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On the Positive Effect of Delay on the Rate of Convergence of a Class of Linear Time-Delayed Systems

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***Abstract*—This paper is a comprehensive study of a long observed phenomenon of increase in the stability margin and so the rate of convergence of a class of linear systems due to time delay. Our results determine (a) in what systems the delay can lead to increase in the rate of convergence, (b) the exact range of time delay for which the rate of convergence is greater than that of the delay free system, and (c) an estimate on the value of the delay that leads to the maximum rate of convergence. We also show that the ultimate bound on the maximum achievable rate of convergence via time delay is e (Euler's number) times the delay free rate. For the special case when the system matrix eigenvalues are all negative real numbers, we expand our results to show that the rate of convergence in the presence of delay depends only on the eigenvalues with minimum and maximum real parts. Moreover, we determine the exact value of the maximum rate of convergence and the corresponding maximizing time delay. The final contribution of this paper is to show the application of our results in analyzing the use of a delayed feedback to increase the rate of convergence of an agreement algorithm for networked systems.**

***Index Terms*—Linear Time-delayed Systems, Relative Stability, Rate of Convergence, Lambert W Function, Accelerated Static Average Consensus**

I. Introduction

Many physical systems have time-delayed states because of communication, computation or processing lags [1], [2]. The stability and convergence properties of time-delayed systems have been studied extensively in the literature. Many of these results point to and highlight the adverse effect of time delay on system behavior, e.g., instability, undesirable oscillations and deficient performance [3], [4], [5], and attempt to address these issues. But, despite the prevailing intuition that links time delay to system sluggishness, performance loss and instability, there are also results that exploit time delay to improve performance for some systems. For example, it is demonstrated that delay is effective on stabilizing biology inspired systems [6], [7]. Some other work introduced delay to a dynamical system in order to stabilize oscillatory behavior [8], [9], tuning vibration absorbers [10], improving robustness [11], controlling flexible structure [12] and steering system trajectory [13], [14]. In this paper, we study the positive effect of the time delay on increasing the stability margin and the rate of convergence of a class of linear time-delayed systems. Our study is based on characterizing the variation of the real part of the right-most root of the characteristic equation of the system using the Lambert W function.

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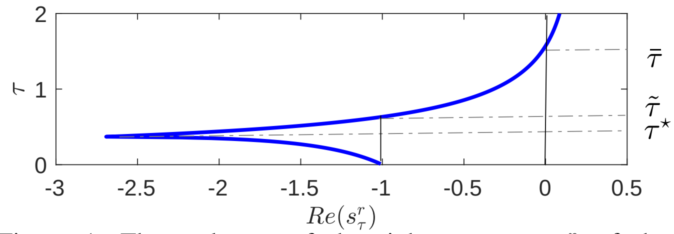


Figure 1: The real part of the rightmost root s_τ^r of the characteristic equation of system $\dot{x} = -x(t-\tau)$ versus time delay in $[0,2]$, i.e., $\tau \in [0,2]$.

For linear systems with delay, various methods are deployed to estimate the rate of convergence of the delayed systems. Lyapunov-based methods [15], [16], matrix measure [17], Riccati equation [18], Hopf bifurcation [19], α-stability [15] and pseudospectral and operator approximation techniques [20] are some of the approaches that are used to estimate the rate of convergence of time-delayed systems. As known in the literature, the rate of convergence of a zero input response of a linear time-invariant system in the absence and presence of time delay is determined by the magnitude of the real part of the rightmost root of the characteristic equation of the system [21], [22]. For a system without delay the rightmost root is the rightmost eigenvalue of the system matrix. For a linear time-delayed system with one fixed time delay, [23] uses the Lambert W function to determine the exact location of the roots of the characteristic equation, and subsequently the rate of convergence of the system. The aforementioned results only obtain or estimate the rate of convergence for a given time delay value.

For linear time-invariant (LTI) time-delayed systems that are exponentially stable in the absence of time delay, the *continuity stability property* theorem [1, Proposition 3.1] indicates that as time delay τ increases the system eventually may become unstable. The critical value of τ for which stability is violated is when the rightmost roots of the characteristic equation are on the imaginary axis–for any delay beyond this critical value the system is unstable. Even though the continuity stability property determines the *admissible* range of delay for which the system is exponentially stable, it does not discuss how the rate of convergence changes with time delay in this admissible range. As it appears, for linear time-delayed systems, the rightmost root(s) are not necessarily traversing towards the right half plane monotonically as time delay increases (see Fig. 1 for an example). Therefore, it is possible to increase the rate of convergence of a system using time delay. Some work in the literature point to increase of stability margin and consequently the rate of convergence of linear systems due to time delay [24], [25], [26], [27]. However, these results provide only sufficient conditions for ranges of time delay that result in increase of the rate of convergence for very specific linear systems. They also do not specify the maximum attainable rate of convergence due to the time delay.

To develop our results we invoke various properties of the Lambert W function (c.f. [28], [29], [30]), and use complex analysis and topology. The Lambert W function has been used to analyze time-delayed systems including stability analysis, eigenvalue assignment and obtaining the rate of convergence for different dynamical systems [23], [28], [31], [32].

Statement of contributions: in this paper we characterize fully how the real part of the rightmost root(s) of the characteristic equation and so the rate of convergence of a class of time-delayed LTI systems changes with delay. We start by showing that the rate of convergence of this class of time-delayed LTI systems can increase with time delay if and only if the argument of the rightmost eigenvalue(s) of the system matrix is strictly between $\frac{3\pi}{4}$ and $\frac{5\pi}{4}$; if a system does not satisfy this condition, its rate of convergence is in fact decreases strictly with delay in its admissible delay range. We then proceed to determine the exact range of time delay for which a system has a rate of convergence greater than the rate of a delay free system. For delays beyond this range we show that the rate of convergence decreases strictly. Our next result is to obtain an estimate on the value of the time delay corresponding to maximum achievable rate. Another interesting result that we establish is to show that the ultimate bound on the maximum achievable rate of convergence via time delay is $\mathrm{e} \approx 2.71828$ times the delay free rate. For the special case when the system matrix eigenvalues are all negative real numbers, the relative ease in mathematical manipulations allows us also to expand our results to show that the rate of convergence in the presence of delay depends only on the eigenvalues with minimum and maximum real parts. Moreover, we determine the exact value of the maximum rate of convergence and the corresponding maximizing time delay. In this special case note that based on our earlier results, the rate of convergence of the system, regardless of the value of its eigenvalues, is guaranteed to have a value greater than the rate of delay free system for some ranges of delay. An interesting application of the theoretical results developed in this paper is to use time-delayed states (outdated information) to accelerate the rate of convergence of linear distributed algorithms for networked systems. An example case of accelerated static average consensus algorithm is demonstrated in this paper. Fast convergence is highly desirable for agreement algorithms in networked system, because of their essential role in enabling many cooperative control problems (see [33] for more details). A preliminary version of this paper, which discusses only the special case of a system with a Hurwitz system matrix that has only real eigenvalues appears in [34]. The proof of the main results in [34] are presented in this paper.

Notation: we let $\mathbb{R}$, $\mathbb{R}_{>0}$, $\mathbb{R}_{\geq 0}$, $\mathbb{Z}$, and $\mathbb{C}$ denote the set of real, positive real, non-negative real, integer, and complex numbers, respectively. For $z \in \mathbb{C}$, $\mathrm{Re}(z)$, $\mathrm{Im}(z)$ represents its real and imaginary parts, respectively. Moreover, $|z|$ and $\arg(z)$ represent, respectively, its magnitude, i.e., $|z| = \sqrt{\mathrm{Re}(z)^2 + \mathrm{Im}(z)^2}$ and $\arg(z) = \text{atan2}(\mathrm{Im}(z), \mathrm{Re}(z))$. Moreover, $\mathrm{conj}(z)$ us the conjugate value of z. We let $\mathbb{C}^l = \{x \in \mathbb{C} | \mathrm{Re}(C) \geq 0\}$ and $\mathbb{C}^l_- = \{x \in \mathbb{C} | \mathrm{Re}(C) > 0\}$. The transpose of a matrix $\mathbf{A} = [a_{ij}] \in \mathbb{R}^{n\times m}$ is $\mathbf{A}^\top$ and its element in i^{th} row and j^{th} column is a_{ij}. The set of eigenvalues of matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ is $\mathrm{eig}(\mathbf{A})$. Given the sets $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A} \subsetneq \mathcal{B}$ means that $\mathcal{A}$ is the strict subset of $\mathcal{B}$. Moreover, $\mathcal{A}\backslash\mathcal{B}$ is the set of all elements of $\mathcal{A}$ that are not elements of $\mathcal{B}$. Finally, for a vector $\mathbf{s} \in \mathbb{R}^n$, $\mathrm{diag}(\mathbf{s})$ is the diagonal matrix whose n diagonal entries are the n elements of the vector $\mathbf{s}$.

Organization: the remainder of this paper is organized as follows. Section II gives a brief review of the relevant aspect of the Lambert W function. Section III presents the problem definition and our objectives. To facilitate our study we define a function, which we call delay rate gain. Section IV studies the properties of the delay rate gain function, paving the way towards our main results, which is given in Section V. Section VI demonstrates the application of our results in studying the rate of convergence of the Laplacian static average consensus algorithm in the presents of delay. Section VI also contains several numerical examples. Section VII gathers our concluding remarks and ideas for future work. Finally, the appendix at the end of the paper contains the proofs of the results of Section IV.

II. Lambert W function

Our work rely on some of the properties of the Lambert W function, which for convenience of the reader are reviewed below (c.f. [28], [29], [30]). For a given $z \in \mathbb{C}$, Lambert W function is defined as the solution of the equation $s\,\mathrm{e}^s = z$, i.e., $s = W(z)$. Except for $z = 0$, which gives $W(0) = 0$, W is a multivalued function with the infinite number of solutions denoted by $W_k(z)$ with $k \in \mathbb{Z}$, where W_k is called the k^{th} branch of W function. $W_k(z)$ can readily be computed in Matlab or Mathematica. For any $z \in \mathbb{R}$, the value of all the branches of the Lambert W function except for some parts of branch 0 and branch -1 are complex (non-zero imaginary part). Zero branch of the Lambert function, W_0 is of special interest in this paper. This branch is an injective function, which has the following properties

$$W_0(-\frac{1}{\mathrm{e}}) = -1, \;\; W_0(0) = 0, \tag{1a}$$

$$\mathrm{Re}(W_0(z)) > -1, \qquad z \in \mathbb{R}\backslash\{-\frac{1}{\mathrm{e}}\}, \tag{1b}$$

$$W_0(z) \in \mathbb{R}, \qquad z \in [-\frac{1}{\mathrm{e}}, \infty), \tag{1c}$$

$$\mathrm{Im}(W_0(z)) \in (-\pi, \pi)\backslash\{0\}, \;\; z \in \mathbb{C}\backslash(-\infty, -\frac{1}{\mathrm{e}}), \tag{1d}$$

$$W_0(\mathrm{conj}(z)) = \mathrm{conj}(W_0(z)), \qquad z \in \mathbb{C}. \tag{1e}$$

The maximum real part of the Lambert W function at a given point $z \in \mathbb{C}$ has the following property (see Fig. 2).

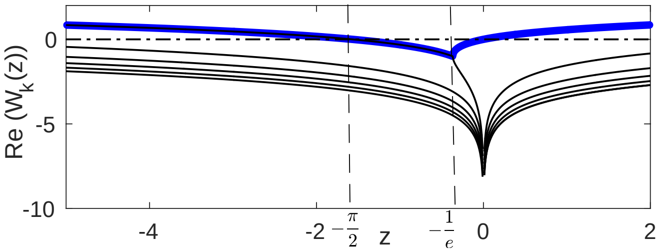


Figure 2: $\mathrm{Re}(W_k(z))$ vs. $z \in [-5, 2]$ for $k = \{-5, -4, \cdots, 4, 5\}$: solid blue line shows 0 branch while the black solid lines show the other branches. Some of the branches overlap. The relation described in Lemma 2.1 is evident in this figure.

Lemma 2.1 (Maximum real part of Lambert W function [28]): *For any $z \in \mathbb{C}$ we have*

$$\mathrm{Re}(W_0(z)) \geq \max\left\{\,\mathrm{Re}(W_k(z))|\; k \in \mathbb{Z}\backslash\{0\}\right\}. \quad (2)$$

The equality holds between branch 0 and -1 over $z \in \mathbb{R}_{\leq 0}$ where we have $\mathrm{Re}(W_0(z)) = \mathrm{Re}(W_{-1}(z))$.

Other properties of the Lambert W function that we use are

$$\frac{\mathrm{d}\,W(z)}{\mathrm{d}\,z} = \frac{1}{z + \mathrm{e}^{W(z)}}, \quad z \in \mathbb{C}\backslash\{\frac{1}{\mathrm{e}}\}, \quad (3a)$$

$$\lim_{z \to 0} \frac{W(z)}{z} = 1, \quad (3b)$$

and $W_0(x)$ around $x=0$ is given by (convergence radius of $\frac{1}{\mathrm{e}}$)

$$W_0(x) = \sum\nolimits_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} x^n. \quad (4)$$

III. Problem formulation

We study the effect of a fixed *time delay* $\tau \in \mathbb{R}_{>0}$ on the rate of convergence of the retarded time-delayed system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t - \tau), \quad (5a)$$

$$\mathbf{x}(t) = \boldsymbol{\phi}(t), \; t \in [-\tau, 0], \quad (5b)$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ is the state variable at time t, and $\boldsymbol{\phi}(t)$ is a specified pre-shape function. Throughout the paper we make the following assumption about the system matrix $\mathbf{A}$.

Assumption: system matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ is Hurwitz, and its eigenvalues are $\{\alpha_1, \cdots, \alpha_n\} \subset \mathbb{C}^l_-$, which are ordered according to $|\,\mathrm{Re}(\alpha_1)| \leq |\,\mathrm{Re}(\alpha_2)| \leq \cdots \leq |\,\mathrm{Re}(\alpha_n)|$. □

The trivial solution $\mathbf{x} \equiv \mathbf{0}$ of (5) is said to be globally exponentially stable if and only if there exists a $\kappa \in \mathbb{R}_{>0}$ and an $\bar{\rho}_\tau \in \mathbb{R}_{>0}$ such that the trajectories of (5) satisfy

$$\|\mathbf{x}(t)\| \leq \kappa\, \mathrm{e}^{-\bar{\rho}_\tau\, t} \sup_{t \in [-\tau, 0]} \|\mathbf{x}(t)\|, \quad t \in \mathbb{R}_{\geq 0}. \quad (6)$$

The exponential stability of (5) can be assessed in terms of the roots of its characteristic equation $\mathcal{F}: \mathbb{C} \to \mathbb{C}$ given by

$$\mathcal{F}(s) = \det\!\big(s\,\mathbf{I} - \mathbf{A}e^{-\tau\, s}\big) = \Pi_{i=1}^{n}(s - \alpha_i e^{-\tau\, s}). \quad (7)$$

Theorem 3.1 (Exponential stability of the time-delayed system (5) [1]): *The linear time-delayed system* (5) *is exponentially stable if and only if* $\{s \in \mathbb{C} |\,\mathrm{Re}(s) \geq 0, \mathcal{F}(s) = 0\} = \{\}$.

The characteristic equation (7) is transcendental and has an infinite number of roots in the complex plane. These roots can be obtained using the Lambert W function. Let $p = \tau s$, then $(s - \alpha_i\, \mathrm{e}^{-\tau\, s}) = 0$ can be written as $p\, \mathrm{e}^{p} = \alpha_i\, \tau$, which specifies the roots of the characteristic equation (7) as

$$\big\{s \in \mathbb{C}\,|\, s = \frac{1}{\tau} W_k(\alpha_i\, \tau), \quad i \in \{1, \ldots, n\},\; k \in \mathbb{Z}\big\}. \quad (8)$$

As expected, as $\tau \to 0$ we recover the eigenvalues of $\mathbf{A}$ as the roots of the characteristic equation (7) (recall (3b)). Since system matrix $\mathbf{A}$ is Hurwitz, *the continuity stability property* theorem for linear time-delayed systems [1, Proposition 3.1] guarantees the existence of a $\epsilon \in \mathbb{R}_{>0}$ such that for all $\tau \in [0, \epsilon)$ the roots of the characteristic equation (here (7)) are all located strictly on the left hand side of the complex plane, i.e., for any $\tau \in [0, \epsilon)$ the exponential stability of the system (5) is preserved. Moreover, the *critical* value of delay $\bar{\tau} \in \mathbb{R}_{>0}$ beyond which the system is unstable corresponds to the smallest value of $\tau \in \mathbb{R}_{>0}$ for which the rightmost root of the characteristic equation (7) is on the imaginary axis. We refer to $[0, \bar{\tau})$ as the *admissible range* of delay. For the system (5) this critical value is as follows.

Lemma 3.1 (Admissible range for delay τ for the system (5) [35]): *The time-delayed system* (5) *is exponentially stable if and only if $\tau \in [0, \bar{\tau}) \subset \mathbb{R}_{\geq 0}$ where*

$$\bar{\tau} = \min\{\,\bar{\tau}_i\}_{i=1}^{n}, \quad \bar{\tau}_i = \big|\mathrm{atan}(\frac{\mathrm{Re}(\alpha_i)}{\mathrm{Im}(\alpha_i)})\big| \big/ |\alpha_i|. \quad (9)$$

If $\{\alpha_1, \cdots, \alpha_n\} \subset \mathbb{R}_{<0}$, then

$$\bar{\tau} = \min\{\,\bar{\tau}_i\}_{i=1}^{n} = \frac{\pi}{2\,|\alpha_n|}, \quad \bar{\tau}_i = \frac{\pi}{2\,|\alpha_i|}. \quad (10)$$

For LTI system (5), with or without delay, the rate of convergence is no worse that the absolute value of the real part of the rightmost solution of the characteristic equation [21], [22]. In the absence of delay, the rightmost solution of the characteristic equation is the right most eigenvalue of the system matrix $\mathbf{A}$, i.e., α_1. In what follows, we refer to $\rho_0 = |\,\mathrm{Re}(\alpha_1)|$ as the rate of convergence of system (5) in the absence of delay. Given (8) and invoking Lemma 2.1, the rate of convergence of the system (5) in its admissible delay range is given as follows.

