we are missing sets with $||F| - \frac{n}{2}| \leq r - 1$, or have sets with $||F| - \frac{n}{2}| \geq r + 1$. Moreover, the stability estimates we obtain can also be tight. For instance, if we replace ℓ sets of size $\frac{n}{2} \pm r$ with sets of size $\frac{n}{2} + r + 1$, Theorem 2.3.1, together with some simple computations, shows that we should gain at least $\ell \binom{n/2+r}{k-2} (k-1)!$ extra k -chains. Moreover, it is easy to check that we gain precisely that many k -chains in the case when our family includes all of the $(\frac{n}{2} + r)$ -sets and none of the $(\frac{n}{2} - r)$ -sets in the shadow of the $(\frac{n}{2} + r + 1)$ -sets which were added.

Similarly, if we replace ℓ sets of size $\frac{n}{2} + r - 1$ with sets of size $\frac{n}{2} + r$, the theorem shows that we must gain at least $\frac{2r-1}{n/2+r} \ell \binom{n/2+r}{k-1} (k-1)!$ extra k -chains. This is tight again, if our family includes all the sets of size $\frac{n}{2} + r$ containing any of the replaced sets. As we remarked earlier, these stability results are new even in the case $k = 2$.

2.4 Larger families

While Theorem 2.1.3 provides a tight bound on the number of k -chains appearing in families contained within the k middle levels, it underestimates the number of k -chains appearing in larger families. This is because in our calculations we assumed every k -chain had steps $(F_i \setminus F_{i-1})$ of size 1, as this maximizes the number of permutations containing the k -chain. However, when we are working with the $k + 1$ middle levels, we also have k -chains with a larger step of size 2, and so we shall have to make our argument more robust in order to handle these chains.

However, there is one additional difficulty. Recall that the number of permutations containing a k -chain $F_1 \subset F_2 \subset \dots \subset F_k$ is given by $|\cap_{i=1}^k S_n[F_i]| = |F_1|! \prod_{i=1}^{k-1} |F_{i+1} \setminus F_i|! (n - |F_k|)!$. If we fix F_k , say, then we would hope that for k -chains involving a larger step, the largest this can be is to have one step of size 2, and have all the other steps have size 1, as this is precisely the type of k -chain that appears in our extremal families. Such k -chains are in $2(|F_k| - k)!(n - |F_k|)!$ permutations.

Unfortunately, a chain with $k - 2$ steps of size 1 and one step of size $|F_k| - k + 1$ (so that $|F_1| = 1$) is contained in $(|F_k| - k + 1)!(n - |F_k|)!$ permutations, which is larger than the bound we require. However, recall that we assign k -chains to either F_1 or F_k , depending on which is further from the middle. Thus, unless $|F_k| = n - 1$, we would assign the above k -chain to F_1 , and so we would be fixing F_1 and not F_k . Hence the one case we need to avoid is having a k -chain starting with a set of size 1 and ending with a set of size $n - 1$.

We shall later provide a separate argument to show that there cannot be any sets of size $n - 1$ in an extremal family, thus bypassing this problem. In the meanwhile, for the purposes of our stability results, we shall assume there are no sets of size $n - 1$. We now require two arguments - one to bound the number of k -chains with larger steps, and one to bound the total number of k -chains.

We introduce the following notation for the remainder of this section. Given n and k , we let $a = \lceil \frac{n+k}{2} \rceil$, so that the k middle levels are those sets of sizes between $a - k$ and $a - 1$, and the $(k + 1)$ st middle level has sets of size a . For a set family \mathcal{F} , we let $\mathcal{C}(\mathcal{F})$ be the set of k -chains in \mathcal{F} . We partition these into two subsets: $\mathcal{C}_1(\mathcal{F})$ are those k -chains $F_1 \subset F_2 \subset \dots \subset F_k$ with $|F_{i+1} \setminus F_i| = 1$ for all $1 \leq i \leq k - 1$, and $\mathcal{C}_2(\mathcal{F}) = \mathcal{C}(\mathcal{F}) \setminus \mathcal{C}_1(\mathcal{F})$ those k -chains with a larger step. We let $C(\mathcal{F})$, $C_1(\mathcal{F})$ and $C_2(\mathcal{F})$ denote the number of k -chains in these subsets respectively. As in Theorem 2.1.3, we will again identify a k -chain with one of its endpoints F_1 or F_k , depending which is further from the middle level, giving the partition $\{\mathcal{C}(F)\}_{F \in \mathcal{F}}$ of $\mathcal{C}(\mathcal{F})$. These sets will again be partitioned into $\mathcal{C}_1(F)$ and $\mathcal{C}_2(F)$, depending on whether or not the k -chains have a step of size at least 2. Finally, $C(F)$, $C_1(F)$ and $C_2(F)$ represent the sizes of the corresponding sets of k -chains.

2.4.1 Counting k -chains with larger steps

We begin by showing that large families must contain a number of k -chains with a step of size at least 2. The following proposition also provides some stability, which we shall require to show that an extremal family cannot contain any sets of size $n - 1$.

Proposition 2.4.1. *Let \mathcal{F} be a set family of size $|\mathcal{F}| = M_k + t_1$, with $1 \leq |F| \leq n - 2$ for all $F \in \mathcal{F}$, and with at least t_2 sets missing from the middle $k - 1$ levels.*

Then

$$C_2(\mathcal{F}) \geq \left(t_1 + \binom{k-1}{a} t_2 \right) \binom{a}{k} \binom{k}{2} (k-1)!.$$

Proof. We prove the statement by induction on $t_1 + t_2 \geq 0$, noting that we must have $t_2 \geq 0$. The base case of $t_1 + t_2 = 0$ is trivial, as in this case the right-hand side is non-positive.

The proof will now run along very similar lines to that of Theorem 2.1.3,

and we shall just make a few changes to count only those chains with a large step. To begin with, when we are counting sets and k -chains in permutations, we only want to consider those k -chains with a large step. To ensure this, we shall not count k -chains that appear consecutively in some permutation. That is, if $\sigma \in S_n$ contains the sets $F_1 \subset F_2 \subset \dots \subset F_s$ for some $s \geq k + 1$, we will count $F_1 \subset F_2 \subset \dots \subset F_{k-1} \subset F_{k+1}$, but not $F_1 \subset F_2 \subset \dots \subset F_k$. Thus every k -chain we consider is bound to have some step of size at least 2. If $s \geq k$, then the number of such chains is $\binom{s}{k} - (s - (k - 1))$, and since $(k-1)s - (\binom{s}{k} - (s - (k - 1))) \leq k^2 - k$, it follows that

$$\begin{aligned} (k^2 - k)n! &\geq (k - 1) \sum_{F \in \mathcal{F}} |S_n[F]| - \sum_{F_1 \subset \dots \subset F_k \in \mathcal{C}_2(\mathcal{F})} |\cap_{i=1}^k S_n[F_i]| \\ &\geq \sum_{F \in \mathcal{F}} \left((k - 1) |S_n[F]| - \sum_{F_1 \subset \dots \subset F_k \in \mathcal{C}_2(F)} |\cap_{i=1}^k S_n[F_i]| \right). \end{aligned}$$

As before, we now seek to maximize the terms $|\cap_{i=1}^k S_n[F_i]|$. Provided we have $1 \leq |F| \leq n - 2$ for all sets $F \in \mathcal{F}$, if we fix one of the endpoints of the chain, the number of permutations it is contained in is maximized when we have one step of size 2, and all the other steps of size 1. Thus we can bound $|\cap_{i=1}^k S_n[F_i]|$ by $2|F_1|!(n - |F_1| - k)!$ or $2(|F_k| - k)!(n - |F_k|)!$. Dividing through by $n!$, we have that

$$k^2 - k \geq \sum_{F \in \mathcal{F}} \frac{1}{\binom{n}{|F|}} \left(k - 1 - \frac{C_2(F)}{\binom{m(F)}{k} \frac{k!}{2}} \right).$$

Now, by definition, we must have $C_2(F) = 0$ for all sets F in the k middle levels; that is, with $a - k \leq |F| \leq a - 1$. Let $\hat{\mathcal{F}} = \{F \in \mathcal{F} : |F| \leq a - k - 1 \text{ or } |F| \geq a\}$ be those sets outside the middle k levels. Thus

$$k^2 - k \geq \sum_{i=a-k}^{a-1} \frac{(k-1)|\mathcal{F}_i|}{\binom{n}{i}} + \sum_{F \in \hat{\mathcal{F}}} \frac{1}{\binom{n}{|F|}} \left(k - 1 - \frac{C_2(F)}{\binom{m(F)}{k} \frac{k!}{2}} \right).$$

We may assume that for every set $F \in \hat{\mathcal{F}}$, $C_2(F) \leq \binom{a}{k} \binom{k}{2} (k-1)! = (k-1) \binom{a}{k} \frac{k!}{2}$, since otherwise we may remove F from \mathcal{F} and are then done by induction. Hence,

since $m(F) \geq a$ for all $F \in \mathcal{F}$, the parenthetical term in the second sum is always non-negative, and so the right-hand side is minimized by replacing $\binom{n}{|F|}$ by $\binom{n}{a}$, giving

$$\begin{aligned} k^2 - k &\geq \sum_{i=a-k}^{a-1} \frac{(k-1)|\mathcal{F}_i|}{\binom{n}{i}} + \sum_{F \in \hat{\mathcal{F}}} \frac{1}{\binom{n}{a}} \left(k-1 - \frac{C_2(F)}{\binom{m(F)}{k} \frac{k!}{2}} \right) \\ &\geq \sum_{i=a-k}^{a-1} \frac{(k-1)|\mathcal{F}_i|}{\binom{n}{i}} + \sum_{F \in \hat{\mathcal{F}}} \frac{1}{\binom{n}{a}} \left(k-1 - \frac{C_2(F)}{\binom{a}{k} \frac{k!}{2}} \right) \\ &= \sum_{i=a-k}^{a-1} \frac{(k-1)|\mathcal{F}_i|}{\binom{n}{i}} + \frac{(k-1)|\hat{\mathcal{F}}|}{\binom{n}{a}} - \frac{C_2(\mathcal{F})}{\binom{n}{a} \binom{a}{k} \frac{k!}{2}}, \end{aligned}$$

and so

$$\frac{C_2(\mathcal{F})}{\binom{n}{a} \binom{a}{k} \frac{k!}{2}} \geq \sum_{i=a-k}^{a-1} \frac{(k-1)|\mathcal{F}_i|}{\binom{n}{i}} - (k^2 - k) + \frac{(k-1)|\hat{\mathcal{F}}|}{\binom{n}{a}}.$$

If the middle k levels were full, then the first sum would equal $k^2 - k$. Since we must have at least t_2 sets missing from the middle $k-1$ levels, the right-hand side is minimized when there are exactly t_2 sets missing, all of size $a-1$. In this case, $|\hat{\mathcal{F}}| = t_1 + t_2$, giving

$$\frac{C_2(\mathcal{F})}{\binom{n}{a} \binom{a}{k} \frac{k!}{2}} \geq \frac{(k-1)t_1}{\binom{n}{a}} + (k-1)t_2 \left(\frac{1}{\binom{n}{a}} - \frac{1}{\binom{n}{a-1}} \right) = \left(t_1 + \left(\frac{2a-n-1}{a} \right) t_2 \right) \frac{k-1}{\binom{n}{a}}.$$

As $a = \lceil \frac{n+k}{2} \rceil$, we have $2a - n - 1 \geq k - 1$, and so multiplying through by $\binom{n}{a} \binom{a}{k} \frac{k!}{2}$ gives the desired bound. \square

2.4.2 Counting all k -chains

As Proposition 2.4.1 offers us some control over the number of chains with large steps, we can now proceed to bound the total number of k -chains in \mathcal{F} . Again, our result provides some stability, as we shall require to forbid sets of size $n-1$.

Theorem 2.4.2. *Let \mathcal{F} be a set family of size $|\mathcal{F}| = M_k + t_1$, with $1 \leq |F| \leq n-2$*

for all $F \in \mathcal{F}$, and with at least t_2 sets missing from the middle $k-1$ levels. Then

$$\begin{aligned} C(\mathcal{F}) &\geq \binom{n}{a-k} \binom{n-a+k}{k-1} (k-1)! \\ &\quad + \left(t_1 + \left(\frac{k-1}{a} \right) t_2 \right) \left(\binom{a}{k-1} + \binom{a}{k} \binom{k}{2} \right) (k-1)!. \end{aligned}$$

Proof. We prove the theorem by induction on $t_1 + t_2 \geq 0$, and again must have $t_2 \geq 0$. The base case of $t_1 + t_2 = 0$ follows from Theorem 2.1.3, as when $|\mathcal{F}| = M_k = M_{k-1} + \binom{n}{a-k}$, the right-hand side above is less than the lower bound for $t = \binom{n}{a-k}$ in Theorem 2.1.3.

We may now assume that any set F not in the middle $k-1$ levels is contained in at most $\left(\binom{a}{k-1} + \binom{a}{k} \binom{k}{2} \right) (k-1)!$ k -chains. If not, then we may remove F from \mathcal{F} , thus decreasing t_1 by 1. Applying the inductive hypothesis to $\mathcal{F} \setminus \{F\}$ and adding the k -chains involving F then gives the requisite number of k -chains.

