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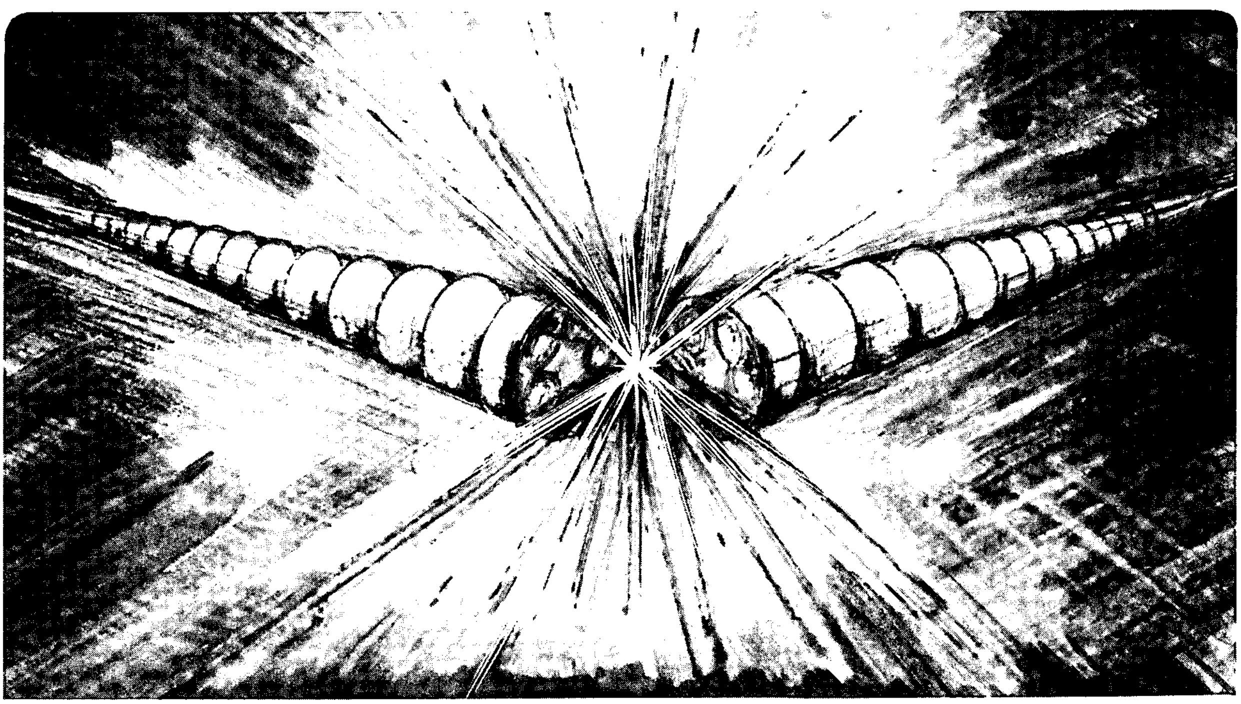

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STABILITY OF THE EQUILIBRIUM HELICAL ORBITS IN FREE-ELECTRON LASERS

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We present a proof of the stability of the equilibrium helical orbits in a free-electron laser with a helical wiggler field and a combined axial guide magnetic field directed in the the conventional and reversed directions.

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1. Background

The stability problem of equilibrium orbits in free-electron lasers (FELs) has aroused extensive interest [1-6]. Because the euqations governing motion of an electron in a combined axial guide magnetic field and a helical wiggler field are nonlinear, and the general solutions are not easy to obtain analytically, Liapunov's first method in ordinary differential equation theory [7], often called linear perturbation analysis, has previously been used to study the stability problem [1,5, 6]. The unstable orbits were established without initial energy perturbation taken into account. Due to breakdown of the first method, however, the "stable orbits" were affirmed without any proof and has caused some confusion in the FEL community [8]. In a recent FEL experiment with a reversed axial guide magnetic field [9], some unique phenomena were observed, which has rekindled concern in equilibrium orbits [10]. Therefore, strictly solving the stability problem has important significance both in theory and in practice. In this work, we present a proof of the stability of equilibrium helical orbits in a FEL with an ideal helical wiggler field and a uniform axial guide magnetic field in the conventional and reversed directions.

