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Supporting Generative Thinking about Number Lines, the Cartesian Plane, and Graphs of Linear Functions

by

Darrell Steven Earnest

A dissertation submitted in partial satisfaction of the

requirements for the degree of

Doctor of Philosophy

in

Education

in the

Graduate Division

of the

University of California, Berkeley

Committee in charge:

Professor Geoffrey B. Saxe, chair
Professor Alan Schoenfeld
Professor George Lakoff

Fall 2012

Supporting Generative Thinking about Number Lines, the Cartesian Plane, and Graphs of Linear Functions

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Darrell Steven Earnest

Abstract

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By

Darrell Steven Earnest

Doctor of Philosophy in Education

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Professor Geoffrey Saxe, Chair

This dissertation explores fifth and eighth grade students' interpretations of three kinds of mathematical representations: number lines, the Cartesian plane, and graphs of linear functions. Two studies were conducted.

In Study 1, I administered the paper-and-pencil Linear Representations Assessment (LRA) to examine students' understanding of the three representations. The LRA had an experimental component that compared performance on routine problems to non-routine problems (problems not amenable to routine solution procedures). I administered the assessment to Grade 5 students (n=126) who had no formal instruction involving function graphs, and I compared their performances with those of Grade 8 students (n=131) enrolled in Algebra 1. A repeated measures ANOVA revealed students in each grade performed better on routine problems compared to non-routine problems, suggesting that routine problems may falsely indicate greater competence. Paired samples t-tests indicated no differences in performance between Grades 5 and 8 students on number line items, though Grade 8 students outperformed fifth graders on Cartesian plane and function graph items. Videotaped interviews with a subset of Grades 5 and 8 students revealed that students in each grade approached tasks across representations in similar ways, suggesting persisting misconceptions. Interviews also revealed patterns unique to each grade.

In Study 2, I examined the efficacy of a tutorial intervention. The intervention introduced written definitions to support principled understandings of the number line, the Cartesian plane, and function graphs. A repeated measures ANOVA that compared pre/posttest scores of Grade 5 students (n=20) to a matched control group (n=20) revealed significant gains from pre- to posttest in the experimental group, with no detectable gains in a control. At posttest, Grade 5 tutorial students performed significantly better on non-routine LRA problems than Grade 8 students who did not receive the tutorial. Video analysis revealed a correlation between tutorial students' appropriate uptake of definitions and gains from pretest to posttest.

Analyses across the two studies indicate that instruction that supports students' coordination of linear and numerical units can support students' learning with understanding. Potential applications include the development of curricula to support

students' learning with understanding related to these representations and teacher professional development interventions.

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My deepest gratitude goes to the subjects of this work. The most fun part of this project was when—once the design was set and the video camera was turned on, zoomed in, and out of the way—I was finally able to chat one-on-one with a 5th or 8th grader about some math. These students taught me much more than I could teach them. I sincerely thank each and every one of them.

Chapter 1: Introduction

Researchers and mathematicians have highlighted the promise of linear representations of quantities to support elementary students' understanding of the number system (Bass, 1998; Saxe et al., 2009, 2010, 2011; Wu, 2005, 2009). Moreover, using the Cartesian plane, a powerful representational context for reasoning about simple mathematical functions (Carraher & Schliemann, 2007; Kaput, Carraher & Blanton, 2008), involves the coordination of two linear dimensions. Yet, little in the way of systematic research on students' reasoning involving these representations or on instruction to support generative understandings exists. In this dissertation, I report a two-part design research project on Grades 5 and 8 students' reasoning in the context of linear representations of quantities with a focus on number lines, the Cartesian plane, and Cartesian graphs of simple mathematical functions. In the first part, I report a study on Grades 5 and 8 students' understanding of linear representations through an analysis of their performances on a linear representations assessment as well as follow-up interviews on targeted assessment items. In the second part, I report on the design of a learning environment to support fifth graders' generative thinking of linear representations, followed by an analysis of the learning trajectories supported by the designed environment.

One Student's Reasoning on a Linear Representation Task

How might a young student conceptually coordinate the linear features of a representation such as a Cartesian plane? Consider the following excerpt from a clinical interview conducted for this study. I provided Curtis, a Grade 5 student at an urban public charter school, a diagram of the Cartesian plane with axes labeled with the numerical units 0 and 2, and asked him to identify the point at the coordinates (5, 5) (see Figure 1, left side). Because two values were provided on each axis, the exact position of every numerical unit along either axis was determined, as was the name of any ordered pair in the plane. Curtis named the point, (4.5, 4.5). In a follow-up interview, Curtis explained his thinking. Addressing the "4" element of the "4.5," Curtis pointed at the horizontal axis to the numerical units "0" and "2" and said, "Because as it says here, it goes to 4. It's going 2, 4." His justification indicated attention to linear measure. Explaining the ".5," again he gestured along the horizontal axis, this time at the midpoint of the linear unit between "4" and the nearest tickmark to the right (see the right side of Figure 1). He said, "the 'point five' comes because there's a little space in here," as he used his pen to inscribe "4.5" below the midpoint.

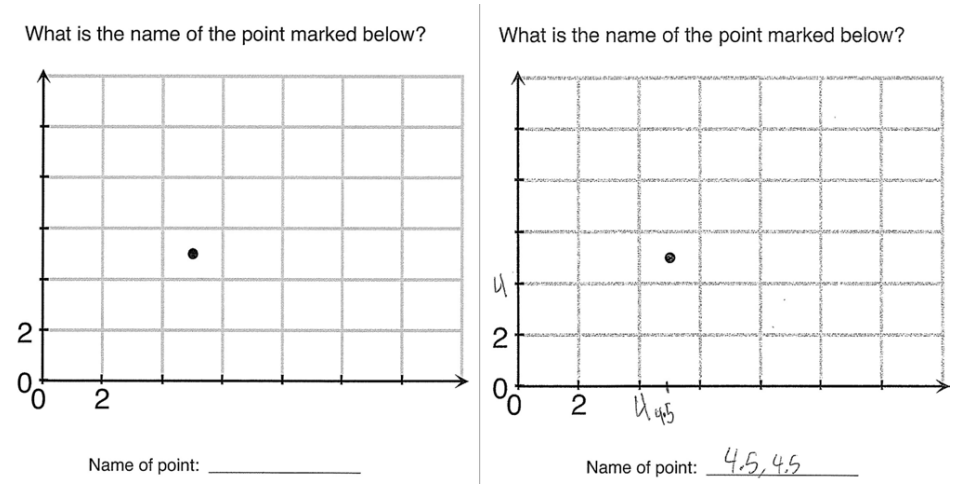


Figure 1: The original problem (left) and Curtis's solution (right)

I presented the correct response to Curtis as a countersuggestion: "Some students said that the point is called (5, 5); what do you think of that?" Curtis exclaimed, "Oh god, you've got to be kidding me!" quickly followed by, "Well, actually, it does have a pretty good point!" Curtis then seemed willing to accept that both (4.5, 4.5) and (5, 5) could be ordered pairs for that point—an impossibility with a Cartesian plane. He stated that it would depend on whether "you're *completely* [emphasis his] doing the 2s, or if you're doing the 2s [and] the 1s" (an interpretation of his meaning is simulated in Figure 2). He clarified that (4.5, 4.5) could be the name "if you're only counting by 2s and you're pretending the 1s don't exist at all." His justification did not coordinate the linear distance of 2 with other linear units along the axis.

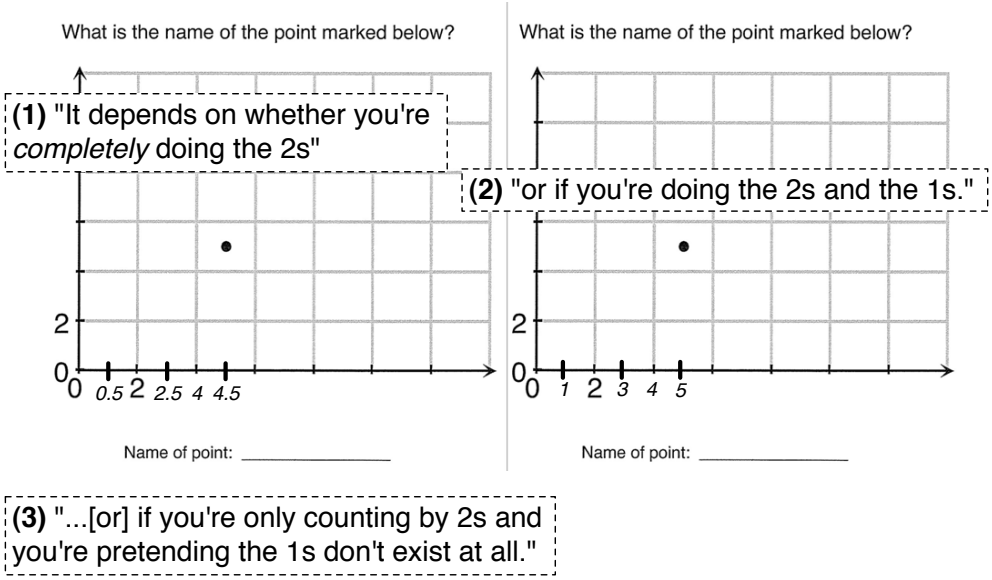
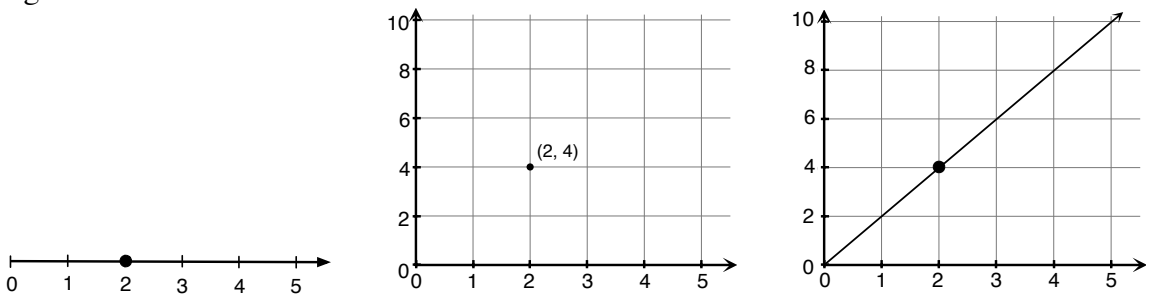


Figure 2: Curtis's annotated response for (4.5, 4.5) (left) and (5, 5) (right)

What are the varied ways that students like Curtis coordinate representational features of the Cartesian grid to determine the name of a point? A first purpose of this dissertation involves documenting ways that students solve problems like the one used with Curtis. In this case, Curtis imposed some structure onto the representation. He named the ordered pair using projections of particular values from each axis. In extending values along each axis, he treated numbers as increasing in value from left to right and from bottom to top on number lines. He

iterated the interval of 2 in order to identify the location of 4, thereby adhering to the provided metric. Nonetheless, he inaccurately named the point. Along the axes, Curtis treated intervals of 2 as unit intervals, thereby resulting in an incorrect name, (4.5, 4.5). One may imagine that as Curtis progresses in school and mathematical functions are introduced on the plane, he may interpret points along that function line in similar ways, though existing research has not yet explored this.

How might instruction better support Grade 5 students like Curtis? A second purpose of this dissertation is to explore an instructional approach to support students' potentially rich and generative understandings. A premise of the instructional design is that developing representational fluency with number lines and the Cartesian plane are important resources for students to make sense of graphs of algebraic functions, such as the progression displayed in Figure 3.



The point 2 on a number line. The point (2, 4) on a Cartesian plane. The function f(*x*)=2x, which passes through (2, 4)

Figure 3: Progression of linear representations of quantities

Linear Representations of Quantities in Mathematics Education

I consider here students' coordination of *numerical units* and *linear units* as they reason about linear representations. Numerical and linear units are two independent systems that must be conceptually coordinated to make meaning of linear representations of quantities (e.g., Carraher, Schliemann, Brizuela & Earnest, 2006; Gravemeijer & Stephan, 2002; Treffers, 1991; Saxe et al., 2009, 2010, in press; Schliemann, Carraher, & Caddle, 2008;). I define numerical units as units of discrete quantity, and they are a focus of instruction beginning in preschool and early primary grades. I define linear units as the congruent segments that are concatenated to constitute a single number line or two perpendicular number lines. These two representational systems may function independently. Numerical units and counting principles may be learned without any linear context, and at the same time, linear units may be analyzed and utilized in contexts without reference to numerical units. Linear representations of quantities require a coordination of these two different types of units.

In this section, I consider the ways a coordination of numerical and linear units present challenges to students as they make sense of each kind of linear representation that I target in my dissertation: number lines, the Cartesian plane, and function graphs. I then consider a potentially useful learning progression for students from number lines to the Cartesian plane, and finally to the introduction of functions on the plane.

Number lines. The number line, a foundational idea in the learning progression that I propose, embodies a coordination of linear units and numerical units. Despite the ubiquity of number lines and their demonstrated utility in elementary mathematics (Bass, 1998; Carraher & Schliemann, 2007; Carraher et al., 2006; Corwin, Russell, & Tierney, 1990; Kaput, 2008; Moses & Cobb, 2001; Saxe, Shaughnessy, Shannon, Langer-Osuna, Chinn & Gearhart, 2007; Wu, 2009), children often do not conceptualize the number line in terms of this coordination. Equal partitioning with consecutive integers inscribed below successive tickmarks is the canonical number line representation (see Figure 4a) and the staple of instruction. Without an understanding of number lines as a coordination of linear and numerical units, children treat as valid number lines that privilege one unit and ignore the other (Saxe et al., 2007; Saxe, Shaughnessy, Gearhart & Haldar, in press). In doing so, points along the line no longer represent magnitudes from zero. For example, children that privilege numerical units independent of linear units may extend consecutive numerical units when linear units are not congruent. In the case of Figure 4b, the result is a number line with intervals of 1 represented as linear units of differing lengths. Alternatively, children may consider a number line with congruent linear units as correct, even if numerical units indicate those congruent linear units represent intervals of differing lengths, as in Figure 4c.

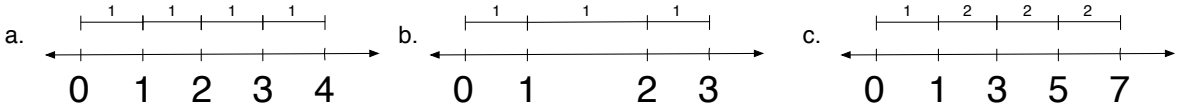


Figure 4: Number lines on which (a) consecutive numerical units are coordinated with congruent linear units, (b) linear units are equal without coordination with numerical units, and (c) numerical units are consistent without coordination with linear units

Cartesian plane. The Cartesian plane has at its core two number lines, one rotated to be perpendicular to the other, with an intersection point at the origin. This linear representation requires a coordination of linear units and numerical units on two independent axes. Little research exists on students' understanding of the plane alone (one exception is Lehrer, Strom & Confrey, 2002). The scale of each axis need not be the same, a property that may remain hidden in the typical unit scaling of axes in mathematics education. Children are often provided with grids with numerical units already inscribed, with the scale typically identical on each axis (Leinhardt, Zasklavsky, & Stein, 1990). Such a treatment does not allow children to grapple with the coordination of linear units and numerical units on the independent axes.

A point in the plane is named based on the intersection of two projections, one from a value on the horizontal axis and the other from a value on the vertical axis. Naming the point requires not only the projection from each axis, but also conceptualizing each of those number lines as a coordination of linear and numerical units. To illustrate the required coordinations, consider the grids presented in Figure 5. Figure 5a presents a grid with an unnamed point in the plane. To name that point, one must also coordinate linear and numerical units on the independent axes before projecting a value from each of those axes. Figure 5b shows such a coordination. The axes have different scales, with a unit interval on the horizontal axis and a multiunit interval of 2 on the vertical axis. A challenge for students is first making this coordination of linear and numerical units on each axis. In Figure 5c the point was also named based on projections from each axis. However, because linear units were treated as unit intervals

on the vertical axis without coordination with numerical units, the value for the vertical axis was misnamed as 4.5.

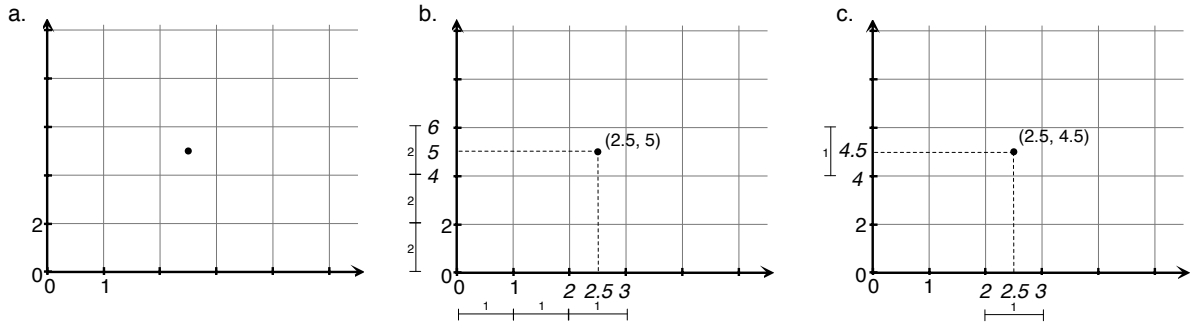


Figure 5: Cartesian plane (a) with unit and multiunits on each axis, (b) solved by coordinating linear and numerical units, and (c) by considering numerical units independent of linear units (c)

Graphs of mathematical functions. The Cartesian plane serves as a representation on which to plot mathematical functions. Function graphs display the joint variation—the value of one quantity in relation to a given value of the other quantity—of quantities represented on the axes. In the case of elementary school students, research has shown that young students can reason meaningfully with graphs of simple mathematical functions with appropriate instructional support (Brizuela & Earnest, 2008; Carraher et al., 2008; Schliemann et al., 2008). While research has provided evidence of young students plotting and interpreting function graphs, research has not yet made clear the conceptual challenges young students encounter as they engage with such representations.

A principal feature of function graphs is slope, the ratio of change of the dependent variable to a unit change in the independent variable. Essential to this definition in the context of linear representations of quantities is a coordination of linear units and numerical units. Slope refers both to a unit change on the horizontal axis—the linear unit corresponding to the numerical inscriptions 0 and 1 (or another interval of 1)—and the corresponding interval of change on the vertical axis. Nonetheless, many students interpret slope as the change in a single variable or in terms of the orientation of the line in the plane independent of linear and numerical units along the axes (Caddle & Earnest, 2008; Lobato, Ellis & Muñoz, 2003; Zaslavsky, Sela & Leron, 2002).

To illustrate, consider the grids in Figure 6. Figure 6a displays a function graph in Quadrant I with the function f(x)=2x. One successful approach to determine the slope using features of the graph requires one to identify a unit change on the horizontal axis, and then project this interval onto the function line. Then, to determine the interval of change along the horizontal axis, one must then project from the function line to the vertical axis. Figure 6b displays the unit change on the horizontal axis and the corresponding change of 2 from the vertical axis. Even though linear units are congruent, the ratio of change is 2 to 1 because of differing scale along either axis. Figure 6c displays the same graph on which each linear unit on the vertical axis was determined to be a unit interval. The ratio of change, 1 to 1, thereby reflects linear units alone rather than the coordination of linear units and numerical units together; the 1 to 1 ratio communicates no mathematical meaning based on the joint variation of the function.

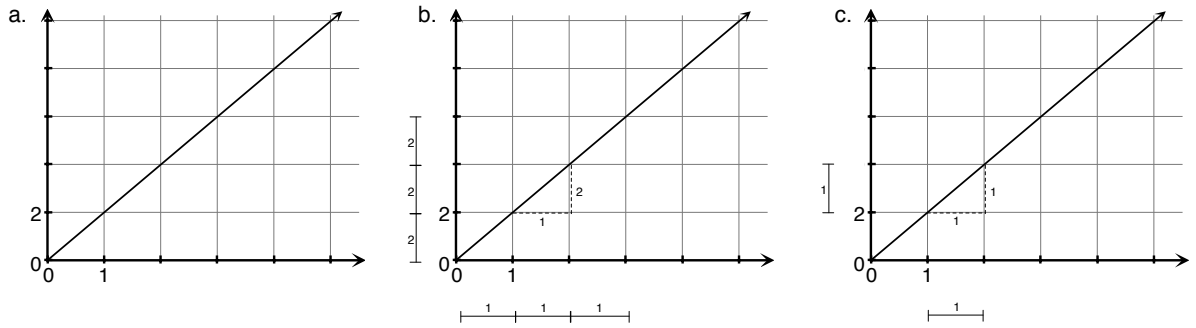


Figure 6: Linear function graphs (a) with unit and multiunits on each axis, (b) with a slope determined by a coordination of linear and numerical units, and (c) with a slope determined by the use of linear units alone

Linear and numerical units across linear representations of quantities. To date, research focused on the number line (Saxe et al., 2010, in press) as a coordination of linear units and numerical units, but not across linear representations of quantities. Research has not yet investigated young students' conceptual coordinations and the challenges they may face across linear representations of quantities. The coordination of linear and numerical units on a single number line provides a conceptual foundation that may productively be extended to the Cartesian plane. This coordination may be further extended to understanding slope in function graphs. While research has indicated that young students, like Curtis, may productively reason with such representations (Carraher et al., 2008; Schliemann et al., 2008), an examination of student reasoning and of productive instruction to support students is understudied. I argue that the coordination of linear and numerical units may be a productive route to support students' rich and generative thinking involved with these linear representations of quantities.

Structure of the Dissertation

In this introductory chapter, I have discussed the coordination of linear units and numerical units as a foundation to linear representations of quantities for an area of mathematics education for which learning and teaching remain understudied. In the first part of my dissertation, I assess students' understanding of three related kinds of representations - number lines, the Cartesian plane, and function graphs. I analyze patterns of reasoning across assessment items for fifth graders, and then to understand whether the patterns of errors in fifth graders persisted at higher grade levels when students were enrolled in algebra, I include a Grade 8 sample. Further, to understand better student reasoning at each grade levels, I interview a subset of Grade 5 and Grade 8 students, probing their reasoning on selected assessment items.

Building on the assessment and interview findings, in the second part of my dissertation, I explore student learning in the context of an instructional approach designed to support Grade 5 students' understandings of the three linear representational contexts. I engage students in a one-on-one tutorial intervention, first, to explore the efficacy of an intervention, and second, to explore children's conceptual coordinations as they use their prior understandings to construct new conceptual coordinations.

Chapter 2: Assessing Students' Understanding of Linear Representations of Quantities

This chapter has three purposes. First, I describe targeted ideas of the Linear Representations Assessment (LRA), an assessment of students' competence with each domain of linear representation of interest: number lines, the Cartesian plane, and graphs of linear functions. Second, I report a study on patterns of students' performance on the LRA. Third, I report data produced in a follow-up to the assessment results, describing interviews in which I probed students' thinking on selected LRA problems.

Design Features of the LRA

In this section I describe the design of the LRA. The LRA was developed to serve two research functions. The first function was to assess students' understanding of linear representations of quantities, using a broad range of items that included number line items, Cartesian plane items, and function graph items. The second function was to contrast student performances on two matched item types that cut across the three representations. One item type included routine problems and the other non-routine problems. The matched items for each problem type were similar in content, but non-routine problems were distinguished from routine problems by an atypical representational design. Typical of problems in mathematics education, routine problems included canonical representations, like those students often see in school and are amenable to memorized or familiar solution approaches. Non-routine problems included "distracters" and required conceptual understanding to construct adequate solutions. For each routine problem, I constructed a non-routine problem that matched in content. To illustrate, Figure 7a asks a student to name a point in the plane. The grid features unit scaling on each axis, with the unnamed point positioned on gridlines at integer values for each axis. Figure 7b addresses the same content, but does so in a non-routine problem format. Atypical of problems in mathematics education, the grid features a multiunit interval along each axis, with the unnamed point not corresponding to gridlines. I first describe assessment features of the LRA, followed by the experimental features.

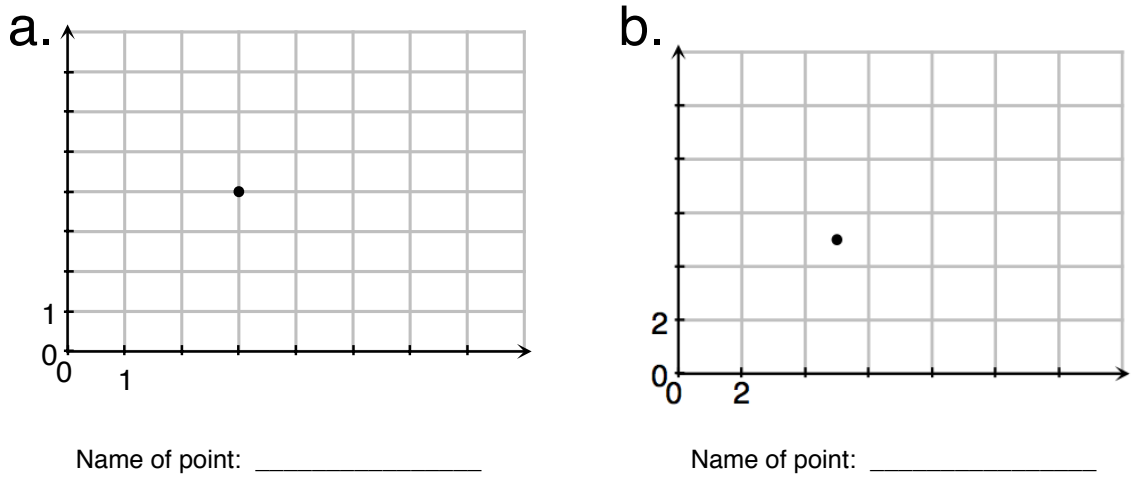


Figure 7: Matched (a) routine and (b) non-routine items on the LRA

The assessment function of the LRA: Evaluating students understanding of linear representations of quantities. The LRA was designed to assess student solutions to tasks involving the three linear representations of quantities. To date, systematic research with a focus across linear representations of quantities has not been conducted. The LRA served as an instrument to document students' conceptual coordinations on targeted ideas.

In this section, I provide further detail on LRA items and the targeted mathematical ideas they address. To do so, I draw upon prior literature and patterns that emerged in the context of my own pilot work on targeted ideas of LRA tasks. Pilot work occurred over a period of sixteen months in Grades 5 through 8, and included videotaped one-on-one interviews and classroom observations with specific tasks. Over the piloting period, items were designed and refined to assess the targeted ideas described below.

Targeted ideas and assessment tasks: Number lines. Targeted mathematical ideas of number lines included interval and origin.[1] I first provide further detail on the LRA's assessment of interval followed by origin using the LRA item, *1, 3; Place 0*, which featured a multiunit interval of 2 and non-congruent linear units, with a prompt to place 0 on the line (Figure 8). The appropriate location of 0 was one unit interval to the left of 1.

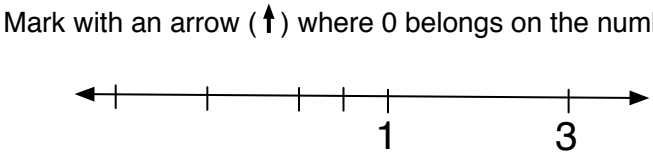


Figure 8: LRA item *1, 3; Place 0* designed to assess interval and origin

Interval. I define a mature understanding of linear interval as an appropriate coordination of linear units with numerical units. One way to make this coordination is to mark the endpoint of congruent linear units with consecutive integers, or values that increase (to the right) or decrease (to the left) by consistent increments. LRA item *1, 3; Place 0* was designed to assess students' varied understandings of interval. When making an appropriate coordination of linear and numerical units, students may first identify the multiunit of 2 using the numerical units 1 and 3 (Figure 9a), and then partition the multiunit in half, thereby locating 2 and identifying two unit intervals. With a unit interval identified, students may iterate it once to the left of 1 to position 0 at the left endpoint of that linear unit (Figure 9b).[2]

[1] A third mathematical idea of number lines, order, is implicitly assessed in these items. Based on literature and extensive piloting, the property of order becomes salient when investigating understanding of rational number (Corwin, Russell, & Tierney, 1990; Saxe, Shaughnessy, Shannon, Langer-Osuna, Chinn & Gearhart, 2007; Wu, 2009) and/or negative integers on the number line (Bass, 1998; Wu, 2002); piloting did not reveal underlying issues with order among only positive integers. For this reason, no task explicitly focused on order, though all implicitly assessed for this.

[2] This solution is not the only possible approach. One may also iterate the multiunit of 2 to the left of 1, and then partition it in half to locate 0.

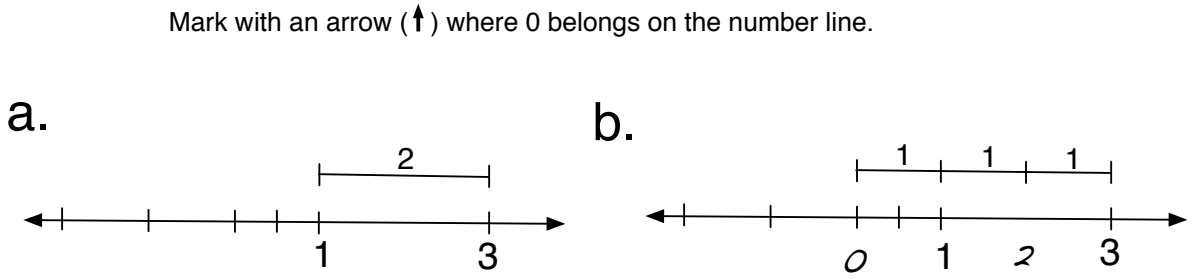


Figure 9: Solution for LRA item *1, 3; Place 0* in which linear units and numerical units are coordinated

Other ways of coordinating of linear and numerical units result in alternative (and incorrect) positions for 0 on the line. I provide two patterns that emerged from prior literature and pilot work. A first pattern involved students attending to numerical units without coordination with the provided interval (Carraher et al., 2006; Saxe et al., 2009; Saxe et al., in press). In this pattern, students position consecutive integers on the line irrespective of their coordination with linear units. For example, pilot interviews revealed upper elementary and middle school students extending consecutive numerical units to the left of 1. The linear unit between 0 and 1 was determined by information given in the problem, namely the arbitrary positioning of tickmarks on the line (Figure 10a). The resulting position corresponded to the actual position of 0.5 on the line.

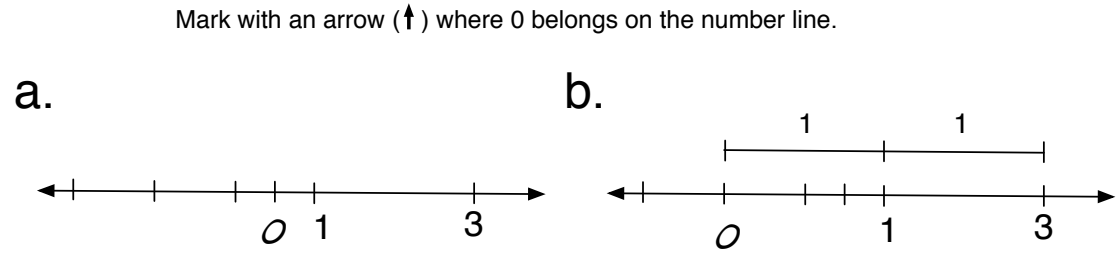


Figure 10: Solutions for item *1, 3; Place 0* in which (a) numerical units are privileged independent of linear units, and (b) linear units are privileged independent of numerical units

A second pattern involved students attending to linear units independent of the numerical units 1 and 3. In pilot work involving item *1, 3; Place 0* with upper elementary and middle school students, a pattern emerged in which students iterated the linear unit from 1 to 3 to the left of 1, yet treated that linear unit as a unit interval of 1 (Figure 10b). The resulting position for 0 corresponded to the actual position of -1 on the line. Item *1, 3; Place 0* was designed to document the pervasiveness of these patterns across students involving coordinating linear units and numerical units.

Zero as origin on the number line. Origin refers to a starting point on the number line and corresponds to the location of 0, with negative numbers to the left and positive numbers to the right. Origin has been identified as a central idea in measurement (Lehrer, 2003) as a starting point for indexing the length of objects when those objects are concatenated end-to-end. LRA item *1, 3; Place 0* was designed to assess students' understanding of the position of zero on the number line. When making an appropriate coordination of linear and numerical units, students

position 0 with respect to the interval from 1 to 3 provided on the line as shown above in Figure 9.

Pilot work involving item *1, 3; Place 0* revealed a pattern in which students positioned 0 independent of any linear unit or numerical unit on the line. In this pattern, students applied their understanding of zero as a starting point. For example, pilot interviews revealed students position 0 at the leftmost tickmark of the number line without considering numerical or linear units (Figure 11). The resulting position corresponded to the actual position of -2 on the line.

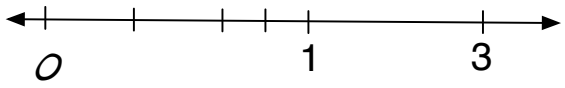


Figure 11: Solution for item *1, 3; place 0* in which zero is positioned as a starting point

Targeted ideas and assessment tasks: The Cartesian plane. Targeted mathematical ideas of the Cartesian plane included the interpreting points long two axes and interpreting points in the plane. I first provide detail on the LRA's assessment of interpreting points along two axes followed by interpreting points in the plane. I illustrate the targeted ideas using two LRA items described below.

Interpreting points on two axes. I define interpreting points on two axes as an appropriate coordination of linear units and numerical units along two independent axes. LRA item *Scale with multiunit intervals* was designed to assess students' varied understanding of interpreting points on two axes (Figure 12a). Unlike routine tasks common in mathematics education on which unit scaling is provided (Leinhardt, Zaslavsky, & Stein, 1990), the task features a grid with multiunit intervals provided on each axis, with a target value to identify along the horizontal axis. When making an appropriate coordination of linear and numerical units, students may first partition the multiunit from 0 to 2 in half, thereby locating 1 and identifying two unit intervals. With a unit interval identified, students may iterate it once to the right of 2 to determine the target value to be 3 (Figure 12b). This coordination is independent of values along the vertical axis.

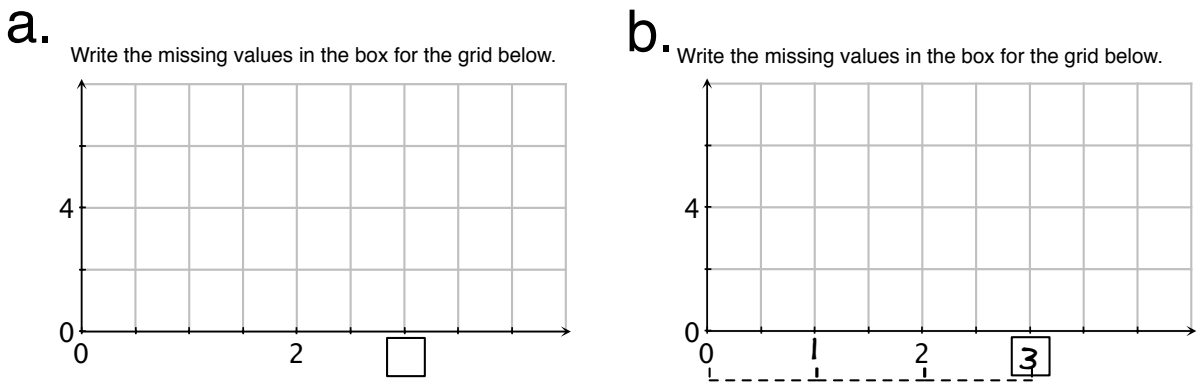


Figure 12: (a) LRA item *Scale with multiunit intervals* for which (b) linear units are coordinated with numerical units to determine the target value

While a review of research revealed few studies on student understanding of interpreting points along two axes, prior research on number lines implies that students may not appropriately coordinate linear and numerical units along two axes (Saxe et al., 2009; Saxe et al., in press). For item *Scale with multiunit intervals,* other ways of coordinating linear and numerical units result

in alternative names for the target value. One pattern that emerged from videotaped pilot interviews involved students positioning consecutive integers at endpoints of congruent linear units to the right of 2 that are not coordinated with the interval from 0 to 2. For example, pilot interview data revealed upper elementary and middle school students naming the next tickmark 3 and then naming the target value 4 (Figure 13). The item *Scale with multiunit intervals* was designed to document the varied patterns across students involving interpreting points along two axes.

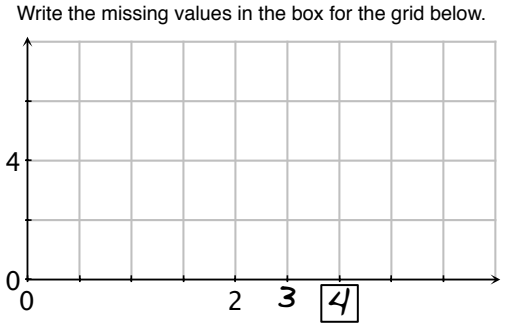


Figure 13: LRA item *Scale with multiunit intervals* in which the target value is named by extending numerical units with provided linear units, but without coordination with the 0 to 2 interval

Interpreting points in the plane. Successful interpretation of points in the plane requires one to project from values along two independent axes to a single point in the plane. In the case where particular values along axes have not been provided or identified, one must first coordinate linear and numerical units in order to identify appropriate values along the axes. LRA item *Identify 5, 5* was designed to assess students' varied understandings of naming a point in the plane. The task features 0 and 2 along horizontal and vertical axes, and asks students to name a point in the plane (Figure 11a). When making an appropriate coordination of linear and numerical units, students may first iterate the interval of 2 to locate 4 and 6, and then partition one of those intervals to identify a unit interval on each axis in order to name the point (5, 5) (Figure 11b).

