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On n-Dependence

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Abstract In this note we develop and clarify some of the basic combinatorial properties of the new notion of n-dependence (for $1 \leq n < \omega$) recently introduced by Shelah [22]. In the same way as dependence of a theory means its inability to encode a bipartite random graph with a definable edge relation, n-dependence corresponds to the inability to encode a random $(n+1)$-partite $(n+1)$-hypergraph with a definable edge relation. We characterize n-dependence by counting φ-types over finite sets (generalizing Sauer-Shelah lemma and answering a question of Shelah from [23]) and in terms of the collapse of random ordered $(n+1)$-hypergraph indiscernibles down to order-indiscernibles (which implies that the failure of n-dependence is always witnessed by a formula in a single free variable).

1 Introduction

Shelah had introduced the notion of a *dependent theory* (also called NIP) in his work on the classification program for first-order theories [20]. Since then dependent theories had attracted a lot of attention due to the purely model theoretic work on generalizations of stability and o-minimality (e.g. [13, 21, 8]), the analysis of some important algebraic examples (e.g. [10]) and connections to combinatorics (e.g. [3]).

More recently, in [23, 22] Shelah had introduced a generalization of dependence called n-dependence, where $1 \leq n < \omega$. The change is that instead of forbidding an encoding of a random bipartite graph with a definable edge relation, one forbids an encoding of a random $(n+1)$-partite $(n+1)$-hypergraph with a definable edge relation (see Definition 2.1). Then dependence corresponds to 1-dependence, and we have an increasing family of classes of theories.

So far, not much is known about n-dependent theories. In [22] Shelah demonstrates some results about connected components for (type)-definable groups in 2-dependent theories (which can be viewed as a form of modularity in certain context,

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see remarks in [12, Section 6.5]). In [11] Hempel shows a finitary version of this result giving a certain "chain condition" for groups definable in n-dependent theories and demonstrating that every n-dependent field is Artin-Schreier closed. Some further questions and statements are mentioned in [23, Section 5(H)]. The aim of this note is essentially to clarify that material and to answer some questions posed there. Here is the outline of the paper.

In Section 2 we define n-dependence of a formula and give some motivating examples of n-dependent theories.

In Section 3 we introduce a generalization of VC-dimension capturing n-dependence and give a corresponding generalization of Sauer-Shelah lemma using bounds on the so-called Zarankiewicz numbers for hypergraphs from combinatorics. As an application we characterize n-dependent theories by counting φ-types over finite sets and give a counterexample to a more optimistic bound asked by Shelah. A preliminary version of the upper bound result has appeared in [24]. The optimality of our result remains open (and is closely connected to the open problem of lower bounds for Zarankiewicz numbers).

In Section 4 we discuss existence of various generalized indiscernibles useful for the study of n-dependence and connections to some results from structural Ramsey theory. In Section 5 we apply these observations to show that a theory is n-dependent if and only if every ordered random $(n+1)$-hypergraph indiscernible is actually just order-indiscernible. The case $n=1$ is due to Scow [18].

Another application of hypergraph indiscernibles is given in Section 6 where we demonstrate that a theory is n-dependent if and only if every formula in a *single* free variable is n-dependent. This is a result due to Shelah [22, Claim 2.6], however the authors found the proof suggested there to be lacking in some details and we use this opportunity to provide a detailed account of Shelah's theorem.

Finally, in the Appendix we verify a claim from Section 4 that the class of ordered partite hypergraphs forms a Ramsey class. This might be folklore, but we feel that a readable account could be beneficial.

2 n-Dependence

The following property was introduced in [23, Section 5(H)] and [22, Definition 2.4].

Definition 2.1 A formula $\varphi(x; y_0, \ldots, y_{n-1})$ has the *n-independence property*, or IP_n (with respect to a theory T), if in some model there is a sequence $(a_{0,i}, \ldots, a_{n-1,i})_{i\in\omega}$ such that for every $s \subseteq \omega^n$ there is b_s such that

$$\models \phi\left(b_s; a_{0,i_0}, \ldots, a_{n-1,i_{n-1}}\right) \Leftrightarrow (i_0, \ldots, i_{n-1}) \in s.$$

Here $x, y_0, \ldots, y_{n-1}$ are possibly tuples of variables. Otherwise we say that $\varphi(x, y_0, \ldots, y_{n-1})$ is n-dependent, or NIP_n. A theory is *n-dependent*, or NIP_n, if it implies that every formula is n-dependent.

We give some motivating examples and remarks.

Example 2.2
1. If T is n-dependent then it is $(n+1)$-dependent. Of course, T is dependent if and only if it is 1-dependent.
2. The theory of a random n-hypergraph is $(n+1)$-dependent, but not n-dependent. Here $(n+1)$-dependence is immediate by quantifier elimination and Proposition 6.5, and n-independence is witnessed by the edge relation.

The same holds for random n-partite n-hypergraphs and for random K_m-free n-hypergraph.

3. Similarly, it follows by the type-counting criterion from Proposition 6.5 that in fact any theory with elimination of quantifiers in which any atomic formula has at most n variables is n-dependent. In particular, any theory eliminating quantifiers in a finite relational language is n-dependent, where n is the maximum of the arities of the relations in the language.
4. A theory T is called quasifinite if there is a function $\nu : \omega \to \omega$ such that every finite subset T_0 of T has a finite model in which the number of k-types is bounded by $\nu(k)$. In particular, every quasifinite theory is pseudofinite and $\aleph_0$-categorical. Quasifinite theories are studied in depth in [7], and in [12, Section 6.5] it is pointed out that every quasi-finite theory is 2-dependent: it is demonstrated in [7] using the classification of finite simple groups that in a quasifinite theory, $\pi_\Delta(m)$ grows at most as 2^m (see Definition 3.10 and Proposition 6.5). An example of a quasifinite theory is the theory of a generic bilinear form on an infinite-dimensional vector space over a finite field (a direct proof that this theory is 2-dependent is given in [11]).
5. On the other hand, any theory of an infinite boolean algebra is n-independent, for all n (see [22, Example 2.10]).
6. By a result of Beyarslan [4], any pseudo-finite field interprets random n-hypergraph, for all n — so it is not n-dependent for any n. More generally, [11] shows that any PAC field which is not separably closed is n-independent, for all n. In view of this (and the well-known conjecture that all supersimple fields are PAC), one could ask if in fact every (super)simple n-dependent field is separably closed.

3 Counting φ-types and a generalization of Sauer-Shelah lemma

3.1 Sauer-Shelah Lemma and Generalized VC-dimension The maximum number of $\varphi(x,y)$-types over finite sets coincides with the value of the shatter function in the theory of VC-dimension in combinatorics (see e.g. [3] for a detailed account of this correspondence). We generalize the notion of VC-dimension and investigate the upper bound of the generalized shatter function. In this subsection, we discuss purely combinatorial topics. The connection with counting φ-types and n-dependence will be discussed in the next subsection (see Lemma 3.13 and Lemma 3.14).

First we recall classical VC-dimension and Sauer-Shelah lemma. Let X be a set and $\mathscr{C} \subseteq \mathscr{P}(X)$ a class of subsets of X. Given a subset $A \subseteq X$ we write $\mathscr{C} \cap A$ to denote the set $\{C \cap A : C \in \mathscr{C}\}$.

Definition 3.1 (Vapnik and Chervonenkis) A subset $A \subseteq X$ is said to be *shattered* by $\mathscr{C}$ if $\mathscr{C} \cap A = \mathscr{P}(A)$. The VC-dimension of $\mathscr{C}$ is defined as

$$\mathrm{VC}(\mathscr{C}) = \sup\{|A| : A \subseteq X \text{ is shattered by } \mathscr{C}\},$$

and the shatter function of $\mathscr{C}$ is defined as

$$\pi_{\mathscr{C}}(m) := \max\{|\mathscr{C} \cap A| : A \subseteq X, |A| = m\}.$$

Observe that $0 \leq \pi_{\mathscr{C}}(m) \leq 2^m$, and $\pi_{\mathscr{C}}(m) = 2^m$ if and only if $m \leq \mathrm{VC}(\mathscr{C})$.

Fact 3.2 1. (Sauer-Shelah lemma) Assume that $\mathrm{VC}(\mathscr{C}) \leq d$. Then

$$\pi_{\mathscr{C}}(m) \leq \sum_{i \leq d} \binom{m}{i}$$

for all $m \geq d$. In particular, $\pi_{\mathscr{C}}(m) \leq \left(\frac{em}{d}\right)^d = O(m^d)$ for all m.

2. There is a class $\mathscr{C} \subseteq \mathscr{P}(X)$ with $\mathrm{VC}(\mathscr{C}) = d$ such that $\pi_{\mathscr{C}}(m) = \sum_{i \leq d} \binom{m}{i}$ for $m \geq d$ (e.g. the class of all subsets of X of size $\leq d$). Hence, the bound given by Sauer-Shelah lemma is tight.