We once again seek to bound the number of k -chains in our family by counting sets and k -chains in permutations, except this time we shall consider all k -chains appearing in the permutations. A permutation with s sets gives rise to $\binom{s}{k}$ k -chains, and since for all s we have $ks - \binom{s}{k} \leq k^2 - 1$, it follows that

$$\begin{aligned} (k^2 - 1)n! &\geq k \sum_{F \in \mathcal{F}} |S_n[F]| - \sum_{F_1 \subset \dots \subset F_k \in \mathcal{C}(\mathcal{F})} |\cap_{i=1}^k S_n[F_i]| \quad (2.3) \\ &= \sum_{F \in \mathcal{F}} \left(k |S_n[F]| - \sum_{F_1 \subset \dots \subset F_k \in \mathcal{C}(F)} |\cap_{i=1}^k S_n[F_i]| \right). \end{aligned}$$

To maximize $|\cap_{i=1}^k S_n[F_i]|$, since we have no sets of size $n-1$, we should again take all the gaps to be as small as possible. Those chains in $\mathcal{C}_1(\mathcal{F})$ all have steps of size 1, while those in $\mathcal{C}_2(\mathcal{F})$ should have one step of size 2, and the rest of size 1. Dividing by $n!$ gives

$$k^2 - 1 \geq \sum_{F \in \mathcal{F}} \frac{1}{\binom{n}{|F|}} \left(k - \frac{C_1(F)}{\binom{m(F)}{k-1} (k-1)!} - \frac{C_2(F)}{\binom{m(F)}{k} \frac{k!}{2}} \right).$$

By definition, if $a - k + 1 \leq |F| \leq a - 1$, we must have $C_1(F) = C_2(F) = 0$, and if $|F| = a - k$, then $C_2(F) = 0$. Thus we have three types of k -chains to consider: those in $\mathcal{C}_1(F)$ for $|F| = a - k$, those in $\mathcal{C}_1(F)$ for $F \in \hat{\mathcal{F}} = \{F \in \mathcal{F} : |F| \leq a - k - 1 \text{ or } |F| \geq a\}$, and those in $\mathcal{C}_2(F)$ for $F \in \hat{\mathcal{F}}$. Splitting our sums thus, we obtain

$$k^2 - 1 \geq \sum_{i=a-k}^{a-1} \frac{k |\mathcal{F}_i|}{\binom{n}{i}} - \frac{\sum_{|F|=a-k} C_1(F)}{\binom{n}{a-k} \binom{n-a+k}{k-1} (k-1)!} + \sum_{F \in \hat{\mathcal{F}}} \frac{1}{\binom{n}{|F|}} \left(1 - \frac{C_1(F)}{\binom{m(F)}{k-1} (k-1)!} \right) + \sum_{F \in \hat{\mathcal{F}}} \frac{1}{\binom{n}{|F|}} \left(k - 1 - \frac{C_2(F)}{\binom{m(F)}{k} \frac{k!}{2}} \right).$$

Since

$$\sum_{|F|=a-k} C_1(F) = C(\mathcal{F}) - \sum_{F \in \hat{\mathcal{F}}} C_1(F) - \sum_{F \in \hat{\mathcal{F}}} C_2(F),$$

we can substitute this expression into the second sum, and redistribute, to obtain

$$\frac{C(\mathcal{F})}{\binom{n}{a-k} \binom{n-a+k}{k-1} (k-1)!} \geq \sum_{i=a-k}^{a-1} \frac{k |\mathcal{F}_i|}{\binom{n}{i}} - (k^2 - 1) + \Sigma_1 + \Sigma_2,$$

where

$$\Sigma_1 = \sum_{F \in \hat{\mathcal{F}}} \frac{1}{\binom{n}{|F|}} - \frac{C_1(F)}{\binom{n}{|F|} \binom{m(F)}{k-1} (k-1)!} + \frac{C_1(F)}{\binom{n}{a-k} \binom{n-a+k}{k-1} (k-1)!}, \text{ and}$$

$$\Sigma_2 = \sum_{F \in \hat{\mathcal{F}}} \frac{1}{\binom{n}{|F|}} \left(k - 1 - \frac{C_2(F)}{\binom{m(F)}{k} \frac{k!}{2}} \right) + \frac{C_2(F)}{\binom{n}{a-k} \binom{n-a+k}{k-1} (k-1)!}.$$

The following lemmas, whose proofs we defer to the end of this subsection, allow us to bound these sums.

Lemma 2.4.3. *For every $F \in \hat{\mathcal{F}}$, we have*

$$\frac{1}{\binom{n}{|F|}} - \frac{C_1(F)}{\binom{n}{|F|} \binom{m(F)}{k-1} (k-1)!} + \frac{C_1(F)}{\binom{n}{a-k} \binom{n-a+k}{k-1} (k-1)!} \geq \frac{\binom{a}{k-1}}{\binom{n}{a-k} \binom{n-a+k}{k-1}}.$$

Lemma 2.4.4. *For every $F \in \hat{\mathcal{F}}$, we have*

$$\frac{1}{\binom{n}{|F|}} \left(k - 1 - \frac{C_2(F)}{\binom{m(F)}{k} \frac{k!}{2}} \right) \geq \frac{1}{\binom{n}{a}} \left(k - 1 - \frac{C_2(F)}{\binom{a}{k} \frac{k!}{2}} \right).$$

We now replace our summands with these lower bounds, obtaining

$$\begin{aligned}
\frac{C(\mathcal{F})}{\binom{n}{a-k} \binom{n-a+k}{k-1} (k-1)!} &\geq \sum_{i=a-k}^{a-1} \frac{k |\mathcal{F}_i|}{\binom{n}{i}} - (k^2 - 1) + \sum_{F \in \hat{\mathcal{F}}} \frac{\binom{a}{k-1}}{\binom{n}{a-k} \binom{n-a+k}{k-1}} \\
&+ \sum_{F \in \hat{\mathcal{F}}} \left(\frac{k-1}{\binom{n}{a}} + C_2(F) \left(\frac{1}{\binom{n}{a-k} \binom{n-a+k}{k-1} (k-1)!} - \frac{1}{\binom{n}{a} \binom{a}{k} \frac{k!}{2}} \right) \right) \\
&= \sum_{i=a-k}^{a-1} \frac{k |\mathcal{F}_i|}{\binom{n}{i}} - (k^2 - 1) + |\hat{\mathcal{F}}| \left(\frac{\binom{a}{k-1}}{\binom{n}{a-k} \binom{n-a+k}{k-1}} + \frac{k-1}{\binom{n}{a}} \right) \\
&+ \frac{C_2(\mathcal{F})}{\binom{n}{a-k} \binom{n-a+k}{k-1} (k-1)!} \left(1 - \frac{2}{n-a+1} \right).
\end{aligned}$$

We can use Proposition 2.4.1 to lower bound $C_2(\mathcal{F})$. Moreover, as $a \geq n-a+k$, it follows that each set in $\hat{\mathcal{F}}$, whose size is not a , has greater weight than any set in the middle k levels. Thus, the right-hand side is minimized when we fill the middle $k-1$ levels as much as possible. If we were to have the full k middle levels, the first sum would be equal to k^2 . However, as we must have at least t_2 sets missing from the middle $k-1$ levels, it is best to have exactly t_2 sets of size $a-1$ missing, resulting in

$$\begin{aligned}
\frac{C(\mathcal{F})}{\binom{n}{a-k} \binom{n-a+k}{k-1} (k-1)!} &\geq 1 - \frac{kt_2}{\binom{n}{a-1}} + (t_1 + t_2) \left(\frac{\binom{a}{k-1}}{\binom{n}{a-k} \binom{n-a+k}{k-1}} + \frac{k-1}{\binom{n}{a}} \right) \\
&+ \frac{(t_1 + \frac{k-1}{a} t_2) \binom{a}{k} \binom{k}{2} (k-1)!}{\binom{n}{a-k} \binom{n-a+k}{k-1} (k-1)!} \left(1 - \frac{2}{n-a+1} \right) \\
&= 1 + A_1 t_1 + A_2 t_2,
\end{aligned}$$

where, after simplifying the binomial expressions, we find

$$\begin{aligned}
A_1 &= \frac{\binom{a}{k-1}}{\binom{n}{a-k} \binom{n-a+k}{k-1}} + \frac{\binom{a}{k} \binom{k}{2}}{\binom{n}{a-k} \binom{n-a+k}{k-1}} + \frac{k-1}{\binom{n}{a}} - \frac{2 \binom{a}{k} \binom{k}{2}}{(n-a+1) \binom{n}{a-k} \binom{n-a+k}{k-1}} \\
&= \frac{\binom{a}{k-1} + \binom{a}{k} \binom{k}{2}}{\binom{n}{a-k} \binom{n-a+k}{k-1}}, \text{ and} \\
A_2 &= \frac{\binom{a}{k-1}}{\binom{n}{a-k} \binom{n-a+k}{k-1}} + \frac{(k-1) \binom{a}{k} \binom{k}{2}}{a \binom{n}{a-k} \binom{n-a+k}{k-1}} + \frac{k-1}{\binom{n}{a}} - \frac{k}{\binom{n}{a-1}} - \frac{2(k-1) \binom{a}{k} \binom{k}{2}}{a(n-a+1) \binom{n}{a-k} \binom{n-a+k}{k-1}} \\
&= \frac{k-1}{a} \left(\frac{\binom{a}{k-1} + \binom{a}{k} \binom{k}{2}}{\binom{n}{a-k} \binom{n-a+k}{k-1}} + \frac{2a-n-k}{\binom{n}{a}} \right) \geq \frac{k-1}{a} \frac{\binom{a}{k-1} + \binom{a}{k} \binom{k}{2}}{\binom{n}{a-k} \binom{n-a+k}{k-1}}.
\end{aligned}$$

Multiplying through by $\binom{n}{a-k} \binom{n-a+k}{k-1} (k-1)!$ gives the desired bound. \square

To complete the proof, we now prove the two lemmas.

Proof of Lemma 2.4.3. Note that $\binom{n}{|F|} \binom{m(F)}{k-1} = \binom{n}{k-1} \binom{n-k+1}{m(F)-k+1}$, and, by the same token, $\binom{n}{a-k} \binom{n-a+k}{k-1} = \binom{n}{k-1} \binom{n-k+1}{a-k}$. Since $a-k = \lfloor \frac{n-k+1}{2} \rfloor$, and, as $F \in \hat{\mathcal{F}}$, we have $m(F) \geq a$, it follows that $\binom{n}{|F|} \binom{m(F)}{k-1} \leq \binom{n}{a-k} \binom{n-a+k}{k-1}$. Thus the left-hand side of the inequality is minimized when we choose $C_1(F)$ as large as possible. By definition, $C_1(F) \leq \binom{m(F)}{k-1} (k-1)!$. Making this substitution gives

$$\frac{1}{\binom{n}{|F|}} - \frac{C_1(F)}{\binom{n}{|F|} \binom{m(F)}{k-1} (k-1)!} + \frac{C_1(F)}{\binom{n}{a-k} \binom{n-a+k}{k-1} (k-1)!} \geq \frac{\binom{m(F)}{k-1}}{\binom{n}{a-k} \binom{n-a+k}{k-1}} \geq \frac{\binom{a}{k-1}}{\binom{n}{a-k} \binom{n-a+k}{k-1}}.$$

\square

Proof of Lemma 2.4.4. If $m(F) = a$, then we have equality, so we may assume $m(F) \geq a+1$. By induction, we can assume that no set is in more than $\left(\binom{a}{k-1} + \binom{a}{k} \binom{k}{2}\right) (k-1)!$ k -chains, giving a bound on $C_2(F)$. Thus

$$C_2(F) \leq \left(\binom{a}{k-1} + \binom{a}{k} \binom{k}{2} \right) (k-1)! \leq \binom{a+1}{k} \binom{k}{2} (k-1)! \leq (k-1) \binom{m(F)}{k} \frac{k!}{2},$$

and so the factor $\left(k-1 - \frac{C_2(F)}{\binom{m(F)}{k} \frac{k!}{2}}\right)$ is non-negative. As $\binom{n}{a} \geq \binom{n}{|F|}$, we thus have

$$\frac{1}{\binom{n}{|F|}} \left(k-1 - \frac{C_2(F)}{\binom{m(F)}{k} \frac{k!}{2}} \right) \geq \frac{1}{\binom{n}{a}} \left(k-1 - \frac{C_2(F)}{\binom{m(F)}{k} \frac{k!}{2}} \right) \geq \frac{1}{\binom{n}{a}} \left(k-1 - \frac{C_2(F)}{\binom{a}{k} \frac{k!}{2}} \right).$$

\square

2.4.3 Forbidding large sets

Given the previous theorem, all that remains is to show that an extremal family cannot contain sets of size $n-1$. The idea behind this is as follows. A set of size $n-1$ has a very large shadow in the $k-1$ middle levels. In order for this set to not contain too many k -chains, we must therefore be missing a lot of sets in the $k-1$

middle levels. By Theorem 2.4.2, it then follows that the remainder of the family must contain many more k -chains than it ought to. The relevant calculations are given below.