Lemma 3.2 (Rate of convergence of (5) for a delay in admissible range [22]): *The exponential rate of convergence of the time-delayed system* (5) *for any $\tau \in [0, \bar{\tau})$, where $\bar{\tau}$ is given in* (9)*, is given by*

$$\rho_\tau = \min\Big\{\; -\frac{1}{\tau}\,\mathrm{Re}(W_0(\alpha_i \tau))\;\Big\}_{i=1}^{n}. \quad (11)$$

As we show below, the rate of convergence ρ_τ in (11) is a continuous function of τ.

Lemma 3.3 (ρ_τ is a continuous function of τ): *The rate of convergence ρ_τ of system* (5) *given by* (11) *is a continuous function of $\tau \in \mathbb{R}_{\geq 0}$,*

Proof 1: For any given $\alpha \in \mathbb{C}$, $\mathrm{Re}(W_0(\alpha\tau))$ is a continuous function of $\tau \in \mathbb{R}_{\geq 0}$. Moreover, for any $\alpha \in \mathbb{C}_-^l$, by virtue of (3b) we have $\lim_{\tau\to 0^-} \frac{\mathrm{Re}(W_0(\alpha\tau))}{\tau} = 1$. Therefore, for every α_i, $i \in \{1,\dots,n\}$, $\frac{\mathrm{Re}(W_0(\alpha_i\tau))}{\tau}$ is continuous over $\tau \in \mathbb{R}_{\geq 0}$. Then, the proof is follows from the fact that the maximum/minimum of continuous functions is a continuous function (c.f. [36, Problem 1.2.13]).

Our objective in this paper is to show that for system (5), it is possible to have $\rho_\tau > \rho_0 = |\,\mathrm{Re}(\alpha_1)|$ for certain values of delay $\tau \in (0,\bar{\tau})$. In particular, we carefully examine the variation of ρ_τ with $\tau \in (0,\bar{\tau})$ to address the following questions: (a) for what systems delay can lead to a higher rate of convergence, (b) for what values of delay $\rho_\tau > \rho_0 = |\,\mathrm{Re}(\alpha_1)|$ (c) what is the maximum value of ρ_τ and the corresponding maximizer τ.

To compare the rate of convergence (11) to the delay free rate, we define the *delay rate gain function* as follows

$$g(x) = \begin{cases} \frac{\mathrm{Re}(W_0(x))}{\mathrm{Re}(x)}, & x \in \mathbb{C}_-^l, \\ 1, & x = 0. \end{cases} \qquad (12)$$

For any $\alpha \in \mathbb{C}_-^l$ using delay rate gain we can write

$$-\frac{1}{\tau}\,\mathrm{Re}(W_0(\alpha\tau)) = g(\alpha\tau)\,|\,\mathrm{Re}(\alpha)|, \quad \tau \in \mathbb{R}_{>0}. \qquad (13)$$

Therefore, the rate of convergence (11) of the system (5) can be expressed also as

$$\rho_\tau = \min\left\{g(\alpha_i\tau)\,|\,\mathrm{Re}(\alpha_i)|\,\right\}_{i=1}^n, \quad \tau \in \mathbb{R}_{>0}. \qquad (14)$$

In the proceeding section, we study the variation of $g(\alpha\tau)$ with $\tau \in \mathbb{R}_{>0}$ for any given $\alpha \in \mathbb{C}_-^l$ with the objective of identifying the ranges of the time delay for which $g(\alpha\tau)$ is greater than one. We use the results then in the Section V to study the variation of ρ_τ with respect to $\tau \in (0,\bar{\tau})$.

IV. Preparatory results: delay rate gain function

In this section, we study some of the properties of the delay rate gain function (12), with the objective of identifying the ranges of the time delay that $g(\alpha\tau)$ is greater than one for a given $\alpha \in \mathbb{C}_-^l$.

Lemma 4.1 (characterizing the solutions of $g(\alpha\tau) = 0$ and $g(\alpha\tau) = 1$): *For a given $\alpha \in \mathbb{C}_-^l$, the delay rate gain* (12) *satisfies*

$$\lim_{\tau\to 0} g(\alpha\,\tau) = 1, \qquad (15a)$$

$$\{\tau \in \mathbb{R}_{\geq 0}\,|\,g(\alpha\tau){=}0\} {=} \{\bar{\tau}\},\ \ \bar{\tau} {=} \left|\mathrm{atan}\left(\frac{\mathrm{Re}(\alpha)}{\mathrm{Im}(\alpha)}\right)\right| \Big/ |\alpha|, \qquad (15b)$$

If $\alpha \in \mathbb{R}_{<0}$, then $\bar{\tau} = \pi/2|\alpha|$. For $\alpha \in \mathbb{R}_{<0}$ we also have

$$\{\tau \in \mathbb{R}_{>0}\,|\,g(\alpha\tau) = 1\} = \{\tilde{\tau}\}, \quad \tilde{\tau} = \tilde{\theta}\cot(\tilde{\theta})/|\alpha|, \qquad (16)$$

where $\tilde{\theta}$ is the unique solution of $\mathrm{e}^{-\theta\cot(\theta)} = \cos(\theta)$ *in* $(0,\pi)$, *which approximately is* 1.01125.

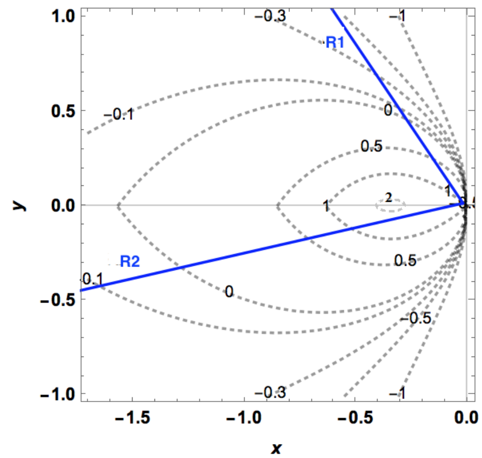


Figure 3: counter plot of $g(\mathsf{x}+\mathsf{y}\mathrm{i})$ for different values of $(\mathsf{x},\mathsf{y}) \in \mathbb{R}_{\leq 0} \times \mathbb{R}$. The lines R1 and R2, atop of the contour plots, show $\mathrm{Re}(\alpha\,\tau)$ versus $\mathrm{Im}(\alpha\,\tau)$, for different values of $\tau \in \mathbb{R}_{\geq 0}$ for, respectively, $\alpha = -3 + 5\mathrm{i}$ ($\arg(\alpha) = \frac{2\pi}{3}$) and $\alpha = -5 - 1.34\mathrm{i}$ ($\arg(\alpha) = \frac{13\pi}{12}$).

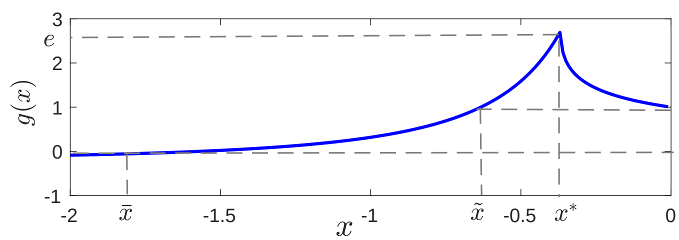


Figure 4: delay rate gain versus $x \in \mathbb{R}_{<0}$. Points $x^\star = -\frac{1}{\mathrm{e}}$, $\tilde{x} \approx -0.63$ and $\bar{x} = -\frac{\pi}{2}$, respectively, correspond to maximum delay, unit delay gain and maximum admissible delay.

The proof of Lemma 4.1 is given in the appendix. Next, given (15a), by using a similar argument to that of the first part of the proof of Lemma 3.3, we state the following result.

Lemma 4.2 ($g(\alpha\tau)$ is a continuous function of τ): *For a given $\alpha \in \mathbb{C}_-^l$, $g(\alpha\tau)$ is a continuous function of $\tau \in \mathbb{R}_{\geq}0$.*

Our next result specifies how the sign of $g(\alpha\tau)$ for a given $\alpha \in \mathbb{C}_-^l$ changes with respect to $\tau \in \mathbb{R}_{>0}$.

Lemma 4.3 (values of τ for which $g(\alpha\tau) > 0$): *For a given $\alpha \in \mathbb{C}_-^l$, $g(\alpha\tau) > 0$ for $\tau \in (0,\bar{\tau})$, $g(\alpha\tau) < 0$ for $\tau \in (\bar{\tau},\infty)$ and $g(\alpha\bar{\tau}) = 0$, where $\bar{\tau}$ is given in* (15b).

The proof of Lemma 4.3 is given in the appendix. The proof relies on studying the properties of the level set (see Fig. 3)

$$\mathcal{C}_c = \{(\mathsf{x},\mathsf{y}) \in \mathbb{R}_{<0} \times \mathbb{R}|\ g(\mathsf{x}+\mathsf{y}\mathrm{i}) = c\}, \qquad (17)$$

and the superlevel set

$$\mathcal{S}_c = \{(\mathsf{x},\mathsf{y}) \in \mathbb{R}_{<0} \times \mathbb{R}|\ g(\mathsf{x}+\mathsf{y}\mathrm{i})\ \geq\ c\}, \qquad (18)$$

of the delay rate gain function for $c = 0$.

In the remainder of this section our objective is to characterize the variation of $g(\alpha\tau)$ from 1 at $\tau = 0$ to 0 at $\tau = \bar{\tau}$

(recall (15)), with the specific objective of identifying the values of $\tau \in (0,\bar{\tau})$ for which $g(\alpha\tau) > 1$. In this regard, first we consider Fig. 4, which shows variation of $g(x)$ versus $x \in \mathbb{R}_{\leq 0}$ over $x \in [0,-2]$. This plot reveals a set of interesting facts as follows.

- At $x = \bar{x} = -\frac{\pi}{2}$ we have $g(\bar{x}) = 0$, while for any $x \in (\bar{x}, 0)$, $g(x) > 0$ and for any $x < \bar{x}$, $g(x) < 0$ (as expected according to Lemma 4.3).
- At $x = x^\star = -\frac{1}{\mathrm{e}}$, the maximum delay rate gain $g(x^\star) = \mathrm{e}$ is attained. For any $x \in (x^\star, 0)$, $g(x)$ strictly decreases from e to 1, while for any $x \in (\bar{x}, x^\star)$, $g(x)$ increases strictly from 0 to e.
- Let $\tilde{x}$ be the non-zero solution of $g(\tilde{x}) = 1$, $x \in \mathbb{R}_{<0}$ (an approximate value of $\tilde{x}$ is -0.63336, see Lemma 4.1 for analytic characterization of $\tilde{x}$ using $\tilde{x} = \alpha\tilde{\tau}$). Then, for any $x \in (\tilde{x}, 0)$ we have $g(x) > 1$. Also, for any $x \in [\bar{x}, \tilde{x}]$, $g(x)$ strictly increases from 0 to 1.

For a given $\alpha \in \mathbb{R}_{<0}$, using the aforementioned observations, one can describe the variation of $g(\alpha\tau)$ over $\tau \in [0,\bar{\tau}]$. We formalize this characterization in the following lemma, whose rigorous proof is given in the appendix.

Lemma 4.4 (variation of $g(\alpha\tau)$ for an $\alpha \in \mathbb{R}_{<0}$ and $\tau \in \mathbb{R}_{>0}$): *Consider the delay rate gain function* (12)*. Let $\alpha \in \mathbb{R}_{<0}$ be given. Recall $\bar{\tau}$ and $\tilde{\tau}$ from Lemma 4.1. Let $\tau^\star = \frac{1}{\mathrm{e}\,|\alpha|}$. Then, the followings hold.*

(a) *For any $\tau \in (0,\tau^\star) \subset (0,\bar{\tau})$, $g(\alpha\tau) > 1$, and $g(\alpha\tau)$ strictly increases from 1 to* e*; $g(\alpha\tau^\star) = \mathrm{e}$; and for any $\tau \in (\tau^\star,\bar{\tau}) \subset (0,\bar{\tau})$, $g(\alpha\tau) > 0$, and $g(\alpha\tau)$ strictly decreases from* e *to* 0*.*
(b) *For any $\tau \in (0,\tilde{\tau}) \subset (0,\bar{\tau})$, $g(\alpha\tau) > 1$; $g(\alpha\tilde{\tau}) = 1$; $g(\alpha\bar{\tau}) = 0$; for any $\tau \in (\tilde{\tau},\bar{\tau})$, $0 < g(\alpha\tau) < 1$; and for $\tau \in (\bar{\tau},\infty)$, $g(\alpha\tau) < 0$.*
(c) *The maximum value of $g(\alpha\tau)$ is* e*, which is attained at $\tau = \tau^\star \in (0,\tilde{\tau})$.*

When α is a complex number, the variation of $g(\alpha\tau)$ with $\tau \in \mathbb{R}_{>0}$ is more involved. As seen in the contour plots in Fig. 3, $g(x)$ attains a value greater than one for some $x = \mathsf{x} + \mathrm{i}\,\mathsf{y}$. Specifically, consider the points on the dashed lines $R1$ and $R2$ on Fig. 3, which depict, respectively, $(\mathrm{Re}(\alpha_{R1}\tau), \mathrm{Im}(\alpha_{R1}\tau))$ and $(\mathrm{Re}(\alpha_{R2}\tau), \mathrm{Im}(\alpha_{R2}\tau))$ for $\tau \in \mathbb{R}_{\geq 0}$, with $\alpha_{R1} = -3+5\mathrm{i}$ and $\alpha_{R2} = -5-1.34\mathrm{i}$. As seen, $g(\alpha_{R1}\tau)$, $\tau \in \mathbb{R}_{>0}$ is always less than one, while $g(\alpha_{R2}\tau)$ is greater than one for $\tau \in (0,\tilde{\tau})$, where $\tilde{\tau}$ satisfies $g(\alpha\tilde{\tau}) = 1$ (see also Fig. 5). Therefore, we expect that a delay rate gain of greater than one is only possible for certain values of $\alpha \in \mathbb{C}_-^l$. In what follows we set off to address (a) for what values of $\alpha \in \mathbb{C}_-^l$, $g(\alpha\tau)$ can have a value greater than 1 for a $\tau \in \mathbb{R}_{\geq 0}$, (b) what values of $\tau \in (0,\bar{\tau})$ correspond to $g(\alpha\tau) > 1$, and (c) what the maximum gain $g(\alpha\tau^\star)$ and the corresponding $\tau^\star$ are. We start our study by the following result that for a given $\alpha \in \mathbb{C}_-^l$, characterizes the sign of $\mathrm{d}g(\alpha\tau)/\mathrm{d}\tau$ for any $\tau \in (0,\bar{\tau})$ (proof is given in the appendix).

Lemma 4.5 (variation of $\mathrm{d}g(\alpha\tau)/\mathrm{d}\tau$ with $\tau \in (0,\bar{\tau}]$ for a $\alpha \in \mathbb{C}_-^l$): *Consider the delay rate gain function* (12)*. Let $\alpha \in \mathbb{C}_-^l$ be given. Recall $\bar{\tau}$ from Lemma 4.1 and $\mathcal{S}_0$ in* (18)*. Let*

$$\Lambda = \Big\{(\mathsf{x},\mathsf{y}) \in \mathbb{R}_{\leq 0}\times\mathbb{R} \Big| \mathsf{x}+\mathsf{y}\,\mathrm{i} = r\,\mathrm{e}^{\phi\,\mathrm{i}},\ r = -\frac{\cos(2\theta)}{\cos(\theta)}\,\mathrm{e}^{-\cos(2\theta)},$$
$$\phi = \theta - \cos(2\theta)\tan(\theta)\,\mathrm{e}^{-\cos(2\theta)},\ \frac{3\pi}{4} \leq \theta \leq \frac{5\pi}{4}\Big\}. \quad (19)$$

Then, the following assertions hold.