Throughout this subsection, we fix (infinite) sets $X_0, \ldots, X_{n-1}$ and $X = \prod_{i<n} X_i$. For a class $\mathscr{C} \subseteq \mathscr{P}(X)$ we define a notion of VC_n-dimension of $\mathscr{C}$.

Definition 3.3 A subset $A \subseteq X$ is said to be a box of $\mathrm{size}(A) = m$ if $A = \prod_{i<n} A_i$ for some $A_i \subseteq X_i$ $(i < n)$ with each $|A_i| = m$. The VC_n-dimension of $\mathscr{C}$ is defined as

$$\mathrm{VC}_n(\mathscr{C}) = \sup\{\mathrm{size}(A) : A \subseteq X \text{ is a box shattered by } \mathscr{C}\},$$

and the corresponding shatter function by

$$\pi_{\mathscr{C},n}(m) := \max\{|\mathscr{C} \cap A| : A \subseteq X \text{ is a box of size } m\}.$$

Remark 3.4 1. $0 \leq \pi_{\mathscr{C},n}(m) \leq 2^{m^n}$.

2. $\pi_{\mathscr{C},n}(m) = 2^{m^n}$ if and only if $m \leq \mathrm{VC}_n(\mathscr{C})$.

We generalize Sauer-Shelah lemma below. First we introduce some notation from extremal graph theory.

Definition 3.5 Let $G^{(n)}(m_0, \ldots, m_{n-1})$ denote an n-partite n-uniform hypergraph such that the i-th part has m_i vertices. If $m_0 = \ldots = m_{n-1} = m$, we simply write $G^{(n)}(m)$. Moreover, let $K^{(n)}(m)$ be the complete n-partite n-uniform hypergraph $G^{(n)}(m)$. (For example, $K^{(2)}(3)$ is the bipartite complete graph $K_{3,3}$.) Then:

- The value $\mathrm{ex}_n(m, K^{(n)}(d))$ is the minimum natural number k satisfying the following: for every (not partite) n-uniform hypergraph G with m-vertices, if G has $\geq k$ edges then G contains $K^{(n)}(d)$ as a subgraph.
- The Zarankiewicz number $z_n(m,d)$ is the minimum natural number z satisfying the following: every $G^{(n)}(m)$ having $\geq z$ edges contains $K^{(n)}(d)$ as a subgraph.

Fact 3.6 [9] For given n and d, let $\varepsilon = \dfrac{1}{d^{n-1}}$. Then there is $k \in \omega$ such that for every $m > k$ we have:

1. $\mathrm{ex}_n(m, K^{(n)}(d)) \leq m^{n-\varepsilon}$,
2. in particular $z_n(m,d) \leq (nm)^{n-\varepsilon}$.

It is known that the bound given above is tight for $n = 2$ and $d = 2, 3$ (see e.g. [17]), but the question about lower bounds is widely open even for graphs in general (the best lower bound for $n = 2$ and $d \geq 5$ is $\Omega(n^{2-2/d} \log(d)^{1/(d^2-1)})$ [5]). For our purposes we will only need the following:

Fact 3.7 [6, Chapter 5.2, Corollary 2.7] There is $k \in \omega$ such that

$$z_2(m,2) > m^{3/2}\left(1 - \frac{1}{m^{1/6}}\right)$$

for every $m > k$.

In particular, we can find $c > 0$ such that $z_2(m,2) \geq cm^{3/2}$ for every $m \in \omega$.

In order to generalize Sauer-Shelah lemma we need the so-called "shifting technique" lemma from combinatorics (see e.g. [16]).

Fact 3.8 (Shifting technique) Let A be any finite set and $\mathscr{C} \subseteq \mathscr{P}(A)$. Then there is $\mathscr{C}' \subseteq \mathscr{P}(A)$ such that:

1. $|\mathscr{C}| = |\mathscr{C}'|$,
2. if $\mathscr{C}'$ shatters $B \subseteq A$ then so does $\mathscr{C}$,
3. if $B \subseteq C \in \mathscr{C}'$ then $B \in \mathscr{C}'$.

Proposition 3.9 Let $\mathscr{C}$ be a class of subsets of X.

1. Assume that $\mathrm{VC}_n(\mathscr{C}) \leq d$. Then $\pi_{\mathscr{C},n}(m) \leq \sum_{i<z} \binom{m^n}{i}$ for $m \geq d$, where $z = z_n(m, d+1)$.
2. In particular, for $m \gg n, d$, we have $\pi_{\mathscr{C},n}(m) \leq 2^{cm^{n-\varepsilon} \log_2 m} \leq 2^{m^{n-\varepsilon'}}$, where $c = n^{n+1-\varepsilon}$, $\varepsilon = \dfrac{1}{(d+1)^{n-1}}$ and $\varepsilon' = \varepsilon'(n,d) > 0$ is small enough.
3. There is a class $\mathscr{C} \subseteq \mathscr{P}(X)$ with $\mathrm{VC}_n(\mathscr{C}) = d$ such that $\pi_{\mathscr{C},n}(m) \geq 2^{z-1}$ where $z = z_n(m, d+1)$.

Note that the first item in the proposition gives Sauer-Shelah lemma where $n = 1$, since $z_1(m, d+1) = d+1$. In addition, by Fact 3.7, we can find a class $\mathscr{C}$ and $c > 0$ such that $\pi_{\mathscr{C},2}(m) \geq 2^{cm^{3/2}}$ for every m. Unfortunately, the inequality $\pi_{\mathscr{C},n}(m) \leq \sum_{i<z} \binom{m^n}{i}$ may not be tight.

Proof of Proposition 3.9 (1): We show that $\pi_{\mathscr{C},n}(m) \leq \sum_{i<z} \binom{m^n}{i}$. Let $A \subseteq X$ be a box of size m. It is enough to show that $|\mathscr{C} \cap A| \leq \sum_{i<z} \binom{m^n}{i}$. Let $\mathscr{C}'$ be given by Lemma 3.8 applied to $\mathscr{C} \cap A$. By the third condition in the lemma, every $B \in \mathscr{C}'$ is shattered by $\mathscr{C}'$, and moreover $\mathscr{C}$ shatters all members in $\mathscr{C}'$ by the second condition. Hence $\mathscr{C}'$ contains no box of size $d+1$ since $\mathrm{VC}_n(\mathscr{C}) \leq d$.

Claim For $B \in \mathscr{C}'$, $|B| < z_n(m, d+1)$.

Proof of the claim Suppose that $|B| \geq z_n(m, d+1)$. Consider the n-partite n-uniform hypergraph $G = (A_0 \sqcup \ldots \sqcup A_{n-1}; B)$ (recall that $A = \prod_{i<n} A_i$ is a box of size m). Then G has a subgraph $G' \cong K^{(n)}(d+1)$ by Fact 3.6. Notice that the set of edges B' of G' (i.e. $B' = E(G')$) is a subset of B, hence B' is shattered by $\mathscr{C}$. However, B' is a box of size $d+1$. This contradicts the fact that $\mathrm{VC}_n(\mathscr{C}) \leq d$. □

Therefore $\mathscr{C}' \subseteq \{B \subseteq A : |B| < z_n(m; d+1)\}$ and so,

$$|\mathscr{C} \cap A| = |\mathscr{C}'| \leq |\{B \subseteq A : |B| < z_n(m; d+1)\}| \leq \sum_{i<z} \binom{m^n}{i}.$$

(2): A straightforward calculation using (1) and Fact 3.6.

(3): Without loss of generality, we may assume $X_i = \omega$ for all $i < n$, since the shatter function $\pi_{\mathscr{C},n}$ is determined locally, i.e. if $X \subseteq X'$ and $\mathscr{C}' = \mathscr{C}$ is a family of subsets of X' then $\pi_{\mathscr{C}',n}(m) = \pi_{\mathscr{C},n}(m)$. For each $m \in \omega$ with $m \geq d$, let $(A_0^m \cup \ldots \cup A_{n-1}^m; E^m)$ be an n-partite n-uniform hypergraph having $(z_n(m, d+1) - 1)$-edges with no subgraph isomorphic to $K^{(n)}(d+1)$. We may assume that X_i is the disjoint union of A_i^m $(m \geq d)$. Let $\mathscr{C} = \bigcup_m \mathscr{P}(E^m)$. Clearly, we have $\pi_{\mathscr{C},n}(m) \geq 2^{z_n(m,d+1)-1}$ and $\mathrm{VC}_n(\mathscr{C}) \geq d$. On the other hand, since every $C \in \mathscr{C}$ is in some $\mathscr{P}(E^m)$, every box B shattered by $\mathscr{C}$ must be a subset of some $\prod_i A_i^m$. This means $\mathrm{VC}_n(\mathscr{C}) \leq d$. □

3.2 VC_n-dimension and n-dependence In this subsection, we translate our situation with an n-dependent formula into the theory of VC$_n$-dimension.