2. Stability criterion

We begin by reminding the reader of stability criterion. Liapunov's first method [7] states that the solution $\boldsymbol{y}=\boldsymbol{0}$ to the autonomous system $(d/dt)\boldsymbol{y}=\boldsymbol{f}(\boldsymbol{y})$ is unstable if there is at least one characteristic root with a positive real part of the linear, first, approximation of the system. It should be emphasized that this is a sufficient condition but not a necessary one. In other words, if there is no root which has a positive real part , we can not affirm that the solution is stable. In such a case, the solution may be unstable, stable, or asymptotically stable, depending upon high-order nonlinear terms. In short, the first method breaks down.

Liapunov's second method [11] states that if there is a function, $V(\mathbf{y})$, which satisfies $V(\mathbf{0})=0$ and $V(\mathbf{y}\neq\mathbf{0})>0$ (positive definite), then $\mathbf{y}=\mathbf{0}$ is unstable when $(d/dt)V(\mathbf{y})$ is positive definite, stable when $(d/dt)V(\mathbf{y})$ is negative semidefinite (≤ 0), (at this time $V(\mathbf{y})$ is the Liapunov function), and asymptotically stable when $(d/dt)V(\mathbf{y})$ is negative definite ($=0$ at $\mathbf{y}=\mathbf{0}$, and <0 at $\mathbf{y}\neq\mathbf{0}$). Here is an example for which the Liapunov's first method breaks down but the second method works.

Suppose we have the nonlinear system of equations

$$\frac{dx}{dt}=-y+ax^3-yz, \tag{1}$$

$$\frac{dy}{dt}=x+ay^3-xz, \tag{2}$$

$$\frac{dz}{dt}=az^3+2xy, \tag{3}$$

with the null solution $x=0$, $y=0$, and $z=0$. The first approximation of the system is given by

$$\frac{dx}{dt}=-y, \tag{4}$$

$$\frac{dy}{dt}=x, \tag{5}$$

$$\frac{dz}{dt}=0, \tag{6}$$

with three characteristic roots: 0, $+i$, and $-i$. According to the first method, there is nothing we can say because no root has a positive real part. Then we turn to the second method. Taking $V(x,y,z)=0.5(x^2+y^2+z^2)$ as a function of the system, we find that $(d/dt)V(x,y,z)=a(x^4+y^4+z^4)$, and $V(x,y,z)$ is a Liapunov function when $a\leq 0$. From this it follows that the null solution is unstable if $a>0$, stable if $a=0$, or asymptotically stable if $a<0$. The purpose of constructing the above example is to remind the reader of the fact that if there are two pure imaginary and one zero characteristic roots for a nonlinear system, which corresponds to the problem we will discuss below in FELs, we can not determine the stability of the null solution according to Liapunov's first method.

3. Nonlinear system of equations in the wiggler and axial guide fields

By use of the helical basis vectors $\hat{\mathbf{e}}_1=\hat{\mathbf{x}}\cos(k_w z)+\hat{\mathbf{y}}\sin(k_w z)$, $\hat{\mathbf{e}}_2=-\hat{\mathbf{x}}\sin(k_w z)+\hat{\mathbf{y}}\cos(k_w z)$, and $\hat{\mathbf{e}}_3=\hat{\mathbf{z}}$, the ideal helical wiggler field and combined axial guide magnetic field in FELs can be written as

$$\mathbf{B}=B_w\hat{\mathbf{e}}_1 \pm B_0\hat{\mathbf{e}}_3, \tag{7}$$

where $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ are the unit vectors in the x, y, and z-directions, k_w is the wiggler wave number, B_w is the wiggler amplitude, and B_0 is the constant guide field. In Eq. (7), $+B_0$ corresponds to the positive guide field FEL and $-B_0$ correspondes to the reversed guide field FEL. The nonlinear system of equations describing motion of an electron in the field is given by [1]

$$\dot{\beta}_1=\left(\beta_3-\frac{\bar{\Omega}_{0\pm}}{\gamma}\right)\beta_2, \tag{8}$$