(a) *For any $\tau \in (0,\bar{\tau}]$, the delay rate gain satisfies $\frac{\mathrm{d}g(\alpha\tau)}{\mathrm{d}\tau} > 0$ if $(\mathrm{Re}(\alpha\tau), \mathrm{Im}(\alpha\tau)) \in \mathrm{int}(\Lambda) = \{(\mathsf{x},\mathsf{y}) \in \mathbb{R}_{\leq 0}\times\mathbb{R}\,|\mathsf{x} + \mathsf{y}\,\mathrm{i} = r\,\mathrm{e}^{\phi\,\mathrm{i}}, 0 < r < -\frac{\cos(2\theta)}{\cos(\theta)}\,\mathrm{e}^{-\cos(2\theta)}, \phi = \theta - \cos(2\theta)\tan(\theta)\,\mathrm{e}^{-\cos(2\theta)},\ , \frac{3\pi}{4} \leq \theta \leq \frac{5\pi}{4}\}$, $\frac{\mathrm{d}g(\alpha\tau)}{\mathrm{d}\tau} = 0$ if $(\mathrm{Re}(\alpha\tau), \mathrm{Im}(\alpha\tau)) \in \Lambda\backslash\{(-\frac{1}{\mathrm{e}}, 0)\}$, and $\frac{\mathrm{d}g(\alpha\tau)}{\mathrm{d}\tau} < 0$ if $(\mathrm{Re}(\alpha\tau), \mathrm{Im}(\alpha\tau)) \in (\mathcal{S}_0\backslash(\mathrm{int}(\Lambda)\cup\Lambda))$.*
(b) *If $\arg(\alpha) \notin (\frac{3\pi}{4}, \frac{5\pi}{4})$, then $\frac{dg(\alpha\tau)}{d\tau} < 0$ for $\tau \in (0,\bar{\tau}]$.*
(c) *If $\arg(\alpha) \in (\frac{3\pi}{4}, \frac{5\pi}{4})$, then $\frac{dg(\alpha\tau)}{d\tau} > 0$ for $\tau \in (0,\tau^\star)$, and $\frac{dg(\alpha\tau)}{d\tau} < 0$ $\tau \in (\tau^\star,\bar{\tau}]$, where $\tau^\star$ is*

$$\tau^\star = -\frac{\cos(2\,\theta^\star)}{|\alpha|\,\cos(\theta^\star)}\,\mathrm{e}^{-\cos(2\,\theta^\star)}, \quad (20)$$

in which $\theta^\star$ is the unique solution of

$$\arg(\alpha) = \theta - \cos(2\,\theta)\tan(\theta)\,\mathrm{e}^{-\cos(2\,\theta)},\ \theta \in (\frac{3\pi}{4}, \frac{5\pi}{4}). \quad (21)$$

At $\tau = \tau^\star$, we have $(\mathrm{Re}(\alpha\tau^\star), \mathrm{Im}(\alpha\tau^\star)) \in \Lambda$. If $\alpha \in \mathbb{R}_{<0}$, then $\tau^\star = \frac{1}{\mathrm{e}\,|\alpha|}$ and

$$\lim_{\tau\to\tau^{\star-}} \frac{dg(\alpha\,\tau)}{d\tau} = +\infty,\quad \lim_{\tau\to\tau^{\star+}} \frac{dg(\alpha\,\tau)}{d\tau} = -\frac{5\,\mathrm{e}^2\,|\alpha|}{3}. \quad (22)$$

Lastly, if $\alpha \in \mathbb{C}_-^l\backslash\mathbb{R}_{<0}$, then $\frac{dg(\alpha\tau)}{d\tau} = 0$ at $\tau = \tau^\star$.

Figure 5 depicts Λ in (19) (red curve) in a (x,y) plane along with the level sets $\mathcal{C}_1$ (green curve) and $\mathcal{C}_0$ (blue curve). Note that Λ is a simple closed curve that divides the space into a bounded interior and an unbounded exterior area. Moreover, Λ is located inside $\mathcal{C}_1$ and between the lines $\mathsf{y} = \pm\mathsf{x}$. We note that $\mathsf{y} = \pm\mathsf{x}$ are also tangent to $\mathcal{C}_1$ at the origin (a rigorous study of these geometric observations is available in the appendix). The bottom plot in Fig. 5 shows also how $g(\mathsf{x}+\mathsf{y}\,\mathrm{i})$ varies along the lines R1 and R2 for points inside $\mathcal{S}_0$. As seen, the delay rate gain is strictly decreasing along R1. However, along R2 it is strictly increasing until R2 intersects Λ, and it is strictly decreasing afterward until R2 intersects $\mathcal{C}_0$.

With Lemma 4.5 at hand, we are now ready to present the main result of this section.

Theorem 4.1 ($g(\alpha\tau)$ vs. $\tau \in \mathbb{R}_{>0}$ for $\alpha \in \mathbb{C}_-^l$): *Consider the delay rate gain function* (12)*. Let $\alpha \in \mathbb{C}_-^l$ be given. Recall $\bar{\tau}$ from Lemma 4.1. Then, the following assertions hold.*

(a) *If $\arg(\alpha) \notin (\frac{3\pi}{4}, \frac{5\pi}{4})$, then $g(\alpha\tau)$ decreases strictly from 1 to 0 for $\tau \in [0,\bar{\tau}]$.*
(b) *If $\arg(\alpha) \in (\frac{3\pi}{4}, \frac{5\pi}{4})$, then: (i) $g(\alpha\tau)$ increases strictly from 1 to $g(\alpha\tau^\star)$ for $\tau \in [0,\tau^\star]$ and decreases strictly from $g(\alpha\tau^\star)$ to 0 for $\tau \in [\tau^\star,\bar{\tau}]$, where $\tau^\star$ is specified in the statement (c) of Lemma 4.5; (ii) $\tilde{\tau} \in (0,\bar{\tau})$ such that $g(\alpha\tilde{\tau}) = 1$ exists and is unique and satisfies $0 < \tau^\star <$*

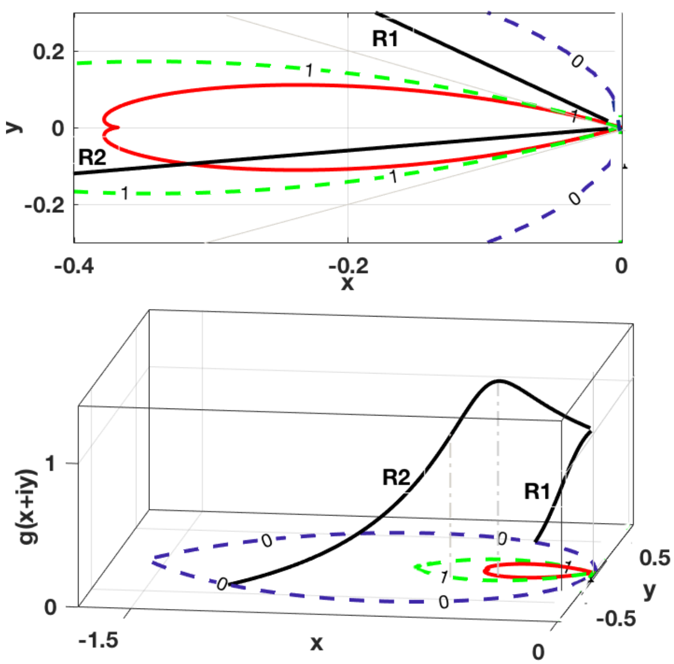


Figure 5: The top plot shows Λ in (19) (red curve) in a (x,y) plane along with the level sets $\mathcal{C}_1$ (green curve) and $\mathcal{C}_0$ (blue curve). The bottom plot shows $g(\mathrm{x}+\mathrm{i}\,\mathrm{y})$ vs. (x,y) for points along the lines R1 and R2 shown in the top.

$\tilde{\tau}<\bar{\tau}$; *(iii)* $g(\alpha\tau)>1$ *for* $\tau\in(0,\tilde{\tau})$, *and* $g(\alpha\tau)<1$ *for* $\tau\in(\tilde{\tau},\bar{\tau}]$.

Proof 2: We recall that $\lim_{\tau\to 0} g(\alpha\tau)=1$, $g(\alpha\bar{\tau})=0$, and $g(\alpha\tau)$ is a continuous function of $\tau\in\mathbb{R}_{\geq 0}$. Then, the proof of the statements (a) and (b) follows respectively from the statements (b) and (c) of Lemma 4.5.

V. Delay effects on the rate of convergence

In this section, we inspect closely how the rate of convergence of the time-delayed system (5) changes with time delay. Our study relies on the properties of the delay rate gain function that are established in the previous section.

Earlier we showed that using the delay rate gain function we can write the rate of convergence (11) in the equivalent form (14). Variation of each component $-\frac{1}{\tau}\,\mathrm{Re}(W_0(\alpha_i\tau))=g(\alpha_i\tau)\,|\,\mathrm{Re}(\alpha_i)|$ of (14) with respect to $\tau\in\mathbb{R}_{>0}$ can be characterized via the results in Lemma 4.4 and Theorem 4.1. Now in this section, we look at these variations collectively to characterize the variation of the rate of convergence of the time-delayed system (5) versus time delay. We start by defining some notations. First, let

$$\mathcal{I}_1=\{k\in\{1,\dots,n\}\,|\,\mathrm{Re}(\alpha_k)=\mathrm{Re}(\alpha_1)\},\tag{23a}$$

$$\mathcal{I}_{\text{in}}=\big\{k\in\{1,\dots,n\}\,|\,\arg(\alpha_k)\in(\frac{3\pi}{4},\frac{5\pi}{4})\big\},\tag{23b}$$

$$\mathcal{I}_{\text{out}}=\{1,\cdots,n\}\backslash\mathcal{I}_{\text{in}}.\tag{23c}$$

Next, recall $\bar{\tau}^i$ from (9), and also let

$$\tilde{\tau}_i=\max\{\tau\in[0,\bar{\tau}_i]\,|\,g(\alpha_i\tau)=1\},\tag{24a}$$

$$\tau_i^\star=\arg\max_{\tau\in[0,\bar{\tau}_i]} g(\alpha_i\tau),\tag{24b}$$

$i\in\{1,\dots,n\}$. Recall form Lemma 4.1 that $g(\alpha_i\bar{\tau}_i)=0$. Moreover, by virtue of Lemma 3.1 the critical value of delay for the time-delayed system (5) is $\bar{\tau}=\min\{\bar{\tau}_i\}_{i=1}^n$. Also from Theorem 4.1 we can obtain the following result.

Corollary 5.1 (relative size of $(\bar{\tau}_i,\tau_i^\star,\tilde{\tau}_i)$): *Consider* $\{\alpha_i\}_{i=1}^n\subset\mathbb{C}_-^l$, *the eigenvalues of the system matrix of* (5). *Then,*

$$0<\tau_i^\star<\tilde{\tau}_i<\bar{\tau}_i,\qquad i\in\mathcal{I}_{in},\tag{25a}$$

$$\tau_i^\star=\tilde{\tau}_i=0,\qquad i\in\mathcal{I}_{out},\tag{25b}$$

where for $i\in\mathcal{I}_{in}$, $\tau_i^\star$ *is a unique point obtained from* (20) *for* $\alpha=\alpha_i$, *and* $\tilde{\tau}_i$ *is the unique solution of* $g(\alpha_i\tau)=1$ *in* $(0,\bar{\tau}_i)$.

Lastly, we let

$$\tilde{\tau}=\max\{\tau\in[0,\bar{\tau}]\,|\,\rho_\tau=|\,\mathrm{Re}(\alpha_1)|\},\tag{26a}$$

$$\tau^\star=\arg\max_{\tau\in[0,\bar{\tau})}\rho_\tau.\tag{26b}$$

With the proper notations at hand, we now present our first result, which specifies what system (5) can have a higher rate of convergence in the presence of the time delay.

Theorem 5.1 (Systems for which rate of convergence can increase by time delay): *Consider the linear time-delayed system* (5) *when* $\{\alpha_i\}_{i=1}^n\subset\mathbb{C}_-^l$. *Recall the admissible delay bound* $\bar{\tau}$ *given by* (9). *Then there always exists a* $\tau\in(0,\bar{\tau})$ *for which* $\rho_\tau>\rho_0=|\,\mathrm{Re}(\alpha_1)|$ *if and only if* $\mathcal{I}_1\subset\mathcal{I}_{\text{in}}$.

Proof 3: If there exists a $j\in\mathcal{I}_1$ that is not in $\mathcal{I}_{\text{in}}$, i.e., $\arg(\alpha_j)\notin(\frac{3\pi}{4},\frac{5\pi}{4})$, then by virtue of Lemma (4.3) and the statement (a) of Theorem 4.1 we know that $g(\alpha_j\tau)<1$ for any $\tau\in\mathbb{R}_{>0}$. Subsequently, since $|\,\mathrm{Re}(\alpha_1)|\leq\cdots\leq|\,\mathrm{Re}(\alpha_n)|$, from the definition of the ρ_τ in (14) we obtain that $\rho_\tau<|\,\mathrm{Re}(\alpha_1)|$ for all $\tau\in(0,\bar{\tau})$. Now, assume that $\mathcal{I}_1\subset\mathcal{I}_{\text{in}}$. Then, by virtue of the statement (b) of Theorem 4.1, we know that $g(\alpha_i\tau)>1$ for any $\tau\in(0,\tilde{\tau}_i)$ for $i\in\mathcal{I}_1$, (recall $\tilde{\tau}_i\in(0,\bar{\tau}_i)$ due to (25)). Subsequently, since $g(0)=1$ and $|\,\mathrm{Re}(\alpha_k)|>|\,\mathrm{Re}(\alpha_1)|$ for $k\in\{1,\cdots,n\}\backslash\mathcal{I}_1$, then by virtue of Lemma 4.2, there exists a $\check{\tau}\in((0,\min\{\tilde{\tau}_i\}_{i\in\mathcal{I}_1})\cap(0,\bar{\tau}))$ such that $g(\alpha_i\tau)>1$, $i\in\{1,\cdots,n\}$, for any $\tau\in(0,\check{\tau})$. Then, the proof of the sufficiency of the theorem statement follows from the definition of ρ_τ in (14) and its continuity with respect to $\tau\in\mathbb{R}_{>0}$ (see Lemma 3.3).

From the proof of the Theorem 5.1 we can deduce that for systems that do not satisfy the theorem's condition, we have $\tilde{\tau}=\tau^\star=0$; but, for those that satisfy the condition these values are non-zero. Our next result specifies for what values of time delay a system, which satisfies the necessary and sufficient condition of Theorem (5.1), experiences an increase in its rate of convergence in the presence of delay. This result also gives the value of $\tilde{\tau}$ and provides an estimate on the value of $\tau^\star$.

Theorem 5.2 (Ranges of delay for which the rate of convergence of (5) increases with delay): *Consider the linear time-delayed system* (5) *when* $\{\alpha_i\}_{i=1}^n\subset\mathbb{C}_-^l$. *Recall the admissible delay bound* $\bar{\tau}$ *given by* (9). *Suppose that* $\mathcal{I}_1\subset\mathcal{I}_{\text{in}}$. *Then, the following assertions hold.*

(a) $\tilde{\tau} = \min\{\eta_i\}_{i=1}^{n}$, *where* η_i *is the unique solution of* $g(\alpha_i\tau) = \frac{|\operatorname{Re}(\alpha_1)|}{|\operatorname{Re}(\alpha_i)|}$ *for* $\tau \in (0,\bar{\tau}_i)$. *Moreover,* $\min\{\tau_i^\star\}_{i=1}^{n} < \tilde{\tau} < \bar{\tau}$.

(b) $\rho_\tau > \rho_0 = |\operatorname{Re}(\alpha_1)|$ *for* $\tau \in (0,\tilde{\tau}) \subset (0,\bar{\tau})$, $\rho_\tau = \rho_0 = |\operatorname{Re}(\alpha_1)|$ *at* $\tau = \tilde{\tau}$ *and* $\rho_\tau < \rho_0 = |\operatorname{Re}(\alpha_1)|$ *for* $\tau \in (\tilde{\tau},\bar{\tau})$. *Moreover,* ρ_τ *decreases strictly with* $\tau \in (\tilde{\tau},\bar{\tau})$.

(c) $\tau^\star \in \big([\min\{\tau_i^\star\}_{i=1}^{n}, \max\{\tau_i^\star\}_{i=1}^{n}] \cap (0,\tilde{\tau})\big) \subset (0,\bar{\tau})$.

Proof 4: For $j \in \mathcal{I}_{\text{out}}$, the statement (a) of Theorem (4.1) guarantees that $g(\alpha_j\tau)$ is strictly decreasing from 1 to 0 for $\tau \in [0,\bar{\tau}_j]$. Thus, for $j \in \mathcal{I}_{\text{out}}$, given the continuity of $g(\alpha_j\tau)$ in $\tau \in [0,\bar{\tau}_j]$ and $\frac{|\operatorname{Re}(\alpha_1)|}{|\operatorname{Re}(\alpha_j)|} < 1$ (recall that $\mathcal{I}_1 \not\subset \mathcal{I}_{\text{out}}$), η_j is a non-zero unique value in $(0,\bar{\tau}_j)$ at which we have $g(\alpha_j\eta_j)|\operatorname{Re}(\alpha_j)| = |\operatorname{Re}(\alpha_1)|$. Moreover, for $j \in \mathcal{I}_{\text{out}}$, we have $g(\alpha_j\eta_j)|\operatorname{Re}(\alpha_j)| < |\operatorname{Re}(\alpha_1)|$ for $\tau \in (\eta_j,\bar{\tau}_j)$ and $g(\alpha_j\eta_j)|\operatorname{Re}(\alpha_j)| > |\operatorname{Re}(\alpha_1)|$ for $\tau \in (0,\eta_j)$. Recall also that $\tau_j^\star = 0$ for $j \in \mathcal{I}_{\text{out}}$. For $j \in \mathcal{I}_{\text{in}}$, the statement (b) of Theorem (4.1) guarantees that $g(\alpha_j\tau)$ is strictly increasing from 1 to its maximum value $g(\alpha_j\tau_j^\star) > 1$ for $\tau \in (0,\tau_j^\star)$ and it is strictly decreasing from $g(\alpha_j\tau_j^\star) > 1$ to zero for $\tau \in [\tau_j^\star,\bar{\tau}_j]$. Thus, for $j \in \mathcal{I}_{\text{in}}$, given the continuity of $g(\alpha_j\tau)$ in $\tau \in [0,\bar{\tau}_j]$ and $\frac{|\operatorname{Re}(\alpha_1)|}{|\operatorname{Re}(\alpha_j)|} \leq 1$ (recall that $\mathcal{I}_1 \subset \mathcal{I}_{\text{in}}$), η_j is a non-zero unique value in $(\tau_j^\star,\bar{\tau}_j)$ at which we have $g(\alpha_j\eta_j)|\operatorname{Re}(\alpha_j)| = |\operatorname{Re}(\alpha_1)|$. Moreover, for $j \in \mathcal{I}_{\text{in}}$, we have $g(\alpha_j\eta_j)|\operatorname{Re}(\alpha_j)| < |\operatorname{Re}(\alpha_1)|$ for $\tau \in (\eta_j,\bar{\tau}_j)$ and $g(\alpha_j\eta_j)|\operatorname{Re}(\alpha_j)| > |\operatorname{Re}(\alpha_1)|$ for $\tau \in (0,\eta_j)$. From the aforementioned observations, the validity of the statements (a) and (b) follows from the continuity of ρ_τ in $\tau \in [0,\bar{\tau}]$, its definition (14) and also noting that the minimum of a set of strictly decreasing functions is also strictly decreasing.