Definition 3.10 Let $\Delta(x, y_0, \ldots, y_{n-1})$ be a set of formulas with $|y_k| = l_k$, and for each $k < n$, let A_k be a (small) set of *tuples* of length l_k in a monster model. A (complete) Δ-type $p(x)$ over $(A_0, \ldots, A_{n-1})$ is a (maximal) consistent subset of $\{\varphi(x, a_0, \ldots, a_{n-1})^{\mathrm{if(i=0)}} : \varphi \in \Delta, a_k \in A_k, i < 2\}$, and $S_\Delta(A_0, \ldots, A_{n-1})$ is the set of complete Δ-types over $(A_0, \ldots, A_{n-1})$. For a natural number $m \in \omega$, we put

$$\pi_\Delta(m) := \sup\{|S_\Delta(B_0, \ldots, B_{n-1})| : |B_0| = \ldots = |B_{n-1}| = m\}.$$

If Δ consists only of a single formula ϕ, then we simply write S_ϕ, π_ϕ, etc.

Remark 3.11 Let $\varphi(x, y_0, \ldots, y_{n-1})$ and $\psi(x, y_0, \ldots, y_{n-1})$ be formulas.

1. $0 \leq \pi_\varphi(m) \leq 2^{m^n}$ for every $m \in \omega$.
2. (a) $\pi_{\neg\varphi}(m) = \pi_\varphi(m)$.
   (b) $\pi_{\varphi\wedge\psi}(m) \leq \pi_{\{\varphi,\psi\}}(m) \leq \pi_\varphi(m) \cdot \pi_\psi(m)$.
3. The following are equivalent:
   (a) φ is n-dependent.
   (b) $\pi_\varphi(m) < 2^{m^n}$ for some $m \in \omega$.
   (c) There is $d \in \omega$ such that $\pi_\varphi(m) = 2^{m^n}$ for $m \leq d$ and $\pi_\varphi(m) < 2^{m^n}$ for $m > d$.

We call the number d in condition (3c) the (dual VC$_n$-)dimension of φ. The dimension of φ will be denoted by $\dim(\varphi)$. In [23, Section 5(H), Question 5.67(1)], Shelah asks whether the following condition $(*)$ is equivalent to n-dependence of $\varphi(x, y_0, \ldots, y_{n-1})$:

$(*)$ There is $k \in \omega$ such that $\pi_\varphi(m) \leq 2^{cm^{n-1}}$ for all $m > k$,

where $c = |x|$. Clearly $(*)$ implies n-dependence, however $(*)$ is too strong to be equivalent to it. In fact, as stated it is trivially false for $n = 1$. But even if we fix the $n = 1$ case by replacing $2^{cm^{n-1}}$ with $2^{cm^{n-1}\log_2 m}$, it is still too strong for larger n, as the following theorem demonstrates.

Theorem 3.12

1. (Weak form of $(*)$) If $\varphi(x, y_0, \ldots, y_{n-1})$ is n-dependent with $\dim(\varphi) \leq d$ then $\pi_\varphi(m) \leq \sum_{i<z} \binom{m^n}{i}$ for $d \leq m$, where $z = z_n(m, d+1)$. In particular, for $m \gg n, d$ we have $\pi_\varphi(m) \leq 2^{cm^{n-\varepsilon}\log_2 m} \leq 2^{m^{n-\varepsilon'}}$ where $c = n^{n+1-\varepsilon}$, $\varepsilon = \dfrac{1}{(d+1)^{n-1}}$ and $\varepsilon' = \varepsilon'(n, d) > 0$ is small enough.
2. (Counterexample for Shelah's question): There are $c > 0$, a theory T and a formula $\varphi(x, y_0, y_1)$ such that φ is 2-dependent and $\pi_\varphi(m) \geq 2^{cm^{3/2}}$ for every m.

We first need to introduce some notation and generalize some standard observations from $n = 1$ to arbitrary n. Let $\varphi(x, y_0, \ldots, y_{n-1})$ be a formula and fix a model M of T. For simplicity of notation we will assume all variables to be of length 1, for example, $|x| = |y_0| = |a_0| = 1$. The class $\mathscr{C}_\varphi$ is defined as

$$\mathscr{C}_\varphi = \{\varphi(b, M^n) : b \in M\} \subseteq 2^{M^n},$$

where $\varphi(b, M^n) = \{(a_0, \ldots, a_{n-1}) \in M^n : M \models \varphi(b, a_0, \ldots, a_{n-1})\}$.

Lemma 3.13 For every finite $A \subseteq M^n$, we have $|S_\varphi(A)| = |\mathscr{C}_\varphi \cap A|$, where $S_\varphi(A)$ is the set of all complete ϕ-types. (A complete ϕ-type over A is a maximal consistent set of formulas of the form $\varphi(x, a_0, \ldots, a_{n-1})$ or $\neg\varphi(x, a_0, \ldots, a_{n-1})$ with $(a_0, \ldots, a_{n-1}) \in A$.)

Proof Note that since A is finite, p is realized in M for every $p \in S_\varphi$. Consider the map from S_φ to $\mathscr{C}_\varphi \cap A$ given by $p(x) \mapsto \varphi(b, M^n) \cap A$ for some $b \models p$. Notice that b and b' satisfy the same type p if and only if for every $(a_0, \ldots, a_{n-1}) \in A$, $\varphi(b, a_0, \ldots, a_{n-1}) \leftrightarrow \varphi(b', a_0, \ldots, a_{n-1})$ holds. Hence the map is well-defined and injective. Moreover, if $\varphi(b, M^n) \cap A \in \mathscr{C}_\varphi \cap A$ then we can find $\mathrm{tp}_\varphi(b/A) \in S_\varphi(A)$. So the map is a bijection. □

The above lemma shows that there is no difference between counting types and counting the size of the restricted class. Hence, by the definition, we have:

Lemma 3.14

1. $\pi_\varphi(m) = \pi_{\mathscr{C}_\varphi, n}(m)$ for every $m \in \omega$.
2. $\dim(\varphi) = \mathrm{VC}_n(\mathscr{C}_\varphi)$. In particular, a formula φ is n-dependent if and only if the VC_n-dimension of $\mathscr{C}_\varphi$ is finite.

Note that π_φ and $\dim(\varphi)$ do not depend on the model inside which they are calculated, thus they are indeed properties of a formula. Now, we give a proof of our theorem.

Proof of Theorem 3.12 (1): Immediate from Proposition 3.9 and Lemma 3.14.

(2): By the second item of Proposition 3.9 and the remark after its statement, for countable sets X_0 and X_1, we can find $c > 0$ and $\mathscr{C} \subseteq \mathscr{P}(X_0 \times X_1)$ such that $\mathrm{VC}_2(\mathscr{C}) = 1$ and $\pi_{\mathscr{C},2}(m) \geq 2^{cm^{3/2}}$ for all m. We may assume that $X_0 = X_1$. With a set $Y = \{b_C : C \in \mathscr{C}\}$, we define a structure $M = (Y \cup X_0, R(x, y_0, y_1))$ by the following: $R(b, a_0, a_1)$ if and only if $(a_0, a_1) \in C \subseteq X_0^2$ and $b = b_C$ for some $C \in \mathscr{C}$. Then, in M, we have $\mathscr{C}_R = \mathscr{C}$, hence $\pi_R(m) = \pi_{\mathscr{C},2}(m)$. □

Corollary 3.15 Let $\varphi(x, y_0, \ldots, y_{n-1})$ and $\psi(x, y_0, \ldots, y_{n-1})$ be n-dependent formulas. Then $\neg\varphi$, $\varphi \wedge \psi$ and $\varphi \vee \psi$ are n-dependent.

Proof Immediate from Remark 3.11 and Theorem 3.12, since

$$\log(\pi_\varphi(m)\pi_\psi(m)) = \log(\pi_\varphi(m)) + \log(\pi_\psi(m)) = O(m^{n-\varepsilon})$$

for some $\varepsilon > 0$. □

4 Generalized indiscernibles

4.1 Ramsey property and hypergraphs In this subsection, we arrange several facts in structural Ramsey theory with hypergraphs. We postpone some of the proofs until the appendix.

Let L_0 be a finite relational language, and let A, B, C be L_0-structures. We denote by $\binom{B}{A}$ the set of all $A' \subseteq B$ such that $A' \cong_{L_0} A$. If A has no non-trivial automorphisms, then $\binom{B}{A}$ is considered as a set of all embeddings $A \to B$. Using this notation, for $k \in \omega$, we write $C \to (B)^A_k$ to denote the following property: for every map $c : \binom{C}{A} \to k$ (called a coloring) there is $B' \in \binom{C}{B}$ such that $c|\binom{B'}{A}$ is constant.