Proposition 2.4.5. *Suppose $n \geq 15$ and $k \leq n - 6$, and let \mathcal{F} be a set family with $|F| = M_k + t$. If we do not have $1 \leq |F| \leq n - 2$ for all $F \in \mathcal{F}$, then*

$$c_k(\mathcal{F}) > \binom{n}{a-k} \binom{n-a+k}{k-1} (k-1)! + t \left(\binom{a}{k-1} + \binom{a}{k} \binom{k}{2} \right) (k-1)!.$$

Proof. The same proof as in Lemma 2.3.2 shows that we may assume we do not have \emptyset or $[n]$ in \mathcal{F} . Hence it suffices to show there are no sets of size $n - 1$ in our family.

Suppose towards contradiction we had some set $F \in \mathcal{F}$ with $|F| = n - 1$. We may, by induction, assume that F is in at most $\left(\binom{a}{k-1} + \binom{a}{k} \binom{k}{2} \right) (k-1)!$ k -chains.

Since F contains $\binom{n-1}{a-1}$ sets of size $a - 1$, there are $\binom{n-1}{a-1} \binom{a-1}{k-2} (k-2)!$ possible k -chains that F might be in which consist of $k - 1$ sets from the $k - 1$ middle levels followed by F . Hence we must be missing a lot of sets from the $k - 1$ middle levels to prevent F from being in too many k -chains. The sets of size $a - 1$ are contained in the most such k -chains, so if we are missing t_2 sets, we must have

$$\left(\binom{n-1}{a-1} - t_2 \right) \binom{a-1}{k-2} (k-2)! \leq \left(\binom{a}{k-1} + \binom{a}{k} \binom{k}{2} \right) (k-1)!.$$

Solving for t_2 gives

$$t_2 \geq \binom{n-1}{a-1} - a \left(1 + \frac{(a-k+1)(k-1)}{2} \right).$$

We now remove from \mathcal{F} all sets of size $n - 1$, thus losing at most n sets, and apply Theorem 2.4.2 with the above value of t_2 . In the theorem, the number of k -chains is governed by the expression $(t_1 + \binom{k-1}{a} t_2)$. We are decreasing t_1 by at most n , but increasing t_2 by at least $\binom{n-1}{a-1} - a \left(1 + \frac{(a-k+1)(k-1)}{2} \right)$, resulting in a net gain in the previous expression of at least

$$\frac{k-1}{a} \left(\binom{n-1}{a-1} - a \left(1 + \frac{(a-k+1)(k-1)}{2} \right) \right) - n.$$

Since $n - k \geq 6$, we have $n - a \geq 3$. If $n - a$ is some constant, then the first term is at least cubic in n , while the term we are subtracting is quadratic, since in this case $a - k + 1$ will also be constant. On the other hand, if $n - a$ is large, the first term will be at least a large power of n , while the term we subtract is at most cubic in n . Given $n \geq 15$, some simple but tedious calculations show that in either case, having a set of size $n - 1$ increases the number of k -chains our family must contain. \square

Note that the condition $k \leq n - 6$ is near-optimal, since if $k = n - 3$, then $M_k = 2^n - 2n - 2$, and so by volume considerations alone there must be extremal families with sets of size at least $n - 1$.

Theorem 2.1.4 now follows easily.

Proof of Theorem 2.1.4. Since $n \geq 15$ and $k \leq n - 6$, Proposition 2.4.5 shows that $1 \leq |F| \leq n - 2$ for all $F \in \mathcal{F}$. We may then apply Theorem 2.4.2 with $t_1 = t$ and $t_2 = 0$ to obtain the bound

$$C(\mathcal{F}) \geq \binom{n}{a-k} \binom{n-a+k}{k-1} (k-1)! + t \left(\binom{n}{k-1} + \binom{a}{k} \binom{k}{2} \right) (k-1)!.$$

Recalling that $a = \lceil \frac{n+k}{2} \rceil$, this is precisely the desired lower bound.

We can again deduce a characterization of the extremal families. Note that we must have $t_2 = 0$ for the above bound to hold, and so the $k - 1$ middle levels must be full. In order to have equality in Lemma 2.4.4, we also needed $m(F) = a$ for all $F \in \hat{\mathcal{F}}$. If $a = \frac{n+k}{2}$, then the remaining $\binom{n}{a} + t$ sets must have size $\frac{n}{2} \pm \frac{k}{2}$. To obtain equality in (2.3), every chain must pass through either k or $k + 1$ sets of \mathcal{F} , and so we must have $\{G \notin \mathcal{F} : |G| = \frac{n}{2} \pm \frac{k}{2}\}$ forming an antichain.

If, on the other hand, $a = \frac{n+k+1}{2}$, then sets of size $a = \frac{n}{2} + \frac{k+1}{2}$ carry greater weight than sets of size $a - k = \frac{n}{2} - \frac{k-1}{2}$. We can then redefine t_2 above to be the number of sets missing in the k middle levels and obtain the same result. Hence it follows that we must have all sets in the k middle levels, with the remaining sets

in $\hat{\mathcal{F}}$ of size $\frac{n}{2} \pm \frac{k+1}{2}$. In order to maintain equality in (2.3), every chain must pass through at most one set $F \in \mathcal{F}$ with $|F| = \frac{n}{2} \pm \frac{k+1}{2}$, and so $\{F \in \mathcal{F} : |F| = \frac{n}{2} \pm \frac{k+1}{2}\}$ must be an antichain.

Thus, once again, the extremal families are exactly the same as those that minimize the number of 2-chains, as given by Theorem 2.1.2. \square

2.5 Concluding remarks and open problems

In this chapter, we have partially answered Kleitman's conjecture by showing that the families that minimize the number of 2-chains also minimize the number of k -chains when they occupy up to the $k+1$ middle levels. While we strongly believe the conjecture is true in general, we suspect new ideas are needed to deal with larger families. As the number of levels grows with respect to k , the number of different types of chains - in terms of the sizes of the steps between sets - grows rapidly, and these would all need to be controlled to obtain a precise result. In this direction, though, the same methods we have used above can be applied to show the following: if we have integers $\alpha_1, \alpha_2, \dots, \alpha_{k-1}$ with $\sum_i \alpha_i = \ell - 1$, then, provided $|\mathcal{F}| \leq M_\ell$ and the largest set in our family has size at most $n - \max_i \alpha_i$, the number of k -chains $F_1 \subset F_2 \subset \dots \subset F_k$ with $|F_{i+1} \setminus F_i| \geq \alpha_i$ is minimized by taking sets in the middle ℓ levels.

Considering the case of 2-chains, this chapter has focused on showing that a family with more than $\binom{n}{\lfloor n/2 \rfloor}$ sets must contain many 2-chains. A closely related problem is to determine whether such a family must have any sets contained in many 2-chains. This type of question has been studied before in other settings. For example, when one is considering the number of triangles in a graph, Erdős showed in [14] that any graph with $\left\lfloor \frac{n^2}{4} \right\rfloor + 1$ edges must contain an edge in at least $\frac{n}{6} + o(n)$ triangles. It is well-known and easy to see that the hypercube, a graph whose vertices are subsets of $[n]$, with two vertices adjacent if they are

comparable and differ in exactly one element, has independence number 2^{n-1} . Chung, Füredi, Graham and Seymour [7] proved any induced subgraph on $2^{n-1} + 1$ vertices contains a vertex of degree at least $(\frac{1}{2} + o(1)) \log_2 n$. It is an open problem to determine whether or not this bound is tight (the corresponding upper bound is $O(\sqrt{n})$), and the answer to this question has ramifications in theoretical computer science.

In the context of Sperner's theorem the above problem has a negative answer, which may be surprising given the previous two examples. For convenience, let us assume $n = 2m + 1$ is odd, and consider the following set family. Let

$$\mathcal{F} = \{F : 1 \notin F, |F| = m\} \cup \{F : 1 \in F, |F| = m + 1\}.$$

This family contains $2\binom{2m}{m} = (1 + \frac{1}{n}) \binom{n}{\lfloor n/2 \rfloor}$ sets, and so we are indeed beyond the Sperner bound. However, it is easy to see that the only pairs of comparable sets are of the form $\{F, \{1\} \cup F\}$ for every $F \in \mathcal{F}$ with $|F| = m$. Hence each set of the family is in only one pair of comparable sets. In fact, for this family we have $c_2(\mathcal{F}) = c_2(n, |\mathcal{F}|)$, so it is possible to have an extremal family with the comparable pairs distributed as evenly as possible. Theorem 2.1.2 shows that any family with $2\binom{2m}{m} + 1$ sets must contain a set in at least two 2-chains. It is an open problem as to whether this is also the largest family without a set that contains two other sets (and hence is the maximum set in two 2-chains). This configuration is known as a 2-fork, and the upper bound, which can be obtained using similar arguments as in Theorem 2.1.3, is $(1 + \frac{2}{n}) \binom{n}{\lfloor n/2 \rfloor}$, as shown by Katona and Tarján [24].

We find most exciting the prospect of studying Erdős–Rademacher-type problems in other settings. Within the context of Sperner's Theorem, a paper of Qian, Engel and Xu [32] studied an extension for multiset families, where the same set may be chosen multiple times. In [10], we derive an Erdős–Rademacher-type strengthening of the Erdős–Ko–Rado Theorem. However, as one can investigate

similar extensions for any extremal result, there is truly no end to the number of directions in which this project can be continued. We hope that further work of this nature will lead to many interesting results and a greater understanding of classical theorems in extremal combinatorics.

Note added in proof: During the preparation of this manuscript, it came to our attention that Dove, Griggs, Kang and Sereni [11] have independently obtained Theorem 2.1.3.

CHAPTER 3

$K_{s,t}$ -saturated bipartite graphs

3.1 Introduction

For two graphs G and H , G is said to be H -saturated if it contains no copy of H as a subgraph, but the addition of any edge missing from G creates a copy of H in G . The saturation number $sat(n, H)$ is defined as the minimum number of edges in an H -saturated graph on n vertices. Notice that the problem of finding the saturation number for H is, in some sense, the dual of the classical Turán problem.

Probably the most natural setup of this problem is when we choose H to be a fixed complete graph K_s . This was first studied by Zykov [36] in the 1940's, and later by Erdős, Hajnal and Moon [17] in 1964. They proved that $sat(n, K_s) = (s-2)n - \binom{s-1}{2}$. Here the upper bound comes from the K_s -saturated graph that has $s-2$ vertices connected to all other vertices. Later, the closely related notion of weak saturation was introduced by Bollobás [6]. A graph G is weakly H -saturated if it is possible to add back the missing edges of G one by one in some order, so that each addition creates a new copy of H . Trivially, if G is H -saturated, then any order satisfies this property, hence G is also weakly H -saturated. Let $w-sat(n, K_s)$ be the minimum number of edges in an n -vertex graph that is weakly K_s -saturated. We then have $w-sat(n, K_s) \leq sat(n, K_s)$. Somewhat surprisingly, one can prove using algebraic techniques (see e.g. [27]) that these two functions are actually equal. On the other hand, the extremal graphs for these problems

are not the same and already for $s = 3$ there are weakly K_3 -saturated graphs (i.e. trees) which are not K_3 -saturated.

The paper by Erdős, Hajnal and Moon also introduced the bipartite saturation problem, where we are looking for the minimum number of edges $\text{sat}(K_{n,n}, H)$ in an H -free n -by- n bipartite graph, such that adding any missing edge between the two color classes creates a new copy of H . (Of course, this definition is only meaningful if H is also bipartite.) They conjectured that $\text{sat}(K_{n,n}, K_{s,s}) = n^2 - (n - s + 1)^2$. Once again, this is seen to be tight by selecting $s - 1$ vertices on each side of the bipartite graph and connecting them to every vertex on the opposite side. In the bipartite setting, one can impose an additional restriction on the problem by ordering the two vertex classes of H and requesting that each missing edge create an H respecting the order: the first class of H lies in the first class of G . For example, let $K_{(s,t)}$ be the complete “ordered” s -by- t bipartite graph with s vertices in the first class and t vertices in the second, then a bipartite graph G is $K_{(s,t)}$ -saturated if each missing edge creates a $K_{s,t}$ with the s -vertex class lying in the first class of G . Indeed, the conjecture of Erdős, Hajnal and Moon was independently confirmed by Wessel [34] and Bollobás [5] a few years later as the special case of the following result: $\text{sat}(K_{(n,n)}, K_{(s,t)}) = n^2 - (n - s + 1)(n - t + 1)$. This was further generalized in the 80s by Alon [1] to complete k -uniform hypergraphs in a k -partite setting using algebraic tools. Alon showed that the saturation and weak saturation bounds are the same in this case as well. For a more detailed discussion of H -saturation in general, we refer the reader to the survey [19] by Faudree, Faudree and Schmitt.

In this chapter we study the unordered case of bipartite saturation. Although this is arguably the most natural setting for the bipartite problem, it did not receive any attention until very recently in [4, 31]. Moshkovitz and Shapira [31] studied the unordered weak saturation number of $K_{s,t}$, $s \leq t$, and showed that $w\text{-sat}(K_{n,n}, K_{s,t}) = (2s - 2 + o(1))n$. Note that, surprisingly, it is much smaller than

the corresponding ordered saturation number and only depends on the size of the smaller part. One might think that a similar gap exists for saturation numbers as well. Moshkovitz and Shapira [31] conjectured that this is not the case, and that ordered and unordered bipartite saturation numbers differ only by an additive constant. More precisely, they made the following conjecture and constructed an example showing that, if true, this bound is tight.