To prove the statement (c) we proceed as follows. From the statement (b) we can conclude that $0 < \tau^\star < \tilde{\tau}$. Given the definition of ρ_τ in (14), we already know that $\rho_{\tau^\star} = \min\{g(\alpha_i\tau^\star)\,|\operatorname{Re}(\alpha_i)|\}_{i=1}^{n} \leq g(\alpha_i\tau^\star)|\operatorname{Re}(\alpha_i)| \leq g(\alpha_i\tau_i^\star)|\operatorname{Re}(\alpha_i)|$, $i \in \{1,\cdots,n\}$. Therefore, $\tau^\star \leq \max\{\tau_i^\star\}_{i=1}^{n}$. If $\mathcal{I}_{\text{out}} \neq \{\}$, because of $\tau_j^\star = 0$, $j \in \mathcal{I}_{\text{out}}$, then $\tau^\star \geq \min\{\tau_i\}_{i=1}^{n}$ is trivial. Now assume $\mathcal{I}_{\text{out}} = \{\}$. In this case if $\tau^\star$ is not equal to any of the $\tau_i^\star$, $i \in \{1,\cdots,n\}$, then in order for $\tau^\star$ to be a maximizer point we should have non-empty $\bar{\mathcal{I}} \subsetneq \{1,\cdots,n\}$ and $\hat{\mathcal{I}} \subsetneq \{1,\cdots,n\}$ such that $\frac{\mathrm{d}g(\alpha_i\tau^\star)}{\mathrm{d}\tau} > 0$ for $i \in \bar{\mathcal{I}}$ and $\frac{\mathrm{d}g(\alpha_i\tau^\star)}{\mathrm{d}\tau} < 0$ for $i \in \hat{\mathcal{I}}$. Consequently, since for $\tau \in (0,\tau_i^\star)$ we have $\frac{\mathrm{d}g(\alpha_i\tau)}{\mathrm{d}\tau} > 0$ and for $\tau \in (\tau_i^\star,\bar{\tau}_i)$ we have $\frac{\mathrm{d}g(\alpha_i\tau)}{\mathrm{d}\tau} < 0$, for $i \in \mathcal{I}_{\text{in}} = \{1,\cdots,n\}$, then we can conclude that $\tau^\star \geq \min\{\tau_i^\star\}_{i=1}^{n}$.

The statement (c) of Theorem 5.2 provides only an estimate on the location of $\tau^\star$. However, by relying on the proof argument of this statement we can narrow down the search for $\tau^\star$ to a set of discrete points as explained in the remark below. In what follows we let

$$\rho_\tau^i = g(\alpha_i\tau)|\operatorname{Re}(\alpha_i)|, \quad i \in \{1,\cdots,n\}.$$

Remark 5.1 (Candidate points for $\tau^\star$, when $\mathcal{I}_1 \subset \mathcal{I}_{\text{in}}$): *Consider system* (5) *when* $\{\alpha_i\}_{i=1}^{n} \subset \mathbb{C}_-^l$ *and* $\mathcal{I}_1 \subset \mathcal{I}_{in}$. *From the proof argument of the statement (c) of Theorem 5.2, it follows that* $\tau^\star$ *is either a point in* $\mathcal{T}^\star = \{\tau_i^\star\}_{i\in\mathcal{J}}$ *where* $\mathcal{J} = \{i \in \{1,\cdots,n\}|\min\{\tau_k^\star\}_{k=1}^{n} \leq \tau_i^\star < \tilde{\tau}\}$ *or an intersection point of a* ρ_τ^i *and a* ρ_τ^j *in* $\tau \in \big([\min\{\tau_k^\star\}_{k=1}^{n}, \max\{\tau_k^\star\}_{k=1}^{n}] \cap (0,\tilde{\tau})\big)$ *where* $\frac{dg(\alpha_i\tau)}{d\tau} > 0$ *and* $\frac{dg(\alpha_j\tau)}{d\tau} < 0$. *Based on this observation, we propose the following procedure to identify the candidate points for* $\tau^\star$. *Let* $\mathcal{J}^r = \{i \in \mathcal{I}_{in}|\tau_i^\star \geq \tilde{\tau}\}$, *and* $\mathcal{J}_j^l = \{i \in \mathcal{J}|\tau_i^\star \geq \tau_j^\star\}$ *for any* $j \in \mathcal{J}$. *We note that for* $i \in \mathcal{J}^r$ *we have* $\frac{dg(\alpha_i\tau)}{d\tau} > 0$ *for any* $\tau \in (0,\tilde{\tau})$, *and for any* $j \in \mathcal{J}$ *we have* $\frac{dg(\alpha_k\tau)}{d\tau} < 0$ *for any* $\tau \in (\tau_j^\star,\tilde{\tau})$ *and also* $\frac{dg(\alpha_k\tau)}{d\tau} > 0$ *for any* $\tau \in (\tau_j^\star,\tau_k^\star) \subset (\tau_j^\star,\tilde{\tau})$, $k \in \mathcal{J}_j^l$. *Now for any* $j \in \mathcal{J}$ *let* $\mathcal{T}_j$ *be the set of intersection points of* ρ_τ^j *with* ρ_τ^k, $k \in (\mathcal{J}_j^l \cup \mathcal{J}^r)$ *for* $\tau \in (\tau_j^\star,\tilde{\tau})$ *(here note that the possible intersection between* ρ_τ^j *and* ρ_τ^k *is in fact located at* $(\tau_j^\star,\tau_k^\star) \subset (\tau_j^\star,\tilde{\tau})$ *for* $k \in \mathcal{I}_j^l$*). Then, following the proof argument of the statement (c) of of Theorem 5.2, we have* $\tau^\star \in ((\underset{j\in\mathcal{J}}{\cup}\mathcal{T}_j) \cup \mathcal{T}^\star)$.

Remark 5.2 (Ultimate bound on the maximum possible increase in the rate of convergence of system (5) in the presence of time delay): *The statement (a) of Lemma 4.4 indicates that for any* $\alpha \in \mathbb{R}_{<0}$, *regardless of the value of* α, *the maximum attainable delay rate gain* $g(\alpha\tau)$ *is equal to* $\mathrm{e} \approx 2.7183$. *The contour plot of* $g(x)$ *in fact shows that the maximum attainable positive value for* $g(x)$ *for any* $x \in \mathbb{C}_-^l$ *is* $\mathrm{e} \approx 2.7183$, *(see Fig. 4; recall here that the statement (c) of Lemma A.1 in the appendix indicates that* $\mathcal{C}_c \subset \operatorname{int}(\mathcal{C}_0 \cup \{0,0\}) \subsetneq \mathcal{S}_0$ *for* $c > 0$*). Therefore, since for any* $\tau \in (0,\bar{\tau})$, $\rho_\tau = \min\{g(\alpha_i\tau)\,|\operatorname{Re}(\alpha_i)|\}_{i=1}^{n}$, *the maximum possible attainable rate because of delay for system* (5) *is* $\mathrm{e}\,|\operatorname{Re}(\alpha_1)| = \mathrm{e}\,\rho_0$. □

Our next result shows that if $\mathcal{I}_1 \not\subset \mathcal{I}_{\text{in}}$, not only $\rho_\tau < \rho_0$ but also ρ_τ is a strictly decreasing function of $\tau \in (0,\bar{\tau})$.

Lemma 5.1 (Rate of convergence when $\mathcal{I}_1 \subset \mathcal{I}_{\text{out}}$): *Consider the linear time-delayed system* (5) *when* $\{\alpha_i\}_{i=1}^{n} \subset \mathbb{C}_-^l$. *Recall the admissible delay bound* $\bar{\tau}$ *given by* (9). *If* $\mathcal{I}_1 \subset \mathcal{I}_{out}$, *then* ρ_τ *decreases strictly from* ρ_0 *to* 0 *for* $\tau \in [0,\bar{\tau}]$.

Proof 5: Recall that $\bar{\tau} = \min\{\bar{\tau}_i\}_{i=1}^{n}$. Also recall the defintion of η_i from the statement (a) of Theorem 4.1. Next, note that due to the statement (a) of Theorem 4.1 for every $i \in \mathcal{I}_{\text{out}}$ we know that $\rho_\tau^i = g(\alpha_i\tau)|\operatorname{Re}(\alpha_i)|$ is strictly decreasing for $\tau \in [0,\bar{\tau})$. We note that because $\mathcal{I}_1 \subset \mathcal{I}_{\text{out}}$, the previous statement holds for $i \in \mathcal{I}_1$. It also means that $\rho_\tau^i = g(\alpha_i\tau)|\operatorname{Re}(\alpha_i)| < |\operatorname{Re}(\alpha_1)|$ for $\tau \in [0,\bar{\tau})$. For $i \in \mathcal{I}_{\text{in}}$, from the proof argument of the statement (a) and (b) of Theorem 5.2 we know that $\rho_\tau^i = g(\alpha_i\tau)|\operatorname{Re}(\alpha_i)|$ is decreasing for $\tau \in (\eta_i,\bar{\tau}_i]$ and also that $\rho_\tau^i = g(\alpha_i\tau)|\operatorname{Re}(\alpha_i)| \geq |\operatorname{Re}(\alpha_1)$ for $\tau \in (0,\eta_i]$, which means that $\rho_\tau^i = g(\alpha_i\tau)|\operatorname{Re}(\alpha_i)| \geq \rho_\tau^j$, $j \in \mathcal{I}_1$ for $\tau \in (0,\eta_i]$. Let $\mathcal{P} = \{\eta \in \mathbb{R}_{>0}|\eta = \eta_i$ if $\eta_i < \bar{\tau}, i \in \mathcal{I}_{\text{in}}\}$. If $\mathcal{P} = \{\}$, then we have $\rho_\tau = \min\{\rho_\tau^i\}_{i=1}^{n} = \min\{\rho_\tau^j\}_{j\in\mathcal{I}_{\text{out}}}$ for any $\tau \in [0,\bar{\tau})$. Therefore, since at delay interval $[0,\bar{\tau})$, ρ_τ is the minimum of strictly decreasing functions, it is also strictly decreasing. When $\mathcal{P} \neq \{\}$, let $\mathcal{P} = \{p_1,\cdots,p_{|\mathcal{P}|}\}$, where $p_m < p_n$ if $m < n$. Also, let $p_0 = 0$, $p_{|\mathcal{P}|+1} = \bar{\tau}$. Then, in light of the earlier observations, at each delay interval $[p_i,p_{i+1})$, $i \in \{0,\cdots,|\mathcal{P}|\}$, we have $\rho_\tau = \min\{\rho_\tau^i\}_{i=1}^{n} = \min\{\rho_\tau^j\}_{j\in\mathcal{K}_i}$, where $\mathcal{K}_i = \mathcal{I}_{\text{out}} \cup \{k \in \mathcal{I}_{\text{in}}|\eta_k \leq p_i\}$. Since at each interval $[p_i,p_{i+1})$, $i \in \{0,\cdots,|\mathcal{P}|\}$, each ρ_τ^j, $j \in \mathcal{K}_i$ is strictly decreasing, therefore ρ_τ is also strictly decreasing in delay interval $[p_i,p_{i+1})$. The proof then follows from

$\cup_{i=1}^{|\mathcal{P}|}[p_i, p_{i+1}) = [0, \bar{\tau})$.

In the remainder of this section we focus on the special case when all the eigenvalue of $\mathbf{A}$ are real negative numbers. In this special case, because of some ease in the mathematical manipulations and since the directional derivative of $g(x)$ with respect to τ along $x = \alpha_i\tau$, for all $i \in \{1, \cdots, n\}$, follows a same pattern, we are able to derive a simpler expressions to compute $\tilde{\tau}$ and $\tau^\star$. We start our study by showing that in this case the rate of convergence ρ_τ depends only on α_1 and α_n.

Lemma 5.2 (When $\{\alpha_i\}_{i=1}^n \subset \mathbb{R}_{<0}$, ρ_τ of system (5) depends only on α_1 and α_n): *Consider the linear time-delayed system* (5) *when $\{\alpha_i\}_{i=1}^n \subset \mathbb{R}_{<0}$. Recall the admissible delay bound $\bar{\tau} = \frac{\pi}{2|\alpha_n|}$ of this system from Lemma 3.1. Then,*

(a) $\rho_\tau = g(\alpha_1\tau)\,|\alpha_1| = -\frac{1}{\tau}\operatorname{Re}(W_0(\alpha_1\tau))$ *for any* $\tau \in (0, \tau_n^\star) \subset (0, \bar{\tau})$,
(b) $\rho_\tau = \min\{g(\alpha_1\tau)\,|\alpha_1|, g(\alpha_n\tau)|\alpha_n|\}$ *for any* $\tau \in ([\tau_n^\star, \tau_1^\star] \cap (0, \bar{\tau}))$,
(c) *If* $\tau_1^\star < \bar{\tau}$, *then* $\rho_\tau = g(\alpha_n\tau)\,|\alpha_n| = -\frac{1}{\tau}\operatorname{Re}(W_0(\alpha_n\tau))$ *for any* $\tau \in (\tau_1^\star, \bar{\tau}) \subset (0, \bar{\tau})$,

where $\tau_1^\star = \frac{1}{|\alpha_1|\,\mathrm{e}}$ *and* $\tau_n^\star = \frac{1}{|\alpha_n|\,\mathrm{e}}$.

Proof 6: We start the proof with some observations. Let $\mathcal{R}_r = [0, \frac{1}{\mathrm{e}}]$ and $\mathcal{R}_l = [\frac{1}{\mathrm{e}}, \frac{\pi}{2})$. Then, note that from the statement (a) of Lemma 4.4, we have that $g(-\xi)$ increases strictly from 1 to e for $\xi \in \mathcal{R}_r$. Therefore,

$$g(-\xi_1) < g(-\xi_2), \quad \text{if } \ (\xi_1 < \xi_2 \text{ and } \xi_1, \xi_2 \in \mathcal{R}_r). \quad (27)$$

Next, let $\mu = g(-\xi)\,\xi$. We note that $\frac{\mathrm{d}\mu}{\mathrm{d}\xi} = \frac{\mathrm{d}g(-\xi)}{\mathrm{d}\xi}\xi + g(-\xi)$. For $\xi \in (\frac{1}{\mathrm{e}}, \frac{\pi}{2})$, from (A.42) and the manipulations leading to it, we can write $\frac{\mathrm{d}\mu}{\mathrm{d}\xi} = \frac{1}{\xi}\,\frac{\mathsf{u}^2(-\mathsf{u}\frac{\cos(\mathsf{u})}{\sin(\mathsf{u})} + \cos(2\,\mathsf{u}))}{(-\mathsf{u}\,\cos(\mathsf{u}) + \sin(\mathsf{u}))^2 + \mathsf{u}^2\,\sin^2(\mathsf{u})} + \frac{1}{\xi}\,\frac{\mathsf{u}\cos(\mathsf{u})}{\sin(\mathsf{u})} = \frac{1}{2\xi}\,\frac{-\mathsf{u}(2\mathsf{u} - \sin(2\mathsf{u}))}{(-\mathsf{u}\,\cos(\mathsf{u}) + \sin(\mathsf{u}))^2 + \mathsf{u}^2\,\sin^2(\mathsf{u})}$, where $\mathsf{u} = \operatorname{Im}(W_0(-\xi))$. Consequently, for $\xi \in (\frac{1}{\mathrm{e}}, \frac{\pi}{2})$, since $\mathsf{u} \in (0, \frac{\pi}{2})$ and therefore $\sin(2\mathsf{u}) \leq 2\mathsf{u}$, we get $\frac{\mathrm{d}\mu}{\mathrm{d}\xi} < 0$. This conclusion along with μ being a continuous function in $\xi \in \mathcal{R}_l$, confirms that

$$g(-\xi_1)\xi_1 > g(-\xi_2)\xi_2, \quad \text{if } (\xi_1 < \xi_2 \text{ and } \xi_1, \xi_2 \in \mathcal{R}_l). \quad (28)$$

Now, to prove the statement (a) we proceed as follows. Since according to the statement (a) of Lemma 4.4, we have $|\alpha_i|\tau_i^\star = \frac{1}{\mathrm{e}}$ for $i \in \{1, \cdots, n\}$, then $0 < \tau_n^\star \leq \tau_{n-1}^\star \leq \cdots \leq \tau_1^\star$. This fact along with $|\alpha_1|\tau \leq \cdots \leq |\alpha_n|\tau$ lead us to conclude from (27) that

$$g(\alpha_1\tau) \leq g(\alpha_2\tau) \leq \cdots \leq g(\alpha_n\tau), \quad \tau \in (0, \tau_n^\star]. \quad (29)$$

Here we used the fact that for $\tau \in (0, \tau_n^\star]$, we have $|\alpha_i|\tau \in \mathcal{R}_r$ and for $\tau \in [\tau_1^\star, \bar{\tau}]$, we have $|\alpha_i|\tau \in \mathcal{R}_l$, $i \in \{1, \cdots, n\}$. Now, given (29), we have $\rho_\tau = \min\{g(\alpha_i\tau)|\alpha_i|\}_{i=1}^n = g(\alpha_i\tau)|\alpha_1|$, which completes the proof of the statement (a).