Definition 4.1 Let K be a set of (the isomorphism types of) L_0-structures and let $A, B \in K$. We say that K has the (A,B)-Ramsey property if for every $k \in \omega$ there is $C \in K$ such that $C \to (B)^A_k$. In addition, if K has the (A,B)-Ramsey property for every $A, B \in K$, then we say that K has Ramsey property (or it is a Ramsey class).

We introduce three Ramsey classes: ordered n-partite sets, ordered n-uniform hypergraphs and ordered n-partite n-uniform hypergraphs.

Let $L_{\mathrm{op}} = \{<, P_0(x), \ldots, P_{n-1}(x)\}$. An ordered n-partite set is an L_{op}-structure A such that A is the disjoint union of $P_0(A), \ldots, P_{n-1}(A)$ and that $<$ is a linear ordering on A with $P_0(A) < \ldots < P_{n-1}(A)$.

Fact 4.2 (see Appendix A, Proposition A.3) Let K be the set of all finite ordered n-partite sets and let $K^* = \{A : A \subseteq B \in K\}$ be the hereditary closure of K. Then K and K^* are Ramsey classes. The Fraïssé limit of K^* will be denoted by $O_{n,p}$.

Let $L_0 = \{R_i\}_{i \in I}$ be a finite relational signature, let n_i be the arity of R_i. A *hypergraph of type* L_0 is a structure $\left(A, \left(R_i^A\right)_{i \in I}\right)$ such that for all $i \in I$:

- $R_i(a_0, \ldots, a_{n_i - 1}) \Rightarrow a_0, \ldots, a_{n_i - 1}$ are distinct,
- $R_i(a_0, \ldots, a_{n_i - 1}) \Rightarrow R\left(a_{\sigma(0)}, \ldots, a_{\sigma(n_i - 1)}\right)$ for any permutation $\sigma \in \mathrm{Sym}(n_i)$.

Thus essentially $R_i^A \subseteq [A]^{n_i}$, the set of subsets of A of size n_i. Let OH_{L_0} be the set of all *(linearly) ordered* L_0-hypergraphs, it is a Fraïssé class and admits a Fraïssé limit — the ordered random L_0-hypergraph, with the order isomorphic to $(\mathbb{Q}, <)$. In particular, an ordered L_0-hypergraph is called an ordered n-uniform hypergraph if $L_0 = \{R(x_0, \ldots, x_{n-1})\}$, and G_n denotes the countable ordered n-uniform random hypergraph. It is proved in [14, 15] and independently in [1] that:

Fact 4.3 For any finite L_0, the class of all ordered L_0-hypergraphs OH_{L_0} is a Ramsey class.

Fix a language $L_{\mathrm{opg}} = \{R(x_0, \ldots, x_{n-1}), <, P_0(x), \ldots, P_{n-1}(x)\}$. A (linearly) ordered n-partite n-uniform hypergraph is an L_{opg}-structure $(A; <, R, P_0, \ldots, P_{n-1})$ such that:

1. $(A; R, P_0, \ldots, P_{n-1})$ is an n-partite n-uniform hypergraph, i.e. A is the (pairwise disjoint) union $P_0 \sqcup \ldots \sqcup P_{n-1}$ such that if $(a_0, \ldots, a_{n-1}) \in R$ then $P_i \cap \{a_0 \ldots a_{n-1}\}$ is a singleton for every $i < n$,
2. $<$ is a linear ordering on A with $P_0(A) < \ldots < P_{n-1}(A)$.

Fact 4.4 (see Proposition A.5 and Lemma A.1) Let K be the set of all finite ordered n-partite n-uniform hypergraphs and let $K^* = \{A : A \subseteq B \in K\}$ be the hereditary closure of K. Then both K and K^* have Ramsey property.

The Fraïssé limit of K^* is called an ordered n-partite n-uniform random hypergraph, denoted by $G_{n,p}$.

Remark 4.5 The first order theories of G_n and $G_{n,p}$ can be axiomatized in the following way:

1. A structure $(M, <, R)$ is a model of $\mathrm{Th}(G_n)$ if and only if:
   - $(M, <, R)$ is an ordered n-uniform hypergraph,
   - $(M, <)$ is DLO,
   - for every finite disjoint sets $A_0, A_1 \subset M^{n-1}$ and $b_0 < b_1 \in M$, there is $b_0 < b < b_1$ such that $R(b, a_{i,1}, \ldots, a_{i,n-1})^{\mathrm{if}(i=0)}$ holds for every $(a_{i,1}, \ldots, a_{i,n-1}) \in A_i$ and $i < 2$.

In particular, an ordered random 1-hypergraph is a dense linear order with a dense co-dense subset.

2. A structure $(M,<,R,P_0,\dots,P_{n-1})$ is a model of $\mathrm{Th}(G_{n,p})$ if and only if:
   - $(M,<,R,P_0,\dots,P_{n-1})$ is an ordered n-partite n-uniform hypergraph,
   - $(P_i(M),<)$ is DLO for each $i<n$,
   - for every $j<n$, any finite disjoint sets $A_0,A_1 \subset \prod_{i\neq j} P_i(M)$ and $b_0<b_1 \in P_j(M)$, there is $b_0<b<b_1$ such that $R(b,a_{i,1},\dots,a_{i,n-1})^{\mathrm{if}(i=0)}$ for every $(a_{i,1},\dots,a_{i,n-1}) \in A_i$ and $i<2$.
3. $G_{n,p}|L_{\mathrm{op}}$ is isomorphic to $O_{n,p}$.
4. $\mathrm{Th}(G_n)$, $\mathrm{Th}(G_{n,p})$, and $\mathrm{Th}(O_{n,p})$ are ω-categorical and admit quantifier elimination.

4.2 Generalized indiscernibles The notion of generalized indiscernibles, which was introduced in [19, Section 2], and was used implicitly by Shelah already in [20], is a good tool to study n-dependence.

Definition 4.6 Let T be a theory in the language L, and let $\mathbb{M}$ be a monster model of T.

1. Let I be a structure in the language L_0. We say that $\bar{a}=(a_i)_{i\in I}$ with $a_i \in \mathbb{M}$ is an I-indiscernible if for all $n\in\omega$ and all $i_0,\dots,i_n$ and $j_0,\dots,j_n$ from I we have:
$$\mathrm{qftp}_{L_0}(i_0,\dots,i_n)=\mathrm{qftp}_{L_0}(j_0,\dots,j_n)\Rightarrow$$
$$\mathrm{tp}_L\left(a_{i_0},\dots,a_{i_n}\right)=\mathrm{tp}_L\left(a_{j_0},\dots,a_{j_n}\right).$$
An I-indiscernible $(a_i)_{i\in I}$ is also called an L_0-indiscernible to clarify the structure on I. (So, for $L_1\subseteq L_0$, $(a_i)_{i\in I}$ is said to be L_1-indiscernible if it is $(I|L_1)$-indiscernible.)
2. For L_0-structures I and J, we say that $(b_i)_{i\in J}$ is *based on* $(a_i)_{i\in I}$ if for any finite set Δ of L-formulas, and for any finite tuple $(j_0,\dots,j_n)$ from J there is a tuple $(i_0,\dots,i_n)$ from I such that:
   - $\mathrm{qftp}_{L_0}(j_0,\dots,j_n)=\mathrm{qftp}_{L_0}(i_0,\dots,i_n)$ and
   - $\mathrm{tp}_\Delta\left(b_{j_0},\dots,b_{j_n}\right)=\mathrm{tp}_\Delta\left(a_{i_0},\dots,a_{i_n}\right)$.
3. [19] Let I be a structure in the language L_0. We say that I has the *modeling property* if given any $\bar{a}=(a_i)_{i\in I}$ there exists an L_0-indiscernible $\bar{b}=(b_i)_{i\in I}$ based on $\bar{a}$.

For a class K of L_0-structures, we say an L_0-structure G is K-universal if for every $A\in K$ there is $A'\subset G$ such that $A\cong A'$.

Fact 4.7 [19] Let K be a class of finite L_0-structures and let G be a countable K-universal L_0-structure such that $A\in K$ for every finite $A\subset G$. Then K is a Ramsey class if and only if G has the modeling property.

Proof We prove left to right for the sake of exposition. Take any finite subsets $A\subset B\subset G$ and a formula $\varphi((x_g)_{g\in A})$. Since $A,B\in K$ and K is a Ramsey class, there is $C\in K$ such that $C\to(B)^A_2$. By the assumption, we may assume $C\subset G$. Hence we can find $(a_g)_{g\in B'}\subset(a_g)_{g\in G}$ with $B'\in\binom{C}{B}$ such that for any $A',A''\in\binom{B'}{A}$, $\varphi((a_g)_{g\in A'})\leftrightarrow\varphi((a_g)_{g\in A''})$. By compactness, we have an L_0-indiscernible $(a'_g)_{g\in G}$ based on $(a_g)_{g\in G}$, since for given $(a_g)_{g\in G}$ the statement "$(x_g)_{g\in G}$ is based on $(a_g)_{g\in G}$" can be expressed by a set of L-formulas. □

The following corollary is our main tool in the next section.

