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A FINITE ELEMENT ELASTICITY COMPLEX IN THREE DIMENSIONS

LONG CHEN AND XUEHAI HUANG

ABSTRACT. A finite element elasticity complex on tetrahedral meshes and the corresponding commutative diagram are devised. The H^1 conforming finite element is the finite element developed by Neilan for the velocity field in a discrete Stokes complex. The symmetric div-conforming finite element is the Hu-Zhang element for stress tensors. The construction of an $H(\text{inc})$-conforming finite element of minimum polynomial degree 6 for symmetric tensors is the focus of this paper. Our construction appears to be the first $H(\text{inc})$-conforming finite elements on tetrahedral meshes without further splitting. The key tools of the construction are the decomposition of polynomial tensor spaces and the characterization of the trace of the inc operator. The polynomial elasticity complex and Koszul elasticity complex are created to derive the decomposition. The trace of the inc operator is induced from a Green's identity. Trace complexes and bubble complexes are also derived to facilitate the construction. Two-dimensional smooth finite element Hessian complex and div div complex are constructed.

1. INTRODUCTION

A Hilbert complex is a sequence of Hilbert spaces connected by a sequence of linear operators satisfying the property: the composition of two consecutive operators vanishes. Let Ω be a bounded domain in $\mathbb{R}^3$. The elasticity complex

$$\boldsymbol{RM} \xrightarrow{\subset} \boldsymbol{H}^1(\Omega;\mathbb{R}^3) \xrightarrow{\text{def}} \boldsymbol{H}(\text{inc},\Omega;\mathbb{S}) \xrightarrow{\text{inc}} \boldsymbol{H}(\text{div},\Omega;\mathbb{S}) \xrightarrow{\text{div}} \boldsymbol{L}^2(\Omega;\mathbb{R}^3) \to \boldsymbol{0} \tag{1}$$

plays an important role in both theoretical and numerical study of linear elasticity, where $\boldsymbol{RM}$ is the space of the linearized rigid body motion, def is the symmetric gradient operator, $\boldsymbol{H}(\text{inc},\Omega;\mathbb{S})$ is the space of symmetric tensor $\boldsymbol{\tau}$ s.t. $\text{inc}\,\boldsymbol{\tau} := -\text{curl}(\text{curl}\,\boldsymbol{\tau})^{\intercal} \in \boldsymbol{L}^2(\Omega;\mathbb{M})$, and $\boldsymbol{H}(\text{div},\Omega;\mathbb{S})$ is the space for the symmetric stress tensor $\boldsymbol{\sigma}$ with $\text{div}\,\boldsymbol{\sigma} \in \boldsymbol{L}^2(\Omega;\mathbb{R}^3)$. We shall present a finite element elasticity complex

$$\boldsymbol{RM} \xrightarrow{\subset} \boldsymbol{V}_h \xrightarrow{\text{def}} \boldsymbol{\Sigma}_h^{\text{inc}} \xrightarrow{\text{inc}} \boldsymbol{\Sigma}_h^{\text{div}} \xrightarrow{\text{div}} \mathcal{Q}_h \to \boldsymbol{0} \tag{2}$$

on a tetrahedral mesh $\mathcal{T}_h$ of Ω. In the complex (2), the H^1-conforming finite element is the finite element $\boldsymbol{V}_h$ developed by Neilan for the velocity field in a finite element Stokes complex [34]. The $\boldsymbol{H}(\text{div};\mathbb{S})$-conforming finite element $\boldsymbol{\Sigma}_h^{\text{div}}$ is the Hu-Zhang element for the symmetric stress tensor [28, 31]. The space $\mathcal{Q}_h$ for $\boldsymbol{L}^2(\Omega;\mathbb{R}^3)$ is simply the discontinuous piecewise polynomial space. Some degrees of freedom of

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Neilan element and Hu-Zhang element are modified for the consideration of the commutative digram. The missing component is an $\boldsymbol{H}(\mathrm{inc};\mathbb{S})$-conforming finite element $\boldsymbol{\Sigma}_h^{\mathrm{inc}}$ which is the focus of this work.

In the solid mechanics application, the most important component in the complex (2) is the finite element $\boldsymbol{\Sigma}_h^{\mathrm{div}}$ for stress tensors. Construction of finite elements for stress tensors can benefit from the structure of the complex. For example, the bubble polynomial elasticity complex is built and used in [5] to construct a finite element for symmetric stress tensors. Here the bubble polynomial spaces are referred to polynomials with vanished traces on the boundary of each tetrahedron. In [28, 31, 17], a precise characterization of $\boldsymbol{H}(\mathrm{div};\mathbb{S})$ bubble polynomial space is given which leads to a stable $\mathbb{P}_k(\mathbb{S})-\mathbb{P}_{k-1}(\mathbb{R}^d)$ stress-displacement finite element pair in arbitrary dimension. Identification of its preceding space $\boldsymbol{\Sigma}_h^{\mathrm{inc}}$ will be helpful for the design of fast solvers and a posteriori error analysis [12] for the mixed formulation of linear elasticity problems. It may also find applications in other fields such as continuum modeling of defects [2] and relativity [19].

Elasticity complex (1) and many more complexes can be derived from the composition of de Rham complexes in the so-called Bernstein-Gelfand-Gelfand (BGG) construction [7]. Finite element complexes for the de Rham complex are well understood and can be derived systematically in the framework of Finite Element Exterior Calculus [4, 6]. It is natural to ask if a finite element elasticity complex can be derived by the BGG construction. One key in the BGG construction is the existence of smooth finite element de Rham complexes. With nodal finite element de Rham complexes, a two-dimensional finite element elasticity complex has been constructed in [21] using the BGG construction which generalizes the first finite element elasticity complex of Arnold and Winther [8].

In three dimensions, however, smooth discrete de Rham complexes are not easy due to the super-smoothness of multivariate splines [25] (cf. superspline in [24, 33]). To relax the super-smoothness, the element can be further split so that inside one element the shape function is not $\mathcal{C}^\infty$ smooth. Such approach leads to the so-called macro elements. In particular, a two-dimensional elasticity strain complex has been constructed on the Clough-Tocher split of a triangle [22], and more recently a finite element elasticity complex has been constructed on the Alfeld split of a tetrahedron [20] based on the smooth finite element de Rham complex [26] on such split.

We shall construct a finite element elasticity complex on a tetrahedral mesh and the corresponding commutative diagram without further splitting. Let K be a polyhedron. We first give a polynomial elasticity complex and a Koszul type complex, which can be summarized as one double-directional complex below:

$$\boldsymbol{RM} \underset{\pi_{RM}}{\overset{\subset}{\rightleftarrows}} \mathbb{P}_{k+1}(K;\mathbb{R}^3) \underset{\boldsymbol{\tau}\cdot\boldsymbol{x}}{\overset{\mathrm{def}}{\rightleftarrows}} \mathbb{P}_k(K;\mathbb{S}) \underset{\boldsymbol{x}\times\boldsymbol{\tau}\times\boldsymbol{x}}{\overset{\mathrm{inc}}{\rightleftarrows}} \mathbb{P}_{k-2}(K;\mathbb{S}) \underset{\mathrm{sym}(\boldsymbol{v}\boldsymbol{x}^{\intercal})}{\overset{\mathrm{div}}{\rightleftarrows}} \mathbb{P}_{k-3}(K;\mathbb{R}^3) \underset{\supset}{\rightleftarrows} 0\,.$$

Several decompositions of polynomial tensor spaces, especially for $\mathbb{P}_k(K;\mathbb{S})$, can be obtained consequently. We then study trace operators for the inc operator since the traces on face and edges are crucial to ensure the $H(\mathrm{inc})$-conformity. To do so, we use a symmetric notation $\mathrm{inc}\,\boldsymbol{\tau} = \nabla\times\boldsymbol{\tau}\times\nabla$ (see Section 2 for notation) and derive the following symmetric Green's identity:

$$
\begin{aligned}
(\nabla\times\boldsymbol{\sigma}\times\nabla,\boldsymbol{\tau})_K-(\boldsymbol{\sigma},\nabla\times\boldsymbol{\tau}\times\nabla)_K &= (\operatorname{tr}_1(\boldsymbol{\sigma}),\operatorname{tr}_2(\boldsymbol{\tau}))_{\partial K}-(\operatorname{tr}_2(\boldsymbol{\sigma}),\operatorname{tr}_1(\boldsymbol{\tau}))_{\partial K}\\
&\quad+\sum_{F\in\mathcal{F}(K)}\sum_{e\in\mathcal{E}(F)}(\boldsymbol{n}\cdot\boldsymbol{\sigma}\times\boldsymbol{n},\boldsymbol{t}_{F,e}\cdot\boldsymbol{\tau})_e\\
&\quad-\sum_{F\in\mathcal{F}(K)}\sum_{e\in\mathcal{E}(F)}(\boldsymbol{t}_{F,e}\cdot\boldsymbol{\sigma},\boldsymbol{n}\cdot\boldsymbol{\tau}\times\boldsymbol{n})_e,
\end{aligned}
$$

where, with Π_F denoting the projection operator to face F,

$$
\begin{aligned}
\operatorname{tr}_1(\boldsymbol{\tau}) &:= \boldsymbol{n}\times\boldsymbol{\tau}\times\boldsymbol{n},\\
\operatorname{tr}_2(\boldsymbol{\tau}) &:= \Pi_F(\boldsymbol{\tau}\times\nabla)\times\boldsymbol{n}+\nabla_F(\boldsymbol{n}\cdot\boldsymbol{\tau}\,\Pi_F).
\end{aligned}
$$

We show $\operatorname{tr}_1(\boldsymbol{\tau}) \in \boldsymbol{H}(\operatorname{div}_F\operatorname{div}_F, F;\mathbb{S})$ and $\operatorname{tr}_2(\boldsymbol{\tau}) \in \boldsymbol{H}(\operatorname{rot}_F, F;\mathbb{S})$, and reveal boundary complexes induced by trace operators; see Section 4.2 for details. Then the edge traces of the face traces $\operatorname{tr}_1(\boldsymbol{\tau})$ and $\operatorname{tr}_2(\boldsymbol{\tau})$ imply the continuity of $\boldsymbol{\tau}|_e$ and $(\nabla\times\boldsymbol{\tau})\cdot\boldsymbol{t}_e$. Further edge degrees of freedom will be derived from the requirement $\operatorname{inc}\boldsymbol{\tau}$ is in the Hu-Zhang finite element space. The face degree of freedom will be based on the decomposition of polynomial tensors of $\boldsymbol{H}(\operatorname{div}_F\operatorname{div}_F, F;\mathbb{S})$ and $\boldsymbol{H}(\operatorname{rot}_F, F;\mathbb{S})$. The volume degree of freedom is from the decomposition of $\mathbb{P}_k(K;\mathbb{S})$ based on the polynomial elasticity complex.

Recently there has been a lot of progress in the construction of finite elements for tensors [11, 21, 22, 14, 18, 16, 29, 20]. Our construction appears to be the first $H(\operatorname{inc})$-conforming finite elements for symmetric tensors on tetrahedral meshes without further splitting. Our finite element spaces are constructed for tetrahedrons but some results, e.g., traces and Green's formulae etc, hold for general polyhedrons. Our approach of constructing finite element for tensors, through decomposition of polynomial space and characterization of trace operators, seems simpler and more straightforward than the BGG construction through smooth finite element de Rham complexes. For example, although macro finite elements are adopted, the finite element elasticity complex in [20] is still smoother than complex (2) in the sense that the space $\boldsymbol{V}_h$ in [20] is H^2-conforming, and $\boldsymbol{\Sigma}_h^{\operatorname{inc}}$ is H^1-conforming. Ours is more natural: $\boldsymbol{V}_h$ is H^1-conforming, and $\boldsymbol{\Sigma}_h^{\operatorname{inc}}$ is $H(\operatorname{inc})$-conforming.

Notation on meshes. Let $\{\mathcal{T}_h\}_{h>0}$ be a regular family of polyhedral meshes of Ω. For each element $K\in\mathcal{T}_h$, denote by $\boldsymbol{n}_K$ the unit outward normal vector to ∂K. In most places, it will be abbreviated as $\boldsymbol{n}$ for simplicity. Denote by $\mathcal{F}(K)$, $\mathcal{E}(K)$ and $\mathcal{V}(K)$ the set of all faces, edges and vertices of K, respectively. Similarly let $\mathcal{E}(F)$ be the set of all edges of face F. For $F\in\mathcal{F}(K)$, its orientation is given by the outwards normal direction $\boldsymbol{n}_{\partial K}$ which also induces a consistent orientation of edge $e\in\mathcal{E}(F)$. Namely the edge vectors $\boldsymbol{t}_{F,e}$ and outwards normal vector $\boldsymbol{n}_{\partial K}$ follows the right hand rule. Then define $\boldsymbol{n}_{F,e}=\boldsymbol{t}_{F,e}\times\boldsymbol{n}_{\partial K}$ as the outwards normal vector of e on the face F.

Let $\mathcal{F}_h$, $\mathcal{E}_h$ and $\mathcal{V}_h$ be the union of all faces, edges and vertices of the partition $\mathcal{T}_h$, respectively. For any $F\in\mathcal{F}_h$, fix a unit normal vector $\boldsymbol{n}_F$ and two unit tangent vectors $\boldsymbol{t}_{F,1}$ and $\boldsymbol{t}_{F,2}$, which will be abbreviated as $\boldsymbol{t}_1$ and $\boldsymbol{t}_2$ without causing any confusions. For any $e\in\mathcal{E}_h$, fix a unit tangent vector $\boldsymbol{t}_e$ and two unit normal vectors $\boldsymbol{n}_{e,1}$ and $\boldsymbol{n}_{e,2}$, which will be abbreviated as $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ without causing any confusions. Those notation are illustrated in Fig. 1. We emphasize that $\boldsymbol{n}_F, \boldsymbol{t}_e, \boldsymbol{n}_{e,1}$, and $\boldsymbol{n}_{e,2}$ are globally defined not depending on the elements.

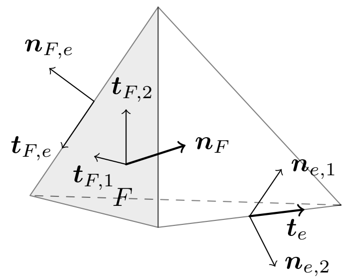


FIGURE 1. Tangent vectors and normal vectors of edges and faces.

The rest of this paper is organized as follows. In Section 2, we present a notation system on the vector and tensor operations. We construct polynomial complexes and derive decompositions of polynomial tensors spaces related to the elasticity complex in Section 3. In Section 4, we discuss traces for the inc operator based on the Green's identity, and present corresponding trace complexes and bubble complexes. Two smooth finite element complexes in two dimensions are devised in Section 5. In Section 6, we construct an $H(\mathrm{inc})$-conforming finite element and a finite element elasticity complex in three dimensions. Finally, a commutative diagram for the finite element elasticity complex is developed in Section 7.

2. VECTOR AND TENSOR OPERATIONS

One complication on the construction of finite elements for tensors is the notation system for tensor operations. In this section, we adapt the notation system used in solid mechanics [32]. In particular, we separate the row and column operations to the right and left sides of the matrix, respectively.

2.1. **Tensor calculus.** Define the dot product and the cross product from the left

$$\boldsymbol{b}\cdot\boldsymbol{A},\quad \boldsymbol{b}\times\boldsymbol{A},$$

which is applied column-wise to the matrix $\boldsymbol{A}$. When the vector is on the right of the matrix

$$\boldsymbol{A}\cdot\boldsymbol{b},\quad \boldsymbol{A}\times\boldsymbol{b},$$

the operation is defined row-wise. Here for cleaner notation, when the vector $\boldsymbol{b}$ is on the right, it is treated as a row-vector $\boldsymbol{b}^{\intercal}$ while when on the left, it is a column vector.

The ordering of performing the row and column products does not matter which leads to the associative rule of the triple products

$$\boldsymbol{b}\times\boldsymbol{A}\times\boldsymbol{c} := (\boldsymbol{b}\times\boldsymbol{A})\times\boldsymbol{c} = \boldsymbol{b}\times(\boldsymbol{A}\times\boldsymbol{c}).$$

Similar rules hold for $\boldsymbol{b}\cdot\boldsymbol{A}\cdot\boldsymbol{c}$ and $\boldsymbol{b}\cdot\boldsymbol{A}\times\boldsymbol{c}$ and thus parentheses can be safely skipped. Another benefit is the transpose of products. For the transpose of product of two objects, we take transpose of each one, switch their order, and add a negative sign if it is the cross product.

For two column vectors $\boldsymbol{u},\boldsymbol{v}$, the tensor product $\boldsymbol{u}\otimes\boldsymbol{v} := \boldsymbol{u}\boldsymbol{v}^{\intercal}$ is a matrix which is also known as the dyadic product $\boldsymbol{u}\boldsymbol{v} := \boldsymbol{u}\boldsymbol{v}^{\intercal}$ with cleaner notation (one

$^\intercal$ is skipped). The row-wise product and column-wise product of $\boldsymbol{uv}$ with another vector $\boldsymbol{x}$ will be applied to the neighboring vector:

$$\boldsymbol{x} \cdot (\boldsymbol{uv}) = (\boldsymbol{x} \cdot \boldsymbol{u})\boldsymbol{v}^\intercal, \quad (\boldsymbol{uv}) \cdot \boldsymbol{x} = \boldsymbol{u}(\boldsymbol{v} \cdot \boldsymbol{x}),$$
$$\boldsymbol{x} \times (\boldsymbol{uv}) = (\boldsymbol{x} \times \boldsymbol{u})\boldsymbol{v}, \quad (\boldsymbol{uv}) \times \boldsymbol{x} = \boldsymbol{u}(\boldsymbol{v} \times \boldsymbol{x}).$$

We treat the Hamilton operator $\nabla = (\partial_1, \partial_2, \partial_3)^\intercal$ as a column vector. For a vector function $\boldsymbol{u} = (u_1, u_2, u_3)^\intercal$, $\operatorname{curl} \boldsymbol{u} = \nabla \times \boldsymbol{u}$, and $\operatorname{div} \boldsymbol{u} = \nabla \cdot \boldsymbol{u}$ are standard differential operations. Define $\nabla \boldsymbol{u} = \nabla \boldsymbol{u}^\intercal = (\partial_i u_j)$ which can be understood as the dyadic product of Hamilton operator ∇ and column vector $\boldsymbol{u}$.

Applying these matrix-vector operations to the Hamilton operator ∇, we get column-wise differentiations $\nabla \cdot \boldsymbol{A}, \nabla \times \boldsymbol{A}$, and row-wise differentiations $\boldsymbol{A} \cdot \nabla, \boldsymbol{A} \times \nabla$. Conventionally, the differentiation is applied to the function after the ∇ symbol. So a more conventional notation is

$$\boldsymbol{A} \cdot \nabla := (\nabla \cdot \boldsymbol{A}^\intercal)^\intercal, \quad \boldsymbol{A} \times \nabla := -(\nabla \times \boldsymbol{A}^\intercal)^\intercal.$$

By moving the differential operator to the right, the notation is simplified and the transpose rule for matrix-vector products can be formally used. Again the right most column vector is treated as a row vector to make the notation cleaner. We introduce the double differential operators as

$$\operatorname{inc} \boldsymbol{A} := \nabla \times \boldsymbol{A} \times \nabla, \quad \operatorname{div} \operatorname{div} \boldsymbol{A} := \nabla \cdot \boldsymbol{A} \cdot \nabla.$$

As the column and row operations are independent, and no product rule of differentials is needed, the ordering of operations is not important and parentheses is skipped. Parentheses will be added when it is necessary.

In the literature, differential operators for matrices are usually applied row-wise to tensors. To distinguish with ∇ notation, we define operators in letters as

$$\begin{aligned} \operatorname{grad} \boldsymbol{u} &:= \boldsymbol{u} \nabla^\intercal = (\partial_j u_i) = (\nabla \boldsymbol{u})^\intercal, \\ \operatorname{curl} \boldsymbol{A} &:= -\boldsymbol{A} \times \nabla = (\nabla \times \boldsymbol{A}^\intercal)^\intercal, \\ \operatorname{div} \boldsymbol{A} &:= \boldsymbol{A} \cdot \nabla = (\nabla \cdot \boldsymbol{A}^\intercal)^\intercal. \end{aligned}$$

Note that the transpose operator $^\intercal$ is involved for tensors and in particular $\operatorname{grad} \boldsymbol{u} \neq \nabla \boldsymbol{u}$, $\operatorname{curl} \boldsymbol{A} \neq \nabla \times \boldsymbol{A}$, $\operatorname{curl} \boldsymbol{A} \neq \boldsymbol{A} \times \nabla$ and $\operatorname{div} \boldsymbol{A} \neq \nabla \cdot \boldsymbol{A}$. For symmetric tensors, $\operatorname{div} \boldsymbol{A} = (\nabla \cdot \boldsymbol{A})^\intercal$, $\operatorname{curl} \boldsymbol{A} = (\nabla \times \boldsymbol{A})^\intercal$.

Integration by parts can be applied to row-wise differentiations as well as column-wise ones. For example, we shall frequently use

$$(\nabla \times \boldsymbol{\tau}, \boldsymbol{\sigma})_K = (\boldsymbol{\tau}, \nabla \times \boldsymbol{\sigma})_K + (\boldsymbol{n} \times \boldsymbol{\tau}, \boldsymbol{\sigma})_{\partial K},$$
$$(\boldsymbol{\tau} \times \nabla, \boldsymbol{\sigma})_K = (\boldsymbol{\tau}, \boldsymbol{\sigma} \times \nabla)_K + (\boldsymbol{\tau} \times \boldsymbol{n}, \boldsymbol{\sigma})_{\partial K}.$$

Similar formulae hold for grad, curl, div operators. Be careful on the possible sign change and the transpose operator when letter differential operators and ∇ operators are mixed together. Chain rules and product rules are also better used in the same category of differential operations (row-wise, column-wise or letter operators).

Denote the space of all 3×3 matrices by $\mathbb{M}$, all symmetric 3×3 matrices by $\mathbb{S}$, and all skew-symmetric 3×3 matrices by $\mathbb{K}$. For any matrix $\boldsymbol{B} \in \mathbb{M}$, we can decompose it into symmetric and skew-symmetric parts as

$$\boldsymbol{B} = \operatorname{sym}(\boldsymbol{B}) + \operatorname{skw}(\boldsymbol{B}) := \frac{1}{2}(\boldsymbol{B} + \boldsymbol{B}^\intercal) + \frac{1}{2}(\boldsymbol{B} - \boldsymbol{B}^\intercal).$$

The symmetric gradient of a vector function $\boldsymbol{u}$ is defined as

$$\operatorname{def} \boldsymbol{u} := \operatorname{sym} \nabla \boldsymbol{u} = \frac{1}{2}(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^{\intercal}) = \frac{1}{2}(\boldsymbol{u}\nabla + \nabla \boldsymbol{u}).$$

In the last identity, the dyadic product is used to emphasize the symmetry in notation. In the context of elasticity, $\operatorname{def} \boldsymbol{u}$ is commonly denoted by $\varepsilon(\boldsymbol{u})$.