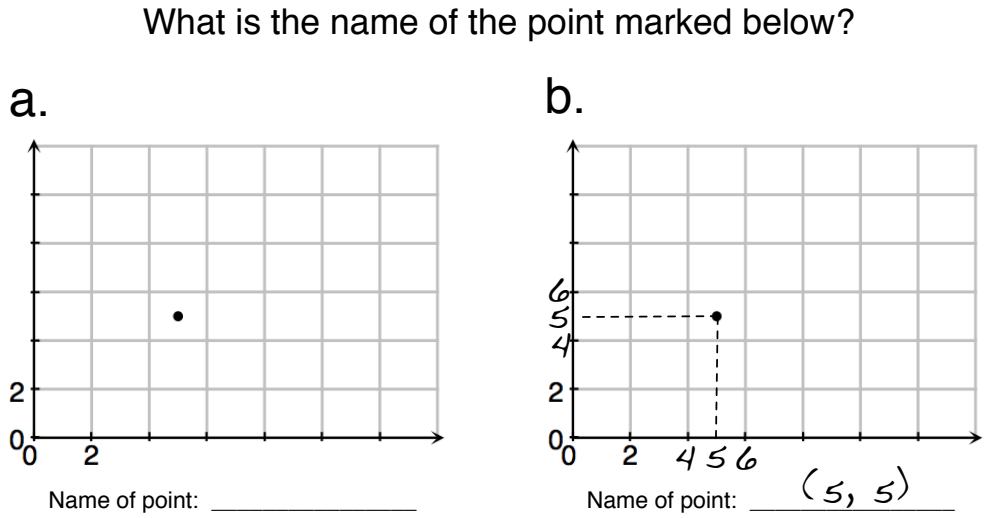


Figure 14: (a) LRA item *Identify 5, 5* for which (b) linear units are coordinated with numerical units along two axes to determine the appropriate name

Other ways of coordinating linear and numerical result in alternative names for the point in the plane.[3] A pattern that emerged from videotaped pilot interviews involving this task, and that is consistent with prior literature on coordinating linear and numerical units on a single number line (Saxe et al., 2009; Saxe et al., in press), involved students extended numerical units to linear grid units without coordination with the intervals of 2 along each axis. For example, pilot interviews revealed upper elementary and middle school students naming tickmarks to the right of 2 on the horizontal axis (and above 2 on the vertical axis) by consecutive integers (Figure 15). Because the corresponding values along each axis were not determined through a coordination of linear and numerical units, students incorrectly named the point (3.5, 3.5). The item *Identify 5, 5* was designed to document the varied patterns across students involving interpreting a point in the plane.

[3] A review of research did not reveal recent assessment studies on students placing points in the plane, though some research focused on identifying points in four quadrants using a computer setting (Moschkovich, Schoenfeld, & Arcavi, 1993) or generating notation for a function given points in the plane, as in computer game settings (Dugdale & Kibbey, 1986). Because this study focuses only on Quadrant I and does not involve generating or interpreting notation for functions, I do not consider these studies here. While an earlier study revealed that among 1400 high school students in the U.K., 90 to 95% of them plotted or interpreted points in the plane successfully (Kerslake, 1977), this study provided values along axes, thereby not assessing students' construction of scale in conjunction with plotting or interpreting points.

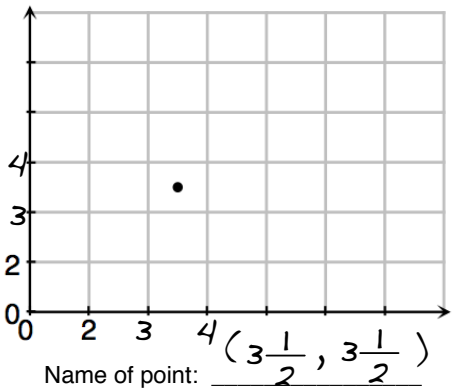


Figure 15: LRA item *Identify 5, 5* solved without coordinating numerical units and linear units with the interval provided

Targeted ideas and assessment tasks: Graphs of linear functions. Targeted mathematical ideas of linear function graphs included interpreting rate of change and coordinating a story context with a graph. I first provide detail on the LRA's assessment of rate of change followed by story-to-graph coordination. I illustrate the targeted ideas using two LRA items described below.

Interpreting rate of change. Rate of change, or slope, of a linear function refers to a ratio of change on the vertical axis to the corresponding unit change on the horizontal axis. LRA item *Rate of change* was designed to assess students' varied interpretations of rate of change in terms of linear and numerical units. The item (Figure 16) features two grids (Quadrant I only) with the same linear function plotted on each. While the functions are mathematically identical, the differing scale across grids results in different orientations of the function lines in the planes. When making an appropriate coordination of linear units and numerical units to determine which (if either) function has a greater rate of change, students may identify particular points along each function line (e.g., comparing the common points (1, 2) and (2, 4)), or determine and compare the value for slope.

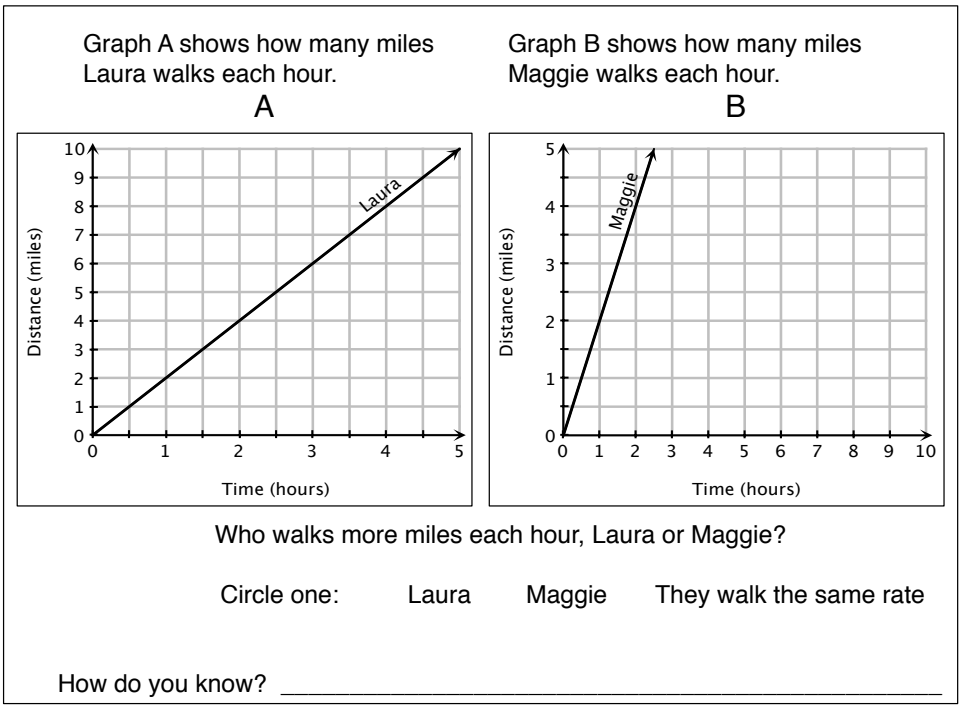


Figure 16: Graphing linear functions task to assess interpretation of rate of change

Prior research indicates children may interpret rate of change in terms of only one variable (Lobato, Ellis, & Muñoz, 2003) or in terms of the orientation of the line in the plane independent of numerical and linear units along the axes (Caddle & Earnest, 2008; Zaslavsky, Sela, & Leron, 2002). Pilot work involving this task corroborated those prior findings (Earnest, 2010) and revealed an additional pattern. First, students compared numerical units along one axis, thereby interpreting rate of change in terms of only one variable (e.g., Laura walked 10 miles total but Maggie walked only 5 miles total). Second, students interpreted the orientation of the line in the plane independent of linear or numerical units (e.g., interpreting Maggie's line as steeper than Laura's line). Third, students assigned consecutive numbers to each linear unit without coordinating with numerical units along the axes (for example, counting over one unit along the horizontal axis as shown in Figure 17). The item *Rate of change* was designed to document the varied patterns across students as they draw upon linear units and numerical units to determine the rate of change.

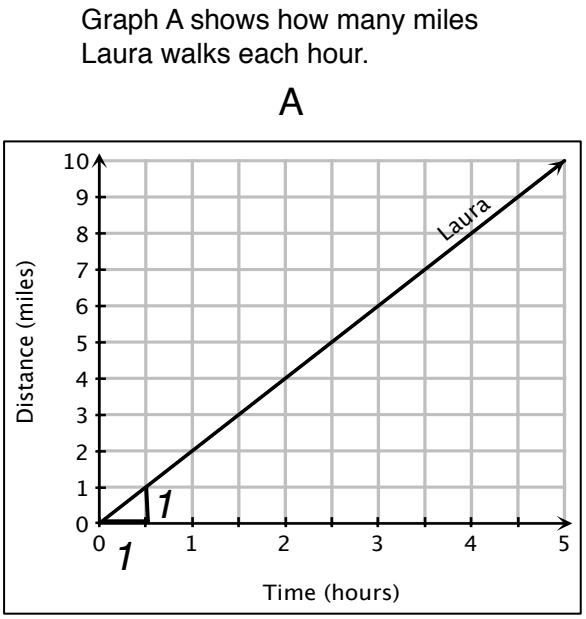


Figure 17: A part of the LRA item *Rate of change* illustrating the use of linear units independent of numerical units to determine the rate of change

Coordinating a story context with a graph. Coordinating a story context with a graph requires one to determine the joint variation—the variation of one variable that depends on the variation of another variable— involved in the story and then to coordinate this with linear and numerical inscriptions on a graph. Story contexts are often used in elementary school instruction involving simple mathematical functions (Carraher & Schliemann, 2007). The LRA item *John and Mary's money* was designed[4] to assess students' use of linear and numerical units when making the story-to-graph coordination. The item features a prompt stating, "Mary always has three times as much money as John. Which of the lines below shows this?" A grid with numerical units along the axes provides three functions from which to choose, Lines A, B, or C. After identifying the joint variation from the prompt, students making an appropriate coordination with numerical and linear units may identify ordered pairs consistent with the joint variation (e.g., (1, 3), (2, 6), (3, 9)) that correspond with one of the given functions, Line B.

[4] This task was adapted slightly from the TERC-Tufts Early Algebra project.

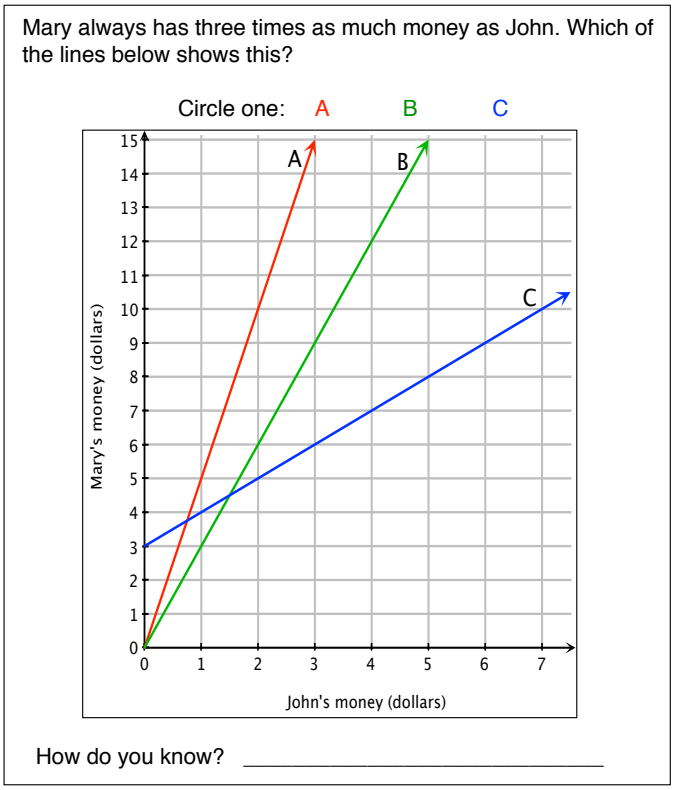


Figure 18: Graphing linear functions task to assess students' coordination of a story context with the graph

While a review of research did not reveal studies on students' story-to-graph coordination in terms of numerical and linear units, pilot work with item *John and Mary's money* revealed students' use of numerical units and linear units. Two patterns emerged. First, students drew upon numerical units to identify three-ness. For example, many students coordinated the 3 in the prompt with a single numerical unit of 3 along the vertical axis, leading to an incorrect choice of Line C. Alternatively, other students chose Line A because an ordered pair, (3, 15), included values that were each multiples of 3. Second, interviews revealed a pattern in which students coordinated the 3 in the prompt with three linear grid units in the plane. This involved identifying three linear grid units from 15 on the vertical axis to Line A in the plane. The item *John and Mary's money* was designed to document students' use of linear and numerical units when engaged in a story-to-graph coordination.

The experimental design function of LRA: Contrasting student performances on routine and non-routine problems. The LRA featured an experimental design element in order to compare students' responses under two conditions. Reviews of literature revealed a dearth of systematic research involving students' understanding of the three linear representations in question. To address this systematically, the LRA assessed students' responses to both *routine* and *non-routine* problems. In this section, I first define routine and non-routine in the context of the LRA, and then describe the experimental feature of the LRA that allowed for contrast of the two problem types.

Defining routine and non-routine problems. I define a routine problem as one that includes typical or traditional representational treatment. Such problem design typifies what students might see in textbooks or on standardized tests. Canonical routine problems are amenable to solution approaches that range from mathematically grounded conceptual coordinations to memorized procedures with an impoverished mathematical foundation. For example, Figure 19a displays a routine Cartesian plane task from the LRA. In the task, unit intervals are provided along each axis, which features congruent linear units across each entire axis. To solve the problem, students might coordinate the linear unit defined by the interval from 0 to 1 together with the numerical inscriptions in order to iterate that unit interval to then name the point (3, 4) (Figure 19b). On the other hand, students may observe the counting pattern of 1s and extend this numerical pattern without necessarily considering coordination with linear units (Figure 19c). The character of these two conceptual coordinations is markedly different; one involves an appropriate coordination of linear units and numerical units with the provided unit interval, while the other extends numerical units without overtly considering the unit interval. On the routine problem, each leads to the same solution.

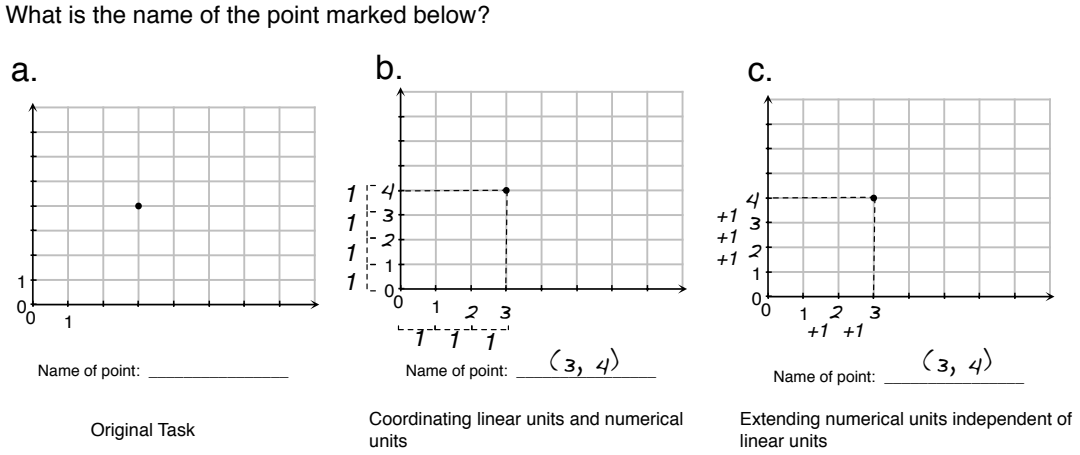


Figure 19: (a) A routine task solved (b) by coordinating linear units and numerical units and (c) by extending consecutive numerical units

I define a non-routine problem as one for which the representation is recognizable as a category of mathematical representation (e.g., recognizable as a number line), yet with a representational feature altered (a "distracter"). This uncommon item design problematizes the underlying mathematics and leads to varied solutions depending on the problem solving approach. For example, Figure 20a displays a non-routine Cartesian plane task from the LRA. In the task, a multiunit of 2 is provided on each axis. To solve the problem accurately, students must coordinate the linear unit defined by the interval from 0 to 2 together with numerical units

in order to iterate that interval of 2 to the right (Figure 20b) and then partition this interval of 2 in half to find the unit interval. Alternatively, students may extend numerical units using tickmarks but without coordinating linear units with the 0 and 2 (Figure 20c). Because of the qualitatively different nature of the solution approaches and the non-routine design, each strategy leads to a unique solution, thereby revealing a student's conceptual coordinations.

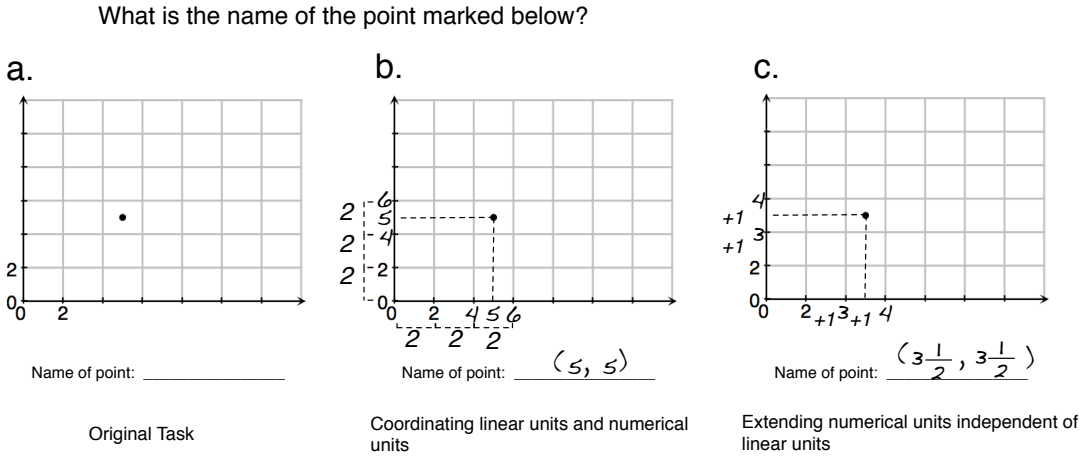


Figure 20: (a) A non-routine task solved (b) by coordinating linear units and numerical units and (c) by extending numerical units

Experimental contrast. The LRA afforded a comparison between student performance on routine versus non-routine problems. To achieve comparability of these two problem types, I developed matching items based on a consideration both of the mathematical aspects and the representational aspects of the problem. Mathematical aspect refers to target mathematics of the item. In the prior two tasks in Figure 19 and Figure 20, for example, I characterize the target mathematics as a coordination of linear and numerical units on two independent axes. Representational aspect refers to inscriptional features that have no bearing on the underlying mathematics. For example, the grid in Figure 19 features unit intervals along each axis, while the grid in Figure 20 features multiunit intervals of 2. Despite these differences in representation, each axis adheres to the same mathematical principles and conventions, namely that since two values are placed on the line, the position of all other values is known. While representational aspects may be inconsequential from a mathematician's perspective, they may be highly relevant from the point of view of children. The experimental condition reveals representational aspects of consequence to children.

In order to systematically compare students' performances on routine tasks as compared to non-routine items, a subset of tasks were matched to include one routine and one non-routine task. In these pairs, mathematical aspects were held constant while one or two representational aspects were altered. In Figure 19 and Figure 20, both routine and non-routine tasks require identifying the name of a point in the plane. Yet the non-routine version features multiunit intervals and a point that does not fall on gridlines, whereas the routine version features unit intervals and a point that falls along gridlines. While the mathematical aspects are held constant, two representational aspects are altered.

The comparison affords particular insights on student reasoning. In my analytic treatment, I determine students that respond correctly to both routine and non-routine problems

to be coordinating linear and numerical units in mathematically accurate ways. Conversely, I determine students that respond inaccurately to both routine and non-routine problems not to be coordinating linear and numerical units in accurate ways. I determine students that respond correctly to a routine problem yet incorrectly to the matched non-routine problem as having partial understandings about the underlying mathematics. Such understandings may lead to success with canonical representational treatments, but do not support success with the non-routine treatment.

Present study: Research questions. Two research questions guided Study 1. First, how do students perform on problems involving the three representations under different conditions? Second, what is the character of students' conceptual coordinations when engaged with such problems?

Methods

Participants. Participants include 126 Grade 5 students and 131 Grade 8 students drawn from elementary and middle schools in Northern California. Two schools were K-8 and provided students for both the Grades 5 and 8 sample. One additional elementary and middle school provided the remaining students. A subset of Grade 5 (n=33) and Grade 8 (n=26) students participated in follow-up clinical interviews. Interview selection criteria are described below.

Piloting. Piloting of assessment tasks was involved one-on-one interviews, tutoring, and classroom observations. Piloting occurred in four phases: (1) one-on-one interviews with Grades 5 and 6 students in February and March of 2009; (2) classroom teaching experiment in collaboration with an in-service Grade 6 teacher with after-class interviews with students, May 2009; (3) one-on-one interviews with students in Grades 5, 6 and 8, Fall 2009 through Spring 2010; and (4) one-on-one interviews with Grades 5 and 8 students, Fall 2010. Throughout these phases, I conferred with research groups at the University of California, Berkeley, to gather different perspectives on the design of tasks. Piloting resulted in the final Linear Representations Assessment.

Assessment. The LRA featured 32 items including (a) ten number lines items, (b) 11 Cartesian plane items, and (c) 11 graphs of linear functions items (see Appendix A). Eighteen tasks were matched—one routine version matched with an analogous non-routine counterpart—with three pairs of matched problems for each representation (see Appendix B).

Most number line tasks were drawn from assessments given as part of the Learning Mathematics through Representations project at UC Berkeley (Saxe et al., 2009, 2010, 2011, in press); remaining tasks were inspired in large part by the same project. All items went through multiple rounds of piloting. The LRA featured three conditions of number line task of varying complexity: (1) unit interval provided, multiunit target; (2) multiunit interval provided, multiunit interval target; and (3) multiunit interval provided, fractional target.

All Cartesian plane tasks on the LRA were designed for this study and went through multiple rounds of piloting. The LRA featured three conditions of Cartesian plane task of varying complexity: (1) unit interval provided on each axis, unit and multiunit interval targets; (2) unit and multiunit interval provided, unit and multiunit targets; (3) multiunit interval provided, fractional targets.

LRA tasks involving linear function graphs were taken from different research projects (Carraher, Schliemann, & Schwartz, 2008; Carraher, Schliemann, & Caddle, 2008; Zaslavsky, Sela & Leron, 2002) and curriculum development projects (Early Algebra, Early Arithmetic, www.earlyalgebra.terc.edu), with other items designed for this project (Earnest, 2010). The LRA featured two conditions of linear function graph tasks: (1) unit interval provided along each axis; and (2) multiunit provided along at least one axis.

To address this potential proximity of students to one another during assessment administration, three orders of the assessment were administered. Each of the 32 items was on each of the assessments. Principles for the ordering included: (a) matched routine and non-routine items should not follow one another; (b) for matched items, the routine item always should appear before than the non-routine; (c) number line, Cartesian plane, and graphing functions tasks should be intermingled. Order distribution to students was determined randomly; the student closest to the teacher's desk was selected for the first order, the student closest to the first would receive the second order, the third the third order, the fourth the first order, etc.

Assessment administration. Teachers administered the Linear Representations Assessment (LRA) with the researcher present in the classroom. The LRA was administered to each participating classroom. Classroom teachers administered the assessment in January or February, 2011. The researcher was present to oversee each administration. Students had 30 minutes to complete the assessment; all students finished. Administration was staggered over four weeks across classrooms due to logistics of data collection at multiple research sites. All assessments were collected and scanned.

Scoring the LRA. The LRA was scored immediately following administration. Each item on the LRA received two codes. I recorded in a FileMaker Pro database a code to record if an item was answered correctly or incorrectly, while also entering students' answers for each item for each student. FileMaker Pro was used for all assessment coding, and the two sets of codes were triangulated to check for errors in coding.

Interviews. All interviews were conducted in one-on-one (interviewer-interviewee) sessions. The location varied depending on the school. In all cases, the interview was conducted in a semi-private location on school grounds. In the interview, the researcher provided a copy of the original assessment along with two pens with differently colored ink, one for the student and one for the researcher.

The number of questions asked during the interview varied based on particular responses from an individual student. In all cases, the interviewer asked the student about one non-routine number line task, one non-routine Cartesian plane task, and one non-routine graphing functions task.

Interviews focused on a predetermined subset of assessment items. These included four number line tasks, four Cartesian plane tasks, and four graphing functions tasks (see Appendix C). The interviewer occasionally made judgment calls to interview students on other questions. For example, in order to minimize any distress that may arise from the student during the interview, the interviewer made in-the-moment decisions to include tasks on which a particular student responded correctly and would likely provide confidently a justification. The goal of this was to make the student as comfortable as possible. In the case that a student appeared to be in distress, the researcher intended to stop the interview; this did not occur.

The interview protocol adhered to the following procedure for each item: (1) The researcher read the question on the paper; (2) the researcher asked the student to read her/his original answer; (3) the researcher asked the student to explain how she/he was thinking about the problem; (4) the researcher presented a countersuggestion to the student. In the case where a student responded correctly, the researcher provided a pre-determined, incorrect countersuggestion with a rationale for that incorrect response. In the case where a student responded incorrectly initially, the researcher provided the correct response as a countersuggestion.

All students with permission were eligible for interviews. Two criteria determined the interview group: (a) common responses, and (b) overall score on the assessment. First, students were selected so that interviews captured the common responses on the assessment; I considered a response to be common if that choice had a frequency of 5 students or greater. Second, I used the overall performance on the assessment (as determined by total number of items correct) to select students across the overall range for the entire sample.

Interviews were recorded using a single videocamera and attached flat microphone. The camera was positioned to focus on the paper and student's hands to record gestures and inscriptions made in the context of a student's explaining or changing a response. All video was compressed into smaller files, and StudioCode was used to analyze the video.

Results

The results are presented in two sections. In the first, I present quantitative results to show that students in each grade performed better on routine problem as compared to matched non-routine problems, and this difference was statistically significant. In the second section, I use interview data to reveal patterns in students' coordinations of linear and numerical units on non-routine tasks.

Performance results on routine and non-routine problems. In this first section, I present quantitative results of the assessment. I include four analyses. First, I begin by determining if there is a difference in student performance based on the three orders distributed among each classroom of students. Second, I compare overall performance on the LRA. Third, I look at performance for each grade on the 18 matched routine and non-routine problems. Fourth, using only the nine non-routine problems, I then analyze performance as a function of grade and representation.

Order effect analysis. Recall that three orders of the assessment were distributed across students in each participating classroom. A one-way analysis of variance controlling for order was used to compare performance of students. The ANOVA did not detect a difference in performance due to order, $F(2, 254) = .958$, $p = .358$. Figure 21 displays means for each order for each of the grades. Because results do not point to a detectable difference in performance as a function of order, the remainder of the analyses do not distinguish based on order.

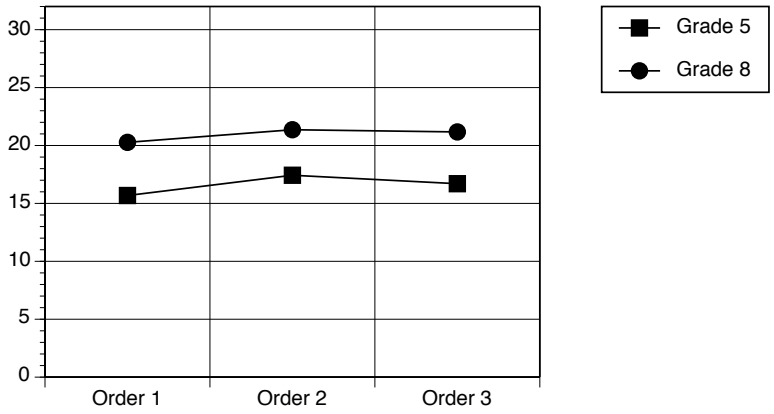


Figure 21: Mean performance on assessment as a function of order and grade

Overall Performance on the LRA. I analyzed overall performance on the LRA to detect any differences based on grade. Each student was given one point for each correct response; incorrect responses were assigned 0 points. Consistent with my expectations, Grade 8 students outperformed Grade 5 students (see Figure 22). Grade 5 students had a mean of 16.61 out of 32 items with a standard deviation of 6.972, while Grade 8 students had a mean performance of 20.95, with a standard deviation of 6.447. An Independent Samples t-test revealed that theses differences were statistically significant ($t(255) = -5.188$, $p < .0001$ (one-tailed).

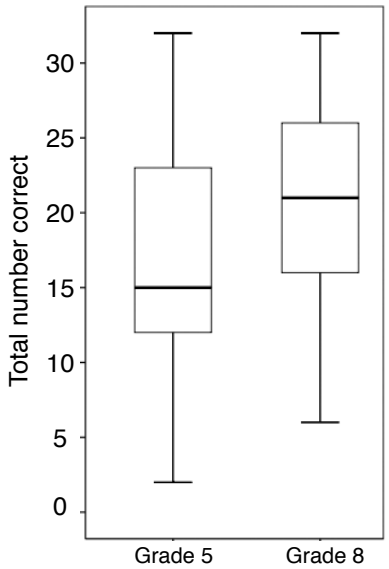


Figure 22: Overall results on the LRA by grade

Performance on 18 matched problems. A subscale of eighteen problems on the assessment were nine routine items matched to nine analogous non-routine items to create nine pairs. Each student was given one point for each correct response; incorrect responses were assigned 0 points. Means and standard deviations are presented in Table 1, and results are presented in boxplots in Figure 23. A Two (problem type) ✕ Two (Grade) Repeated Measures Analysis of Variance (ANOVA) revealed main effects for problem type ($F(1, 255) = 375.948$, p

< .0001) and for grade (F(1, 255) = 44.113, $p < .0001$). There was no significant interaction between problem type and grade, $F(1, 255) = 375.948$, $p = 0.805$. I conducted post-hoc analyses for problem type for each of the grades. Grade 5 students were significantly more successful at routine problems than non-routine problems, t (125) = 14.240, $p < 0.0001$. Likewise, Grade 8 students were also significantly more successful at routine problems than non-routine problems, t (130) = 13.247, $p < 0.0001$. Regardless of grade, there was a significant decrease in score from routine to non-routine tasks. Given the significant decrease in performance, non-routine tasks are a focus of further analysis.

Table 1: Means and standard deviations by grade for 18 matched problems on the LRA

Grade	*Problem Type*	*Mean*	*Standard Deviation*
5	Routine	6.198	1.811
	Non-routine	3.849	2.433
8	Routine	7.672	1.237
	Non-routine	5.382	2.476

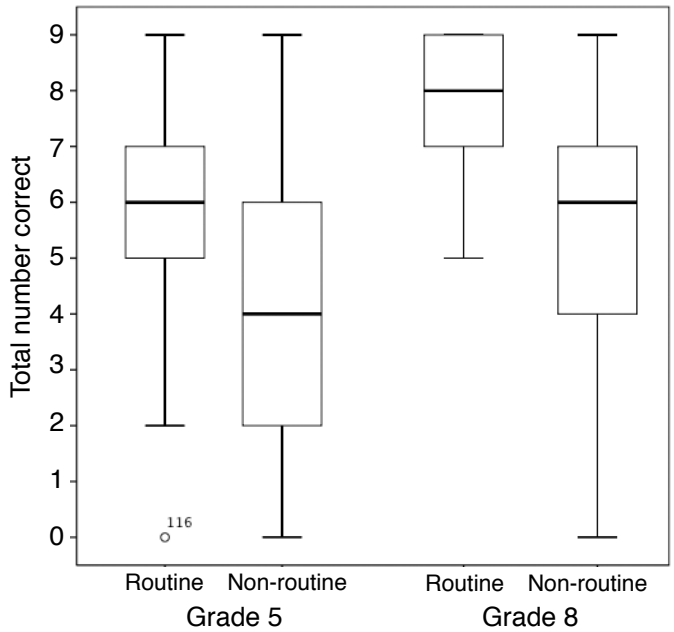


Figure 23: Boxplots showing performance on the LRA by grade and problem type

Performance on non-routine problems by representation and grade. I now turn to an analysis of performance on the nine non-routine problems as a function of representation and grade. A Three (representation) ✕ Two (grade) Repeated Measures ANOVA revealed a main effects of representation (F(2, 510) = 40.570, $p < .0001$) and grade (F(1, 255) = 25.023, $p <$ .0001), and a significant interaction between representation and grade, F(2, 510 = 7.5400), $p =$ 0.0001. One-way repeated measures ANOVAs determined that performance differed statistically significantly across representations for both Grade 5 (F(2, 124) = 35.977, $p < .0001$) and Grade 8 (F(2, 129) = 9.545, $p < .0001$).

Follow-up analyses treat each grade separately for each of the three representations. Using a Paired Samples T-Test, I first analyze non-routine problem performance within each

grade and between each representation. Recall that each representation had three non-routine problems included in this analysis. Table 2 presents means and standard deviations for performances by grade on each of the three representations. Grade 5 students showed a significant difference in performance between number line tasks and Cartesian plane tasks, $t(125) = 4.489$, $p < 0.0001$, a significant difference between Cartesian plane tasks and graphing linear functions tasks, $t(125) = 4.523$, $p < 0.0001$, and a significant difference between number line tasks and graphing linear function tasks, $t(125) = 7.689$, $p < 0.0001$. Grade 5 students' performances showed a significant decrease with each subsequent linear representation. Figure 24 presents means on non-routine problems by representation and grade.

Table 2: Means and standard deviations by grade and representation for non-routine problems on the LRA

Grade	Representation	Mean	Standard Deviation
5	Number line	1.722	1.143
	Cartesian plane	1.254	0.995
	Graphs of functions	0.873	0.971
8	Number line	1.931	1.053
	Cartesian plane	1.886	0.997
	Graphs of functions	1.565	1.016

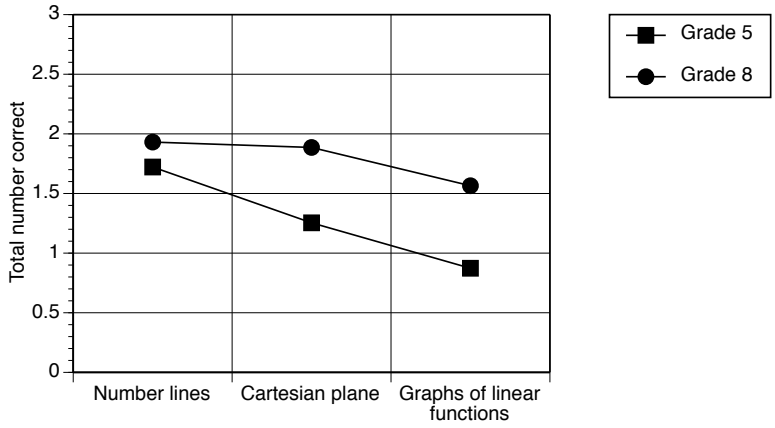


Figure 24: Non-routine problem performance by representation and grade

Grade 8 students showed no statistical difference in performance between number line tasks and Cartesian plane tasks $t(130) = 0.492$, $p = 0.624$, though there was a significant difference between Cartesian plane tasks and graphing linear functions tasks, $t(130) = 3.876$, $p < 0.0001$, and between number line tasks and graphing linear functions tasks, $t(130) = 3.752$, $p < 0.0001$.

Finally, I analyzed performance on each representation as a function of grade. I used Independent Samples T-Tests to analyze non-routine problem performance for each representation across the two grades. On non-routine number line tasks, there was no detectable difference in performance between students in Grade 5 and Grade 8, $t(255) = -1.525$, $p = 0.128$ (two-tailed). Non-routine problem performance by representation and grade was significantly different for Cartesian plane tasks and graphing linear functions tasks. On non-routine Cartesian

plane tasks, Grade 5 students performed at statistically lower levels than Grade 8 students, $t(255) = -5.080$, $p < 0.0001$, as they did for graphing linear functions tasks, $t(255) = -5.576$, $p < 0.0001$.

These analyses reveal that students' performance decreased on from routine to non-routine problems in each grade. Furthermore, Grade 5 students perform significantly better on number line tasks as compared to Cartesian plane tasks, whereas there was no detectable difference in performance among Grade 8 students for the same two representations. There was no detectable difference by grade in performance on number line items, though there was a significant difference on Cartesian plane and graphing functions items. I now turn to qualitative data to understand students' conceptual coordinations as they interpreted these items.

The character of student understanding. In this second section, I present qualitative results using interview data. In addition to clarifying quantitative results from the first section, a goal is to address the second research question regarding the character of students' conceptual coordinations in order to reveal patterns of understanding as they relate to the coordination of linear and numerical units.

Problems were selected for analyses that were both targets of interviews and that represented the targeted ideas of the LRA. I provide data on common[5] responses for target items; then, using interviews conducted with a subset of students, I also analyze students' justifications in terms of the coordination of linear and numerical units. Since only a subset of students was interviewed, the interviews do not reflect the conceptual coordinations of all students. Nonetheless, since selection criteria for interviewees considered common responses across a range of students with varied assessment scores, I treat interviews as an estimate of the overall character of student thinking on the LRA.

In this section, I first present a number line task, *1, 3; Place 0*, that assessed student understanding of interval and origin. I then turn to two Cartesian plane tasks. The first, *Scale with multiunit intervals*, assessed the identification of points along two axes. The second, *Identify 5, 5*, assessed the identification of a point in the plane. I then turn to two graphing functions tasks. The first, *Rate of change*, assessed students' use of linear and numerical units to determine rate of change. The second, *John and Mary's money*, assessed story-to-graph coordination. As displayed in Figure 25, many students in each grade responded incorrectly across these five problems, providing a rich source of data to document and understand students' conceptual coordinations as they engaged with these problems.

[5] For the purposes of this analysis, I define a common response as one that has a frequency count of 5 or greater.