To prove the statement (b) we proceed as follows. For a $\tau \in [\tau_n^\star, \tau_1^\star]$, let $\mathcal{I}_r = \{i \in \{1, \cdots, n\} || \alpha_j|\tau \in \mathcal{R}_r\}$ and $\mathcal{I}_l = \{i \in \{1, \cdots, n\} || \alpha_j|\tau \in \mathcal{R}_l\}$. Surely, for any $\tau \in [\tau_n^\star, \tau_1^\star]$, $1 \in \mathcal{R}_r$ and $n \in \mathcal{R}_l$, however for $j \in \{2, \cdots, n-1\}$, $|\alpha_j|\tau$, depending on its value, can be in either in $\mathcal{R}_l$ or $\mathcal{R}_r$. Following the same proof argument of the statement (a), then for any $\tau \in [\tau_n^\star, \tau_1^\star]$, we have $\min\{g(\alpha_i\tau)|\alpha_i|\}_{i\in\mathcal{I}_l} = g(\alpha_1\tau)|\alpha_1|$. To complete the proof of the statement (b), by taking into account the definition of ρ_τ in (14), next we show that for any $\tau \in [\tau_n^\star, \tau_1^\star]$, we have $\min\{g(\alpha_i\tau)|\alpha_i|\}_{i\in\mathcal{I}_r} = g(\alpha_n\tau)|\alpha_n|$. For this, we note that since $0 < \tau_n^\star \leq \tau_{n-1}^\star \leq \cdots \leq \tau_1^\star$ and $|\alpha_1|\tau \leq \cdots \leq |\alpha_n|\tau$, we can conclude from (28) that for any $\{j, k\} \subset \mathcal{I}_r$ such that $j > k$ we have

$$g(\alpha_j\tau)|\alpha_j|\tau \leq g(\alpha_k\tau)|\alpha_k|\tau, \quad \tau \in [\tau_1^\star, \tau_n^\star]. \quad (30)$$

Therefore, we can write $\min\{g(\alpha_i\tau)|\alpha_i|\}_{i\in\mathcal{I}_r} = g(\alpha_n\tau)|\alpha_n|$.

Proof of the statement (c) follows directly from the proof of the statement (b), by noting that for $\tau \in [\frac{1}{\mathrm{e}\,|\alpha_1|}, \frac{\pi}{2\,|\alpha_n|}) \subset (0, \bar{\tau})$, we have $\alpha_i\tau \in \mathcal{R}_l$ for all $i \in \{1, \ldots, n\}$.

Our next result shows that when all the eigenvalues $\{\alpha_i\}_{i=1}^n$ of $\mathbf{A}$ are negative real numbers, $\tilde{\tau}$ which according to the statement (a) of Theorem 5.2 is equal to $\min\{\eta_i\}_{i=1}^n$ is in fact given by $\min\{\eta_1, \eta_n\}$. The result below also gives a close form solution for $\tau^\star$ and its corresponding $\rho_\tau^\star$.

Theorem 5.3 (Rate of convergence of (5) with and without delay when $\{\alpha_i\}_{i=1}^n \subset \mathbb{R}_{<0}$): *Consider the linear time-delayed system* (5) *when $\{\alpha_i\}_{i=1}^n \subset \mathbb{R}_{<0}$. Recall the admissible delay bound $\bar{\tau} = \frac{\pi}{2|\alpha_n|}$ of this system from Lemma 3.1. Then,*

(a) $\tilde{\tau} = \min\{\tilde{\tau}_1, \eta_n\}$, *where η_n is defined in the statement (a) of Theorem 5.2. Moreover,* $\tau_n^\star < \tilde{\tau} < \bar{\tau}$.
(b) *the maximum rate of convergence of*

$$\rho_\tau^\star = \mathrm{e}^{\frac{\arccos(\frac{\alpha_1}{\alpha_n})}{\sqrt{(\frac{\alpha_n}{\alpha_1})^2 - 1}}}\ |\alpha_1|, \quad (31)$$

is attained at

$$\tau^\star = \frac{\arccos(\frac{\alpha_1}{\alpha_n})}{|\alpha_1|\sqrt{\frac{\alpha_n}{\alpha_1}^2 - 1}}\,\mathrm{e}^{-\frac{\arccos(\frac{\alpha_1}{\alpha_n})}{\sqrt{(\frac{\alpha_n}{\alpha_1})^2 - 1}}}, \quad (32)$$

where $\tau_1^\star = \frac{1}{\mathrm{e}\,|\alpha_1|}$ *and* $\tau_n^\star = \frac{1}{\mathrm{e}\,|\alpha_n|}$. *Moreover,* $\tau^\star \in ([\tau_n^\star, \tau_1^\star] \cap [\tau_n^\star, \bar{\tau}))$.

Proof 7: Lemma 5.2 showed that

$$\rho_\tau = \min\{g(\alpha_1\tau)\,|\alpha_1|, g(\alpha_n\tau)\,|\alpha_n|\}, \quad \tau \in (0, \bar{\tau}). \quad (33)$$

We will use this fact to prove our statements.

To prove the statement (a), first note that by definition we have $g(\alpha_1\tilde{\tau}_1) = 1$ and $g(\alpha_1\eta_n) = \frac{\alpha_1}{\alpha_n}$. Therefore, we have $g(\alpha_1\tilde{\tau}_1)|\alpha_1| = g(\alpha_1\eta_n)|\alpha_n| = |\alpha_1|$. Next, note that in the proof of the statement (a) of Theorem 5.2 we have already shown that η_n satisfies $\tau_n^\star < \eta_n < \bar{\tau}_n = \bar{\tau}$, which means $\tau \in (\eta_n, \bar{\tau}) \subset [\tau_n^\star, \bar{\tau})$. Therefore, by virtue of the statement (a) of Lemma 4.4, we have $g(\alpha_n\tau)|\alpha_n| < g(\alpha_n\eta_n)|\alpha_n| = |\alpha_1|$ for $\tau \in (\eta_n, \bar{\tau}) \subset [\tau_n^\star, \bar{\tau})$. Subsequently, from (33), we conclude that $\rho_\tau < |\alpha_1|$ for $\tau \in (\eta_n, \bar{\tau})$. Next, note that for any $\tau \in (\tilde{\tau}_1, \bar{\tau})$ by virtue of the statement (b) of Lemma 4.4, we have $g(\alpha_1\tau) < 1$ and consequently, $g(\alpha_1\tau)|\alpha_1| < |\alpha_1|$. Therefore, because of (33), we have also the guarantee that $\rho_\tau < \rho_0 = |\alpha_1|$ for any $\tau \in (\tilde{\tau}_1, \bar{\tau})$. As a result, we can conclude from (33) that $\rho < \rho_0$ for $\tau \in (\min\{\tilde{\tau}_1, \eta_n\}, \bar{\tau})$. Using Lemma 4.4, we have the guarantees that $g(\alpha_n\tau) > 1$ for $\tau \in (0, \tau_n^\star]$ and $g(\alpha_n\tau)$ is strictly decreasing for $\tau \in [\tau_n^\star, \bar{\tau}_n)$. Therefore, $g(\alpha_n\tau) > \frac{\alpha_1}{\alpha_n}$ for $\tau \in (0, \eta_n)$ (note here that $\tau_n^\star <$

$\tilde{\tau}_n \leq \eta_n < \bar{\tau}$). From Lemma 4.4, we also know that $g(\alpha_1\tau) > 1$ for $\tau \in (0, \tilde{\tau}_1)$. As a result, we can conclude from (33) that $\rho_\tau > |\alpha_1|$ for $\tau \in (0, \min\{\tilde{\tau}_1, \eta_n\})$. This completes the proof of statement (a).

Using (33), we proceed to prove our statement (b) as follows. First we consider the case that $\alpha_1 = \alpha_n$ in which the rate of convergence at $\tau \in (0, \bar{\tau})$ is $\rho_\tau = g(\alpha_1\tau)\,|\alpha_1|$. Here, by virtue of Lemma 4.4 one can see that the maximum rate of $\rho^\star_\tau = \mathrm{e}\,|\alpha_1|$ is attained at $\tau^\star = \frac{1}{\mathrm{e}\,|\alpha_1|}$ (here note that $\tau^\star_1 = \tau^\star_n = \tau^\star$). Then, for the case of $\alpha_1 = \alpha_n$ the proof of the statement (b) follows from $\lim_{\frac{\alpha_n}{\alpha_1}\to 1}\big(\frac{\arccos(\frac{\alpha_1}{\alpha_n})}{\sqrt{(\frac{\alpha_n}{\alpha_1})^2-1}}\big) = 1$.

Next, we consider the case where $|\alpha_n| > |\alpha_1|$. From the proof of the statement (a), we know that $\tau^\star$ should satisfy $\tau^\star \in (0, \tilde{\tau})$. Also, recall that $\tau^\star_n < \tilde{\tau}$. Since $\tau^\star_n < \tau^\star_1$, by virtue of the statement (a) of Lemma 4.4 we know that both $g(\alpha_1\tau)$ and $g(\alpha_n\tau)$ are strictly increasing for $\tau \in [0, \tau^\star_n)$. Therefore, from (33) we can conclude that ρ_τ is also strictly increasing in $\tau \in [0, \tau^\star_n)$. Hence, $\tau^\star \geq \tau^\star_n$. If $\tau^\star_1 < \bar{\tau}$, then by virtue of the statement (a) of Lemma 4.4 we know that both $g(\alpha_1\tau)$ and $g(\alpha_n\tau)$ are strictly decreasing for $\tau \in [0, \tau^\star_n)$. Therefore, from (33) we can conclude that ρ_τ is also strictly decreasing in $\tau \in (\tau^\star_1, \bar{\tau})$. Hence, $\tau^\star \leq \tau^\star_1$. If $\tau^\star_1 > \bar{\tau}$, we know that $\rho_\tau < 0$ and the system is unstable for $\tau > \tau^\star_1$.

So far we have shown that $\tau^\star \in ([\tau^\star_n, \tau^\star_1] \cap [\tau^\star_n, \bar{\tau}))$. From the discussions for far we also know that $g(\alpha_1\tau)|\alpha_1|$ is strictly increasing for $\tau \in [\tau^\star_n, \tau^\star_n]$, and $g(\alpha_n\tau)|\alpha_n|$ is strictly decreasing in $[\tau^\star_n, \bar{\tau})$. Therefore, from (33) we conclude that at $\tau^\star$ we have $g(\alpha_1\tau^\star)|\alpha_1| = g(\alpha_n\tau^\star)|\alpha_n|$, or equivalently (recall (12)) when

$$\mathrm{Re}(W_0(\alpha_1\,\tau^\star)) = \mathrm{Re}(W_0(\alpha_n\,\tau^\star)). \tag{34}$$

Because for $\tau \in ([\tau^\star_n, \tau^\star_1] \cap [\tau^\star_n, \bar{\tau}))$, we have $\alpha_1\tau^\star \in [\alpha_1 \min\{\frac{1}{|\alpha_1|\,\mathrm{e}}, \frac{\pi}{2|\alpha_n|}\}, -\frac{\alpha_1}{\alpha_n\,\mathrm{e}}] \subset [-\frac{1}{\mathrm{e}}, 0)$, then $W_0(\xi_1)$ is a negative real number, i.e., $\mathrm{Re}(W_0(\alpha_1\,\tau^\star)) = W_0(\alpha_1\,\tau^\star) = \mathsf{w}_1 \in \mathbb{R}_{<0}$. Subsequently, from (34) we have $W_0(\alpha_n\tau^\star) = \mathsf{w}_1 + \mathrm{i}\,\mathsf{u}$ for some $\mathsf{u} \in (-\pi, \pi)$. Therefore, we can write

$$\begin{aligned} W_0(\alpha_1\,\tau^\star) &= \mathsf{w}_1 &&\rightarrow\ \mathsf{w}_1\,\mathrm{e}^{\mathsf{w}_1} = \alpha_1\,\tau^\star,\\ W_0(\alpha_n\tau^\star) &= \mathsf{w}_1 + \mathrm{i}\,\mathsf{u} &&\rightarrow\ (\mathsf{w}_1 + \mathrm{i}\,\mathsf{u})\,\mathrm{e}^{\mathsf{w}_1+\mathrm{i}\,\mathsf{u}} = \alpha_n\tau^\star, \end{aligned}$$

which by eliminating $\tau^\star$ gives

$$\mathsf{w}_1\cos(\mathsf{u}) - \mathsf{u}\sin(\mathsf{u}) = \frac{\alpha_n}{\alpha_1}\mathsf{w}_1, \text{ and } \mathsf{u}\cos(\mathsf{u}) + \mathsf{w}_1\sin(\mathsf{u}) = 0.$$

Then, using $\cos(\mathsf{u})^2 + \sin(\mathsf{u})^2 = 1$, we obtain $\cos(\mathsf{u}) = \frac{\alpha_1}{\alpha_n}$ and $\sin(\mathsf{u}) = \frac{(\alpha_1^2-\alpha_n^2)\,\mathsf{w}_1}{\alpha_n\alpha_1\,\mathsf{u}}$. Subsequently, because $\frac{\alpha_n}{\alpha_1} \in (1, \infty)$, we obtain $\mathsf{u} = \arccos(\frac{\alpha_1}{\alpha_n}) \in (0, \pi) \subset (-\pi, \pi)$ and $\sin(\mathsf{u}) = \frac{1}{|\alpha_n|}\sqrt{\alpha_n^2 - \alpha_1^2}$ and $\mathsf{w}_1 = -\frac{\arccos(\frac{\alpha_1}{\alpha_n})}{\sqrt{(\frac{\alpha_n}{\alpha_1})^2-1}}$. Then, by virtue of $\mathsf{w}_1\,\mathrm{e}^{\mathsf{w}_1} = \alpha_1\,\tau^\star$, (32) is confirmed. Finally, (31) is confirmed by $\rho^\star_\tau = -g(\alpha_1\tau^\star)\,\alpha_1 = -\frac{1}{\tau^\star}\mathrm{Re}(W_0(\alpha_1\tau^\star)) = -\frac{\mathsf{w}_1}{\tau^\star}$.

It is interesting to note that the suprimum value of $\arccos(\gamma)/\sqrt{\frac{1}{\gamma^2} - 1}$ for $\gamma \in (0, 1)$ is 1. Therefore, $\rho^\star_\tau$ in (31) is always less than or equal to $\mathrm{e}\,|\alpha_1| = \mathrm{e}\,\rho_0$, regardless of value of $\alpha_1 \in \mathbb{R}_{<0}$ and $\alpha_n \in \mathbb{R}_{<0}$. This observation is in accordance with Remark 5.2.