Conjecture 3.1.1. *Let $1 \leq s \leq t$ be integers. Then there is an n_0 such that if $n \geq n_0$ and G is a $K_{s,t}$ -saturated n -by- n bipartite graph, then G contains at least $(s + t - 2)n - \left\lfloor \left(\frac{s+t-2}{2}\right)^2 \right\rfloor$ edges.*

In this chapter we prove the following result which confirms the above conjecture up to a small additive constant.

Theorem 3.1.2. *Let $1 \leq s \leq t$ be fixed and $n \geq t$. Then*

$$\text{sat}(K_{n,n}, K_{s,t}) \geq (s + t - 2)n - (s + t - 2)^2.$$

The proof is presented in Section 3.2. In Section 3.3, we show that if the conjecture is true, it has many extremal examples. Finally, in Section 3.4, we prove Conjecture 4.1.1 in the first open case of $K_{2,3}$ -saturation.

3.2 Lower bounds on the saturation number

Let G be a bipartite graph with vertex class U and U' of size n . Assume $1 \leq s \leq t \leq n$ and suppose G is $K_{s,t}$ -saturated, i.e. each missing edge between U and U' creates a new $K_{(s,t)}$ or a new $K_{(t,s)}$ when added to G . Here $K_{(a,b)}$ refers to a complete bipartite graph with a vertices in U and b vertices in U' .

Let us start with the following, easy special case of Theorem 3.1.2.

Proposition 3.2.1. *Suppose a $K_{s,t}$ -saturated graph has minimum degree $\delta < t - 1$. Then it contains at least $n(t + s - 2) - (s + t - 2)^2$ edges.*

Proof. The $K_{s,t}$ -saturated property ensures that each vertex has at least $s - 1$ neighbors, so we actually have $s - 1 \leq \delta < t - 1$. Let u_0 be a vertex of degree δ ; we may assume that $u_0 \in U$. Then adding any missing edge u_0u' to G (where $u' \in U' - N(u_0)$) should create a new $K_{(t,s)}$ because it cannot create a $K_{(s,t)}$. For such a u' , let $S_{u'} \subseteq U$ be the $t - 1$ vertices other than u_0 in the t -class of this $K_{(t,s)}$, and define $V \subseteq U$ to be the union of these $S_{u'}$. Then all vertices in V have at least $s - 1$ neighbors in $N(u_0)$ and all vertices in $U' - N(u_0)$ have at least $t - 1$ neighbors in V . Now we can count the number of edges in G as follows:

$$\begin{aligned}
e(U, U') &= e(V, N(u_0)) + e(V, U' - N(u_0)) + e(U - V, U') \\
&\geq (s - 1)|V| + (t - 1)(n - |N(u_0)|) + \delta(n - |V|) \\
&\geq (s - 1)|V| + (t - 1)(n - t + 2) + (s - 1)(n - |V|) \\
&\geq n(t + s - 2) - (t - 1)(t - 2) \\
&\geq n(t + s - 2) - (s + t - 2)^2.
\end{aligned}$$

□

The case when $\delta \geq t - 1$ is considerably more complicated. We introduce the following structure to count the edges of G (see Figure 1). The *core* of this structure is a set $\tilde{A}_0 = A_0 \cup A'_0$ with $A_0 \subseteq U$ and $A'_0 \subseteq U'$ satisfying the following technical property:

- there are vertices $u_0 \in A_0$ and $u'_0 \in A'_0$ such that their neighborhoods are also contained in the core.

Next, we build the shell around the core: starting with $\tilde{A} = \tilde{A}_0$, we iteratively add any vertex v to \tilde{A} that has at least $t - 1$ neighbors in it. In other words, $\tilde{A} = A \cup A'$ is the smallest set containing \tilde{A}_0 such that any vertex $v \in G - \tilde{A}$ has fewer than $t - 1$ neighbors in \tilde{A} . Here $A_0 \subseteq A \subseteq U$ and $A'_0 \subseteq A' \subseteq U'$. We use the variables $x_0 = |A_0|$, $x'_0 = |A'_0|$, $x = |A|$ and $x' = |A'|$ to denote the sizes of the corresponding sets. Obviously $x_0 \leq x$ and $x'_0 \leq x'$.

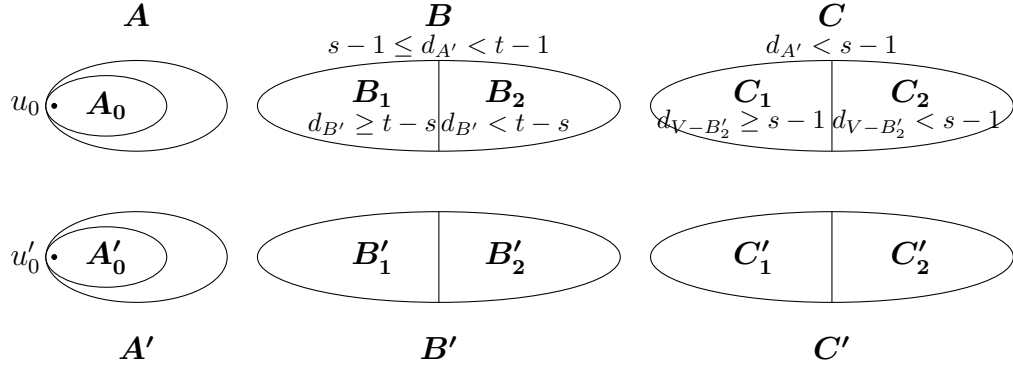


Figure 3.1: the structure for counting the edges

The following, rather scary, lemma is the key to our lower bounds on the saturation numbers. It shows that we can find about $n(s + t - 2)$ edges in a $K_{s,t}$ -saturated graph, provided we have a small enough core.

Lemma 3.2.2. *Assuming $\delta \geq t - 1$, suppose the core spans $e = e(A_0, A'_0)$ edges.*

Then G has at least

$$n(s + t - 2) - (x_0 + x'_0)(t - 1) - \left\lfloor \frac{(s - 1)^2}{4} \right\rfloor + e + \min\{(t - s)x, (t - s)x'\}$$

edges.

Proof. By the construction of \tilde{A} , we know that it spans at least $e + (t - 1)(x + x' - x_0 - x'_0)$ edges. Indeed, each vertex we added to the shell brings at least $t - 1$ new edges. The idea is to count $t - 1$ edges from the remaining vertices on one side of the graph, say $U' - A'$, and then to find $s - 1$ new (yet uncounted) edges from the other side, $U - A$. Of course, if a vertex in $U - A$ has at least $s - 1$ neighbors in A' , then these edges are guaranteed to be new.

So let us continue with our definition of the structure. We know that any vertex in $U - A$ has fewer than $t - 1$ neighbors in A' . We break this set into two parts by defining B to be the set of vertices in $U - A$ having at least $s - 1$ neighbors in A' , and C to be those having fewer than $s - 1$ neighbors in A' . Similarly we

break $U' - A'$ into two sets B' and C' based on the size of the neighborhood in A . We need to break B further into two parts B_1 and B_2 , by defining B_1 to be the set of vertices having at least $t - s$ neighbors in B' . Similarly, let B'_1 be the set of vertices in B' having at least $t - s$ neighbors in B (see Figure 1).

Note that any vertex in B'_1 already has $t - 1$ neighbors in $A \cup B$ (at least $s - 1$ in A and at least $t - s$ in B), but this is not necessarily true for B'_2 . This, together with our strategy to find $s - 1$ new edges from the vertices in C motivates our last partitioning: We now break C into two parts C_1 and C_2 , where C_1 is the set of those vertices in C which have at least $s - 1$ neighbors outside B'_2 , and $C_2 = C - C_1$. We similarly define $C'_1 = \{v \in C' : |N(v) - B_2| \geq s - 1\}$, where $N(v)$ is the neighborhood of v , and $C'_2 = C' - C'_1$.

An observation here, which will prove to be crucial when counting the edges, is that C_2 and C'_2 span a complete bipartite graph. Indeed, suppose there is a missing edge vv' in G , with $v \in C_2$ and $v' \in C'_2$. Adding this edge creates a $K_{(s,t)}$ or a $K_{(t,s)}$, suppose it is a $K_{(s,t)}$. Then v' is connected to all the $s - 1$ vertices other than v in the s -vertex class of this $K_{(s,t)}$. But v' is in C'_2 , so it has at most $s - 2$ neighbors outside B_2 , consequently there is a vertex $w \in B_2$ in the s -class. Similarly, using that v is in C_2 , we find at least $t - s$ vertices of the t -class in B'_2 . But then $w \in B_2$ has at least $t - s$ neighbors in $B'_2 \subseteq B'$, which contradicts the definition of B_2 . The same argument leads to a contradiction if the edge creates a $K_{(t,s)}$, hence we can conclude that there is no missing edge between C_2 and C'_2 .

On another note, observe that adding the edge u_0v' , where u_0 is the vertex in A_0 defined in the property of the core and v' is any vertex in C' , cannot create a $K_{(s,t)}$. Indeed, if it created a $K_{(s,t)}$, then all the vertices of the t -class except v' are neighbors of u_0 , so they are sitting in the core, A'_0 . This means that each vertex in the s -class is connected to at least $t - 1$ vertices in the core, hence the whole s -class is in A . But then v' has at least $s - 1$ neighbors in A , contradicting $v' \in C'$. So we see that adding u_0v' creates a $K_{(t,s)}$. Then, all the vertices of the

s -class of this copy of $K_{(t,s)}$ except v' are in A'_0 , therefore the vertices of the t -class have at least $s - 1$ neighbors in A' . Hence all of them are in $A \cup B$, implying that every $v' \in C'$ has at least $t - 1$ neighbors in $A \cup B$. The same argument shows that each $v \in C$ has at least $t - 1$ neighbors in $A' \cup B'$.

Lemma 3.2.2 will now follow from the following claim, possibly applied to the graph with the two vertex classes switched.

Claim 3.2.3. *Assuming $\delta \geq t - 1$, suppose $|C_2| \leq |C'_2|$. Then*

$$e(U, U') \geq n(s + t - 2) - (x_0 + x'_0)(t - 1) - \left\lfloor \frac{(s - 1)^2}{4} \right\rfloor + e + (t - s)x.$$

Proof. Let $y = |C_2|$ and $y' = |C'_2|$, and let us count the edges in G . We noted above that each vertex in B'_1 has at least $t - 1$ neighbors in $A \cup B$, so $e(A \cup B, B'_1) \geq (t - 1)|B'_1|$. By assumption, each vertex in B'_2 has degree at least $t - 1$, hence $e(A \cup B \cup C, B'_2) \geq (t - 1)|B'_2|$. We have also shown that each vertex in C' has at least $t - 1$ neighbors in $A \cup B$, so $e(A \cup B, C') \geq (t - 1)|C'|$. This so far means that

$$e(A \cup B, B'_1) + e(A \cup B \cup C, B'_2) + e(A \cup B, C') \geq (t - 1)(n - x'). \quad (3.1)$$

Now look at what we have left from the other side: By definition, any vertex in B has at least $s - 1$ neighbors in A' , so $e(B, A') \geq (s - 1)|B|$. We also defined C_1 so that its vertices have at least $s - 1$ neighbors outside B'_2 , this gives $e(C_1, A' \cup B'_1 \cup C') \geq (s - 1)|C_1|$. As we noted above, the vertices of C_2 are all connected to the vertices of C'_2 , so $e(C_2, C'_2) = yy'$. Using the fact that $y(s - 1 - y) \leq \left\lfloor \frac{(s-1)^2}{4} \right\rfloor$ (y is an integer), we get that

$$\begin{aligned} e(B, A') + e(C_1, A' \cup B'_1 \cup C') + e(C_2, C'_2) &\geq (s - 1)(n - x - y) + yy' & (3.2) \\ &\geq (s - 1)(n - x) - (s - 1)y + y^2 \\ &\geq (s - 1)(n - x) - \left\lfloor \frac{(s - 1)^2}{4} \right\rfloor. \end{aligned}$$

We have also seen that $e(A, A')$ is at least $e + (t - 1)(x + x' - x_0 - x'_0)$.

It is easy to check that we never counted an edge more than once above, hence

$$\begin{aligned} e(U, U') &\geq (t-1)(n-x') + (s-1)(n-x) + (t-1)(x+x'-x_0-x'_0) \\ &\quad + e - \left\lfloor \frac{(s-1)^2}{4} \right\rfloor \\ &= n(t+s-2) + (t-s)x - (x_0+x'_0)(t-1) + e - \left\lfloor \frac{(s-1)^2}{4} \right\rfloor, \end{aligned}$$

what we wanted to show. □

□

□

We state the following immediate corollary of this claim, which we need in Section 4.

Corollary 3.2.4. *If we have equality in Claim 3.2.3, then the following statements hold:*

- any vertex in $B'_1 \cup C'$ has exactly $t-1$ neighbors in $A \cup B$,
- any vertex in B has exactly $s-1$ neighbors in A' ,
- the vertices in C_1 have exactly $s-1$ neighbors outside B'_2 , and
- $y(s-1) - yy' = \lfloor (s-1)^2/4 \rfloor$.

Now we are ready to prove our general theorem, which is tight up to an additive constant. Let us emphasize, however, that since our methods do not give the exact result, we will not make any effort to optimize the constant error term.