Corollary 4.8 Let G be one of the following structures G_n, $G_{n,p}$ or $O_{n,p}$, and let $\bar{a} = (a_g)_{g\in G}$ be given. Then there is a G-indiscernible $(b_g)_{g\in G}$ based on $\bar{a}$.

Proof Combining Fact 4.3, Fact 4.4, Fact 4.2 and Fact 4.7. □

We see the most basic application of the above corollary.

Remark 4.9 (Existence of an L_{op}-indiscernible witness) In the definition of IP_n, the index set of a witness of IP_n is ω^n. By compactness, we can replace ω^n by any $P_0 \times \ldots \times P_{n-1}$ with infinite sets P_i $(i < n)$. Put $G = P_0 \sqcup \ldots \sqcup P_{n-1}$ and note that it can be seen as an L_{op}-structure. In this situation, we say that $(a_g)_{g\in G}$ is a witness of IP_n for ϕ if for any two disjoint subsets X_0 and X_1 of $P_0 \times \ldots \times P_{n-1}$ we have that

$$\{\varphi(x, a_{g_0}, \ldots, a_{g_{n-1}})\}_{(g_0,\ldots,g_{n-1})\in X_0} \cup \{\neg\varphi(x, a_{g_0}, \ldots, a_{g_{n-1}})\}_{(g_0,\ldots,g_{n-1})\in X_1}.$$

is consistent. Furthermore, observe that if $(b_g)_{g\in O_{n,p}}$ is an L_{op}-indiscernible based on $(a_g)_{g\in O_{n,p}}$, then $(b_g)_{g\in O_{n,p}}$ is also an witness of IP_n since the L_{op}-isomorphism $X_0X_1 \cong_{L_{\mathrm{op}}} Y_0Y_1$ implies that Y_0 and Y_1 are disjoint subsets of $P_0 \times \ldots \times P_{n-1}$ as well.

5 IP_n and random hypergraph indiscernibles

We recall that, as before, L_{opg} and L_{op} denote the languages $\{<, R, P_0, \ldots, P_{n-1}\}$ and $\{<, P_0, \ldots, P_{n-1}\}$, respectively. In this section, we give characterizations of n-dependence using L_{opg}-indiscernibles and L_{op}-indiscernibles.

5.1 Basic properties of IP_n and indiscernible witnesses We begin with some easy remarks on n-dependence.

Remark 5.1
1. A theory T is n-dependent if and only if $T(A)$ is n-dependent for every parameter set A. In fact, if $\varphi(x, y_0, \ldots, y_{n-1}, A)$ has IP_n in $T(A)$ witnessed by $(a_g)_{g\in O_{n,p}}$, then $\psi(x, z_0, \ldots, z_{n-1})$ has IP_n witnessed by $(b_g)_{g\in O_{n,p}}$ where $z_i = y_i w$, $\psi(x, z_0, \ldots, z_{n-1}) = \varphi(x, y_0, \ldots, y_{n-1}, w)$ and $b_g = a_g A$.
2. Let $x \subseteq w$ and $y_i \subseteq z_i$ $(i < n)$ be variables. If $\varphi(x, y_0, \ldots, y_{n-1})$ is n-dependent then so is $\psi(w, z_0, \ldots, z_{n-1}) = \varphi(x, y_0, \ldots, y_{n-1})$. In other words, n-dependence is preserved under adding dummy variables.
3. Suppose that T admits quantifier elimination. If there is no atomic formula having IP_n, then T is NIP_n. This follows from Corollary 3.15.

For an n-partite n-uniform hypergraph $(G, R, P_0, \ldots, P_{n-1})$, we say a formula $\varphi(x_0, \ldots, x_{n-1})$ encodes G if there is a G-indexed set $(a_g)_{g\in G}$ such that

$$\models \varphi(a_{g_0}, \ldots, a_{g_{n-1}}) \Leftrightarrow R(g_0, \ldots, g_{n-1})$$

for every $g_i \in P_i$.

Proposition 5.2 Let $\varphi(x, y_0, \ldots, y_{n-1})$ be a formula. The following are equivalent.

1. φ has IP_n.
2. φ encodes every $(n+1)$-partite $(n+1)$-uniform hypergraph G.
3. φ encodes $G_{n+1,p}$ as a partite hypergraph.
4. φ encodes $G_{n+1,p}$ as a partite hypergraph by a $G_{n+1,p}$-indiscernible $(a_g)_{g\in G_{n+1,p}}$.

Proof (1)⇒ (2): By compactness, it is enough to check it for every finite hypergraph G with $|P_0(G)| = \ldots = |P_n(G)| = k$. Let $(a_g)_{g\in O_{n,p}}$ be a witness of IP_n of φ. Let $V_i \subset P_i(O_{n,p})$ be a k-point subset for each $i < n$. For simplicity, we consider $V_i = P_{i+1}(G)$. For $g \in P_0(G)$, let $X_g = \{(g_0,\ldots,g_{n-1}) : G \models R(g,g_0,\ldots,g_{n-1}), g_i \in V_i\}$. By the definition of IP_n, we can find b_g such that

$$\varphi\left(b_g, a_{g_0},\ldots,a_{g_{n-1}}\right) \Leftrightarrow (g_0,\ldots,g_{n-1}) \in X_g.$$

Then letting $a_g = b_g$ for $g \in P_0(G)$, we have that $(a_g)_{g\in G}$ witnesses that φ encodes G.

(2)⇒(3): Trivial.

(3)⇒(4): Suppose that $(a_g)_{g\in G_{n+1,p}}$ witnesses (3). By Corollary 4.8, there is a $G_{n+1,p}$-indiscernible $(b_g)_{g\in G_{n+1,p}}$ based on $(a_g)_{g\in G_{n+1,p}}$, which then also witnesses that φ encodes $G_{n+1,p}$ as a partite hypergraph.

(4)⇒(1): Since $G_{n+1,p}$ is random, the set $\{a_g : g \in P_i(G_{n+1,p}), i > 0\}$ witnesses IP_n for φ. □

As any permutation of parts of a countable partite random hypergraph is an automorphism, we have that n-dependence is preserved under rearranging the order of the variables (in particular, one can exchange the roles of the free variable and a parameter variable):

Corollary 5.3 Let $\varphi(x,y_0,\ldots,y_{n-1})$ be a formula. Suppose that $(w,z_0,\ldots,z_{n-1})$ is any permutation of the sequence $(x,y_0,\ldots,y_{n-1})$. Then $\psi(w,z_0,\ldots,z_{n-1})$ is n-dependent if and only if $\varphi(x,y_0,\ldots,y_{n-1})$ is n-dependent, where

$$\psi(w,z_0,\ldots,z_{n-1}) = \varphi(x,y_0,\ldots,y_{n-1}).$$

.

5.2 Characterizations of NIP_n by collapsing indiscernibles Recall that $(G_n,<,R)$ is a countable ordered n-uniform random hypergraph, and $(G_{n,p},<,R,P_0,\ldots,P_{n-1})$ is a countable ordered n-partite n-uniform random hypergraph. In this subsection we prove the following theorem.

Theorem 5.4 The following are equivalent:

1. T is n-dependent,
2. every $G_{n+1,p}$-indiscernible is actually L_{op}-indiscernible,
3. every G_{n+1}-indiscernible is actually order indiscernible, i.e. $\{<\}$-indiscernible.

Remark 5.5
1. When $n = 1$, this is due to Scow [18].
2. In the theorem, (2)⇒(1) follows immediately from Proposition 5.2, since if φ encodes $G_{n+1,p}$ by $(a_g)_{g\in G_{n+1,p}}$, then $(a_g)_{g\in G_{n+1,p}}$ cannot be an L_{op}-indiscernible.
3. This characterization suggests that strongly minimal theories may be considered as "0-dependent" theories.

First we discuss (3)⇒(2). Let P_i be the i-th part of $G_{n+1,p}$. Since P_i is order isomorphic to $\mathbb{Q}$, we may assume $P_i = \{g^i_q : q \in \mathbb{Q}\}$ with $g_q < g_p \Leftrightarrow q < p$. Let G^*_{n+1} be an ordered $(n+1)$-uniform hypergraph defined as

- $G^*_{n+1} = \{h_q : h_q = (g^0_q,\ldots,g^n_q), q \in \mathbb{Q}\}$,
- $\{h_{q_0},\ldots,h_{q_n}\} \in R(G^*_{n+1}) \Leftrightarrow R(g^0_{q_0},\ldots,g^n_{q_n})$ for $q_0 < \ldots < q_n$,
- $h_q < h_p \Leftrightarrow q < p$

for every $q, p \in \mathbb{Q}$. The hypergraph G^*_{n+1} is clearly K-universal where K is the class of all finite ordered n-uniform hypergraphs.