VI. Demonstrative example: static average consensus problem in networked systems

We demonstrate our results by studying the effect of delay on the static average consensus algorithm (c.f. [33]) for a group of N networked agents interacting over a strongly connected and weight-balanced directed graph (or simply digraph), similar to the ones shown in Fig. 6[1]. An arrow from agent i to agent j means that agent i can obtain information from agent j. The set of all agents that can send information to agent i are called its out-neighbors.A digraph is strongly connected if there is a directed path from every agent i to every agent j in the graph. Let $\mathbf{W} = [\mathsf{w}_{ij}] \in \mathbb{R}^{N\times N}$ be the adjacency matrix of a given digraph, defined according to $\mathsf{w}_{ii} = 0$, $\mathsf{w}_{ij} > 0$ if agent j can send information to agent i, and $\mathsf{w}_{ij} = 0$ otherwise. A digraph of N agents is weight-balanced if and only if $\sum_{j=1}^N \mathsf{w}_{ij} = \sum_{j=1}^N \mathsf{w}_{ji}$ for any $i \in \{1, \cdots, N\}$. Let every agent in this network have a local reference value $\mathsf{r}^i \in \mathbb{R}$, $i \in \{1, \cdots, N\}$. The static average consensus problem consists of designing a distributed algorithm that enables each agent to obtain $\frac{1}{N}\sum_{j=1}^N \mathsf{r}^j$ by using the information it only receives from its out-neighbors. As shown in [33], for strongly connected and weight-balanced digraphs, the Laplacian dynamics

$$\dot{x}^i(t) = u^i = \sum\nolimits_{j=1}^N \mathsf{w}_{ij}(x^j(t) - x^i(t)), \quad x^i(0) = \mathsf{r}^i,$$

$i \in \{1, \cdots, N\}$, is guaranteed to satisfy $x^i \rightarrow \frac{1}{N}\sum_{j=1}^N \mathsf{r}^j$, as $t \rightarrow \infty$. Using the aggregated state vector $\mathbf{x} = [x^1, \cdots, x^N]^\top$ the compact form of the Laplacian dynamics above in the presence of delay $\tau \in \mathbb{R}_{>0}$ is (c.f. [33])

$$\begin{aligned} &\dot{\mathbf{x}}(t) = -\mathbf{L}\,\mathbf{x}(t-\tau), \qquad (35)\\ &x^i(t) = \phi^i(t) \in \mathbb{R}, \ \ t \in [-\tau, 0], \ \phi^i(0) = \mathsf{r}^i, \ \ i \in \{1, \cdots, N\}, \end{aligned}$$

where $\mathbf{L} = \mathrm{diag}(\mathbf{W}\mathbf{1}_N) - \mathbf{W}$, see Fig. 6 for examples. For strongly connected and weight-balanced digraphs, we have $\mathrm{rank}(\mathbf{L}) = N-1$, $\mathbf{L}\mathbf{1}_N = \mathbf{0}$, $\mathbf{1}_N^\top\mathbf{L} = \mathbf{0}$. Moreover, $\mathbf{L}$ has one simple zero eigenvalue $\lambda_1 = 0$ and the rest of its eigenvalues $\{\lambda_j\}_{j=2}^N$ has negative real parts [37]. Next, consider the change of variable $\mathbf{y} = \mathbf{T}^\top\mathbf{x}$ with $\mathbf{T} = \begin{bmatrix}\frac{1}{\sqrt{N}}\mathbf{1}_N & \mathbf{R}\end{bmatrix}$, where $\mathbf{R}$ is such that $\mathbf{T}^\top\mathbf{T} = \mathbf{T}\mathbf{T}^\top = \mathbf{I}_N$. Then the Laplacian dynamics can be represented in the following equivalent form

$$\dot{y}_1(t) = 0, \qquad y_1(0) = \frac{1}{\sqrt{N}}\sum\nolimits_{i=1}^N \mathsf{r}^i, \tag{36a}$$

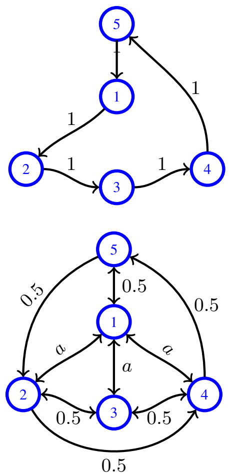
$$\dot{\mathbf{y}}_{2:N}(t) = \mathbf{A}\mathbf{y}_{2:N}(t-\tau). \tag{36b}$$

where $\mathbf{y} = [y_1^\top\ \mathbf{y}_{2:N}^\top]^\top$ and $\mathbf{A} = -(\mathbf{R}^\top\mathbf{L}\mathbf{R})$. The matrix $-\mathbf{R}^\top\mathbf{L}\mathbf{R}$ is Hurwitz with eigenvalues $\{\alpha_j\}_{j=1}^n = \{\lambda_i\}_{i=2}^N \subset \mathbb{C}^l_-$, $n = N-1$. Evidently, here we have

$$\begin{aligned} \lim_{t\to\infty}\mathbf{x}(t) =& \frac{1}{\sqrt{N}}\lim_{t\to\infty} y_1(t)\mathbf{1}_N + \lim_{t\to\infty}\mathbf{R}\mathbf{y}_{2:N}(t)\\ =&(\frac{1}{N}\sum\nolimits_{i=1}^N \mathsf{r}^i)\mathbf{1}_N + \mathbf{R}\lim_{t\to\infty}\mathbf{y}_{2:N}(t). \end{aligned}$$

Therefore, the correctness and the convergence rate of the average consensus algorithm (35) are determined, respectively,

[1] Please see [37] for graph related terminologies and definitions.



(a) $$\mathbf{L}=\begin{bmatrix}1 & -1 & 0 & 0 & 0\\ 0 & 1 & -1 & 0 & 0\\ 0 & 0 & 1 & -1 & 0\\ 0 & 0 & 0 & 1 & -1\\ -1 & 0 & 0 & 0 & 1\end{bmatrix}$$

(b) $$\mathbf{L}=\begin{bmatrix}3a+0.5 & -a & -a & -a & -0.5\\ -a & a+1 & -0.5 & -0.5 & 0\\ -a & -0.5 & a+1 & -0.5 & 0\\ -a & 0 & -0.5 & a+1 & -0.5\\ -0.5 & -0.5 & 0 & 0 & 1\end{bmatrix}$$

Figure 6: Strongly connected and weight-balanced networks with their corresponding Laplacian matrices.

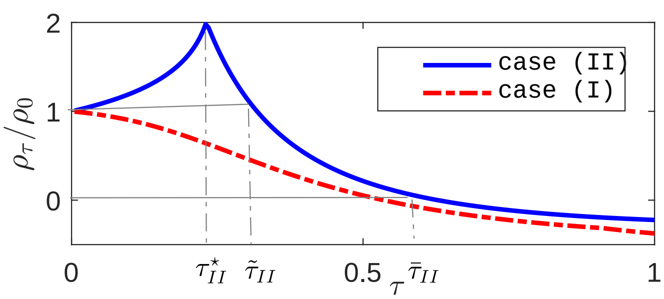


Figure 7: Normalized rate of convergence, ρ_τ/ρ_0 versus time delay for the cases (I) and (II). According to the plot $\tau^\star_{II} = 0.23$, $\tilde{\tau}_{II} = 0.32$ and $\bar{\tau}_{II} = 0.63$.

by exponential stability and the convergence rate of (36b). Since the time-delayed system (36b) is in the form of our system of interest (5), the effect of delay and how it can potentially be used to accelerate the rate of convergence of the algorithm (35) can be fully analyzed by the results described in Section V.

In what follows, we present three demonstrative examples of executing algorithm (35) over the networks shown in Fig. 6. We report our numerical values in two digit precision. As mentioned earlier, the convergence rate of algorithm (35), is determined by the convergence rare of the time-delayed system (36b). For the networks shown in Fig. 6, the eigenvalues of $\mathbf{A} = -\mathbf{R}^\top\mathbf{L}\mathbf{R}$ are as follows (I) $\text{eig}(\mathbf{A}) = \{\alpha_j\}_{j=1}^4 = \{-0.69+0.95\,\text{i}, -0.69-0.95\,\text{i}, -1.80+0.58\,\text{i}, -1.80-0.58\,\text{i}\}$ for the network of Fig. 6(a), (II) $\text{eig}(\mathbf{A}) = \{\alpha_j\}_{j=1}^4 = \{-1.50, -1.50, -2.00, -2.50\}$ for the network of Fig. 6(b) when $a = 0.50$ and (III) $\text{eig}(\mathbf{A}) = \{\alpha_j\}_{j=1}^4 = \{-1.05, -1.47+0.18\,\text{i}, -1.47-0.18\,\text{i}, -1.70\}$ for the network of Fig. 6(b) when $a = 0.20$.

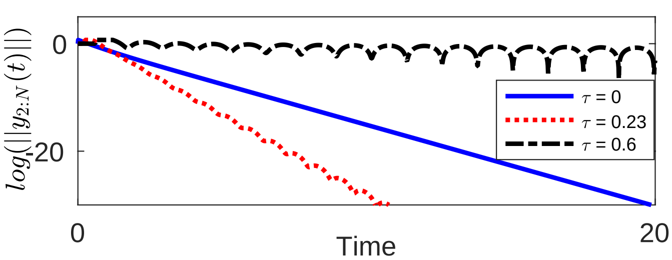


Figure 8: Norm of time response of the system (36) for case (II) in logarithmic scale for different value of the time delay: $\tau = 0$, $\tau = \tau^\star = 0.23$ and $\tau = 0.60$.

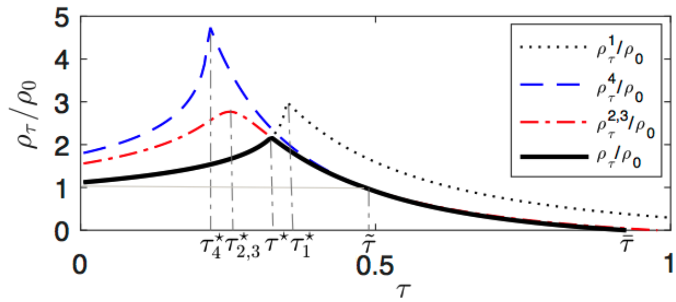


Figure 9: The Normalized rate of convergence versus time delay for different modes of system (36) for case (III). ρ^1_τ, ρ^2_τ, ρ^3_τ and ρ^4_τ are the rate of convergence corresponding to $\alpha_1 = -1.05$, $\alpha_{2,3} = -1.47 \pm 0.18\,\text{i}$ and $\alpha_4 = -1.70$, respectively. Recall that according to (11) we have $\rho_\tau/\rho_0 = \min\{\rho^i_\tau/\rho_0\}_{i=1}^4$, which its normalized value is shown by the thick black curve.

Fig. 7 shows the rate of convergence ρ_τ as defined in (11) versus time delay for the cases (I) and (II). In the case (I), we have $\{\alpha_i\}_{i=1}^4 \subset \mathbb{C}^-$, $\mathcal{I}_{\text{in}} = \{3,4\}$ and $\mathcal{I}_1 = \mathcal{I}_{\text{out}} = \{1,2\}$ where $\mathcal{I}_1$, $\mathcal{I}_{\text{in}}$ and $\mathcal{I}_{\text{out}}$ are defined by (23). Therefore, since $\mathcal{I}_1 \subset \mathcal{I}_{\text{out}}$, as predicted by Lemma 5.1, ρ_τ decreases strictly with delay delay until it reaches 0 at $\bar{\tau} = 0.51$, as shown in Fig. 7.

For the case (II) we have $\{\alpha_i\}_{i=1}^4 \subset \mathbb{R}^-$ and also $\mathcal{I}_1 = \mathcal{I}_{\text{in}} = \{1,2,3,4\}$. Hence, as predicted by Theorem 5.1, there exists a $\tilde{\tau} \in \mathbb{R}_{>0}$ such that $\rho_\tau > \rho_0$ for $\tau \in (0,\tilde{\tau}) \subset (0,\bar{\tau})$. In this case, $\bar{\tau} = \frac{\pi}{2|\alpha_4|} = 0.63$ (marked as $\bar{\tau}_{II}$ on x-axis of Fig. 7). Moreover, $\tilde{\tau}$, following the statement (a) of Theorem 5.3, is the minimum of can be obtained as $\tilde{\tau} = \min\{\tilde{\tau}_1 = 0.71, \eta_4 = 0.32\} = 0.32$, which is exactly the same value that one reads on Fig. 7, marked as $\tilde{\tau}_{II}$ on x-axis. Also, the maximum rate of convergence is attained at $\tau^\star = 0.23$ (marked as $\tau^\star_{II}$ on x-axis of Fig. 7), which can be obtained from (32) in the statement (b) of Theorem 5.3. At $\tau^\star$, the maximum attainable rate of convergence can be obtained from (31) in the statement (b) of Theorem 5.3 to be $\rho_\tau = 1.98\rho_0$, which matches the value one reads on Fig. 7. Fig. 8 shows the norm of response of system (36b) in logarithmic scale for different values of time delay for case (II). As seen in the figure, the rate of convergence for $\tau = \tau^\star = 0.23$ is greater than delay free case. For $\tau = 0.60$ which is close to the critical value of time delay $\bar{\tau} = 0.63$, the rate of convergence is near zero.

For case (III), we have $\{\alpha_i\}_{i=1}^4 \in \mathbb{C}^-$ and $\mathcal{I}_1 = \{1\} \subset \mathcal{I}_{\text{in}} = \{1,2,3,4\}$. Therefore, according to Theorem 5.1, we expect existence of $\tilde{\tau} \in \mathbb{R}_{>0}$ such that $\rho_\tau > \rho_0$ for

$\tau \in (0, \tilde{\tau}) \subset (0, \bar{\tau})$, which is in accordance with the trend one observes for ρ_τ in Fig. 9. Also, as seen in Fig. 9, we have $\rho_\tau = \min\{\rho_\tau^i\}_{i=1}^4 = \min\{\rho_\tau^1, \rho_\tau^{2,3}\}$ for any $\tau \in [0, \bar{\tau}]$. Here, $\bar{\tau} = 0.92$, and $\tilde{\tau} = 0.46$, which as expected from the statement (a) of Theorem 5.2, is the minimum of $\eta_1 = 0.59$, $\eta_2 = 0.46$, $\eta_3 = 0.46$ and $\eta_4 = 0.47$. Moreover, as expected from Theorem 5.2, the value of $\tau^\star$ satisfies $\tau^\star \in [\tau_4^\star, \tau_1^\star] \cap [0, \tilde{\tau})$ as shown in Fig. 9 where $\tau_1^\star = \frac{1}{|\alpha_1|\,\mathrm{e}} = 0.35 = \min\{\tau_i^\star\}_{i=1}^4$ and $\tau_4^\star = \frac{1}{|\alpha_4|\,\mathrm{e}} = 0.21 = \max\{\tau_i^\star\}_{i=1}^4$. In this case the maximum attainable rate is $\rho_{\tau^\star} = 1.92\rho_0$.

VII. Conclusion and future work

In this paper, we examined the effect of a fixed time delay on the rate of convergence of a class of time-delayed LTI systems. Our work was motivated by a long observed phenomenon that, contrary to intuition, delay in fact can lead to increase in stability margin and so the rate of convergence of some systems. In this work, for the class of time-delayed LTI systems that we studied, we address the following fundamental questions (a) what systems can experience increase in their rate of convergence due to delay (b) for what values of delay the rate of convergence is increased due to delay (c) what is the maximum achievable rate due to delay and its corresponding maximizing delay value. An interesting result that we established was to show that the ultimate bound on the maximum achievable rate of convergence via time delay is $\mathrm{e} \approx 2.71828$ times the delay free rate. Our analysis relied on use of the Lambert W function to specify the rightmost root of the characteristic equation of our time-delayed LTI system of interest. An important application of the theoretical results developed in this paper can be use of out-dated information to accelerate convergence of distributed linear algorithms in networked systems. An important set of distributed algorithms that are in the form of LTI systems is the agreement or consensus algorithms. Agreement algorithms in network systems play a crucial role in facilitating many cooperative task. Therefore, their fast convergence is always a desirable factor. An example case of accelerated static average consensus algorithm via the use of an out-dated feedback is demonstrated in this paper. As is known, and also expected, the convergence rate of agreement algorithms depend on network topology. The detailed analysis we offered in this paper can be of great value in topology design to optimize the agreement speed when out-dated feedbacks are used. Our future work is focused on both expanding our results to a wider class of time-delayed LTI systems and exploring the application of our theoretical results in design of fast-converging distributed algorithms for networked systems.

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Appendix

This appendix contains the proof of the lemmas of Section IV.

Proof 8 (Proof of Lemma 4.1): *Given the definition of* $g(x)$ *in* (13)*, the proof of* (15a) *follows directly from* (3b).

To validate (15b)*, we proceed as follows. Note that* $g(\alpha\tau)=0$ *requires* $\mathrm{Re}(W_0(\alpha\tau))=0$ *which implies that* $W_0(\alpha\tau)=\mathsf{u}\,\mathrm{i}$ *for some non-zero* $\mathsf{u}\in(-\pi,\pi)$*. Following the definition of the Lambert W function, then, we can write*

$$\mathsf{u}\,\mathrm{e}^{\mathsf{u}\mathrm{i}}=(\mathrm{Re}(\alpha)\tau+\mathrm{i}\,\mathrm{Im}(\alpha)\tau)\Leftrightarrow\begin{cases}-\mathsf{u}\sin(\mathsf{u})=\mathrm{Re}(\alpha)\tau,\\ \mathsf{u}\cos(\mathsf{u})=\mathrm{Im}(\alpha)\tau,\end{cases}$$

$$\Leftrightarrow\begin{cases}\mathsf{u}^2=\tau^2|\alpha|^2,\\ \tan(\mathsf{u})=\frac{\mathrm{Re}(\alpha)}{\mathrm{Im}(\alpha)},\end{cases}$$

which for $\mathsf{u}\in(-\pi,\pi)$*, after eliminating* u *gives* $\tau=|atan(\frac{\mathrm{Re}(\alpha)}{\mathrm{Im}(\alpha)})|/|\alpha|$ *as the unique solution for* $g(\alpha\tau)=0$*. For* $\alpha\in\mathbb{R}_{<0}$*, we have* $atan(\frac{\mathrm{Re}(\alpha)}{\mathrm{Im}(\alpha)})=\frac{\pi}{2}$*, which means that* $\bar{\tau}=\frac{\pi}{2|\alpha|}$*. Finally, to validate* (16) *we proceed as follows.* $g(\alpha\tau)=1$ *means that* $\mathrm{Re}(W_0(\alpha\tau))=\mathrm{Re}(\alpha\tau)$*. Then,* $W_0(\alpha\tau)=\mathrm{Re}(\alpha\tau)+\mathrm{i}\,\theta$ *for some non-zero* $\theta\in(-\pi,\pi)$*. Then, we obtain the value of* θ *as follows. Invoking definition of Lambert function, we have*

$$(\mathrm{Re}(\alpha\tau)+\theta\,\mathrm{i})\,\mathrm{e}^{\mathrm{Re}(\alpha\tau)+\theta\,\mathrm{i}}=\mathrm{Re}(\alpha\tau),$$

$$\Leftrightarrow\begin{cases}\theta\cos\theta+\mathrm{Re}(\alpha\tau)\sin\theta &=0,\\ \mathrm{e}^{\mathrm{Re}(\alpha\tau)}\big(\mathrm{Re}(\alpha\tau)\cos\theta-\theta\sin\theta\big) &=\mathrm{Re}(\alpha\tau),\end{cases}$$

which using some trigonometric manipulations can also be stated equivalently as

$$\mathrm{Re}(\alpha\tau)=-\theta\cot(\theta) \tag{A.37a}$$

$$\mathrm{e}^{-\theta\cot(\theta)}=\cos(\theta). \tag{A.37b}$$

For $\theta\in(-\pi,\pi)$*,* (A.37b) *has two distinct solutions* $\theta\approx\pm1.01125$*. Thus, the proof of* (16) *follows from* (A.37a).