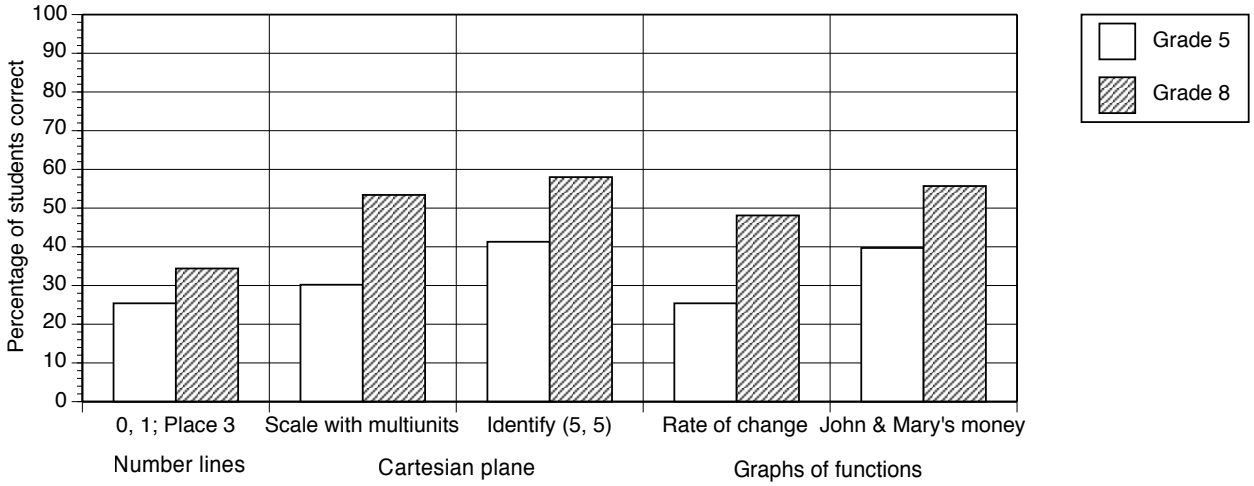


Figure 25: Percentage correct on target LRA items

Patterns of student understanding: Number line task. The number line item *1, 3; Place 0* assessed the coordination of linear and numerical units on a single number line. This item features three differently sized linear units along the line (Figure 26). Tickmarks are provided at the points -2, -1, 0, and 0.5. No tickmark is provided at the location of 2, requiring students first to locate a unit interval on the line in order to find the appropriate location for 0. The pass rate on this problem was 25.4% among Grade 5 students and 34.4% among Grade 8 students. Figure 26 presents the item with a coding scheme (top) together with the breakdown of responses by grade. The correct location, Tickmark C, is indicated with a box. I first provide data on the breakdown of response choices across students, and then I elaborate on students' coordination of linear and numerical units using the interview data. The most common incorrect response in both grades was to position "0" at Tickmark D, immediately to the left of the numerical unit "1"; 32.5% of Grade 5 students and 34.4% of Grade 8 did this. The second most common incorrect response was different for Grades 5 and 8 students. For Grade 5 students, 19.8% positioned 0 at Tickmark A, while 15.3% of Grade 8 students positioned "0" at Tickmark B.

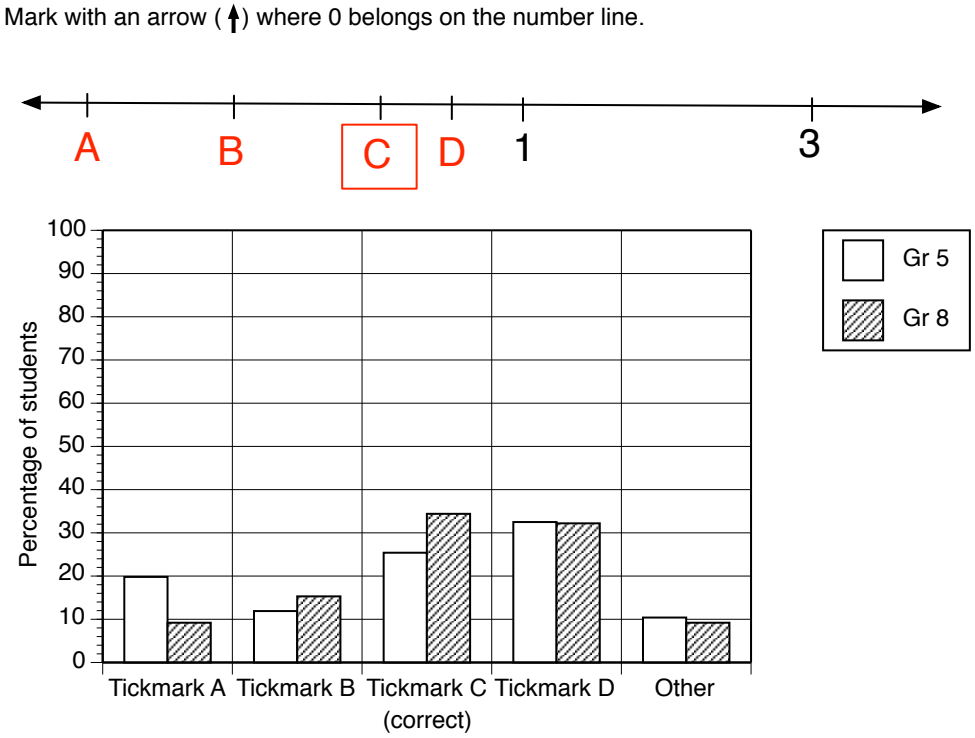


Figure 26: Item *1, 3; Place 0* with coding scheme (top) and percentage of students' solutions (bottom)

Interviews with Grade 5 (n=19) and Grade 8 (n=16) students revealed four distinct categories of conceptual coordinations: (a) coordination of linear units and numerical units; (b) linear units privileged; (c) numerical units privileged; (d) leftmost point as origin (linear and numerical units ignored), with some students coded as "other." Table 3 compares students' locations for 0 within each of the four categories.[6] Each conceptual coordination was determined to be 100%; for example, the first row of Table 3 indicates that of 13 students coded as coordinating linear and numerical units, 8% (one student) chose Tickmark B while 92% (12 students) chose Tickmark C. In Table 3, the value 1 indicates that 100% of interviewees coded as that conceptual coordination all located 0 at the same position on the number line (such proportions are reflected in three of the four categories). If students changed their response during the interview, coding reflects their oral justification up until the point at which they changed their response in the interview. If students changed their response upon being presented the task in the interview, that student was coded as "other."

[6] Proportions in Table 3 combine interview data for Grades 5 and 8. Similarities and differences between grades are discussed in the prose.

Table 3: Proportions of students' answer choices by coordination type on Item *1, 3; Place 0*

	Answer choice			
Conceptual coordination	A	B	C (correct)	D
Coordinated linear and numerical units (n=13)		.08	.92	
Linear units privileged (n=7)		1		
Numerical units privileged (n=5)				1
Origin (n=5)	1			
Other (n=5)	.2	.8		

For those students that coordinated linear units with numerical units on the line, 92% of these interviewees (n=12) accurately positioned 0 at Tickmark C. I first present detail from interviews for those placing 0 at Tickmark, followed by detail for the student placing 0 at Tickmark B. For those placing 0 at Tickmark C, students coordinated the linear units and numerical units on the line in order to find the unit interval—in all cases, this was enacted by first locating "2" on the line—and then iterating this unit interval to the left of 1 to locate "0" at Tickmark C. For example, ID 5420 gestured to the tickmark she drew in at the midpoint between "1" and "3" and stated (Figure 27), "I made this mark, that would be 2. And it *<the distance between 1 and 2>* had that much space. So I measured that much space, and 0 would be here *<indicates Tickmark C>*." Grade 8 interview responses had a similar character. One student, ID 8815, indicated attention to the spaces while gesturing to linear units on the line. He stated, "I put 0 because, just like, the amount of space… So the space in between the two numbers." All interviewees that accurately located 0 at Tickmark C were all coded as coordinating linear units and numerical units.

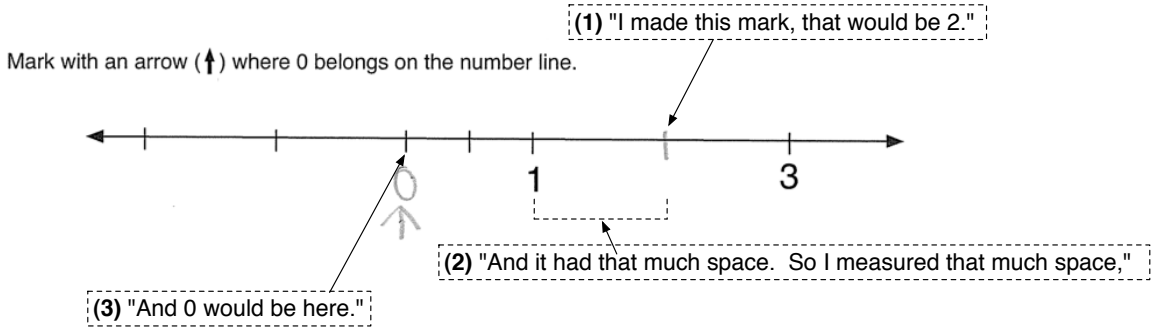


Figure 27: ID 5420's annotated response to *1, 3; Place 0*

One Grade 8 student, ID 8611, that was coded as coordinating linear units and numerical units placed 0 at Tickmark B. In her justification, she indicated congruent linear units while at the same time drawing attention to a pattern of odd numbers she found in the numerical units. She stated, "Because right here *<Tickmark B>* 0… It would be like odd numbers, so the lines

would be the same." Her focus is both on the pattern of odd numerical units while at the same time stating that the linear units ("the lines") would be the same. As noted below, all other students that located 0 at Tickmark B privileged linear units on the line independent of numerical units.

Seven interviewees—four in Grade 5 and three in Grade 8—were coded as privileging linear units in their justifications. Of these students, 100% positioned 0 at Tickmark B. In these cases, students used the linear unit from 1 to 3—while treating this as a unit interval—and iterated that linear unit to the left to position 0 at Tickmark B. For example, ID 8729 stated, "I just sort of looked at the spacing for this one. And it looked liked these were not equal spaces, so I wouldn't put 0 there <*indicating Tickmark D*>, and I would put 0 here <*indicating Tickmark B*>." Of these seven students, the majority (4) changed their response to the correct answer, Tickmark C, after providing an initial justification for Tickmark B. For example, ID 5203 stated (Figure 28), "I put it here <*indicating Tickmark B*> because the numbers are all evenly spaced. So it goes, 0, 1, 3… Oh, I get it. I think it should be 0 here <*indicating Tickmark C*>, and then one, *two*, three," emphasizing the two as he spoke.

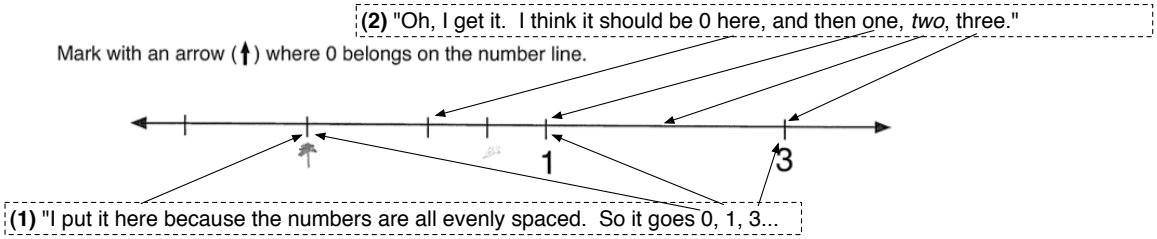


Figure 28: ID 5203's annotated response to *Item 1, 3; Place 0*

Six students were coded as privileging numerical units, all of whom were in Grade 8.[7] Of these students, all positioned 0 at Tickmark D. Students privileging numerical units to place 0 at Tickmark D coordinated the "1" provided on the line with placing 0 at the next available tickmark to the left. In these efforts, all students adhered to the order principle of number lines while ignoring linear units on the line. For example, ID 8617 positioned 0 at Tickmark D and stated, "I just marked it because it was next to the 1. And zero is supposed to go here so it can go 0, 1, 2, 3 <*gesturing to each number on the line*>."

I coded five students (four in Grade 5 and one in Grade 8) as using their understanding of origin—or 0 as a starting point on the line—to position 0. Of those interviewees, 100% positioned 0 at Tickmark A. In the justifications, students' positioning of 0 was independent of both linear units and numerical units already on the line. For example, ID 5405 stated, "I marked that there because I just thought it was the start" (Figure 29). Other students coded as origin provided similar justifications that referenced Tickmark A as a starting point.

[7] No Grade 5 students that located 0 at Tickmark D were interviewed for this study. Because there was no detectable difference in performance on number line tasks as a function of grade, I assume Grade 8 students' coordinations are consistent with those students in Grade 5 that also chose Tickmark D.

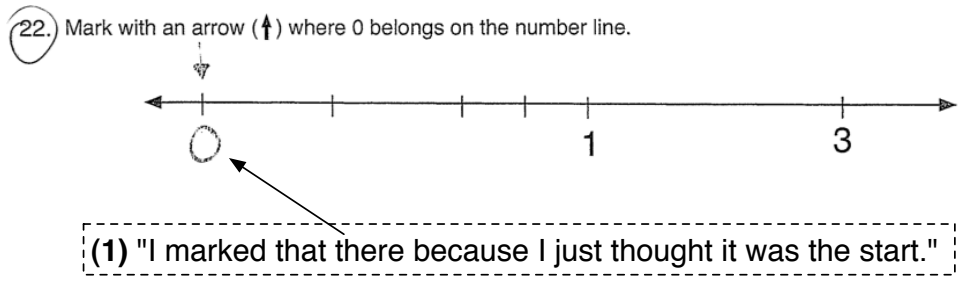


Figure 29: ID 5405's annotated response to Item *1, 3; Place 0*

Five additional students were coded as "other". These students changed their response when presented with the task during the interviews.

This analysis of interview data revealed the following trends. First, the coordination of linear units and numerical units was almost always consistent with accurate placement of the target number on the number line. Second, students in both grades privileged either linear units or numerical units. In all cases, these students placed 0 at an incorrect location. Third, many students in Grade 5 (19.8%) and a smaller percentage of Grade 8 students (9.2%) positioned 0 at Tickmark A; interviews indicate that all students doing this applied a partial understanding of origin, specifically that the leftmost point of a number line is the start point.

Patterns of understanding: Cartesian plane tasks. I now turn to two Cartesian plane items using assessment and interview data. Grades 5 and 8 students performed differently on these tasks according to quantitative findings; this implies interview data would reveal the character of such differences. I present here two items: *Scale with multiunit intervals* and *Identify 5, 5*.

Scale with multiunit intervals. I first analyze the item *Scale with multiunit intervals*, which featured a horizontal axis with 0 and 2, thereby providing an interval of 2. A box is positioned at the location for 3 (see Figure 30). Unlike prior tasks, gridlines are provided at intervals of 0.5 along the horizontal axis. The vertical axis features 0 and 4 on it, thereby defining a multiunit interval of 4, with a box at the location of 6. Gridlines are provided at intervals of 2. I focus here on the response for the horizontal axis. The accurate response of "3" was provided by 41.3% and 60.3% of Grades 5 and 8 students, respectively. This item featured a write-in response (as opposed to multiple choice), and for that reason many different responses are represented in the data. With percentages presented as Grade 5 and Grade 8 (as shown at the bottom of Figure 30), incorrect responses included: 2.5 (4.8% and 11.5%), 4 (38.9% and 18.3%), 6 (5.6% and 4.6%), and other (9.5% and 5.4%). I provide here further detail on response patterns across students.

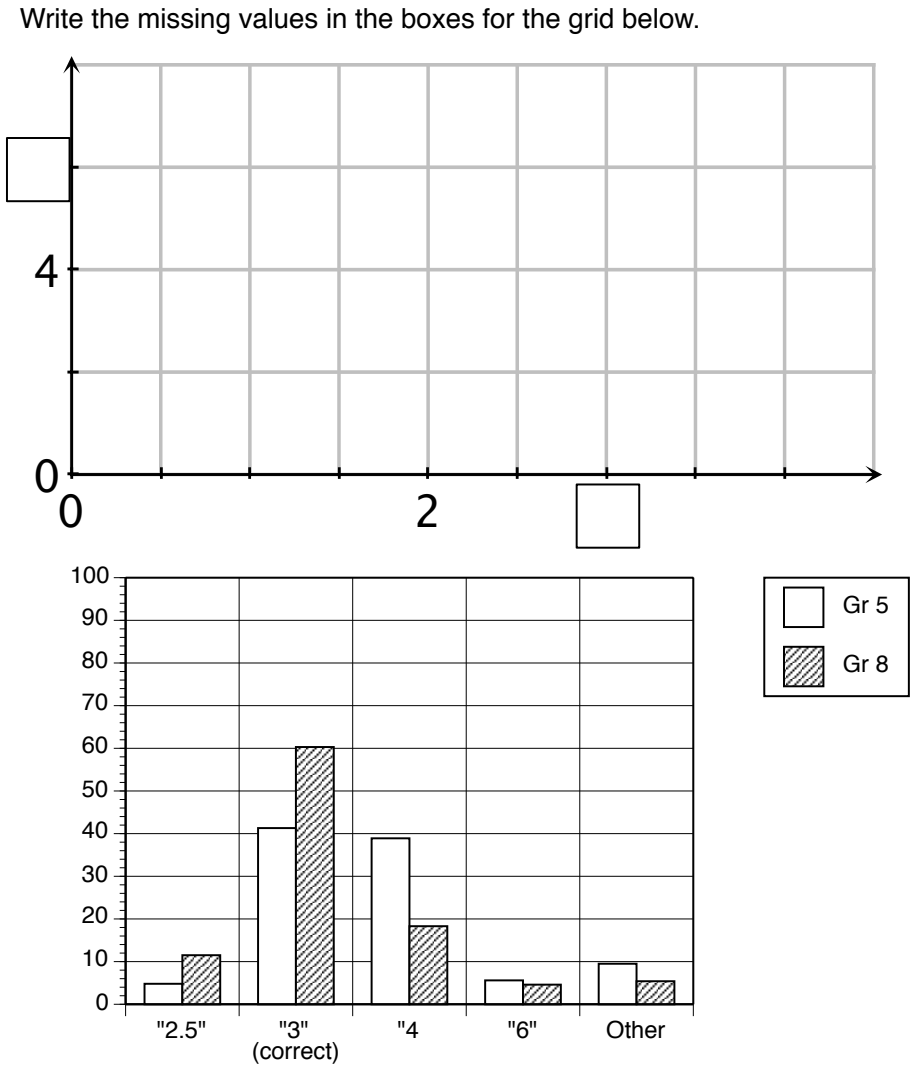


Figure 30: Item *Scale with multiunit intervals* (top) and percentage of students' solutions for the horizontal axis (bottom)

Interviews with Grade 5 (n=21) and Grade 8 (n=10) students revealed four distinct categories of conceptual coordinations: (a) coordination of linear and numerical units; (b) numerical units privileged; (c) matched axes; (d) partial coordination of linear and numerical units; with some students coded as "other" (typically these students changed their response as they began to explain). Table 4 displays the proportions of students' answer choices by these four categories; the value 1 indicates that 100% of interviewees coded as that conceptual coordination wrote the same value. If students changed their responses during the interview, coding reflects their oral justification up until the point at which they changed their response.

Table 4: Proportion of coordination types by answer choice on Item *Scale with multiunit axes*

	Responses				
Conceptual coordination	2.5	3 (correct)	4	6	other
Coordinated linear and numerical units (n=9)		1			
Numerical units privileged (n=4)			.75	.25	
Matched axes (n=5)			.6	.2	.2
Partial coordination (n=5)	1				
Other (n=8)	.375	.375	.125		.125

I first present the coordination of linear units and numerical units. All nine students (six Grade 5 and three Grade 8) coded as making this coordination accurately wrote "3" in the box. No cases of coordinating linear units and numerical units resulted in another response. For example, ID 5213 explained (Figure 31), "I looked here and said, this is 2, so this [distance from 0 to 2] needs to divide into two equal parts. So this is 1 <*indicating the accurate position for 1*>. So then 1, 2, 3 <*indicating the 3 in the box.*>. Other students responding with 3 followed similar patterns.

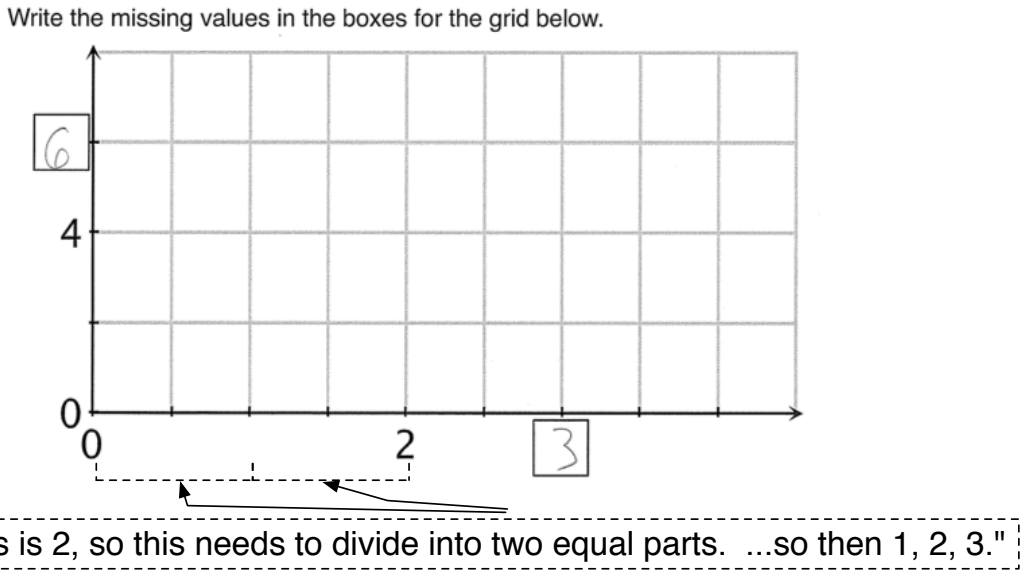


Figure 31: ID 5213's annotated response to Item *Scale with multiunit intervals*

A second coordination category privileged numerical units. Four interviewees—three Grade 5 and one Grade 8—were assigned this category. All produced incorrect responses.. Three interviewees coded as this wrote "4" as the value on the horizontal axis, with one Grade 5 student writing "6." In three of the four cases, students extended consecutive numerical units from 2 by counting on by each tickmark (to write "4" in the box). For example, ID 5103 wrote

"4" in the box. In the interview, he first indicated the 2 and then assigned a consecutive integer to tickmarks moving to the right, stating, "It has a 2 here. So I thought it would be 2, 3, 4." ID 5109 explained, "So I started counting 2, then I got to 4." In the remaining case, ID 5421 observed the numerical units 0 and 2, then counted on by 2 using each tickmark to the right of 2: "They started counting by 2s, and made a 4 here <*to the right of "2"*>, then a 6 <*in the box*>."

A third coordination category matched numerical units across the two axes. All produced incorrect responses. Five interviewees—two Grade 5 students and three Grade 8 students—were coded as Matched Axes. Of these students, the majority (60%) wrote "4" in the box along the horizontal axis, with one student writing "6" and another writing "I don't know." In these cases, students' justifications for values along the horizontal axis were inextricably linked with numerical units along the vertical axis, indicating that for these students the two number lines were not independent from one another. For example, ID 8624, who wrote "4" along the horizontal axis and "6" along the vertical axis, indicated that she coordinated a *counting on by 2s* strategy along both axes. Beginning with the vertical axis, she stated (Figure 32), "Since I saw how the 4 started off here, it went up by 2s." Then, indicating the horizontal axis, she explained, "So I decided to go up by 2s here." In going up by 2s, she ignored the tickmark between "2" and the box. Similarly, ID 8815, who had the same solution as ID 8624, indicated she went up by 2s along each axis, beginning with the horizontal axis (Figure 33): "I added. 0 plus 2 is 2, then 2 plus 2 is 4. And I was thinking that it <*indicating the tickmark on the vertical axis between 0 and 4*> would be 2 here because it's the same. Adding 2. Like, 4 plus 2 equals 6."

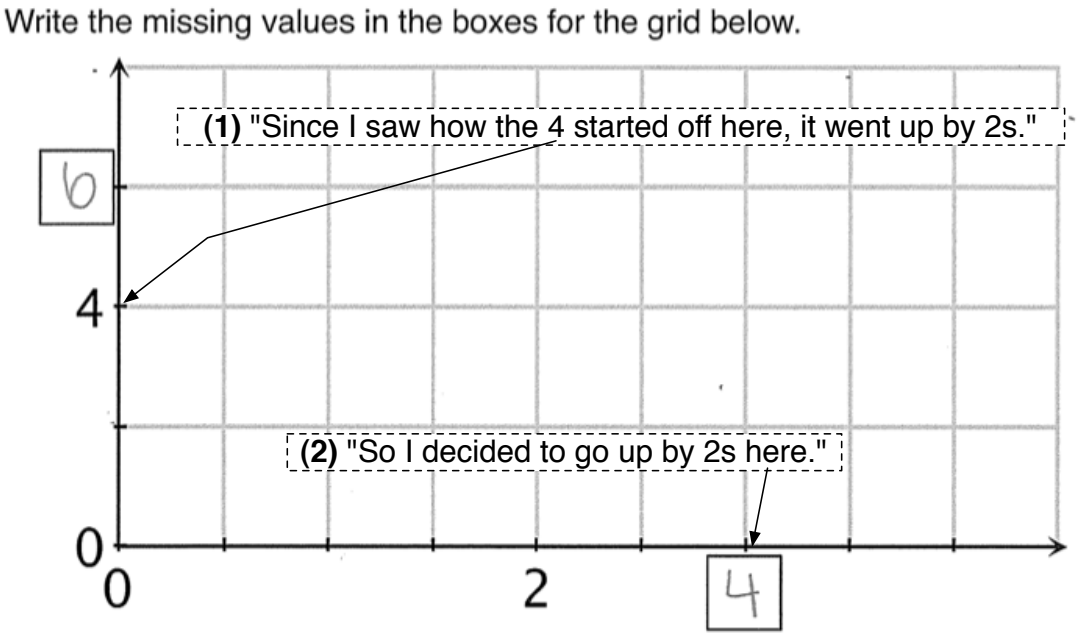


Figure 32: ID 8624's annotated response to Item *Scale with multiunit intervals*

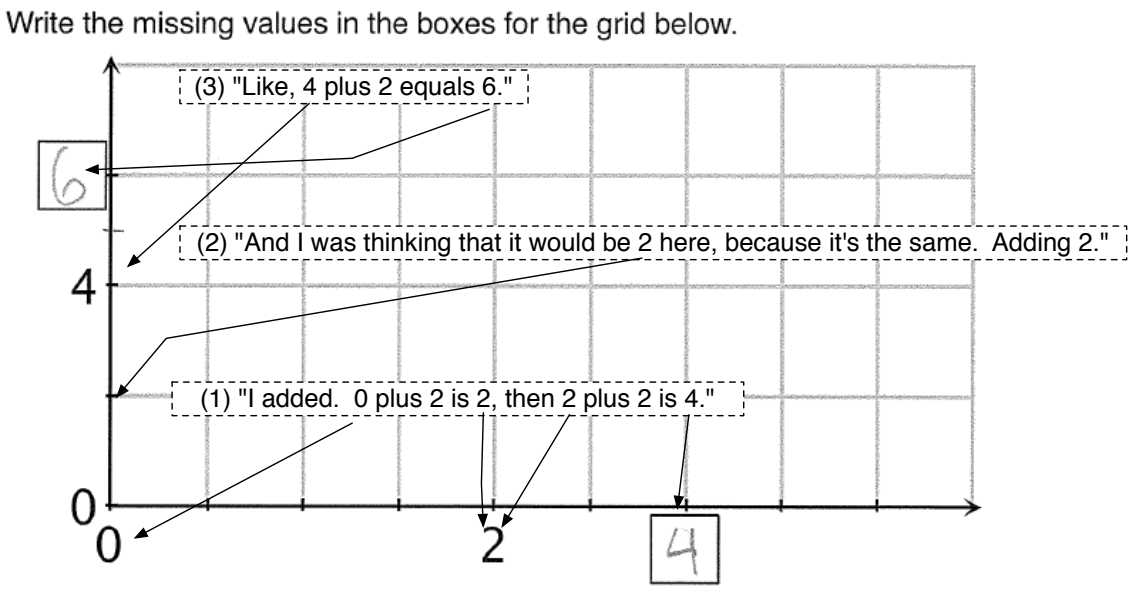


Figure 33: ID 8815's annotated response to Item *Scale with multiunit intervals*

Two students coded as Matched Axes expressed confusion about the value on the horizontal axis; in both cases, each student first accurately identified the value on the vertical axis by identifying a counting pattern of 2. For example, ID 5207 wrote "6" on the vertical axis and "I don't know" below the box for the horizontal axis (Figure 34). Explaining the "6" along the vertical axis, she stated, "I think it <*the tickmark between 0 and 4*> was a 2, and I thought it was going up by 2s." Turning to the horizontal axis, she identified the 2, stating, "Well, there was only a 2. And I didn't really get how this [pattern along the vertical axis] would relate to the 2 <*indicating the 2 on the horizontal axis*>, so I didn't really get it." ID 8619 wrote in "6" on both axes. In his explanation (Figure 35), he first stated broadly, "I counted by 2s." Clarifying questions indicated that this student coordinated the extension of numerical units on both axes. About the counting by 2s, the interviewer asked, "Where?" The student gestured to the vertical axis and stated, "Here. It was 2, 4, 6." When asked about the horizontal axis, also a "6", he stated, "I put the 6 here because I started counting 2, 4, 6. But I don't know why the 2 was there <*gesturing to the position of 2*> instead of here <*indicating a position closer to 0*>." This particular student coordinated linear and numerical units along the vertical axis. Nonetheless, he then repeated the numerical pattern on the horizontal axis, yet with expressed dismay over the relationship between this pattern and the given position of 2 along the axis. Another grade 5 student, ID 5101, mentioned a "counting by 2s" pattern but indicated she did not recall her rationale for solving the problem. In her written work, she wrote 4 along the horizontal axis and 8 along the vertical axis (Figure 36). Of note is that she connected two values on the horizontal axis with unique values on the vertical axis, and in each case the numerical unit on the vertical axis is double that on the horizontal. These inscriptions indicate she may have found a pattern across axes.

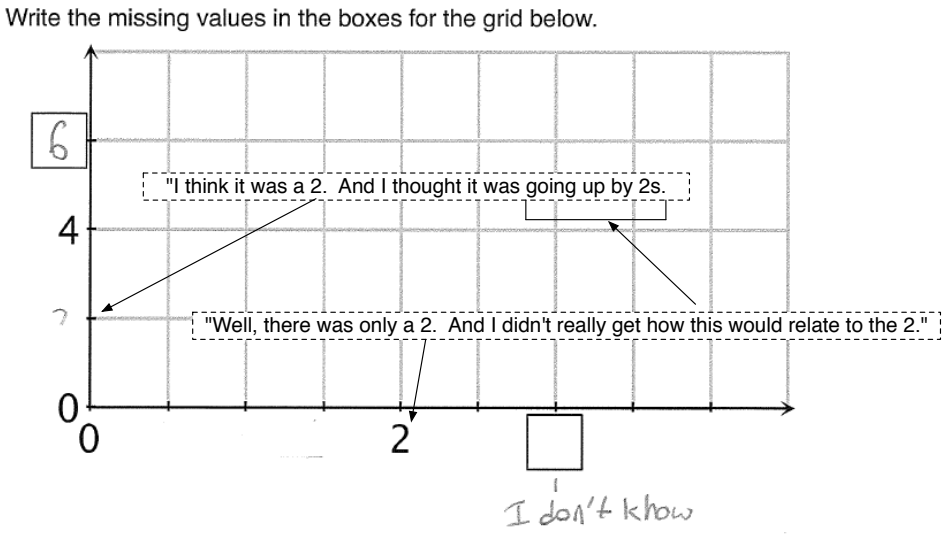


Figure 34: ID 5207's annotated response on item *Scale with multiunit axes*

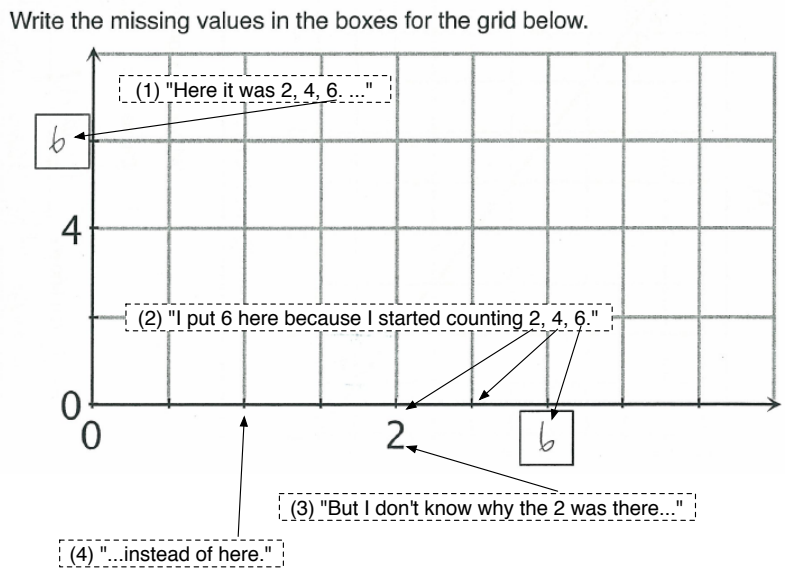


Figure 35: ID 8619's annotated response to Item *Scale with multiunit axes*

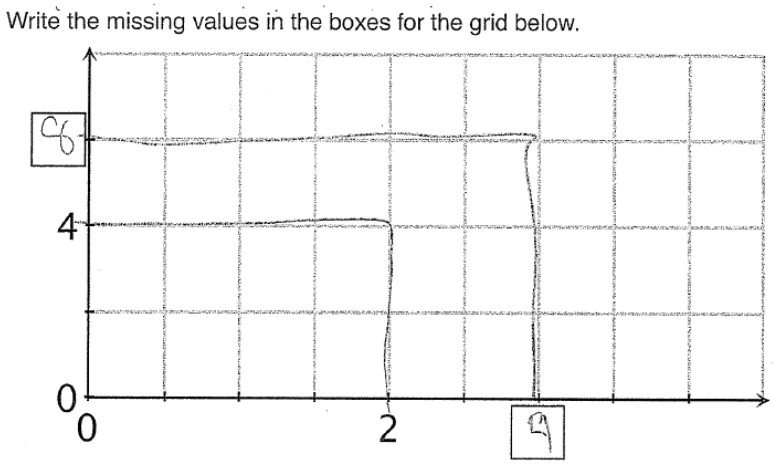


Figure 36: ID 5101's response to item *Scale with multiunit axes*

A fourth category featured a partial coordination of linear units and numerical units. Students coded as partial coordination made some efforts to coordinate linear units and numerical units, yet treated each interval of 2 as a unit interval. Four Grade 5 students and two Grade 8 students were coded as partial coordination. Each student wrote "2.5" as the missing value. For example, ID 5209 wrote 2 ½ in the box, and in her justification identified the non-labeled tickmarks between 0 and 2 as decimals (Figure 37): "…I was thinking that 0 <*indicating 0 on the horizontal axis, then counting tickmarks to the right*> decimal, decimal, decimal, 2." Each time the student said "decimal," she pointed to the next consecutive tickmark until reaching the 2 already on the graph. She elaborated further on the first two tickmarks to the right of zero, followed by the first two tickmarks to the right of 2: "And this would be a quarter, and another quarter. And then the same here," indicating that each tickmark to the right of 2 represented a quarter. She thereby treated this interval as a unit interval. This partial coordination is corroborated with justifications from other students that wrote the same value. For example, ID 5215 explained "2 ½" by stating "Let's say 0 takes 1, 2, 3, 4 hash marks <*indicates each hash mark as he counts*>. This [interval after 2] takes 1, 2, 3, 4 [hash marks], and this [box] is the second one. So it's 2 ½." (Figure 38 shows ID 5215's work with an illustration for his count of tickmarks). By calling the value in the box 2 ½, students adopted a perspective of a number line as a series of unit intervals that count on by 2s.