Proof of (3)⇒(2) of Theorem 5.4 Let $(a_g)_{g\in G_{n+1,p}}$ be a $G_{n+1,p}$-indiscernible which is not L_{op}-indiscernible. We construct a G_{n+1}-indiscernible which is not order indiscernible. Assume that $G_{n+1} = \{g^i_q : i < n+1, q \in \mathbb{Q}\}$ as discussed above. By the assumption there are $A \cong_{L_{\mathrm{op}}} B \subset G_{n+1,p}$ such that $\mathrm{tp}((a_g)_{g\in A}) \neq \mathrm{tp}((a_g)_{g\in B})$. Without loss of generality, we may assume that if $g^i_q, g^j_p \in A$ and $i < j$ then $q < p$, and the same for B. For $h_q \in G^*_{n+1}$, let $b_{h_q} = (a_{g^0_q}, \dots, a_{g^n_q})$ and consider G^*_{n+1}-indexed set $(b_h)_{h\in G^*_{n+1}}$. Let $A^* = \{h_q : g^i_q \in A\}$ and $B^* = \{h_q : g^i_q \in B\}$. Then we have $\mathrm{tp}((b_h)_{h\in A^*}) = \mathrm{tp}((b_h)_{h\in X})$ whenever $A^* \cong_{<,R} X \subset G^*_{n+1}$ (and the same holds for B^*). Applying Fact 4.7 to G^*_{n+1}, we have a G^*_{n+1}-indiscernible $(b'_h)_{h\in G^*_{n+1}}$ based on $(b_h)_{h\in G^*_{n+1}}$. By the construction, $(b'_h)_{h\in G^*_{n+1}}$ is not order indiscernible. Finally, by compactness, we can find $(c_g)_{g\in G_{n+1}}$ that is G_{n+1}-indiscernible but not order indiscernible. □

Now we work towards the converse. Although the remaining part is only (1)⇒(3), we see both (1)⇒(3) and (1)⇒(2) with the same method because a proposition proved in the second one is used in the next section. So let (G_*, L_{o*}, L_{g*}) be either $(G_n, \{<\}, \{<, R\})$ or $\big(G_{n,p}, L_{\mathrm{op}}, L_{\mathrm{opg}}\big)$.

Let $V \subset G_*$ be a finite set and $g_0, \dots, g_{n-1}, g'_0, \dots, g'_{n-1} \in G_* \setminus V$ such that $R(g_0, \dots, g_{n-1}) \not\leftrightarrow R(g'_0, \dots, g'_{n-1})$. Then $W = g_0 \dots g_{n-1}V$ is said to be V-*adjacent* to $W' = g'_0 \dots g'_{n-1}V$ if

- $W \cong_{L_{o*}} W'$,
- for every nonempty $\bar{v} \in V$ with $|\bar{v}| = k$ and $i_0, \dots, i_{n-k-1} < n$

$$R(g_{i_0}, \dots, g_{i_{n-k-1}}, \bar{v}) \leftrightarrow R(g'_{i_0}, \dots, g'_{i_{n-k-1}}, \bar{v}).$$

Recall that R is a symmetric relation, so we do not care about order permutations of substituted elements. W is said to be *adjacent* to W' if there is $V \subset W \cap W'$ such that W is V-adjacent to W'. Roughly speaking, W is adjacent to W' if W can be made isomorphic to W' by adding or deleting an edge.

Lemma 5.6 Let $W, W' \subset G_*$ be subsets such that $W \cong_{L_{o*}} W'$. Then there is a sequence $W = W_0, W_1, \dots, W_k$ such that W_{i+1} is adjacent to W_i for every $i < k$ and $W_k \cong_{L_{g*}} W'$.

Proof The proof is the same for both $(G_n, \{<\}, \{<, R\})$ and $\big(G_{n,p}, L_{\mathrm{op}}, L_{\mathrm{opg}}\big)$. We only prove the statement for $\big(G_{p,n}, L_{\mathrm{op}}, L_{\mathrm{opg}}\big)$. If $W \cong_{L_{\mathrm{opg}}} W'$ then there is nothing to show, so we may assume that $W \not\cong_{L_{\mathrm{opg}}} W'$. Consider any $g_i \in W$ $(i < n)$ such that $g_i \in P_i$, and let $V = W \setminus \{g_0, \dots, g_{n-1}\}$. By Remark 4.5, we can find $g'_0 \in G_{n,p}$ such that $g_0 g_1 \dots g_{n-1}V$ is V-adjacent to $g'_0 g_1 \dots g_{n-1}V$. This means that we can change the existence of any edge by moving a vertex, and get the required sequence. □

Lemma 5.7 Let $V \subset G_*$ be a finite set and let $g_0 < \dots < g_{n-1} \in G_* \setminus V$ with $R(g_0, \dots, g_{n-1})$. Then there are infinite sets $X_0 < \dots < X_{n-1} \subseteq G_*$ such that

- $(G'; <, R) \cong (G_{n,p}; <, R)$ where $G' = X_0 \dots X_{n-1}$,
- for any $g'_i \in X_i$ $(i < n)$, either $W \cong_{L_{\mathrm{opg}}} W'$ or W is V-adjacent to W', where $W = g_0 \dots g_{n-1}V$ and $W' = g'_0 \dots g'_{n-1}V$.

Proof Again, the same argument works for both cases; we deal with partite graphs. Let $g_0, g_1, \ldots$ be an enumeration of $G_{p,n}$. We choose $G' = \{h_i\}_{i\in\omega}$ by recursion on i. First set $h_i = g_i$ for $i < n$. Suppose now that we have already obtained $h_0, \ldots, h_{m-1}$ for some $m \geq n$. Since $G_{n,p}$ is random, we can find $h_m \in G_{n,p}$ such that $h_m V \cong_{L_{\mathrm{op}}} g_i V$ with i taken in such a way that $g_m \in P_i$, that $g_0 \ldots g_m \cong_{L_{\mathrm{opg}}} h_0 \ldots h_m$, and for every nonempty $\bar{v} \in V$

$$R(g_m, g_{i_0}, \ldots, g_{i_{k-1}}, \bar{v}) \leftrightarrow R(h_m, h_{i_0}, \ldots, h_{i_{k-1}}, \bar{v}).$$

Finally, note that $X_i = P_i(G')$ satisfies the requirements. ☐

Now we prove that the existence of a G_*-indiscernible which is not L_{o*}-indiscernible implies IP_n. We carefully discuss how to find a witness of IP_n.

Proposition 5.8 Suppose that there is a $G_{n+1,p}$-indiscernible $(a_g)_{g\in G_{n+1,p}}$ that is not L_{op}-indiscernible. Then there are a finite set $V \subset G_{n+1,p}$, an $L(A)$-formula $\varphi(x, y_0, \ldots, y_{n-1}, A)$ with $A = (a_g)_{g\in V}$ and a subgraph $G' \subset G_{n+1,p}$ with $G' \cong_{L_{\mathrm{opg}}} G_{n+1,p}$ such that $\varphi(x, y_0, \ldots, y_{n-1})$ encodes G' by $(a_g)_{g\in G'}$. In particular, the formula $\varphi(x, y_0, \ldots, y_{n-1}, A)$ has IP_n. Moreover, for every $W, W' \subset G'$, we have $WV \cong_{L_{\mathrm{op}}} W'V$ whenever $W \cong_{L_{\mathrm{op}}} W'$.

Proof Since $(a_g)_{g\in G_{n+1,p}}$ is not L_{op}-indiscernible, there are some subsets $W, W' \subset G$ with $W \cong_{L_{\mathrm{op}}} W'$, and an L-formula $\varphi(x_0, \ldots, x_{m-1})$ such that

$$\models \varphi((a_g)_{g\in W}) \wedge \neg\varphi((a_{g'})_{g'\in W'}).$$

Without loss, we may assume by Lemma 5.6 that W is V-adjacent to W' for some subset V such that $W = g_0 g_1 \ldots g_n V$, $W' = g'_0 g'_1 \ldots g'_n V$, and $R(g_0, \ldots, g_n) \wedge \neg R(g'_0, \ldots, g'_n)$ holds. Now, let $G' \subset G_{n+1,p}$ be a subgraph obtained after applying Lemma 5.7 to V and $g_0 \ldots g_n$. Then for every $h_i \in P_i(G')$ $(i < n+1)$, we have

$$R(h_0, \ldots, h_n) \Leftrightarrow h_0 \ldots h_n V \cong_{L_{\mathrm{opg}}} W$$

and

$$\neg R(h_0, \ldots, h_n) \Leftrightarrow h_0 \ldots h_n V \cong_{L_{\mathrm{opg}}} W'.$$

Since $(a_g)_{g\in G_{n+1,p}}$ is $G_{n+1,p}$-indiscernible, we have that $\varphi(a_{h_0}, \ldots, a_{h_n}, A)$ holds if and only if $R(h_0, \ldots, h_n)$ holds, where $A = (a_g)_{g\in V}$. Thus, the fact that the relation R is random on G' implies that $\varphi(x, y_0, \ldots, y_{n-1}, A)$ has IP_n. Finally, the moreover part is immediate from the definition of G'. ☐

Proposition 5.9 Suppose that there is a G_{n+1}-indiscernible which is not $<$-indiscernible. Then T has IP_n.