$$\dot{\beta}_2=-\left(\beta_3-\frac{\bar{\Omega}_{0\pm}}{\gamma}\right)\beta_1-\frac{\bar{\Omega}_w}{\gamma}\beta_3, \tag{9}$$

$$\dot{\beta}_3=\frac{\bar{\Omega}_w}{\gamma}\beta_2, \tag{10}$$

where β_1, β_2, and β_3 are, respectively, the velocity components normalized to the light speed c in free space on the helical basis vectors, $\gamma=(1-\beta_1^2-\beta_2^2-\beta_3^2)^{-1/2}$ is the relativistic factor, $\bar{\Omega}_{0\pm}=\pm|e|B_0/(m_0ck_w)$, and $\bar{\Omega}_w=|e|B_w/(m_0ck_w)$ with e the electron charge and m_0 the electron rest mass. In Eqs. (8)-(10), $\dot{\chi}\equiv d\chi/d\tau$ with $\tau=k_wct$.

Suppose that $(\beta_{10},0,\beta_{30})$ is the helical orbit solution to Eqs.(8)-(10), where β_{10} and β_{30} are constants determined by initial conditions. By setting $\beta_1=\beta_{10}+\xi$, $\beta_2=\zeta$, and $\beta_3=\beta_{30}+\eta$, where (ξ,ζ,η) stands for the perturbation solution caused by perturbation of the initial conditions, the stability problem of the helical orbit of Eqs. (8)-(10) is turned into the one of the null solution of the following nonlinear system

$$\dot{\xi}=(\eta+\alpha_{\pm})\zeta, \quad (11)$$

$$\dot{\zeta}=-(\eta+\alpha_{\pm})\xi+\frac{\overline{\Omega}_w}{\gamma_0}\left(\frac{\beta_{30}}{\alpha_{0\pm}}-\frac{\gamma_0}{\gamma}\right)\eta+\frac{\overline{\Omega}_w}{\gamma_0\alpha_{0\pm}}\beta_{30}^2\left(1-\frac{\gamma_0}{\gamma}\right), \quad (12)$$

$$\dot{\eta}=\frac{\overline{\Omega}_w}{\gamma}\zeta, \quad (13)$$

where $\alpha_{\pm}=\beta_{30}-\overline{\Omega}_{0\pm}/\gamma$, $\alpha_{0\pm}=\beta_{30}-\overline{\Omega}_{0\pm}/\gamma_0$, and $\gamma_0=\gamma(\xi,\zeta,\eta=0)=(1-\beta_{10}^2-\beta_{30}^2)^{-1/2}$ is the relativistic factor of the electron moving on the helical orbit. In obtaining Eqs. (11)-(13), the helical-orbit relation $\alpha_{0\pm}\beta_{10}+\overline{\Omega}_w\beta_{30}/\gamma_0=0$ was used.

4. Stability of the equilibrium helical orbits in FELs

Just as we mentioned above, the stability problem of the helical orbits in FELs is equivalent to the one of the null solution of the system of Eqs. (11)-(13). Below we will use Liapunov's first and second methods to determine the stability.

First we turn to Liapunov's first method. The first approximation of the system of Eqs. (11)-(13) is given by

$$\dot{\xi}=\alpha_{0\pm}\zeta, \quad (14)$$

$$\dot{\zeta}=-\left[\alpha_{0\pm}+\left(\frac{\overline{\Omega}_w}{\alpha_{0\pm}}\right)^2\beta_{30}^3\right]\xi+\frac{\overline{\Omega}_w}{\gamma_0\alpha_{0\pm}}\left(\frac{\overline{\Omega}_{0\pm}}{\gamma_0}+\gamma_0^2\beta_{30}^3\right)\eta, \quad (15)$$

$$\dot{\eta}=\frac{\overline{\Omega}_w}{\gamma_0}\zeta. \quad (16)$$

The first approximation has three characteristic roots: $\sigma_1=0$, $\sigma_2=+\sqrt{D_{\pm}}$, and $\sigma_3=-\sqrt{D_{\pm}}$, where

$$D_{\pm}=\frac{\overline{\Omega}_w^2\,\overline{\Omega}_{0\pm}}{\gamma_0^3\alpha_{0\pm}}-\alpha_{0\pm}^2. \quad (17)$$

Here the signs $\pm$ correspond to the conventional and reversed guide field FELs respectively. Obviously, the helical orbit is unstable when $D_{\pm}>0$. For the reversed guide field FEL, however, $D_{-}<0$ because $\bar{\Omega}_{0-}\leq 0$ and $\alpha_{0-}>0$. The situation is the same as the example we took in Sec. 2. From this, we can see that Liapunove's first method can not be used to determine the stability for the reversed guide field FEL.