Theorem 3.2.5. *If $G = (U, U', E)$ is a $K_{s,t}$ -saturated bipartite graph with n vertices on each side, then it contains at least $(s+t-2)n - (s+t-2)^2$ edges.*

Proof. Following Lemma 3.2.2, our plan is to find an appropriate core.

By Proposition 3.2.1, we may assume that the minimum degree of our graph is at least $t-1$. Suppose for contradiction that G contains fewer than $(s+$

$t - 2)n - (s + t - 2)^2$ edges. Then there is a vertex $u_0 \in U$ of degree at most $s + t - 3$. Moreover, there is a non-adjacent vertex $u'_0 \in U' - N(u_0)$ of degree at most $s + t - 3$ as well, since otherwise the number of edges in G would be at least $(n - (s + t - 3))(s + t - 2) > (s + t - 2)n - (s + t - 2)^2$, contradicting our assumption. Set $A_0 = \{u_0\} \cup N(u'_0)$ and $A'_0 = \{u'_0\} \cup N(u_0)$, and define $\tilde{A}_0 = A_0 \cup A'_0$ to be the core.

Using the above notation, we see that $x_0 = |A_0| = 1 + |N(u'_0)| \leq s + t - 2$ and $x'_0 = |A'_0| = 1 + |N(u_0)| \leq s + t - 2$. Since u_0 and u'_0 are not adjacent, we can add the edge $u_0u'_0$ to create a new $K_{s,t}$. Notice that all the vertices of this $K_{s,t}$ are adjacent to either u_0 or u'_0 , hence they all lie in the core. Consequently, the core spans $e = e(A_0, A'_0) \geq st - 1$ edges. Now applying Lemma 3.2.2 we get

$$\begin{aligned}
e(U, U') &\geq n(s + t - 2) - (x_0 + x'_0)(t - 1) + \min\{(t - s)x, (t - s)x'\} \\
&\quad - \left\lfloor \frac{(s - 1)^2}{4} \right\rfloor + e \\
&\geq n(s + t - 2) - (x_0 + x'_0)(t - 1) + \min\{(t - s)x_0, (t - s)x'_0\} \\
&\quad - \left\lfloor \frac{(s - 1)^2}{4} \right\rfloor + st - 1 \\
&\geq n(s + t - 2) - (s + t - 2)^2 + st - 1 - \left\lfloor \frac{(s - 1)^2}{4} \right\rfloor \\
&\geq n(s + t - 2) - (s + t - 2)^2.
\end{aligned}$$

This contradicts the assumption, thus proving the theorem. \square

3.3 Extremal graphs

As we mentioned in the introduction, Moshkovitz and Shapira [31] constructed a $K_{s,t}$ -saturated n -by- n bipartite graph showing that the bound of the Conjecture 4.1.1, if true, is tight. It appears that this example is not unique. In this section we describe a general family of such graphs which contains the example by Moshkovitz and Shapira as a special case (when $l = 1$).

Example. As usual, we denote the two sides of the bipartite graph by U and U' , where $|U| = |U'| = n$. Let us break each class into two major parts: $U = V \cup W$ and $U' = V' \cup W'$, where $|V| = |V'| = \lfloor \frac{t+s-2}{2} \rfloor$ (assume n is large enough). Suppose W and W' are further broken into some parts W_1, \dots, W_l and W'_1, \dots, W'_l where $|W_i| = |W'_i| \geq t - s$ for all i . The construction of an extremal graph G goes as follows.

First include in G all the edges between V and V' , making it a complete bipartite graph. Also, for every i , choose the edges between W_i and W'_i to span an arbitrary $(t - s)$ -regular graph. It remains to describe the edges going between different type of classes.

We do not include any edge between W_i and W'_j for any $i \neq j$. Instead, choose arbitrary sets $S' \subseteq V'$ and $S_1, \dots, S_l \subseteq V$ of size $s - 1$, and take all edges going between W_i and S' as well as the edges between S_i and W'_i , for all i . A straightforward computation shows that the number of edges in this G is exactly the number in the conjecture. We claim that G is $K_{s,t}$ -saturated.

Let us see what happens when we add a missing edge uu' to G . If $u' \in W'$, i.e. $u' \in W'_i$ for some i , then let N be the set of its $t - s$ neighbors in W_i . Since $u \in U - N - S_i$, the set $S_i \cup \{u\} \cup N \cup S' \cup \{u'\}$ then forms a $K_{(t,s)}$. On the other hand, if $u' \in V'$, then $u \in W_i$ for some i . Let N' be the set of the $t - s$ neighbors of u in W'_i , then $u' \in U' - N' - S'$ and hence the set $S_i \cup \{u\} \cup S' \cup N' \cup \{u'\}$ forms a $K_{(s,t)}$. This proves the saturation property.

The asymmetric structure of the above example comes from the relaxation of the $l = 1$ case, which corresponds to the construction of Moshkovitz and Shapira. When all the vertices in W' are connected to the same subset of V of size $s - 1$, adding an edge between W and W' creates both a $K_{(s,t)}$ and a $K_{(t,s)}$. Our example exploits the freedom we had in choosing the edges between W' and V . In our case, when $l > 1$, adding an edge between W_i and W'_j with $S_i \neq S_j$ creates only a $K_{(t,s)}$.

The existence of such asymmetric examples provides further difficulties in proving an exact result.

3.4 The $K_{2,3}$ case

For $t = s$, Conjecture 4.1.1 trivially follows from the ordered result by Bollobás [5]. The other extreme is also easy to handle. When $s = 1$, the $K_{1,t}$ -saturated property merely means that the vertices of degree less than $t - 1$ span a complete bipartite graph. Then it is a simple exercise to show (see [31]) that the conjecture holds in this case as well.

Thus the first open case is $s = 2$ and $t = 3$, where the conjecture asserts that any $K_{2,3}$ -saturated graph contains at least $3n - 2$ edges. We note that there are many saturated graphs on $3n - 2$ edges. In fact, there are many such examples which are even $K_{(2,3)}$ -saturated: Just take a vertex $v' \in U'$ that is connected to everything in U , and make sure that every other vertex in U' has degree 2.

In this section we prove the matching lower bound. A brief summary of the coming theorem can be phrased as follows. By finding an appropriate core, our techniques from Section 3.2 easily give a $3n - 3$ lower bound. The rest of the proof is then a series of small structural observations, ultimately ruling out the possibility that a $K_{2,3}$ -saturated graph with $3n - 3$ edges exists.

Theorem 3.4.1. *If $G = (U, U'; E)$ is a $K_{2,3}$ -saturated bipartite graph with $n \geq 4$ vertices in each part, then it has at least $3n - 2$ edges.*

Proof. As a first step, we show in the spirit of Proposition 3.2.1 that it is enough to consider graphs of minimum degree 2.

Lemma 3.4.2. *If G contains fewer than $3n - 2$ edges, then it has minimum degree 2. Moreover, it contains two non-adjacent vertices $u_0 \in U$ and $u'_0 \in U'$ of degree 2.*

Proof. The saturation property ensures that each vertex has at least one neighbor. Suppose there is a vertex u of degree 1 – wlog $u \in U$ – and let $u' \in U'$ be its neighbor. Take any vertex $v' \in U'$ other than u' , then adding the edge uv' cannot create a $K_{(2,3)}$, so it must create a $K_{(3,2)}$, with the 2-vertex class being $\{u', v'\}$. For any such v' , let $U_{v'} \subseteq U$ be the 3-class of this $K_{(3,2)}$, so $U_{v'}$ consists of u and two neighbors of v' . We count the two edges between v' and $U_{v'}$ for each $v' \in U'$, $v' \neq u'$ to get a total of $2n - 2$ different edges.

Now let $X = \cup U_{v'}$, then every vertex in X is connected to u' because each of the above $K_{(3,2)}$'s contains u' . This gives $|X|$ new edges. On the other hand, we still have not encountered any edges touching $U - X$. But since we know that each vertex has at least one neighbor, we surely have at least $n - |X|$ new edges. This is already a total of $3n - 2$ edges in G , contradicting our assumption.

Therefore the minimum degree is at least 2, but in fact it is exactly two, as otherwise we would have at least $3n$ edges in the graph. Let u_0 have degree 2 – we may assume $u_0 \in U$. If every non-adjacent vertex in U' has at least 3 neighbors, then we have $2 \cdot 2 + 3(n - 2) = 3n - 2$ edges incident to U' , again a contradiction. Hence there is a $u'_0 \in U'$ of degree 2 that is not adjacent to u_0 , and we are done. \square

Suppose G is a counterexample to our theorem, and apply Lemma 3.4.2 to get two non-adjacent vertices u_0 and u'_0 of degree 2. Denote the neighbors of u_0 by $u'_1, u'_2 \in U'$, the neighbors of u'_0 by $u_1, u_2 \in U$, and let $A_0 = \{u_0, u_1, u_2\}$ and $A'_0 = \{u'_0, u'_1, u'_2\}$ be the core of the structure we described at the beginning of Section 3.2. Using this core we will also construct the sets $A, B = B_1 \cup B_2, C = C_1 \cup C_2$ and $A', B' = B'_1 \cup B'_2, C' = C'_1 \cup C'_2$ as defined by the structure.

Assume that $|C_2| \leq |C'_2|$ and apply Claim 3.2.3 with $s = 2$ and $t = 3$ to the structure of core $\tilde{A}_0 = A_0 \cup A'_0$. These choices for s and t significantly simplify

the bound we get from this claim:

$$e(U, U') \geq 3n - 6 \cdot 2 - 0 + e + x = 3n - 12 + e + x.$$

Using that the addition of the edge $u_0u'_0$ creates a $K_{2,3}$ inside the core (as all neighbors of u_0 and u'_0 are in \tilde{A}_0), it is easy to check that $e = e(A_0, A'_0) \geq 6$. We also know that $x \geq x_0 = 3$, so $e(U, U') \geq 3n - 3$. Then these inequalities together with Corollary 3.2.4 imply that if $e(U, U') = 3n - 3$ then G satisfies the following five properties:

1. $e = e(A_0, A'_0) = 6$ and $x = |A| = 3$,
2. any vertex in $B'_1 \cup C'$ has exactly 2 neighbors in $A \cup B$,
3. any vertex in B has exactly 1 neighbor in A' ,
4. the vertices in C_1 have exactly 1 neighbor outside B'_2 , and
5. for $y = |C_2|$ and $y' = |C'_2|$ (with $0 \leq y \leq y'$) we have $y(s - 1) - yy' = y(1 - y') = 0$, so either $y = 0$ or $y = y' = 1$.

The following lemma supplements the fifth property and shows that C_2 must be empty and C'_2 must be non-empty, by taking care of the case $y = y' = 0$ and $y = y' = 1$.

Lemma 3.4.3. *If $|C_2| = |C'_2|$ then G spans at least $3n - 2$ edges.*

Proof. As G is a counterexample, by Lemma 3.4.2 it has minimum degree 2. Since $y = y'$, we may apply Claim 3.2.3 and Corollary 3.2.4 to G 's "mirror", with U and U' switched, and observe that the five properties hold for this mirror graph as well. Then the first property gives $x = 3$, $x' = 3$ and $e = 6$. So $A = \{u_0, u_1, u_2\}$ and $A' = \{u'_0, u'_1, u'_2\}$ (i.e. any vertex not in the core has at most one neighbor in it), and the core spans 6 edges. By symmetry we can assume that adding the edge $u_0u'_0$ creates a $K_{(2,3)}$ on the set $\{u_0, u_1, u'_0, u'_1, u'_2\}$, so the missing edges are

$u_0u'_0$, $u_2u'_1$ and $u_2u'_2$. We also assumed that there is no vertex of degree 1, so u_2 must have some neighbor v' in B' . Note that v' has exactly one neighbor in A , in particular it is not connected to u_1 .

Now let us see what happens when we add the edge u_0v' . We cannot create a $K_{(2,3)}$, because that would use both u'_1 and u'_2 , but their only common neighbor other than u_0 is u_1 (recall that no vertex outside A' can have 2 neighbors in A), which is not connected to v' . So it must be a $K_{(3,2)}$, and it is not using u_2 , as u_2 has no common neighbor with u_0 . But then the $K_{(3,2)}$ contains two neighbors of v' that are not in A , but are connected to a vertex in A' . Then, by definition, these neighbors are in B . So $v' \in B'_1$ has at least two neighbors in B and one in A , and this contradicts the second property. \square

From now on we assume that C_2 is empty and C'_2 is non-empty. Then the fourth property also implies that the vertices in $C = C_1$ have exactly one neighbor outside B'_2 . Moreover, the third property tells us that each vertex in B has exactly one neighbor in A' .

Lemma 3.4.4. *All vertices in B are connected to the same vertex in A' .*

Proof. Break B into parts based on the neighbor in A' by putting the vertices in B connected to $w' \in A'$ into the set $B_{w'}$. We claim that vertices in different parts do not share common neighbors, or in other words, any vertex $v' \in B' \cup C'$ has all its neighbors in B contained in the same part $B_{w'}$.

Indeed, any vertex in B' has at most one neighbor in B : this is true by definition for the vertices in B'_2 , and follows from the second property for B'_1 (every vertex in B'_1 has a neighbor in A). Now look at the vertices in C' . An easy observation in Lemma 3.2.2 shows that adding the edge u_0v' for $v' \in C'$ cannot create a $K_{(2,3)}$. So it creates a $K_{(3,2)}$, and this $K_{(3,2)}$ must contain the two neighbors of v' in B and a neighbor w'_0 of u_0 in A' . Hence both neighbors of v' are in $B_{w'_0}$, establishing the claim.