Proof The proof is the same as for Proposition 5.8. ☐

6 Reduction to 1 variable

Recall that $(G_{n,p}; <, R, P_0, \ldots, P_{n-1})$ is a countable ordered n-partite n-uniform random hypergraph.

A standard characterization of dependence of a formula in terms of finite alternation on an infinite indiscernible sequence (see e.g. [2, Proposition 4]) can be easily reformulated using Ramsey and compactness in the following way:

Remark 6.1 Let R be a dense co-dense subset of $\mathbb{Q}$. The following are equivalent:

1. $\varphi(x, y)$ has IP.

2. There are b and $(a_i)_{i\in\mathbb{Q}}$ such that:
   (a) $(a_i)_{i\in\mathbb{Q}}$ is order indiscernible,
   (b) $\models \varphi(b,a_i)$ if and only if $i\in R$, for all $i\in\mathbb{Q}$,
   (c) in addition, $(a_i)_{i\in\mathbb{Q}}$ is $(<,R)$-indiscernible over b.

We give an appropriate generalization for n-dependence (recall that an ordered random 1-hypergraph is just a dense linear order with a dense co-dense subset).

Lemma 6.2 The following are equivalent.

1. $\varphi(x,y_0,\dots,y_{n-1})$ has IP_n.
2. There are b and $(a_g)_{g\in G_n}$ such that:
   (a) $(a_g)_{g\in G_{n,p}}$ is L_{op}-indiscernible,
   (b) $\models \varphi(b,a_{g_0},\dots,a_{g_{n-1}})$ if and only if $R(g_0,\dots,g_{n-1})$, for all $g_i\in P_i$.
3. There are b and $(a_g)_{g\in G_{n,p}}$ satisfying the conditions (2a), (2b) and
   (c) $(a_g)_{g\in G_{n,p}}$ is $G_{n,p}$-indiscernible over b.

Proof (3) $\Rightarrow$ (2): Trivial.

(2) $\Rightarrow$ (1): We show that $(a_g)_{g\in G_{n,p}}$ witnesses the n-independence of φ. Let X_0 and X_1 be any disjoint finite subsets of $P_0\times\dots\times P_{n-1}$. As $G_{n,p}$ is random, there are subsets $X_0'X_1'\cong_{L_{\mathrm{op}}} X_0X_1$ of $P_0\times\dots\times P_{n-1}$ such that $X_0'\subset R$ and $X_1'\subset R^c=\Pi_i P_i\setminus R$. By the assumption, observe that $\varphi(b,a_{g_0},\dots,a_{g_{n-1}})$ holds only when $(g_0,\dots,g_{n-1})\in X_0'$, and hence

$$\models \exists x \bigwedge_{i<2}\ \bigwedge_{(g_0,\dots,g_{n-1})\in X_i} \varphi(x,a_{g_0},\dots,a_{g_{n-1}})^{\mathrm{if(i=0)}}$$

since $(a_g)_{g\in G_n}$ is L_{op}-indiscernible.

(1) $\Rightarrow$ (3): Suppose that $\varphi(x,y_0,\dots,y_{n-1})$ has IP_n. By Remark 4.9, we may assume the existence of an L_{op}-indiscernible $(a'_g)_{g\in G_{n,p}}$ witnessing that φ has IP_n. Then there is some b such that $\models\varphi(b,a'_{g_0},\dots,a'_{g_{n-1}})$ if and only if $R(g_0,\dots,g_{n-1})$, for all $g_i\in P_i$. Let $(a_g)_{g\in G_{n,p}}$ be a $G_{n,p}$-indiscernible based on $(a'_g)_{g\in G_{n,p}}$ over b. Clearly, any such sequence satisfies conditions (b) and (c). To see that $(a_g)_{g\in G_{n,p}}$ is an L_{op}-indiscernible sequence, consider some finite subsets W and V of $G_{n,p}$ with $W\cong_{L_{\mathrm{op}}} V$, and a formula $\theta((x_g)_{g\in W})$. We show that $\theta((a_g)_{g\in W})$ holds if and only if $\theta((a_g)_{g\in V})$ holds. Since $(a_g)_{g\in G_{n,p}}$ is based on $(a'_g)_{g\in G_{n,p}}$, there is $W'V'\cong_{L_{\mathrm{opg}}} WV$ such that

$$\theta((a_g)_{g\in W})\leftrightarrow\theta((a'_g)_{g\in W'}) \text{ and } \theta((a_g)_{g\in V})\leftrightarrow\theta((a'_g)_{g\in V'})$$

hold. Now, the fact that $W'\cong_{L_{\mathrm{op}}} V'$ yields that $\theta((a'_g)_{g\in W'})$ holds if and only if $\theta((a'_g)_{g\in V'})$ holds, as desired. □

Proposition 6.3 The following are equivalent:

1. Every L-formula $\phi(x,y_0,\dots,y_{n-1})$ with $|x|\le m$ is n-dependent.
2. For any $(a_g)_{g\in G_{n,p}}$ and b with $|b|=m$, if $(a_g)_{g\in G_{n,p}}$ is $G_{n,p}$-indiscernible over b and L_{op}-indiscernible (over $\emptyset$), then it is L_{op}-indiscernible over b.

Proof (1) $\Rightarrow$ (2): Let $(a_g)_{g\in G_{n,p}}$ be $G_{n,p}$-indiscernible over b with $|b|=m$, and set $a'_g=ba_g$. Thus $(a'_g)_{g\in G_{n,p}}$ is $G_{n,p}$-indiscernible (over $\emptyset$). Suppose, towards a contradiction, that $(a_g)_{g\in G_{n,p}}$ is not L_{op}-indiscernible over b; in other words $(a'_g)_{g\in G_{n,p}}$ is not L_{op}-indiscernible (over $\emptyset$). By Proposition 5.8, there is a subgraph $G'\subset G_{n,p}$, a finite set $V\subset G_{n,p}$, and a formula $\psi(y'_0,\dots,y'_{n-1},\bar z')$ such that

1. $G' \cong_{L_{\text{opg}}} G_{n,p}$,
2. $R(g_0, \ldots, g_{n-1})$ holds if and only if $\psi(a'_{g_0}, \ldots, a'_{g_{n-1}}, (a'_g)_{g \in V})$ for every $g_i \in P_i(G')$,
3. for every $W, W' \subset G'$ we have $WV \cong_{L_{\text{op}}} W'V$ whenever $W \cong_{L_{\text{op}}} W'$.

Now, observe that each variable y'_i is of the form xy_i where x corresponds to the tuple b, and the variable $\bar{z}'$ is of the form $x\bar{z}$. Let $\varphi(x, z_0, \ldots z_{n-1})$ be the formula $\psi(y'_0, \ldots, y'_{n-1}, \bar{z}')$ where $z_i = y_i\bar{z}$. Moreover, for each $g \in G'$, let $c_g = a_g(a_{g'})_{g' \in V}$ and note that $\varphi(b, c_{g_0}, \ldots, c_{g_{n-1}})$ holds if and only if $R(g_0, \ldots, g_{n-1})$ does. As the sequence $(a_g)_{g \in G_{n,p}}$ is L_{op}-indiscernible, so is $(c_g)_{g \in G'}$, and therefore the formula $\varphi(x, y_0, \ldots y_{n-1})$ has IP_n by Lemma 6.2.

(2) $\Rightarrow$ (1): Immediate from Lemma 6.2. □

The following theorem is from [22, Section 2]. We are following the same strategy as the proof there, however the authors felt that a more detailed account could be provided.

Theorem 6.4 (Shelah) A complete theory T is n-dependent if and only if every formula $\varphi(x, y_0, \ldots, y_{n-1})$ with $|x| = 1$ is n-dependent.