We define an isomorphism from $\mathbb{R}^3$ to the space of skew-symmetric matrices $\mathbb{K}$ as follows: for a vector $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)^{\intercal} \in \mathbb{R}^3$,

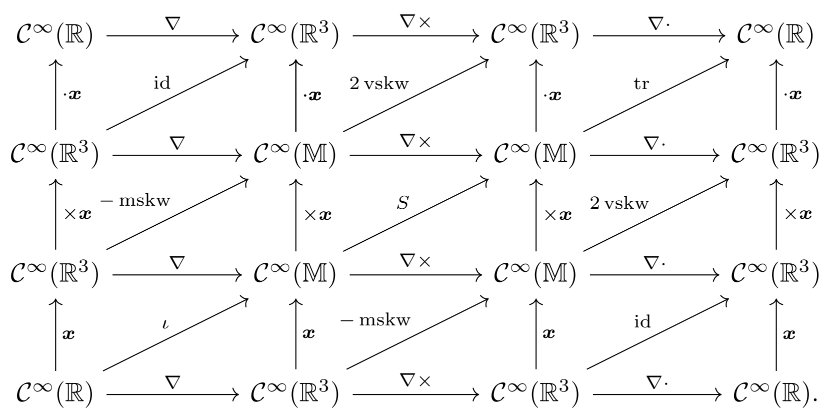
$$\operatorname{mskw} \boldsymbol{\omega} := \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}.$$

Obviously $\operatorname{mskw} : \mathbb{R}^3 \to \mathbb{K}$ is a bijection. We define $\operatorname{vskw} : \mathbb{M} \to \mathbb{R}^3$ by $\operatorname{vskw} := \operatorname{mskw}^{-1} \circ \operatorname{skw}$. Using these notation, we have the decomposition

$$\operatorname{grad} \boldsymbol{v} = \operatorname{def} \boldsymbol{v} + \frac{1}{2} \operatorname{mskw}(\nabla \times \boldsymbol{v}). \tag{3}$$

2.2. **Identities on tensors.** We shall present identities based on diagram (4) and refer to [7] for a unified proof. Let $S\boldsymbol{\tau} = \boldsymbol{\tau}^{\intercal} - \operatorname{tr}(\boldsymbol{\tau})\boldsymbol{I}$ and $\iota : \mathbb{R} \to \mathbb{M}$ by $\iota v = v\boldsymbol{I}$.

(4)



The north-east diagonal operator is the Poisson bracket $[\mathrm{d}, \kappa] = \mathrm{d}((\cdot)\kappa) - (\mathrm{d}(\cdot))\kappa$ for $\mathrm{d} = \nabla, \nabla\times, \nabla\cdot$ being applied from the left and the Koszul operator $\kappa = \boldsymbol{x}, \times\boldsymbol{x}, \cdot\boldsymbol{x}$ applied from the right. For example, we have

$$\begin{aligned} \nabla \times (\boldsymbol{\tau} \cdot \boldsymbol{x}) - (\nabla \times \boldsymbol{\tau}) \cdot \boldsymbol{x} &= 2 \operatorname{vskw} \boldsymbol{\tau}, \qquad \text{block } (1,2), \\ \nabla(\boldsymbol{u} \times \boldsymbol{x}) - (\nabla \boldsymbol{u}) \times \boldsymbol{x} &= -\operatorname{mskw} \boldsymbol{u}, \quad \text{block } (2,1). \end{aligned} \tag{5}$$

The parallelogram formed by the north-east diagonal and the horizontal operators is anticomutative. For example, we will use the following identities:

$$\begin{aligned} &\operatorname{tr}(\nabla \times \boldsymbol{\tau}) = -\nabla \cdot 2 \operatorname{vskw}(\boldsymbol{\tau}), \qquad \text{block } (1,2), \\ &2\operatorname{vskw} \nabla \boldsymbol{u} = -\nabla \times \boldsymbol{u}, \\ &\quad \nabla \times \boldsymbol{u} = \nabla \cdot \operatorname{mskw}(\boldsymbol{u}). \end{aligned} \tag{6}$$

Taking transpose, we can get similar formulae for row-wise differential operators. By replacing ∂_i by x_i, we can get the anticomutativity of the parallelograms formed by the vertical and the north-east diagonal operators. For example, (6) becomes

$$\operatorname{tr}(\boldsymbol{\tau} \times \boldsymbol{x}) = -2 \operatorname{vskw}(\boldsymbol{\tau}) \cdot \boldsymbol{x}. \tag{7}$$

2.3. **Tensors on surfaces.** Given a plane F with normal vector $\boldsymbol{n}$, for a vector $\boldsymbol{v} \in \mathbb{R}^3$, we define its projection to plane F

$$\Pi_F \boldsymbol{v} := (\boldsymbol{n} \times \boldsymbol{v}) \times \boldsymbol{n} = \boldsymbol{n} \times (\boldsymbol{v} \times \boldsymbol{n}) = -\boldsymbol{n} \times (\boldsymbol{n} \times \boldsymbol{v}) = (\boldsymbol{I} - \boldsymbol{n}\boldsymbol{n}^{\intercal})\boldsymbol{v},$$

which is called the tangential component of $\boldsymbol{v}$. The vector

$$\Pi_F^{\perp} \boldsymbol{v} := \boldsymbol{n} \times \boldsymbol{v} = (\boldsymbol{n} \times \Pi_F)\boldsymbol{v}$$

is called the tangential trace of $\boldsymbol{v}$, which is a rotation of $\Pi_F \boldsymbol{v}$ on F (90° counterclockwise with respect to $\boldsymbol{n}$). Note that $\Pi_F = \boldsymbol{I} - \boldsymbol{n}\boldsymbol{n}^{\intercal}$ is a 3×3 symmetric matrix. With a slight abuse of notation, we use Π_F to denote the piecewise defined projection to the boundary of K.

We treat the Hamilton operator $\nabla = (\partial_1, \partial_2, \partial_3)^{\intercal}$ as a column vector and define

$$\nabla_F := \Pi_F \nabla, \quad \nabla_F^{\perp} := \Pi_F^{\perp} \nabla.$$

We have the decomposition

$$\nabla = \nabla_F + \boldsymbol{n}\, \partial_n.$$

For a scalar function v,

$$\begin{aligned} \nabla_F v &= \Pi_F(\nabla v) = -\boldsymbol{n} \times (\boldsymbol{n} \times \nabla v), \\ \nabla_F^{\perp} v &= \boldsymbol{n} \times \nabla v = \boldsymbol{n} \times \nabla_F v, \end{aligned}$$

are the surface gradient of v and surface curl, respectively. For a vector function $\boldsymbol{v}$, $\nabla_F \cdot \boldsymbol{v}$ is the surface divergence:

$$\operatorname{div}_F \boldsymbol{v} := \nabla_F \cdot \boldsymbol{v} = \nabla_F \cdot (\Pi_F \boldsymbol{v}).$$

By the cyclic invariance of the mix product and the fact $\boldsymbol{n}$ is constant, the surface rot operator is

$$\operatorname{rot}_F \boldsymbol{v} := \nabla_F^{\perp} \cdot \boldsymbol{v} = (\boldsymbol{n} \times \nabla) \cdot \boldsymbol{v} = \boldsymbol{n} \cdot (\nabla \times \boldsymbol{v}), \tag{8}$$

which is the normal component of $\nabla \times \boldsymbol{v}$. The tangential trace of $\nabla \times \boldsymbol{v}$ is

$$\boldsymbol{n} \times (\nabla \times \boldsymbol{v}) = \nabla(\boldsymbol{n} \cdot \boldsymbol{v}) - \partial_n \boldsymbol{v} = \nabla_F(\boldsymbol{n} \cdot \boldsymbol{v}) - \partial_n(\Pi_F \boldsymbol{v}). \tag{9}$$

By definition, for a vector function $\boldsymbol{v}$,

$$\operatorname{rot}_F \boldsymbol{v} = \nabla_F^{\perp} \cdot \boldsymbol{v} = -\nabla_F \cdot (\boldsymbol{n} \times \boldsymbol{v}), \quad \operatorname{div}_F \boldsymbol{v} = \nabla_F \cdot \boldsymbol{v} = \nabla_F^{\perp} \cdot (\boldsymbol{n} \times \boldsymbol{v}).$$

We define, for a vector function $\boldsymbol{v}$,

$$\begin{aligned} &\nabla_F \boldsymbol{v} := \nabla_F \boldsymbol{v}^{\intercal} = \Pi_F \nabla \boldsymbol{v}^{\intercal}, \quad \operatorname{grad}_F \boldsymbol{v} = \boldsymbol{v} \nabla_F = (\nabla_F \boldsymbol{v})^{\intercal}, \\ &\nabla_F^{\perp} \boldsymbol{v} := \nabla_F^{\perp} \boldsymbol{v}^{\intercal} = \boldsymbol{n} \times (\nabla \boldsymbol{v}^{\intercal}), \quad \operatorname{curl}_F \boldsymbol{v} := \boldsymbol{v} \nabla_F^{\perp} = (\nabla_F^{\perp} \boldsymbol{v})^{\intercal}, \\ &\operatorname{def}_F \boldsymbol{v} := \operatorname{sym}(\nabla_F \boldsymbol{v}), \quad \operatorname{sym}\operatorname{curl}_F \boldsymbol{v} := \operatorname{sym}(\operatorname{curl}_F \boldsymbol{v}). \end{aligned}$$

For a tensor function $\boldsymbol{\tau}$,

$$\begin{aligned} \operatorname{div}_F \boldsymbol{\tau} := \boldsymbol{\tau} \cdot \nabla_F = (\nabla_F \cdot \boldsymbol{\tau}^{\intercal})^{\intercal}, &\quad \operatorname{div}_F \operatorname{div}_F \boldsymbol{\tau} := \nabla_F \cdot \boldsymbol{\tau} \cdot \nabla_F, \\ \operatorname{rot}_F \boldsymbol{\tau} := \boldsymbol{\tau} \cdot (\boldsymbol{n} \times \nabla) = (\nabla_F^{\perp} \cdot \boldsymbol{\tau}^{\intercal})^{\intercal}, &\quad \operatorname{rot}_F \operatorname{rot}_F \boldsymbol{\tau} := \nabla_F^{\perp} \cdot \boldsymbol{\tau} \cdot \nabla_F^{\perp}. \end{aligned}$$

Although we define the surface differentiation through the projection, it can be verified that the definition is intrinsic in the sense that it depends only on the function value on the surface F. Namely $\nabla_F v = \nabla_F(v|_F)$, $\nabla_F \cdot \boldsymbol{v} = \nabla_F \cdot \Pi_F \boldsymbol{v}$, $\nabla_F \boldsymbol{v} = \nabla_F(\boldsymbol{v}|_F)$ and thus Π_F is sometimes skipped after ∇_F.

3. Polynomial Complexes

In this section we consider polynomial elasticity complexes on a bounded and topologically trivial domain $D \subset \mathbb{R}^3$. Without loss of generality, we assume $\mathbf{0} = (0,0,0) \in D$ which can be easily satisfied by changing of variable $\boldsymbol{x} - \boldsymbol{x}_c$ with an arbitrary $\boldsymbol{x}_c \in D$.

Given a non-negative integer k, let $\mathbb{P}_k(D)$ stand for the set of all polynomials in D with the total degree no more than k, and $\mathbb{P}_k(D;\mathbb{X})$ denote the tensor or vector version for $\mathbb{X} = \mathbb{S}, \mathbb{K}, \mathbb{M}$, or $\mathbb{R}^3$. Similar notation will be applied to a two-dimensional face F and one-dimensional edge e. Let Q_D^k be the L^2-orthogonal projection operator onto $\mathbb{P}_k(D;\mathbb{X})$. Let $\mathbb{H}_k(D;\mathbb{X}) := \mathbb{P}_k(D;\mathbb{X})\backslash\mathbb{P}_{k-1}(D;\mathbb{X})$ be the space of homogeneous polynomials of degree k.

Recall that $\dim \mathbb{P}_k(D) = \binom{k+d}{d}$ for a d-dimensional domain D, $\dim \mathbb{M} = 9, \dim \mathbb{S} = 6$, and $\dim \mathbb{K} = 3$. We list a useful result in [16]

$$\mathbb{P}_k(D) \cap \ker(\ell I + \boldsymbol{x}\cdot\nabla) = 0 \tag{10}$$

for any positive number ℓ, where I is the identity operator.

3.1. Polynomial elasticity complex.

The polynomial de Rham complex is

$$\mathbb{R} \xrightarrow{\subseteq} \mathbb{P}_{k+1}(D) \xrightarrow{\nabla} \mathbb{P}_k(D;\mathbb{R}^3) \xrightarrow{\nabla\times} \mathbb{P}_{k-1}(D;\mathbb{R}^3) \xrightarrow{\nabla\cdot} \mathbb{P}_{k-2}(D) \to 0. \tag{11}$$

As D is topologically trivial, complex (11) is also exact, which means the range of each map is the kernel of the succeeding map.

For later use, recall the following polynomial elasticity complex in [5, (2.6)]

$$\boldsymbol{RM} \xrightarrow{\subseteq} \mathbb{P}_{k+1}(D;\mathbb{R}^3) \xrightarrow{\operatorname{def}} \mathbb{P}_k(D;\mathbb{S}) \xrightarrow{\operatorname{inc}} \mathbb{P}_{k-2}(D;\mathbb{S}) \xrightarrow{\operatorname{div}} \mathbb{P}_{k-3}(D;\mathbb{R}^3) \to \mathbf{0}, \tag{12}$$

where the linearized rigid body motion

$$\boldsymbol{RM} = \{\boldsymbol{a}\times\boldsymbol{x} + \boldsymbol{b} : \boldsymbol{a},\boldsymbol{b} \in \mathbb{R}^3\} = \{\boldsymbol{N}\boldsymbol{x} + \boldsymbol{b} : \boldsymbol{N} \in \mathbb{K}, \boldsymbol{b} \in \mathbb{R}^3\}. \tag{13}$$

Complex (12) is an exact sequence for a topologically trivial domain D. In the following, we give a more precise characterization of the div operator.

Lemma 3.1. $\operatorname{div} : \operatorname{sym}(\boldsymbol{x}\mathbb{P}_{k-3}(D;\mathbb{R}^3)) \to \mathbb{P}_{k-3}(D;\mathbb{R}^3)$ *is bijective.*

Proof. As $\operatorname{div}(\operatorname{sym}(\boldsymbol{x}\mathbb{P}_{k-3}(D;\mathbb{R}^3))) \subseteq \mathbb{P}_{k-3}(D;\mathbb{R}^3)$ and $\dim \operatorname{sym}(\boldsymbol{x}\mathbb{P}_{k-3}(D;\mathbb{R}^3)) = \dim \mathbb{P}_{k-3}(D;\mathbb{R}^3)$, it is sufficient to prove $\operatorname{sym}(\boldsymbol{x}\mathbb{P}_{k-3}(D;\mathbb{R}^3)) \cap \ker(\operatorname{div}) = \{\mathbf{0}\}$. That is: for any $\boldsymbol{q} \in \mathbb{P}_{k-3}(D;\mathbb{R}^3)$ satisfying $\operatorname{div}\operatorname{sym}(\boldsymbol{x}\boldsymbol{q}^\intercal) = \mathbf{0}$, we are going to prove $\boldsymbol{q} = \mathbf{0}$.

By a direct computation,

$$\begin{aligned}
\operatorname{div}(\boldsymbol{q}\boldsymbol{x}^\intercal) &= (\boldsymbol{q}\boldsymbol{x}^\intercal)\cdot\nabla = \boldsymbol{q}(\boldsymbol{x}\cdot\nabla) + (\boldsymbol{q}\nabla)\cdot\boldsymbol{x} = 3\boldsymbol{q} + (\operatorname{grad}\boldsymbol{q})\cdot\boldsymbol{x},\\
\operatorname{div}(\boldsymbol{x}\boldsymbol{q}^\intercal) &= (\boldsymbol{x}\boldsymbol{q}^\intercal)\cdot\nabla = \boldsymbol{x}(\boldsymbol{q}\cdot\nabla) + (\boldsymbol{x}\nabla)\cdot\boldsymbol{q} = (\operatorname{div}\boldsymbol{q})\boldsymbol{x} + \boldsymbol{q},\\
2\operatorname{div}\operatorname{sym}(\boldsymbol{x}\boldsymbol{q}^\intercal) &= 4\boldsymbol{q} + (\operatorname{grad}\boldsymbol{q})\cdot\boldsymbol{x} + (\operatorname{div}\boldsymbol{q})\boldsymbol{x}.
\end{aligned}$$

It follows from $\operatorname{div}\operatorname{sym}(\boldsymbol{x}\boldsymbol{q}) = \mathbf{0}$ that

$$4\boldsymbol{q} + (\operatorname{grad}\boldsymbol{q})\cdot\boldsymbol{x} = -(\operatorname{div}\boldsymbol{q})\boldsymbol{x}. \tag{14}$$

Since $\operatorname{div}((\operatorname{grad}\boldsymbol{q})\cdot\boldsymbol{x}) = (I + \boldsymbol{x}\cdot\operatorname{grad})\operatorname{div}\boldsymbol{q}$, applying the divergence operator on both side of (14) yields

$$(5I + \boldsymbol{x}\cdot\operatorname{grad})\operatorname{div}\boldsymbol{q} = -(3I + \boldsymbol{x}\cdot\operatorname{grad})\operatorname{div}\boldsymbol{q}.$$

Hence we acquire from (10) that $\operatorname{div}\boldsymbol{q}=0$, and (14) reduces to

$$4\boldsymbol{q}+(\operatorname{grad}\boldsymbol{q})\cdot\boldsymbol{x}=\boldsymbol{0}\quad\forall\,\boldsymbol{x}\in D.$$

Applying (10) again gives $\boldsymbol{q}=\boldsymbol{0}$. □

3.2. **Koszul elasticity complex.** Recall the Koszul complex

$$0\to\mathbb{P}_{k-2}(D)\xrightarrow{\phi\boldsymbol{x}}\mathbb{P}_{k-1}(D;\mathbb{R}^3)\xrightarrow{\boldsymbol{u}\times\boldsymbol{x}}\mathbb{P}_{k}(D;\mathbb{R}^3)\xrightarrow{\boldsymbol{v}\cdot\boldsymbol{x}}\mathbb{P}_{k+1}(D)\to 0,\tag{15}$$

which is also an exact sequence.

Define operator $\boldsymbol{\pi}_{RM}:\mathcal{C}^1(D;\mathbb{R}^3)\to\boldsymbol{RM}$ as

$$\boldsymbol{\pi}_{RM}\boldsymbol{v}:=\boldsymbol{v}(\boldsymbol{0})+\frac{1}{2}(\operatorname{curl}\boldsymbol{v})(\boldsymbol{0})\times\boldsymbol{x}.$$

By a direct calculation $\nabla\times(\boldsymbol{a}\times\boldsymbol{x})=2\boldsymbol{a}$ and the definition of $\boldsymbol{RM}$, cf. (13), it holds

$$\boldsymbol{\pi}_{RM}\boldsymbol{v}=\boldsymbol{v}\quad\forall\,\boldsymbol{v}\in\boldsymbol{RM}.\tag{16}$$

Lemma 3.2. *The following polynomial space sequence*

$$\boldsymbol{0}\xrightarrow{\subset}\mathbb{P}_{k-3}(D;\mathbb{R}^3)\xrightarrow{\operatorname{sym}(\boldsymbol{v}\boldsymbol{x})}\mathbb{P}_{k-2}(D;\mathbb{S})\xrightarrow{\boldsymbol{x}\times\boldsymbol{\tau}\times\boldsymbol{x}}\mathbb{P}_{k}(D;\mathbb{S})\xrightarrow{\boldsymbol{\tau}\cdot\boldsymbol{x}}\mathbb{P}_{k+1}(D;\mathbb{R}^3)\xrightarrow{\boldsymbol{\pi}_{RM}}\boldsymbol{RM}\longrightarrow\boldsymbol{0}\tag{17}$$

is a complex and is exact.

Proof. We first verify (17) is a complex. For any $\boldsymbol{v}\in\mathbb{P}_{k-3}(D;\mathbb{R}^3)$ and $\boldsymbol{\tau}\in\mathbb{P}_{k-2}(D;\mathbb{S})$, we have

$$\begin{aligned}\boldsymbol{x}\times\operatorname{sym}(\boldsymbol{v}\boldsymbol{x})\times\boldsymbol{x}&=\frac{1}{2}\boldsymbol{x}\times(\boldsymbol{x}\boldsymbol{v}+\boldsymbol{v}\boldsymbol{x})\times\boldsymbol{x}=\boldsymbol{0},\\(\boldsymbol{x}\times\boldsymbol{\tau}\times\boldsymbol{x})\cdot\boldsymbol{x}&=\boldsymbol{0}.\end{aligned}$$

As $\boldsymbol{\tau}\in\mathbb{P}_k(D;\mathbb{S})$, (5) implies $\nabla\times(\boldsymbol{\tau}\cdot\boldsymbol{x})=(\nabla\times\boldsymbol{\tau})\cdot\boldsymbol{x}$, we get $\boldsymbol{\pi}_{RM}(\boldsymbol{\tau}\cdot\boldsymbol{x})=\boldsymbol{0}$. Thus (17) is a complex.

We now verify the exactness.

1. If $\boldsymbol{x}\times\boldsymbol{\tau}\times\boldsymbol{x}=\boldsymbol{0}$ and $\boldsymbol{\tau}\in\mathbb{P}_{k-2}(D;\mathbb{S})$, then $\boldsymbol{\tau}=\operatorname{sym}(\boldsymbol{v}\boldsymbol{x})$ for some $\boldsymbol{v}\in\mathbb{P}_{k-3}(D;\mathbb{R}^3)$.

For any $\boldsymbol{\tau}\in\mathbb{P}_{k-2}(D;\mathbb{S})$ satisfying $\boldsymbol{x}\times(\boldsymbol{\tau}\times\boldsymbol{x})=\boldsymbol{0}$, by the exactness of Koszul complex (15), there exists $\widetilde{\boldsymbol{v}}\in\mathbb{P}_{k-2}(D;\mathbb{R}^3)$ such that $\boldsymbol{\tau}\times\boldsymbol{x}=\boldsymbol{x}\widetilde{\boldsymbol{v}}$. By (7), as $\boldsymbol{\tau}$ is symmetric, $\boldsymbol{\tau}\times\boldsymbol{x}$ is trace-free. Then it follows $\widetilde{\boldsymbol{v}}\cdot\boldsymbol{x}=\operatorname{tr}(\boldsymbol{x}\widetilde{\boldsymbol{v}})=\operatorname{tr}(\boldsymbol{\tau}\times\boldsymbol{x})=0$. Then there exists $\boldsymbol{v}_1\in\mathbb{P}_{k-3}(D;\mathbb{R}^3)$ such that $\widetilde{\boldsymbol{v}}=\boldsymbol{v}_1\times\boldsymbol{x}$. As a result, we have

$$(\boldsymbol{\tau}-\boldsymbol{x}\boldsymbol{v}_1)\times\boldsymbol{x}=\boldsymbol{\tau}\times\boldsymbol{x}-\boldsymbol{x}(\boldsymbol{v}_1\times\boldsymbol{x})=\boldsymbol{\tau}\times\boldsymbol{x}-\boldsymbol{x}\widetilde{\boldsymbol{v}}=\boldsymbol{0}.$$

Again there exists $\boldsymbol{v}_2\in\mathbb{P}_{k-3}(D;\mathbb{R}^3)$ such that $\boldsymbol{\tau}=\boldsymbol{x}\boldsymbol{v}_1+\boldsymbol{v}_2\boldsymbol{x}$. By the symmetry of $\boldsymbol{\tau}$, it holds $\boldsymbol{\tau}=\operatorname{sym}(\boldsymbol{x}(\boldsymbol{v}_1+\boldsymbol{v}_2))$.

2. If $\boldsymbol{\tau}\cdot\boldsymbol{x}=\boldsymbol{0}$ and $\boldsymbol{\tau}\in\mathbb{P}_k(D;\mathbb{S})$, then $\boldsymbol{\tau}=\boldsymbol{x}\times\boldsymbol{\sigma}\times\boldsymbol{x}$ for some $\boldsymbol{\sigma}\in\mathbb{P}_{k-2}(D;\mathbb{S})$.