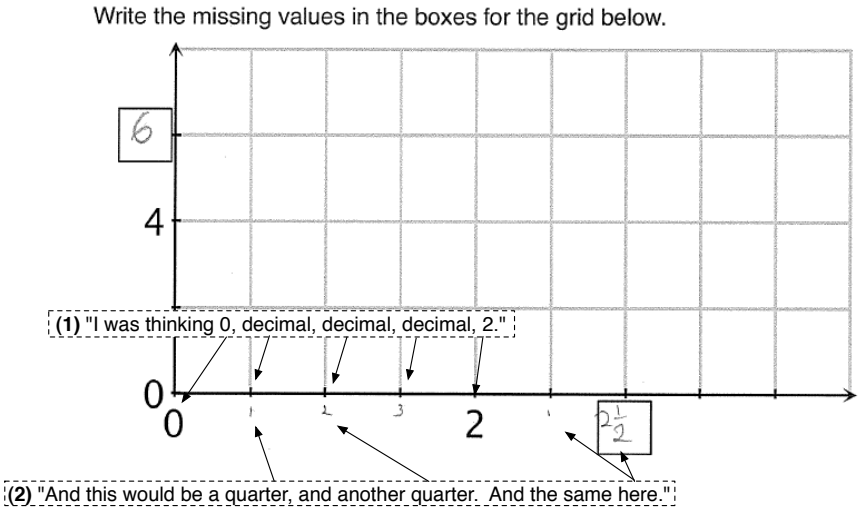


Figure 37: ID 5209's annotated response to item *Scale with multiunit intervals*

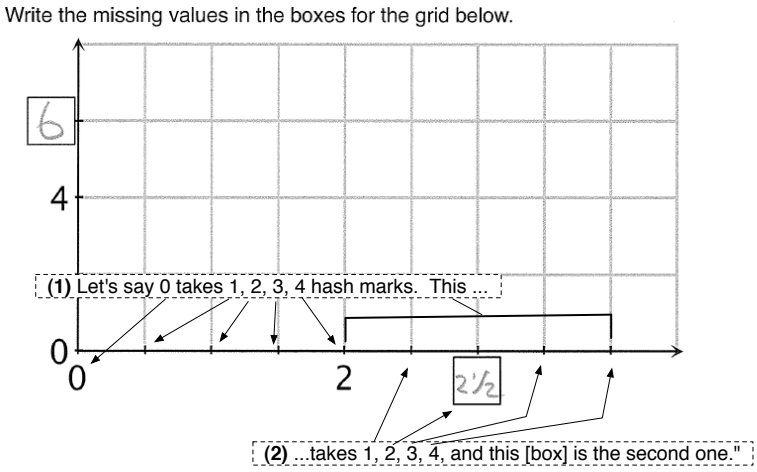


Figure 38: ID 5215's annotated response to item *Scale with multiunit intervals*

Analysis of interview data for the item *Scale with multiunit intervals* revealed the following trends. First, the coordination of linear units and numerical units was consistent with accurate naming of the missing value along the horizontal axis. Second, students in both grades privileged numerical units while ignoring linear units; in all cases, this resulted in an incorrect value in the box. Third, some students matched numerical units across the two axes, thereby treating them as dependent on one another. Fourth, some students partially coordinated linear units with numerical units, but in this treated each multiunit of 2 as a unit interval. Interview data did not reveal differential patterns as a function of grade.

Identify 5, 5. I now turn to a second item, *Identify 5, 5*, which featured a multiunit interval of 2 along each axis and required students to name a point that did not fall along a gridline. Figure 39 displays solutions by grade; 41.3% and 58% of Grades 5 and 8 students responded accurately with (5, 5). The most common incorrect response in both grades was (3.5, 3.5), which 18.3% and 17.6% of Grades 5 and 8 students wrote. In addition, a minority of Grade 8 students, 8.4%, provided the response (4.5, 4.5), as did 3.2% of Grade 5 students. A large minority of responses appeared idiosyncratic in that each had a frequency fewer than 5; these responses were

coded as other. This category, however, accounted for 30.2% and 14.5% of Grades 5 and 8 students. I provide here further detail on response patterns across students.

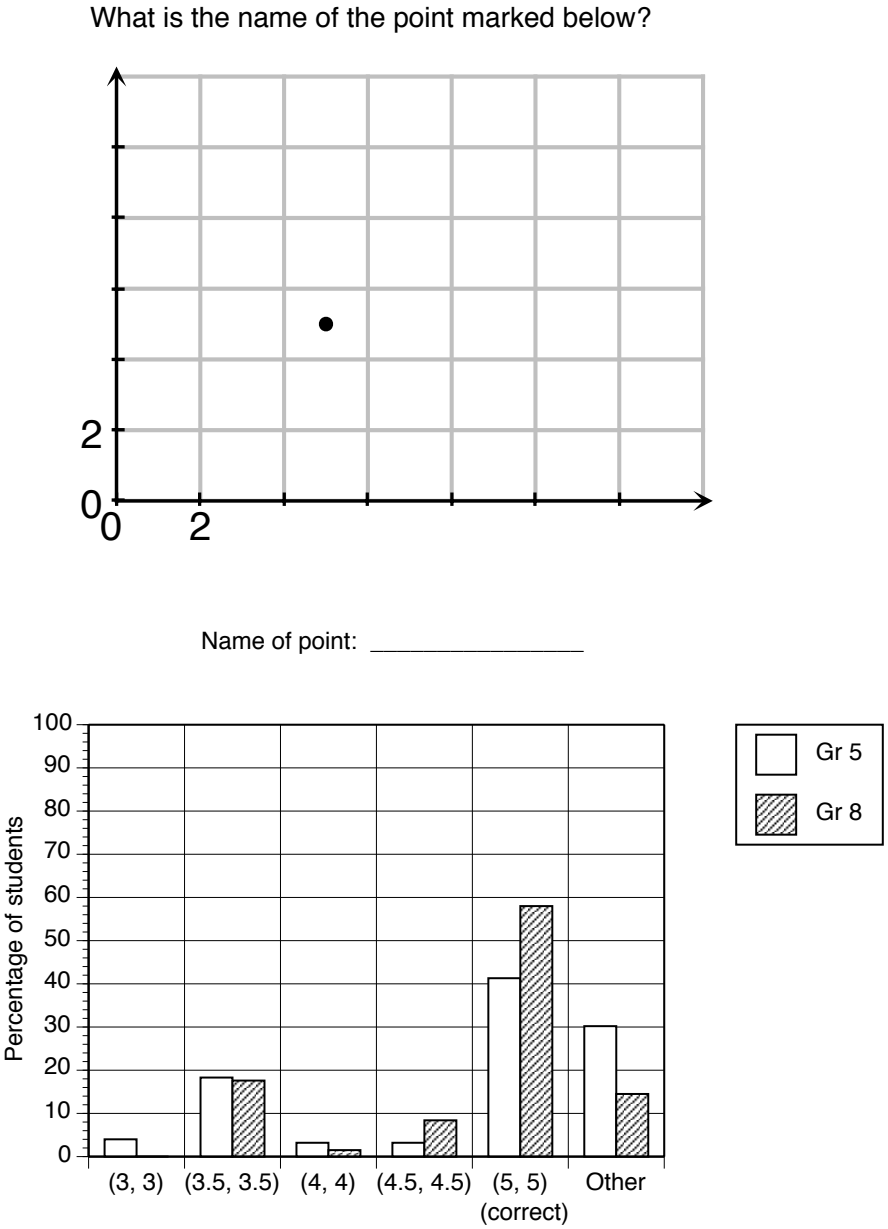


Figure 39: Item *Identify (5, 5)* (top) and percentage of students' solutions (bottom)

Interviews with Grade 5 (n=20) and Grade 8 (n=11) students revealed three categories of conceptual coordinations: (a) coordination of linear units and numerical units; (b) numerical units privileged; (c) partial coordination of linear units and numerical units; with remaining students coded as "other". These conceptual coordinations were associated with particular response choices. Table 5 displays the proportions of students' coordinations by answer choice. For students coded as a, b, or c, all named a point in the plane based on linear projections from each axis.

Table 5: Proportion of coordination types by answer choice on item *Identify 5, 5*

	Responses			
Both grades	3.5, 3.5	4.5, 4.5	5, 5 (correct)	Other
Coordinated linear and numerical units (n=11)			1	
Numerical units privileged (n=9)	1			
Partial coordination (n=4)		1		
Other (n=7)		.29	.29	.43

For those students that coordinated linear units with numerical units, 100% of these interviewees (n=11) accurately named the point (5, 5). This included eight Grade 5 students and three Grade 8 students. These students either first located the unit interval by partitioning the interval from 0 to 2 in half, and then iterated the unit interval to locate 5 on each axis; or, students iterated the multiunit of 2 to the right to locate 4, and in some cases 6, along each axis, and then found the unit interval to locate 5 on each axis. For example, a Grade 5 student, ID 5219, explained the horizontal axis (Figure 40), stating, "Here it is 0 and it jumps right to 2 without a line in the middle, and the same here <*indicating the interval from 2 to the next tickmark*>. So right in the middle [of 0 and 2] should be a 1, and then that would be a 3, and then 4, and then right there would be 5. And the same over here <*indicating the vertical axis*>." Similarly, Grade 8 students responding with (5, 5) indicated the role of linear units together with numerical units in their explanation. For example, ID 8611 stated, "I was thinking the 0 and the 2. So we skipped one. So the one must be in the in between. So I thought the 3 must be right there [between 2 and the next tickmark]. And the 4 right here, then the 6. Then right here [between 4 and 6] the 5. So it would be 5 and 5."

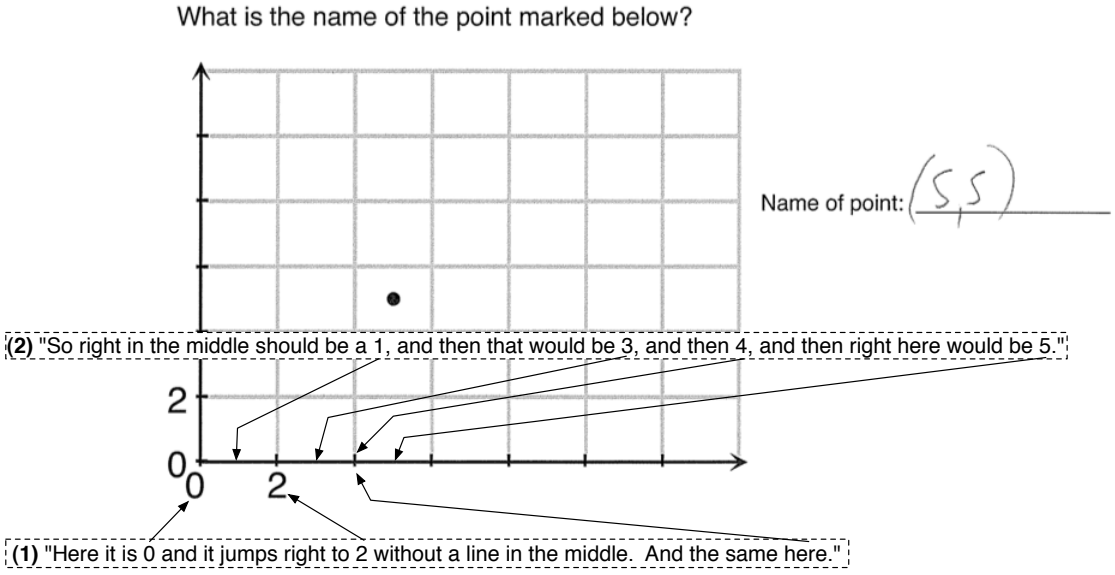


Figure 40: ID 5219's annotated response to Item *Identify 5, 5*

A second category emerging from interviewees was to privilege numerical units (n=9) without attention to linear units. All interviewees coded as privileging numerical units named the point (3.5, 3.5), including six Grade 5 students and three Grade 8 students. One Grade 5 student, ID 5207, captured eloquently the spirit of other interviewees on this answer: "I thought it was the next number, was 3. Then it was 4. And 5, and 6. That little dot was in the middle, and it was right in between 4 and 3." She articulated a solution strategy that her peers shared; namely, extending numerical units by counting on from 2, and naming the point based on its middle-ness between 3 and 4. The three Grade 8 students likewise shared similar explanations. ID 8617 counted on from the 2 on either axis, stating, "Because it's going 2, 3. You have to make it into a decimal because it's in the middle." Interviewed students with this response counted on from 2 using consecutive integers. They did not, in any overt way, coordinate the 0 and 2 along either axis. Instead, they used the 2 to count on by ones, using each tickmark as a place for each consecutive tickmark.

A third category that emerged from four interviewees (two Grade 5 students and two Grade 8 students) was a partial coordination of linear units and numerical units. In all four cases, interviewees named the point (4.5, 4.5). In all four justifications, students counted on by 2s using tickmarks and then treated the multiunit intervals of 2 as unit intervals. One Grade 8 student, ID 8610, explained the point (4.5, 4.5) as, "Because it goes by 2s. It's 2, 4, and one half." In a similar way, ID 5314 indicated the 2 along the horizontal axis and stated, "So that's a 2. That <*midway between 2 and the tickmark to the right*> would be 2 ½, and that <*the tickmark to the right*> would be 4, because it's counting by 2s. So it would be 4 ½ here. It goes up, and 4 ½ over here, and they meet at the 4 ½ point on both of [the axes]" (Figure 41). Students responding (4.5, 4.5) on this task made some effort to coordinate linear and numerical units, yet at the same time treated each of linear unit as a unit interval rather than an interval of 2.

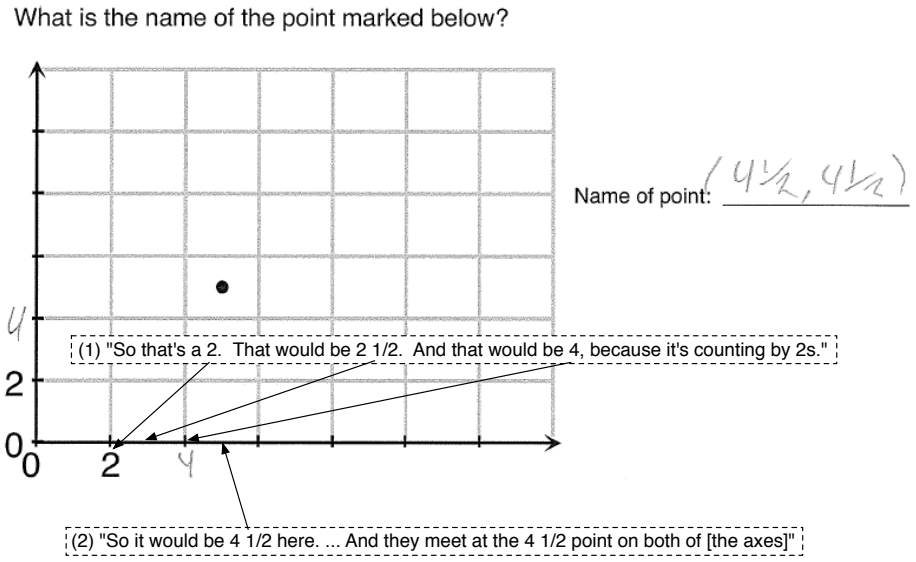


Figure 41: ID 5314's annotated response to Item *Identify 5, 5*

Analysis of interview data for the item *Identify 5, 5* revealed the following trends. First, the coordination of linear units and numerical units was consistent with accurate naming of the point. Second, students in each grade privileged numerical units; in each case, this resulted in mis-identification of the point in the plane. Third, some students partially coordinated linear

units with numerical units, but treated each linear unit as the unit interval. Fourth, students in each grade named a point in the plane based on linear projections from each axis.

Patterns of understanding: Graphing linear functions tasks. I now turn to the final linear representation of quantities in this study, graphs of linear functions. I present here an analyses on two problems: *Rate of change* and *John and Mary's money.*

Rate of change. I present here interview data on the item *Rate of Change*, which featured two grids each with a linear function (see Figure 42). Each grid has a different scale and features the same f(x) = 2x function, yet the different scale renders as different the orientation of the line in the plane. The prompt asked, "Who walks more miles each hour?" Students could choose among three options: *Graph A: Laura*, *Graph B: Maggie*, and *They walk the same rate.* On this problem, 25.4% and 48.1% of Grades 5 and 8 students chose the correct option, *They walk the same rate*. Among incorrect responses, Grade 5 students were likely to choose *Graph A: Laura* (49.2%) over *Graph B: Maggie* (24.6%), whereas incorrect responses from Grade 8 students were split between *Graph A: Laura* (22.9%) and *Graph B: Maggie* (28.2%). Figure 42 provides frequencies of responses for item *Rate of change*. I provide here further detail on response patterns across students.

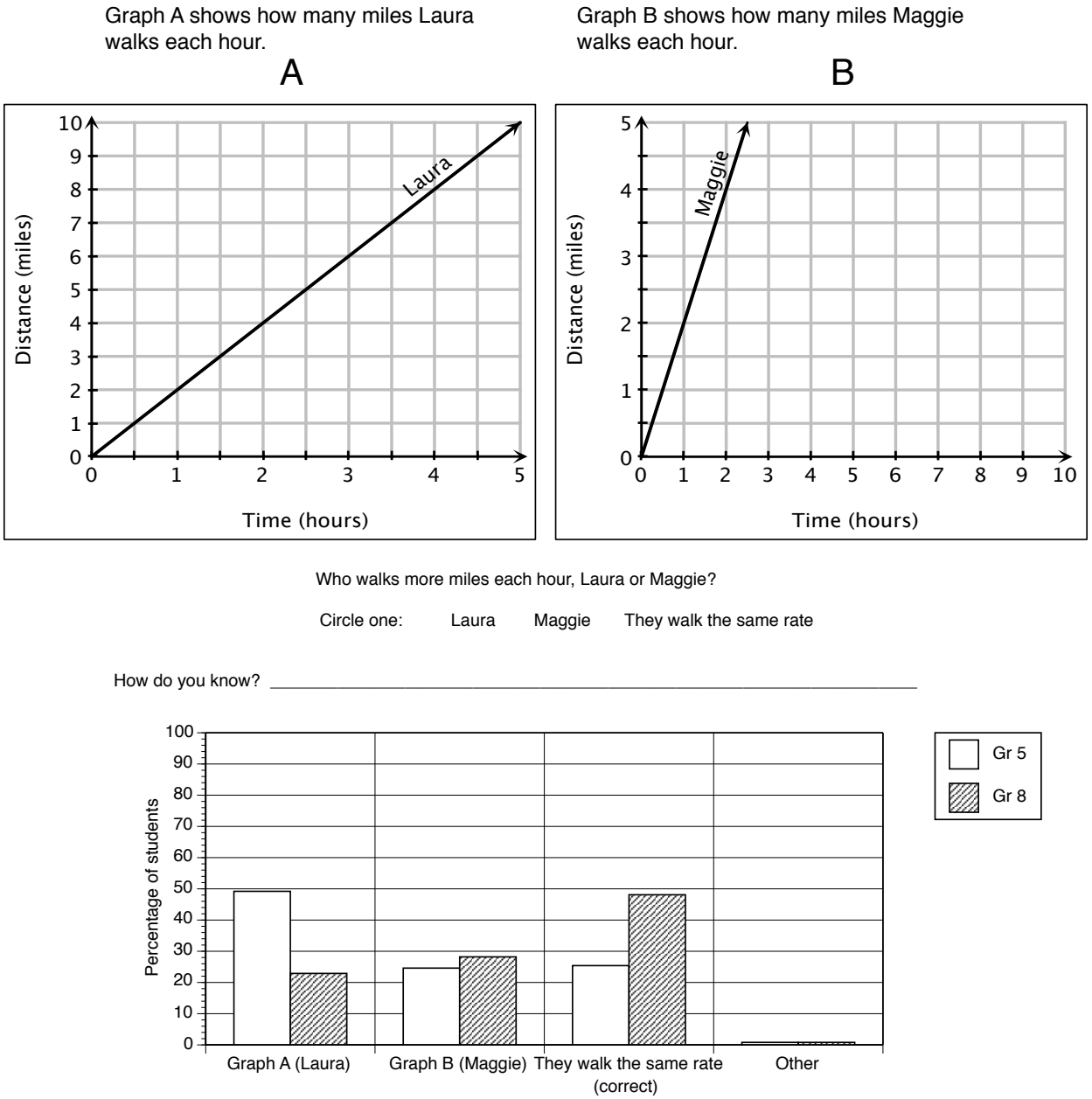


Figure 42: Item *Rate of change* (top) and percentage of students' solutions (bottom)

Interviews with Grade 5 (n=24) and Grade 8 (n=16) students revealed four distinct categories of conceptual coordinations: (a) coordination of linear units and numerical units; (b) linear units privileged; (c) numerical units (one quantity only) privileged; (d) and orientation of the line in the plane; with some students coded as "other." Table 6 displays the proportions of answer choices by coordination; the value 1 indicates that 100% of interviewees coded as that conceptual coordination all chose the same response. If students changed their response during the interview, coding reflects their oral justification up until the point at which they changed their response.

Table 6: Proportion of coordination types by answer choice on item *Rate of change*

Conceptual coordination	Laura	Maggie	They walk the same rate (correct)	Other
Coordinated linear and numerical units (n=17)			1	
Linear units privileged (n=6)	.5	.5		
Numerical units (one quantity only) privileged (n=7)	.71	.29		
Orientation of the line (n=5)	.4	.6		
Other (n=5)	.4	.6		

I first present the coordination of linear units and numerical units. One hundred percent of students coded as making this coordination (13 Grade 5 students and four Grade 8 students) accurately chose *They walk the same rate*. No cases of coordinating linear units and numerical units resulted in another response choice. Data indicate three ways in which students coordinated linear and numerical units. First, some students compared a single data point (e.g., (1, 2)) on each graph. For example, ID 5116 first explained Laura's graph: "Since you can see that it stopped here <*indicating the point (1, 2)*>, and <*moving over to Maggie's graph*> I saw at 1 [hour] it stops here <*indicating the point (1, 2) on the function line*>, then goes straight across to 2." Second, other students compared the point on the function line that corresponded to the greatest value along the *y*-axis. For example, ID 5216 explained, "Laura walks 10 miles every 5 minutes, and Maggie walks 5 minutes every 2 and a half miles (sic). They <*referring to the two graphs*> are just different numbers, but they're equivalent. It's just that the graph is showing different numbers." Third, other students determined the distance walked for each hour. For example, ID 5314 determined the rate of change: "Because they walk about 2 times as many miles as the time it took them." In each of these cases, numerical units along the horizontal axes had to be coordinated with precise points along the function line to find corresponding numerical units on the vertical axis; to do so, students had to coordinate numerical units with corresponding linear units on each grid.

A second coordination category was to privilege linear units independent of numerical units. Six interviewees (two Grade 5 students and four Grade 8 students) were coded as privileging linear units. In each grade, half of the students coordinated linear units to choose *Graph A: Laura*, while the other half coordinated linear units to choose *Graph B: Maggie*. In all cases regardless of answer choice, students treated linear gridline units as unit intervals independent of numerical inscriptions along the axes. For example, ID 5421, who chose *Graph A: Laura*, determined that Maggie walked ¼ of a mile each hour; she did this by partitioning the first horizontal gridline unit into four congruent linear units of ¼, and then treating each linear unit along the vertical axis as a unit interval (Figure 43). In the case of the horizontal axis, treating each linear unit as a unit interval was accurate. In the case of the vertical axis, this was inaccurate; first linear unit is an interval of 0.5 (see Figure 43, right side). Other students privileged linear units to choose *Graph B: Maggie*. For example, ID 8730 treated each linear unit along the horizontal axis of Graph A (left) as the unit interval, ignoring numerical units inscribed

along that axis. After correctly asserting that Maggie walks 2 miles in one hour, he stated that Laura walks a mile per hour. He stated (Figure 44), "I kind of looked at their, the lines, and their formation… How it goes through [Laura's] every margin, like every coordinate. So each coordinate <*indicating where the function line intersects with two gridlines*>, it <*the function line*> goes through all of them like that kind of, like straight…. So in an hour she walks one mile." Like the prior student, ID 8730 does not consider values along the axes in her response; instead, she treats inscriptions in the plane independently from numerical units, with her efforts coordinating gridlines with the function line. Similarly, ID 8724 wrote as her explanation, "Maggie's graph has a slope of 2, so she walks 2 miles each hour, and Laura's graph has a slope of 1, so she walks 1 mile each hour." In the interview, she further explained, "To find the slope it's rise over run. So when it meets a point for Laura, it goes up one and across one. So it goes up 1 each time. And this one <*indicating Maggie's graph*> is in units of 2, so when you go up one you go over 2 (sic), so it's a slope of 2" (Figure 45). This student treated each linear unit on Graph A as a unit interval of 1.

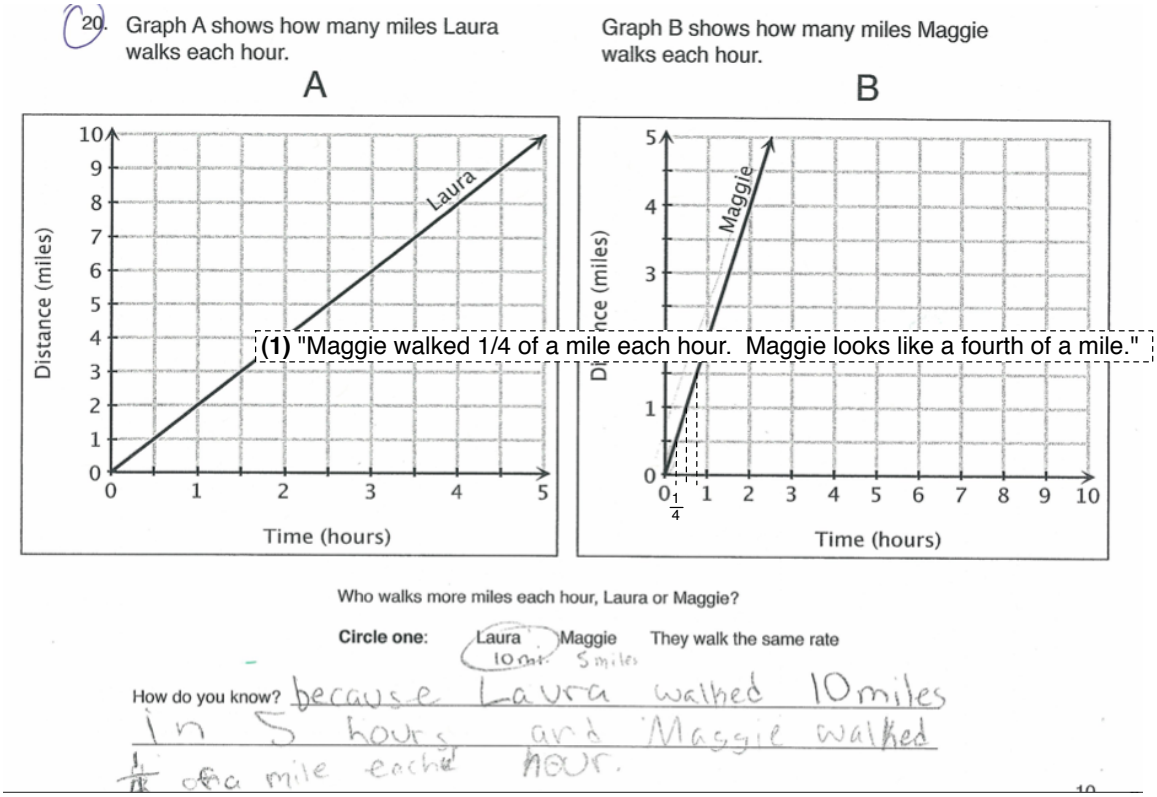


Figure 43: ID 5421's annotated solution for Item *Rate of change*

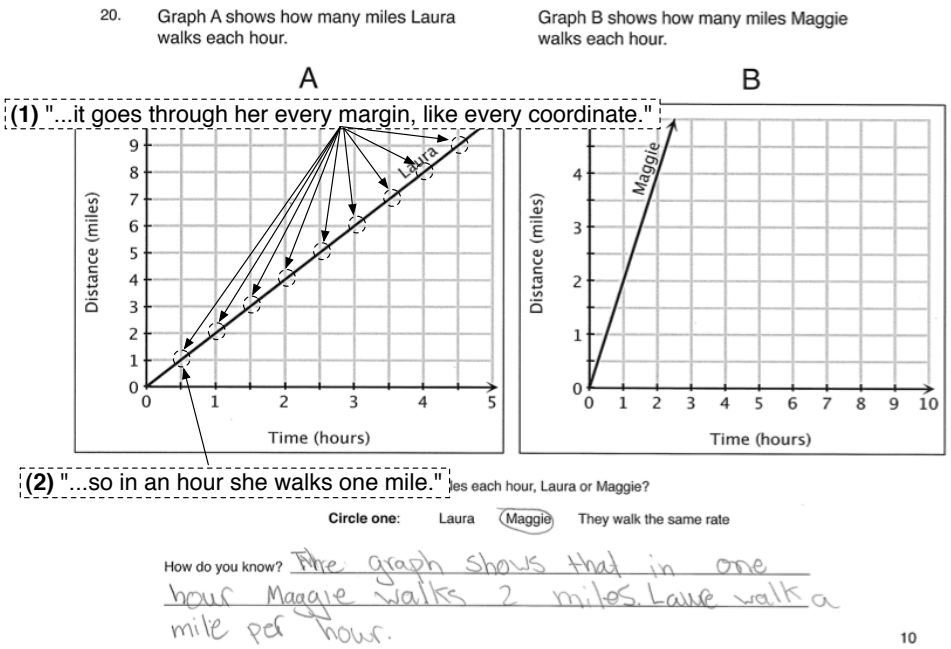


Figure 44: ID 8730's annotated response to Item *Rate of change*

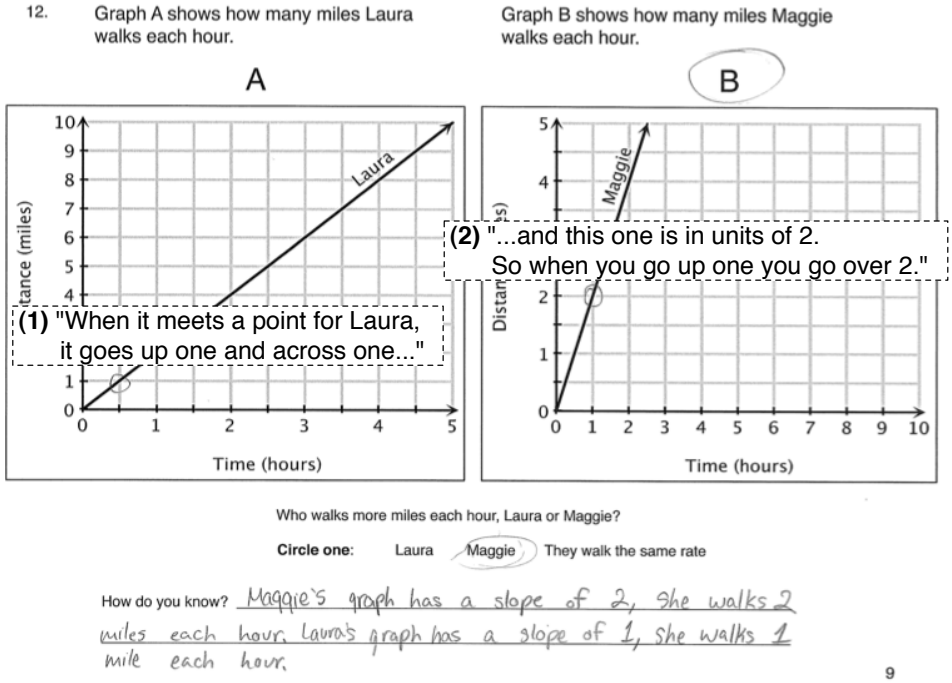


Figure 45: ID 8724's annotated response to item *Rate of change*

A third category was to coordinate numerical units for one quantity only. Of the seven interviewees coded as this (four Grade 5 and three Grade 8 students), the majority (71%) chose *Graph A: Laura*, with the remainder choosing *Graph B: Maggie.* Students coded as this compared one of three quantities across the two grids: maximum value along the vertical axis (10

and 5), maximum value along the horizontal axis (5 and 10), or the value along the horizontal axis at the greatest point on the function line (5 and 2.5). For example, ID 5115 chose *Graph A: Laura*, and stated in the interview, "She [Maggie] didn't do it the same as Laura. She [Maggie] did 5 miles. Laura does 10 miles," indicating quantities along the vertical axis only. Similarly, ID 8812, who also selected *Graph A: Laura*, explained that, "…so Maggie walked only 5 miles, and Laura walked 10. So I decided that Laura walked more." Alternatively, ID 5103, who also chose *Graph A: Laura*, privileged values along the horizontal axis when coordinated with the endpoint of the function line (Figure 46). In the interview, he explained, "I knew that it <*indicating Laura's graph*> was more because it <*Maggie's graph*> stops at 2 <*gestures from the endpoint of the function line down to the horizontal axis and incorrectly identified this point as "2"*>, and this one <*indicating Laura's graph*> goes to 5." In this case, the student coordinated the inscription of the function with values along the horizontal axis, but without considering values along the vertical axis. Across these seven students, they treated a point in the plane as representing one quantity only.

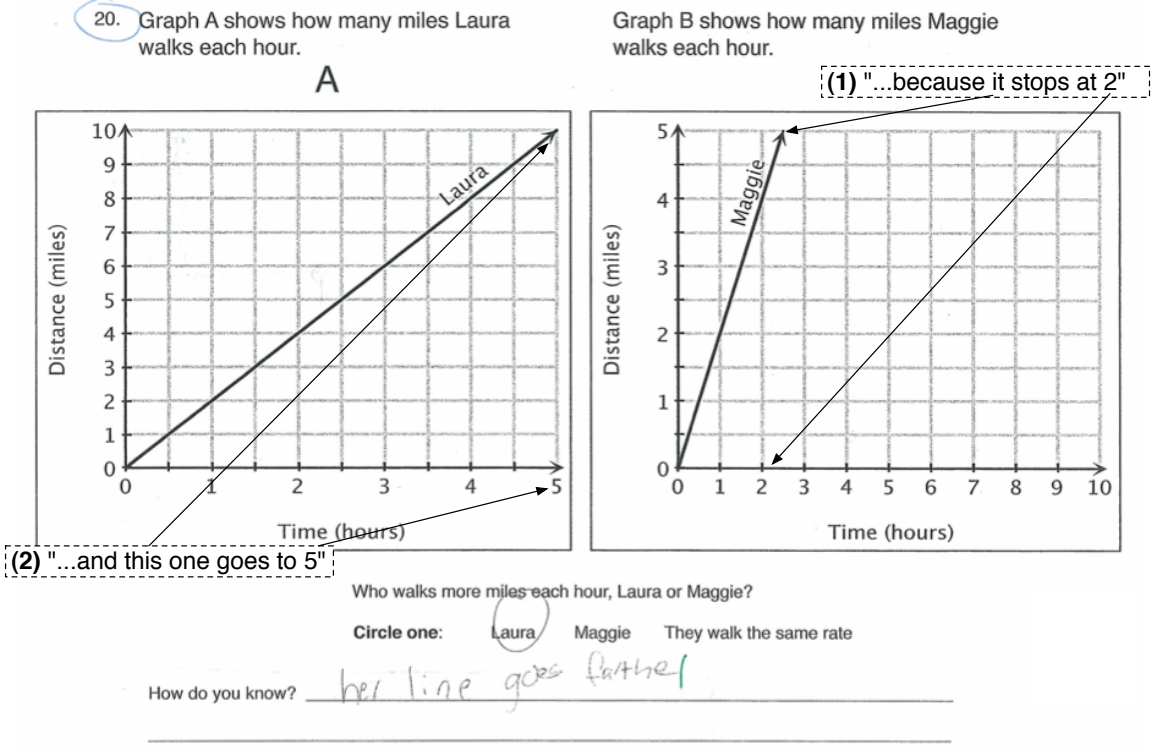


Figure 46: ID 5103's annotated response to Item *Rate of change*

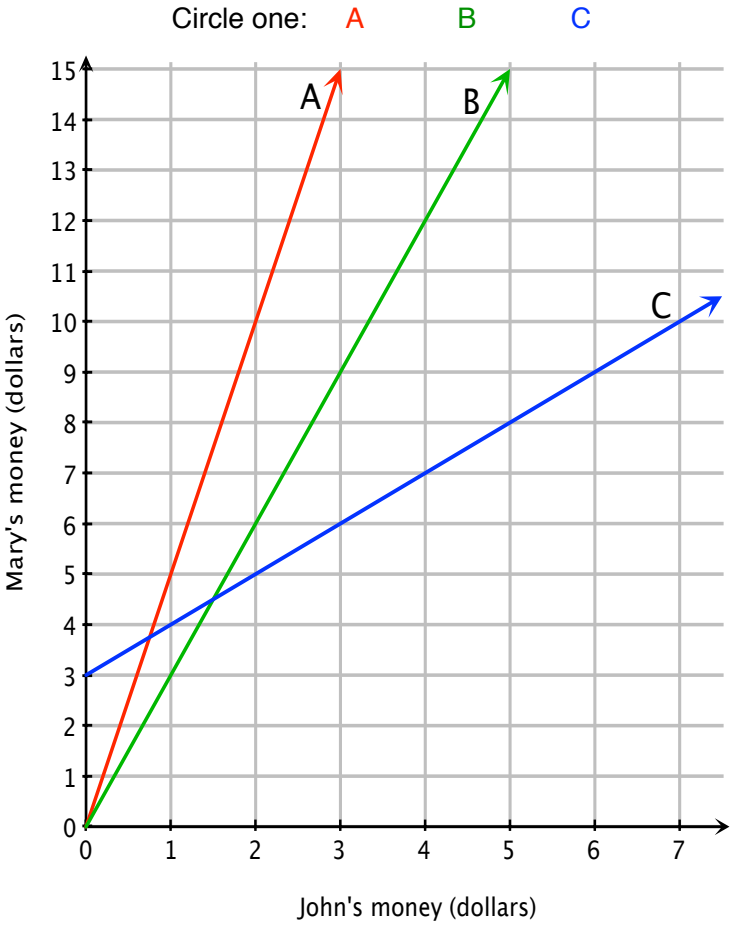
A fourth and final coordination category was to use the orientation of the function line in the plane. Five interviewees were coded as this, one Grade 5 student and four Grade 8 students. The Grade 5 student chose *Graph B: Maggie*, while Grade 8 students were evenly divided between *Graph A: Laura* and *Graph B: Maggie*. While only five students were coded as this, the trend suggests that Grade 8 students were more likely to interpret the orientation of the line in the plane over Grade 5 students, especially when considering that the overall *n* for Grade 5 interviewees on the item *Rate of change* (n = 24) was 50% more than that for Grade 8 interviewees (n = 16). In all cases, students either referred to the orientation of the line in terms of relative steepness, or referred to the length of the function line in the plane. In all cases, students coordinated the function line with the overall plane but independent of numerical

inscriptions along axes. For example, ID 8603, who chose *Graph B: Maggie*, pointed to the orientation of the line by her use of the word *steepness*: "Because it's the steepness. I guess it's because [Maggie] did it more fast than [Laura] did." Similarly, the Grade 5 student also chose *Graph B: Maggie* and stated, "Because Maggie's is high and short." Other students used the length of the function line to justify *Graph A: Laura*. For example, ID 8617 stated, "She [Laura] walks longer than Maggie."