As we noted above, C' is not empty, so take a vertex $v'_1 \in C'$ and assume that the neighbors of v'_1 in B are in $B_{w'_1}$. We will show that $B = B_{w'_1}$. Suppose not, i.e. there is a $v_1 \in B_{w'_2}$ with $w'_1 \neq w'_2$. Then the edge $v_1v'_1$ is missing; let us see what happens when we add that edge. We create a $K_{(2,3)}$ or a $K_{(3,2)}$, so in any case there are vertices $v_2 \in U$ and $v'_2 \in U'$ such that $v_1v'_2$, $v_2v'_2$ and $v_2v'_1$ are all edges of G . Here v_2 cannot be in A , as it is connected to $v'_1 \in C'$. It is not in B either, since then one of v'_1 and v'_2 would have neighbors in both $B_{w'_1}$ and $B_{w'_2}$. So $v_2 \in C = C_1$ (since C_2 is empty). Now the fourth property says that v_2 has exactly 1 neighbor outside B'_2 . Since $v'_1 \in C'$, v'_2 must be in B'_2 . But the vertices in B'_2 have no neighbors in B so $v_1v'_2$ cannot be an edge, giving a contradiction. \square

Note that this lemma implies that one of the two neighbors of u_0 – say u'_1 – is not connected to any vertex in B , and therefore it is adjacent to at least two vertices in $A = A_0$. We also recall that the core only spans six edges. It is time to analyze what happens in the core when we add the edge $u_0u'_0$. It might create a $K_{(3,2)}$ or a $K_{(2,3)}$, but the obtained graph is inside the core in both cases.

Case 1: $u_0u'_0$ creates a $K_{(3,2)}$.

If this $K_{(3,2)}$ used u'_2 , then the core of G would contain more than 6 edges: 5 from the $K_{(3,2)}$ and 2 other edges incident to u'_1 , which is impossible. So u'_1 is connected to both u_1 and u_2 , while u'_2 is not connected to any of them. Note, however, that u'_2 is connected to all vertices in B .

Let v be any vertex in $U - A$. When we add the edge vu'_0 to G , we create a $K_{(2,3)}$ or a $K_{(3,2)}$, so there is a vertex v' connected to both v and u_1 or u_2 . Then v' is not in A' , since v is only connected to u'_2 in A' , but both $u_1u'_2$ and $u_2u'_2$ are missing. Thus $v' \in B'$ (it has a neighbor in A , so it is not in C'). When we add the edge u_0v' , we cannot create a $K_{(2,3)}$, because that would use both u'_1 and u'_2 , which only share u_0 as their common neighbor. So it creates a $K_{(3,2)}$ using one of u'_1 and u'_2 . It cannot be u'_1 , because then the 3-class of the $K_{(3,2)}$ is exactly A ,

making v' have 2 neighbors in A . Thus, by definition, $v' \in A'$ which contradicts $v' \in B'$. But it cannot be u'_2 either, because then v' would have two neighbors in B , which together with a neighbor in A that v' must have, contradicts the second property. So this case is impossible.

Case 2: $u_0u'_0$ creates a $K_{(2,3)}$.

Then one of u_1 and u_2 – say u_1 – is connected to both u'_1 and u'_2 , and the other is connected to neither. But then u'_1 has exactly two neighbors, u_0 and u_1 , and the set $\bar{A} = \{u_0, u_1, u'_1, u'_2\}$ spans four edges. This means that we can apply Lemma 3.2.2 taking \bar{A} as the core, and u_0 and u'_1 being its “distinguished” vertices with their neighborhoods also sitting in the core. One can check that all conditions are satisfied, and with our new values of $x \geq x_0 = 2$, $x' \geq x'_0 = 2$ and $e = 4$, we get that G has at least $3n - 8 - 0 + 4 + 2 = 3n - 2$ edges. This contradiction finishes the proof of the theorem. \square

3.5 Concluding remarks

Although we could slightly improve the error term in Theorem 4.1.2, it seems that more ideas are needed to prove the full conjecture. We also note that our methods can be used to provide an asymptotically tight estimate on the minimum number of edges in a $K_{s,t}$ -saturated unbalanced bipartite graph (i.e., with parts of size m and n). Determining the precise value in this unbalanced case might be even more challenging, although we believe that a straightforward modification of the extremal construction from the balanced case is tight here as well.

Bipartite saturation results were generalized to the hypergraph setting in [1, 31], where G and H are assumed to be k -partite k -uniform hypergraphs, and G is H -saturated if any new hyperedge meeting one vertex from each color class creates a new copy of H . It would be interesting to extend our results to get an asymptotically tight bound for the unordered k -partite hypergraph saturation

problem.

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CHAPTER 4

Maximizing the number of independent sets of a fixed size

4.1 Introduction

Given a finite graph G , let $i_t(G)$ be the number of independent sets of size t in a graph, and let $i(G) = \sum_{t \geq 0} i_t(G)$ be the total number of independent sets. There are many extremal results on $i(G)$ and $i_t(G)$ over families of graphs with various degree restrictions. Kahn [21] and Zhao [35] studied the maximum number of independent sets in a d -regular graph. Relaxing the regularity constraint to a minimum degree condition, Galvin [20] conjectured that the number of independent sets in an n -vertex graph with minimum degree $\delta \leq \frac{n}{2}$ is maximized by a complete bipartite graph $K_{\delta, n-\delta}$. This conjecture was recently proved (in stronger form) by Cutler and Radcliffe [8] for all n and δ , and they characterized the extremal graphs for $\delta > \frac{n}{2}$ as well.

One can further strengthen Galvin's conjecture by asking whether the extremal graphs also simultaneously maximize the number of independent sets of size t , for all t . This claim unfortunately is too strong, as there are easy counterexamples for $t = 2$. On the other hand, no such examples are known for $t \geq 3$. Moreover, in this case Engbers and Galvin [16] made the conjecture:

Conjecture 4.1.1. *For every $t \geq 3$ and $\delta \leq n/2$, the complete bipartite graph $K_{\delta, n-\delta}$ maximizes the number of independent sets of size t , over all n -vertex graphs with minimum degree at least δ .*

Engbers and Galvin [16] proved conjecture 4.1.1 for $\delta = 2$ and $\delta = 3$, and for all $\delta > 3$, they proved it when $t \geq 2\delta + 1$. Alexander, Cutler, and Mink [2] proved it for the entire range of t for bipartite graphs, but it appeared nontrivial to extend the result to general graphs. The first result for all graphs and all t was obtained by Law and McDiarmid [28], who proved the statement for $\delta \leq n^{1/3}/2$. This was improved by Alexander and Mink [3], who required that $\frac{(\delta+1)(\delta+2)}{3} \leq n$. In this chapter, we completely resolve conjecture 4.1.1.

Theorem 4.1.2. *Let $\delta \leq n/2$. For every $t \geq 3$, every n -vertex graph G with minimum degree at least δ satisfies $i_t(G) \leq i_t(K_{\delta, n-\delta})$, and when $t \leq \delta$, $K_{\delta, n-\delta}$ is the unique extremal graph.*

4.2 Proof

We will work with the complementary graph, and count cliques instead of independent sets. Cutler and Radcliffe [8] also discovered that the complement was more naturally amenable to extension; we will touch on this in our concluding remarks. Let us define some notation for use in our proof. A t -clique is a clique with t vertices. For a graph $G = (V, E)$, \overline{G} is its complement, and $k_t(G)$ is the number of t -cliques in G . For any vertex $v \in V$, $N(v)$ is the set of the neighbors of v , $d(v)$ is the degree of v , and $k_t(v)$ is the number of t -cliques which contain vertex v . Note that $\sum_{v \in V} k_t(v) = tk_t(G)$. We also define $G + H$ as the graph consisting of the disjoint union of two graphs G and H . By considering the complementary graph, it is clear that our main theorem is equivalent to the following statement.

Proposition 4.2.1. *Let $1 \leq b \leq \Delta + 1$. For all $t \geq 3$, $k_t(G)$ is maximized by $K_{\Delta+1} + K_b$, over $(\Delta+1+b)$ -vertex graphs with maximum degree at most Δ . When $t \leq b$, this is the unique extremal graph, and when $b < t \leq \Delta + 1$, the extremal graphs are $K_{\Delta+1} + H$, where H is an arbitrary b -vertex graph.*

Remark. When $b \leq 0$, the number of t -cliques in graphs with maximum degree

at most Δ is trivially maximized by the complete graph. On the other hand, when $b > (\Delta + 1)$, the problem becomes much more difficult, and our investigation is still ongoing. This chapter focuses on the first complete segment $1 \leq b \leq \Delta + 1$, which, as mentioned in the introduction, was previously attempted in [3, 16, 28].

Although our result holds for all $t \geq 3$, it turns out that the main step is to establish it for the case $t = 3$ using induction and double-counting. Afterward, a separate argument will reduce the general $t > 3$ case to this case of $t = 3$.

Lemma 4.2.2. *Proposition 4.2.1 is true when $t = 3$.*

Proof. We proceed by induction on b . The base case $b = 0$ is trivial. Now assume it is true for $b - 1$. Suppose first that $k_3(v) \leq \binom{b-1}{2}$ for some vertex v . Applying the inductive hypothesis to $G - v$, we see that

$$k_3(G) \leq k_3(G - v) + k_3(v) \leq \binom{\Delta + 1}{3} + \binom{b - 1}{3} + \binom{b - 1}{2} \leq \binom{\Delta + 1}{3} + \binom{b}{3},$$

and equality holds if and only if $G - v$ is optimal and $k_3(v) = \binom{b-1}{2}$. By the inductive hypothesis, $G - v$ is $K_{\Delta+1} + H'$, where H' is a $(b - 1)$ -vertex graph. The maximum degree restriction forces v 's neighbors to be entirely in H' , and so $G = K_{\Delta+1} + H$ for some b -vertex graph H . Moreover, since $k_3(v) = \binom{b-1}{2}$ we get that for $b \geq 3$, H is a clique.

This leaves us with the case where $k_3(v) > \binom{b-1}{2}$ for every vertex v , which forces $b \leq d(v) \leq \Delta$. We will show that here, the number of 3-cliques is strictly suboptimal. The number of triples (u, v, w) where uv is an edge and vw is not an edge is clearly $\sum_{i=1}^n d(v)(n - 1 - d(v))$. Also, every set of 3 vertices either contributes 0 to this sum (if either all or none of the 3 edges between them are present), or contributes 2 (if they induce exactly 1 or exactly 2 edges). Therefore,

$$2 \left[\binom{n}{3} - (k_3(G) + k_3(\overline{G})) \right] = \sum_{v \in V} d(v)(n - 1 - d(v)).$$

Rearranging this equality and applying $k_3(\overline{G}) \geq 0$, we find

$$k_3(G) \leq \binom{n}{3} - \frac{1}{2} \sum_{v \in V} d(v)(n-1-d(v)). \quad (4.1)$$

Since we already bounded $b \leq d(v) \leq \Delta$, and $b + \Delta = n - 1$ by definition, we have $d(v)(n-1-d(v)) \geq b\Delta$. Plugging this back into (4.1) and using $n = (\Delta + 1) + b$,

$$k_3(G) \leq \binom{n}{3} - \frac{nb\Delta}{2} = \binom{\Delta + 1}{3} + \binom{b}{3} - \frac{b(\Delta + 1 - b)}{2} < \binom{\Delta + 1}{3} + \binom{b}{3},$$

because $b \leq \Delta$. This completes the case where every vertex has $k_3(v) > \binom{b-1}{2}$. \square

We reduce the general case to the case of $t = 3$ via the following variant of the celebrated theorem of Kruskal-Katona [23, 26], which appears as Exercise 31b in Chapter 13 of from Lovász's book [29]. Here, the generalized binomial coefficient $\binom{x}{k}$ is defined to be the product $\frac{1}{k!}(x)(x-1)(x-2)\cdots(x-k+1)$, which exists for non-integral x .

Theorem 4.2.3. *Let $k \geq 3$ be an integer, and let $x \geq k$ be a real number. Then, every graph with exactly $\binom{x}{2}$ edges contains at most $\binom{x}{k}$ cliques of order k .*

We now use Lemma 4.2.2 and Theorem 4.2.3 to finish the general case of Proposition 4.2.1.

Lemma 4.2.4. *If Proposition 4.2.1 is true for $t = 3$, then it is also true for $t > 3$.*

Proof. Fix any $t \geq 4$. We proceed by induction on b . The base case $b = 0$ is trivial. For the inductive step, assume the result is true for $b - 1$. If there is a vertex v such that $k_3(v) \leq \binom{b-1}{2}$, then by applying Theorem 4.2.3 to the subgraph induced by $N(v)$, we find that there are at most $\binom{b-1}{t-1}$ cliques of order $t - 1$ entirely contained in $N(v)$. The t -cliques which contain v correspond bijectively to the $(t - 1)$ -cliques in $N(v)$, and so $k_t(v) \leq \binom{b-1}{t-1}$. The same argument used at the beginning of Lemma 4.2.2 then correctly establishes the bound and characterizes the extremal graphs.