Proof We show by induction that if the condition (2) of Proposition 6.3 holds for $m = 1$, then it holds for all $m \in \omega$. Let $\bar{b} = b_0 \ldots b_m$ be a tuple with $|b_i| = 1$, and let $(a_g)_{g \in G_{n,p}}$ be given such that $(a_g)_{g \in G_{n,p}}$ is $G_{n,p}$-indiscernible over $\bar{b}$. Note that $(a_g)_{g \in G_{n,p}}$ is L_{op}-indiscernible over b_m, as otherwise it is not L_{op}-indiscernible over b_m but $G_{n,p}$-indiscernible over b_m, contradicting the inductive assumption. Now, consider the sequence $(b_m a_g)_{g \in G_{n,p}}$ and notice that it is clearly $G_{n,p}$-indiscernible over $b_0 \ldots b_{m-1}$. Applying the inductive assumption again, we conclude that $(b_m a_g)_{g \in G_{n,p}}$ is L_{op}-indiscernible over $b_0 \ldots b_{m-1}$, which implies that $(a_g)_{g \in G_n}$ is L_{op}-indiscernible over $b_0 \ldots b_m$, as desired. □

Finally, we summarize the basic properties of n-dependent theories established throughout the paper, giving a criterion for the n-dependence of a theory.

Proposition 6.5

1. Boolean combinations preserve n-dependence (Corollary 3.15).
2. Permuting variables preserves n-dependence (Corollary 5.3).
3. Failure of n-dependence of a theory is witnessed by a formula in a single free variable (Theorem 6.4).
4. If T eliminates quantifiers, then in order to check that T is n-dependent it is enough to check that every atomic formula in a single free variable is n-dependent, e.g. by checking that the number of ϕ-types is not maximal (combining (1) and (3) above).

Appendix A Ramsey property for hypergraphs

In this section we verify that the two classes of structures considered in the previous sections have Ramsey property. First we see that the class of finite ordered n-partite sets is Ramsey, and then that the class of all finite (linearly) ordered n-partite n-uniform hypergraphs is also Ramsey. Basic notation and definitions are already given in Section 4.1, so we don't repeat them.

For a given class K of L-structures, let K^* be the hereditary closure of K, i.e. $K^* = \{A : A \subseteq B, B \in K\}$.

Lemma A.1 Let K be a set of L-structures satisfying Ramsey property. Suppose that every $A \in K$ has no non-trivial automorphisms, i.e. $Aut(A) = \{id_A\}$. If the hereditary closure K^* of K has the amalgamation property, then K^* has Ramsey property.

Proof Let $A, B \in K^*$. Fix an extension $A \subseteq A_0 \in K$ and consider a structure $A'_0 \cong A_0$. Note that, in general, $\binom{A'_0}{A}$ is not a singleton. However, by the assumption, we can recognize a unique $A' \subset A'_0$ which corresponds to $A \subset A_0$ through the unique isomorphism $A'_0 \cong A_0$. By applying amalgamation property, we have an extension $B_0 \in K$ of B such that for every $A' \in \binom{B}{A}$ there is an extension $A' \subseteq A'_0 \in \binom{B_0}{A_0}$ which is isomorphic to the extension $A \subseteq A_0$. Since K has Ramsey property, we can find $C \in K$ such that $C \to (B_0)^{A_0}_k$. It is easy to check that $C \to (B)^A_k$, since any coloring $c : \binom{C}{A} \to k$ induces a coloring $\tilde{c}(A'_0) = c(A')$ for $A' \subset A'_0 \in \binom{C}{A_0}$, hence there is $B'_0 \subset C$ on which $\tilde{c}$ is constant. Clearly, for a $B' \subset B_0$, c is constant on $\binom{B'}{A}$. □

For L-structures A and B, let $A \oplus B$ be an $L \cup \{P_0(x), P_1(x)\}$-structure such that $P_0 = A$, $P_1 = B$ and $A \oplus B = P_0 \sqcup P_1$. Let K_0 and K_1 be two classes of L-structures. We define a class $K_0 \oplus K_1$ of $L \cup \{P_0(x), P_1(x)\}$-structures by

$$K_0 \oplus K_1 := \{A_0 \oplus A_1 : A_0 \in K_0, A_1 \in K_1\}.$$

Lemma A.2 If K_0 and K_1 have Ramsey property, then so does $K_0 \oplus K_1$.

Proof Let $A_0 \oplus A_1, B_0 \oplus B_1 \in K_0 \oplus K_1$. Fix $C_1 \in K_1$ such that $C_1 \to (B_1)^{A_1}_k$. Let $m = \left|\binom{C_1}{A_1}\right|$. We can find $C^0_0, \ldots, C^m_0 = C_0 \in K$ such that $C^0_0 \to (B_0)^{A_0}_k$ and $C^{i+1}_0 \to (C^i_0)^{A_0}_k$ for $i \leq m$. We show that $C_0 \oplus C_1 \to (B_0 \oplus B_1)^{A_0 \oplus A_1}_k$. Let $c : \binom{C_0 \oplus C_1}{A_0 \oplus A_1} \to k$ be a coloring. Then for each $A'_1 \in \binom{C_1}{A_1}$, we have an induced coloring $c_{A'_1} : \binom{C_0}{A_0} \to k$ such that $c_{A'_1}(A'_0) = c(A'_0 \oplus A'_1)$. By the construction, we can find $B'_0 \in \binom{C_0}{B_0}$ such that $c_{A'_1}$ is constant on $\binom{B'_0}{A_0}$ for every $A'_1 \in \binom{C_1}{A_1}$. Then, the values of $c_{A'_1}$ on $\binom{B'_0}{A_0}$ define a coloring $\tilde{c} : \binom{C_1}{A_1} \to k$ by $\tilde{c}(A'_1) = c_{A'_1}(A'_0)$ where $A'_0 \subset B'_0$. Hence there is $B'_1 \in \binom{C_1}{B_1}$ such that $\tilde{c}$ is constant on $\binom{B'_1}{A_1}$. Therefore, c is constant on $\binom{B'_0 \oplus B'_1}{A_0 \oplus A_1}$. □

The classical Ramsey theorem implies that the class of all finite linearly ordered sets has Ramsey property. Therefore, with the above lemmas, we have the following:

Proposition A.3 Let K be the set of finite ordered n-partite sets. Then both K and its hereditary closure K^* have Ramsey property.

Next we'll prove that the set of finite ordered n-partite n-uniform hypergraphs has Ramsey property. Let R be an n-place relation for some $n \geq 1$. Recall that an ordered n-uniform hypergraph is an L-structure A such that R is symmetric and irreflexive on A and that $<$ is a linear ordering on A. Our starting point is the following well-known fact:

Fact A.4 (Nesétril, Rödl [14, 15]; Abramson, Harrington [1]) Let K be the set of all finite ordered n-uniform hypergraphs. Then K has Ramsey property.

Recall that $L_{\mathrm{opg}} = \{R(x_0,\ldots,x_{n-1}),<,P_0(x),\ldots,P_{n-1}(x)\}$ and that an ordered n-partite n-uniform hypergraph is an L_{opg}-structure A satisfying the following:

1. $A|\{R,P_0,\ldots,P_{n-1}\}$ is an n-partite n-uniform hypergraph,
2. $<$ is a total ordering on A satisfying $P_0(A) < \ldots < P_{n-1}(A)$.

Proposition A.5 Let K be the set of finite ordered n-partite n-uniform hypergraphs. Then K has Ramsey property.

Proof Since the general case is similar, we assume $n = 2$ for simplicity. Fix $A,B \in K$ and $k \in \omega$. Let $A_0 = A|\{R,<\}$ and $B_0 = B|\{R,<\}$ respectively. Then there is an ordered graph C_0 such that $C_0 \to (B_0)^{A_0}_k$. For a given ordered graph $X_0 = \{v_0 < \ldots < v_{m-1}\}$, let $\tilde{X}_0$ be an ordered bipartite graph such that

- $P_i(\tilde{X}_0) = \{w^i_0 < \ldots < w^i_{m-1}\}$,
- $R(\tilde{X}_0) \ni (w^0_i, w^1_j)$ if and only if $i < j$ and $R(v_i,v_j)$ in X_0.

Claim $\tilde{C}_0 \to (B)^A_k$.

Suppose that X is a bipartite graph, with

$$P_0(X) = \{v_0 < \ldots < v_{l-1}\} \text{ and } P_1(X) = \{v_l < \ldots < v_{m-1}\}.$$

For the ordered graph $X_0 = X|\{R,<\}$, put

$$\bar{X}_0 = \{w^0_0 < \ldots < w^0_{l-1}\} \cup \{w^1_l < \ldots < w^1_{m-1}\} \subset \tilde{X}_0.$$

Then $\bar{X}_0$ is a bipartite subgraph of $\tilde{X}_0$. One can easily check that $\bar{X}_0 \cong X$. With this fact in mind, let $c : \binom{\tilde{C}_0}{A} \to k$ be any coloring. Then there is an induced coloring $\tilde{c} : \binom{C_0}{A_0} \to k$ such that $\tilde{c}(A'_0) = c(\bar{A}'_0)$ for all $A'_0 \in \binom{C_0}{A_0}$. Let $B'_0 \in \binom{C_0}{A_0}$ be such that $\tilde{c}$ is constant on $\binom{B'_0}{A_0}$. Then $\bar{B}'_0 \in \binom{\tilde{C}_0}{B}$ satisfies the required condition. □

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