For the conventional guide field FEL where $\bar{\Omega}_{0+}\geq 0$, from $D_{+}>0$ and $\alpha_{0+}>0$ we obtain a regime in which the helical orbit is unstable. The regime is described by

$$\bar{\Omega}_{0+} + \left(\bar{\Omega}_{0+}\bar{\Omega}_{w}^{2}\right)^{\frac{1}{3}} > \gamma_0\beta_{30} > \bar{\Omega}_{0+} > 0. \tag{18}$$

The ustable regime only occurs when the axial guide field is non-zero.

Now we turn to Liapunov's second method to further determine the stability. Let us examine the following function

$$V(\xi,\zeta,\eta)=[F(\xi,\zeta,\eta)]^2 + [G(\xi,\zeta,\eta)]^2 + [H(\xi,\zeta,\eta)]^2, \tag{19}$$

where

$$F(\xi,\zeta,\eta)=\frac{\beta_{30}^{2}}{\alpha_{0\pm}}\left(\frac{\gamma}{\gamma_0}-1\right), \tag{20}$$

$$G(\xi,\zeta,\eta)=\frac{1}{2}\left[\xi^2+\zeta^2-\left(\frac{\gamma\beta_{30}}{\gamma_0\alpha_{0\pm}}-1\right)\eta^2\right]-\frac{\beta_{30}^{2}}{\alpha_{0\pm}}\left(\frac{\gamma}{\gamma_0}-1\right)\eta, \tag{21}$$

and

$$H(\xi,\zeta,\eta)=\frac{1}{2}\eta^2+\alpha_{\pm}\eta-\frac{\bar{\Omega}_w}{\gamma}\xi. \tag{22}$$

If we can show that $V(\xi,\zeta,\eta)$ satisfies two conditions: (1) $V(\xi,\zeta,\eta)$ is positive definite, and (2) $\dot{V}(\xi,\zeta,\eta)$ is negative semidefinite, then $V(\xi,\zeta,\eta)$ is a Liapunov function. Further, we can affirm that the helical orbit is stable.

Using Eqs. (11)-(13), we obtain $\dot{V}(\xi,\zeta,\eta)=0$, i. e., negative semidefinite, so that $V(\xi,\zeta,\eta)$ meets the second condition of Liapunov function. Obviously, from Eq. (19) we see that $V(\xi,\zeta,\eta)\geq 0$ for any value of ξ, ζ, and η, and $V(\xi,\zeta,\eta)=0$ when $\xi=0$, $\zeta=0$, and $\eta=0$. The next thing we should do is to determine the condition under which the equation $V(\xi,\zeta,\eta)=0$ has no root except the one: $\xi=0$, $\zeta=0$, and $\eta=0$. To this end, we define

$$DE_{\pm}(\xi,\zeta,\eta)=\frac{\overline{\Omega}_{0\pm}}{\gamma_0\alpha_{0\pm}}-\frac{\gamma_0^2\left(\frac{1}{2}\eta+\alpha_{0\pm}\right)^2}{\overline{\Omega}_w^2}. \tag{23}$$

If $V(\xi,\zeta,\eta)=0$, then $F(\xi,\zeta,\eta)=0$, $G(\xi,\zeta,\eta)=0$, and $H(\xi,\zeta,\eta)=0$ must hold. From Eqs. (20)-(22), we obtain

$$\gamma-\gamma_0=0, \tag{24}$$

$$\frac{1}{2}\left[\xi^2+\zeta^2-\left(\frac{\gamma\beta_{30}}{\gamma_0\alpha_{0\pm}}-1\right)\eta^2\right]-\frac{\beta_{30}^2}{\alpha_{0\pm}}\left(\frac{\gamma}{\gamma_0}-1\right)\eta=0, \tag{25}$$

$$\frac{1}{2}\eta^2+\alpha_{\pm}\eta-\frac{\overline{\Omega}_w}{\gamma}\xi=0. \tag{26}$$