For any $\boldsymbol{\tau}\in\mathbb{P}_k(D;\mathbb{S})$ satisfying $\boldsymbol{\tau}\cdot\boldsymbol{x}=\boldsymbol{0}$, by the exactness of Koszul complex (15), there exists $\boldsymbol{\tau}_1\in\mathbb{P}_{k-1}(D;\mathbb{M})$ such that $\boldsymbol{\tau}=\boldsymbol{\tau}_1\times\boldsymbol{x}$. By the symmetry of $\boldsymbol{\tau}$, it holds

$$(\boldsymbol{x}\cdot\boldsymbol{\tau}_1)\times\boldsymbol{x}=\boldsymbol{x}\cdot(\boldsymbol{\tau}_1\times\boldsymbol{x})=\boldsymbol{x}\cdot\boldsymbol{\tau}=(\boldsymbol{\tau}\cdot\boldsymbol{x})^{\intercal}=\boldsymbol{0}.$$

Thus there exists $q\in\mathbb{P}_{k-1}(D)$ satisfying $\boldsymbol{x}\cdot\boldsymbol{\tau}_1=q\boldsymbol{x}$, i.e. $\boldsymbol{x}\cdot(\boldsymbol{\tau}_1-q\boldsymbol{I})=\boldsymbol{0}$. Again there exists $\boldsymbol{\tau}_2\in\mathbb{P}_{k-2}(D;\mathbb{M})$ satisfying $\boldsymbol{\tau}_1=q\boldsymbol{I}+\boldsymbol{x}\times\boldsymbol{\tau}_2$. Hence

$$\boldsymbol{\tau}=q\boldsymbol{I}\times\boldsymbol{x}+\boldsymbol{x}\times\boldsymbol{\tau}_2\times\boldsymbol{x}.$$

It follows from the symmetry of $\boldsymbol{\tau}$ that

$$\boldsymbol{\tau} = \operatorname{sym}\boldsymbol{\tau} = \operatorname{sym}(q\boldsymbol{I}\times\boldsymbol{x} + \boldsymbol{x}\times\boldsymbol{\tau}_2\times\boldsymbol{x}) = \operatorname{sym}(\boldsymbol{x}\times\boldsymbol{\tau}_2\times\boldsymbol{x}) = \boldsymbol{x}\times\operatorname{sym}\boldsymbol{\tau}_2\times\boldsymbol{x}.$$

Here we use the fact that $\boldsymbol{x}\times\operatorname{skw}\boldsymbol{\tau}_2\times\boldsymbol{x}\in\mathbb{P}_k(D;\mathbb{K})$.

3. $\mathbb{P}_k(D;\mathbb{S})\cdot\boldsymbol{x} = \mathbb{P}_{k+1}(D;\mathbb{R}^3)\cap\ker(\pi_{RM})$.
   As a result of step 1,

$$\dim(\boldsymbol{x}\times\mathbb{P}_{k-2}(D;\mathbb{S})\times\boldsymbol{x}) = \dim\mathbb{P}_{k-2}(D;\mathbb{S}) - \dim\mathbb{P}_{k-3}(D;\mathbb{R}^3) = \frac{1}{2}k(k-1)(k+4).$$

Then we get from step 2 that

$$\begin{aligned}\dim(\mathbb{P}_k(D;\mathbb{S})\cdot\boldsymbol{x}) &= \dim\mathbb{P}_k(D;\mathbb{S}) - \dim(\boldsymbol{x}\times\mathbb{P}_{k-2}(D;\mathbb{S})\times\boldsymbol{x})\\ &= (k+3)(k+2)(k+1) - \frac{1}{2}k(k-1)(k+4)\\ &= \frac{1}{2}(k+4)(k+3)(k+2) - 6. \end{aligned} \tag{18}$$

It follows from (16) that $\boldsymbol{\pi}_{RM}\mathbb{P}_{k+1}(D;\mathbb{R}^3) = \boldsymbol{RM}$, and by (18),

$$\dim(\mathbb{P}_k(D;\mathbb{S})\cdot\boldsymbol{x}) + \dim\boldsymbol{RM} = \dim\mathbb{P}_{k+1}(D;\mathbb{R}^3).$$

Therefore the complex (17) is exact. ☐

Remark 3.3. *Another Koszul elasticity complex*

$$\boldsymbol{0}\xrightarrow{\subset}\mathbb{H}_{k-3}(D;\mathbb{R}^3)\xrightarrow{\mathcal{K}_3}\mathbb{H}_{k-2}(D;\mathbb{S})\xrightarrow{\mathcal{K}_2}\mathbb{H}_k(D;\mathbb{S})\xrightarrow{\mathcal{K}_1}\mathbb{H}_{k+1}(D;\mathbb{R}^3)\longrightarrow\boldsymbol{RM}\longrightarrow\boldsymbol{0}$$

has been constructed in [23, Section 3.2] *by using different Koszul operators*

$$\begin{aligned}\mathcal{K}_1\boldsymbol{\tau} &= \frac{1}{k+1}\boldsymbol{\tau}\cdot\boldsymbol{x} - \frac{1}{k(k+1)}\boldsymbol{x}\times(\nabla\times\boldsymbol{\tau})\cdot\boldsymbol{x},\\ \mathcal{K}_2\boldsymbol{\tau} &= \frac{1}{k(k+1)}\boldsymbol{x}\times\boldsymbol{\tau}\times\boldsymbol{x},\\ \mathcal{K}_3\boldsymbol{v} &= \frac{1}{k}\operatorname{sym}(\boldsymbol{v}\boldsymbol{x}) - \frac{1}{k(k+1)}\operatorname{sym}((\boldsymbol{x}\boldsymbol{v}^{\intercal}\times\boldsymbol{x})\times\nabla),\end{aligned}$$

which satisfy homotopy identities

$$\begin{aligned}\mathcal{K}_1\operatorname{def}\boldsymbol{v} &= \boldsymbol{v} \quad \forall\ \boldsymbol{v}\in\mathbb{H}_{k+1}(D;\mathbb{R}^3), k\geq 1,\\ \operatorname{def}\mathcal{K}_1\boldsymbol{\tau} + \mathcal{K}_2\operatorname{inc}\boldsymbol{\tau} &= \boldsymbol{\tau} \quad \forall\ \boldsymbol{\tau}\in\mathbb{H}_k(D;\mathbb{S}),\\ \operatorname{inc}\mathcal{K}_2\boldsymbol{\tau} + \mathcal{K}_3\operatorname{div}\boldsymbol{\tau} &= \boldsymbol{\tau} \quad \forall\ \boldsymbol{\tau}\in\mathbb{H}_{k-2}(D;\mathbb{S}),\\ \operatorname{div}\mathcal{K}_3\boldsymbol{v} &= \boldsymbol{v} \quad \forall\ \boldsymbol{v}\in\mathbb{H}_{k-3}(D;\mathbb{R}^3).\end{aligned}$$

Ours are simpler but without homotopy identities.

3.3. **Decomposition of polynomial tensor spaces.** Combining the two complexes (12) and (17) yields

$$\boldsymbol{RM}\underset{\pi_{RM}}{\overset{\subset}{\rightleftarrows}}\mathbb{P}_{k+1}(D;\mathbb{R}^3)\underset{\boldsymbol{\tau}\cdot\boldsymbol{x}}{\overset{\operatorname{def}}{\rightleftarrows}}\mathbb{P}_k(D;\mathbb{S})\underset{\boldsymbol{x}\times\boldsymbol{\tau}\times\boldsymbol{x}}{\overset{\operatorname{inc}}{\rightleftarrows}}\mathbb{P}_{k-2}(D;\mathbb{S})\underset{\operatorname{sym}(\boldsymbol{v}\boldsymbol{x})}{\overset{\operatorname{div}}{\rightleftarrows}}\mathbb{P}_{k-3}(D;\mathbb{R}^3)\underset{\supset}{\rightleftarrows}\boldsymbol{0}\,. \tag{19}$$

Although there are no homotopy identities, from (19), we can derive the following space decompositions which play a vital role in the design of degrees of freedom.

Lemma 3.4. *We have the following space decompositions*

$$\mathbb{P}_{k+1}(D;\mathbb{R}^3) = (\mathbb{P}_k(D;\mathbb{S})\cdot\boldsymbol{x}) \oplus \boldsymbol{RM}, \tag{20}$$

$$\mathbb{P}_k(D;\mathbb{S}) = \operatorname{def}\mathbb{P}_{k+1}(D;\mathbb{R}^3) \oplus (\boldsymbol{x}\times\mathbb{P}_{k-2}(D;\mathbb{S})\times\boldsymbol{x}), \tag{21}$$

$$\mathbb{P}_{k-2}(D;\mathbb{S}) = \operatorname{inc}\mathbb{P}_k(D;\mathbb{S}) \oplus \operatorname{sym}(\mathbb{P}_{k-3}(D;\mathbb{R}^3)\boldsymbol{x}). \tag{22}$$

Proof. The decomposition (20) is trivial by the exactness of (17).

For any $\boldsymbol{q}\in\mathbb{P}_{k+1}(D;\mathbb{R}^3)$ satisfying $\operatorname{def}\boldsymbol{q}\in\boldsymbol{x}\times\mathbb{P}_{k-2}(D;\mathbb{S})\times\boldsymbol{x}$, we have

$$(\nabla\boldsymbol{q}+(\nabla\boldsymbol{q})^\intercal)\cdot\boldsymbol{x} = 2(\operatorname{def}\boldsymbol{q})\cdot\boldsymbol{x} = \boldsymbol{0}.$$

Since $(\nabla\boldsymbol{q})\boldsymbol{x} = \nabla(\boldsymbol{x}^\intercal\boldsymbol{q})-\boldsymbol{q}$, we get

$$(\nabla\boldsymbol{q})^\intercal\cdot\boldsymbol{x}+\nabla(\boldsymbol{x}^\intercal\boldsymbol{q}) = \boldsymbol{q}. \tag{23}$$

Noting that

$$\boldsymbol{x}\cdot(\nabla\boldsymbol{q})^\intercal\cdot\boldsymbol{x} = \boldsymbol{x}\cdot(\operatorname{def}\boldsymbol{q})\cdot\boldsymbol{x} = 0,$$

we obtain from (23) that

$$(\boldsymbol{x}\cdot\nabla)(\boldsymbol{x}^\intercal\boldsymbol{q}) = \boldsymbol{x}^\intercal\boldsymbol{q}.$$

Hence $\boldsymbol{x}^\intercal\boldsymbol{q}$ is a linear function. In turn, it follows from (23) that $\boldsymbol{q}\in\mathbb{P}_1(D;\mathbb{R}^3)$, which together with the fact $\boldsymbol{x}^\intercal\boldsymbol{q}$ is linear implies $\boldsymbol{q}\in\boldsymbol{RM}$. Thus (21) follows from the fact that the dimensions on two sides of (21) coincide.

By Lemma 3.1, the sum in (22) is a direct sum. Thus the decomposition (22) follows. ☐

3.4. **Polynomial complexes in two dimensions.** We have similar polynomial complexes in two dimensions. Here we collect some which will appear as the trace complexes on face F of a polyhedron. Let $\boldsymbol{n}$ be a normal vector of F. For $\boldsymbol{x}\in F$, denote by $\boldsymbol{x}^\perp = \boldsymbol{n}\times\boldsymbol{x}$. Set $\boldsymbol{RT} := \mathbb{P}_0(F;\mathbb{R}^2)+\boldsymbol{x}\mathbb{P}_0(F)$. For a scalar function v,

$$\pi_1 v := v(0,0)+\boldsymbol{x}\cdot\nabla_F v(0,0).$$

Again, here without loss of generality, we assume $(0,0)\in F$ and in general the $\boldsymbol{x}$ in the results presented below can be replaced by $\boldsymbol{x}-\boldsymbol{x}_c$ with an arbitrary $\boldsymbol{x}_c\in F$.

The following div div polynomial complexes has been established in [14]:

$$\boldsymbol{RT} \underset{\boldsymbol{x}}{\overset{\subset}{\rightleftarrows}} \mathbb{P}_{k+1}(F;\mathbb{R}^2) \underset{\boldsymbol{\tau}\cdot\boldsymbol{x}^\perp}{\overset{\operatorname{sym}\operatorname{curl}_F}{\rightleftarrows}} \mathbb{P}_k(F;\mathbb{S}) \underset{v\boldsymbol{x}\boldsymbol{x}^\intercal}{\overset{\operatorname{div}_F\operatorname{div}_F}{\rightleftarrows}} \mathbb{P}_{k-2}(F) \underset{\supset}{\rightleftarrows} 0\,, \tag{24}$$

which implies the following decomposition

- $\mathbb{P}_{k+1}(F;\mathbb{R}^2) = (\mathbb{P}_k(F;\mathbb{S})\cdot\boldsymbol{x}^\perp)\oplus\boldsymbol{RT}$.
- $\mathbb{P}_k(F;\mathbb{S}) = \operatorname{sym}\operatorname{curl}_F\,\mathbb{P}_{k+1}(F;\mathbb{R}^2)\oplus\mathbb{P}_{k-2}(F)\boldsymbol{x}\boldsymbol{x}^\intercal$.
- $\operatorname{div}_F\operatorname{div}_F : \mathbb{P}_{k-2}(F)\boldsymbol{x}\boldsymbol{x}^\intercal\to\mathbb{P}_{k-2}(F)$ is a bijection.

The following two-dimensional Hessian polynomial complex and its Koszul complex can be also found in [14, Section 3.1]

$$\mathbb{P}_1(F) \underset{\pi_1}{\overset{\subset}{\rightleftarrows}} \mathbb{P}_{k+1}(F) \underset{\boldsymbol{x}\cdot\boldsymbol{\tau}\cdot\boldsymbol{x}}{\overset{\nabla_F^2}{\rightleftarrows}} \mathbb{P}_{k-1}(F;\mathbb{S}) \underset{\operatorname{sym}(\boldsymbol{x}^\perp\boldsymbol{v})}{\overset{\operatorname{rot}_F}{\rightleftarrows}} \mathbb{P}_{k-2}(F;\mathbb{R}^2) \underset{\supset}{\rightleftarrows} \boldsymbol{0}\,. \tag{25}$$

The implied decompositions are

- $\mathbb{P}_{k+1}(F) = (\boldsymbol{x}\cdot\mathbb{P}_{k-1}(F;\mathbb{S})\cdot\boldsymbol{x})\oplus\mathbb{P}_1(F)$.
- $\mathbb{P}_{k-1}(F;\mathbb{S}) = \nabla_F^2\,\mathbb{P}_{k+1}(F)\oplus\operatorname{sym}(\boldsymbol{x}^\perp\mathbb{P}_{k-2}(F;\mathbb{R}^2))$.
- $\operatorname{rot}_F : \operatorname{sym}(\boldsymbol{x}^\perp\mathbb{P}_{k-2}(F;\mathbb{R}^2))\to\mathbb{P}_{k-2}(F;\mathbb{R}^2)$ is a bijection.

4. Traces and Bubble Complexes

Besides the decomposition of polynomial spaces, another key of our construction is the characterization of the trace operator. We first derive a symmetric form of Green's identity for the inc operator from which we define two traces. We show the traces of spaces in the elasticity complex form two complexes on each face and will call them trace complexes. On the other hand, the kernel of traces in the polynomial space are called bubble polynomial function spaces, abbreviated as bubble spaces, which also form a complex and is called the bubble complexes. We also present several bubble complexes on each face.

When defining and studying the traces, we consider smooth enough functions not in the most general Sobolev spaces setting. The precise Sobolev spaces for the traces of the inc operator are not easy to identify and not necessary for our purposes, as the shape function is a polynomial being smooth inside the element.

4.1. Green's identity.

Consider $\boldsymbol{\sigma}, \boldsymbol{\tau} \in \boldsymbol{H}^2(K;\mathbb{S})$. By the symbolical symmetry, we expect the following symmetric form of the Green's identity

$$(\nabla\times\boldsymbol{\sigma}\times\nabla, \boldsymbol{\tau})_K - (\nabla\times\boldsymbol{\tau}\times\nabla, \boldsymbol{\sigma})_K = (\operatorname{tr}_1(\boldsymbol{\sigma}), \operatorname{tr}_2(\boldsymbol{\tau}))_{\partial K} - (\operatorname{tr}_1(\boldsymbol{\tau}), \operatorname{tr}_2(\boldsymbol{\sigma}))_{\partial K},$$

which belongs to a class of second Green's identities. For the scalar Laplacian operator, it reads as: for $u, v \in H^2(K)$,

$$-(\Delta u, v)_K + (\Delta v, u)_K = (\operatorname{tr}_1(u), \operatorname{tr}_2(v))_{\partial K} - (\operatorname{tr}_1(v), \operatorname{tr}_2(u))_{\partial K}.$$

where $\operatorname{tr}_1(u) = u$ is the Dirichlet trace and $\operatorname{tr}_2(v) = \partial_n v$ is the Neumann trace. For the double curl operator, we have a similar formula: for $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{H}^2(K;\mathbb{R}^3)$,

$$(\nabla\times(\nabla\times\boldsymbol{u}), \boldsymbol{v})_K - (\nabla\times(\nabla\times\boldsymbol{v}), \boldsymbol{u})_K = -(\operatorname{tr}_1(\boldsymbol{u}), \operatorname{tr}_2(\boldsymbol{v}))_{\partial K} + (\operatorname{tr}_1(\boldsymbol{v}), \operatorname{tr}_2(\boldsymbol{u}))_{\partial K}.$$

where $\operatorname{tr}_1(\boldsymbol{u}) = (\boldsymbol{n}\times\boldsymbol{u})\times\boldsymbol{n}$ is the tangential component of $\boldsymbol{u}$ (Dirichlet type) and $\operatorname{tr}_2(\boldsymbol{u}) = \boldsymbol{n}\times(\nabla\times\boldsymbol{u})$ is the Neumann type trace.

As $\boldsymbol{\sigma}$ is symmetric, $(\nabla\times\boldsymbol{\sigma})^{\intercal} = -\boldsymbol{\sigma}\times\nabla$. Therefore $(\nabla\times(\cdot), (\cdot)\times\nabla)$ is a symmetric bilinear form on $\boldsymbol{H}^1(K,\mathbb{S})$, i.e.,

$$(\nabla\times\boldsymbol{\sigma}, \boldsymbol{\tau}\times\nabla)_K = (\nabla\times\boldsymbol{\tau}, \boldsymbol{\sigma}\times\nabla)_K.$$

Applying integration by parts, we have

$$(\nabla\times\boldsymbol{\sigma}, \boldsymbol{\tau}\times\nabla)_K = (\nabla\times\boldsymbol{\sigma}\times\nabla, \boldsymbol{\tau})_K + (\nabla\times\boldsymbol{\sigma}, \boldsymbol{\tau}\times\boldsymbol{n})_{\partial K}, \tag{26}$$

$$(\nabla\times\boldsymbol{\tau}, \boldsymbol{\sigma}\times\nabla)_K = (\nabla\times\boldsymbol{\tau}\times\nabla, \boldsymbol{\sigma})_K + (\nabla\times\boldsymbol{\tau}, \boldsymbol{\sigma}\times\boldsymbol{n})_{\partial K}. \tag{27}$$

The difference between (26) and (27) implies the following Green's identity

$$(\nabla\times\boldsymbol{\sigma}\times\nabla, \boldsymbol{\tau})_K - (\boldsymbol{\sigma}, \nabla\times\boldsymbol{\tau}\times\nabla)_K = (\boldsymbol{\sigma}\times\boldsymbol{n}, \nabla\times\boldsymbol{\tau})_{\partial K} - (\nabla\times\boldsymbol{\sigma}, \boldsymbol{\tau}\times\boldsymbol{n})_{\partial K}.$$

But in this form, the trace $\boldsymbol{\sigma}\times\boldsymbol{n}$ and $\nabla\times\boldsymbol{\sigma}$ are still linearly dependent.

We further expand the boundary term into tangential and normal parts

$$(\boldsymbol{\sigma}\times\boldsymbol{n}, \nabla\times\boldsymbol{\tau})_{\partial K} = (\boldsymbol{n}\times\boldsymbol{\sigma}\times\boldsymbol{n}, \boldsymbol{n}\times(\nabla\times\boldsymbol{\tau}))_{\partial K} + (\boldsymbol{n}\cdot\boldsymbol{\sigma}\times\boldsymbol{n}, \boldsymbol{n}\cdot(\nabla\times\boldsymbol{\tau}))_{\partial K}.$$

Recall that, on one face $F \in \mathcal{F}(K)$, $\boldsymbol{n}\cdot(\nabla\times\boldsymbol{\tau}) = \nabla_F^{\perp}\cdot\Pi_F\boldsymbol{\tau} = -\nabla_F\cdot(\boldsymbol{n}\times\boldsymbol{\tau})$. Then we get from the integration by parts on face F that

$$
\begin{aligned}
(\boldsymbol{n}\cdot\boldsymbol{\sigma}\times\boldsymbol{n}, \nabla_F\cdot(\boldsymbol{n}\times\boldsymbol{\tau}))_F &= -(\nabla_F(\boldsymbol{n}\cdot\boldsymbol{\sigma}\times\boldsymbol{n}), \boldsymbol{n}\times\boldsymbol{\tau})_F \\
&\quad + \sum_{e\in\mathcal{E}(F)} (\boldsymbol{n}\cdot\boldsymbol{\sigma}\times\boldsymbol{n}, \boldsymbol{n}_{F,e}\cdot(\boldsymbol{n}\times\boldsymbol{\tau}))_e \\
&= (\nabla_F(\boldsymbol{n}\cdot\boldsymbol{\sigma}\,\Pi_F), \boldsymbol{n}\times\boldsymbol{\tau}\times\boldsymbol{n})_F \\
&\quad - \sum_{e\in\mathcal{E}(F)} (\boldsymbol{n}\cdot\boldsymbol{\sigma}\times\boldsymbol{n}, \boldsymbol{t}_{F,e}\cdot\boldsymbol{\tau})_e,
\end{aligned}
$$

where recall that $\boldsymbol{t}_{F,e} = \boldsymbol{n}\times\boldsymbol{n}_{F,e}$. Therefore we can write the boundary term as

$$
\begin{aligned}
(\boldsymbol{\sigma}\times\boldsymbol{n}, \nabla\times\boldsymbol{\tau})_{\partial K} &= (\boldsymbol{n}\times\boldsymbol{\sigma}\times\boldsymbol{n}, \boldsymbol{n}\times(\nabla\times\boldsymbol{\tau}))_{\partial K} \\
&\quad - \sum_{F\in\mathcal{F}(K)} (\nabla_F(\boldsymbol{n}\cdot\boldsymbol{\sigma}\,\Pi_F), \boldsymbol{n}\times\boldsymbol{\tau}\times\boldsymbol{n})_F \\
&\quad + \sum_{F\in\mathcal{F}(K)}\sum_{e\in\mathcal{E}(F)} (\boldsymbol{n}\cdot\boldsymbol{\sigma}\times\boldsymbol{n}, \boldsymbol{t}_{F,e}\cdot\boldsymbol{\tau})_e,
\end{aligned}
$$

and by symmetry

$$
\begin{aligned}
(\boldsymbol{\tau}\times\boldsymbol{n}, \nabla\times\boldsymbol{\sigma})_{\partial K} &= (\boldsymbol{n}\times\boldsymbol{\tau}\times\boldsymbol{n}, \boldsymbol{n}\times(\nabla\times\boldsymbol{\sigma}))_{\partial K} \\
&\quad - \sum_{F\in\mathcal{F}(K)} (\nabla_F(\boldsymbol{n}\cdot\boldsymbol{\tau}\,\Pi_F), \boldsymbol{n}\times\boldsymbol{\sigma}\times\boldsymbol{n})_F \\
&\quad + \sum_{F\in\mathcal{F}(K)}\sum_{e\in\mathcal{E}(F)} (\boldsymbol{n}\cdot\boldsymbol{\tau}\times\boldsymbol{n}, \boldsymbol{t}_{F,e}\cdot\boldsymbol{\sigma})_e.
\end{aligned}
$$

The difference of these two terms suggests us to define

$$
\begin{aligned}
\mathrm{tr}_1(\boldsymbol{\tau}) &:= \boldsymbol{n}\times\boldsymbol{\tau}\times\boldsymbol{n}, \\
\widetilde{\mathrm{tr}}_2(\boldsymbol{\tau}) &:= \boldsymbol{n}\times(\nabla\times\boldsymbol{\tau})\Pi_F + \nabla_F(\boldsymbol{n}\cdot\boldsymbol{\tau}\,\Pi_F).
\end{aligned}
$$

We can simplify the trace $\widetilde{\mathrm{tr}}_2(\boldsymbol{\tau})$ as follows. Apply $\Pi_F(\cdot)\Pi_F$ to the tangential trace of $\nabla\times\boldsymbol{\tau}$, cf. (9), to get

$$
\Pi_F(\boldsymbol{n}\times(\nabla\times\boldsymbol{\tau}))\Pi_F = \nabla_F(\boldsymbol{n}\cdot\boldsymbol{\tau}\Pi_F) - \Pi_F\partial_n\boldsymbol{\tau}\Pi_F. \tag{28}
$$

Because $\widetilde{\mathrm{tr}}_2(\boldsymbol{\tau})$ is integrated on the face with a tangential symmetric matrix $\boldsymbol{n}\times\boldsymbol{\sigma}\times\boldsymbol{n}$, it can be further simplified to $\mathrm{sym}\,\widetilde{\mathrm{tr}}_2(\boldsymbol{\tau})$. Therefore we define

$$
\mathrm{tr}_2(\boldsymbol{\tau}) := \mathrm{sym}\,\widetilde{\mathrm{tr}}_2(\boldsymbol{\tau}) = 2\mathrm{def}_F(\boldsymbol{n}\cdot\boldsymbol{\tau}\,\Pi_F) - \Pi_F\partial_n\,\boldsymbol{\tau}\,\Pi_F, \tag{29}
$$

which is a symmetric matrix on each face. Such trace has been identified in [5].