To prove the rest of the results in Section IV we rely on studying the derivative of $g(x)$ along $x=\alpha\tau$ with respect to $\tau\in\mathbb{R}_{>0}$ for a given $\alpha\in\mathbb{C}$. Using (3a) and (13), the derivative of delay rate gain function along $x=\alpha\tau\neq-\frac{1}{\mathrm{e}}$ with respect to time delay $\tau\in\mathbb{R}_{>0}$ can be written as

$$\frac{\mathrm{d}\,g(\alpha\tau)}{\mathrm{d}\tau}=\frac{1}{\mathrm{Re}(\alpha)}\Big[-\frac{1}{\tau^2}\mathrm{Re}(W_0(\alpha\tau))+\frac{1}{\tau}\mathrm{Re}(\frac{\alpha}{\alpha\tau+\mathrm{e}^{W_0(\alpha\tau)}})\Big],$$

which can also be represented as

$$\begin{aligned}\frac{\mathrm{d}\,g(\alpha\tau)}{\mathrm{d}\tau}&=-\frac{1}{\mathrm{Re}(\alpha)\tau^2}\mathrm{Re}(\frac{\alpha\tau\,W_0(\alpha\tau)}{\alpha\tau+\mathrm{e}^{W_0(\alpha\tau)}})\\&=-\frac{1}{\mathrm{Re}(\alpha)\tau^2}\mathrm{Re}(\frac{W_0^2(\alpha\tau)}{1+W_0(\alpha\tau)}).\end{aligned}$$

Let $W_0(\alpha\tau)=\mathsf{w}+\mathrm{i}\,\mathsf{u}$, where $\mathsf{u}\in(-\pi,\pi)$. Then, for $x=\alpha\tau\neq-\frac{1}{\mathrm{e}}$, we can write

$$\begin{aligned}\frac{\mathrm{d}\,g(\alpha\tau)}{\mathrm{d}\tau}&=-\frac{1}{\mathrm{Re}(\alpha)\tau^2}\frac{\mathsf{w}^3+\mathsf{w}^2-\mathsf{u}^2+\mathsf{w}\mathsf{u}^2}{(\mathsf{w}+1)^2+\mathsf{u}^2}\\&=-\frac{1}{\mathrm{Re}(\alpha)\tau^2}\frac{(\mathsf{w}^2+\mathsf{u}^2)\,\mathsf{w}+(\mathsf{w}^2-\mathsf{u}^2)}{(\mathsf{w}+1)^2+\mathsf{u}^2}.\end{aligned} \tag{A.38}$$

At $\alpha\tau=-\frac{1}{\mathrm{e}}$, the right and the left derivative of $g(\alpha\tau)$ along $x=\alpha\tau$ are obtained as follows. For $x=\alpha\tau\in[-\frac{1}{\mathrm{e}},0]$, $W_0(x)\in\mathbb{R}$. Thus, by setting $\mathsf{u}=0$, (A.38) gives

$$\frac{\mathrm{d}\,g(\alpha\tau)}{\mathrm{d}\tau}=\frac{1}{|\alpha|\,\tau^2}\frac{\mathsf{w}^2}{(\mathsf{w}+1)},\qquad\tau\in(0,\frac{1}{\mathrm{e}\,|\alpha|}), \tag{A.39}$$

thus

$$\lim_{\tau\to\frac{1}{\mathrm{e}\,|\alpha|}^-}\frac{\mathrm{d}\,g(\alpha\,\tau)}{\mathrm{d}\tau}=\lim_{\mathsf{w}\to-1^+}\mathrm{e}^2\,|\alpha|\,\frac{\mathsf{w}^2}{(\mathsf{w}+1)}=+\infty. \tag{A.40}$$

For any $x=\alpha\tau\in(-\infty,-\frac{1}{\mathrm{e}})$, $W_0(x)=\mathsf{w}+\mathrm{i}\,\mathsf{u}$ is a complex number with $\mathsf{u}\in(0,\pi)$, and satisfies

$$(\mathsf{w}+\mathrm{i}\,\mathsf{u})\,\mathrm{e}^{\mathsf{w}+\mathsf{u}\,\mathrm{i}}=\alpha\tau\Leftrightarrow\begin{cases}\mathrm{e}^{\mathsf{w}}\,(\mathsf{w}\cos(\mathsf{u})-\mathsf{u}\sin(\mathsf{u}))=\alpha\tau,\\ \mathrm{e}^{\mathsf{w}}\,(\mathsf{u}\cos(\mathsf{u})+\mathsf{w}\sin(\mathsf{u}))=0.\end{cases}$$

Therefore, for $\alpha\tau\in(-\infty,-\frac{1}{\mathrm{e}})$, for which we always have $\mathsf{u}\neq0$, we have $\mathsf{w}=-\mathsf{u}\cos(\mathsf{u})/\sin(\mathsf{u})$ and

$$\frac{\mathrm{d}\,g(\alpha\,\tau)}{\mathrm{d}\tau}=\frac{1}{|\alpha|\,\tau^2}\frac{\mathsf{u}^2(-\mathsf{u}\frac{\cos(\mathsf{u})}{\sin(\mathsf{u})}+\cos(2\,\mathsf{u}))}{(-\mathsf{u}\,\cos(\mathsf{u})+\sin(\mathsf{u}))^2+\mathsf{u}^2\,\sin^2(\mathsf{u})}. \tag{A.41}$$

Using the L'Hospital's rule [38, Theorem 5.5.2], we can then colcude that

$$\lim_{\tau\to\frac{1}{\mathrm{e}\,|\alpha|}^+}\frac{\mathrm{d}\,g(\alpha\,\tau)}{\mathrm{d}\tau}= \tag{A.42}$$

$$\mathrm{e}^2\,|\alpha|\lim_{\mathsf{u}\to0}\frac{\frac{\mathsf{u}^2}{\sin^2(\mathsf{u})}(-\mathsf{u}\frac{\cos(\mathsf{u})}{\sin(\mathsf{u})}+\cos(2\mathsf{u}))}{(-\mathsf{u}\frac{\cos(\mathsf{u})}{\sin(\mathsf{u})}+1)^2+\mathsf{u}^2}=-\frac{5\,\mathrm{e}^2\,|\alpha|}{3}.$$

Next is an intermediate result that we use in the proof Lemma 4.3 and Lemma 4.5. To establish this result, we rely on the Jordan Curve Theorem, which states that a simple and closed curve divides the plane into an "interior" region bounded by the curve and an "exterior" region containing all of the nearby and far away exterior points [39].

Lemma A.1 (Some of the properties of level set $\mathcal{C}_0$ and superlevel set $\mathcal{S}_0$): *Consider the level set* $\mathcal{C}_0$ (17) *and the superlevel set* $\mathcal{S}_0$ (18). *Let* $\bar{\mathcal{C}}_0 = \mathcal{C}_0 \cup \{(0,0)\}$ *and* $\bar{\mathcal{S}}_0 = \mathcal{S}_0 \cup \{(0,0)\}$. *Then, the following assertions hold.*

(a) $\bar{\mathcal{C}}_0$ *is a simple closed curve in* $\mathbb{R}^2$ *that is symmetric about the* x *axis and intersects the* x *axis at only two points* $\mathsf{x} = 0$ *and* $\mathsf{x} = -\frac{\pi}{2}$. *Moreover, it passes through the origin tangent to the* y *axis.*
(b) $\bar{\mathcal{S}}_0 = \bar{\mathcal{C}}_0 \cup \text{int}(\bar{\mathcal{C}}_0)$, *and is a compact convex subset of* $\mathbb{R}^2$.
(c) $\mathcal{C}_c \subset \text{int}(\bar{\mathcal{C}}_0) \subset \bar{\mathcal{S}}_0$ *for* $c > 0$ *and* $\mathcal{C}_c \subset \text{ext}(\bar{\mathcal{C}}_0)$ *for* $c < 0$.

Proof 9: For a $c \in \mathbb{R}$, by definition, $g(x) = c$ for any $x = (\mathsf{x} + \mathsf{y}\,\mathrm{i}) \in \mathbb{C}_-^l$ is equivalent to $\mathrm{Re}(W_0(\mathsf{x} + \mathsf{y}\mathrm{i})) = c\mathsf{x}$. In other words, $W_0(\mathsf{x} + \mathsf{y}\,\mathrm{i}) = c\,\mathsf{x} + \mathsf{u}\,\mathrm{i}$, where $\mathsf{u} \in (-\pi, \pi)$. Thereby, given property (1e) of the Lambert W_0 function, each level set $\mathcal{C}_c$, $c \in \mathbb{R}$, is symmetric about the real axis. Moreover, from the definition of the Lambert W function we can write

$$(c\mathsf{x} + \mathsf{u}\,\mathrm{i})\,\mathrm{e}^{c\mathsf{x} + \mathsf{u}\mathrm{i}} = \mathsf{x} + \mathsf{y}\mathrm{i} \Leftrightarrow \begin{cases} \mathrm{e}^{c\mathsf{x}}\,(c\mathsf{x}\cos(\mathsf{u}) - \mathsf{u}\sin(\mathsf{u})) = \mathsf{x}, \\ \mathrm{e}^{c\mathsf{x}}\,(\mathsf{u}\cos(\mathsf{u}) + c\mathsf{x}\sin(\mathsf{u})) = \mathsf{y}. \end{cases} \tag{A.43}$$

For $c = 0$, by eliminating u in (A.43) via trigonometric manipulations, we can characterize $\mathcal{C}_0$ by

$$\mathcal{C}_0 = \Big\{(\mathsf{x}, \mathsf{y}) \in \mathbb{R}_{<0} \times \mathbb{R} \,\Big|\, \frac{\mathsf{x}}{\mathsf{y}} = \pm\tan\big(\sqrt{\mathsf{x}^2 + \mathsf{y}^2}\,\big), \quad 0 < (\mathsf{x}^2 + \mathsf{y}^2) < \pi^2\Big\}.$$

In polar coordinates, any $(\mathsf{x}, \mathsf{y}) \in \mathcal{C}_0$, reads as $\mathsf{x} = r\cos(\theta)$ and $\mathsf{y} = r\sin(\theta)$ with

$$\begin{cases} \tan(r) = \pm\cot(\theta), & \theta \in (\frac{\pi}{2}, \frac{3\pi}{2}), \\ 0 < r < \pi, \end{cases}$$

Therefore, $\mathcal{C}_0 = \Big\{(\mathsf{x}, \mathsf{y}) \in \mathbb{R}_{<0} \times \mathbb{R} \,\Big|\, \mathsf{x} = r\cos(\theta), \mathsf{y} = \pm r\sin(\theta),\ r = \theta - \frac{\pi}{2},\ \theta \in (\pi/2, \pi]\Big\}$. Evidently,

$$\bar{\mathcal{C}}_0 = \Big\{(\mathsf{x}, \mathsf{y}) \in \mathbb{R}_{\leq} 0 \times \mathbb{R} \,\Big|\, \mathsf{x} = r\cos(\theta),\ \mathsf{y} = \pm r\sin(\theta), \quad r = \theta - \frac{\pi}{2},\ \theta \in [\pi/2, \pi]\Big\}. \tag{A.45}$$

From (A.45), for any point on the upper half (resp. positive y) and lower half (resp. negative y) of $\bar{\mathcal{C}}_0$, in polar coordinates, r is a continuous and a bounded function of θ and $\frac{dr}{d\theta} = 1$ exists on $\theta \in (\pi/2, \pi)$ (resp. $\theta \in (\pi, 3\pi/2)$). Therefore, $\bar{\mathcal{C}}_0$ is a simple closed curve. Next, note that on $\bar{\mathcal{C}}_0$, as $\theta \to \frac{\pi}{2}^+$, and due to symmetry also as $\theta \to \frac{3\pi}{2}^-$, it follows that $r \to 0$. Therefore, $\bar{\mathcal{C}}_0$ passes through the origin tangent to the y axis. Also, at $\theta = \frac{\pi}{2}$ and $\theta = \pi$ we have, respectively, $(\mathsf{x}, \mathsf{y}) = (0, 0)$ and $(\mathsf{x}, \mathsf{y}) = (-\frac{\pi}{2}, 0)$. As a result, assertion (a) holds.

Since $\bar{\mathcal{C}}_0$ is a simple closed curve, it follows from the Jordan Curve Theorem that $\bar{\mathcal{C}}_0$ divides the plane into an interior region bounded by $\mathcal{C}_0$ and an exterior region containing all of the nearby and far away exterior points. Moreover, note that the curvature of the upper half of $\bar{\mathcal{C}}_0$ in a (x, y) plane is $\kappa = \frac{r^2 + 2r_\theta^2 - r r_{\theta\theta}}{(r^2 + r_\theta^2)^{\frac{3}{2}}} = \frac{r^2 + 2}{(r^2 + 1)^{\frac{3}{2}}} > 0$, where $(.)_\theta = \partial(.)/\partial\theta$ [40]. Therefore the upper half curve of $\mathcal{C}_0$ is a convex curve. Consequently, $\mathcal{Z} = \bar{\mathcal{C}}_0 \cup \text{int}(\bar{\mathcal{C}}_0)$ is a compact convex set (recall that $\mathcal{C}_0$ is symmetric about x axis). To complete the proof of the statement (b), we show that $\bar{\mathcal{S}}_0 = \mathcal{Z}$.

Consider an $\alpha \in \mathbb{C}_-^l$ and recall $\bar{\tau}$ in (15b). Because $(0, 0) \in \text{bd}(\mathcal{Z})$ and $(\mathrm{Re}(\alpha\bar{\tau}), \mathrm{Im}(\alpha\bar{\tau})) \in \text{bd}(\mathcal{Z})$ (recall $g(\alpha\bar{\tau}) = 0$), it follows from the compact convexity of $\mathcal{Z}$ that $(\mathrm{Re}(\alpha\tau), \mathrm{Im}(\alpha\tau)) \in \text{int}(\mathcal{C}_0)$ for $\tau \in (0, \bar{\tau})$ and $(\mathrm{Re}(\alpha\bar{\tau}), \mathrm{Im}(\alpha\bar{\tau})) \in \text{ext}(\mathcal{C}_0)$ for $\tau \in (\bar{\tau}, \infty)$. Therefore, given that $g(\alpha\tau)$ is a continuous function of $\tau \in \mathbb{R}_{\geq 0}$ (see Lemma 4.2) and $g(\alpha\tau)$ at $\tau = 0$ and $\tau = \bar{\tau}$ is equal to, respectively, 1 and 0, we can conclude that $g(\alpha\tau) > 0$ for any $\tau \in (0, \bar{\tau})$. This means that at any $(\mathrm{Re}(\alpha\tau), \mathrm{Im}(\alpha\tau)) \in \text{int}(\mathcal{C}_0)$, we have $g(\alpha\tau) > 0$. Consequently, $\mathcal{Z} = \mathcal{S}_0$, which completes the proof of statement (b).

From validity of the statement (b), we can readily deduce that $\mathcal{C}_c \subset \text{int}(\mathcal{C}_0) \subset \mathcal{S}_0$ for $c > 0$. To complete the proof of the statement (c), we recall from the proof of the statement (b) that for any $\alpha \in \mathbb{C}_-^l$, $(\mathrm{Re}(\alpha\tau), \mathrm{Im}(\alpha\tau)) \in \text{ext}(\mathcal{C}_0)$ for $\tau \in (\bar{\tau}, \infty)$, i.e., $g(\alpha\tau) \neq 0$ for $\tau \in (\bar{\tau}, \infty)$. Then, combined with the fact that $g(\alpha\tau)$ is a continuous function of $\tau \in \mathbb{R}_{\geq 0}$ and also that at $\bar{\tau}$ from (A.38) we have (recall that $\mathrm{Re}(\alpha) < 0$)

$$\frac{\mathrm{d}g(\alpha\bar{\tau})}{\mathrm{d}\tau} = \frac{1}{\mathrm{Re}(\alpha)\bar{\tau}^2}\,\frac{\bar{\mathsf{u}}^2}{1 + \bar{\mathsf{u}}^2} < 0, \tag{A.46}$$

we can conclude that $g(\alpha\tau) < 0$ for $\tau \in (\bar{\tau}, \infty)$. This means that $\mathcal{C}_c \subset \text{ext}(\bar{\mathcal{C}}_0)$ for $c < 0$. To arrive at (A.46), we relied on the knowledge that $g(\alpha\bar{\tau}) = 0$ indicates that $\mathrm{Re}(W_0(\alpha\bar{\tau})) = 0$, therefore $W_0(\alpha\bar{\tau}) = 0 + \bar{\mathsf{u}}\,\mathrm{i}$ for some $\bar{\mathsf{u}} \in (-\pi, \pi)$.

The topological properties of the the $\mathcal{C}_0$ level set, which are established in Lemma A.1 is evident in Fig. (3). Using the results of Lemma A.1, we proceed next to present the proof of Lemma 4.3.