If some $k_3(v) = \binom{\Delta}{2}$, then the maximum degree condition implies that the graph contains a $K_{\Delta+1}$ which is disconnected from the remaining $b \leq \Delta + 1$ vertices, and the result also easily follows. Therefore, it remains to consider the case where all $\binom{b-1}{2} < k_3(v) < \binom{\Delta}{2}$, in which we will prove that the number of t -cliques is strictly suboptimal. It is well-known and standard that for each fixed k , the binomial coefficient $\binom{x}{k}$ is strictly convex and increasing in the real variable x on the interval $x \geq k-1$. Hence, $\binom{k}{k} = 1$ implies that $\binom{x}{k} < 1$ for all $k-1 < x < k$, and so Theorem 4.2.3 then actually applies for all $x \geq k-1$. Thus, if we define $u(x)$ to be the positive root of $\binom{u}{2} = x$, i.e., $u(x) = \frac{1+\sqrt{1+8x}}{2}$, and let

$$f_t(x) = \begin{cases} 0 & \text{if } u(x) < t-2 \\ \binom{u(x)}{t-1} & \text{if } u(x) \geq t-2, \end{cases} \quad (4.2)$$

the application of Kruskal-Katona in the previous paragraph establishes that $k_t(v) \leq f_t(k_3(v))$.

We will also need that $f_t(x)$ is strictly convex for $x > \binom{t-2}{2}$. For this, observe that by the generalized product rule,

$$f'_t(x) = u' \cdot [(u-1)(u-2) \cdots (u-(t-2)) + \cdots + u(u-1) \cdots (u-(t-3))],$$

which is $u'(x)$ multiplied by a sum of $t-1$ products. Since $u'(x) = \frac{2}{\sqrt{1+8x}}$, for any constant C , $(u')(u-C) = 1 - \frac{2C-1}{\sqrt{1+8x}}$. Note that this is a positive increasing function when $C \in \{1, 2\}$ and $x > \binom{t-2}{2}$. In particular, since $t \geq 4$, each of the $t-1$ products contains a factor of $(u-1)$ or $(u-2)$, or possibly both; we can then always select one of them to absorb the (u') factor, and conclude that $f'_t(x)$ is the sum of $t-1$ products, each of which is composed of $t-2$ factors that are positive increasing functions on $x > \binom{t-2}{2}$. Thus $f_t(x)$ is strictly convex on that domain, and since $f_t(x) = 0$ for $x \leq \binom{t-2}{2}$, it is convex everywhere.

If $t = \Delta + 1$, there will be no t -cliques in G unless G contains a $K_{\Delta+1}$, which must be isolated because of the maximum degree condition; we are then finished as before. Hence we may assume $t \leq \Delta$ for the remainder, which in particular

implies that $f_t(x)$ is strictly convex and strictly increasing in the neighborhood of $x \approx \binom{\Delta}{2}$. Let the vertices be v_1, \dots, v_n , and define $x_i = k_3(v_i)$. We have $tk_t(G) = \sum_{v \in V} k_t(v) \leq \sum_{i=1}^n f_t(x_i)$, and so it suffices to show that $\sum f_t(x_i) < t\binom{\Delta+1}{t} + t\binom{b}{t}$ under the following conditions, the latter of which comes from Lemma 4.2.2.

$$\binom{b-1}{2} < x_i < \binom{\Delta}{2}; \quad \sum_{i=1}^n x_i \leq 3\binom{\Delta+1}{3} + 3\binom{b}{3}. \quad (4.3)$$

To this end, consider a tuple of real numbers (x_1, \dots, x_n) which satisfies the conditions. Although (4.3) constrains each x_i within an open interval, we will perturb the x_i within the closed interval which includes the endpoints, in such a way that the objective $\sum f_t(x_i)$ is nondecreasing, and we will reach a tuple which achieves an objective value of exactly $t\binom{\Delta+1}{t} + t\binom{b}{t}$. Finally, we will use our observation of strict convexity and monotonicity around $x \approx \binom{\Delta}{2}$ to show that one of the steps strictly increased $\sum f_t(x_i)$, which will complete the proof.

First, since the upper limit for $\sum x_i$ in (4.3) is achievable by setting $\Delta + 1$ of the x_i to $\binom{\Delta}{2}$ and b of the x_i to $\binom{b-1}{2}$, and $f_t(x)$ is nondecreasing, we may replace the x_i 's with another tuple which has equality for $\sum x_i$ in (4.3), and all $\binom{b-1}{2} \leq x_i \leq \binom{\Delta}{2}$. Next, by convexity of $f_t(x)$, we may push apart x_i and x_j while conserving their sum, and the objective is nondecreasing. After a finite number of steps, we arrive at a tuple in which all but at most one of the x_i is equal to either the lower limit $\binom{b-1}{2}$ or the upper limit $\binom{\Delta}{2}$, and $\sum x_i = 3\binom{\Delta+1}{3} + 3\binom{b}{3}$. However, since this value of $\sum x_i$ is achievable by $\Delta + 1$ many $\binom{\Delta}{2}$'s and b many $\binom{b-1}{2}$'s, this implies that in fact, the tuple of x_i 's has precisely this form. (To see this, note that by an affine transformation, the statement is equivalent to the fact that if n and k are integers, and $0 \leq y_i \leq 1$ are n real numbers which sum to k , all but one of which is at an endpoint, then exactly k of the y_i are equal to 1 and the rest are equal to 0.) Thus, our final objective is equal to

$$(\Delta + 1)\binom{\Delta}{t-1} + b\binom{b-1}{t-1} = t\binom{\Delta+1}{t} + t\binom{b}{t},$$

as claimed. Finally, since some x_i take the value $\binom{\Delta}{2}$, the strictness of $f_t(x)$'s monotonicity and convexity in the neighborhood $x \approx \binom{\Delta}{2}$ implies that at some stage of our process, we strictly increased the objective. Therefore, in this case where all $\binom{b-1}{2} < k_3(v) < \binom{\Delta}{2}$, the number of t -cliques is indeed sub-optimal, and our proof is complete. \square

4.3 Concluding remarks

The natural generalization of Proposition 4.2.1 considers the maximum number of t -cliques in graphs with maximum degree Δ and $n = a(\Delta + 1) + b$ vertices, where $0 \leq b < \Delta + 1$. In the language of independent sets, this question was also proposed by Engbers and Galvin [16]. The case $a = 0$ is trivial, and Proposition 4.2.1 completely solves the case $a = 1$. We believe that also for $a > 1$ and $t \geq 3$, $k_t(G)$ is maximized by $aK_{\Delta+1} + K_b$, over $(a(\Delta + 1) + b)$ -vertex graphs with maximum degree at most Δ .

An easy double-counting argument shows that it is true when $b = 0$. When $a \geq 2$ and $b > 0$, the problem seems considerably more delicate. Nevertheless, the same proof that we used in Lemma 4.2.4 (*mutatis mutandis*) shows that the general case $t > 3$ of this problem can be reduced to the case $t = 3$. Therefore, the most intriguing and challenging part is to show that $aK_{\Delta+1} + K_b$ maximizes the number of triangles over all graphs with $(a(\Delta + 1) + b)$ vertices and maximum degree at most Δ . We will provide some partial results on this main case in the next chapter, but our investigation is still ongoing.

CHAPTER 5

Generalizations on the number of independent sets

5.1 Introduction

In Chapter 4, we proved the number of independent sets of size $t \geq 3$ is maximized by $K_{\delta, n-\delta}$ when the minimum degree of the graph is at least $\delta \leq \frac{n}{2}$. In the complement, this means the number of cliques of size $t \geq 3$ is maximized by taking $K_{\Delta+1} + K_b$ when the maximum degree of the graph is at most $\Delta \geq \frac{n}{2}$ and $n = \Delta + 1 + b$. A natural generalization of this result, which was conjectured by Engbers and Galvin [16] is for the case $n = a(\Delta + 1) + b$:

Conjecture 5.1.1. *For all $n, \Delta \in \mathbb{N}$ and $n = a(\Delta+1)+b$ with $0 \leq b \leq \Delta$, $a, b \in \mathbb{N}$, let G be a graph with n vertices with maximum degree at most Δ and $t \geq 3$ be an integer,*

$$k_t(G) \leq k_t(aK_{\Delta+1} \cup K_b).$$

According to the same proof of Lemma 4.2.4, it is equivalent to prove this conjecture for $t = 3$.

Conjecture 5.1.2. *For all $n, \Delta > 6 \in \mathbb{N}$ and $n = a(\Delta + 1) + b$ with $0 \leq b \leq \Delta$, $a, b \in \mathbb{N}$, let G be a graph with n vertices with $\Delta(G) \leq \Delta$ then*

$$k_3(G) \leq k_3(aK_{\Delta+1} \cup K_b).$$

Quite recently, Cutler and Radcliffe [9] proved the conjecture for $r \leq 6$ and established a weaker form of the conjecture to narrow down the range of a . In

this chapter, a simpler way to narrow down the range of a is presented in section 2. Then in section 3 we prove the conjecture for all a , and some range of b .

Theorem 5.1.3. *Let G be a graph with n vertices and $\Delta(G) \leq \Delta$, here $n = a(\Delta + 1) + b$, $\Delta > 6$. If graph G maximizes the number of triangles, and $b \leq \frac{a}{2}$ or $\Delta + 1 \leq b + \frac{a}{4}$ or $a \geq \frac{4\sqrt{3}\Delta}{9}$, then G contains a $K_{\Delta+1}$.*

Theorem 5.1.4. *If $\Delta + 1 - \epsilon\Delta \leq b \leq \Delta$, where $\epsilon = (\frac{3}{64})^2$, $aK_{\Delta+1} \cup K_b$ maximizes the number of triangles in the graph with $n = a(\Delta + 1) + b$ vertices and maximum degree Δ .*

5.2 Proof of Theorem 5.1.3

In addition to the notations in Chapter 4, we define $d(u, v)$ to be the common degree of two vertices u and v , $d_S(u)$ to be the degree of u in the vertex set S . We will prove a proposition at first and Theorem 5.1.3 will be a corollary of this proposition.

Proposition 5.2.1. *Assume $\Delta > 6$ and graph G maximizes the number of triangles with maximum degree Δ , if there exists an vertex u such that $k_3(u) \geq \binom{\Delta}{2} - \frac{\Delta}{4} + \frac{1}{2}$, then u is included in a clique of size $\Delta + 1$.*

Proof. Assume there exists an vertex u with $k_3(u) \geq \binom{\Delta}{2} - \frac{\Delta}{4} + \frac{1}{2}$ which is not included in a $K_{\Delta+1}$, then there exists at most $\Delta/4 - 1/2$ non-edges in the neighbor of u . Let A be the set of vertices in $N(u)$ with at least one non-edge in $N(u)$ and let $|A| = x \leq \Delta/2 - 1$. Let v_1 be a vertex in A with minimal non-degree t in A . Assume v_1 is not connected with v_2 in A and v_2 has non-degree d in A .

Consider the following process, for each of v_1, v_2 , if it has one edge going out of $N(u)$, we just delete it, otherwise we do nothing. Then we can connect v_1, v_2 without violating the maximum degree condition. The number of triangles we are destroying in the process is at most $2x$. But the number of new triangles we are

creating by connecting v_1, v_2 is at least $\Delta - t - d$. As the number of non-edges in A is at most $\Delta/4 - 1/2$ and the minimal non-degree in A is t , by counting the number of non-degrees, it follows: $tx + (d - t) \leq \Delta/2 - 1$. Thus

$$\begin{aligned} \Delta - t - d - 2x &\geq 2tx + 2(d - t) + 2 - t - d - 2x \\ &= d + 2x(t - 1) - 3t + 2 \\ &\geq t + 2(t + 1)(t - 1) - 3t + 2 \\ &= 2t^2 - 2t \geq 0 \end{aligned}$$

Here equality holds when $d = t = 1, x = 2$ and $tx + (d - t) = \Delta/2 - 1$, then $\Delta = 6$. However, $\Delta > 6$, the equality will not hold and $\Delta - t - d > 2x$. It means after the process, the number of triangles in the graph increases, thus contradicting the optimality. Therefore for any vertex u with $T(u) \geq \binom{\Delta}{2} - \frac{\Delta}{4} + \frac{1}{2}$, it is included in a clique of size $K_{\Delta+1}$. \square

Now we will prove Theorem 5.1.3 by using Proposition 5.2.1.