We present another form of tr_2 which is obtained by taking the transpose of the second term in $\widetilde{\mathrm{tr}}_2(\boldsymbol{\tau})$ and more useful than (29).

Lemma 4.1. *For any sufficiently smooth and symmetric tensor $\boldsymbol{\tau}$, it holds*

$$
\mathrm{tr}_2(\boldsymbol{\tau}) = \boldsymbol{n}\times(\nabla\times\boldsymbol{\tau})\Pi_F + (\Pi_F\boldsymbol{\tau}\cdot\boldsymbol{n})\nabla_F \tag{30}
$$

$$
= \Pi_F(\boldsymbol{\tau}\times\nabla)\times\boldsymbol{n} + \nabla_F(\boldsymbol{n}\cdot\boldsymbol{\tau}\,\Pi_F). \tag{31}
$$

Proof. We take the transpose of (28) and use the symmetry of $\boldsymbol{\tau}$ to get

$$
\Pi_F((\boldsymbol{\tau}\times\nabla)\times\boldsymbol{n})\Pi_F = (\Pi_F\boldsymbol{\tau}\cdot\boldsymbol{n})\nabla_F - \Pi_F\partial_n\boldsymbol{\tau}\Pi_F. \tag{32}
$$

The difference of (32) and (28) implies

$$\Pi_F(\boldsymbol{n}\times(\nabla\times\boldsymbol{\tau}))\Pi_F + (\Pi_F\boldsymbol{\tau}\cdot\boldsymbol{n})\nabla_F = \Pi_F((\boldsymbol{\tau}\times\nabla)\times\boldsymbol{n})\Pi_F + \nabla_F(\boldsymbol{n}\cdot\boldsymbol{\tau}\Pi_F).$$

As $\operatorname{tr}_2(\boldsymbol{\tau}) = \operatorname{sym}\widetilde{\operatorname{tr}}_2(\boldsymbol{\tau})$, we obtain (30). As $\boldsymbol{\tau}$ is symmetric, taking transpose, we obtain (31). ☐

We are in the position to summarize the symmetric form of Green's identity.

Theorem 4.2 (Symmetric Green's identity for the inc operator)**.** *Let K be a polyhedron, and let $\boldsymbol{\sigma},\boldsymbol{\tau}\in\boldsymbol{H}^2(K;\mathbb{S})$. Then we have*

$$\begin{aligned}(\nabla\times\boldsymbol{\sigma}\times\nabla,\boldsymbol{\tau})_K-(\boldsymbol{\sigma},\nabla\times\boldsymbol{\tau}\times\nabla)_K &= (\operatorname{tr}_1(\boldsymbol{\sigma}),\operatorname{tr}_2(\boldsymbol{\tau}))_{\partial K}-(\operatorname{tr}_2(\boldsymbol{\sigma}),\operatorname{tr}_1(\boldsymbol{\tau}))_{\partial K}\\ &\quad+\sum_{F\in\mathcal{F}(K)}\sum_{e\in\mathcal{E}(F)}(\boldsymbol{n}\cdot\boldsymbol{\sigma}\times\boldsymbol{n},\boldsymbol{t}_{F,e}\cdot\boldsymbol{\tau})_e\\ &\quad-\sum_{F\in\mathcal{F}(K)}\sum_{e\in\mathcal{E}(F)}(\boldsymbol{t}_{F,e}\cdot\boldsymbol{\sigma},\boldsymbol{n}\cdot\boldsymbol{\tau}\times\boldsymbol{n})_e.\end{aligned}\tag{33}$$

As both $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ are symmetric, by taking transpose of the boundary terms, we can get another equivalent version of Green's identity. For example, the edge term can be $-(\boldsymbol{n}\times\boldsymbol{\sigma}\cdot\boldsymbol{n},\boldsymbol{\tau}\cdot\boldsymbol{t}_{F,e})_e$.

When the domain Ω is decomposed into a polyhedral mesh, for piecewise smooth function to be in $\boldsymbol{H}(\operatorname{inc},\Omega;\mathbb{S})$, it suffices that the edge terms across different elements are canceled.

Lemma 4.3. *Let $\boldsymbol{\tau}\in\boldsymbol{L}^2(\Omega;\mathbb{S})$ such that*

(i) $\boldsymbol{\tau}|_K\in\boldsymbol{H}(\operatorname{inc},K;\mathbb{S})$ *for each polyhedron $K\in\mathcal{T}_h$;*
(ii) $\operatorname{tr}_1(\boldsymbol{\tau})|_F\in\boldsymbol{L}^2(F;\mathbb{S})$ *is single-valued for each $F\in\mathcal{F}_h^i$;*
(iii) $\operatorname{tr}_2(\boldsymbol{\tau})|_F\in\boldsymbol{L}^2(F;\mathbb{S})$ *is single-valued for each $F\in\mathcal{F}_h^i$;*
(iv) $\boldsymbol{\tau}|_e\in\boldsymbol{L}^2(e;\mathbb{S})$ *is single-valued for each $e\in\mathcal{E}_h^i$,*

then $\boldsymbol{\tau}\in\boldsymbol{H}(\operatorname{inc},\Omega;\mathbb{S})$.

Proof. Take any $\boldsymbol{\sigma}\in\mathcal{C}_0^\infty(\Omega;\mathbb{S})$. Sum the Green's identify (33) over $K\in\mathcal{T}_h$ to get

$$\begin{aligned}&(\boldsymbol{\tau},\nabla\times\boldsymbol{\sigma}\times\nabla)-\sum_{K\in\mathcal{T}_h}(\nabla\times\boldsymbol{\tau}\times\nabla,\boldsymbol{\sigma})_K\\ &=\sum_{K\in\mathcal{T}_h}(\operatorname{tr}_1(\boldsymbol{\sigma}),\operatorname{tr}_2(\boldsymbol{\tau}))_{\partial K}-\sum_{K\in\mathcal{T}_h}(\operatorname{tr}_2(\boldsymbol{\sigma}),\operatorname{tr}_1(\boldsymbol{\tau}))_{\partial K}\\ &\quad+\sum_{K\in\mathcal{T}_h}\sum_{F\in\mathcal{F}(K)}\sum_{e\in\mathcal{E}(F)}\big((\boldsymbol{n}\cdot\boldsymbol{\sigma}\times\boldsymbol{n},\boldsymbol{t}_{F,e}\cdot\boldsymbol{\tau})_e-(\boldsymbol{t}_{F,e}\cdot\boldsymbol{\sigma},\boldsymbol{n}\cdot\boldsymbol{\tau}\times\boldsymbol{n})_e\big).\end{aligned}\tag{34}$$

We note that $\operatorname{tr}_1(\boldsymbol{\tau})$ is independent of the choice of the direction of normal vectors but $\operatorname{tr}_2(\boldsymbol{\tau})$ is an odd function of $\boldsymbol{n}$ in the sense that $\operatorname{tr}_2(\boldsymbol{\tau};-\boldsymbol{n})=-\operatorname{tr}_2(\boldsymbol{\tau};\boldsymbol{n})$. Therefore if $\operatorname{tr}_1(\boldsymbol{\tau})$ and $|\operatorname{tr}_2(\boldsymbol{\tau})|$ are single-valued on face F, the face terms in (34) will be canceled out when integrated over a mesh of the domain Ω.

The edge vector $\boldsymbol{t}_{F,e}$ in (34) is the orientation of edge e induced by the outwards normal vector $\boldsymbol{n}_{\partial K}$ of the face F with respect to K. Therefore, for an interior face $F=K\cap K'$, $\boldsymbol{t}_{F(K),e}=-\boldsymbol{t}_{F(K'),e}$, where $F(K)$ means $F\in\mathcal{F}(K)$ with normal vector $\boldsymbol{n}_{\partial K}$. Hence if $\boldsymbol{\tau}$ is single-valued on edge e, the edge terms in (34) will be canceled out when integrated over a mesh of the domain Ω.

Then (34) reduces to

$$(\boldsymbol{\tau}, \nabla \times \boldsymbol{\sigma} \times \nabla) = \sum_{K \in \mathcal{T}_h} (\nabla \times \boldsymbol{\tau} \times \nabla, \boldsymbol{\sigma})_K,$$

which implies the distribution $\nabla \times \boldsymbol{\tau} \times \nabla \in \boldsymbol{L}^2(\Omega; \mathbb{S})$. So $\boldsymbol{\tau} \in \boldsymbol{H}(\mathrm{inc}, \Omega; \mathbb{S})$. □

Remark 4.4. *The continuity of $\boldsymbol{\tau}\ |_e$, which implies $\boldsymbol{\tau}(\delta)$ is also continuous at vertices, is a sufficient condition for the cancelation of edge terms but may not be necessary.*

Lemma 4.3 is implicitly contained in [5, Section 6] but the Green's identity (33) and the form of $\mathrm{tr}_2(\boldsymbol{\tau})$, cf. (30), seem new. When the domain is smooth, the edge jump can be replaced by a curvature term, cf. [1, Theorem 3.16], where a different Green's formula on smooth domains is derived.

4.2. **Trace complexes.** For a vector $\boldsymbol{v} \in \mathbb{R}^3$, define the tangential trace and the normal trace as

$$\mathrm{tr}_1(\boldsymbol{v}) := \boldsymbol{v} \times \boldsymbol{n}, \quad \mathrm{tr}_2(\boldsymbol{v}) := \boldsymbol{v} \cdot \boldsymbol{n}.$$

For a smooth and symmetric tensor $\boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div}, K; \mathbb{S})$, define the normal-normal trace and the normal-tangential trace as

$$\mathrm{tr}_1(\boldsymbol{\sigma}) := \boldsymbol{n} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{n}, \quad \mathrm{tr}_2(\boldsymbol{\sigma}) := \boldsymbol{n} \times \boldsymbol{\sigma} \cdot \boldsymbol{n}.$$

Then we will have the following trace complexes

$$\begin{array}{ccccccccc}
\boldsymbol{a} \times \boldsymbol{x} + \boldsymbol{b} & \xrightarrow{\subset} & \boldsymbol{v} & \xrightarrow{\mathrm{def}} & \boldsymbol{\tau} & \xrightarrow{\mathrm{inc}} & \boldsymbol{\sigma} & \xrightarrow{\mathrm{div}} & \boldsymbol{p} \\
\downarrow{\scriptstyle \mathrm{tr}_1} & & \downarrow{\scriptstyle \mathrm{tr}_1} & & \downarrow{\scriptstyle \mathrm{tr}_1} & & \downarrow{\scriptstyle \mathrm{tr}_1} & & \\
a_F \boldsymbol{x}_F + \boldsymbol{b}_F & \xrightarrow{\subset} & \boldsymbol{v} \times \boldsymbol{n} & \xrightarrow{\mathrm{sym\,curl}_F} & \boldsymbol{n} \times \boldsymbol{\tau} \times \boldsymbol{n} & \xrightarrow{\mathrm{div}_F \mathrm{div}_F} & \boldsymbol{n} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{n} & \longrightarrow & 0
\end{array}, \tag{35}$$

and

$$\begin{array}{ccccccccc}
\boldsymbol{a} \times \boldsymbol{x} + \boldsymbol{b} & \xrightarrow{\subset} & \boldsymbol{v} & \xrightarrow{\mathrm{def}} & \boldsymbol{\tau} & \xrightarrow{\mathrm{inc}} & \boldsymbol{\sigma} & \xrightarrow{\mathrm{div}} & \boldsymbol{p} \\
\downarrow{\scriptstyle \mathrm{tr}_2} & & \downarrow{\scriptstyle \mathrm{tr}_2} & & \downarrow{\scriptstyle \mathrm{tr}_2} & & \downarrow{\scriptstyle \mathrm{tr}_2} & & \\
\boldsymbol{a}_F \cdot \boldsymbol{x}_F + b_F & \xrightarrow{\subset} & \boldsymbol{v} \cdot \boldsymbol{n} & \xrightarrow{\nabla_F^2} & \mathrm{tr}_2(\boldsymbol{\tau}) & \xrightarrow{\nabla_F^{\perp} \cdot} & \boldsymbol{n} \cdot \boldsymbol{\sigma} \times \boldsymbol{n} & \longrightarrow & \boldsymbol{0}
\end{array}. \tag{36}$$

In (35) and (36), we present the concrete form instead of Sobolev spaces as we will work mostly on polynomial functions which are smooth enough to define the trace pointwise. In the next two lemmas we will verify the commutative diagrams (35) and (36). Some results can be found in [5, Section 5].

Lemma 4.5. *For any sufficiently smooth vector function $\boldsymbol{v}$, we have*

$$\boldsymbol{n} \times (\mathrm{def}\, \boldsymbol{v}) \times \boldsymbol{n} = \mathrm{sym\,curl}_F(\boldsymbol{v} \times \boldsymbol{n}), \tag{37}$$

$$\mathrm{tr}_2(\mathrm{def}\, \boldsymbol{v}) = \nabla_F^2(\boldsymbol{v} \cdot \boldsymbol{n}). \tag{38}$$

Proof. Using our notation, the first identity (37) is straightforward:

$$\boldsymbol{n} \times (\nabla \boldsymbol{v}) \times \boldsymbol{n} = \nabla_F^{\perp}(\boldsymbol{v} \times \boldsymbol{n}) = (\mathrm{curl}_F(\boldsymbol{v} \times \boldsymbol{n}))^{\intercal}.$$

Then apply the sym operator to get (37).

Let $\boldsymbol{\tau} = \operatorname{def} \boldsymbol{v}$. Using (31), $\nabla \times \nabla \boldsymbol{v} = \boldsymbol{0}$ and (9), it follows that

$$
\begin{aligned}
\operatorname{tr}_2(\boldsymbol{\tau}) &= \Pi_F(\boldsymbol{\tau} \times \nabla) \times \boldsymbol{n} + \nabla_F(\boldsymbol{n} \cdot \boldsymbol{\tau}\, \Pi_F) \\
&= \frac{1}{2}\Pi_F(\nabla \boldsymbol{v} \times \nabla) \times \boldsymbol{n} + \frac{1}{2}\nabla_F(\partial_n(\Pi_F \boldsymbol{v}) + \nabla_F(\boldsymbol{v} \cdot \boldsymbol{n})) \\
&= \frac{1}{2}\nabla_F\big(\nabla_F(\boldsymbol{v} \cdot \boldsymbol{n}) - \partial_n \Pi_F \boldsymbol{v}\big) + \frac{1}{2}\nabla_F(\partial_n(\Pi_F \boldsymbol{v}) + \nabla_F(\boldsymbol{v} \cdot \boldsymbol{n})) \\
&= \nabla_F^2(\boldsymbol{v} \cdot \boldsymbol{n}),
\end{aligned}
$$

as required. □

We then verify the second block.

Lemma 4.6. *For any sufficiently smooth and symmetric tensor $\boldsymbol{\tau}$, it holds that*

$$
\boldsymbol{n} \cdot (\nabla \times \boldsymbol{\tau} \times \nabla) \cdot \boldsymbol{n} = \operatorname{div}_F \operatorname{div}_F(\boldsymbol{n} \times \boldsymbol{\tau} \times \boldsymbol{n}), \tag{39}
$$

$$
\boldsymbol{n} \cdot (\nabla \times \boldsymbol{\tau} \times \nabla) \times \boldsymbol{n} = \nabla_F^{\perp} \cdot \operatorname{tr}_2(\boldsymbol{\tau}). \tag{40}
$$

Proof. The first identity is from a direct computation

$$
\boldsymbol{n} \cdot (\nabla \times \boldsymbol{\tau} \times \nabla) \cdot \boldsymbol{n} = \nabla_F^{\perp} \cdot \Pi_F\, \boldsymbol{\tau}\, \Pi_F \cdot \nabla_F^{\perp} = \operatorname{div}_F \operatorname{div}_F(\boldsymbol{n} \times \boldsymbol{\tau} \times \boldsymbol{n}).
$$

To prove the second identity, we use the trace representation form (31) and the fact $\nabla_F^{\perp} \cdot \nabla_F = 0$ to get

$$
\boldsymbol{n} \cdot (\nabla \times \boldsymbol{\tau} \times \nabla) \times \boldsymbol{n} = \nabla_F^{\perp} \cdot (\boldsymbol{\tau} \times \nabla) \times \boldsymbol{n} = \nabla_F^{\perp} \cdot \operatorname{tr}_2(\boldsymbol{\tau}).
$$

□

4.3. **Continuity on edges.** In order to construct an $H(\operatorname{inc})$-conforming finite element, the trace complexes inspire us to adopt $\boldsymbol{H}(\operatorname{div}_F \operatorname{div}_F, F; \mathbb{S})$ conforming finite element to discretize $\boldsymbol{n} \times \boldsymbol{\tau} \times \boldsymbol{n}$, and $\boldsymbol{H}(\operatorname{rot}_F, F; \mathbb{S})$-conforming finite element to discretize $\operatorname{tr}_2(\boldsymbol{\tau})$. The trace for $\boldsymbol{v}_F \in \boldsymbol{H}(\operatorname{rot}_F, F; \mathbb{S})$ is $\boldsymbol{v}_F \cdot \boldsymbol{t}$ on ∂F. Two trace operators for $\boldsymbol{H}(\operatorname{div}_F \operatorname{div}_F, F; \mathbb{S})$ are identified in [14, Lemma 2.1] and will be recalled below.

Lemma 4.7 (Green's identity for the two-dimensional div div operator [14])**.** *Let F be a polygon, and let $\boldsymbol{\tau} \in \mathcal{C}^2(F; \mathbb{S})$ and $v \in H^2(F)$. Then we have*

$$
\begin{aligned}
(\operatorname{div}_F \operatorname{div}_F \boldsymbol{\tau}, v)_K = (\boldsymbol{\tau}, \nabla_F^2 v)_K &- \sum_{e \in \mathcal{E}(F)} \sum_{\delta \in \partial e} \operatorname{sign}_{e,\delta}(\boldsymbol{t}_{F,e} \cdot \boldsymbol{\tau} \cdot \boldsymbol{n}_{F,e})(\delta) v(\delta) \\
&- \sum_{e \in \mathcal{E}(F)} \big[\operatorname{tr}_{e,1}(\boldsymbol{\tau}), \partial_{n_{F,e}} v)_e - (\operatorname{tr}_{e,2}(\boldsymbol{\tau}), v)_e \big],
\end{aligned} \tag{41}
$$

where

$$
\begin{aligned}
\operatorname{sign}_{e,\delta} &:= \begin{cases} 1, & \text{if } \delta \text{ is the end point of } e, \\ -1, & \text{if } \delta \text{ is the start point of } e, \end{cases} \\
\operatorname{tr}_{e,1}(\boldsymbol{\tau}) &:= \boldsymbol{n}_{F,e} \cdot \boldsymbol{\tau} \cdot \boldsymbol{n}_{F,e}, \\
\operatorname{tr}_{e,2}(\boldsymbol{\tau}) &:= \partial_t(\boldsymbol{t}_{F,e} \cdot \boldsymbol{\tau} \cdot \boldsymbol{n}_{F,e}) + \boldsymbol{n}_{F,e} \cdot \operatorname{div}_F \boldsymbol{\tau}.
\end{aligned}
$$

The trace of $\boldsymbol{\tau} \in \boldsymbol{H}(\operatorname{rot}_F, F; \mathbb{S})$ is $\boldsymbol{\tau} \cdot \boldsymbol{t}$ and denoted by $\operatorname{tr}_{e,3}(\boldsymbol{\tau}) = \boldsymbol{\tau} \cdot \boldsymbol{t}$.