Analysis of the interview data for the item *Rate of change* revealed the following trends. First, coordinating linear units and numerical units was consistent with the correct response choice, *They both walk the same rate*. No other coordination resulted in the correct response. Second, students in both grades privileged linear units independent of numerical inscriptions along the axes. In all cases, students chose an incorrect response choice. Third, students coordinated numerical units across grids, but with a focus on just one quantity. Fourth, students interpreted the orientation of the line in the plane independent of any numerical units along the axes. Data imply that Grade 8 students are more likely to interpret the orientation of the line over Grade 5 students.

Story context. I present here a final task used to assess students' conceptual coordinations when presented with a story problem. Item *John and Mary's money* begins with the prompt: "Mary always has 3 times as much money as John. Which of the lines below shows this?" Below this is a grid on which are plotted three different functions (all of which are restricted for positive values): Line A ($f(x)=5x$), Line B ($f(x)=3x$), and Line C ($f(x)=x+3$). Figure 47 displays the task and the percentage correct by grade on this task. In order to solve the problem, students needed to coordinate the joint variation in the story ("three times as much") with numerical and linear units on the grid. On this problem, 39.7% and 55.7% of Grades 5 and 8 students accurately identified Line B as showing this story context. I provide here further detail on response patterns across students. The most popular choice among students in both grades was the correct response. With an almost equivalent percentage of students in each grade, Line C was the most common incorrect response among both Grade 5 (31.7%) and Grade 8 (31.3%) students. The final option, Line A, still had more than ten percent of students in each grade selecting it, with 15.1% of Grade 5 students and 11.5% of Grade 8 students. Students that did not provide a response were coded as "Other."

Mary always has three times as much money as John. Which of the lines below shows this?



How do you know?

_____

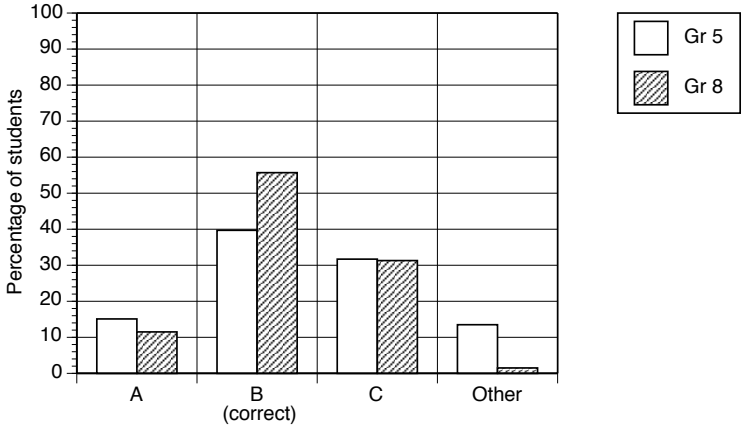


Figure 47: Item *John and Mary's money* (top) and percentage of students' solutions (bottom)

Interviews with Grade 5 (n=21) and Grade 8 (n=17) students revealed four distinct categories of conceptual coordinations: (a) joint variation of numerical units along two axes; (b) linear units privileged; (c) numerical units privileged; and (d) orientation of the line in the plane;

with some students coded as "other." Note that because the problem design featured consecutive numerical units after each congruent linear unit, this item did not directly assess a coordination of linear units with numerical units.

Table 7: Proportion of coordination types by answer choice on item *John and Mary's money*

Conceptual coordination	A	B (correct)	C	Other
Joint variation of numerical units (n=9)		1		
Linear units privileged (n=1)	1			
Numerical units privileged (n=14)	.14		.79	.07
Orientation of the line (n=10)	.2	.8		
Other (n=4)	1		3	

I first present the category coordination Joint variation of numerical units along two axes. All students (n=9) coded as coordinating numerical units along two axes chose the correct response, *Line B.* These students successfully coordinated the joint variation from the prompt with quantities along each axis to identify Line B. Students did this either by identifying a single point or multiple points along the line that reflected the appropriate joint variation. For example, ID 5129 located the point (5, 15) that fell along Line B, and, returning to the prompt, explained, "3 times 5 is 15." Meanwhile, ID 5304 made a general statement about the joint variation displayed in Line B, stating: "B … is always three times as much money." Note that many students coded as Orientation of the line also chose the correct response, as described below.

A second coordination category, privileging linear units, was assigned to only one interviewee, ID 5203. This student chose *Line A* as representing the relation between John and Mary's money. ID 5203 identified three "spaces" –linear units—on the graph between 15 on the vertical axis and the position of the endpoint for Line A (Figure 48). He stated, "There are 3 spaces up here… I just think that means that three times as much money."

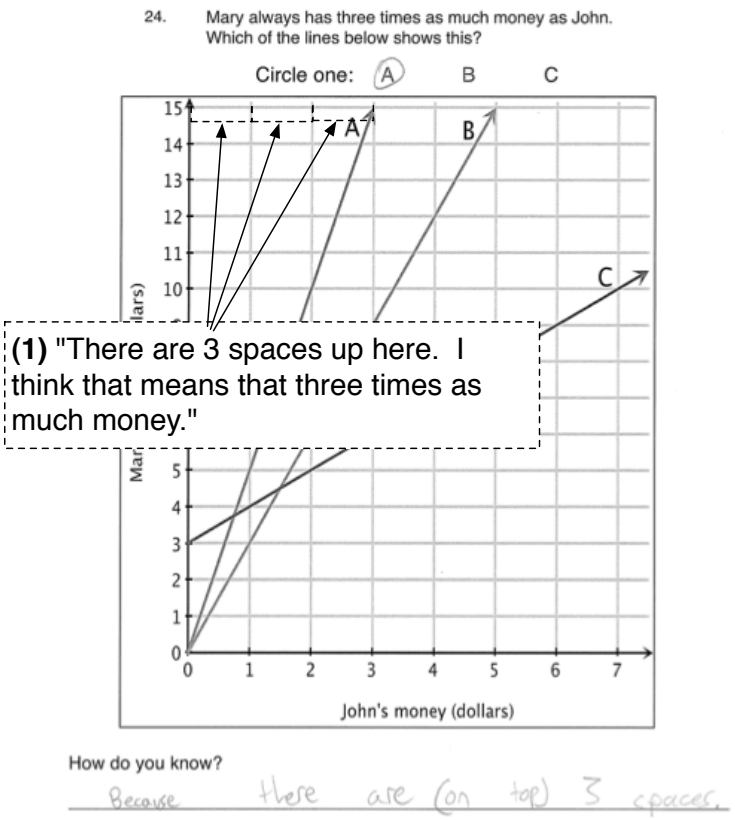


Figure 48: ID 5203's annotated response to Item *John and Mary's money*

A third category was to privilege numerical units independent of any linear unit. Fourteen students were assigned this code, including six Grade 5 students and eight Grade 8 students. Most students coded as privileging numerical units chose *Line C*, with a small minority—two Grade 8 students—choosing *Line A*. ID 5311 explained his decision for Line C by stating, "Mary's money has 3 dollars," as he indicated the *y*-intercept at 3 for *Line C*. Six of the Grade 8 students interviewed provided similar rationales. Other students also justified Line C by identifying numerical units along the horizontal and vertical axes that were multiples of 3. For example, ID 8614 stated (Figure 49), "I thought if Mary has always like that <*gesturing over the length of Line C*>. So it's like 3, then it goes up again to 3, that's 6. Then it goes up again 3, that's 9," with a gesture to the numerical unit along the horizontal axis with each multiple of 3. The two Grade 8 students using numerical units to choose *Line A* each identified the numerical unit 3 on the horizontal axis as corresponding to the end of *Line A* in the plane.

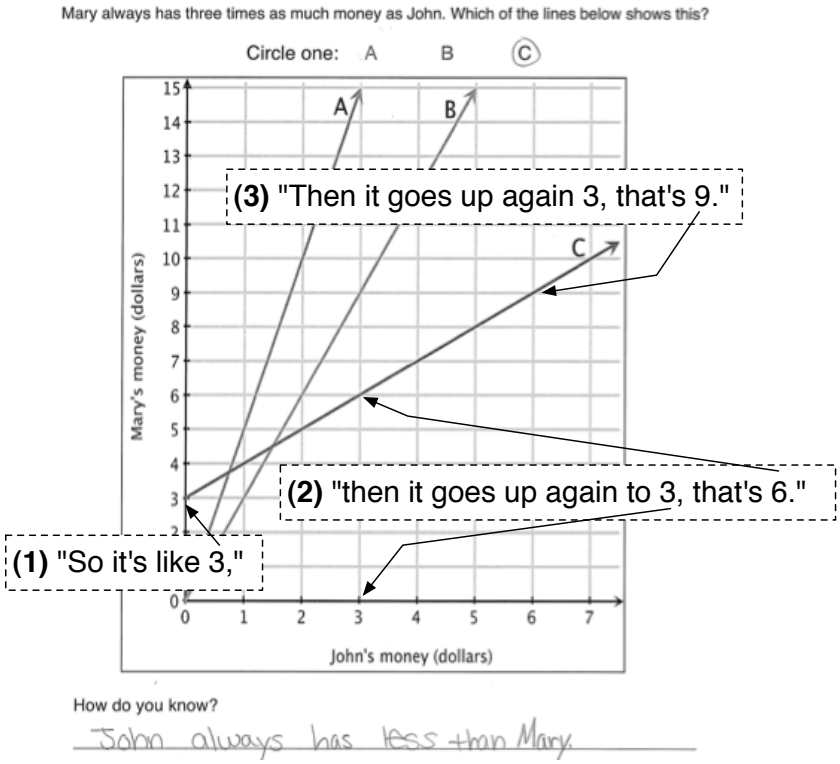


Figure 49: ID 8614's annotated response to Item John and Mary's money

A fourth and final coordination category was to use the orientation of the function line in the plane independent of numerical values along the axes. Ten students—three in Grade 5 and seven in Grade 8—were coded as this. In all cases but two Grade 8 students, students used the orientation of the line to choose the accurate response, *Line B*. Nonetheless, their use of the orientation of the line did not consider the joint variation in the story. Rather, in each case, students referred to a sense of what graphs typically look like, or, a prototype for any graph. For example, ID 5205 chose *Line B*, and stated, "I chose B. If it shows that John has more money, it would be going more this way <*traces her pen across Line B*>." Her response did not draw any attention to the joint variation from the problem. Similar general statements were provided by other students assigned this code. ID 5103 referred to the orientation of Line B in the plane and compared this to Line C, stating, "Mostly on charts that say they have money, like that <*sketches his own graph, see* Figure 50*a*> if they have a lot. And they go low if they don't have enough. And if it's over here they only have a little bit <*adds to his sketch, see* Figure 50*b*>." Other students coded as Orientation of the line and chose Line B referred to looking for the line that showed (a) *bigness* (ID 5101, ID 8604), (b) *going up faster* (ID 8610), (c) being *longer* (ID 8623, ID 8617), or (d) the line going *more out* (ID 8812). In their explanations, these students referred only to the shape of the graph independent of values along the axes or connecting specific aspects of the function line to the functional relation in the prompt. For the two Grade 8 students that used the orientation of the line to select *Line A*, both of them referred to the steepness of the line independent of any reference to numerical units.

(1) "Mostly on charts that say they have money, like that if they have a lot"

(2) "And they go low if they don't have enough."

Figure 50: ID 5103's example for (1) graphs that display information on money with (2) his counterexample for graphs that "don't have enough" money

Analysis of the interview data for the item *John and Mary's money* revealed the following trends. First, students that considered the joint variation as indicated by the prompt all chose the correct response, *Line B*. Second, one student privileged linear units as indicated by gridlines independent from any numerical units. Third, students considered numerical units—typically the "3" along the vertical axis—independent of any other inscriptions. Fourth, students considered the orientation of the line independent of numerical units. Students did this either by referring to the steepness of the line in the plane, or by referencing a prototypic graph.

Summarizing problem profiles. In this section, I provided profiles of students' responses for five challenging non-routine problems from the LRA. Analyses across the five targeted tasks reveal patterns in students' conceptual coordinations involving linear units and numerical units. I review these patterns here. First, the coordination of linear units and numerical units allowed students to select the correct response in almost all cases, including number line, Cartesian plane, and graphing functions tasks. Second, across tasks students privileged linear units or numerical units. Each of these units are core components to understanding graphs, yet at the same time are not sufficient in and of themselves to interpret accurately linear representations of quantities. I argue that the non-routine design of these tasks revealed the overgeneralizations students made regarding linear units or numerical units. Third, students in each grade named a point in the plane using projections from each axis. Fourth, on Cartesian plane tasks and graphing functions tasks with linear units defined by gridlines in the plane, students partially coordinated linear units and numerical units, but in these cases treated linear units as representing unit intervals. Fifth, on graphing tasks specifically, students sometimes privileged just one quantity instead of coordinating quantities along each axis. Sixth, students considered the orientation of the line—either in terms of steepness or in terms of a prototype of a linear function in the plane—independent of numerical values along the axes. Data suggest that Grade 8 students were more prone to this than Grade 5 students.

Discussion

As students solved the LRA problems, they generated conceptual coordinations between linear units and numerical units. In this discussion, I consider properties of these coordinations. I first consider findings of the assessment study to identify patterns across students as they solved problems featuring linear representations of quantities. Second, I consider the findings in terms of identifying instructional goals for a tutorial sequence seeking to support students' generative thinking involving these representations.

Patterns of student understanding. In this first section, I consider patterns that emerged from assessment and interview data. I first consolidate trends across data points for Grade 5 students to consider the resources on which students may build productively. The trends upon which I draw include students' (1) adherence to the order principle; (2) treatment of 0 as unique; (3) efforts to extend numerical units; (4) efforts to use congruent linear units on the line; and (5) naming a point in the plane based on projections from each axis. While none of these trends are sufficient in and of themselves, each indicates conceptual resources students brought to these tasks. I also consider differences between grades based on findings from the assessments and interviews.

Adherence to the order principle. The first trend was adherence to the order principle. There were no instances in this data corpus in which students violated the order principle, which I define as numbers increase from left to right and decrease from right to left (and, for the vertical axis, increase moving up and decrease moving down). While problem design featured only Quadrant I and did not assess for order, the order principle stands out as an intuition and resource.

Treatment of 0 as unique. Grade 5 students also gave special treatment to the placement of the number 0. Interviews indicated younger students may place 0 at the beginning of the number line because of zero's identity as a 'start point'. Zero, in fact, does have a special role on the number line as a point of demarcation between positive and negative integers. Furthermore, with respect to linear measure, measures in the real world typically begin with a zero-point. In this data, students' treatment of zero as unique was at times not coordinated with other points along a number line. Nonetheless, the unique role emerged as a potential resource students brought.

Efforts to extend numerical units or linear units. The third and fourth trends reflect conceptual coordinations of linear units and numerical units as students engaged with LRA items. Results of number line tasks and Cartesian plane tasks indicated that many students responding incorrectly extended numerical units independent of a provided interval. On Cartesian plane items, the design of which typically featured congruent linear units, a trend emerged in which students counted on by numerical units (e.g., 0, 2, 4), but then treated each interval as a unit interval. In this case, students do, in fact, coordinate linear and numerical units on the line, though only partially. While many students appeared to draw on numerical or linear units in promising ways, they demonstrated a need for further support to coordinate linear and numerical units appropriately along the axes.

Furthermore, a trend emerged in which students made efforts to use equal linear units on tasks that provided unequal linear units, though not always in coordination with numerical inscriptions along the line. This trend indicates students have intuitions about the role of equal linear units on number lines, and need further support to move towards normative mathematical understandings as applied to equal linear units—namely, coordinating these linear units with numerical inscriptions.

Naming a point in the plane based on projections from each axis. A trend that emerged from tasks in which students identified a point in the plane is that students' overwhelmingly named a point based on projections from each axis. Many students, however, did not first appropriately coordinate linear units and numerical units, resulting in an incorrect name for the point. Nonetheless, data indicate that students in Grade 5 conceptualize a grid consistent with conventions for naming a point in a two-dimensional axis.

Differences by grade. The sample for the assessment study included Grade 8 students as a way to determine patterns among Grade 5 students that may persist through Grade 8. I also use this contrast to consider patterns among Grade 5 students that did not emerge in Grade 8 data, and patterns among Grade 8 students that did not emerge in Grade 5 data.

In both grades, students displayed a need for instructional support in order to coordinate linear units and numerical units. Furthermore, patterns among students in both grades indicate points in the plane as independent of values along the axes (for example, assigning numerical value to linear grid units). Based on these trends, I conjecture that providing support in these areas for Grade 5 students may address patterns that exist not only among that population, but also ones that persist through Grade 8.

Additionally, one particular trend emerged that was unique to the Grade 8 population: ascribing meaning to the orientation of the line independent of numerical units along the axes. Steepness of the line in the plane is a useful strategy to determine rate of change; however, Grade 8 students' overgeneralization of the meaning of steepness indicate they interpret inscriptions in the plane independent of numerical units along the axes. Furthermore, their calculations of slope relied on a non-normative conceptualization of linear units in the plane (or, gridline units) independent of numerical units along the axes.

Three instructional goals for a tutorial intervention. In order to clarify instructional goals for a task sequence, I consider here how instruction may build productively on the resources for task design mentioned above. Data from this study clarified instructional goals by pointing to the following areas for support: (1) coordinating linear units and numerical units in principled ways; (2) treating two axes as each representing a different quantities; and (3) treating a graph as a model of joint variation of a story context.

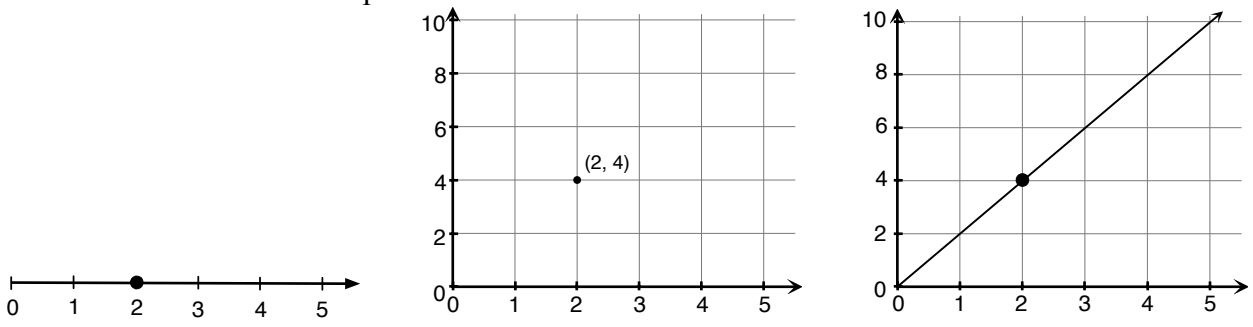
1. Coordinating linear and numerical units. The first instructional goal is to support students' principled coordination of linear units and numerical units. Data in this chapter have revealed students privileging either numerical units or linear units. Both approaches display concerted efforts to name points along a number line or in the plane, yet each may be insufficient to accurately solve the problem. I attribute other patterns of understanding to be rooted in this lack of coordination of linear units and numerical units, such as assigning numerical values based on linear grid units. I propose that instruction focusing on the coordination of linear and numerical units would support rich and generative understandings across these linear representations.

2. Treating two axes as representing different quantities. A second instructional goal that builds off the first is to support students in treating each axis as representing *two different quantities*. Cartesian plane tasks revealed that some students made efforts to apply one numerical pattern to both axes. In these cases, students are not treating axes as representing different quantities; rather, they use axes in conjunction with each other in order to create or continue number patterns. Furthermore, graphing tasks revealed a pattern in which students privileged values along one axis at the expense of the other. I conjecture that initial work with two axes in which each represents a different quantity would (a) create a foundation on which students could then draw when interpreting functions plotted in the plane, and (b) address a pattern among Grades 5 and 8 students in which they chose a graph based on the prototypic appearance.

3. Treating a graph as a model of joint variation of a story context. A final instructional goal is to support students in coordinating a story context with inscriptions in the plane. Students in this study made concerted efforts to do just this, as evidenced in many of the incorrect

strategies in which students located the three-ness of the story context to aspects of three-ness (either in a numerical unit or in linear units) in the provided graph. I conjecture that exploration and modeling with problems featuring the joint variation of a simple mathematical function would support this work. Furthermore, recall that students chose an answer based on a prototype for a graph, and that this choice was independent of linear or numerical units. I conjecture that such approaches may be addressed through instruction highlighting linear and numerical units. I conjecture that such instruction would support students' understanding of any point along a function line as representing two quantities, and that each of those quantities are determined by coordinations of linear and numerical units along two axes.

To conclude this discussion, I return to a premise of this study that arose in Chapter 1 regarding the progression of linear representations of quantities. A goal of this study is to support students' generative understandings. The resources and instructional goals identified using results of the assessment and interview study underscore the promise of a progression of linear representations of quantities to support young students' understanding of linear function graphs. I return to the example of the number line as a coordination of linear and numerical units, a foundational property of the line that may be extended productively across linear representations of quantities. Figure 51 below shows the three representations. Beginning on the right, the graph displays a linear function in the plane. Any point along the function line represents two quantities. Each of those quantities is determined by projections from a named point from each axis, as shown in the Cartesian plane (middle). In turn, the name of each of those axis values is determined through a coordination of linear and numerical units, though each axis may or may not have different scales. I propose that students' generative understandings of linear function graphs have critical foundations in these properties of a single number line. I explore this in the context of instruction in Chapter 3.



The point 2 on a number line. | The point (2, 4) on a Cartesian plane. | The function f(*x*)=2x, which passes through (2, 4)

Figure 51: Linear and numerical units across linear representations of quantities

Chapter 3: Tutorial Experiment

Chapter 3 has three purposes. First, I describe three design features of the pedagogy designed to support fifth grade students' generative thinking on the three linear representations of quantities. Second, I analyze the efficacy of the tutorial intervention. Third, I report the character of students' understanding as they progressed through the tutorial.

Design Features of the Tutorial Intervention

Three design features were coordinated in order to develop the tutorial intervention. These features included: (1) modeling, (2) communication, and (3) the use of definitions. Each of the three design features builds on results of the assessment study (Chapter 2) involving the coordination of linear and numerical units that provided analyses of fifth graders' conceptual resources as well as areas in which they need support (Table 8).

Table 8: Summary of findings from the assessment study (Chapter 2)

	Number Lines	**Cartesian Plane**	**Graphs of Linear Functions**
Resources	• Adherence to the order principle • Treatment of 0 as unique • Efforts to extend numerical units • Efforts to extend linear units	• Naming a point based on projections from each axis	
Areas to support	• Coordinating linear units and numeric units in principled ways	• Treating two axes as representing different quantities	• Treating a graph as a model of joint variation of a story context

Modeling. The tutorial treated each of the three linear representations of quantities as mathematical models of real-world phenomena to support students' generative understandings: (1) the number line as a model of physical magnitudes recorded on a line; (2) the Cartesian plane as a model of physical magnitudes recorded on two perpendicular number lines; and (3) a function graph as a model of the relation between an independent and dependent variable on the plane.

Number line as a model of physical magnitudes. The number line was treated as a model for indexing physical magnitudes on an open line. Replicating the conceptual framing of a prior study (Saxe et al., 2010), a set of physical manipulatives, Cuisenaire rods (C-rods), was introduced to support modeling activity. As with any physical object in the world, C-rods are not inherently units of length and, therefore, they must be quantified in conceptual activity. Each of ten colors corresponds to a unique C-rod length (in centimeters) with multiplicative relations among the different rods. For example, any red rod has a 2-centimeter length, while any purple rod has a 4-centimeter length. Multiple red rods may be concatenated (or a single red rod iterated) and treated in terms of red rod units. At the same time, since one purple rod is the same length as two red rods, one purple may be treated as equivalent to the length of two reds.

For the number line to serve as a model of the physical lengths, C-rods are recorded on an open line to generate a number line representation. For example, students can record the length of four red rods when conceptualizing those rods as a concatenated sequence of four lengths that can be put in alignment with a line. At the same time, the line must be treated as a vehicle to accumulate linear concatenated segments that begin at the origin and proceed to the right.

In the tutorial sequence, initial problem sets establish the number line as a model of physical magnitude. Students begin through exploration of the physical objects and relations among those objects, for example by comparing rods with equivalent lengths (e.g., multiple red rods) or different lengths (e.g., comparing one purple rod to two red rods). Following this, the physical magnitudes are recorded on an open number line, with units defined by the length of specified rods (e.g., recording the length of 4 red rods on a number line). Subsequent problem sets define units on the line, with C-rod lengths fitting into particular intervals (e.g., an interval from 2 to 4 defined on the line and fitted to a purple rod).

Cartesian plane as a model of physical magnitudes on two axes. I describe here two different sets of physical magnitudes used in Cartesian plane modeling activity, first to extend a single number line to perpendicular axes, and second to establish points in the plane as an index of two coordinated physical magnitudes. When introduced, the axes of the Cartesian plane were treated as models for indexing physical magnitudes on two perpendicular number lines, with a point in the plane as an index of each of those lengths. In the case of the two axes, each axis may serve as a model for the same physical magnitude, for example when each axis is an index of red rod length. On the other hand, each axis may serve as an index of objects of different lengths, for example when the horizontal axis indexes light green rods while the vertical axis indexes red rods. The rods serve as an object to support the conceptual treatment of scale on each axis as independent from the other.

In the tutorial, the definition of a unit as represented by the model shifts along five stages (Figure 52): (a) units defined by the rods and rod relations, to (b) units defined by rods and indexed on an open number line; to (c) units defined by line with rods fitted to unit and/or multiunit intervals; to (d) units defined on perpendicular axes with rods fitted to unit and/or multiunit intervals; and to (e) units defined on perpendicular axes (no rods). Eventually, the rods are removed from use in the tutorial, with units determined by the axes alone without the support of a physical object.

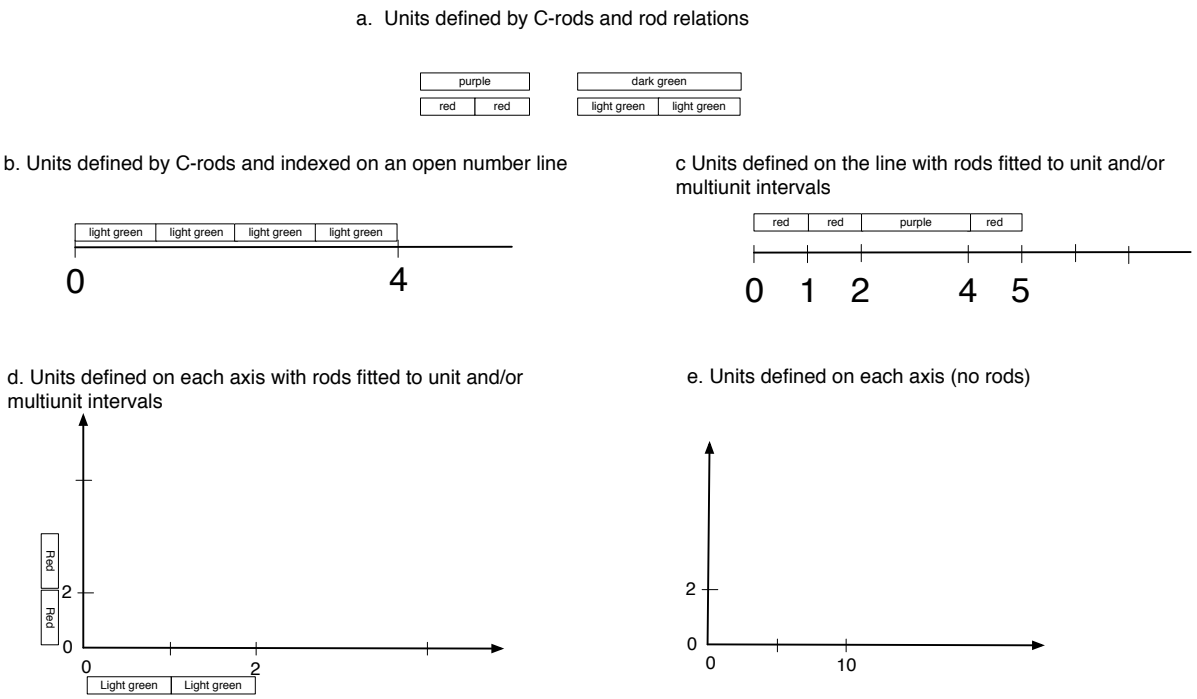


Figure 52: Five phases of C-rod use in the tutorial intervention

In the tutorial intervention, modeling activity shifts to a treatment of a point in the plane as representing two physical magnitudes. Corresponding to real-world horizontal and vertical dimensions, the phenomena involved a cat walking a particular distance, and the water he has in a container measured in inches. Using this phenomena, a point in the plane is treated as an index of those two physical magnitudes. In this case, those separate quantities correspond to a real-world horizontal dimension (distance in yards) and a real world vertical dimension (height of water in a container).

Function graph as a model of independent and dependent variables. A function graph was treated as a model of joint variation in the world. The relation involved an independent variable—a cat walking distance measured in yards—and a dependent variable –the water earned (measured in inches) as a function of distance walked. As with any real world phenomenon, there is nothing inherently numeric or jointly varying about a cat that receives water as a function of distance and, therefore, this must be quantified in conceptual activity. For example, the distance walked can be coordinated with the amount of water poured into a container, so that the amount of water in the container jointly varies with the cat's distance.

The joint variation of the independent and dependent variables must be coordinated with quantification of the plane to generate a graph of a function. For example, joint variation—the relation of the distance a cat walks to the amount of water in a container—can be put in alignment with a Cartesian plane. The plane must be treated as a vehicle to display all possible instantiations of the joint variation, with points in the plane representing joint variation as indexed along either axis. Eventually, the multiple instantiated points of the joint variation can be represented with a line. Figure 53 displays (a) the joint variation of independent and dependent variables, (b) the quantification of a point in the plane, and (c) the coordination of the joint variation to inscriptions in the plane.

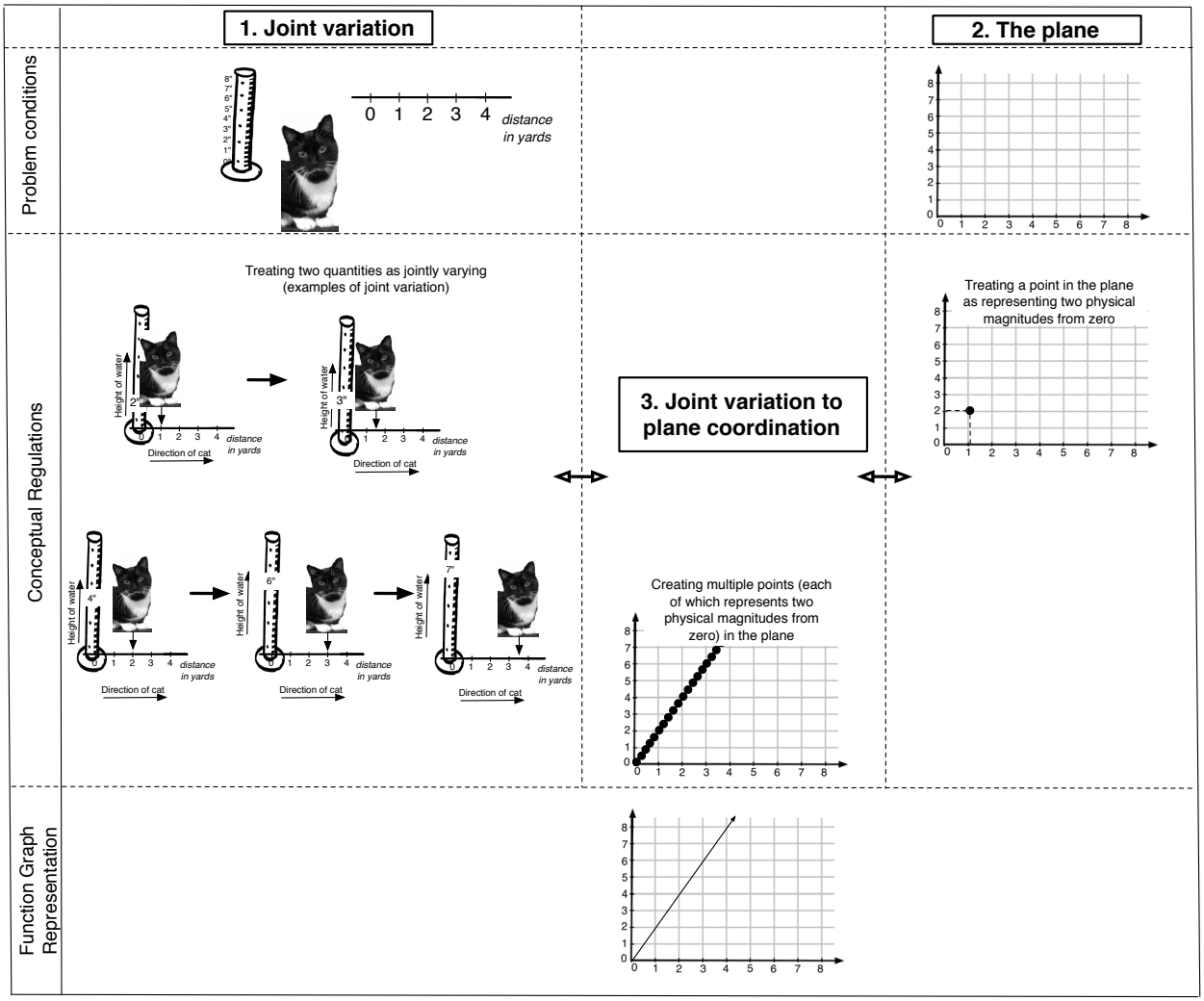


Figure 53: Three strands of conceptual activity regulating the construction of the plane as a record of joint variation

Communication. The tutorial engaged students in communication as a central means to support their generative understandings. Mathematics is a social practice that is supported through communication and is inseparable from human social activity (Leont'ev, 1981; Sfard, 2008, Vygotsky, 1986). In accordance with this, the tutorial was designed to require communication and exploit communicative goals between tutor and tutee. The tutorial introduces a communication game in which tutor and tutee are each game players. This design was used in the context of number lines in a prior study (Saxe et al., 2010) that led to learning gains with a large effect size.

Featuring a recurring activity structure, the game required tutor and tutee to construct number line and graph representations based on directives (printed on a series of playing cards) while an occluding screen blocked players' views from one another. After construction, players compare their work and engage in communication, analysis and justification. Discrepant point placements motivate tutor and tutee to communicate mathematical ideas to one another in order to resolve the discrepancies. With a goal of achieving identical solutions, game players agree to construct subsequent number lines or graphs in a particular way. As tutor and tutee produced

new agreements, the agreements were inscribed onto an agreement sheet for both players to regulate their own constructions on subsequent problems.

Definitions embedded in a task sequence. The tutorial design made mathematical definitions explicit in the context of game play. As mentioned above, game players agreed to construct number lines in particular ways; these agreements were orchestrated by the tutor in pre-determined locations and based on underlying mathematical definitions of the linear representations. In prior work that embedded definitions for positive and negative integers in such a game structure, student take-up of definitions was correlated with learning gains (Saxe et al., 2010). The definitions—or "agreements," as they were called in the tutorial—for positive integers were replicated for the present study, and are displayed in Table 9. They include: (1) order, (2) unit interval, (3) scale, (4) multiplicative relations, and (5) irrelevance of tickmarks. Written on a sheet of paper in language friendly to a Grade 5 student, the definitions allowed tutees to regulate their own constructions and draw upon agreements in their oral justifications of their work. When breaches in communication occurred (or rather, when players had discrepant point placement), players relied on prior agreements to justify his or her construction. Each of the five definitions used in Saxe et al. targets resources identified in Chapter 2 and provides support for a normative coordination of linear units and numeric units.

Table 9: Number line definitions used in Saxe et al. (2010)

Number line agreements
1. Order: Numbers increase from left to right and decrease from right to left
2. Unit Interval: The distance between counting numbers has to be the same
3. Scale: We need to agree on which rod to use
4. Multiplicative relations: The distance between numbers that skip count has to be the same
5. Irrelevance of tickmarks: Every number has a place, but not all need to be shown

In addition to the five definitions (Saxe et al., 2010) in Table 9, five additional definitions were introduced to support the transition to the Cartesian plane and then introduction of functions in the plane. Recall from Chapter 1 that the targeted mathematical ideas of the Cartesian plane included identifying points along two axes and points in the plane, and those of function graphs included rate of change and story-to-graph coordination. Definitions reflect these targeted mathematical ideas or entailments of those ideas. Table 10 displays the full list of ten definitions, which in addition to the five mentioned in Table 9 includes: (6) point placement, (7) scale on two axes, (8) irrelevance of gridlines, (9) linear functions, and (10) story-to-graph coordination. As in the prior five definitions, the full set of 10 definitions seeks to build on student resources identified in Chapter 2 as well as providing support for generative understandings. My working hypothesis was that a student's mindful use of the definitions would lead to a greater success on problem sets, and that the mindful application of these definitions through the tutorial sequence would support a generative understanding of the entailments of linear representations of quantities.

Table 10: Ten definitions used in the tutorial intervention

Number line agreements	Cartesian plane agreements	Function graph agreements
1. Order: Numbers increase from left to right and decrease from right to left	6. Point Placement: One point shows two amounts	9. Linear functions: We can connect points to show many points
2. Unit Interval: The distance between counting numbers has to be the same	7. Scale on two axes: Each number line can count by different amounts, but both MUST follow all agreements	10. Story-to-graph coordination: The story decides what the graph looks like
3. Scale: We need to agree on which rod to use	8. Irrelevance of gridlines: We don't need gridlines to find a point, but sometimes they can be helpful	
4. Multiplicative relations: The distance between numbers that skip count has to be the same		
5. Irrelevance of tickmarks: Every number has a place, but not all need to be shown		

Research questions. Two research questions guided the present study. First, does the tutorial intervention support gains? Second, if the intervention is found to be effective, what is the character of students' successes and struggles as they solve problems involving linear functions on the plane?