Substituting Eqs. (24) and (26) into Eq. (25) to eliminate ξ, we obtain

$$\zeta^2=\eta^2 DE_{\pm}(\xi,\zeta,\eta). \tag{27}$$

Eq. (27) is the necessary condition under which Eqs. (24)-(26) hold. Clearly, if $DE_{\pm}(\xi,\zeta,\eta)<0$, Eq. (27) holds only when $\xi=0$, $\zeta=0$, and $\eta=0$, that is, $V(\xi,\zeta,\eta)=0$ does not have any other root. Thus we come to the conclusion: if $DE_{\pm}(\xi,\zeta,\eta)<0$, then $V(\xi,\zeta,\eta)$ is positive definite and it is a Liapunov function for the nonlinear system of Eqs. (11)-(13).

For the reversed guide field FEL where $\overline{\Omega}_{0-}\leq 0$ and $\alpha_{0-}>0$, we obtain a sufficient condition that $DE_{-}(\xi,\zeta,\eta)<0$ holds in the neighbourhood $(\xi^2+\zeta^2+\eta^2)^{1/2}<2\beta_{30}$ when $\overline{\Omega}_{0-}=0$, and in the one $(\xi^2+\zeta^2+\eta^2)^{1/2}<+\infty$ when $\overline{\Omega}_{0-}<0$. Hence the equilibrium helical orbit is always stable.

For the conventional guide field FEL where $\overline{\Omega}_{0+}\geq 0$, we have to classify two cases by $\alpha_{0+}>0$ and $\alpha_{0+}<0$. For $\alpha_{0+}>0$ ($\gamma_0\beta_{30}>\overline{\Omega}_{0+}$), we find a sufficient condition that $DE_{+}(\xi,\zeta,\eta)<0$ holds in the neighbourhood

$$\sqrt{\xi^2+\zeta^2+\eta^2}<\frac{2}{\gamma_0}\left[\left(\gamma_0\beta_{30}-\overline{\Omega}_{0+}\right)-\sqrt{\frac{\overline{\Omega}_{0+}\overline{\Omega}_w^2}{\left(\gamma_0\beta_{30}-\overline{\Omega}_{0+}\right)}}\right] \tag{28}$$

when $D_+<0$ given by Eq. (17). From $\alpha_{0+}>0$ and $D_+<0$, we get the stable low-guide field regime

$$\gamma_0\beta_{30} > \overline{\Omega}_{0+} + \left(\overline{\Omega}_{0+}\overline{\Omega}_w^2\right)^{\frac{1}{3}} \geq 0. \tag{29}$$

For $\alpha_{0+}<0$ ($\gamma_0\beta_{30} < \overline{\Omega}_{0+}$ and $\overline{\Omega}_{0+}>0$, high-guide field regime), we also obtain a sufficient condintion that $DE_+(\xi,\zeta,\eta)<0$ holds in the neighbourhood $(\xi^2+\zeta^2+\eta^2)^{1/2}<+\infty$. Hence, the equilibrium helical orbits are all stable in the low- and high-guide field regimes for the conventional guide field FEL. From Eq. (28), however, we find that the low-guide field regime has a smaller stable neighbourhood and the neighbourhood decreases with increasing axial guide field. That means that for a FEL near resonance in the low-guide field regime, large signal interaction may make electron helical orbits unstable and result in decreasing efficiency.

5. Conclusions

We have strictly studied the stability of equilibrium helical orbits in the conventional and reversed axial guide field FELs. For the reversed guide field FEL, helical orbits are always stable and the stable neighbourhood is infinite. For the conventional guide field FEL, the stable high-guide field regime also has a infinite stable neighbourhood, but the stable low-guide field regime has a smaller stable neighbourhood and the neighbourhood decreases with increase of the axial guide field.

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