Lemma 4.8. *Let $F \in \mathcal{F}(K)$, $e \in \mathcal{E}(F)$, $\boldsymbol{t}$ be the unit tangential vector of e, and $\boldsymbol{n}_{F,e} := \boldsymbol{t} \times \boldsymbol{n}$. For any sufficiently smooth and symmetric tensor $\boldsymbol{\tau}$, we have on edge e that*

$$\operatorname{tr}_{e,1}(\operatorname{tr}_1(\boldsymbol{\tau})) = -\boldsymbol{t} \cdot \boldsymbol{\tau} \cdot \boldsymbol{t}, \tag{42}$$

$$\operatorname{tr}_{e,2}(\operatorname{tr}_1(\boldsymbol{\tau})) = \partial_t(\boldsymbol{n}_{F,e} \cdot \boldsymbol{\tau} \cdot \boldsymbol{t}) - \boldsymbol{n} \cdot (\nabla \times \boldsymbol{\tau}) \cdot \boldsymbol{t}, \tag{43}$$

$$\operatorname{tr}_{e,3}(\operatorname{tr}_2(\boldsymbol{\tau})) = \boldsymbol{n} \times (\nabla \times \boldsymbol{\tau}) \cdot \boldsymbol{t} + \partial_t(\Pi_F \boldsymbol{\tau} \cdot \boldsymbol{n}). \tag{44}$$

Proof. Let us compute

$$(\boldsymbol{\tau} \times \boldsymbol{n}) \cdot \boldsymbol{n}_{F,e} = \boldsymbol{\tau} \cdot (\boldsymbol{n} \times \boldsymbol{n}_{F,e}) = \boldsymbol{\tau} \cdot \boldsymbol{t}. \tag{45}$$

Then identity (42) follows. The identity (43) follows from

$$\partial_t(\boldsymbol{t} \cdot (\boldsymbol{n} \times \boldsymbol{\tau} \times \boldsymbol{n}) \cdot \boldsymbol{n}_{F,e}) = \partial_t(\boldsymbol{n}_{F,e} \cdot \boldsymbol{\tau} \cdot \boldsymbol{t}),$$

and

$$\begin{aligned} \boldsymbol{n}_{F,e} \cdot \operatorname{div}_F(\boldsymbol{n} \times \boldsymbol{\tau} \times \boldsymbol{n}) &= \nabla_F \cdot (\boldsymbol{n} \times \boldsymbol{\tau} \times \boldsymbol{n}) \cdot \boldsymbol{n}_{F,e} \\ = -\boldsymbol{n} \cdot \nabla \times ((\boldsymbol{\tau} \times \boldsymbol{n}) \cdot \boldsymbol{n}_{F,e}) &= -\boldsymbol{n} \cdot (\nabla \times \boldsymbol{\tau}) \cdot \boldsymbol{t}, \end{aligned}$$

which holds from (8) and (45). The identity (44) is a direct consequence of (30). □

Those formulae on the edge trace suggest the continuity of $\boldsymbol{\tau} \cdot \boldsymbol{t}$ and $(\nabla \times \boldsymbol{\tau}) \cdot \boldsymbol{t}$ on edges. As we mentioned before, in view of conformity, it is sufficient to impose the whole tensor $\boldsymbol{\tau}$ is continuous on edges. The continuity of $(\nabla \times \boldsymbol{\tau}) \cdot \boldsymbol{t}$ is not surprising as $\nabla \times \boldsymbol{\tau} \times \nabla \in \boldsymbol{H}(\operatorname{div}; \mathbb{S})$ and thus the normal trace $((\nabla \times \boldsymbol{\tau}) \times \nabla) \cdot \boldsymbol{n} = (\nabla \times \boldsymbol{\tau}) \cdot \nabla_F^{\perp} \in \boldsymbol{L}^2(F; \mathbb{R}^3)$. Namely $\nabla \times \boldsymbol{\tau} \in \boldsymbol{H}(\operatorname{rot}_F, F; \mathbb{R}^{3\times 2})$, and the trace of $\boldsymbol{H}(\operatorname{rot}_F, F; \mathbb{R}^{3\times 2})$ implies the continuity of $(\nabla \times \boldsymbol{\tau}) \cdot \boldsymbol{t}$ on edges.

The following result on the vanishing edge trace is an easy consequence of formulae (42)-(44).

Corollary 4.9. *If $\boldsymbol{\tau}|_e = \boldsymbol{0}$ and $(\nabla \times \boldsymbol{\tau}) \cdot \boldsymbol{t}|_e = \boldsymbol{0}$ for all $e \in \mathcal{E}(K)$, then $\operatorname{tr}_{e,1}(\operatorname{tr}_1(\boldsymbol{\tau})) = \operatorname{tr}_{e,2}(\operatorname{tr}_1(\boldsymbol{\tau})) = \operatorname{tr}_{e,3}(\operatorname{tr}_2(\boldsymbol{\tau})) = 0$.*

4.4. **Bubble complexes.** We give a characterization of bubble functions following [5]. Let K be a tetrahedron with vertices $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, $\boldsymbol{x}_3$ and $\boldsymbol{x}_4$. We label the face opposite to $\boldsymbol{x}_i$ as the i-th face F_i, and denote by $\boldsymbol{n}_i$ the unit outwards normal vector of face F_i. Set $\boldsymbol{N}_{i,j} := \operatorname{sym}(\boldsymbol{n}_k \boldsymbol{n}_l^{\intercal}) = \frac{1}{2}(\boldsymbol{n}_k \boldsymbol{n}_l^{\intercal} + \boldsymbol{n}_l \boldsymbol{n}_k^{\intercal})$, where $(ijkl)$ is a permutation cycle of (1234). Then it is shown in [19, 10] that the 6 symmetric tensors $\{\boldsymbol{N}_{i,j}, i, j = 1, 2, 3, 4, i < j\}$ form a basis of $\mathbb{S}$.

Define a tangential-tangential bubble function space of tensorial polynomials of degree k as

$$\mathbb{B}_{K,k}^{\operatorname{tr}_1} := \mathbb{P}_k(K; \mathbb{S}) \cap \ker(\operatorname{tr}_1).$$

It is easy to verify $\operatorname{tr}_1(\lambda_i \lambda_j \boldsymbol{N}_{i,j}) = \boldsymbol{0}$, where λ_i is the barycentric coordinate of $\boldsymbol{x}$ corresponding to vertex $\boldsymbol{x}_i$. Since the dimension of $\mathbb{B}_{K,k}^{\operatorname{tr}_1}$ is $k(k^2-1)$ (cf. [5, Lemma 6.1]), we have

$$\mathbb{B}_{K,k}^{\operatorname{tr}_1} = \mathbb{P}_{k-2}(K) \otimes \{\lambda_i \lambda_j \boldsymbol{N}_{i,j}\} = \sum_{1 \le i < j \le 4} \mathbb{P}_{k-2}(K) \lambda_i \lambda_j \boldsymbol{N}_{i,j}.$$

Define an $\boldsymbol{H}(\operatorname{inc}, K; \mathbb{S})$ bubble function space of polynomials of degree k as

$$\mathbb{B}_{K,k}^{\operatorname{inc}} := \mathbb{P}_k(K; \mathbb{S}) \cap \ker(\operatorname{tr}_1) \cap \ker(\operatorname{tr}_2) = \mathbb{B}_{K,k}^{\operatorname{tr}_1} \cap \ker(\operatorname{tr}_2).$$

According to Lemma 6.2 in [5], for any $\boldsymbol{\tau}\in\mathbb{B}_{K,k}^{\mathrm{inc}}$, it holds $\boldsymbol{\tau}\mid_e=\mathbf{0}$ for all $e\in\mathcal{E}(K)$. Thus

$$\boldsymbol{\tau}\in\sum_{1\le i<j\le 4}\lambda_i\lambda_j\big(\lambda_k\mathbb{P}_{k-3}(K)+\lambda_l\mathbb{P}_{k-3}(K)\big)\boldsymbol{N}_{i,j}.$$

Although there is no precise characterization of $\mathbb{B}_{K,k}^{\mathrm{inc}}$, it is shown in [5] that the dimension of $\mathbb{B}_{K,k}^{\mathrm{inc}}$ is k^3-6k^2+11k.

Furthermore the bubble polynomial elasticity complex with $k\ge 4$ is established in [5, Lemma 7.1] and [31, Lemma 3.2]

$$\mathbf{0}\xrightarrow{\subset} b_K\mathbb{P}_{k-3}(K;\mathbb{R}^3)\xrightarrow{\operatorname{def}}\mathbb{B}_{K,k}^{\mathrm{inc}}\xrightarrow{\operatorname{inc}}\mathbb{B}_{K,k-2}^{\mathrm{div}}\xrightarrow{\operatorname{div}}\mathbb{P}_{k-3}(K;\mathbb{R}^3)/\boldsymbol{RM}\to\mathbf{0},\tag{46}$$

where $b_K=\lambda_1\lambda_2\lambda_3\lambda_4$ is the volume bubble polynomial and $\mathbb{B}_{K,k}^{\mathrm{div}}=\mathbb{P}_k(K;\mathbb{S})\cap\boldsymbol{H}_0(\operatorname{div},K;\mathbb{S})$ is the $\boldsymbol{H}(\operatorname{div};\mathbb{S})$ bubble function space and is characterized in [31]

$$\mathbb{B}_{K,k}^{\mathrm{div}}=\sum_{0\le i<j\le 3}\lambda_i\lambda_j\mathbb{P}_{k-2}(K)\boldsymbol{T}_{i,j},\quad k\ge 2\tag{47}$$

with $\boldsymbol{T}_{i,j}:=\boldsymbol{t}_{i,j}\boldsymbol{t}_{i,j}^{\intercal}$ and $\boldsymbol{t}_{i,j}:=\boldsymbol{x}_j-\boldsymbol{x}_i$.

Similarly we also have two-dimensional bubble complexes on face F. The bubble function space $\mathbb{B}_{F,k}^{\operatorname{div}\operatorname{div}}:=\mathbb{P}_k(F;\mathbb{S})\cap\boldsymbol{H}_0(\operatorname{div}_F\operatorname{div}_F,F;\mathbb{S})$ is

$$\{\boldsymbol{\tau}\in\mathbb{P}_k(F;\mathbb{S}):\operatorname{tr}_{e,1}(\boldsymbol{\tau})=\operatorname{tr}_{e,2}(\boldsymbol{\tau})=0,\forall\, e\in\mathcal{E}(F),\ \ \boldsymbol{\tau}(\delta)=\mathbf{0}\ \ \forall\,\delta\in\mathcal{V}(F)\}.$$

We present the results below and a proof of (48) can be found in [14].

$$\mathbf{0}\xrightarrow{\subset} b_F\mathbb{P}_{k-2}(F;\mathbb{R}^2)\xrightarrow{\operatorname{sym}\operatorname{curl}_F}\mathbb{B}_{F,k}^{\operatorname{div}\operatorname{div}}\xrightarrow{\operatorname{div}_F\operatorname{div}_F}\mathbb{P}_{k-2}(F)/\mathbb{P}_1(F)\to 0.\tag{48}$$

For the two-dimensional Hessian polynomial complex, let $\mathbb{B}_{F,k-1}^{\mathrm{rot}}:=\mathbb{P}_{k-1}(F;\mathbb{S})\cap\boldsymbol{H}_0(\operatorname{rot}_F,F;\mathbb{S})$, we have

$$0\xrightarrow{\subset} b_F^2\mathbb{P}_{k-5}(F)\xrightarrow{\nabla_F^2}\mathbb{B}_{F,k-1}^{\mathrm{rot}}\xrightarrow{\operatorname{rot}_F}\mathbb{P}_{k-2}(F;\mathbb{R}^2)/\boldsymbol{RT}\to\mathbf{0},\tag{49}$$

which is a rotation of the two-dimensional elasticity bubble complex established in [8].

At the end of this section, we present two results on the characterization for the dual spaces of bubble spaces. The first one is also included in [15].

Lemma 4.10. *Assume finite-dimensional Hilbert spaces $\mathbb{B}_1,\mathbb{B}_2,\ldots,\mathbb{B}_n$ with the inner product $(\cdot,\cdot)$ form an exact Hilbert complex*

$$0\xrightarrow{\subset}\mathbb{B}_1\xrightarrow{\mathrm{d}_1}\ldots\mathbb{B}_i\xrightarrow{\mathrm{d}_i}\ldots\mathbb{B}_n\to 0,$$

where $\mathbb{B}_i\subseteq\ker(\operatorname{tr}(\mathrm{d}_i))$ for $i=1,2,\cdots,n-1$. Then the bubble space $\mathbb{B}_i$, for $i=1,\ldots,n-1$, is uniquely determined by the DoFs

$$\langle\,\mathrm{d}_i v,q\rangle\quad\forall\, q\in(\,\mathrm{d}_i\mathbb{B}_i)',\tag{50}$$

$$(v,q)\quad\forall\, q\in\mathbb{Q}\cong(\,\mathrm{d}_{i-1}\mathbb{B}_{i-1})',\tag{51}$$

where $\langle\cdot,\cdot\rangle$ is the duality pair and the isomorphism $\mathbb{Q}\to(\,\mathrm{d}_{i-1}\mathbb{B}_{i-1})'$ is given by $p\to(p,\cdot)$ for $p\in\mathbb{Q}$.

Proof. By the splitting lemma in [27] (see also Theorem 2.2 in [13]),

$$\mathbb{B}_i=\mathrm{d}_i^*\,\mathrm{d}_i\mathbb{B}_i\oplus\mathrm{d}_{i-1}\mathbb{B}_{i-1},\tag{52}$$

where d_i^* is the adjoint of d_i with respect to the inner product $(\cdot,\cdot)$. Since d_i^* restricted to $\mathrm{d}_i\mathbb{B}_i$ is injective, the number of DoFs (50)-(51) is same as $\dim\mathbb{B}_i$.

Assume $v \in \mathbb{B}_i$ and all the DoFs (50)-(51) vanish. By the decomposition (52), there exist $v_1 \in \mathbb{B}_i$ and $v_2 \in \mathbb{B}_{i-1}$ such that $v = \mathrm{d}_i^* \mathrm{d}_i v_1 + \mathrm{d}_{i-1} v_2$. The vanishing (50) yields $\mathrm{d}_i v = 0$, that is $\mathrm{d}_i \mathrm{d}_i^*(\mathrm{d}_i v_1) = 0$. Noting that $\mathrm{d}_i \mathrm{d}_i^* : \mathrm{d}_i \mathbb{B}_i \to \mathrm{d}_i \mathbb{B}_i$ is isomorphic, we get $\mathrm{d}_i v_1 = 0$ and thus $v = \mathrm{d}_{i-1} v_2$. Now apply the vanishing (51) to get $v = 0$. ☐

Corollary 4.11. *Assume $\mathbb{B} = \mu\mathbb{P}$, where $\mu \geq 0$ and $\mu \neq 0$, and $\mathrm{d}^* : \mathbb{Q} \to \mathbb{P}$ is isomorphic. Then*

(1) $\mathbb{P} \cong \mathbb{B}'$: for $v \in \mathbb{B}$, if $(v, p) = 0$ for all $p \in \mathbb{P}$, then $v = 0$.
(2) $\mathbb{Q} \cong (\mathrm{d}\mathbb{B})'$: for $v \in \mathrm{d}\mathbb{B}$, if $(v, q) = 0$ for all $q \in \mathbb{Q}$, then $v = 0$.

Proof. (1) By assumption $v = \mu w$ for some $w \in \mathbb{P}$. Then choose $p = w$ to get $(\mu w, w) = 0$ which implies $w = 0$.
(2) By assumption $v = \mathrm{d}(\mu w) \in \mathrm{d}\mathbb{B}$ with $w \in \mathbb{P}$. Notice that $(v, q) = 0$ implies

$$(\mu w, \mathrm{d}^* q) = 0 \quad \forall\, q \in \mathbb{Q}.$$

As $\mathrm{d}^* : \mathbb{Q} \to \mathbb{P}$ is isomorphic, we can find $q \in \mathbb{Q}$ s.t. $\mathrm{d}^* q = w$ and $(\mu w, w) = 0$ which implies $w = 0$ and consequently $v = 0$. ☐

The Koszul complex will play a vital role to find the space $\mathbb{Q}$. Hereafter let $\{\mathcal{T}_h\}_{h>0}$ be a regular family of simplical meshes of a two- or three-dimensional domain Ω.

5. Finite Element Complexes in Two Dimensions

In this section we will construct a smooth finite element Hessian complex and a smooth finite element div div complex in two dimensions and construct commutative diagrams. Assume $\Omega \subset \mathbb{R}^2$ in this section.

5.1. Smooth finite element Hessian complex in two dimensions.

First we construct a finite element Hessian complex in two dimensions, which is smoother than a rotation of the elasticity complex established in [12, (2.3)]. For an integer $k \geq 6$, we shall also construct the following commutative diagram

$$\begin{array}{ccccccccc} \mathbb{P}_1(F) & \xrightarrow{\subset} & \mathcal{C}^\infty(F) & \xrightarrow{\nabla_F^2} & \mathcal{C}^\infty(F;\mathbb{S}) & \xrightarrow{\mathrm{rot}_F} & \mathcal{C}^\infty(F;\mathbb{R}^2) & \longrightarrow & \mathbf{0} \\ & & \downarrow I_F^{\mathrm{hess}} & & \downarrow I_F^{\mathrm{rot}} & & \downarrow I_F^{\mathrm{grad}} & & \\ \mathbb{P}_1(F) & \xrightarrow{\subset} & \mathbb{P}_{k+1}(F) & \xrightarrow{\nabla_F^2} & \mathbb{P}_{k-1}(F;\mathbb{S}) & \xrightarrow{\mathrm{rot}_F} & \mathbb{P}_{k-2}(F;\mathbb{R}^2) & \longrightarrow & \mathbf{0}. \end{array} \tag{53}$$

Recall the Argyris element in [3, 9]. Take $\mathbb{P}_{k+1}(F)$ as the shape function space. The degrees of freedom are given by

$$v(\delta), \nabla_F v(\delta), \nabla_F^2 v(\delta) \quad \forall\, \delta \in \mathcal{V}(F), \tag{54}$$

$$(v, q)_e \quad \forall\, q \in \mathbb{P}_{k-5}(e), e \in \mathcal{E}(F), \tag{55}$$

$$(\partial_n v, q)_e \quad \forall\, q \in \mathbb{P}_{k-4}(e), e \in \mathcal{E}(F), \tag{56}$$

$$(\nabla_F^2 v, \boldsymbol{q})_F \quad \forall\, \boldsymbol{q} \in \boldsymbol{x}\boldsymbol{x}^{\intercal}\mathbb{P}_{k-5}(F). \tag{57}$$

The last DoF (57) is based on (50), the characterization of $(\mathrm{d}\mathbb{B})'$ for $\mathrm{d} = \nabla_F^2$ and $\mathbb{B} = b_F^2 \mathbb{P}_{k-5}(F)$, and the isomorphism $\operatorname{div}_F \operatorname{div}_F : \mathbb{P}_{k-5}(F)\boldsymbol{x}\boldsymbol{x}^{\intercal} \to \mathbb{P}_{k-5}(F)$, cf. (24). One can also use $\mathbb{P} \cong \mathbb{B}'$ to replace (57) by $(v, q)_F$ for all $q \in \mathbb{P}_{k-5}(F)$. The choice (57) is for the commutative diagram (53).

Then consider an $H(\mathrm{rot})$-conforming element for symmetric tensors. Take $\mathbb{P}_{k-1}(F;\mathbb{S})$ as the shape function space. The degrees of freedom are given by

$$\boldsymbol{\tau}(\delta), \mathrm{rot}_F\, \boldsymbol{\tau}(\delta) \quad \forall\, \delta \in \mathcal{V}(F), \tag{58}$$

$$(\boldsymbol{\tau}\boldsymbol{t}, \boldsymbol{q})_e \quad \forall\, \boldsymbol{q} \in \mathbb{P}_{k-3}(e;\mathbb{R}^2), e \in \mathcal{E}(F), \tag{59}$$

$$(\mathrm{rot}_F\, \boldsymbol{\tau}, \boldsymbol{q})_e \quad \forall\, \boldsymbol{q} \in \mathbb{P}_{k-4}(e;\mathbb{R}^2), e \in \mathcal{E}(F), \tag{60}$$

$$(\boldsymbol{\tau}, \boldsymbol{q})_F \quad \forall\, \boldsymbol{q} \in \boldsymbol{x}\boldsymbol{x}^{\intercal}\mathbb{P}_{k-5}(F), \tag{61}$$

$$(\mathrm{rot}_F\, \boldsymbol{\tau}, \boldsymbol{q})_F \quad \forall\, \boldsymbol{q} \in \mathbb{P}_{k-6}(F;\mathbb{S}) \cdot \boldsymbol{x}^{\perp} = \mathbb{P}_{k-5}(F;\mathbb{R}^2)\backslash \boldsymbol{RT}. \tag{62}$$

Again (61) is based on (51) and the characterization of $(\,\mathrm{d}\mathbb{B})'$ for $\mathrm{d} = \nabla_F^2$ and $\mathbb{B} = b_F^2\mathbb{P}_{k-5}(F)$, and (62) is on (50). Note that due to (58) and (60), $\mathrm{rot}_F\, \boldsymbol{\tau}$ is continuous on ∂F and thus the last polynomial space $\mathbb{P}_{k-2}(F;\mathbb{R}^2)$ in (53) is the continuous Lagrange element not discontinuous one. This is smoother than the rotation of the $H(\mathrm{div})$-conforming element for symmetric tensors in [28].

Lemma 5.1. *The degrees of freedom* (58)-(62) *are uni-solvent for* $\mathbb{P}_{k-1}(F;\mathbb{S})$.

Proof. The number of degrees of freedom (58)-(62) is

$$15 + 6(2k-5) + \frac{3}{2}(k-3)(k-4) - 3 = \frac{3}{2}k(k+1),$$

which equals to $\dim \mathbb{P}_{k-1}(F;\mathbb{S})$.

Take $\boldsymbol{\tau} \in \mathbb{P}_{k-1}(F;\mathbb{S})$ and assume all the degrees of freedom (58)-(62) vanish. The vanishing degrees of freedom (58)-(60) imply $\boldsymbol{\tau}\boldsymbol{t}|_{\partial F} = \boldsymbol{0}$ and $(\mathrm{rot}_F\, \boldsymbol{\tau})|_{\partial F} = \boldsymbol{0}$. Apply the integration by parts to get

$$(\mathrm{rot}_F\, \boldsymbol{\tau}, \boldsymbol{q})_F = (\boldsymbol{\tau}, \mathrm{sym}\,\mathrm{curl}_F\, \boldsymbol{q})_F = 0 \quad \forall\, \boldsymbol{q} \in \boldsymbol{RT}.$$

Then it follows from the vanishing DoF (62) and the uni-solvence of Lagrange element that $\mathrm{rot}_F\, \boldsymbol{\tau} = \boldsymbol{0}$. Thanks to the bubble complex (49), there exists $v \in \mathbb{P}_{k-5}(F)$ such that $\boldsymbol{\tau} = \nabla_F^2(b_F^2 v)$. Finally $\boldsymbol{\tau} = \boldsymbol{0}$ follows from the vanishing DoF (61) and the isomorphism $\mathrm{div}_F\,\mathrm{div}_F : \mathbb{P}_{k-2}(F)\boldsymbol{x}\boldsymbol{x}^{\intercal} \to \mathbb{P}_{k-2}(F)$. ☐

Let $I_F^{\mathrm{hess}} : \mathcal{C}^{\infty}(F) \to \mathbb{P}_{k+1}(F)$ be the nodal interpolation operator based on the degrees of freedom (54)-(57), $I_F^{\mathrm{rot}} : \mathcal{C}^{\infty}(F;\mathbb{S}) \to \mathbb{P}_{k-1}(F;\mathbb{S})$ be the nodal interpolation operator based on the degrees of freedom (58)-(62), and $I_F^{\mathrm{grad}} : \mathcal{C}^{\infty}(F;\mathbb{R}^2) \to \mathbb{P}_{k-2}(F;\mathbb{R}^2)$ be the canonical Lagrange interpolation operator based on the degrees of freedom

$$\begin{aligned} \boldsymbol{v}(\delta) &\quad \forall\, \delta \in \mathcal{V}(F), \\ (\boldsymbol{v}, \boldsymbol{q})_e &\quad \forall\, \boldsymbol{q} \in \mathbb{P}_{k-4}(e;\mathbb{R}^2), e \in \mathcal{E}(F), \\ (\boldsymbol{v}, \boldsymbol{q})_F &\quad \forall\, \boldsymbol{q} \in \mathbb{P}_{k-5}(F;\mathbb{R}^2). \end{aligned}$$

Lemma 5.2. *The diagram* (53) *is commutative.*

Proof. Consider $\boldsymbol{\tau} \in \mathcal{C}^{\infty}(F;\mathbb{S})$ and $\boldsymbol{q} \in \mathbb{P}_{k-5}(F;\mathbb{R}^2)$. For $\boldsymbol{q} \in \mathbb{P}_{k-5}(F;\mathbb{R}^2)\backslash\boldsymbol{RT}$

$$(I_F^{\mathrm{grad}}\,\mathrm{rot}_F\, \boldsymbol{\tau} - \mathrm{rot}_F(I_F^{\mathrm{rot}}\boldsymbol{\tau}), \boldsymbol{q})_F = (\mathrm{rot}_F(\boldsymbol{\tau} - I_F^{\mathrm{rot}}\boldsymbol{\tau}), \boldsymbol{q})_F = 0.$$