Methods

Participants. Student performance on the Linear Representations Assessment (LRA) (Chapter 2) was used as a criterion to select a lower performing sample of students for participation in the tutorial study. Forty students with scores at the lower two-thirds of the distribution on the LRA (below 19 items correct out of 32 total items) were matched in pairs and then randomly assigned to a tutorial group (n=20) or control group (n=20). Pairs were determined based on (a) identical pre-test score, meaning that both had the same number correct on the 32-item pre-test, and (b) membership in the same classroom. For three of the 20 pairs, one student had a total score one point higher or lower than the partner; in two of the cases, the higher student was assigned to the control group, and in one case the higher student was assigned to the tutorial group[8]. Students that were interviewed as a part of the assessment study were not considered for the tutorial study[9].

[8] Pairs included: 5121 (Score, 17; tutorial group) and 5109 (16, control); 5119 (11, control) and 5106 (10, tutorial); and 5401 (14, control) and 5423 (13, tutorial).

[9] Due to a clerical error, one student, ID 5101, participated in both the interview group for the assessment study and was assigned to the experimental group for the tutorial study.

Piloting. As reported in Chapter 2, piloting of the tutorial task sequence, along with assessment items on the LRA, occurred in four phases: (1) one-on-one tutorials with Grades 5 and 6 students in February and March of 2009; (2) classroom teaching experiment in collaboration with an in-service Grade 6 teacher with after-class student interviews; (3) one-on-one tutoring and interviews with students in Grades 5, 6 and 8, Fall 2009 through Spring 2010; and (4) one-on-one tutoring with Grade 5 students and interviews with Grades 5 and 8 students. I conferred with research groups at the University of California, Berkeley, to gather different perspectives on the design of tasks and interactions in interviews, tutorials, and classroom sessions. Each of the piloting phases is described in Chapter 2. Piloting resulted in the developed task sequence, including the design of tasks, the phrasing of problem cards, the sequence and wording of agreements, and the design of wrap-up problems.

Assessment. The 32-item LRA (Appendix A) described and reported in Chapter 2 was used as a pre-test. The assessment featured problems involving three linear representations of quantities: ten number line items, 11 Cartesian plane items, and 11 graphs of linear functions items. Eighteen tasks were matched—nine routine problems with nine analogous non-routine problems. Among the nine pairs of matched items, there were three pairs of items for each linear representation of quantities. Three orders of the assessment were generated (see Chapter 2 for analysis of order effect). Participants were re-administered the LRA as a post-test within several days after intervention (tutorial group) or a similar duration of time (control group).

Tutorial materials. The following materials were used in the intervention (Figure 54): (1) a removable screen; (2) a pre-printed number line or Cartesian grid for each of the 66 problems. The student number line or grid was printed on paper, and the tutor number line or grid was printed on transparencies; (3) two markers, one for the student and one for the tutor; (4) Two colors of pairings of Cuisenaire Rods, each with a 2:1 length relation: (Red (length = 2 cm) and purple (length = 4cm) as well as light green (length = 3 cm) and dark green (length = 6cm)); (5) 22 sets of problem cards with three parallel versions in each set (66 problem cards in all); (6) a blank piece of paper (on which the tutor wrote the 'agreements'); (7) modeling materials, including a cat picture glued onto a popsicle stick, a large number line, an empty container marked vertically in inches; and a container filled with water; and (8) wrap-up problems printed on sheets of paper.

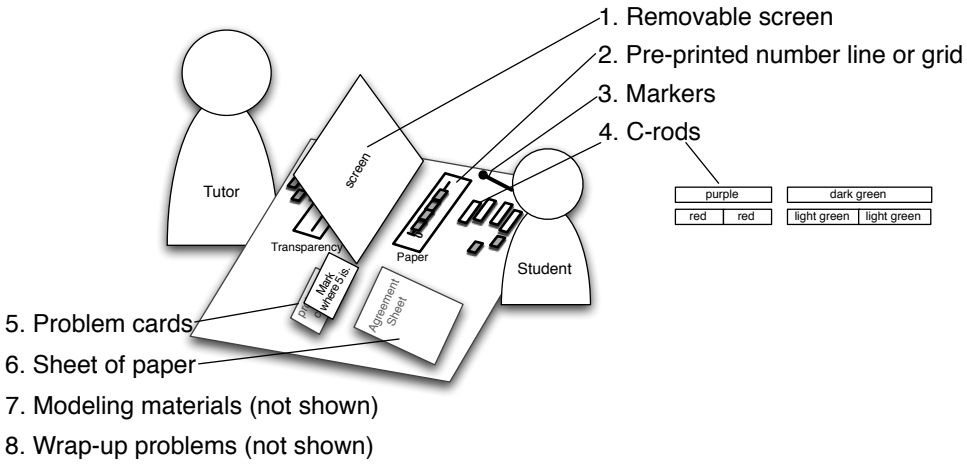


Figure 54: Tutorial materials

Tutorial overview. For each student, three tutorial sessions took place within a two week period, each with a duration of approximately 45 minutes. Each session was videotaped and all student work was collected and scanned. This overview covers three parts: procedures for problem administration, procedures to move on to the next problem, and provisions for relative amounts of tutoring.

Procedures for problem administration. A standard, multi-phased procedure was used as the tutor and student engaged with successive problems, as depicted in Figure 55. The game players—tutor and tutee—were seated at a table or desk in a semi-private location in a classroom or hallway. The multi-phased procedure was designed as part of the LMR project (Saxe et al., 2010). The five phases included: (a) problem card phase; (b) screen down phase; (c) construction phase; (d) screen up, positions match? phase; and (e) introduction of a written agreement phase.

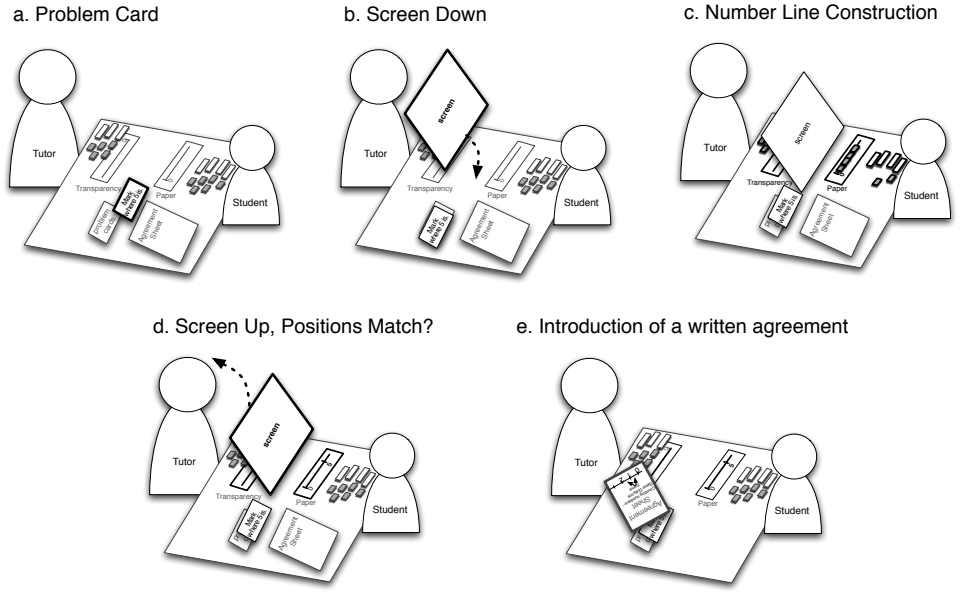


Figure 55: Procedures for problem administration, adapted form Saxe et al. (2010)

a. Problem card phase. In the problem card phase, the tutor presented to the student a card from the card deck, which was on the table in view of both players. The student and tutor alternated reading aloud the card. The tutor provided a number line or coordinate grid that

corresponded to the particular problem card. When applicable, the tutor provided the student with necessary C-rods (problem sets are further described below in the procedures).

b. Screen down phase. In the screen down phase, the tutor placed a screen on the desk so that players' views were occluded from one another. The screen was placed at a slight oblique angle so that the tutor could surreptitiously peer over the screen to see the student's work area.

c. Number line construction phase. In the construction phase, the student and tutor constructed their interpretation of the playing card directives onto the number line or grid keyed to that card.

d. Screen up, positions match? phase. In the next phase, the screen was removed to allow players to again see each others' work spaces with completed constructions on their number line or grid. The tutor overlaid the transparency on the student's paper in order to compare constructions. The tutor and student first acknowledged whether the constructions were similar or different. The tutor then asked the student to explain her/his construction, after which the tutor explained his construction to the student. If constructions were different from one another, the tutor then asked, "What could we do next time to have our work look exactly the same?" In this discussion, the tutor then encouraged them to agree on a particular way to try their subsequent constructions. During game play, a video camera was positioned to record the student's activity and interactions with the tutor.

e. Introduction of a written definition. At pre-determined points in the tutorial, the tutor orchestrated in discussion the introduction of an "agreement" based on one of the ten definitions above. Through communication, the tutor engaged the tutee in co-construction of the agreement based on their shared solutions. The tutor wrote the agreement on the Agreement Sheet, a blank piece of paper visible to both players during all phases of game play.

Procedures to move on to the next problem. When the points did not match, the tutor initiated and supported joint reflection on sources of the discrepancy and ways of coordinating action on subsequent problems by following several heuristics, as appropriate. These procedures are identical to those described in Saxe et al. (2010).

1. Refer back to agreements. The tutor asked the student if he or she had considered any of the agreements when placing the point and, if so, how. Then, drawing upon the prior agreements to justify the placement, the tutor explained the placement of his/her own point (e.g., "I did mine this way because our agreements say…").

2. Refer back to problem. Sometimes a student's inappropriate construction resulted from his/her responding to a problem different from that printed on the playing card. In such cases, the tutor re-oriented the child to the problem card, stating "I did mine this way because the card said to find…".

3. Anticipate the next problem. After drawing upon the agreements or referring to the problem card, the tutor asked the student what each could do differently on the next problem card to try to place points at the same location.

Provisions for relative amounts of tutoring. Students were provided with three opportunities to pass 19 of the 22 problem sets. Each problem set had three parallel versions—A, B, and C[10] (Figure 56). The tutor proceeded differently as a function of whether or not the tutor and student successfully matched constructions. If a student passed at Version A for a given problem set, the tutor concluded the multi-phased procedures by moving on to Version A of the

[10] Three problem sets (Problem Sets I, II, and VII) required all students to complete each of the three versions.

next problem set. In cases when a student did not pass at Version A for a given problem set, the tutor concluded the multi-phased procedures by introducing the problem card for Version B of the same problem set. In all, students had three opportunities to pass at a given problem set. Students that did not pass at Version C moved on to the next problem set. Thus the tutor navigated through the different problem versions, drawing cards of the same problem set or the next problem set depending on the student's success.

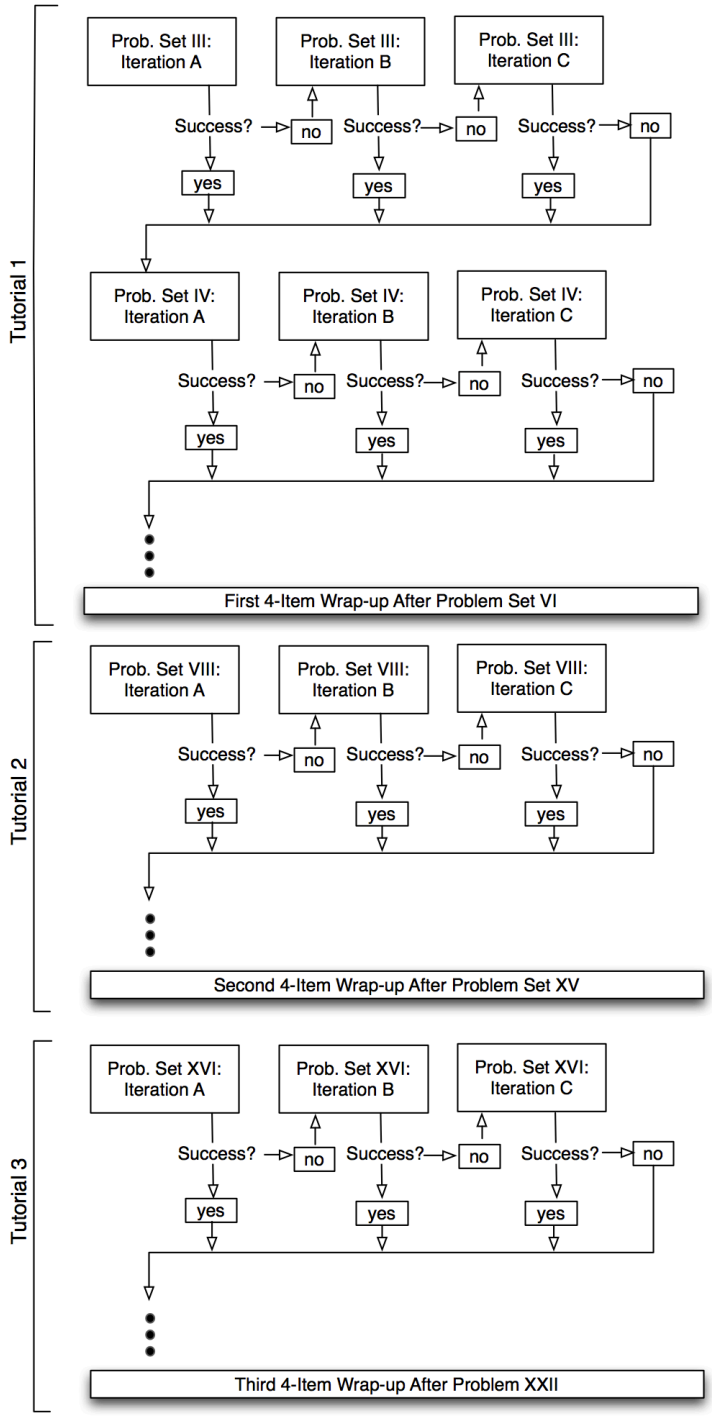


Figure 56: Sequencing of problem sets, iterations, and wrap-up items in the two tutorial sessions (adapted from Saxe et al., 2010)

Tutorial problem sets. The 22 problem sets used over the three tutorial sessions are provided in Appendix D. Earlier problems involved coordinating linear and numeric units on a number line (Problem Sets I-VI). Interim problems established the coordinate plane as two number lines with one rotated to be perpendicular to the other, and required players to quantify both axes and the plane (Problem Sets VII-XII). Later problems involved function graphs (Problem Sets XIII-XXII).

Each problem set served one of two functions with respect to content: introduce or practice. Problem sets alternated these two functions in order to introduce content and then provide students an opportunity to practice that content with slightly more difficult numbers. Odd numbered problem sets introduce content, whereas even numbered problem sets practice the same content.

Number lines: Problem Sets I-VI. The initial six problem sets involved the number line representation and introduced the game context to students, including the various physical materials, guidelines for game play, and the use of written agreements (Figure 57). Content focused on linear unit and numerical unit relations on the line. Five definitions were introduced in the context of number lines. I provide detail here about the problem card, the design of the representation, modeling used, and agreements introduced.

Problem Focus & Number		Problem Statement and Corresponding Grid	Rods (Physical Magnitude)	Number line/Grid
Intro to rod use	I	Mark where 3 is.	No rods used.	No units defined on the line.
	II	Mark where 4 is. 0	Units (student's choice of rod)	No units defined on the line. 0
Unit to multiunits quantified in rods	III	Mark where 6 reds is. Use BOTH the light green and dark green rods. 0	Units (student's choice of rod) purple; red | red	No units defined on the line. 0
	IV	Mark where 6 reds is. Use BOTH the light green and dark green rods. 0	Units (student's choice of rod) dark green; light green | light green	No units defined on the line. 0
Unit to multiunit quantified in rods and on the line	V	Mark where 5 reds is. You can use the red and purple rods. 1	Units (student's choice of rod) purple; red | red	Unit and multiunit lengths defined on the line. red purple 1
	VI	Mark where 4 light greens is. You can use the light green and dark green rods. 2	Units (student's choice of rod) dark green; light green | light green	Unit and multiunit lengths defined on the line. Lt green Dark green Lt green 2

Figure 57: Problem focus, problem number, and problem conditions (problem statement, rods used, and line used)

Problem Sets I and II—Introductory problems: The Order Agreement and the Same Distance Agreement. Problem Sets I and II introduced the game format, with content focused on properties of order and interval. Problem Set I focused on the order principle of number lines. In Problem I.A, the problem card instructed players to "Mark 3," with an equally partitioned line provided without any values. With each student, the tutor intentionally violated the order principle; the tutor ordered numbers on the line from right to left[11] (Figure 58). The resulting

[11] For cases in which the student ordered numbers from right to left, the tutorial protocol indicated the tutor would order numbers from left to right in order for conflict to arise; no participant in the present study did this.

discrepancy led to the need to negotiate an agreement about the *Order* of value on the number line. The agreement was written as the word "Order," with "greater" and "lesser" also written next to arrows pointing to the right and left, respectively.

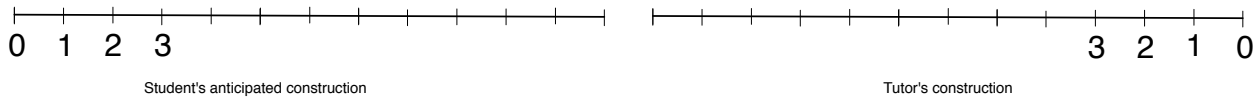


Figure 58: Student's anticipated construction and tutor's construction on Problem Set I

Focusing on the unit interval principle of number lines, Problem Set II introduced two written agreements. In Problem II.A, for which the problem card stated, "Mark 4," a line was provided featuring a tickmark labeled "0" at the left of the line. I anticipated students would construct a number line with equal intervals based on prior work (Saxe et al., 2010). The tutor constructed a line that deliberately violated the unit interval principle (Figure 59). The resulting discrepancy led to negotiating an agreement about *Unit Interval* on number lines, spoken out loud by the tutor as, "The distance between counting numbers has to be the same." The agreement was written as "Counting Numbers – Same Distance," and was accompanied with a drawing of a number line. Problem Sets I and II were the only occasions in the tutorial in which the tutor provided an incorrect solution, and two of only three problem sets for which all students completed all three versions. In problem II.B, the tutor introduced the C-rods in order to mark 4 again on a new number line. After observing the rod color chosen by the student, the tutor chose a different rod length, leading to discrepant placements of 4 on the line. This set the context to introduce the Rod Agreement, in which players agreed that rods need to be used in the same way. In problem II.C, players each used the rod color of the student's choice.

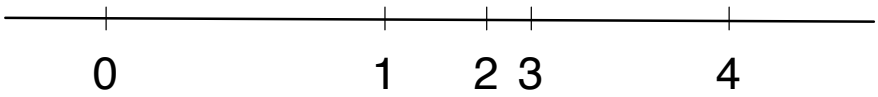


Figure 59: Tutor's incorrect construction for Problem II.A

Problem Sets III and IV: The Skip Counting Agreement and problems involving unit and multi-unit relations. Problem Sets III and IV focused on the coordination of linear units and numerical units. The problems involved 2:1 multiunit to unit relations with the lengths of C-rods and recording these rods on the line. In Problem III.A, the problem card instructed players to "Mark the length of 6 reds using the purple rods," and the number line provided was marked only with zero. To solve the problem, students needed to coordinate rod relations to record the purple length as an interval of 2. Building on the *Unit Interval* agreement, the *Multiunit Interval* agreement was introduced. It was spoken out loud by the tutor as, "the distance between numbers that skip count also have to be the same." The agreement was written as "Skip Counting Numbers – Same Distance," and was accompanied with a drawing of a number line to illustrate this. Problem Set IV provides an opportunity to apply this same agreement. This problem set used another pairing of rods with a 2:1 multiunit (dark green rod) to unit (light green rod) relationship.

Problem Sets V and VI: The Every Number Has a Place Agreement and coordinating unit and multiunit relations on non-routine number lines. Problem Sets V and VI used non-routine problem design to support students' continued coordination of linear units with numeric units. The problems involved 2:1 rod relations, first with red to purple rods (Problem Set V) and then

dark green to light green rods (Problem Set VI). In Problem V.A, the problem card instructed players, "Mark where 5 is. You can use the red and purple rods." The number line provided featured two intervals, one fitted to a red rod and the other to the purple, with "0" and "1" fixed on the line (see Figure 52c or Figure 57). To solve the problem, students needed to coordinate linear distance on the line with numerical values, and to treat rods as representing either an interval of one (red rod) or two (purple rod). To promote coordination of linear units and numerical units on number lines with unequal partitioning, the *Every Number has a Place but Not All Need to be Shown* agreement was introduced. It was spoken and written using the same wording. Problem Set VI provided an opportunity to apply this same agreement using the dark green and light green rods.

The Cartesian plane: Problem Sets VIII-XII. Five interim problem sets introduced the Cartesian plane representation (Figure 60). Content involved scale on two perpendicular axes and quantifying points in the plane. Three definitions were introduced: the A Point Shows Two Amounts Agreement; Each Number Line Can Count by Different Numbers Agreement; and We Don't Need Gridlines Agreement.

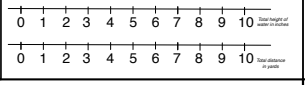
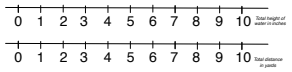
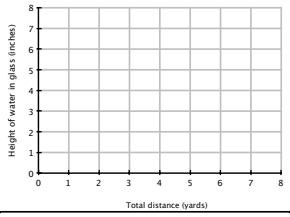
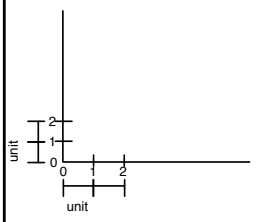
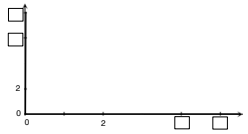
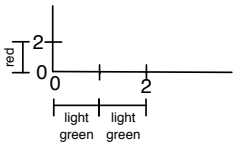
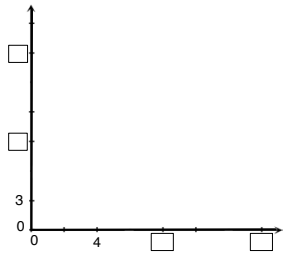
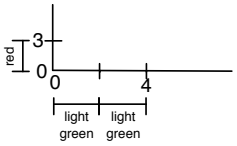
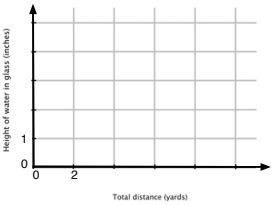
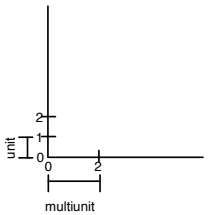
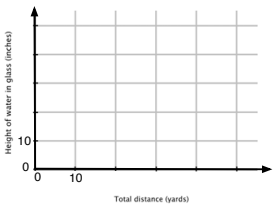
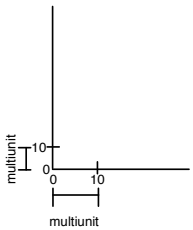
Problem Focus & Number		Problem Statement and Corresponding Grid	Rods (Physical Magnitude)	Number line/Grid
Transition to Cartesian plane	VII	Mark that Beetle walks 6 yards and earns 3 inches of water in his glass. Mark that Rudy walks 3 yards and earns 4 inches of water in his glass. Mark that Viggo walks 2 yards and earns 3 inches of water in his glass. 	No rods used.	Unit length defined on the number lines. 
Point placement	VIII	Mark that Beetle walks 2 yards and earns 6 inches of water in his glass. Mark that Rudy walks 2 yards and earns 4 inches of water in his glass. Mark that Viggo walks 1 yards and earns 6 inches of water in his glass. 	No rods used.	Unit length defined on the axes. 
Scale on two axes	IX	Someone forgot to write in the values on each number line. Fill them in! 	Horizontal: units (light green) Vertical: multiunits (red)	Unit and multi-unit lengths defined on axes. 
	X	Someone forgot to write in the values on each number line. Fill them in! 	Horizontal: multiunits (light green) Vertical: multiunits (red)	Unit and multi-unit lengths defined on axes. 
Gridlines	XI	Mark that Beetle walks 3 yards and earns 5 inches of water in his glass. 	No rods used.	Unit length defined on the axes. 
	XII	Mark that Rudy walks 11 yards and earns 15 inches of water in his glass. 	No rods used.	Unit length defined on the axes. 

Figure 60: Problem focus, problem number, and problem conditions (problem statement, rods used, and line used) for Cartesian plane problems

Problem Sets VII and VIII: A Point Shows Two Amounts Agreement and creating the coordinate plane. Problem sets VII and VIII introduced the Cartesian plane and points in the plane. Problem Set VII was atypical in that all students were administered all three versions of the problem set, similar to Problem Sets I and II; Problem Set VIII followed the format as the other nineteen problem sets with three versions and differentiated amounts of tutoring.

Problem Set VII introduced a double number line to then rotate one to be perpendicular to the other to create the plane (see Carraher et al., 2008, for a similar treatment in a classroom teaching experiment). Players plotted multiple ordered pairs on a double number line to thereby create a representation difficult for tutees to parse or analyze quickly (Figure 61). The difficulty set a context for a different way to represent those ordered pairs. The problem card stated, "Beetle the cat earns water in his glass when he runs. His glass is measured in inches so you can see exactly how much water he has. Mark that he walks 6 yards and earns 3 inches of water." The double number lines provided were canonical lines with unit partitioning. Three props were introduced: a container marked vertically in inches, a number line with units identified in yards, and a second container filled with water. Immediately after plotting quantities on each number line, the tutor turned over the card for Problem VII.B, which stated, "Use the same lines to mark that Rudy the Cat walks 3 yards and earns 4 inches of water." This was followed by a card for problem VII.C, "Use the same lines to mark that Viggo the cat walks 2 yards and earns 3 inches of water." Engaging the tutee in discussion regarding the difficulty parsing the original ordered pairs from the finished number lines, the tutor suggested rotating one number line to be perpendicular to the other, and then used a single point to show the two quantities. Distributing a prepared grid to each player, the tutor then suggested that they each re-solve the three directives on the playing cards the three cards on the new grid. This led to an agreement, *One point shows two amounts*, which was spoken and written in this way. Problem Set VIII provided an opportunity to practice plotting points in the plane.

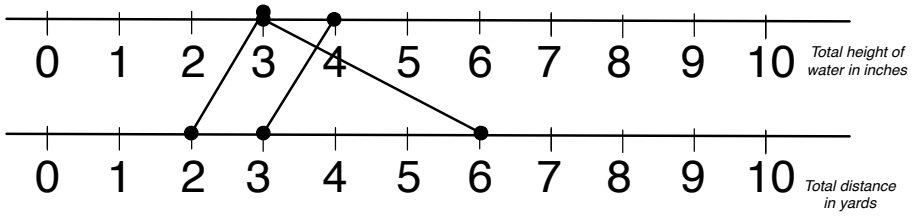


Figure 61: Double number line in Problem Set VII

Problem Sets IX and X, The Each Number Line Can Count by Different Numbers Agreement and scale on two axes. Problem Sets IX and X used a non-routine problem design to support multiplicative relations along two axes. For all problems in problem sets IX and X, directives on the problem cards stated, "Someone forgot to finish writing the values on each axis. Fill them in!" With units defined on each axis, the grid featured unequal partitioning along both the vertical and horizontal axes, with intervals along each axis fitted to either the light green rod or the red rod. To solve the problem, students needed to coordinate linear units on the line with numerical values, and treat rods as representing either an interval of 2 (light green rod on the horizontal axis) or an interval of 1 (red rod on the vertical axis). To promote coordination of linear distance and numerical values on two axes, an agreement was introduced, spoken and written as: *each number line can count by different numbers, but both MUST follow all agreements*. Problem Set X provided students an opportunity to apply this same agreement,

though in this case, the rods represented an interval of 4 (light green rod on the horizontal axis) or an interval of 3 (red rod on the vertical axis).

Problem Sets XI and XII: The We Don't Need Gridlines to Find a Point Agreement and plotting points off of gridlines. Problem Sets XI and XII used non-routine problem design to support treatment of any point in the plane—not just those falling along gridlines—as representing two amounts, each of which indexes a value along each axis. In Problem XI.A, the problem card stated, "Mark that Beetle walks 3 yards and earns 5 inches of water in his glass. You may use the rods." The grid provided featured a multiunit interval of 2 on the horizontal axis, and a unit interval on the vertical axis. To solve the problem, students needed to treat each axis as continuous, with coordination of linear units and numerical units supporting point placement off of a gridline. To support students' in considering non-gridline points as representing two values, an agreement was spoken and written as, *We don't need gridlines to find a point, but sometimes they can be helpful*. Problem Set XII provided an opportunity to apply this same agreement with larger numbers: an interval of 10 is provided on each axis, and players were instructed to "Mark that Rudy walks 11 yards and earns 15 inches of water in his glass."

Graphing linear functions: Problem Sets XIII-XXII. The final ten problem sets involved graphing functions in the coordinate plane (Figure 62 and Figure 63). The goal of these problem sets was to introduce a functional relation to students, and then to plot that joint variation in the plane. Two final definitions were introduced in this section: the We Can Connect Points to Show Many Points Agreement and the Story Decides What the Graph Looks Like Agreement.

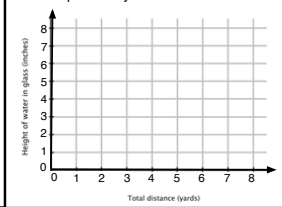
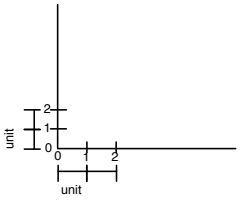
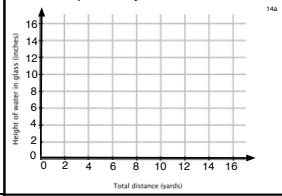
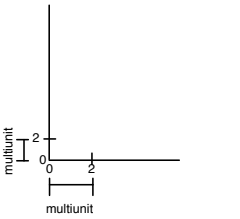
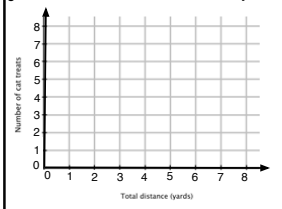
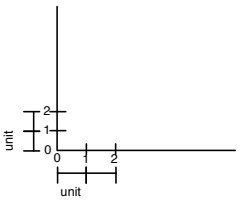
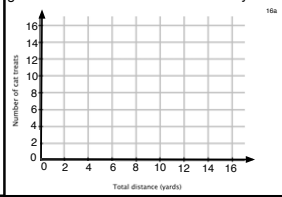
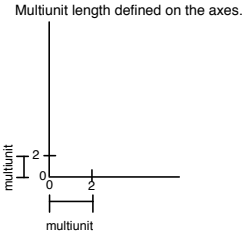
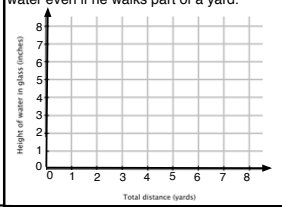
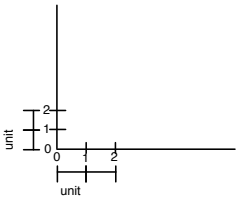
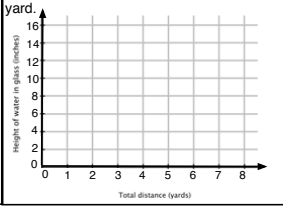
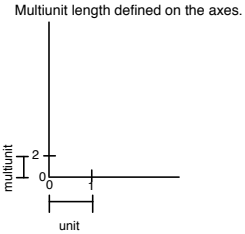
Problem Focus & Number		Problem Statement and Corresponding Grid	Rods (Physical Magnitude)	Number line/Grid
Introducing linear functions	XIII	Beetle starts with no water. Mark that he earns 2 inches of water in his glass for each yard he runs. He gets water even if he walks part of a yard. 	No rods used.	Unit length defined on the axes. 
	XIV	Rudy starts with no water. Mark that he earns 4 inches of water in his glass for each yard he runs. He receives water even if he walks part of a yard. 	No rods used.	Multiunit length defined on the axes. 
Piecewise functions	XV	Beetle starts with no cat treats. Mark that he earns 2 treats for each yard he runs. He gets his treats once he finishes each yard. 	No rods used.	Unit length defined on the axes. 
	XVI	Rudy starts with no cat treats. Mark that he earns 2 treats for each yard he runs. He gets his treats once he finishes each yard. 	No rods used.	Multiunit length defined on the axes. 
Linear functions with a positive y-intercept	XVII	Beetle starts with 2 inches of water in his glass. Mark that he earns 2 inches of water in his glass for each yard he walks. He gets water even if he walks part of a yard. 	No rods used.	Unit length defined on the axes. 
	XVIII	Beetle starts with 2 inches of water in his glass. Mark that he earns 2 inches of water in his glass for each yard he walks. He gets water even if he walks part of a yard. 	No rods used.	Multiunit length defined on the axes. 

Figure 62: Problem focus, problem number, and problem conditions (problem statement, rods used, and line used) for the first set of graphing functions problems

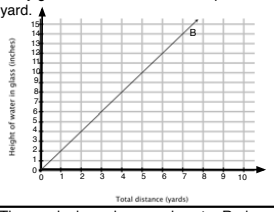
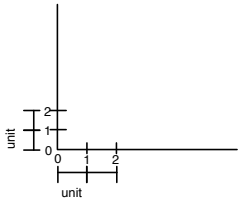
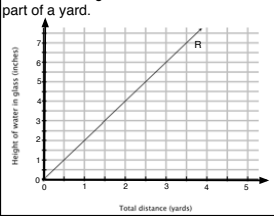
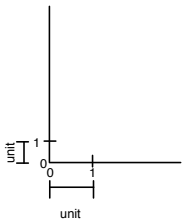
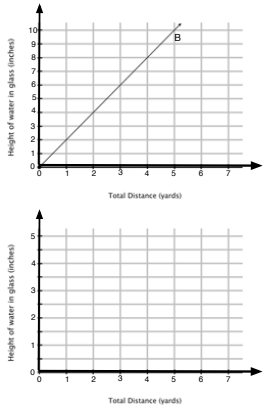
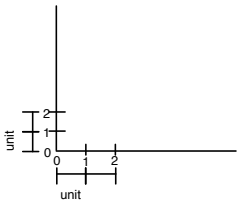
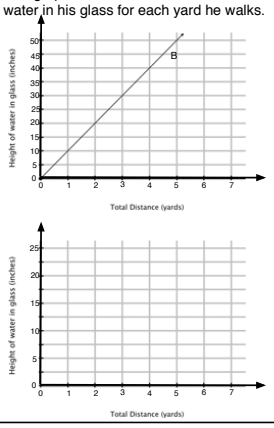
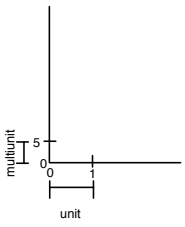
Problem Focus & Number		Problem Statement and Corresponding Grid	Rods (Physical Magnitude)	Number line/Grid
Comparing functions	XIX	The graph shows how much water Beetle earns. Show on the same graph that Rudy starts with no water. He earns 6 inches of water in his glass each yard he walks. Rudy gets water even if he walks part of a yard. 	No rods used.	Unit length defined on the axes. 
	XX	The graph shows how much water Rudy earns. Show on the same graph that Beetle starts with no water. He earns 1 inches of water in his glass each yard he walks. Beetle gets water even if he walks part of a yard. 	No rods used.	Multiunit length defined on the axes. 
Comparing functions across grids with different scales	XXI	The graph shows that Beetle earns 2 inches of water each yard he walks. Show on the new graph that Beetle still earns 2 inches of water in his glass for each yard he walks. 	No rods used.	Unit length defined on the axes. 
	XXII	The graph shows that Beetle earns 2 inches of water each yard he walks. Show on the new graph that Beetle still earns 2 inches of water in his glass for each yard he walks. 	No rods used.	Unit and multiunit length defined on the axes. 

Figure 63: Problem focus, problem number, and problem conditions (problem statement, rods used, and grid used) for second set of graphing functions problems

Problem Sets XIII and XIV: The We Can Connect Points to Show Many Points Agreement and introducing linear functions. Problem sets XIII and XIV featured the introduction of linear functions. The initial problem card in problem XIII.A stated, "Beetle starts with no water. Mark that he earns 2 inches of water in his glass for each yard he runs. He gets water even if he walks part of a yard." Before distributing the corresponding grid, the tutor and tutee used the props to enact the functional relation stated on the card. Using the picture of a cat glued onto a popsicle stick, the tutee represented the cat walking. At the same time, the tutor filled the graduated container according to the problem card and where the tutee positioned the cat. For example, as the tutee moved the cat to 1 yard, the tutor filled the container to 2 inches of water.

The tutor then distributed the grid, which featured unit scaling on both horizontal and vertical axes and gridlines for each integer. To solve the problem, students needed to represent positive values for the underlying function on the problem card (in this case, $f(x)=2x$). To support an understanding of a function line as being constituted by infinitely many points, the *We Can Connect Points to Show Many Points* agreement was introduced. When spoken, the tutor emphasized the idea of infinity—though not the word—by repeating the word *many*: "We can connect points to show many, many, many, many, many points." Problem Set XIV provided an opportunity to practice, with multiunit distances of 2 identified on both horizontal and vertical axes.