Proof of Theorem 5.1.3. Assume there is no $K_{\Delta+1}$ in the optimal graph, then according to Proposition 5.2.1, what we need to prove is the existence of a vertex u with $k_3(u) \geq \binom{\Delta}{2} - \frac{\Delta}{4} + \frac{1}{2}$. Prove by contradiction, assume all vertices are not involved in at most $\binom{\Delta}{2} - \frac{\Delta}{4} + \frac{1}{4}$ triangles, then

$$k_3(G) \leq \left(\binom{\Delta}{2} - \frac{\Delta}{4} + \frac{1}{4} \right) (a(\Delta + 1) + b) / 3$$

As this graph is optimal, $K_3(G) \geq a \binom{\Delta+1}{3} + \binom{b}{3}$, thus we can get

$$a \binom{\Delta + 1}{3} + \binom{b}{3} \leq \left(\binom{\Delta}{2} - \frac{\Delta}{4} + \frac{1}{2} \right) (a(\Delta + 1) + b) / 3$$

After some simplification, it is equivalent with

$$a(\Delta + 1)(\Delta - 1) \leq b(2(\Delta - 1)(\Delta - 1) - 2(b - 1)(b - 2)) < 2b(\Delta + 1 - b)(\Delta + b - 2).$$

However, this is always incorrect when $b \leq \frac{a}{2}$ or $\Delta + 1 - b \leq \frac{a}{4}$ or $a \geq \frac{4\sqrt{3}\Delta}{9}$, which is a contradiction. Thus it finishes our proof. \square

5.3 Proof of Theorem 5.1.4

We restate the theorem here first:

Theorem 5.3.1. *If $\Delta + 1 - \epsilon\Delta \leq b \leq \Delta + 1$, where $\epsilon = (\frac{3}{64})^2$, $aK_{\Delta+1} \cup K_b$ maximizes the number of triangles in the graph with $n = a(\Delta + 1) + b$ vertices and maximum degree Δ .*

The proof of the theorem relies heavily on the edge density of the neighbors of vertices. Here we define a vertex u has a *dense* neighbor if $k_3(u) \geq \frac{13}{16} \binom{\Delta}{2}$. Before proving the theorem, we will prove a useful lemma and a nice proposition, which are the most important steps in the proof of the theorem 5.3.1.

Lemma 5.3.2. *For two vertices u, v in a graph with maximum degree Δ , $0 \leq \alpha \leq 1/2$, $k_3(u), k_3(v) \geq (1 - \alpha + \alpha^2) \binom{\Delta}{2}$, then $d(u, v) > (1 - \alpha)\Delta$ or $d(u, v) < \alpha\Delta$.*

Proof. For two vertices u, v such that $k_3(u), k_3(v) \geq (1 - \alpha + \alpha^2) \binom{\Delta}{2}$, we can count the number of edges in $N(u)$ and $N(v)$. Let $A = N(u) \cap N(v)$ and $x = |A| = d(u, v)$, the by the maximum degree condition, the sum of the number of edges in $N(u)$ and $N(v)$ can be at most $2 \binom{\Delta-x}{2} + x(\Delta - 2) - e(A) + e(A)$.

Meanwhile, by the condition $k_3(u), k_3(v) \geq (1 - \alpha + \alpha^2) \binom{\Delta}{2}$, the sum of the number of edges in $N(u)$ and $N(v)$ is at least $2(1 - \alpha + \alpha^2) \binom{\Delta}{2}$. Hence we can get

$$2 \binom{\Delta - x}{2} + x(\Delta - 2) \geq 2(1 - \alpha + \alpha^2) \binom{\Delta}{2}.$$

It implies

$$\Delta^2 - \Delta x + x^2 > (1 - \alpha + \alpha^2)\Delta^2$$

Hence we can get $d(u, v) > (1 - \alpha)\Delta$ or $d(u, v) < \alpha\Delta$ □

Proposition 5.3.3. *If all vertices have dense neighbors in the graph and $\Delta > 6$, then $aK_{\Delta+1} + K_b$ maximizes the number of triangles in the graphs with maximum degree Δ .*

Proof. We will prove by induction on n . The base case $n \leq 2(\Delta + 1)$ comes from Lemma 4.2.2. For inductive step, we can assume there is no clique of size $\Delta + 1$ in the graph and all vertices have dense neighbors.

Now we build equivalence classes between all the vertices from the extremal graph. For any two vertices u, v they are in the same equivalent class if $d(u, v) > \frac{3\Delta}{4}$. We will verify this definition is consistent. For u, v in the same equivalence class, and v, w in the same equivalence class, we shall prove u, w are also in the same equivalence class.

As $d(u, v) > \frac{3\Delta}{4}$ and $d(v, w) > \frac{3\Delta}{4}$, $d(u, w) > 2\frac{3\Delta}{4} - \Delta = \frac{\Delta}{2}$. Applying Lemma 5.3.2 with $\alpha = \frac{1}{4}$, $d(u, w) > \frac{3\Delta}{4}$ and u, w are also in the same equivalence class.

Thus the binary relation forms equivalence classes. Then we can partition the vertex set V into a disjoint union of equivalence classes. For any two vertices u, v in different equivalent classes, $d(u, v) \leq \frac{3\Delta}{4}$. As u, v both have dense neighbors, by Lemma 5.3.2, $d(u, w) > \frac{3\Delta}{4}$ or $d(u, w) < \frac{\Delta}{4}$, then $d(u, v) < \frac{\Delta}{4}$, which implies $d(u, v) \leq \frac{\Delta-1}{4}$. For u, v in the same equivalent class, $d(u, v) \geq \frac{3\Delta+1}{4}$. By using this fact, we shall prove the following lemma:

Lemma 5.3.4. *For any equivalent class S in the extremal graph, S either is a clique or contains at most one vertex with degree at most $\Delta - 1$ in S .*

Proof of Lemma 5.3.4. Prove by contradiction, assume S is not a clique and contains more than one vertex with degree at most $\Delta - 1$ in S . Then we can divide S into two parts: $A = \{v \in S, d_S(v) = \Delta\}$, $B = S \setminus A$ and $|B| \geq 1$. We shall prove that B should be a clique. Otherwise choose u, v which are not connected in B , we can delete at most one edge going out of S from u, v each, and then connect u and v without violating the maximum degree condition. In this process, we will creat at least $\frac{3\Delta+1}{4} - 2$ new triangles and destroy at most $2\frac{\Delta-1}{4} = \frac{\Delta-1}{2}$ triangles. Since $\frac{3\Delta+1}{4} - 2 > \frac{\Delta-1}{2}$, the number of triangles increases after the process, which is a contradiction.

Hence B is a complete graph. As S is not a clique, A can not be empty. As

for a vertex $u_1 \in B$ and $v_1 \in A$, $d_S(v_1) > d_S(u_1)$, and B is a complete graph, there exists one missing edge containing $u_1 \in B$. We can assume u_1 is not connected with $v_1 \in A$, then for another vertex $u_2 \in B$, we claim $N(v_1) \subset N(u_2)$. This is because if there exists $v_2 \in N(v_1) \setminus N(u_2)$, then $v_2 \in A$. We can delete the edge v_1v_2 and connect two edges v_1u_1 and v_2u_2 and then delete at most one edge going out of S from each of u_1, u_2 , then the number of triangles we are creating is at least $2 \times \frac{3\Delta+1}{4} - I(u_1 \sim v_2) - I(u_2 \sim v_1)$, where $I(u_1 \sim v_2), I(u_2 \sim v_1)$ are indicator variables to test whether the even in the bracket is true or not. The number of triangles we are destroying is at most $2 \times \frac{\Delta-1}{4} + \Delta - 1 - I_{u_1 \sim v_2} - I_{u_2 \sim v_1} + I_{u_1 \sim v_2} I_{u_2 \sim v_1} < 2 \frac{3\Delta+1}{4} - I_{u_1 \sim v_2} - I_{u_2 \sim v_1}$. Thus after the process the number of triangles increases in the graph, which is a contradiction.

□

According to Lemma 5.3.4, we can divide our proof of proposition 5.3.2 into two cases.

Case 1: There exists at least one equivalent class S with one vertex u with degree at most $\Delta - 1$ in S . We claim $d_S(u) = \Delta - 1$. Otherwise we assume $d_S(u) < \Delta - 1$, as all other vertices in S have degree Δ , let N be the set of non-neighbors of u in S , then $N \neq \emptyset$. Pick $v_1 \in N$ and v_2 a neighbor of v_1 , if v_2 is not connected with u , then we delete v_1v_2 and at most two edges going outside of S of u , then connect v_1, u and v_2, u . The number of triangles we are destroying is at most $\Delta + \frac{\Delta-1}{4} \times 2$, while the number of triangles we are creating is at least $\frac{3\Delta+1}{4} \times 2$. It is easy to verify that the number of triangles in the graph increases after the process, which is a contradiction.

Thus u is connected with all neighbors of v_1 , which means $d_S(u) = \Delta$, contradiction. Therefore, $d_S(u) = \Delta - 1$.

Now if there exists one neighbor of u outside of S , we can delete it without decreasing the number of triangles and changing the density condition. Then S is

disconnected with $V \setminus S$ and we can use induction on n to finish the proof.

Case 2: All the equivalent classes are cliques. Assume we have equivalent classes S_1, S_2, \dots, S_l of size a_1, a_2, \dots, a_l . We claim $\frac{3\Delta}{4} + 1 \leq a_i \leq \Delta$. The right hand side inequality comes from the fact that there exists no $K_{\Delta+1}$ in the graph. For any vertex u with degree x , the number of missing degree in the neighbor of u is at most $2\left(\binom{x}{2} - \frac{13}{16}\binom{\Delta}{2}\right)$, so there are at most $\frac{2\left(\binom{x}{2} - \frac{13}{16}\binom{\Delta}{2}\right)}{\Delta-1-(\Delta-1)/4}$ neighbors which are not in the same equivalent class as u . So the number of vertices in the same equivalent class as u is at least $x - \frac{2\left(\binom{x}{2} - \frac{13}{16}\binom{\Delta}{2}\right)}{3(\Delta-1)-4} \geq \Delta - \frac{2\left(\binom{\Delta}{2} - \frac{13}{16}\binom{\Delta}{2}\right)}{3(\Delta-1)-4} = \frac{3\Delta}{4}$. Therefore, for any a_i we have $a_i \geq \frac{3\Delta}{4} + 1$.

As the number of triangles in equivalent classes is $\binom{a_i}{3}$, and for every edge between two equivalent classes, it is contained in at most $\Delta/4$ triangles and each crossing triangles contains at least 2 edges between equivalent classes, we can conclude that the number of triangles in the graph is at most $\sum_{i=1}^l \binom{a_i}{3} + (\Delta - a_i + 1)a_i \frac{\Delta}{4} \times \frac{1}{2}$. By considering the following inequality from basic calculus: if $\Delta + 1 > a_i \geq a_j > 0$ and $a_i + a_j > \Delta/2 + 2$, then

$$\begin{aligned} & \binom{a_i}{3} + (\Delta - a_i + 1)a_i \frac{\Delta}{8} + \binom{a_j}{3} + (\Delta - a_j + 1)a_j \frac{\Delta}{8} \\ & < \binom{a_i + 1}{3} + (\Delta - a_i)(a_i + 1) \frac{\Delta}{8} + \binom{a_j}{3} + (\Delta - a_j + 2)(a_j - 1) \frac{\Delta}{8} \end{aligned}$$

In order to maximize $\sum_{i=1}^l \binom{a_i}{3} + (\Delta - a_i)a_i \frac{\Delta}{8}$, we can keep shifting a_i, a_j apart, until reaching 0 or $\Delta + 1$. Thus

$$\sum_{i=1}^l \binom{a_i}{3} + (\Delta - a_i)a_i \frac{\Delta}{8} \leq a \binom{\Delta + 1}{3} + \binom{b}{3}$$

which finishes the proof. \square

Now we will prove Theorem 5.3.1.

Proof of Theorem 5.3.1. Since $(\Delta + 1)(1 - \epsilon)b \leq \Delta$, $\epsilon\Delta \geq 1$. We will prove the theorem by contradiction, assume the optimal graph G has more triangles than

$aK_{\Delta+1} + K_b$, then:

$$\begin{aligned} \sum_{v \in V} \left(\binom{\Delta}{2} - k_3(v) \right) &= n \binom{\Delta}{2} - 3K_3(G) \\ &< b \left(\binom{\Delta}{2} - \binom{b-1}{2} \right) \leq b(\Delta + 1 - b)(\Delta - 1) \\ &\leq \epsilon \Delta^2 (\Delta - 1) \end{aligned}$$

On the other hand, let $\alpha = 2$ and $M = \{v \in V, k_3(v) \leq (1 - \alpha\sqrt{\epsilon})\binom{\Delta}{2}\}$, $|M| = x$, then $\sum_{v \in V} \left(\binom{\Delta}{2} - k_3(v) \right) \geq x\alpha\sqrt{\epsilon}\binom{\Delta}{2}$. Hence the following inequality is obtained:

$$x\alpha\sqrt{\epsilon}\binom{\Delta}{2} < \epsilon\Delta^2(\Delta - 1)$$

which implies $x < \frac{2\sqrt{\epsilon}\Delta}{\alpha}$. Now we will remove all vertices in M and edges which are connected with M . For every vertex u in the rest of the graph G' , the number of triangles containing u in G' is at least $\binom{\Delta}{2}(1 - \alpha\sqrt{\epsilon}) - x(\Delta - 1) = \binom{\Delta}{2}(1 - \alpha\sqrt{\epsilon} - 4\frac{\sqrt{\epsilon}}{\alpha}) = \binom{\Delta}{2}(1 - 4\sqrt{\epsilon})$. According to Theorem 5.3.3, as $1 - 4\sqrt{\epsilon} = \frac{13}{16}$, all vertices in graph G' are dense, the number of triangles in this graph is at most $a\binom{\Delta}{3} + \binom{b-x}{3}$. Moreover, the number of triangles we deleted is at most $x(1 - \alpha\sqrt{\epsilon})\binom{\Delta}{2}$, so the number of triangles in the original graph is at most $a\binom{\Delta}{3} + \binom{b-x}{3} + x(1 - \alpha\sqrt{\epsilon})\binom{\Delta}{2} < a\binom{\Delta+3}{3} + \binom{b}{3}$, contradiction.

□

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