When $\boldsymbol{q} \in \boldsymbol{RT}$, then using integration by parts

$$(I_F^{\mathrm{grad}}\,\mathrm{rot}_F\, \boldsymbol{\tau} - \mathrm{rot}_F(I_F^{\mathrm{rot}}\boldsymbol{\tau}), \boldsymbol{q})_F = (\boldsymbol{\tau} - I_F^{\mathrm{rot}}\boldsymbol{\tau}, \mathrm{sym}\,\mathrm{curl}_F\, \boldsymbol{q})_F = 0,$$

as $\operatorname{sym}\operatorname{curl}_F \boldsymbol{q} = \boldsymbol{0}$ and $((\boldsymbol{\tau} - I_F^{\rm rot}\boldsymbol{\tau})\boldsymbol{t}, \boldsymbol{q})_e = 0$ for $\boldsymbol{q}|_e \in \mathbb{P}_1(e;\mathbb{R}^2) \subset \mathbb{P}_{k-4}(e;\mathbb{R}^2)$. It is easy to see that $(I_F^{\rm grad}\operatorname{rot}_F \boldsymbol{\tau} - \operatorname{rot}_F(I_F^{\rm rot}\boldsymbol{\tau}))|_{\partial F} = \boldsymbol{0}$. So we have verified

$$I_F^{\rm grad}\operatorname{rot}_F \boldsymbol{\tau} = \operatorname{rot}_F(I_F^{\rm rot}\boldsymbol{\tau}) \quad \forall\, \boldsymbol{\tau} \in \mathcal{C}^\infty(F;\mathbb{S}). \tag{63}$$

For $v \in \mathcal{C}^\infty(F)$, apparently $(\nabla_F^2(I_F^{\rm hess}v))(\delta) = (I_F^{\rm rot}(\nabla_F^2 v))(\delta) = (\nabla_F^2 v)(\delta)$ for each $\delta \in \mathcal{V}(F)$, and by the integration by parts,

$$(\boldsymbol{t}\cdot\nabla_F^2(I_F^{\rm hess}v), \boldsymbol{q})_e = (\boldsymbol{t}\cdot\nabla_F^2 v, \boldsymbol{q})_e = (\boldsymbol{t}\cdot I_F^{\rm rot}(\nabla_F^2 v), \boldsymbol{q})_e \quad \forall\, \boldsymbol{q} \in \mathbb{P}_{k-3}(e;\mathbb{R}^2)$$

on each $e \in \mathcal{E}(F)$. By (63),

$$\operatorname{rot}_F\left(I_F^{\rm rot}(\nabla_F^2 v) - \nabla_F^2(I_F^{\rm hess}v)\right) = \operatorname{rot}_F\left(I_F^{\rm rot}(\nabla_F^2 v)\right) = I_F^{\rm grad}\operatorname{rot}_F(\nabla_F^2 v) = \boldsymbol{0}.$$

For $\boldsymbol{q} \in \boldsymbol{x}\boldsymbol{x}^\intercal\mathbb{P}_{k-5}(F)$,

$$\left(I_F^{\rm rot}(\nabla_F^2 v) - \nabla_F^2(I_F^{\rm hess}v), \boldsymbol{q}\right)_F = \left(\nabla_F^2(v - I_F^{\rm hess}v), \boldsymbol{q}\right)_F = 0.$$

Then by the unisolvence result, cf. Lemma 5.1,

$$I_F^{\rm rot}(\nabla_F^2 v) = \nabla_F^2(I_F^{\rm hess}v) \quad \forall\, v \in \mathcal{C}^\infty(F). \tag{64}$$

Combining (63) and (64) yields the commutative diagram (53). ∎

Next we show the smooth finite element Hessian complex in two dimensions. For an integer $k \geq 6$, define global finite element spaces

$$V_h^{\rm hess} := \{v_h \in H^2(\Omega) : v_h|_F \in \mathbb{P}_{k+1}(F) \text{ for each } F \in \mathcal{F}_h, \text{ all the degrees of freedom (54)-(56) are single-valued}\},$$

$$\boldsymbol{\Sigma}_h^{\rm rot} := \{\boldsymbol{\tau}_h \in \boldsymbol{H}(\operatorname{rot},\Omega;\mathbb{S}) : \boldsymbol{\tau}_h|_F \in \mathbb{P}_{k-1}(F;\mathbb{S}) \text{ for each } F \in \mathcal{F}_h, \text{ all the degrees of freedom (58)-(60) are single-valued}\},$$

$$\boldsymbol{V}_h^{\rm grad} := \{\boldsymbol{q}_h \in \boldsymbol{H}^1(\Omega;\mathbb{R}^2) : \boldsymbol{q}_h|_K \in \mathbb{P}_{k-2}(F;\mathbb{R}^2) \text{ for each } F \in \mathcal{F}_h\}.$$

Note that $\operatorname{rot}\boldsymbol{\Sigma}_h^{\rm rot} \subset \boldsymbol{H}^1(\Omega;\mathbb{R}^2)$.

Counting the dimensions of these spaces, we have

$$\begin{aligned}
\dim V_h^{\rm hess} &= 6\#\mathcal{V}_h + (2k-7)\#\mathcal{E}_h + \frac{1}{2}(k-3)(k-4)\#\mathcal{F}_h,\\
\dim \boldsymbol{\Sigma}_h^{\rm rot} &= 5\#\mathcal{V}_h + (4k-10)\#\mathcal{E}_h + \frac{3}{2}(k-3)(k-4)\#\mathcal{F}_h - 3\#\mathcal{F}_h,\\
\dim \boldsymbol{V}_h^{\rm grad} &= 2\#\mathcal{V}_h + (2k-6)\#\mathcal{E}_h + (k-3)(k-4)\#\mathcal{F}_h.
\end{aligned}$$

Theorem 5.3. *The finite element Hessian complex in two dimensions*

$$\mathbb{P}_1(\Omega) \xrightarrow{\subset} V_h^{\rm hess} \xrightarrow{\nabla^2} \boldsymbol{\Sigma}_h^{\rm rot} \xrightarrow{\operatorname{rot}} \boldsymbol{V}_h^{\rm grad} \to \boldsymbol{0} \tag{65}$$

is exact.

Proof. The inclusion $\nabla^2 V_h^{\rm hess} \subseteq \boldsymbol{\Sigma}_h^{\rm rot}$ follows from (54) and (58), and $\operatorname{rot}\boldsymbol{\Sigma}_h^{\rm rot} \subseteq \boldsymbol{V}_h^{\rm grad}$ holds from (58) and (60). Hence (65) is a complex.

For $\boldsymbol{\tau}_h \in \boldsymbol{\Sigma}_h^{\rm rot} \cap \ker(\operatorname{rot})$, there exists $v_h \in H^2(\Omega)$ such that $\boldsymbol{\tau}_h = \nabla^2 v_h$ and $v_h|_F \in \mathbb{P}_{k+1}(F)$ for each $F \in \mathcal{F}_h$. Thanks to DoF (58), $v_h \in V_h^{\rm hess}$. Thus $\boldsymbol{\Sigma}_h^{\rm rot} \cap \ker(\operatorname{rot}) = \nabla^2 V_h^{\rm hess}$. Then by the Euler's identity,

$$\begin{aligned}
\dim(\operatorname{rot}\boldsymbol{\Sigma}_h^{\rm rot}) &= \dim\boldsymbol{\Sigma}_h^{\rm rot} - \dim V_h^{\rm hess} + 3\\
&= -\#\mathcal{V}_h + (2k-3)\#\mathcal{E}_h + (k-3)(k-4)\#\mathcal{F}_h - 3\#\mathcal{F}_h + 3\\
&= 2\#\mathcal{V}_h + (2k-6)\#\mathcal{E}_h + (k-3)(k-4)\#\mathcal{F}_h = \dim\boldsymbol{V}_h^{\rm grad},
\end{aligned}$$

which implies $\operatorname{rot} \boldsymbol{\Sigma}_h^{\mathrm{rot}} = \boldsymbol{V}_h^{\mathrm{grad}}$. ∎

5.2. **Smooth finite element** $\operatorname{div}\operatorname{div}$ **complex in two dimensions.** Next we construct a finite element $\operatorname{div}\operatorname{div}$ complex in two dimensions, which is smoother than those in [14, 30]. We also have the following commutative diagram

$$\begin{array}{ccccccccc} \boldsymbol{RT} & \xrightarrow{\subset} & C^{\infty}(F;\mathbb{R}^2) & \xrightarrow{\operatorname{sym}\operatorname{curl}_F} & C^{\infty}(F;\mathbb{S}) & \xrightarrow{\operatorname{div}_F\operatorname{div}_F} & C^{\infty}(F) & \longrightarrow & 0 \\ & & \downarrow I_F^{\operatorname{sym}\operatorname{curl}} & & \downarrow I_F^{\operatorname{div}\operatorname{div}} & & \downarrow I_F^{\operatorname{grad}} & & \\ \boldsymbol{RT} & \xrightarrow{\subset} & \mathbb{P}_{k+1}(F;\mathbb{R}^2) & \xrightarrow{\operatorname{sym}\operatorname{curl}_F} & \mathbb{P}_k(F;\mathbb{S}) & \xrightarrow{\operatorname{div}_F\operatorname{div}_F} & \mathbb{P}_{k-2}(F) & \longrightarrow & 0. \end{array} \tag{66}$$

We start from the Argyris element for smooth finite element $\operatorname{div}\operatorname{div}$ complex in two dimensions. Choose $\mathbb{P}_{k+1}(F;\mathbb{R}^2)$ as the shape function space. The degrees of freedom are given by

$$\boldsymbol{v}(\delta), \nabla_F \boldsymbol{v}(\delta), \nabla_F^2 \boldsymbol{v}(\delta) \quad \forall\, \delta \in \mathcal{V}(F), \tag{67}$$

$$(\boldsymbol{v}, \boldsymbol{q})_e \quad \forall\, \boldsymbol{q} \in \mathbb{P}_{k-5}(e;\mathbb{R}^2), e \in \mathcal{E}(F), \tag{68}$$

$$(\partial_n \boldsymbol{v}, \boldsymbol{q})_e \quad \forall\, \boldsymbol{q} \in \mathbb{P}_{k-4}(e;\mathbb{R}^2), e \in \mathcal{E}(F), \tag{69}$$

$$(\operatorname{sym}\operatorname{curl}_F \boldsymbol{v}, \boldsymbol{q})_F \quad \forall\, \boldsymbol{q} \in \operatorname{sym}(\boldsymbol{x}_F^{\perp}\mathbb{P}_{k-5}(F;\mathbb{R}^2)). \tag{70}$$

In order to have the commutative diagram, the last DoF (70) is based on characterization of $(\mathrm{d}\mathbb{B})'$ for $\mathrm{d} = \operatorname{sym}\operatorname{curl}_F$, $\mathbb{B} = b_F^2\mathbb{P}_{k-5}(F;\mathbb{R}^2)$, and the bijection $\operatorname{rot}_F : \operatorname{sym}(\boldsymbol{x}^{\perp}\mathbb{P}_{k-5}(F;\mathbb{R}^2)) \to \mathbb{P}_{k-5}(F;\mathbb{R}^2)$.

Now we construct smooth $\operatorname{div}\operatorname{div}$ element for symmetric tensors. Take $\mathbb{P}_k(F;\mathbb{S})$ as the space of shape functions. Choose the following degrees of freedom

$$\boldsymbol{\tau}(\delta), \nabla_F \boldsymbol{\tau}(\delta), (\operatorname{div}_F\operatorname{div}_F \boldsymbol{\tau})(\delta) \quad \forall\, \delta \in \mathcal{V}(F), \tag{71}$$

$$(\boldsymbol{\tau}, \boldsymbol{q})_e \quad \forall\, \boldsymbol{q} \in \mathbb{P}_{k-4}(e;\mathbb{S}), e \in \mathcal{E}(F), \tag{72}$$

$$(\operatorname{div}_F\operatorname{div}_F \boldsymbol{\tau}, q)_e \quad \forall\, q \in \mathbb{P}_{k-4}(e), e \in \mathcal{E}(F), \tag{73}$$

$$(\operatorname{tr}_{e,2}(\boldsymbol{\tau}), q)_e \quad \forall\, q \in \mathbb{P}_{k-3}(e), e \in \mathcal{E}(F), \tag{74}$$

$$(\boldsymbol{\tau}, \boldsymbol{q})_F \quad \forall\, \boldsymbol{q} \in \operatorname{sym}(\boldsymbol{x}_F^{\perp}\mathbb{P}_{k-5}(F;\mathbb{R}^2)), \tag{75}$$

$$(\operatorname{div}_F\operatorname{div}_F \boldsymbol{\tau}, q)_F \quad \forall\, q \in \mathbb{P}_{k-5}(F)\backslash\mathbb{P}_1(F). \tag{76}$$

Lemma 5.4. *The degrees of freedom (71)-(76) are uni-solvent for $\mathbb{P}_k(F;\mathbb{S})$.*

Proof. The number of degrees of freedom (71)-(76) is

$$30 + 12(k-3) + 3(k-2) + \frac{3}{2}(k-3)(k-4) - 3 = \frac{3}{2}(k+1)(k+2),$$

which equals to $\dim \mathbb{P}_k(F;\mathbb{S})$.

Take $\boldsymbol{\tau} \in \mathbb{P}_k(F;\mathbb{S})$ and assume all the degrees of freedom (71)-(76) vanish. The vanishing degrees of freedom (71)-(74) imply $\boldsymbol{\tau}|_{\partial F} = \boldsymbol{0}$, $(\operatorname{div}_F\operatorname{div}_F\boldsymbol{\tau})|_{\partial F} = 0$ and $\operatorname{tr}_{e,2}(\boldsymbol{\tau})|_{\partial F} = 0$. We get from the Green's identity (41) that

$$(\operatorname{div}_F\operatorname{div}_F\boldsymbol{\tau}, q)_F = 0 \quad \forall\, q \in \mathbb{P}_1(F),$$

which together with the vanishing (76) and the uni-solvence of the Lagrange element that $\operatorname{div}_F\operatorname{div}_F \boldsymbol{\tau} = 0$. By the bubble complex (48), there exists $\boldsymbol{w} \in \mathbb{P}_{k+1}(F;\mathbb{R}^2)$ such that $\boldsymbol{\tau} = \operatorname{sym}\operatorname{curl}_F \boldsymbol{w}$. By the homogeneous boundary conditions, $\boldsymbol{\tau} = \operatorname{sym}\operatorname{curl}_F(b_F^2\boldsymbol{v})$ with $\boldsymbol{v} \in \mathbb{P}_{k-5}(F;\mathbb{R}^2)$. Finally $\boldsymbol{\tau} = \boldsymbol{0}$ follows from the vanishing (75) and the polynomial complex (25). ∎

Let $I_F^{\mathrm{sym\,curl}}: \mathcal{C}^\infty(F;\mathbb{R}^2) \to \mathbb{P}_{k+1}(F;\mathbb{R}^2)$ be the nodal interpolation operator based on the degrees of freedom (67)-(70), and $I_F^{\mathrm{div\,div}}: \mathcal{C}^\infty(F;\mathbb{S}) \to \mathbb{P}_k(F;\mathbb{S})$ be the nodal interpolation operator based on the degrees of freedom (71)-(76).

Lemma 5.5. *The diagram* (66) *is commutative.*

Proof. For $\boldsymbol{\tau} \in \mathcal{C}^\infty(F;\mathbb{S})$, clearly we have

$$\big(I_F^{\mathrm{grad}}(\mathrm{div}_F\,\mathrm{div}_F\,\boldsymbol{\tau}) - \mathrm{div}_F\,\mathrm{div}_F(I_F^{\mathrm{div\,div}}\boldsymbol{\tau})\big)|_{\partial F} = 0.$$

Then by the Green's identity (41), for $q \in \mathbb{P}_1(F)$ we have

$$(I_F^{\mathrm{grad}}(\mathrm{div}_F\,\mathrm{div}_F\,\boldsymbol{\tau}) - \mathrm{div}_F\,\mathrm{div}_F(I_F^{\mathrm{div\,div}}\boldsymbol{\tau}), q)_F = (\mathrm{div}_F\,\mathrm{div}_F(\boldsymbol{\tau} - I_F^{\mathrm{div\,div}}\boldsymbol{\tau}), q)_F = 0.$$

For $q \in \mathbb{P}_{k-5}(F)\backslash\mathbb{P}_1(F)$,

$$(I_F^{\mathrm{grad}}(\mathrm{div}_F\,\mathrm{div}_F\,\boldsymbol{\tau}) - \mathrm{div}_F\,\mathrm{div}_F(I_F^{\mathrm{div\,div}}\boldsymbol{\tau}), q)_F = (\mathrm{div}_F\,\mathrm{div}_F(\boldsymbol{\tau} - I_F^{\mathrm{div\,div}}\boldsymbol{\tau}), q)_F = 0.$$

Hence we get from the uni-solvence of the Lagrange element that

$$\mathrm{div}_F\,\mathrm{div}_F(I_F^{\mathrm{div\,div}}\boldsymbol{\tau}) = I_F^{\mathrm{grad}}\,\mathrm{div}_F\,\mathrm{div}_F\,\boldsymbol{\tau} \quad \forall\, \boldsymbol{\tau} \in \mathcal{C}^\infty(F;\mathbb{S}). \tag{77}$$

For $\boldsymbol{v} \in \mathcal{C}^\infty(F;\mathbb{R}^2)$, set $\boldsymbol{\tau}_1 = \mathrm{sym\,curl}_F(I_F^{\mathrm{sym\,curl}}\boldsymbol{v})$ and $\boldsymbol{\tau}_2 = I_F^{\mathrm{div\,div}}(\mathrm{sym\,curl}_F\,\boldsymbol{v})$ for simplicity. Obviously

$$(\boldsymbol{\tau}_1 - \boldsymbol{\tau}_2)(\delta) = 0, \quad \nabla(\boldsymbol{\tau}_1 - \boldsymbol{\tau}_2)(\delta) = \boldsymbol{0} \quad \forall\, \delta \in \mathcal{V}(F).$$

For $\boldsymbol{q} \in \mathbb{P}_{k-4}(e;\mathbb{S})$ and $e \in \mathcal{E}(F)$,

$$(\boldsymbol{\tau}_1, \boldsymbol{q})_e = (\mathrm{sym\,curl}_F\,\boldsymbol{v}, \boldsymbol{q})_e = (\boldsymbol{\tau}_2, \boldsymbol{q})_e.$$

For $q \in \mathbb{P}_{k-3}(e)$,

$$\begin{aligned}(\partial_t(\boldsymbol{t}^\intercal\boldsymbol{\tau}_1\boldsymbol{n}) + \boldsymbol{n}^\intercal\,\mathrm{div}_F\,\boldsymbol{\tau}_1, q)_e &= (\partial_{tt}(I_F^{\mathrm{sym\,curl}}\boldsymbol{v}\cdot\boldsymbol{t}), q)_e = (\partial_{tt}(\boldsymbol{v}\cdot\boldsymbol{t}), q)_e \\ &= (\partial_t(\boldsymbol{t}^\intercal\boldsymbol{\tau}_2\boldsymbol{n}) + \boldsymbol{n}^\intercal\,\mathrm{div}_F\,\boldsymbol{\tau}_2, q)_e.\end{aligned}$$

And for $\boldsymbol{q} \in \mathrm{sym}(\boldsymbol{x}_F^\perp \otimes \mathbb{P}_{k-5}(F;\mathbb{R}^2))$,

$$(\boldsymbol{\tau}_1, \boldsymbol{q})_F = (\mathrm{sym\,curl}_F\,\boldsymbol{v}, \boldsymbol{q})_F = (\boldsymbol{\tau}_2, \boldsymbol{q})_F.$$

By (77),

$$\mathrm{div}_F\,\mathrm{div}_F(\boldsymbol{\tau}_2) = I_F^{\mathrm{grad}}\,\mathrm{div}_F\,\mathrm{div}_F(\mathrm{sym\,curl}_F\,\boldsymbol{v}) = 0 = \mathrm{div}_F\,\mathrm{div}_F\,\boldsymbol{\tau}_1.$$

Then by Lemma 5.4,

$$\mathrm{sym\,curl}_F(I_F^{\mathrm{sym\,curl}}\boldsymbol{v}) = I_F^{\mathrm{div\,div}}(\mathrm{sym\,curl}_F\,\boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in \mathcal{C}^\infty(F;\mathbb{R}^2). \tag{78}$$

Combining (77) and (78) yields the commutative diagram (66). ☐

Next we show the smooth finite element div div complex in two dimensions. Define global finite element spaces

$$\begin{aligned}\boldsymbol{V}_h^{\mathrm{sym\,curl}} &:= \{\boldsymbol{v}_h \in \boldsymbol{H}^1(\Omega;\mathbb{R}^2) : \boldsymbol{v}_h|_F \in \mathbb{P}_{k+1}(F;\mathbb{R}^2) \text{ for each } F \in \mathcal{F}_h, \text{ all the} \\ &\qquad \text{degrees of freedom (67)-(69) are single-valued}\}, \\ \boldsymbol{\Sigma}_h^{\mathrm{div\,div}} &:= \{\boldsymbol{\tau}_h \in \boldsymbol{H}(\mathrm{div\,div},\Omega;\mathbb{S}) : \boldsymbol{\tau}_h|_F \in \mathbb{P}_k(F;\mathbb{S}) \text{ for each } F \in \mathcal{F}_h, \text{ all the} \\ &\qquad \text{degrees of freedom (71)-(74) are single-valued}\}, \\ V_h^{\mathrm{grad}} &:= \{q_h \in H^1(\Omega) : q_h|_K \in \mathbb{P}_{k-2}(F) \text{ for each } F \in \mathcal{F}_h\}.\end{aligned}$$

Counting the dimensions of these spaces, we have

$$\dim \boldsymbol{V}_h^{\operatorname{sym}\operatorname{curl}} = 12\#\mathcal{V}_h + (4k-14)\#\mathcal{E}_h + (k-3)(k-4)\#\mathcal{F}_h,$$

$$\dim \boldsymbol{\Sigma}_h^{\operatorname{div}\operatorname{div}} = 10\#\mathcal{V}_h + (5k-14)\#\mathcal{E}_h + \frac{3}{2}(k-3)(k-4)\#\mathcal{F}_h - 3\#\mathcal{F}_h,$$

$$\dim V_h^{\mathrm{grad}} = \#\mathcal{V}_h + (k-3)\#\mathcal{E}_h + \frac{1}{2}(k-3)(k-4)\#\mathcal{F}_h.$$

Theorem 5.6. *The finite element* $\operatorname{div}\operatorname{div}$ *complex in two dimensions*

$$\boldsymbol{RT} \xrightarrow{\subset} \boldsymbol{V}_h^{\operatorname{sym}\operatorname{curl}} \xrightarrow{\operatorname{sym}\operatorname{curl}} \boldsymbol{\Sigma}_h^{\operatorname{div}\operatorname{div}} \xrightarrow{\operatorname{div}\operatorname{div}} V_h^{\mathrm{grad}} \to 0 \tag{79}$$

is exact.

Proof. The inclusion $\operatorname{sym}\operatorname{curl} \boldsymbol{V}_h^{\operatorname{sym}\operatorname{curl}} \subseteq \boldsymbol{\Sigma}_h^{\operatorname{div}\operatorname{div}}$ follows from (67)-(69) and (71)-(72), and $\operatorname{div}\operatorname{div} \boldsymbol{\Sigma}_h^{\operatorname{div}\operatorname{div}} \subseteq V_h^{\mathrm{grad}}$ holds from (71) and (73). Hence (79) is a complex.