Problem Sets XV and XVI: The Story Decides What the Graph Looks Like Agreement and piecewise functions. Problem Sets XV and XXVI featured a non-linear functional relation in the form of a step function[12] to provide a contrast to points in the plane connected by a line. A goal of such a contrast was to provide a new context to coordinate precise placement of points with indices along axes. The grids provided were identical to the sequence in Problem Sets XIII and XVI. The problem card for problem XV.A stated, "Beetle starts with no cat treats. Mark that he earns 2 cat treats for each yard he runs. He gets his treats once he finishes each yard." In this case, the final sentence on the card was intended to support a step—versus a linear—function (Figure 64). Once again, the tutor and tutee modeled the functional relation on the problem card. The tutee moved the cat along the yard line; once the tutee moved the cat across an integer, the tutor placed two manipulatives on the table. After modeling and addressing any questions regarding the joint variation and the problem card, players solved the problem. In order to do so, they needed to show on the graph that for the distance between 0 to just before 1 yard, the cat had 0 treats; for the distance between 1 and just before 2 yards, the cat had 2 treats, and so on. To support the relationship of the context to inscriptions in the plane, the tutor introduced the *Story Decides what the Graph Looks Like* Agreement. Problem Set XVI provided an opportunity to apply the agreement, with a grid that was scaled by 2s on each axis.

[12] A step function is another functional relation in which the dependent variable remains constant within each of a series of adjacent intervals but changes in value from one interval to the next.

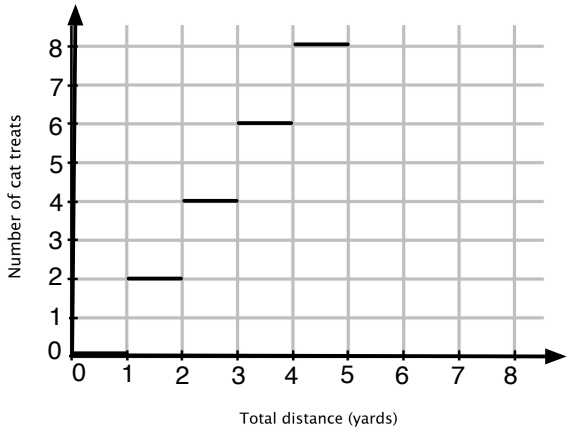


Figure 64: Step function for Problem XV.A

Problem Sets XVII and XVIII: Linear functions with a positive y-intercept. Problem Sets XVII and XVIII featured linear functions with a non-zero y-intercept. A goal of this content was to address a common problem for students beginning to graph functions. The problem card directive drew attention to the invariant—the amount of water with which Beetle the Cat starts—as well as the variant—the increasing amount of water he earns with each additional yard. In Problem XVII.A, the problem card stated, "Beetle starts with 2 inches of water in his glass. Mark that he earns 2 inches of water in his glass for each yard he walks. He gets water even if he walks part of a yard." The grid provided featured unit scaling on each axis. To solve the problem, students needed to consider the invariant and the variant as well as the relation between the two. For example, students needed to coordinate that Beetle started with 2 inches of water, but then also received 2 inches of water with each yard he walked; if Beetle walked 1 yard, he earned 2 inches of water as a function of distance walked, yet would have 4 inches of water total. Problem XVIII.A featured a similar problem card. The grid for this problem featured unit scaling on the horizontal axis and multiunit intervals of 2 on the vertical axis. For these problem sets and all subsequent problem sets, no new agreement was introduced. Instead, these problem sets served as an opportunity to apply and extend prior agreements in more complex problems.

Problem Sets XIX and XX: Comparing functions. Problem Sets XIX and XX engaged tutees in comparing two linear functions on a single grid. A goal of this content was to support students in comparing two linear functions in the same grid. In Problem XIX.A, the problem card stated, "The graph shows how much water Beetle earns. Show on the same graph that Rudy starts with no water. He earns 6 inches of water in his glass each yard he walks. Rudy gets water even if he walks part of a yard." The grid featured a linear function ($f(x)=2x$) and was marked with a "B" to stand for Beetle (see Figure 63). Unit intervals were provided on both horizontal and vertical axes. To solve the problem, students needed to plot the function independent from the linear function already on the grid. Problem set XX had problem cards with almost identical wording, and a grid featuring a linear function already on it. Unlike the prior problem set, on Problem XX.A both vertical and horizontal axes had gridlines and tickmarks for intervals of 0.5, although only positive integers and 0 are inscribed.

Problem Sets XXI and XXII: Comparing functions across grids with different scales. The final two problem sets in the task sequence engaged students comparing a single function on two grids with different scaling. The goal of this content was to coordinate the joint variation of the functional relation on grids with different scales. In problem XXI.A, the problem card stated,

"The graph shows that Beetle earns 2 inches of water each yard he walks. Show on the new graph that Beetle still earns 2 inches of water in his glass for each yard he walks." A single sheet of paper was provided that featured two grids. The top grid displayed the linear function, with each axis featuring unit scaling. The bottom grid, on which players were instructed by the playing card to inscribe the same function, featured unit scaling on the horizontal axis, and intervals of 0.5 on the vertical axis. To solve the problem, students must coordinate their inscriptions in the plane with values along the axes rather than relying on linear grid units. Problem XXII.A was similar, though the values along the axes were different. In this problem set, the top grid (with a linear function plotted on it) featured multiunit intervals of 5. The bottom grid, on which players were instructed to re-inscribe the same function, the vertical axis featured intervals of 2.5, though only multiples of 5 are inscribed along the axis.

Wrap-ups. All three tutorial sessions terminated with a 'wrap-up' phase. During the wrap up problem phase, the tutor presented the student with the wrap-up problems on a single sheet of paper, and removed all other tutorial materials. As the student solved each wrap-up problem, the tutor asked the student for a justification and, if not volunteered by the student, asked if any agreements were used. Three sets of two to four wrap-up problems were linked to the mathematical content of the respective tutorial session in order to consolidate and synthesize content from that session through supporting agreement usage. The wrap-up problems consisted of non-routine number lines and graphs printed on paper. Wrap-up problems had two formats. The first featured number lines or graphs with boxes for the student to indicate whether the number line or graph is "correct" or "incorrect" and to use the agreements to justify their answer orally. The second type involved matching a story to a graph. For both types, if students answered accurately and used a relevant agreement to justify the answer, the tutor moved on to the next wrap-up problem. If the student responded correctly and did not provide the relevant agreement to support it, the tutor asked explicitly about the relevant agreement. If the student did not indicate the target agreement, the tutor asked the tutee specifically about it. The complete set of problems used in the wrap-up problems is contained in Appendix E.

Results

The results are presented in two sections corresponding to the two research questions (see Table 11). First, I examine if the tutorial intervention supported gains in order to determine tutorial efficacy. To do this, I compare performance on the assessment for tutorial and control groups before and after intervention (or similar duration of time for the control group). Second, I examine what student performances in the tutorial indicate about their successes and hurdles. Within this, I examine correlation between students' uptake of agreements and learning gains. Following this, I analyze pathways through the tutorial problem sets and patterns in coordination efforts across a subset of tasks for both successful and unsuccessful responses. I conclude with a case illustrating one student's pathway through one part of the tutorial to provide further insight into the dynamics of learning in this instructional context.

Table 11: Overview of research questions, participants, data sources and analyses

Research Question	*Participants*	*Data Sources*	*Analyses*
1. Does the tutorial intervention support gains?	• Experimental and control Grade 5 students • Grade 8 assessment students (Chapter 2)	Pre- and post assessments	• ANOVA contrasting tutorial and control gains
2. What do student performances in the tutorial indicate about their successes and hurdles?	Experimental students	• Written student work from tutorial problem sets • Video of tutorial sessions • Pre- and post-assessments	• Correlational analyses of appropriate/ inappropriate agreement use and learning gains • Analysis of pathways through the tutorial problem sets • Constructions students produced in their successful and unsuccessful solutions to Problem Sets XIII and XIV • Case of a student's tutorial interactions, agreement use, and learning progress through Problem Sets XIII and XIV

Effects of the tutorial intervention. To determine whether the tutorial supported students' understanding of linear representations of quantities, I contrasted pre- and post-test scores of experimental and control students. The group contrast served two purposes. First, the contrast provided a control for regression to the mean, which is a potential threat to validity; recall that students in either group were the lower achieving two-thrids of students from the assessment study (Chapter 2), and thus there is a small chance that gains from pre- to post-assessment could be an artifact of sampling. Second, students were paired based on identical pre-assessment score and membership in the same classroom; the random assignment of either pair to the tutorial or control group served to control for possible practice effects and classroom experiences that may influence post-assessment performance. I first present results on all 32 items of the assessment for each group. I then focus on each group's performance for the 18 matched problems for the pre- and post-test. I conclude this section by comparing the Grade 5 experimental students' performance on the 18 matched problems with the Grade 8 students' performance on the same problems, as described in the assessment study (Chapter 2).

Overall assessment results for experimental and control groups. Figure 65 displays pre- and post-assessment performance of the tutorial and control groups. At pretest, groups showed a similar distribution of scores (means and standard deviations are provided in Table 12). Mean scores at pretest were 13.10 (*sd*=2.99) and 13.05 (*sd*=2.98) out of 32 items for the experimental and control groups, respectively. The experimental group had much greater gains on posttest. For the experimental group, the gain was 10.15 points (*sd*=4.36) whereas for the control, the mean gain was 1.20 points (*sd*=5.21). For the tutorial group, the gain represented a shift of 2.33 standard deviations.

Table 12: Means and standard deviations for tutorial study (experimental and control groups)

	Pre-test		Post-test	
	Mean	Standard Deviation	Mean	Standard Deviation
Experimental	13.10	2.99	23.25	4.36
Control	13.05	2.98	14.25	5.21

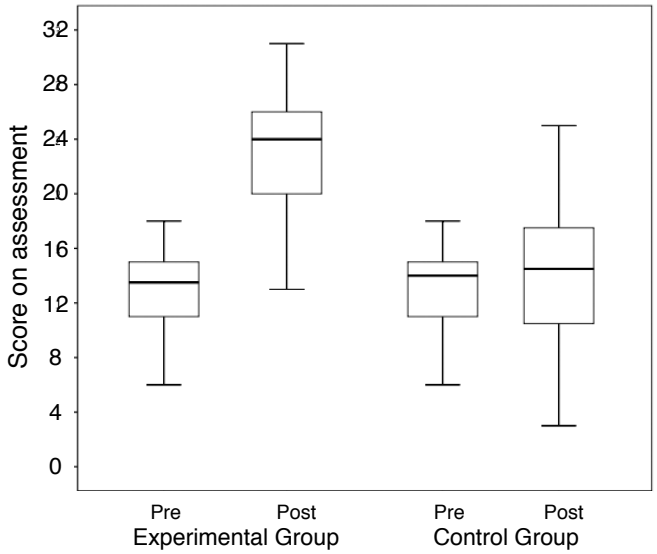


Figure 65: Boxplots of pre- to post-assessment performances for control and tutorial groups

A 2 (group) X 2 (time) ANOVA revealed a main effect for time of test administration (pre or post) ($F(1, 38) = 90.028, p < .0001$), indicating a difference in performance as a function of time, as well as a main effect for group ($F(1, 38) = 16.478, p < .0001$), indicating a difference in performance as a function of group. Analyses also revealed an interaction between group and time ($F(1, 38) = 55.980, p < .0001$), and follow-up analyses revealed a significant difference between experimental students' pre- and post-test performance ($t(19) = 17.058, p < .0001$), but no difference between pre- and post-test for the control ($t(19) = 1.156, p = 0.262$). The experimental group showed a statistically significant gain, with an effect size from pre- to post-test ($d = 2.71$) found to exceed Cohen's (1988) convention for a large effect ($d = .80$).

Matched problem performance for tutorial and control groups. I now present results on the 18 matched problems on the pre- and post-test by group. Figure 66 displays the pre- and post-test performance on the matched problems (routine and non-routine) by group (experimental and control). A 2 (time) X 2 (problem type) X 2 (group) ANOVA showed a main effect of time with students being significantly more likely to achieve a higher test score at post-test than at pre-test ($F(1, 38) = 75.199, p < .0001$), and a main effect for problem type, with students achieving a greater number of correct items on routine problems as compared to non-routine problems ($F(1, 38) = 102.912, p < .0001$). In addition, there was a significant interaction between time (pre-test or post-test) and group (experimental or control), $F(1, 38) = 44.496, p < .0001$, suggesting that performance on pre-test and post-test differed as a function of group. There was also a three-way interaction among problem type, group, and time ($F(1, 38) = 4.113, p = .05$). There was no interaction between problem type and tutorial group ($p = .252$) or time and problem type ($p = .066$).

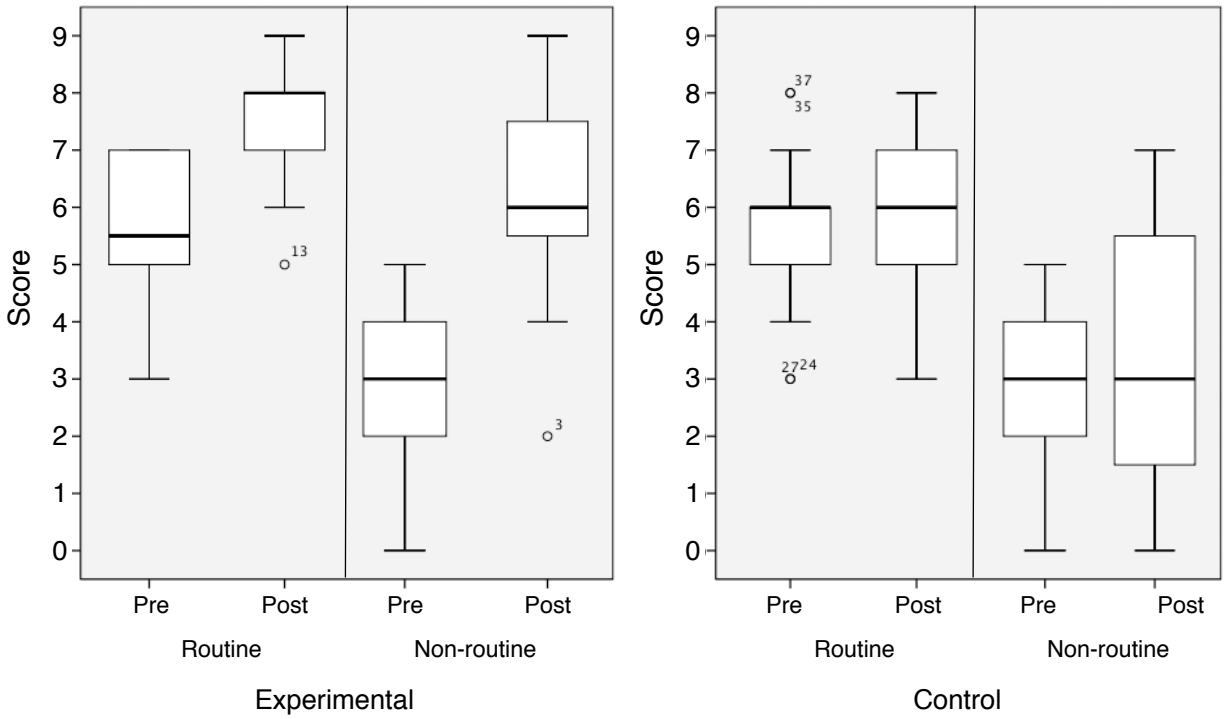


Figure 66: Boxplots for experimental and control groups displaying performance on routine and non-routine problems at pre-test and post-test

Follow-up analyses looked at experimental and control groups separately. For the experimental group, a 2 (time) X 2 (problem type) ANOVA revealed a main effect for time ($F(1, 19) = 197.078, p < .0001$), with students being significantly more likely to achieve a greater number of correct items at post-test as compared to pre-test. There was also a main effect for problem type (routine or non-routine) ($F(1, 19) = 90.611, p < .0001$), with students being significantly more likely to achieve a greater number of correct items on routine problems as compared to non-routine problems. In addition, there was an interaction between time and problem type ($F(1, 19) = 6.489, p = 0.020$). A paired samples t-test revealed a significant gain in performance for routine problems ($t(19) = -8.826, p < 0.0001$), with an effect size from pre- to post-test ($d = 1.85$) found to exceed Cohen's (1988) convention for a large effect ($d = .80$). For non-routine problems, a paired samples t-test also revealed a significant gain in performance ($t(19) = -8.340, p < 0.0001$) that also had a large effect size from pre- to post-test ($d = 2.31$). For the control group, analyses revealed a main effect for problem type ($F(1, 19) = 41.113, p < .0001$), indicating that control group students achieved a greater number of correct items on routine problems as compared to non-routine problems. Time was not found to have a main effect ($p = .247$), indicating that performance was not different as a function of pre-test or post-test. There was no significant interaction in performance based on problem type and time ($F(1, 19) = .011, p = .917$). These analyses suggest that that the experimental group achieved significantly higher scores at post-test on both routine and non-routine problems, whereas the control group did not demonstrate differences in performance based on time.

Matched problem performance for Grade 5 experimental students and Grade 8 students. Quantitative results thus far reveal that the experimental group displayed gains from pre- to post-assessment when compared to a similar control group using, first, all assessment

problems and, second, the sub-scale of matched problems. Chapter 2 presented findings on the LRA for Grade 8 students. Analyses reported above showed a large effect size from pre- to post-test for the experimental group. I now consider another way to evaluate the magnitude of the effect: were the Grade 5 experimental gains equivalent to performance of Grade 8 performances on the same assessment? Recall that Grade 5 experimental students were among the lower performing two-thirds of the entire sample. In order to further explore the magnitude of the effect, I analyze performance of the experimental group of lower performing Grade 5 students with the lower performing two-thirds of Grade 8 students.

Figure 67 displays boxplots for performance for the two groups by problem type: the Grade 5 experimental group *after* intervention and the lower performing two-thirds of Grade 8 students (n=90, highest possible total score of 24 items correct). Note that because of the cut point, the most items correct for a Grade 5 student was 19, whereas for Grade 8 students it was 24. A 2 (problem type) X 2 (group) ANOVA revealed a main effect for problem type ($F(1, 108) = 92.255$, $p < .0001$), indicating that students achieved a greater number of correct items on routine problems as compared to non-routine problems, as well as a main effect for group, ($F(1, 108) = 7.348$, $p = 0.008$). Analyses also revealed an interaction ($F(1, 108) = 14.420$, $p < 0.0001$) between problem type and group. Follow-up analyses looked at performance on each problem type separately to compared performances of Grades 5 and 8 students. For routine problems there was no detectable difference in performance ($t(108) = .413$, $p = .680$ (two-tailed)). For non-routine problems, analyses revealed a significant difference based on group ($t(108) = 3.653$, $p < 0.0001$ (two-tailed)). This suggests that among lower-performing students in Grade 8 compared to Grade 5 experimental students, there was no detectable difference in performance on routine problems, and that Grade 5 tutorial students after intervention outperformed Grade 8 students.[13]

[13] A 2 (problem type) X 2 (group) ANOVA was also conducted to compare Grade 5 experimental students at post-test with the entire sample of Grade 8 students. Analyses revealed a main effect for problem type ($F(1, 149) = 60.454$, $p < .0001$) and an interaction between problem type and group ($F(1, 149) = 4.598$, $p = .034$), but no main effect for group ($F(1, 149) = .176$, $p = 0.676$). Follow-up analyses revealed no detectable difference in performance for either routine ($t(149) = -.986$, $p = .326$) or non-routine ($t(149) = 1.143$, $p = .255$) problems.

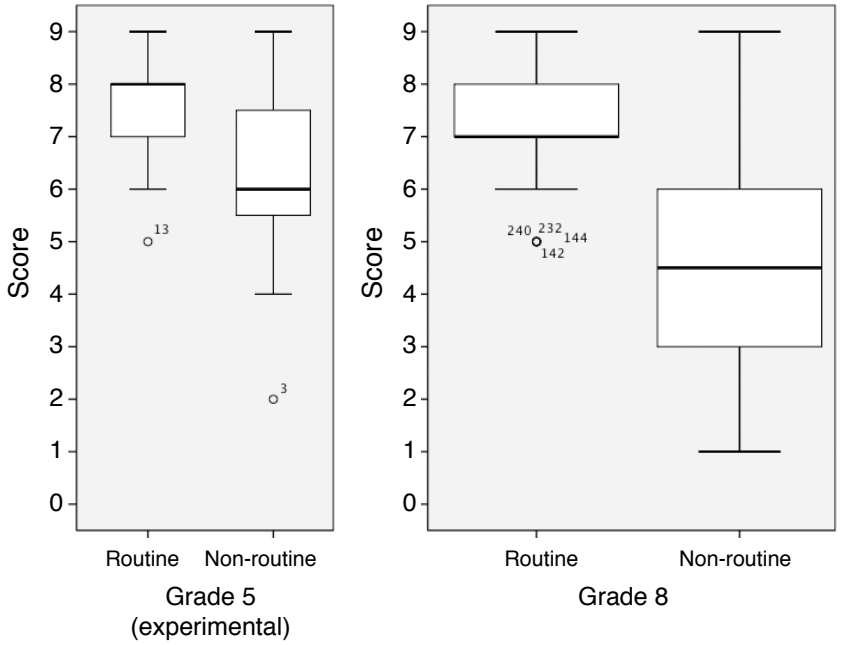


Figure 67: Boxplots for performance on non-routine problems for Grade 5 experimental students and the lower performing two-thirds of Grade 8 students

Student performances in the tutorial. Given that the analyses of efficacy revealed that tutorial students had learning gains as a measure of pre- to post-assessment, I now turn to analyses of the character of students' performance during the tutorial sessions. This section features four separate analyses. The first is concerned with the uptake of agreements during the tutorial session, and specifically whether tutorial students' mindful use of agreements predicted learning gains. The second is a qualitative analysis of pathways through the tutorial, by which I mean a comparison of the 20 students' individual performances on tutorial problem sets. This analysis points to areas that in particular were sources of success or struggle across students. The third analysis examines students' conceptual coordinations in two particular areas of the tutorial on which most or all students required substantial tutorial support. The final analysis is a case study of one student and her coordination efforts as she solved problems.

Mindful use of agreements and learning gains. The purpose of this analysis is to understand if the uptake of agreements supported student learning. I expected that over the course of the three tutorial sessions, students would vary in whether they incorporated mindful use of the agreements in reasoning about linear representations of quantities. Further, I expected that developing active use of agreements would support gains in learning and, therefore, serve as a predictor for learning gains. To produce an index of agreement use in the tutorial, I replicated an analysis conducted in Saxe et al. (2010) in which video and written work from the wrap-up problems was used to produce an index of agreement use. Recall that for each wrap-up problem, students evaluated the adequacy of a problem keyed to particular agreements. For each wrap-up problem, I coded and analyzed (1) the correctness of students' initial judgments, (2) the extent of spontaneous appropriate and inappropriate agreement use, and (3) the relation between appropriate and inappropriate agreement usage and learning gains.

Students' initial judgments of wrap-up problems. Figure 68 shows the distribution of students' initial correct judgments by wrap-up problem (see Appendix E for all wrap-up problems). Students' performances varied by problem. The number of students administered a given problem varied on each wrap-up problem due to constraints of time that varied from session to session. Across problems, students had more difficulty with (a) problems involving interpreting a particular context together with a linear function graph (Problems 7 and 10), and (b) non-routine representations (Problem 2).

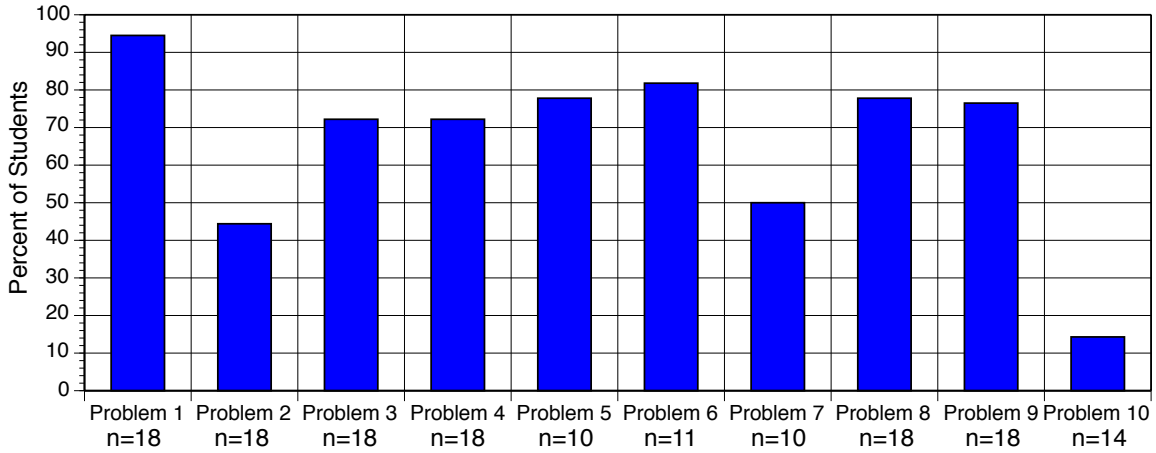


Figure 68: Percent of students producing correct initial evaluations of wrap up problems

Spontaneous appropriate and inappropriate agreement use. After students provided an initial judgment about a wrap-up problem, the tutor asked for a justification. I coded the agreements that students referenced in their justifications and whether students used these agreements appropriately or inappropriately. I coded *appropriate agreement use* when students spontaneously and correctly applied an agreement in order to argue appropriately for a given response to the wrap-up problem. To illustrate, wrap-up problem 5 features coordinate axes with non-congruent linear units and different scales, and offered students to indicate whether the axes had been marked correctly or incorrectly. One student, ID 5111, received an *appropriate agreement use* code. She wrote in tickmarks and values to create congruent linear units on each axis, and stated that she knew where "40" should be along the horizontal axis by referring to the agreement that states that every number has a place but not all need to be shown (Figure 69). To clarify further, she wrote in all values along the horizontal axis to create equal intervals, thereby coordinating the prior agreement with the agreement for multiunit intervals.

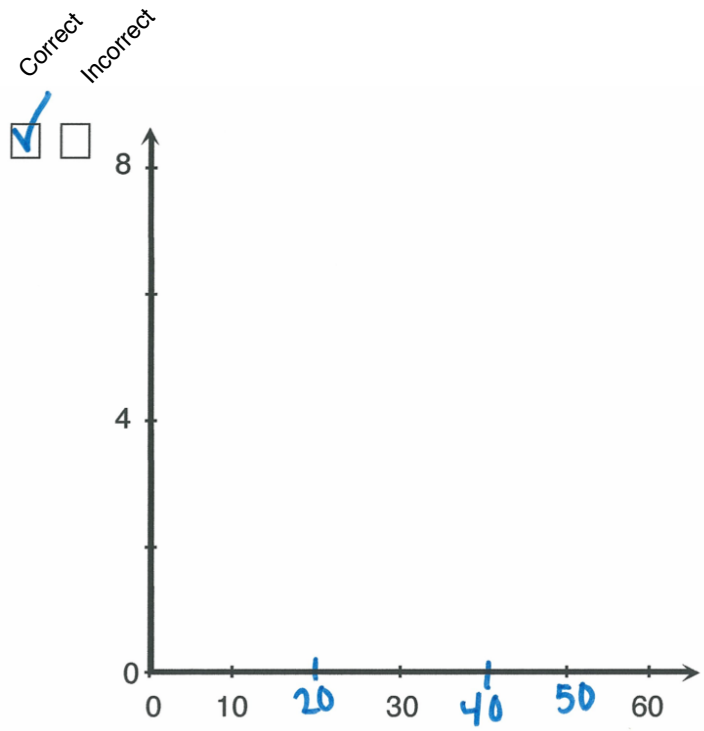


Figure 69: ID 5111's Wrap Up Problem 5 coded as appropriate agreement use

Inappropriate agreement use was coded when students applied an agreement incorrectly to a wrap-up problem, drawing on an agreement inappropriately to support a judgment that an adequately drawn linear representation was incorrect, or that an inadequate linear representation was correct. To illustrate, again consider wrap-up problem 5. ID 5202 received an *inappropriate agreement use* code. She determined that the vertical axis was marked correctly, but then identified two differently sized intervals along the horizontal axis, an interval of 10 (from 0 to 10) and an interval of 20 (from 10 to 30), invoking an inappropriate application of the multiunit interval agreement. These two different intervals indicated to her that the axes were not marked correctly.

Relation between agreement use and learning gains. The purpose of this analysis was to determine whether mindful use of agreements predicted learning gains. I first analyze the appropriate use of agreements and gain scores, and then analyze inappropriate use of agreements and gain scores. Like Saxe at al. (2010), I used the frequencies of appropriate and inappropriate student agreement use over the course of the wrap-up problems as estimates of the character of agreement use over the tutorial sessions. For each analysis—appropriate agreement use and learning gains, and then inappropriate agreement use and learning gains—involved all wrap up problems. Figure 70 displays appropriate and inappropriate agreement usage as frequency counts across the ten wrap up problems for each experimental student. In total, students were coded as *appropriate agreement use* 164 times over the ten wrap up problems, while they were coded as *inappropriate agreement use* only 52 times. Fifteen of the tutorial students were more likely to be coded for appropriate agreement use more often than inappropriate use. Three students (ID 5110, ID 5119, and ID 5128) had identical frequencies for appropriate and inappropriate use, though in all three cases students drew upon the agreements (appropriately or inappropriately) eight or fewer times. In only two cases, a student had a higher frequency of

inappropriate use (ID 5133 and ID 5410). Both students received two of the three lowest scores on the pre-assessment.

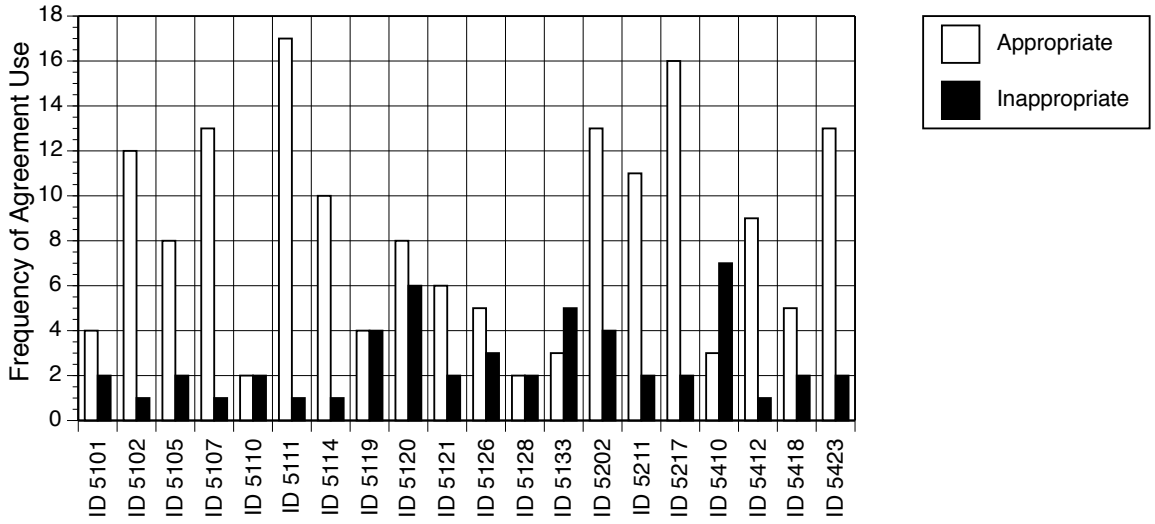


Figure 70: Frequency of appropriate and inappropriate agreement use

To determine whether agreement use predicted learning gains, I correlated the two measures of agreement use (appropriate and inappropriate) with gains, controlling for pre-assessment score. Analyses revealed a correlation between students who used agreements appropriately in the wrap-up sessions and higher gain scores (r (N = 20) = .456, p = .049). No correlation was found between inappropriate agreement use and gain scores (r (N = 20) = .001, p = .998).

Pathways through the tutorial. I next examine individual pathways through the tutorial sessions. In the tutorial, students required differential amounts of support as they engaged in tutorial problem sets, consistent with any typical classroom with diverse learners. Recall that the design feature of three parallel versions for each problem set allowed me to operationalize the amount of support students needed to succeed with a problem set or move on to the next one (in cases for which students did not pass). Accordingly, students may pass at version A, B, or C, or alternatively, not pass at C. To understand whether the tutorial intervention supported student learning, I coded the iteration at which students passed: Pass at A, Pass at B, Pass at C, or No Pass.

Figure 71 displays the twenty pathways through the tutorial sessions based on the version at which students passed. Divided in three sections based on representation, the pathways reveal trends across students. For example, Problem Set XIV shows much variation, with different students passing at A, B, or C, and still others not passing. Alternatively, on the prior Problem Set XIII, students in general required more support as compared to other problem sets, with no students passing at A. In this section, I provide detail on each of the three representations using Figure 70 to anchor the descriptions. I then provide a more detailed analysis of Problem Sets XIII and XIV in which linear function graphs are first introduced and practiced.

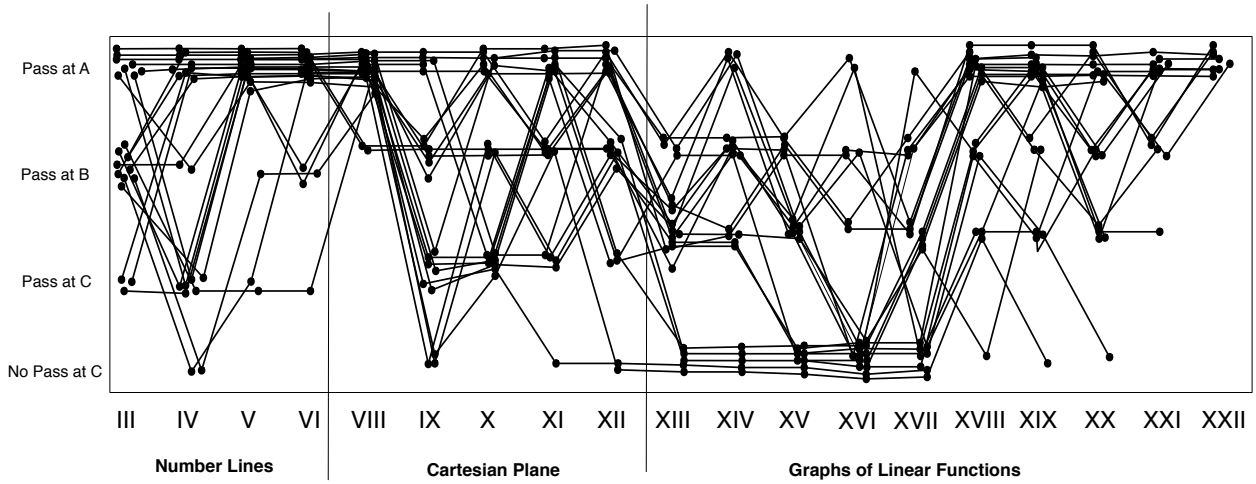


Figure 71: Pathways through the tutorial sequence

Number lines: Problem sets III - VI. The left side of Figure 71 displays pathways across number line problems[14]. I first describe the first pair of problem sets, PS III and PS IV, and then the second pair, PS V and PS VI. Results of these four problem sets are displayed in Figure 72.

[14] Because all students completed all three versions of Problem Sets 1, 2, and 7, these are not featured in Figure 71

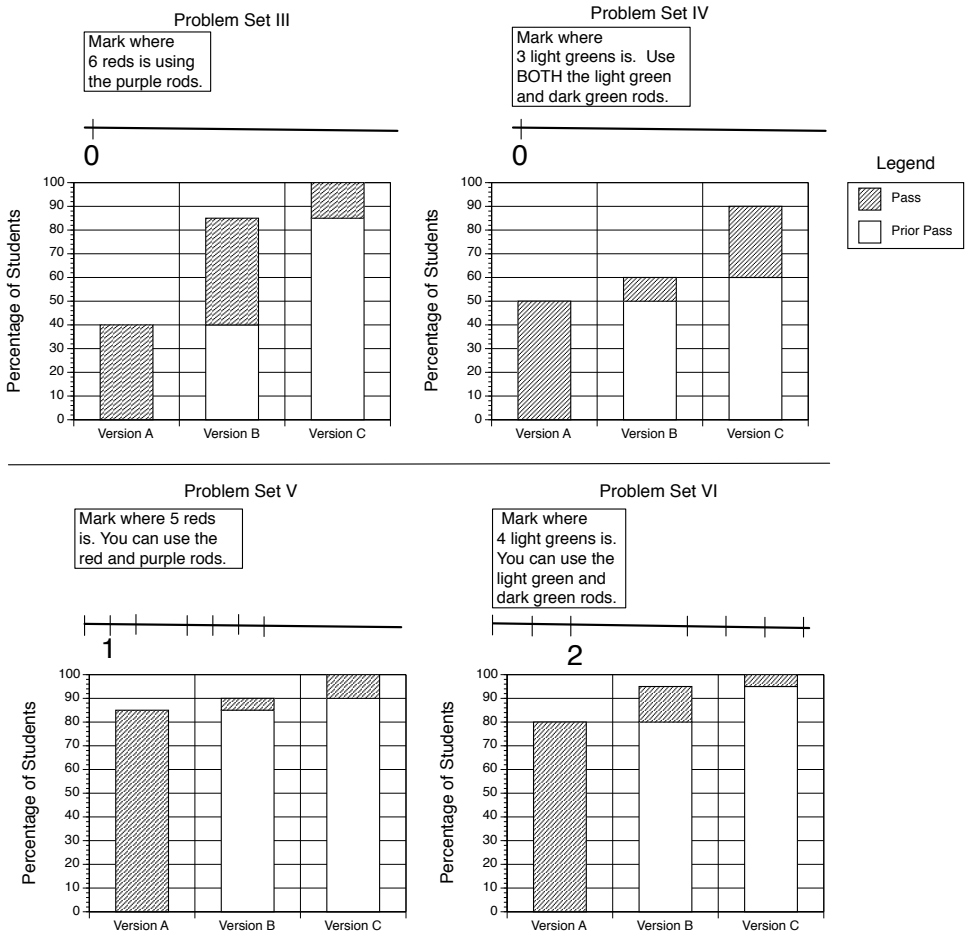


Figure 72: Accumulating percentages of students passing over iterations of number line problems

Problem set III engaged students in representing multiple unit intervals (as represented by red Cuisenaire rods) as a one multiunit interval (a purple Cuisenaire rod), and introduced the multiunit interval agreement. Problem set IV allowed students to practice this content. All students passed Problem Set III at or before version C was administered, with the majority passing at A or B. Problem Set IV had a slightly higher number of students passing at A, and thereby requiring the minimum amount of tutor support. Two students in the tutorial group did not pass Problem Set IV, though each of these students passed on subsequent number line problem sets.