Proof 10 (Proof of Lemma 4.3): *Recall from the statement (b) of Lemma A.1 that* $\mathcal{S}_0$ *is a compact convex set. Therefore, since* $(\mathrm{Re}(\alpha\bar{\tau}), \mathrm{Im}(\alpha\bar{\tau})) \in bd(\bar{\mathcal{S}}_0)$ *(recall* $g(\alpha\bar{\tau}) = 0$*) and* $(0, 0) \in bd(\bar{\mathcal{S}}_0)$*, then* $(\mathrm{Re}(\alpha\tau), \mathrm{Im}(\alpha\tau)) \in \text{int}(\bar{\mathcal{C}}_0)$ *for* $\tau \in (0, \bar{\tau})$ *and* $(\mathrm{Re}(\alpha\bar{\tau}), \mathrm{Im}(\alpha\bar{\tau})) \in \text{ext}(\bar{\mathcal{C}}_0)$ *for* $\tau \in (\bar{\tau}, \infty)$. *Then, the proof follows from the statement (c) of Lemma A.1.*

The proof of Lemma 4.3 can also be deduced from the *continuity stability property* theorem [1, Proposition 3.1] for linear delayed systems. In this regard, consider the dynamical system $\dot{x} = \begin{bmatrix} \mathrm{Re}(\alpha) & \mathrm{Im}(\alpha) \\ -\mathrm{Im}(\alpha) & \mathrm{Re}(\alpha) \end{bmatrix} x(t - \tau)$, whose eigenvalues are α and $\mathrm{conj}(\alpha)$. The real part of the rightmost root of the characteristic equation of this system is given by $\mathrm{Re}(s_\tau^r) = g(\alpha\tau)\,\mathrm{Re}(\alpha)$ (recall (8) and (13)). It follows from the *continuity stability property* theorem [1, Proposition 3.1] and Lemma 3.1 that $\mathrm{Re}(s_\tau^r) \in \mathbb{R}_{<0}$ if and only $\tau \in [0, \bar{\tau})$. Therefore, $g(\alpha\tau) > 0$ if and only if $\tau \in [0, \bar{\tau})$, which along with the fact $g(\alpha\bar{\tau}) = 0$ validates the statement of Lemma 4.3.

Next, we prove Lemma 4.4.

Proof 11 (Proof of Lemma 4.4): *At* $\tau^\star = 1/(\mathrm{e}\,|\alpha|)$, *because* $W_0(-\mathrm{e}^{-1}) = -1$, *we have* $g(\alpha\tau^\star) = \mathrm{Re}(W_0(-\mathrm{e}^{-1}))/(-\mathrm{e}^{-1}) = \mathrm{e}$. *Next, note that from* (15a), (15b) *and Lemma 4.2, we know, respectively, that* $\lim_{\tau\to 0} g(\alpha\tau) = 1$, $\tau = \bar{\tau}$ *is the unique solution of* $g(\alpha\tau) = 0$ *for* $\tau \in \mathbb{R}_{>0}$, *and* $g(\alpha\tau)$ *is a continuous function of* $\tau \in \mathbb{R}_{>0}$. *Therefore, to complete the proof of statement (a), we show next that* $\frac{d\,g(x)}{d\tau} > 0$ *for* $\tau \in (0, \tau^\star)$ *and* $\frac{d\,g(x)}{d\tau} < 0$ *for* $\tau \in (\tau^\star, \bar{\tau})$. *For* $x = \alpha\tau \in [-\frac{1}{\mathrm{e}}, 0]$, *we have* $W_0(x) \in \mathbb{R}$, *and as such by setting* $\mathsf{u} = 0$ *from* (A.38) *we obtain*

$$\frac{d\,g(\alpha\tau)}{d\tau} = \frac{1}{|\alpha|\,\tau^2}\,\frac{\mathsf{w}^2}{(\mathsf{w}+1)} > 0, \qquad \tau \in (0, \tau^\star). \tag{A.47}$$

Moreover, from (A.40) *we have*

$$\lim_{\tau\to\tau^{\star -}} \frac{d\,g(\alpha\,\tau)}{d\tau} = +\infty. \tag{A.48}$$

For any $x = \alpha\tau \in (-\infty, -\frac{1}{\mathrm{e}})$, *recall that* $\frac{d\,g(\alpha\,\tau)}{d\tau}$ *is given by* (A.42). *Because for* $\alpha\tau \in [\frac{-\pi}{2}, -\frac{1}{\mathrm{e}})$ *we have* $\mathsf{u} \in (0, \frac{\pi}{2}]$, *we can confirm that* $(-\mathsf{u}\frac{\cos(\mathsf{u})}{\sin(\mathsf{u})} + \cos(2\,\mathsf{u})) < 0$ *and therefore, we obtain*

$$\frac{d\,g(\alpha\,\tau)}{d\tau} < 0, \qquad \tau \in (\tau^\star, \bar{\tau}]. \tag{A.49}$$

Next, note that from (A.42) *we have*

$$\lim_{\tau\to\tau^{\star +}} \frac{d\,g(\alpha\,\tau)}{d\tau} = -\frac{5\,\mathrm{e}^2\,|\alpha|}{3} < 0. \tag{A.50}$$

In light of the observations above, the proof of the statement (b) follows from (15a), (16), (A.47), (A.49) *and the continuity of* $g(\alpha\tau)$ *for* $\tau \in \mathbb{R}_{>0}$. *Finally, statement (c) is deduced from statements (a) and (b), along with* (A.48) *and* (A.50). *Note that since* $g(\alpha\tau^\star) = \mathrm{e} > 1$, *we have* $\tau^\star \in [0, \tilde{\tau})$.

Next, we use the results of Lemma A.1 to establish the proof of Lemma 4.5, which characterizes the variation of $\frac{d\,g(x)}{d\tau}$ along $x = \alpha\tau$ for $\tau \in \mathbb{R}_{>0}$ such that $(\mathrm{Re}(x), \mathrm{Im}(x)) \in \mathcal{S}_0$.

Proof 12 (Proof of Lemma 4.5): *By virtue of Lemma* (4.3), *we know that* $g(\alpha\tau) > 0$ *for* $\tau \in (0, \bar{\tau})$, *and* $g(\alpha\bar{\tau}) = 0$. *Then, it follows from the definition of delay rate gain and* $\mathrm{Re}(\alpha) < 0$ *that* $\mathrm{Re}(W_0(\alpha\tau)) < 0$ *for* $\tau \in (0, \bar{\tau})$. *Consequently, for* $\tau \in (0, \bar{\tau}]$, *we have* $(\mathrm{Re}(\alpha\tau), \mathrm{Im}(\alpha\tau)) \in \mathcal{S}_0$ *and* $(\mathrm{Re}(W_0(\alpha\tau)), \mathrm{Im}(W_0(\alpha\tau))) \in \mathbb{R}_{\le 0} \times (-\pi, \pi)$.

Next, recall that the derivative of $g(x)$ *with respect to* τ *along* $x = \alpha\tau$, *except at* $x = -\frac{1}{\mathrm{e}}$, *is given by* (A.38), *where we denoted* $W_0(\alpha\tau) = \mathsf{w} + \mathsf{u}\,\mathrm{i}$, *which satisfies* $\mathsf{u} \in (-\pi, \pi)$. *Since* $\mathrm{Re}(\alpha) < 0$, *it is perceived from* (A.38) *that the sign of* $\frac{d\,g(x)}{d\tau}$ *along* $x = \alpha\tau$ *is defined solely by the sign of* $\psi = (\mathsf{w}^2 + \mathsf{u}^2)\,\mathsf{w} + (\mathsf{w}^2 - \mathsf{u}^2)$, *the nominator of* (A.38).

Using the polar coordinates and a set of simple trigonometric manipulations, we identity the points $(\mathsf{w}, \mathsf{u}) \in \mathbb{R}_{\le 0} \times (-\pi, \pi)$ *for which* ψ *retains a zero, a positive and a negative value, as, respectively,* Γ, Γ^+ *and* Γ^-, *where*

$$\Gamma = \Big\{(\mathsf{w}, \mathsf{u}) \in \mathbb{R}_{\le 0} \times (-\pi, \pi) \,\Big|\, \mathsf{w} = \mathsf{R}\cos(\theta),\ \mathsf{u} = \mathsf{R}\sin(\theta),\ 3\pi/4 \le \theta \le 5\pi/4\,,\ \mathsf{R} = -\cos(2\theta)/\cos(\theta)\Big\},$$

$$\Gamma^+ = \Big\{(\mathsf{w}, \mathsf{u}) \in \mathbb{R}_{\le 0} \times (-\pi, \pi) \,\Big|\, \mathsf{w} = \mathsf{R}\cos(\theta),\ \mathsf{u} = \mathsf{R}\sin(\theta),\ 3\pi/4 \le \theta \le 5\pi/4\,,\ \mathsf{R} < -\cos(2\theta)/\cos(\theta)\Big\},$$

$$\Gamma^- = (\mathbb{R}_{\le 0} \times (-\pi, \pi)) \setminus (\Gamma \cup \Gamma^+).$$

Therefore, for $x = \alpha\tau$ *with* $\tau \in (0, \bar{\tau}]$ *we have*

$$\frac{d\,g(x)}{d\tau} > 0 \ \ \textit{if} \ \ (\mathrm{Re}(W_0(x)), \mathrm{Im}(W_0(x))) \in \Gamma^+,$$
$$\frac{d\,g(x)}{d\tau} = 0 \ \ \textit{if} \ \ (\mathrm{Re}(W_0(x)), \mathrm{Im}(W_0(x))) \in (\Gamma\setminus\{(-1, 0)\}),$$
$$\frac{d\,g(x)}{d\tau} < 0 \ \ \textit{if} \ \ (\mathrm{Re}(W_0(x)), \mathrm{Im}(W_0(x))) \in \Gamma^-.$$

Here we used $W_0(-\frac{1}{\mathrm{e}}) = -1$.

Now, let $\alpha\tau = r\,\mathrm{e}^{\phi\,\mathrm{i}}$ *and* $W_0(\alpha\tau) = \mathsf{R}\,\mathrm{e}^{\theta\,\mathrm{i}} = \mathsf{R}\cos(\theta) + \mathsf{R}\sin(\theta)\,\mathrm{i}$, *in which* $-\pi < \mathsf{R}\sin(\theta) < \pi$ *by definition. Then, using the relation* $W_0(\alpha\tau)\,\mathrm{e}^{W_0(\alpha\tau)} = \alpha\tau$ *we obtain that*

$$r = \mathsf{R}\,\mathrm{e}^{\mathsf{R}\cos(\theta)},$$
$$\phi = \theta + \mathsf{R}\,\sin(\theta).$$

Then, recalling the definition of Γ, Γ^+ *and* Γ^-, *the proof of part (a) follows from the observation that for* $x = \alpha\tau$, $\tau \in (0, \bar{\tau}]$ *we have*

$$(\mathrm{Re}(\alpha\tau), \mathrm{Im}(\alpha\tau)) \in \mathrm{int}(\Lambda) \quad \Rightarrow \quad (\mathsf{w}, \mathsf{u}) \in \Gamma^+,$$
$$(\mathrm{Re}(\alpha\tau), \mathrm{Im}(\alpha\tau)) \in \Lambda \quad \Rightarrow \quad (\mathsf{w}, \mathsf{u}) \in \Gamma\ ,$$
$$(\mathrm{Re}(\alpha\tau), \mathrm{Im}(\alpha\tau)) \in (\mathcal{S}_0\setminus(\mathrm{int}(\Lambda) \cup \Lambda)) \quad \Rightarrow \quad (\mathsf{w}, \mathsf{u}) \in \Gamma^-,$$

where $(\mathsf{w}, \mathsf{u}) = (\mathrm{Re}(W_0(\alpha\tau)), \mathrm{Im}(W_0(\alpha\tau)))$. *Recall here that* W_0 *function is injective.*

To prove statement (b) and (c) we proceed as follows. We note that in (19) r *and* ϕ *both are a differentiable and a bounded function of* $\theta \in [\frac{3\pi}{4}, \frac{5\pi}{4}]$. *Moreover,*

$$\frac{d\,\phi}{d\,\theta} = 1 + 2\tan(\theta)\,\mathrm{e}^{-\cos 2\theta}\sin(2\theta)\,(1 - \cos(2\theta)) - \cos(2\theta)\,\mathrm{e}^{-\cos(2\theta)}\,(1 + \tan^2(\theta)) > 0, \tag{A.51}$$

for any $\theta \in [\frac{3\pi}{4}, \frac{5\pi}{4}]$. *Therefore,* ϕ *in* (19) *has a one-to-one correspondence with* $\theta \in [\frac{3\pi}{4}, \frac{5\pi}{4}]$, *which in turn indicates that* r *in* (19) *has a one-to-one correspondence with* ϕ. *Moreover, since* $\phi(\frac{3\pi}{4}) = \frac{3\pi}{4}$ *and* $\phi(\frac{5\pi}{4}) = \frac{5\pi}{4}$, *we have* $\phi \in [\frac{3\pi}{4}, \frac{5\pi}{4}]$. *In addition, we can also conclude that* $\frac{dr}{d\phi} = \frac{dr}{d\theta}/\frac{d\phi}{d\theta}$ *exists and is finite at every* $\theta \in [\frac{3\pi}{4}, \frac{5\pi}{4}]$. *Combined with* r *satisfying* $r(\frac{3\pi}{4}) = r(\frac{5\pi}{4}) = 0$, *we can then conclude that* Λ *is a simple closed curve. Therefore, it follows from the Jordan Curve theorem that* $\mathrm{int}(\Lambda) \cup \Lambda$ *is a connected compact subset of* $\mathbb{R}^2$. *In light of the preceding observations, we make the following conclusions. The ray* $(\mathrm{Re}(\alpha\tau), \mathrm{Im}(\alpha\tau))$, $\tau \in (0, \bar{\tau}]$, *intersects* Λ *if and only if* $\frac{3\pi}{4} <$ *arg*$(\alpha) < \frac{5\pi}{4}$. *Therefore, if arg*$(\alpha) \notin (\frac{3\pi}{4}, \frac{5\pi}{4})$, *we have* $(\mathrm{Re}(\alpha\tau), \mathrm{Im}(\alpha\tau)) \in (\mathcal{S}_0\setminus(\mathrm{int}(\Lambda) \cup \Lambda))$ *for* $\tau \in (0, \bar{\tau}])$. *Then, the proof of the statement (b) follows from the statement (a). On the other hand, if arg*$(\alpha) \in (\frac{3\pi}{4}, \frac{5\pi}{4})$, *then due to the one-to-one correspondence between* r *and* ϕ, *ray* $(\mathrm{Re}(\alpha\tau), \mathrm{Im}(\alpha\tau))$, $\tau \in (0, \bar{\tau}]$ *intersects* Λ *at a unique point. Let this point correspond to* $\tau^\star \in (0, \bar{\tau}]$, *i.e.,* $(\mathrm{Re}(\alpha\tau^\star), \mathrm{Im}(\alpha\tau^\star)) \in \Lambda$. *Then,* (20) *is deduced from the definition of* Λ *in* (19). *Next, note that from compactness of* $\mathrm{int}(\Lambda) \cup \Lambda)$ *and the fact that* $(\mathrm{Re}(\alpha\tau), \mathrm{Im}(\alpha\tau))$, $\tau \in (0, \bar{\tau}]$, *intersects* Λ *at a unique point,* *it follows that* $(\mathrm{Re}(\alpha\tau), \mathrm{Im}(\alpha\tau)) \in \mathrm{int}(\Lambda)$ *for* $\tau \in (0, \tau^\star)$ *and* $(\mathrm{Re}(\alpha\tau), \mathrm{Im}(\alpha\tau)) \in (\mathcal{S}_0 \backslash (\mathrm{int}(\Lambda) \cup \Lambda))$. *Thereby, by virtue of the statement (a) we conclude that* $\frac{dg(\alpha\tau)}{d\tau} > 0$ *for* $\tau \in (0, \tau^\star)$, *and* $\frac{dg(\alpha\tau)}{d\tau} < 0$ $\tau \in (\tau^\star, \bar{\tau}]$. *If* $\alpha \in \mathbb{R}_{<0}$, *then* $\arg(\alpha) = \pi$. *Therefore,* $\tau^\star$ *in* (20) *becomes equal to* $\frac{1}{\mathrm{e}\,|\alpha|}$, *and consequently,* (22) *follows from* (A.40) *and* (A.42). *Lastly, when* $\alpha \notin \mathbb{R}_{<0}$, *since* $(\mathrm{Re}(\alpha\tau^\star), \mathrm{Im}(\alpha\tau^\star)) \in \Lambda \backslash \{(-\frac{1}{\mathrm{e}}, 0)\}$, $\frac{dg(\alpha\tau)}{d\tau} = 0$ *at* $\tau = \tau^\star$ *is deduced from the statement (a).*

Figure 5 depicts Λ in (19) (red curve) in a (x, y) plane along with the level sets $\mathcal{C}_1$ (green curve) and $\mathcal{C}_0$ (blue curve). As one can expect (given (15), Lemma 4.2 and the statement (a) of Lemma 4.5), Λ is located inside $\mathcal{C}_1$ and between the lines $\mathsf{y} = \pm\mathsf{x}$. It is interesting to note that lines $\mathsf{y} = \pm\mathsf{x}$ are also tangent to $\mathcal{C}_1$ at the origin. This observation can be verified as follows. Consider $(\mathsf{x}_1, \mathsf{y}_1) \in \mathcal{C}_1$ and let $\mathsf{x}_1 + \mathsf{y}_1\,\mathrm{i} = \mathsf{r}\,\mathrm{e}^{\mathrm{i}\theta}$. Then, for $(\mathsf{x}_1, \mathsf{y}_1)$ in the close neighborhood of the origin from (15a)) and (4) we expect that $\lim_{r\to 0} -\cos(2\,\theta) + \frac{3}{2}\mathsf{r}\cos(3\,\theta) - \frac{8}{3}\mathsf{r}^2\cos(4\,\theta) + \cdots = 0$. This limit is possible only if $\theta \to \frac{3\pi}{4}$ and $\theta \to \frac{5\pi}{4}$ (solution of $\cos(2\,\theta) = 0$ for $\theta \in [\pi/2, 3\pi/2]$). This verifies that as $(\mathsf{x}_1 + \mathsf{y}_1\,\mathrm{i}) \to 0$ on $\mathcal{C}_1$, $\mathcal{C}_1$ become tangent to the lines $\mathsf{y} = \pm\mathsf{x}$.