For $\boldsymbol{\tau}_h \in \boldsymbol{\Sigma}_h^{\operatorname{div}\operatorname{div}} \cap \ker(\operatorname{div}\operatorname{div})$, there exists $\boldsymbol{v}_h \in \boldsymbol{H}^1(\Omega;\mathbb{R}^2)$ such that $\boldsymbol{\tau}_h = \operatorname{sym}\operatorname{curl} \boldsymbol{v}_h$ and $\boldsymbol{v}_h|_F \in \mathbb{P}_{k+1}(F;\mathbb{R}^2)$ for each $F \in \mathcal{F}_h$. Thanks to DoFs (71)-(72), $\boldsymbol{v}_h \in \boldsymbol{V}_h^{\operatorname{sym}\operatorname{curl}}$. Thus $\boldsymbol{\Sigma}_h^{\operatorname{div}\operatorname{div}} \cap \ker(\operatorname{div}\operatorname{div}) = \operatorname{sym}\operatorname{curl} \boldsymbol{V}_h^{\operatorname{sym}\operatorname{curl}}$. And by the Euler's identity,

$$\begin{aligned}\dim(\operatorname{div}\operatorname{div} \boldsymbol{\Sigma}_h^{\operatorname{div}\operatorname{div}}) &= \dim \boldsymbol{\Sigma}_h^{\operatorname{div}\operatorname{div}} - \dim \boldsymbol{V}_h^{\operatorname{sym}\operatorname{curl}} + 3 \\ &= -2\#\mathcal{V}_h + k\#\mathcal{E}_h + \frac{1}{2}(k-3)(k-4)\#\mathcal{F}_h - 3\#\mathcal{F}_h + 3 \\ &= \#\mathcal{V}_h + (k-3)\#\mathcal{E}_h + \frac{1}{2}(k-3)(k-4)\#\mathcal{F}_h = \dim V_h^{\mathrm{grad}},\end{aligned}$$

which implies $\operatorname{div}\operatorname{div} \boldsymbol{\Sigma}_h^{\operatorname{div}\operatorname{div}} = V_h^{\mathrm{grad}}$. ☐

The finite element $\operatorname{div}\operatorname{div}$ complex (79) is smoother than that in [14, 30] as $V_h^{\mathrm{grad}} \subset H^1(\Omega)$ and $\boldsymbol{V}_h^{\operatorname{sym}\operatorname{curl}} \subset \boldsymbol{H}^2(\Omega;\mathbb{R}^2)$.

6. Finite Element Elasticity Complex

In this section we present a finite element elasticity complex. In the complex, the H^1-conforming finite element is a variant of the finite element developed by Neilan for the Stokes complex [34] with modified DoFs. The $\boldsymbol{H}(\operatorname{div};\mathbb{S})$-conforming finite element is the Hu-Zhang element [28, 31] with modified DoFs. The missing component is $\boldsymbol{H}(\operatorname{inc};\mathbb{S})$-conforming finite element which is the focus of this section.

6.1. H^1 conforming finite element for vectors.

Recall the H^1-conforming finite element for vectors by Neilan in [34]. The space of shape functions is chosen as $\mathbb{P}_{k+1}(K;\mathbb{R}^3)$ for $k+1 \geq 7$. The degrees of freedom are

$$\boldsymbol{v}(\delta), \nabla \boldsymbol{v}(\delta), \nabla^2 \boldsymbol{v}(\delta) \quad \forall\, \delta \in \mathcal{V}(K), \tag{80}$$

$$(\boldsymbol{v}, \boldsymbol{q})_e \quad \forall\, \boldsymbol{q} \in \mathbb{P}_{k-5}(e;\mathbb{R}^3), e \in \mathcal{E}(K), \tag{81}$$

$$(\partial_{n_i} \boldsymbol{v}, \boldsymbol{q})_e \quad \forall\, \boldsymbol{q} \in \mathbb{P}_{k-4}(e;\mathbb{R}^3), e \in \mathcal{E}(K), i = 1, 2, \tag{82}$$

$$(\operatorname{sym}\operatorname{curl}_F(\boldsymbol{v} \times \boldsymbol{n}), \boldsymbol{q})_F \quad \forall\, \boldsymbol{q} \in \operatorname{sym}(\boldsymbol{x}_F^{\perp} \mathbb{P}_{k-5}(F;\mathbb{R}^2)), F \in \mathcal{F}(K), \tag{83}$$

$$(\nabla_F^2(\boldsymbol{v} \cdot \boldsymbol{n}), \boldsymbol{q})_F \quad \forall\, \boldsymbol{q} \in \boldsymbol{x}\boldsymbol{x}^{\intercal} \mathbb{P}_{k-5}(F), F \in \mathcal{F}(K), \tag{84}$$

$$(\operatorname{def} \boldsymbol{v}, \boldsymbol{q})_K \quad \forall\, \boldsymbol{q} \in \operatorname{sym}(\mathbb{P}_{k-3}(K;\mathbb{R}^3)\boldsymbol{x}^{\intercal}). \tag{85}$$

The face moments in the original Neilan's element is $(\boldsymbol{v}, \boldsymbol{q})_F$, $\forall\, \boldsymbol{q} \in \mathbb{P}_{k-5}(F;\mathbb{R}^3)$. We split it into two $(\mathrm{d}\mathbb{B})'$ forms as $\boldsymbol{v} \times \boldsymbol{n} \in \boldsymbol{H}(\operatorname{sym}\operatorname{curl}_F, F)$ and $\boldsymbol{v} \cdot \boldsymbol{n} \in H^2(F)$

cf. the trace complexes (35) and (36) and DoFs (57) and (70). The unisovlence of this variant is easy. The Neilan element has extra super-smoothness at vertices and edges. Note that the normal derivative is only continuous on edges not on faces and thus this element is only in H^1 not H^2. To construct an H^2-conforming element on tetrahedron, the degree of polynomial will be higher, i.e. $k+1 \geq 9$; see Zhang [35].

6.2. $H(\mathrm{div})$ **conforming finite element for symmetric tensors.** Recall the $\boldsymbol{H}(\mathrm{div})$-conforming finite element for symmetric tensors in [31]. The space of shape functions is chosen as $\mathbb{P}_{k-2}(K;\mathbb{S})$ for $k-2 \geq 4$. The degrees of freedom are

$$\boldsymbol{\tau}(\delta) \quad \forall\, \delta \in \mathcal{V}(K), \tag{86}$$

$$(\boldsymbol{n}_i^{\intercal}\boldsymbol{\tau}\boldsymbol{n}_j, q)_e \quad \forall\, q \in \mathbb{P}_{k-4}(e), e \in \mathcal{E}(K), i,j = 1,2, \tag{87}$$

$$(\boldsymbol{n}_i^{\intercal}\boldsymbol{\tau}\boldsymbol{t}, q)_e \quad \forall\, q \in \mathbb{P}_{k-4}(e), e \in \mathcal{E}(K), i = 1,2, \tag{88}$$

$$(\boldsymbol{\tau}\boldsymbol{n}, \boldsymbol{q})_F \quad \forall\, \boldsymbol{q} \in \mathbb{P}_{k-5}(F;\mathbb{R}^3), F \in \mathcal{F}(K), \tag{89}$$

$$(\mathrm{div}\,\boldsymbol{\tau}, \boldsymbol{q})_K \quad \forall\, \boldsymbol{q} \in \mathbb{P}_{k-3}(K;\mathbb{R}^3)\backslash \boldsymbol{RM}, \tag{90}$$

$$(\boldsymbol{\tau}, \boldsymbol{q})_K \quad \forall\, \boldsymbol{q} \in \boldsymbol{x} \times \mathbb{P}_{k-6}(K;\mathbb{S}) \times \boldsymbol{x}. \tag{91}$$

The boundary degree of freedom (86)-(89) will determine the trace $\boldsymbol{\tau}\boldsymbol{n}$ uniquely by the unisolvence of the Lagrange elements. Due to the characterization of $\boldsymbol{H}(\mathrm{div}, K;\mathbb{S})$ bubble function, cf. (47), $\mathbb{B}^{\mathrm{div}}_{K,k-2}$ is uniquely determined by DoF $(\boldsymbol{\tau}, \boldsymbol{q})_K$ for $\boldsymbol{q} \in \mathbb{P}_{k-4}(K;\mathbb{S})$, i.e. $(\mathbb{B}^{\mathrm{div}}_{K,k-2})' \cong \mathbb{P}_{k-4}(K;\mathbb{S})$. The unisovlence will be a consequence of the characterization of the dual of $\mathbb{B}^{\mathrm{div}}_{K,k-2}$ based on Lemma 4.10.

Lemma 6.1. *The bubble space* $\mathbb{B}^{\mathrm{div}}_{K,k-2}$ *is uniquely determined by DoFs* (90)-(91) *and the subspace* $\mathrm{inc}\,\mathbb{B}^{\mathrm{inc}}_{K,k}$ *is determined by DoF* (91).

Proof. By Lemma 4.10, we can determine $\mathbb{B}^{\mathrm{div}}_{K,k-2}$ by two parts: one part for $\mathrm{div}\,\mathbb{B}^{\mathrm{div}}_{K,k-2}$ and the other part for $\mathrm{inc}\,\mathbb{B}^{\mathrm{inc}}_{K,k}$. Thanks to the bubble complex (46), $\mathrm{div}\,\mathbb{B}^{\mathrm{div}}_{K,k-2} = \mathbb{P}_{k-3}(K;\mathbb{R}^3)\backslash \boldsymbol{RM}$ which motivates DoF (90).

For $\boldsymbol{\tau} \in \mathrm{inc}\,\mathbb{B}^{\mathrm{inc}}_{K,k}$, $\boldsymbol{\tau} \perp \mathrm{def}\,\boldsymbol{H}^1(K;\mathbb{R}^3)$ through integration by parts. If (91) vanishes for $\boldsymbol{\tau}$, from the space decomposition $\mathbb{P}_{k-4}(K;\mathbb{S}) = \mathrm{def}\,\mathbb{P}_{k-3}(K;\mathbb{R}^3) \oplus (\boldsymbol{x} \times \mathbb{P}_{k-6}(K;\mathbb{S}) \times \boldsymbol{x})$ in (21), we conclude $\boldsymbol{\tau} \perp \mathbb{P}_{k-4}(K;\mathbb{S})$ and thus $\boldsymbol{\tau} = \boldsymbol{0}$. So we have proved $(\mathrm{inc}\,\mathbb{B}^{\mathrm{inc}}_{K,k})' \cong \boldsymbol{x} \times \mathbb{P}_{k-6}(K;\mathbb{S}) \times \boldsymbol{x}$, which indicates DoF (91). □

6.3. $H(\mathrm{inc})$ **conforming finite element for symmetric tensors.** With previous preparations, we can construct an $H(\mathrm{inc})$ conforming finite element now. Take

$\mathbb{P}_k(K;\mathbb{S}), k \geq 6$, as the space of shape functions. The degrees of freedom are

$$\boldsymbol{\tau}(\delta), \nabla\boldsymbol{\tau}(\delta) \quad \forall\, \delta \in \mathcal{V}(K), \tag{92}$$
$$(\nabla\times\boldsymbol{\tau}\times\nabla)(\delta) \quad \forall\, \delta \in \mathcal{V}(K), \tag{93}$$
$$(\boldsymbol{\tau}, \boldsymbol{q})_e \quad \forall\, \boldsymbol{q} \in \mathbb{P}_{k-4}(e;\mathbb{S}), e \in \mathcal{E}(K), \tag{94}$$
$$(\nabla\times\boldsymbol{\tau}\cdot\boldsymbol{t}, \boldsymbol{q})_e \quad \forall\, \boldsymbol{q} \in \mathbb{P}_{k-3}(e;\mathbb{R}^3), e \in \mathcal{E}(K), \tag{95}$$
$$(\boldsymbol{n}_i^{\intercal}(\nabla\times\boldsymbol{\tau}\times\nabla)\boldsymbol{n}_j, q)_e \quad \forall\, q \in \mathbb{P}_{k-4}(e), e \in \mathcal{E}(K), i,j = 1,2, \tag{96}$$
$$(\boldsymbol{n}_i^{\intercal}(\nabla\times\boldsymbol{\tau}\times\nabla)\boldsymbol{t}, q)_e \quad \forall\, q \in \mathbb{P}_{k-4}(e), e \in \mathcal{E}(K), i = 1,2, \tag{97}$$
$$(\operatorname{div}_F\operatorname{div}_F\operatorname{tr}_1(\boldsymbol{\tau}), q)_F \quad \forall\, q \in \mathbb{P}_{k-5}(F)\backslash\mathbb{P}_1(F), F \in \mathcal{F}(K), \tag{98}$$
$$(\operatorname{tr}_1(\boldsymbol{\tau}), \boldsymbol{q})_F \quad \forall\, \boldsymbol{q} \in \operatorname{sym}(\boldsymbol{x}^{\perp}\mathbb{P}_{k-5}(F;\mathbb{R}^2)), F \in \mathcal{F}(K), \tag{99}$$
$$(\operatorname{rot}_F\operatorname{tr}_2(\boldsymbol{\tau}), \boldsymbol{q})_F \quad \forall\, \boldsymbol{q} \in \mathbb{P}_{k-6}(F;\mathbb{S})\cdot\boldsymbol{x}^{\perp}, F \in \mathcal{F}(K), \tag{100}$$
$$(\operatorname{tr}_2(\boldsymbol{\tau}), \boldsymbol{q})_F \quad \forall\, \boldsymbol{q} \in \boldsymbol{x}\boldsymbol{x}^{\intercal}\mathbb{P}_{k-5}(F), F \in \mathcal{F}(K), \tag{101}$$
$$(\operatorname{inc}\boldsymbol{\tau}, \boldsymbol{q})_K \quad \forall\, \boldsymbol{q} \in \boldsymbol{x}\times\mathbb{P}_{k-6}(K;\mathbb{S})\times\boldsymbol{x}, \tag{102}$$
$$(\boldsymbol{\tau}, \boldsymbol{q})_K \quad \forall\, \boldsymbol{q} \in \operatorname{sym}(\boldsymbol{x}\mathbb{P}_{k-3}(K;\mathbb{R}^3)). \tag{103}$$

We first show the trace is uniquely determined by the degree of freedom (92)-(101) on the boundary.

Lemma 6.2. *Let $F \in \mathcal{F}(K)$ and $\boldsymbol{\tau} \in \mathbb{P}_k(K;\mathbb{S})$. If all the degrees of freedom (92)-(101) on face F vanish, then $\operatorname{tr}_1(\boldsymbol{\tau}) = \boldsymbol{0}$ and $\operatorname{tr}_2(\boldsymbol{\tau}) = \boldsymbol{0}$ on face F.*

Proof. We split our proof into several steps. For the ease of notation, denote $\boldsymbol{\sigma} = \nabla\times\boldsymbol{\tau}\times\nabla \in \mathbb{P}_{k-2}(K;\mathbb{S})$.

Step 1. Traces on edges vanish. By the vanishing degrees of freedom (92), (94), and (95), $\boldsymbol{\tau}\,|_e = \boldsymbol{0}$ and $(\nabla\times\boldsymbol{\tau}\cdot\boldsymbol{t})\,|_e = \boldsymbol{0}$ for any edge $e \in \mathcal{E}(F)$. Then it follows from Corollary 4.9, $\operatorname{tr}_{e,1}(\operatorname{tr}_1(\boldsymbol{\tau})) = \operatorname{tr}_{e,2}(\operatorname{tr}_1(\boldsymbol{\tau})) = \operatorname{tr}_{e,3}(\operatorname{tr}_2(\boldsymbol{\tau})) = 0$. Hence $\operatorname{tr}_1(\boldsymbol{\tau}) \in \mathbb{B}_{F,k}^{\operatorname{div}\operatorname{div}}$ and $\operatorname{tr}_2(\boldsymbol{\tau}) \in \mathbb{B}_{F,k-1}^{\operatorname{rot}}$.

By the vanishing degree of freedom (93), (96), and (97) for $\boldsymbol{\sigma}$, we know all components of $\boldsymbol{\sigma}\,|_e$, except $\boldsymbol{t}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{t}$, are zero.

Step 2. $\operatorname{tr}_1(\boldsymbol{\tau})$ vanishes. Using (39), $\operatorname{div}_F\operatorname{div}_F(\operatorname{tr}_1(\boldsymbol{\tau}))\,|_{\partial F} = \boldsymbol{n}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{n}\,|_{\partial F} = 0$. Thanks to Lemma 5.4, it follows from the vanishing (92), (98) and (99) that $\operatorname{tr}_1(\boldsymbol{\tau}) = \boldsymbol{n}\times\boldsymbol{\tau}\times\boldsymbol{n} = \boldsymbol{0}$.

Step 3. $\operatorname{tr}_2(\boldsymbol{\tau})$ vanishes. Apply (40) to acquire $\operatorname{rot}_F(\operatorname{tr}_2(\boldsymbol{\tau}))\,|_{\partial F} = \boldsymbol{n}\cdot\boldsymbol{\sigma}\times\boldsymbol{n}\,|_{\partial F} = \boldsymbol{0}$. By Lemma 5.1, it follows from the vanishing (100) and (101) that $\operatorname{tr}_2(\boldsymbol{\tau}) = \boldsymbol{0}$. $\square$

Now we are in the position to present the unisolvence.

Theorem 6.3. *The degrees of freedom (92)-(103) are unisolvent for $\mathbb{P}_k(K;\mathbb{S})$.*

Proof. We count the number of degrees of freedom (92)-(103) by the dimension of the sub-simplex

$$
\begin{aligned}
&4 \text{ vertices} : 4 \times (6 + 3 \times 6 + 6) = 120;\\
&6 \text{ edges} : 6[6(k-3) + 3(k-2) + 3(k-3) + 2(k-3)] = 6[14(k-3) + 3];\\
&4 \text{ faces} : 4\Big[\binom{k-3}{2} - 3 + 2\binom{k-3}{2} + 2\binom{k-3}{2} - 3 + \binom{k-3}{2}\Big]\\
&\qquad\qquad = 4\big[3(k-3)(k-4) - 6\big];\\
&1 \text{ volume} : 6\binom{k-1}{3} - 3\binom{k}{3} + 6 + 3\binom{k}{3} = (k-1)(k-2)(k-3) + 6\\
&\qquad\qquad = k^3 - 6k^2 + 11k.
\end{aligned}
$$

The total dimension is $k^3 + 6k^2 + 11k + 6$, which agrees with $\dim \mathbb{P}_k(K;\mathbb{S})$.

Take any $\boldsymbol{\tau} \in \mathbb{P}_k(K;\mathbb{S})$ and suppose all the degrees of freedom (92)-(103) vanish. We are going to prove it is zero.

First of all, by Lemma 6.2, we conclude $\operatorname{tr}_1(\boldsymbol{\tau}) = \boldsymbol{0}$ and $\operatorname{tr}_2(\boldsymbol{\tau}) = \boldsymbol{0}$ and thus $\boldsymbol{\tau} \in \boldsymbol{H}_0(\operatorname{inc}, K;\mathbb{S})$, which immediately induces $\operatorname{inc}\boldsymbol{\tau} \in \operatorname{inc}\mathbb{B}^{\text{inc}}_{K,k}$. Then by Lemma 6.1, vanishing (102) implies $\operatorname{inc}\boldsymbol{\tau} = \boldsymbol{0}$.

By the complex for bubble function spaces (46), there exists $\boldsymbol{v} \in \mathbb{P}_{k-3}(K;\mathbb{R}^3)$ such that $\boldsymbol{\tau} = \operatorname{def}(b_K\boldsymbol{v})$. Lastly by the characterization of $(\,\mathrm{d}\mathbb{B})'$ for $\mathrm{d} = \operatorname{def}, \mathbb{B} = b_K\mathbb{P}_{k-3}(K;\mathbb{R}^3)$, and the isomorphism $\operatorname{div} : \operatorname{sym}(\boldsymbol{x}\mathbb{P}_{k-3}(K;\mathbb{R}^3)) \to \mathbb{P}_{k-3}(K;\mathbb{R}^3)$, cf. Lemma 3.1, we conclude $\boldsymbol{v} = \boldsymbol{0}$.

∎

6.4. **Finite element elasticity complex in three dimensions.** For an integer $k \geq 6$, define global finite element spaces

$$
\begin{aligned}
\boldsymbol{V}_h &:= \{\boldsymbol{v}_h \in \boldsymbol{H}^1(\Omega;\mathbb{R}^3) : \boldsymbol{v}_h|_K \in \mathbb{P}_{k+1}(K;\mathbb{R}^3) \text{ for each } K \in \mathcal{T}_h, \text{ all the}\\
&\qquad \text{degrees of freedom (80)-(84) are single-valued}\},\\
\boldsymbol{\Sigma}_h^{\text{inc}} &:= \{\boldsymbol{\tau}_h \in \boldsymbol{L}^2(\Omega;\mathbb{S}) : \boldsymbol{\tau}_h|_K \in \mathbb{P}_k(K;\mathbb{S}) \text{ for each } K \in \mathcal{T}_h, \text{ all the}\\
&\qquad \text{degrees of freedom (92)-(101) are single-valued}\},\\
\boldsymbol{\Sigma}_h^{\text{div}} &:= \{\boldsymbol{\tau}_h \in \boldsymbol{H}(\operatorname{div},\Omega;\mathbb{S}) : \boldsymbol{\tau}_h|_K \in \mathbb{P}_{k-2}(K;\mathbb{S}) \text{ for each } K \in \mathcal{T}_h, \text{ all the}\\
&\qquad \text{degrees of freedom (86)-(89) are single-valued}\},\\
\mathcal{Q}_h &:= \{\boldsymbol{q}_h \in \boldsymbol{L}^2(\Omega;\mathbb{R}^3) : \boldsymbol{q}_h|_K \in \mathbb{P}_{k-3}(K;\mathbb{R}^3) \text{ for each } K \in \mathcal{T}_h\}.
\end{aligned}
$$

By Lemma 6.2, $\boldsymbol{\tau}_h\ |_e$ is single-valued for $e \in \mathcal{E}_h$ and $\operatorname{tr}_1(\boldsymbol{\tau}_h), \operatorname{tr}_2(\boldsymbol{\tau}_h)$ are single-valued on each $F \in \mathcal{F}_h$, therefore by Lemma 4.3, $\boldsymbol{\Sigma}_h^{\text{inc}} \subset \boldsymbol{H}(\operatorname{inc},\Omega;\mathbb{S})$.

Counting the dimensions of these spaces, we have

$$\dim \boldsymbol{V}_h = 30\#\mathcal{V}_h + (9k-30)\#\mathcal{E}_h + \frac{3}{2}(k-3)(k-4)\#\mathcal{F}_h + \frac{1}{2}k(k-1)(k-2)\#\mathcal{T}_h,$$
$$\dim \boldsymbol{\Sigma}_h^{\rm inc} = 30\#\mathcal{V}_h + (14k-39)\#\mathcal{E}_h + \left(3k^2-21k+30\right)\#\mathcal{F}_h + (k^3-6k^2+11k)\#\mathcal{T}_h,$$
$$\dim \boldsymbol{\Sigma}_h^{\rm div} = 6\#\mathcal{V}_h + (5k-15)\#\mathcal{E}_h + \frac{3}{2}(k-3)(k-4)\#\mathcal{F}_h + (k-1)(k-2)(k-3)\#\mathcal{T}_h,$$
$$\dim \mathcal{Q}_h = \frac{1}{2}k(k-1)(k-2)\#\mathcal{T}_h.$$

Lemma 6.4. *Let $\boldsymbol{\tau} \in \boldsymbol{\Sigma}_h^{\rm inc}$ and $\operatorname{inc}\boldsymbol{\tau} = \boldsymbol{0}$. Then there exists $\boldsymbol{v} \in \boldsymbol{V}_h$ satisfying $\boldsymbol{\tau} = \operatorname{def}\boldsymbol{v}$.*

Proof. By the polynomial elasticity complex (12) and the elasticity complex (1), there exists $\boldsymbol{v} = (v_1, v_2, v_3)^{\intercal} \in \boldsymbol{H}^1(\Omega;\mathbb{R}^3)$ s.t. $\boldsymbol{\tau} = \operatorname{def}\boldsymbol{v}$ and $\boldsymbol{v}|_K \in \mathbb{P}_{k+1}(K;\mathbb{R}^3)$ for each $K \in \mathcal{T}_h$. We are going to show such $\boldsymbol{v} \in \boldsymbol{V}_h$ by verifying the continuity of degrees of freedom (80)-(84). As an H^1 function, $\boldsymbol{v}$ is continuous at vertices, edges and faces. The focus is on the derivatives of $\boldsymbol{v}$.

Due to the additional smoothness of $\boldsymbol{\Sigma}_h^{\rm inc}$, $\nabla(\operatorname{def}\boldsymbol{v})(\delta)$ is single-valued at each vertex $\delta \in \mathcal{V}_h$, and $(\operatorname{def}\boldsymbol{v})\,|_e$ and $(\nabla\times(\operatorname{def}\boldsymbol{v})\cdot\boldsymbol{t})\,|_e$ are single-valued on each edge $e \in \mathcal{E}_h$. Next we show that $\nabla\boldsymbol{v}$ is single-valued on each edge $e \in \mathcal{E}_h$. By (3),

$$\partial_t \boldsymbol{v} = (\nabla\boldsymbol{v})^{\intercal}\cdot\boldsymbol{t} = (\operatorname{def}\boldsymbol{v})\cdot\boldsymbol{t} + \frac{1}{2}\operatorname{mskw}(\nabla\times\boldsymbol{v})\cdot\boldsymbol{t} = (\operatorname{def}\boldsymbol{v})\cdot\boldsymbol{t} + \frac{1}{2}(\nabla\times\boldsymbol{v})\times\boldsymbol{t},$$

$$\nabla\times(\operatorname{def}\boldsymbol{v})\cdot\boldsymbol{t} = \frac{1}{2}\partial_t(\nabla\times\boldsymbol{v}) \tag{104}$$

on edge e with the unit tangential vector $\boldsymbol{t}$. Hence $(\nabla\times\boldsymbol{v})\times\boldsymbol{t}$ and $\partial_t(\nabla\times\boldsymbol{v})$ are single-valued across e. Taking any face $F \in \mathcal{F}_h^i$ shared by $K_1, K_2 \in \mathcal{T}_h$, it follows from the single-valued $(\nabla\times\boldsymbol{v})\times\boldsymbol{t}|_{\partial F}$ that $(\nabla\times\boldsymbol{v})|_{K_1}$ and $(\nabla\times\boldsymbol{v})|_{K_2}$ coincide with each other at the three vertices of F. Thus $(\nabla\times\boldsymbol{v})(\delta)$ is single-valued at each vertex $\delta \in \mathcal{V}_h$, which together with the single-valued $\partial_t(\nabla\times\boldsymbol{v})$ on $\mathcal{E}_h$ implies that $\nabla\times\boldsymbol{v}$ is single-valued on each edge $e \in \mathcal{E}_h$. Since $(\nabla\boldsymbol{v})^{\intercal} = \operatorname{def}\boldsymbol{v} + \frac{1}{2}\operatorname{mskw}(\nabla\times\boldsymbol{v})$, $\nabla\boldsymbol{v}$ is single-valued on each edge $e \in \mathcal{E}_h$.

By the identity

$$\partial_{ij}v_k = \partial_i((\operatorname{def}\boldsymbol{v})_{jk}) + \partial_j((\operatorname{def}\boldsymbol{v})_{ki}) - \partial_k((\operatorname{def}\boldsymbol{v})_{ij}) \quad \text{for } i,j,k = 1,2,3,$$

the tensor $\nabla^2\boldsymbol{v}(\delta)$ is single-valued at each vertex $\delta \in \mathcal{V}_h$ as $\nabla(\operatorname{def}\boldsymbol{v})(\delta)$ is single-valued. Therefore $\boldsymbol{v} \in \boldsymbol{V}_h$. ∎

Theorem 6.5. *The finite element elasticity complex*

$$\boldsymbol{RM} \xrightarrow{\subset} \boldsymbol{V}_h \xrightarrow{\operatorname{def}} \boldsymbol{\Sigma}_h^{\rm inc} \xrightarrow{\operatorname{inc}} \boldsymbol{\Sigma}_h^{\rm div} \xrightarrow{\operatorname{div}} \mathcal{Q}_h \to \boldsymbol{0} \tag{105}$$

is exact.

Proof. The inclusion $\operatorname{def}\boldsymbol{V}_h \subseteq \boldsymbol{\Sigma}_h^{\rm inc}$ follows from (104) and (37)-(38), and $\operatorname{inc}\boldsymbol{\Sigma}_h^{\rm inc} \subseteq \boldsymbol{\Sigma}_h^{\rm div}$ holds from (39)-(40) and Lemma 6.2. The proof of $\operatorname{div}\boldsymbol{\Sigma}_h^{\rm div} = \mathcal{Q}_h$ can be found in [28, 31]. Hence (105) is a complex. And

$$\begin{aligned}\dim(\boldsymbol{\Sigma}_h^{\mathrm{div}}\cap\ker(\mathrm{div})) &= \dim\boldsymbol{\Sigma}_h^{\mathrm{div}} - \dim\mathcal{Q}_h \\ &= 6\#\mathcal{V}_h + (5k-15)\#\mathcal{E}_h + \frac{3}{2}(k-3)(k-4)\#\mathcal{F}_h \\ &\quad + \frac{1}{2}(k-1)(k-2)(k-6)\#\mathcal{T}_h.\end{aligned}$$

It follows from Lemma 6.4 that $\operatorname{def}\boldsymbol{V}_h = \boldsymbol{\Sigma}_h^{\mathrm{inc}}\cap\ker(\mathrm{inc})$. Thus

$$\begin{aligned}\dim(\mathrm{inc}\,\boldsymbol{\Sigma}_h^{\mathrm{inc}}) &= \dim\boldsymbol{\Sigma}_h^{\mathrm{inc}} - \dim\operatorname{def}\boldsymbol{V}_h = \dim\boldsymbol{\Sigma}_h^{\mathrm{inc}} - \dim\boldsymbol{V}_h + 6 \\ &= (5k-9)\#\mathcal{E}_h + \frac{3}{2}(k^2-7k+8)\#\mathcal{F}_h \\ &\quad + \frac{1}{2}(k^3-9k^2+20k)\#\mathcal{T}_h + 6.\end{aligned}$$

Then we get from the Euler's identity that

$$\dim(\boldsymbol{\Sigma}_h^{\mathrm{div}}\cap\ker(\mathrm{div})) - \dim(\mathrm{inc}\,\boldsymbol{\Sigma}_h^{\mathrm{inc}}) = 6\#\mathcal{V}_h - 6\#\mathcal{E}_h + 6\#\mathcal{F}_h - 6\#\mathcal{T}_h + 6 = 0.$$

Therefore $\boldsymbol{\Sigma}_h^{\mathrm{div}}\cap\ker(\mathrm{div}) = \mathrm{inc}\,\boldsymbol{\Sigma}_h^{\mathrm{inc}}$. □

Remark 6.6. *The finite element elasticity complex in* [20] *holds for* $k\geq 4$*, while in complex* (105) $k\geq 6$ *is required. The space* $\boldsymbol{V}_h$ *in complex* (105) *is* H^1*-conforming, while the corresponding space in* [20] *is* H^2*-conforming, and* $\boldsymbol{\Sigma}_h^{\mathrm{inc}}$ *in* [20] *is* H^1*-conforming. Although macro finite elements are adopted, the finite element elasticity complex in* [20] *is still smoother than complex* (105).

7. Commutative Diagram

In this section, we will show the canonical interpolation operators based on DoFs for the finite element elasticity complex (105) commutes with the differential operators.

Let $I_K^{\mathrm{div}}:\mathcal{C}^\infty(K;\mathbb{S})\to\mathbb{P}_{k-2}(K;\mathbb{S})$ be the nodal interpolation operator based on DoFs (86)-(91), $I_K^{\mathrm{inc}}:\mathcal{C}^\infty(K;\mathbb{S})\to\mathbb{P}_{k}(K;\mathbb{S})$ be the nodal interpolation operator based on DoFs (92)-(103), and $I_K^{\mathrm{def}}:\mathcal{C}^\infty(K;\mathbb{S})\to\mathbb{P}_{k+1}(K;\mathbb{S})$ be the nodal interpolation operator based on DoFs (80)-(85). Recall $Q_K := Q_K^{k-3}$ is the L^2-projection. Here for the ease of notation, we skip the degree of polynomial which will be clear in the context.

Lemma 7.1. *It holds*

$$\mathrm{div}(I_K^{\mathrm{div}}\boldsymbol{\tau}) = Q_K\,\mathrm{div}\,\boldsymbol{\tau}\quad\forall\,\boldsymbol{\tau}\in\mathcal{C}^\infty(K;\mathbb{S}). \tag{106}$$

Proof. For $\boldsymbol{\tau}\in\mathcal{C}^\infty(K;\mathbb{S})$ and $\boldsymbol{q}\in\boldsymbol{RM}$, employing the integration by parts,

$$(Q_K\,\mathrm{div}\,\boldsymbol{\tau} - \mathrm{div}(I_K^{\mathrm{div}}\boldsymbol{\tau}),\boldsymbol{q})_K = (\mathrm{div}(\boldsymbol{\tau}-I_K^{\mathrm{div}}\boldsymbol{\tau}),\boldsymbol{q})_K = 0.$$

For $\boldsymbol{q}\in\mathbb{P}_{k-3}(K;\mathbb{R}^3)\backslash\boldsymbol{RM}$, by (90),

$$(Q_K\,\mathrm{div}\,\boldsymbol{\tau} - \mathrm{div}(I_K^{\mathrm{div}}\boldsymbol{\tau}),\boldsymbol{q})_K = (\mathrm{div}(\boldsymbol{\tau}-I_K^{\mathrm{div}}\boldsymbol{\tau}),\boldsymbol{q})_K = 0.$$

Combining the last two identities gives (106). □

Note that the canonical interpolation for the original Hu-Zhang element using $(\boldsymbol{\tau},\boldsymbol{q})_K$ for $\boldsymbol{q}\in\mathbb{P}_{k-4}(K;\mathbb{S})$ as interior DoFs will not satisfy the property (106).

Lemma 7.2. *It holds*

$$\operatorname{inc}(I_K^{\rm inc}\boldsymbol{\tau}) = I_K^{\rm div}(\operatorname{inc}\boldsymbol{\tau}) \quad \forall\, \boldsymbol{\tau}\in\mathcal{C}^\infty(K;\mathbb{S}). \tag{107}$$

Proof. For ease of presentation, set $\boldsymbol{\sigma} = I_K^{\rm div}(\operatorname{inc}\boldsymbol{\tau}) - \operatorname{inc}(I_K^{\rm inc}\boldsymbol{\tau}) \in \mathbb{P}_{k-2}(K;\mathbb{S})$. By (106), we get

$$\operatorname{div}\boldsymbol{\sigma} = \operatorname{div}\left(I_K^{\rm div}(\operatorname{inc}\boldsymbol{\tau})\right) = Q_K \operatorname{div}(\operatorname{inc}\boldsymbol{\tau}) = \mathbf{0}.$$

Thanks to DoFs (93) and (96)-(97), the DoFs (86)-(88) of $\boldsymbol{\sigma}$ vanish. Then apply the integration by parts to get

$$(\operatorname{div}_F\operatorname{div}_F(\boldsymbol{n}\times(\boldsymbol{\tau}-I_K^{\rm inc}\boldsymbol{\tau})\times\boldsymbol{n}), q)_F = 0 \quad \forall\, q\in\mathbb{P}_1(F),$$
$$(\operatorname{rot}_F(\operatorname{tr}_2(\boldsymbol{\tau}-I_K^{\rm inc}\boldsymbol{\tau})), \boldsymbol{q})_F = 0 \quad \forall\, q\in\boldsymbol{RT}.$$

For $\boldsymbol{q}\in\mathbb{P}_{k-5}(F;\mathbb{R}^3)$ and $F\in\mathcal{F}(K)$, it follows from (39)-(40), and DoFs (98), (100) that

$$\begin{aligned}(\boldsymbol{\sigma}\boldsymbol{n},\boldsymbol{q})_F &= ((I_K^{\rm div}(\operatorname{inc}\boldsymbol{\tau}))\boldsymbol{n},\boldsymbol{q})_F - (\operatorname{inc}(I_K^{\rm inc}\boldsymbol{\tau})\boldsymbol{n},\boldsymbol{q})_F = (\operatorname{inc}(\boldsymbol{\tau}-I_K^{\rm inc}\boldsymbol{\tau})\boldsymbol{n},\boldsymbol{q})_F\\ &= (\boldsymbol{n}\cdot\operatorname{inc}(\boldsymbol{\tau}-I_K^{\rm inc}\boldsymbol{\tau})\cdot\boldsymbol{n},\boldsymbol{n}\cdot\boldsymbol{q})_F + (\boldsymbol{n}\times\operatorname{inc}(\boldsymbol{\tau}-I_K^{\rm inc}\boldsymbol{\tau})\cdot\boldsymbol{n},\boldsymbol{n}\times\boldsymbol{q})_F\\ &= (\operatorname{div}_F\operatorname{div}_F(\boldsymbol{n}\times(\boldsymbol{\tau}-I_K^{\rm inc}\boldsymbol{\tau})\times\boldsymbol{n}),\boldsymbol{n}\cdot\boldsymbol{q})_F\\ &\quad - (\operatorname{rot}_F(\operatorname{tr}_2(\boldsymbol{\tau}-I_K^{\rm inc}\boldsymbol{\tau})),\boldsymbol{n}\times\boldsymbol{q})_F = 0.\end{aligned}$$

For $\boldsymbol{q}\in\boldsymbol{x}\times\mathbb{P}_{k-6}(K;\mathbb{S})\times\boldsymbol{x}$,

$$(\boldsymbol{\sigma},\boldsymbol{q})_K = (\operatorname{inc}\boldsymbol{\tau} - \operatorname{inc}(I_K^{\rm inc}\boldsymbol{\tau}),\boldsymbol{q})_K = 0.$$

Therefore we conclude (107) from the uni-solvence of $H(\operatorname{div})$-conforming finite element for symmetric tensors. ∎

Lemma 7.3. *It holds*

$$\operatorname{def}(I_K^{\rm def}\boldsymbol{v}) = I_K^{\rm inc}(\operatorname{def}\boldsymbol{v}) \quad \forall\, \boldsymbol{v}\in\mathcal{C}^\infty(K;\mathbb{R}^3). \tag{108}$$

Proof. For ease of presentation, set $\boldsymbol{\tau} = I_K^{\rm inc}(\operatorname{def}\boldsymbol{v}) - \operatorname{def}(I_K^{\rm def}\boldsymbol{v}) \in \mathbb{P}_k(K;\mathbb{S})$. By (107), we get

$$\operatorname{inc}\boldsymbol{\tau} = \operatorname{inc}\left(I_K^{\rm inc}(\operatorname{def}\boldsymbol{v})\right) = I_K^{\rm div}(\operatorname{inc}(\operatorname{def}\boldsymbol{v})) = \mathbf{0}.$$

Then DoF (102) vanishes. This also means DoFs (93) and (96)-(97) vanish. By the definitions of $I_K^{\rm inc}$ and $I_K^{\rm def}$, we have for $\delta\in\mathcal{V}(K)$,

$$\nabla^i\boldsymbol{\tau}(\delta) = \nabla^i\operatorname{def}(\boldsymbol{v}-I_K^{\rm def}\boldsymbol{v}) = \mathbf{0}, \quad i=0,1,$$

and for $e\in\mathcal{E}(K)$,

$$(\boldsymbol{\tau},\boldsymbol{q})_e = (\operatorname{def}(\boldsymbol{v}-I_K^{\rm def}\boldsymbol{v}),\boldsymbol{q})_e = 0 \quad \forall\, \boldsymbol{q}\in\mathbb{P}_{k-4}(e;\mathbb{S}).$$

For $\boldsymbol{q}\in\mathbb{P}_{k-3}(e;\mathbb{R}^3)$ and $e\in\mathcal{E}(K)$, it follows

$$(\nabla\times\boldsymbol{\tau}\cdot\boldsymbol{t},\boldsymbol{q})_e = (\nabla\times\operatorname{def}(\boldsymbol{v}-I_K^{\rm def}\boldsymbol{v})\cdot\boldsymbol{t},\boldsymbol{q})_e = -\frac{1}{2}(\nabla\times(\boldsymbol{v}-I_K^{\rm def}\boldsymbol{v}),\partial_t\boldsymbol{q})_e = 0.$$

Hence DoFs (92) and (94)-(95) vanish.

Employing (37), it holds $\operatorname{tr}_1(\operatorname{def}(\boldsymbol{v}-I_K^{\rm def}\boldsymbol{v})) = \operatorname{sym}\operatorname{curl}_F((\boldsymbol{v}-I_K^{\rm def}\boldsymbol{v})\times\boldsymbol{n})$. Then for $\boldsymbol{q}\in\mathbb{P}_{k-5}(F)\backslash\mathbb{P}_1(F)$ and $F\in\mathcal{F}(K)$,

$$\begin{aligned}(\operatorname{div}_F\operatorname{div}_F\operatorname{tr}_1(\boldsymbol{\tau}),\boldsymbol{q})_F &= (\operatorname{div}_F\operatorname{div}_F\operatorname{tr}_1(\operatorname{def}(\boldsymbol{v}-I_K^{\rm def}\boldsymbol{v})),\boldsymbol{q})_F\\ &= (\operatorname{div}_F\operatorname{div}_F\operatorname{sym}\operatorname{curl}_F((\boldsymbol{v}-I_K^{\rm def}\boldsymbol{v})\times\boldsymbol{n}),\boldsymbol{q})_F = 0.\end{aligned}$$

We get for $\boldsymbol{q} \in \operatorname{sym}(\boldsymbol{x}^{\perp}\mathbb{P}_{k-5}(F;\mathbb{R}^2))$ that

$$(\operatorname{tr}_1(\boldsymbol{\tau}), \boldsymbol{q})_F = (\operatorname{tr}_1(\operatorname{def}(\boldsymbol{v} - I_K^{\operatorname{def}}\boldsymbol{v})), \boldsymbol{q})_F = (\operatorname{sym}\operatorname{curl}_F((\boldsymbol{v} - I_K^{\operatorname{def}}\boldsymbol{v}) \times \boldsymbol{n}), \boldsymbol{q})_F = 0.$$

Employing (38), it holds $\operatorname{tr}_2(\operatorname{def}(\boldsymbol{v} - I_K^{\operatorname{def}}\boldsymbol{v})) = \nabla_F^2((\boldsymbol{v} - I_K^{\operatorname{def}}\boldsymbol{v}) \cdot \boldsymbol{n})$. Then for $\boldsymbol{q} \in \mathbb{P}_{k-6}(F;\mathbb{S}) \cdot \boldsymbol{x}^{\perp}$,

$$\begin{aligned}(\operatorname{rot}_F \operatorname{tr}_2(\boldsymbol{\tau}), \boldsymbol{q})_F &= (\operatorname{rot}_F \operatorname{tr}_2(\operatorname{def}(\boldsymbol{v} - I_K^{\operatorname{def}}\boldsymbol{v})), \boldsymbol{q})_F \\ &= (\operatorname{rot}_F \nabla_F^2((\boldsymbol{v} - I_K^{\operatorname{def}}\boldsymbol{v}) \cdot \boldsymbol{n}), \boldsymbol{q})_F = 0.\end{aligned}$$

We obtain for $\boldsymbol{q} \in \boldsymbol{x}\boldsymbol{x}^{\intercal}\mathbb{P}_{k-5}(F)$ that

$$(\operatorname{tr}_2(\boldsymbol{\tau}), \boldsymbol{q})_F = (\operatorname{tr}_2(\operatorname{def}(\boldsymbol{v} - I_K^{\operatorname{def}}\boldsymbol{v})), \boldsymbol{q})_F = (\nabla_F^2((\boldsymbol{v} - I_K^{\operatorname{def}}\boldsymbol{v}) \cdot \boldsymbol{n}), \boldsymbol{q})_F = 0.$$

Thus DoFs (98)-(101) vanish.

For $\boldsymbol{q} \in \operatorname{sym}(\boldsymbol{x}\mathbb{P}_{k-3}(K;\mathbb{R}^3))$, it follows

$$(\boldsymbol{\tau}, \boldsymbol{q})_K = (\operatorname{def}\boldsymbol{v} - \operatorname{def}(I_K^{\operatorname{def}}\boldsymbol{v}), \boldsymbol{q})_K = 0.$$

Finally we conclude (108) from the uni-solvence of $H(\operatorname{inc})$-conforming finite element for symmetric tensors, i.e. Theorem 6.3. ☐

Define global commutative projection operators $I_h^{\operatorname{def}} : \mathcal{C}^{\infty}(\Omega;\mathbb{R}^3) \to \boldsymbol{V}_h$, $I_h^{\operatorname{inc}} : \mathcal{C}^{\infty}(\Omega;\mathbb{S}) \to \boldsymbol{\Sigma}_h^{\operatorname{inc}}$, $I_h^{\operatorname{div}} : \mathcal{C}^{\infty}(\Omega;\mathbb{S}) \to \boldsymbol{\Sigma}_h^{\operatorname{div}}$ and $Q_h : \mathcal{C}^{\infty}(\Omega;\mathbb{R}^3) \to \mathcal{Q}_h$ as follows: for each $K \in \mathcal{T}_h$,

$$\begin{aligned}(I_h^{\operatorname{def}}\boldsymbol{v})|_K &:= I_K^{\operatorname{def}}(\boldsymbol{v}|_K), \quad (I_h^{\operatorname{inc}}\boldsymbol{\tau})|_K := I_K^{\operatorname{inc}}(\boldsymbol{\tau}|_K), \\ (I_h^{\operatorname{div}}\boldsymbol{\tau})|_K &:= I_K^{\operatorname{div}}(\boldsymbol{\tau}|_K), \quad (Q_h\boldsymbol{v})|_K := Q_K(\boldsymbol{v}|_K).\end{aligned}$$

Then combining (106), (107) and (108) implies the following commutative diagram

$$\begin{array}{ccccccccccc}\boldsymbol{RM} & \xrightarrow{\subset} & \mathcal{C}^{\infty}(\Omega;\mathbb{R}^3) & \xrightarrow{\operatorname{def}} & \mathcal{C}^{\infty}(\Omega;\mathbb{S}) & \xrightarrow{\operatorname{inc}} & \mathcal{C}^{\infty}(\Omega;\mathbb{S}) & \xrightarrow{\operatorname{div}} & \mathcal{C}^{\infty}(\Omega;\mathbb{R}^3) & \longrightarrow & \boldsymbol{0} \\ & & \big\downarrow I_h^{\operatorname{def}} & & \big\downarrow I_h^{\operatorname{inc}} & & \big\downarrow I_h^{\operatorname{div}} & & \big\downarrow Q_h & & \\ \boldsymbol{RM} & \xrightarrow{\subset} & \boldsymbol{V}_h & \xrightarrow{\operatorname{def}} & \boldsymbol{\Sigma}_h^{\operatorname{inc}} & \xrightarrow{\operatorname{inc}} & \boldsymbol{\Sigma}_h^{\operatorname{div}} & \xrightarrow{\operatorname{div}} & \mathcal{Q}_h & \longrightarrow & \boldsymbol{0}.\end{array}$$

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