Problem set V engaged students in coordinating unit and multiunit intervals on non-routine number lines with unequal partitioning, and problem set VI allowed students to practice this content. For each of these problem sets, the majority of students—17 and then 16 for problem sets V and VI, respectively—passed at version A. All tutorial students passed of both problem sets by version C.

The Cartesian plane: Problem sets VII - XII. The middle section of Figure 71 above displays pathways across Cartesian plane problems. I first describe problem set VIII, followed by the two pairs of problem sets (Problem sets IX and X and problem sets XI and XII). Results of these four problem sets are displayed in Figure 73.

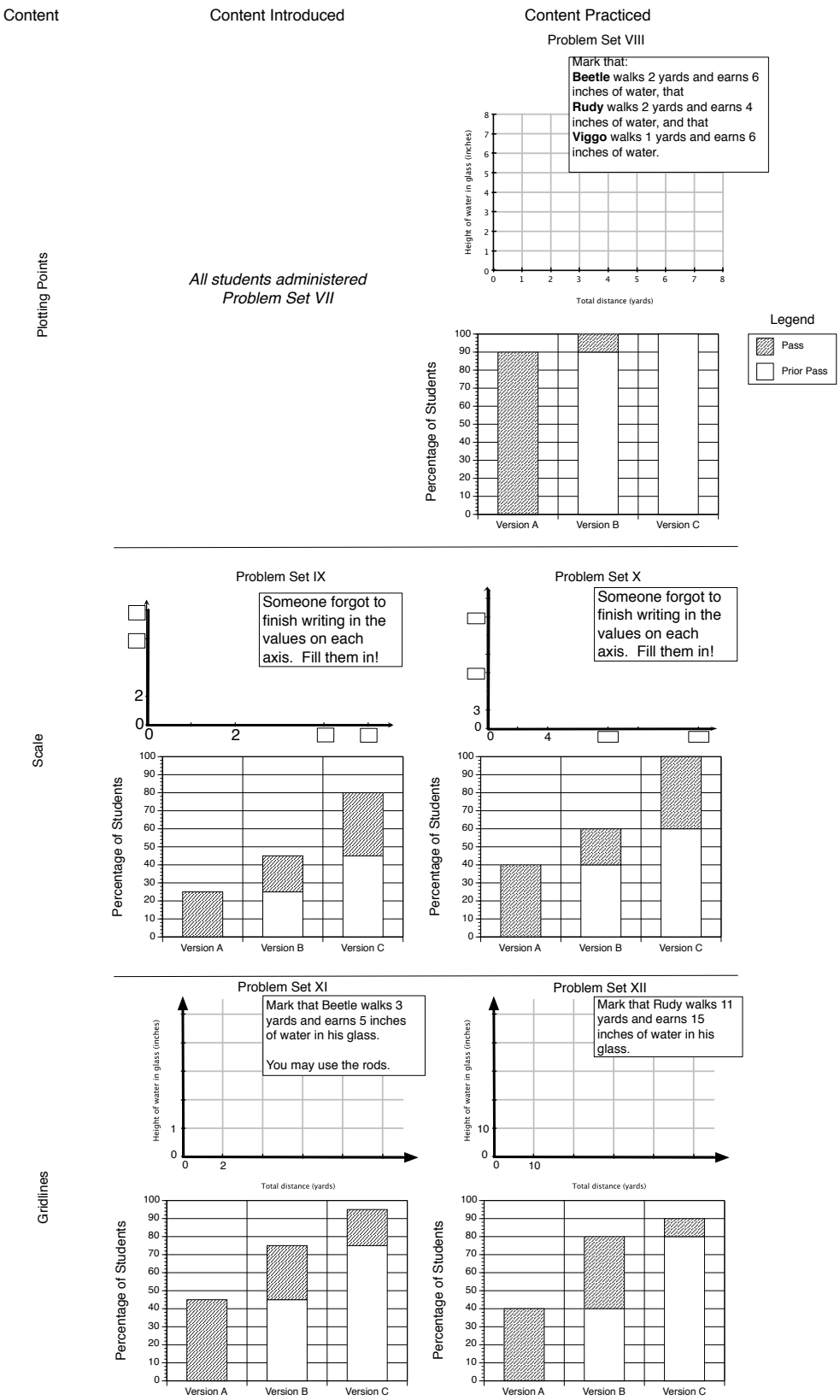


Figure 73: Accumulating percentages of students passing over iterations of Cartesian plane problem sets

Problem Set VIII was designed so that students could practice content introduced in the prior problem set. Problem Set VII, not represented in Figure 73 because all students were administered Versions A, B, and C, is the moment in the tutorial when one number line is rotated to be perpendicular to a second number line to create the coordinate grid. Following this, Problem Set VIII provides students an opportunity to practice plotting a point in the plane. On this particular problem set, tutorial students were overwhelmingly successful. The vast majority passed on Version A, with only two students needing to move on to Version B. On version A, both of these students inverted at least one of the three ordered pair provided on the problem card (e.g., incorrectly plotting the point (2, 6) at (6, 2)). Each of these students passed at Version B. Across problem sets, students by far had the most success with Problem Set VIII.

Problem Set IX introduced non-unit scaling, and was supported through the use of Cuisenaire rods. As shown in Figure 71, the amount of support shifted greatly between problem sets VIII and IX. In fact, this marks the sharpest disconnect between any two consecutive problem sets, in part due to the overwhelming success on Problem Set VIII. On problem set IX, most students (n=7) passed at Version C, contrasting sharply with the 18 students that passed at Version A of Problem Set VIII. Furthermore, more students were administered version C of Problem Set IX than on any prior problem set.

Problem Set X provided students an opportunity to practice the non-unit scaling introduced in the prior problem set. In the first version of the problem set, the horizontal axis featured numeric inscriptions "0" and "4" with a tickmark marking the location for 2 but without a corresponding numeric inscription. In general, the group needed slightly less tutor support as compared to the prior Problem Set IX. All students passed by version C.

Problem Set XI, which brings back the supportive real-world phenomenon of Beetle the cat and his water, problematizes the role of linear grid units. In the first version of the problem set, for which the problem card states a specified distance walked and amount of water for Beetle, gridlines are provided for each integer along the vertical axis, but for only even integers along the horizontal axis with the target point not corresponding to a vertical gridline. Nine students passed at Version A of this problem set, while 6 additional students passed at Version B and 4 more at Version C. Only one student did not pass Problem Set XI.

Students practiced this once again in Problem Set XII on a grid with multiunit intervals of 10 identified. Unlike Problem Set XI, for which one value in the ordered pair fell along a gridline, neither value in the ordered pair falls along gridlines. The amount of tutoring required by students was comparable to that of the prior problem set. Eight students passed at problem XI.A, one less than the prior problem set, with another 8 passing at XI.B and 2 passing at XI.C. Two students did not pass at Problem Set XII, one of whom also did not pass at Problem Set XI.

Graphs of linear functions: Problem sets XIII-XXII. The right side of Figure 71 above displays pathways across problem sets focusing on linear function graphs. Across the three linear representations of quantities, students required the most tutoring support for this area. Results of these ten problem sets are displayed in Figure 74 and Figure 75.

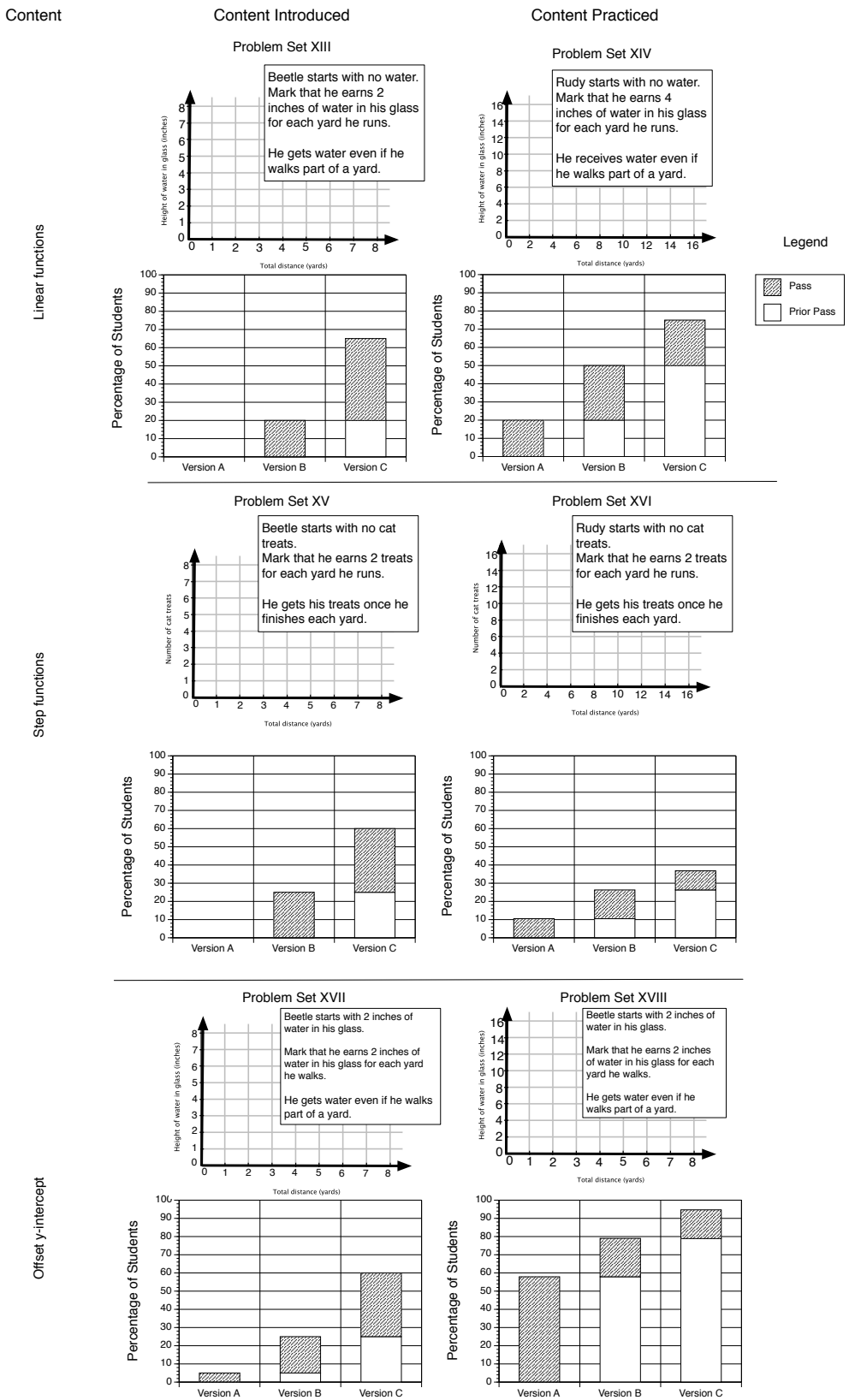


Figure 74: Accumulating percentages of students passing over iterations of graphing functions problem sets XIII - XVIII

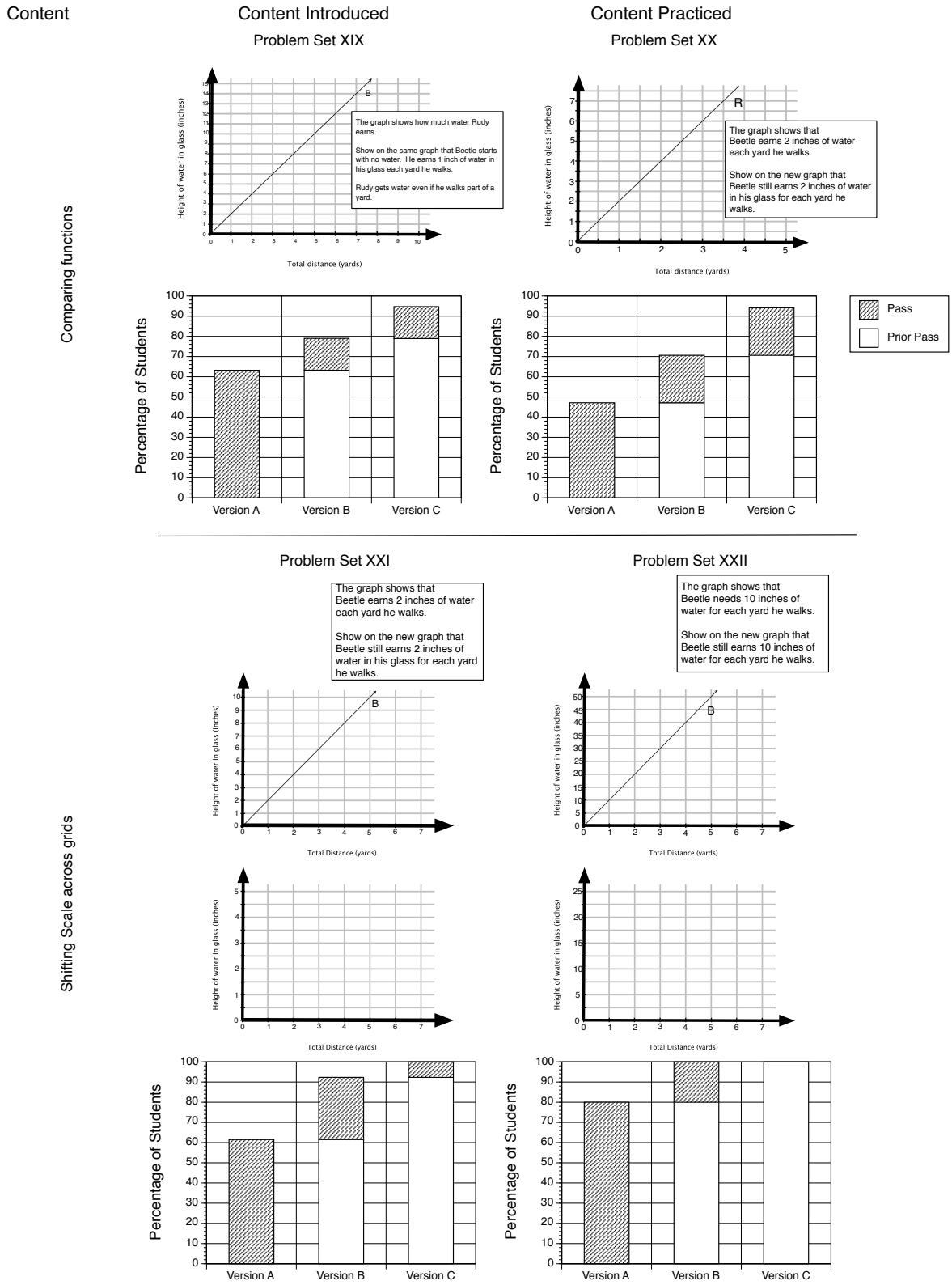


Figure 75: Accumulating percentages of students passing over iterations of graphing functions problem sets XIX - XXII

Problem Set XIII introduced a linear functional relation involving Beetle the cat and the water he earns as a function of distance walked. Upon introduction, tutor and tutee used props (a cat picture glued on a popsicle stick and a water container marked in inches) to illustrate the functional relation. Because functions were being introduced here, I anticipated students would require more tutor support than prior problem sets. No student passed XIII.A. Four students passed at XIII.B and another 8 students passed at XIII.C though 40% of students did not pass at problem set XIII.

Problem Set XIV provided students an opportunity to practice plotting a linear function on a grid with multiunit intervals of 2 identified on both horizontal and vertical axes. As compared to the problem set XIII, students required less tutor support in order to pass at this problem set. Four students passed at XIV.A, with another 6 passing at XIV.B and 5 passing at XIV.C. Five students did not pass this problem set; all five students also did not pass on the prior problem set XIII. The section below provides an in depth analysis of Problem Sets XIII and XIV.

Problem Set XV introduces a piecewise function as a contrast to a linear function, and with a design intent of providing support for coordinating a story context with inscriptions in the plane. Similar to the introduction of a linear function in problem XIII.A, no student passed at XV.A. Twelve students passed at either XV.B or XV.C, and 8 students did not pass at problem set XV.

Problem Set XVI allowed students to practice plotting a step function, though the bar charts in Figure 75 above show that fewer students passed at XVI as compared to XV. Performance from problem set XV to XVI was atypical as compared to every other problem set pair. A reason that this problem set pair is atypical is that fewer students passed at the practice problem set XVI (7 students passing by XVI.C), as compared to the introductory problem set XV (12 students passing by XV.C). Five students passed at XVI.B and another 7 passed at XVI.C, while 8 students did not pass.

Problem sets XVII and XVIII transition back to linear functions, with a focus on functions with an offset y-intercept on the vertical axis. Problem Set XVII presented students with a grid with unit intervals identified on the horizontal and vertical axes. The offset is presented in the problem card as a starting amount of water that Beetle the cat had in his glass, which students then had to coordinate with the change amount. One student passed at XVII.A, which was an improvement as compared to the introductory problem in sets XIII and XV. Another 4 students passed at XVII.B and 7 passed at XVII.C. A total of 8 students did not pass at this problem set.

Performance on problem Set XVIII, which allowed students to practice plotting functions with an offset on the vertical axis, marked a significant shift in performance on the linear functions problem sets. Despite the increased complexity with the provided grid, which featured a unit interval on the horizontal axis and a multiunit interval of 2 along the vertical axis, eleven students passed at XVIII.A. In other words, more than half of the sample –and the highest amount thus far in the linear functions problem sets—required the minimum amount of tutor support in order to pass. An additional 4 students passed at XVIII.B and 3 more passed at XVIII.C, with only one student in the sample that did not pass by XVIII.C. Because of the amount of time available, one student did not participate in Problem Set XVIII or subsequent problem sets.

Problem Sets XIX and XX required students to compare linear functions. The majority of students, 12 total, passed at XIX.A. Three more passed at XIX.B, and another 3 passed at

XIX.C. Only one student did not pass at problem set XIX, while one student was not administered this problem set due to time constraints.

Problem Set XX allowed students to practice comparing linear functions. Unlike prior problem sets, intervals of 0.5 were identified by tickmarks and gridlines, though only zero and positive integers were provided along the axes. Recall that the grid featured a linear function, and the problem card asked players to plot an additional function. Seventeen of the 20 tutorial students were administered problem set XX. Eight students passed at XX.A, while 4 passed at XX.B and another 4 passed at XX.C. Only one student did not pass this problem set.

Problem Sets XXI and XXII, the final problem sets of the tutorial sequence, presented students with a function in one grid and engaged students in plotting the same function in a second grid with a different scale. A large minority of students were not administered this problem set due to time constraints. For the 13 students that participated, 8 passed at XXI.A, 4 at XXI.B, and the final student at XXI.C. All students administered this problem set passed by XXI.C.

Problem Set XXII, which allowed student to practice the same content with slightly more complex values, was administered only to 10 students. All 10 of these students passed by version B of the problem set. On the prior problem set, these same 10 students all passed at either XXII.A or XXII.B.

Summarizing students' pathways through the tutorial sequence. Figure 76 represents the mean pass point across all 20 students. Students were coded as 1 for passing at version A, 2 for passing at version B, 3 for passing at version C, and 4 for no pass. Students that were not administered a particular problem set were not included in the mean score for that problem set. Thick vertical lines mark transitions between representations. An addition vertical line, just before problem set XX, marks the point in the tutorial when the *n* fell below 19 students. Beginning with problem set XX, one could argue that only students who required less overall tutoring would have time to get to these final problem sets, thereby favoring higher performing students. Figure 76 displays the mean pass points for PS XVII and XVIII with a downward sloping line that represents a decrease in overall tutoring support that continues through problem set XIX. Figure 76 confirms that students required the most tutor support upon introduction of linear functions. I now turn to an analysis of the introduction of linear functions to further understand students' conceptual coordinations.

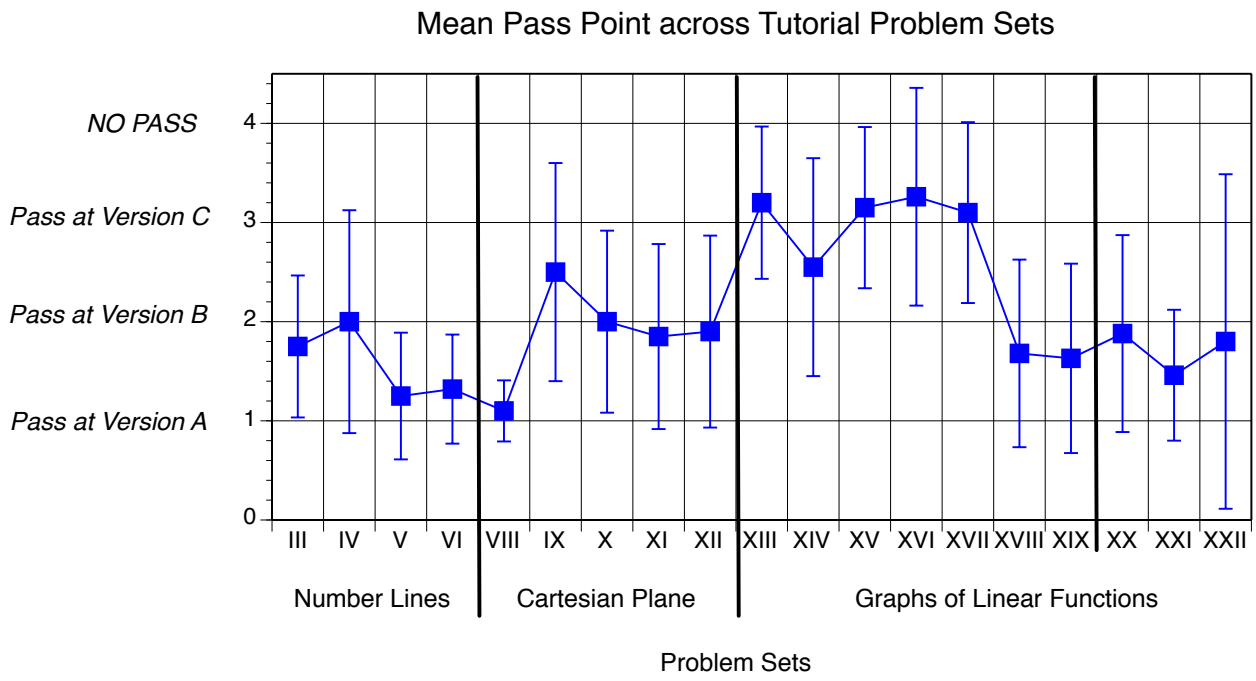


Figure 76: Mean pass points across tutorial problem sets

Students' conceptual coordinations upon introduction of linear functions. To analyze the process of student learning in the tutorial, I focus on the introduction of linear functions, an area requiring the most support as compared to other problem sets. I focus exclusively on linear functions in the plane for two reasons. First, I focus on linear functions in response to research that indicates young children can reason meaningfully with function graphs, yet without evidence of the processes by which students accomplish such task. Second, this are in the tutorial required the most tutor support, suggesting that this area in particular may offer key insights into how students negotiated unfamiliar content and representations. Analysis shows that problem solving approaches featured much variety among solutions coded as Pass and also among those coded as No Pass. First, I focus on students coded as Pass and then on students coded as No Pass.

Passing coordinations. For the problem versions used in Problem Sets XIII and XIV, three categories emerged from the data for students that passed. Recall that Problem XIII.A stated, "Beetle starts with no water. Mark that he earns 2 inches of water in his glass for each yard he runs. He gets water even if he walks part of a yard," with a grid featuring unit scaling on both axes; Problem XIV.A involved a cat earning 4 inches of water for each yard, with a grid featuring multiunit intervals of 2 on both axes. As I will elaborate, these students coordinated some or all inscriptions in the plane with values along the axes in one of three ways: (a) repeated point-with-line construction; (b) coordinated data points followed by a line; and (c) one or more coordinated data points with gridline pattern extension followed by a line. Figure 77 uses Problem XIII.A to illustrate these coordinations, while Table 13 displays the proportion of students who used these coordination types for their final (passing) version of Problem Sets XIII and XIV, respectively. The *n*s reflect the number of students who passed at any of the three versions of each problem; students that did not receive a Pass on either Problem Set XIII or XIV are not represented in this table.

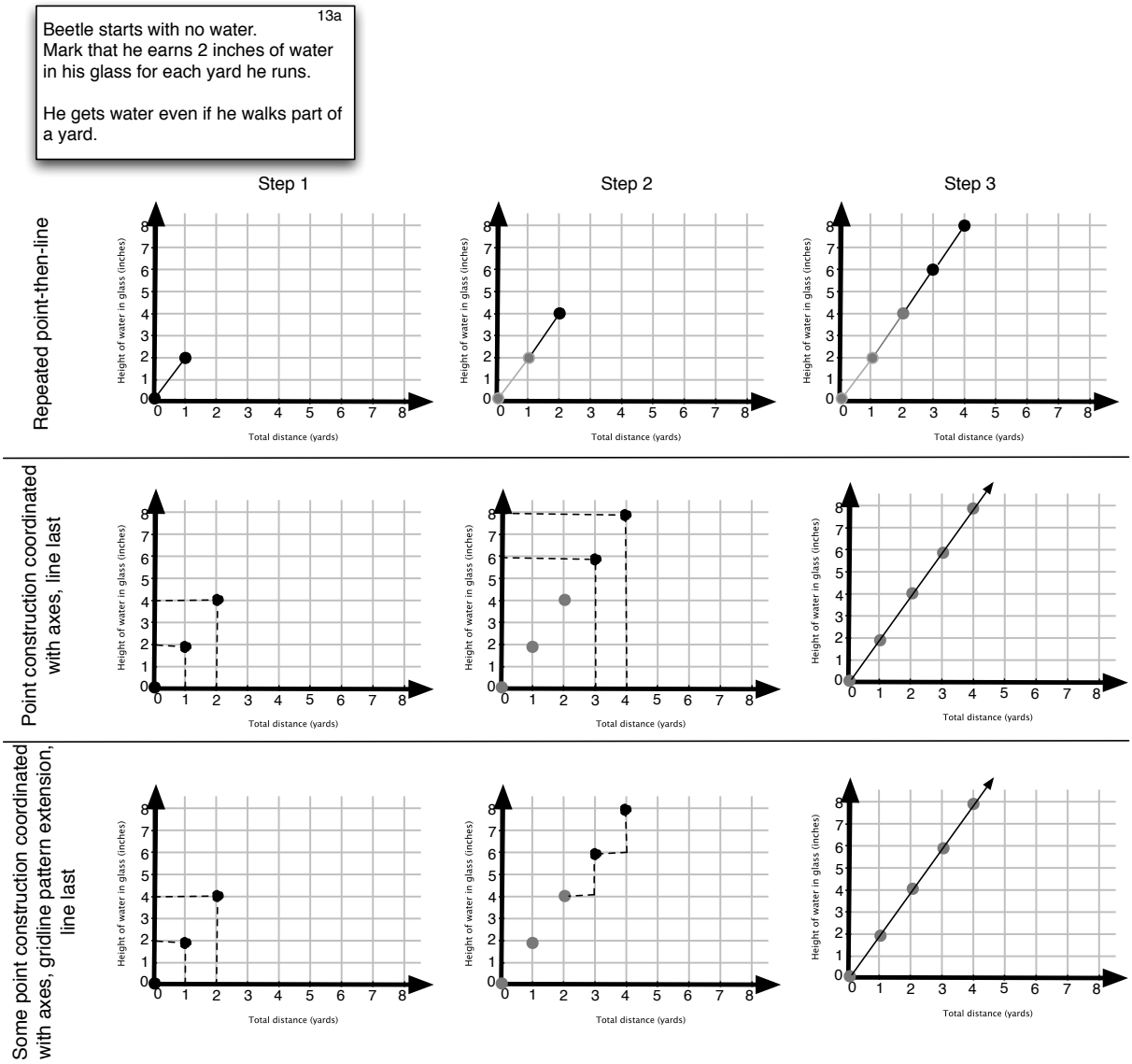


Figure 77: Exemplar coordinations coded as Pass on PS XIII and PS XIV

Table 13: Proportion of coordinations coded as Pass on Problem Sets XIII and XIV

	Type of Coordination	XIII n=11	XVI n=15
a	Repeated point-then-line	.55	.27
b	Point construction coordinated with axes, line last	.18	.20
c	Some point construction coordinated with axes, gridline pattern extension, line last	.27	.40
d	Unclear (either 2 or 3)	--	.13

The first type is a *repeated point-then-line* construction (Figure 77, top). Some students tightly coordinated the inscription of the line with point inscriptions in the plane. These students began at the origin and alternated the inscription of a function line with inscriptions of points

along that line. Inscriptions were closely coordinated with quantities along each axis. For Problem Set XIII, the majority of passing students (55%) coordinated representational features in this manner, yet this type of coordination dropped among passing performances in Problem Set XIV (27%).

The second type is *point construction coordinated with axes, line last* (Figure 77, middle). These students used instantiations to generate and plot ordered pairs in the plane. For each point, students projected from values along each axis. After plotting several points, the student connected points with a line. The inscription of the line was generated using points in the plane. About one fifth of passing students used such an approach on both Problem Set XIII and XIV.

The third type of passing construction involved three steps: *Some points coordinated with each axis, gridline pattern extension, then line last* (Figure 77, bottom). A third passing coordination involved students using one or more instantiations to generate and plot ordered pairs in the plane. After plotting one or more points, students shifted their approach by extending the emerging pattern of points together with linear grid units (e.g., moving over 1 and up 2 by using gridlines). This extension did not coordinate point placement with projections from quantities along the axes. For Problem Set XIII, 27% of students did this, and this increased to 40% in Problem Set XIV. For Problem Set XIV, this was the most common strategy among the passing coordinations. As profiled below, many students coded as No Pass also used linear grid units, yet without success.

No Pass Coordinations. In the No Pass coordinations, the performances judged to be No Pass fell into six categories. The first two categories include solutions that captured the joint variation with one or more precise points, while the final four include solutions with no accurate point placement. The six categories include: (a) joint variation established, with no contradictory inscriptions; (b) joint variation established but not consistent (some contradictory inscriptions); (c) appropriate joint variation not established, different functional relation plotted; (d) appropriate joint variation not established, change in one quantity applied to change in both quantities; (e) appropriate joint variation not established, change in one quantity accurate, no pattern in the other; and (f) other or idiosyncratic constructions. Figure 78 uses Problem XIII.A to illustrate these coordinations, and Table 14 displays the proportion of students who used these coordination types for any construction coded as No Pass for all versions of Problem Sets XIII and XIV, respectively. The *n*s reflect the total number of constructions across students coded as No Pass.

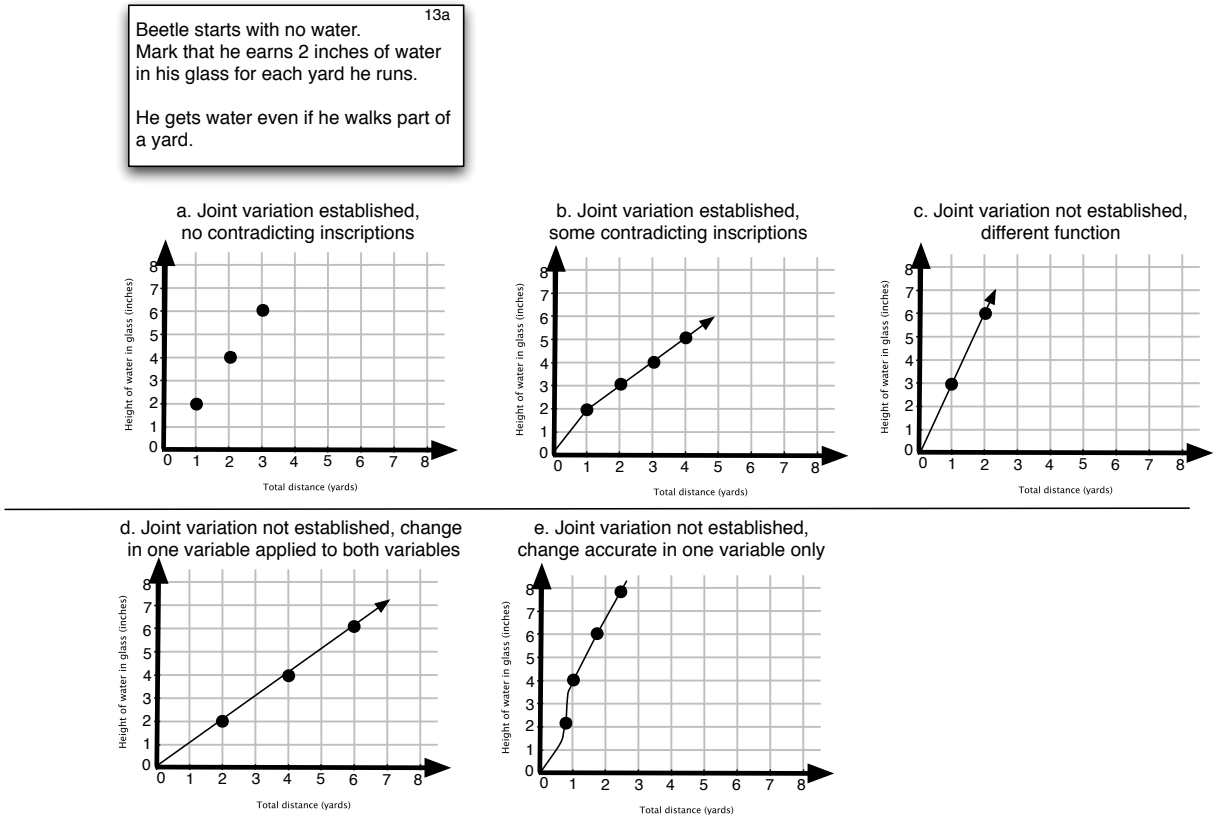


Figure 78: Exemplar coordinations coded as No Pass on PS XIII and PS XIV

Table 14: Proportion of coordinations coded as No Pass on Problem Sets XIII and XIV

	Type of Coordination	XIII n=37	XIV n=28
a	Joint variation established; NO inscriptions that contradict the function	.27	.14
b	Joint variation established; Some contradicting inscriptions	.08	.21
c	Joint variation NOT established; Different function	.05	.18
d	Joint variation NOT established; Change in one variable is applied to both variables	.08	.07
e	Joint variation NOT established; Change is accurate in one variable; no pattern in the other	.27	.25
f	Other/ idiosyncratic	.24	.14

The first type is *joint variation established, no contradicting inscriptions* (Figure 78a). This No Pass performance featured inscriptions in the plane made by the student that captured the jointly varying quantities described on the playing card. In addition, students coded as this did not have inscriptions in the plane that contradicted the functional relation. All students receiving this code accurately plotted points but did not inscribe a function line in the plane.

The second type is *joint variation established, some contradicting inscriptions* (Figure 78b). This category of no pass performance reflects students' efforts to capture the joint variation as presented on the problem card, but also with inscriptions in the plane that contradict that joint variation. These contradicting inscriptions typically followed a gridline pattern extension of moving to the right and then up by one gridline unit each.

A third type is *joint variation not established, different function* (Figure 78c). Students plotted a function different from the functional relation described on the playing card.

A fourth type is *joint variation not established, change in one variable applied to both variables* (Figure 78d). A minority of students applied the change in one of the quantities to both quantities.

A fifth type is *joint variation not established, change is accurate in one variable only* (Figure 78e). A quarter of students coded as No Pass for both Problem Set XIII and XIV coordinated inscriptions in the plane with change of one quantity, but did not consider the other quantity.

The final category of No Pass solution was coded as o*ther/idiosyncratic*. Some solutions coded as No Pass did not correspond to a pattern emerging across solution efforts, or indicate that students elected to make no inscriptions in the plane.

A Student's Trajectory through Problems XIII and XIV. The prior analyses of pre-test to post-test gains on matched problems, the role of agreement use in pre-test to post-test gains, pathways through the tutorial, and the character of student solutions upon introduction of linear function graphs all reveal how participation in the tutorial supported student learning. In this section, I report a case study to provide insight into a student's trajectory as she worked through a sequence of two problem sets. I had two guiding purposes for a case study analysis of the dynamic aspects of the tutorial sessions. The first was to highlight continuities and discontinuities in a student's trajectory of solutions. The second was to understand the way a student's trajectory of solution approaches was interwoven with discrepant solutions, and the ways tutor and student invoked agreements in efforts to reconcile discrepancies. The case study is Jana, and I focus on her efforts to graph linear functions as she moved through Problem Sets XIII and XIV.

I chose Jana as the case for three reasons. First, Jana followed a typical trajectory as she progressed through the tutorial problems. Second, her sores on the pre- and post-test were just above the mean for the tutorial group, and her change from pre- to post-test was just below the mean, indicating typicality (Table 15). Third, Jana showed gains on a non-routine LRA item, *Rate of change*, which posed a significant challenge among students in Grades 5 and 8. This particular item required a coordination of linear units and numerical units in order to determine that two functions on separate grids have the same rate of change. Figure 79 shows the percentage correct on the LRA item *Rate of change* among assessment only participants (left) and tutorial participants (right). Note that among experimental students, who were among the lower performing two-thirds of Grade 5 students, zero students responded accurately to this task at pre-test (Figure 79, right); after intervention, 45% of students responded accurately, matching the performance profile among Grade 8 students in the assessment study (48.1%). I begin by analyzing Jana's performance on item *Rate of change*, followed by an analysis of her progress through Problem Sets XIII and XIV.

Table 15: Jana's and the tutorial group's performance on pre and post-assessment

	Jana	Mean for tutorial group
Pre-assessment Score	14	13.1
Post-assessment Score	24	23.25
Change from pre to post	10	10.15

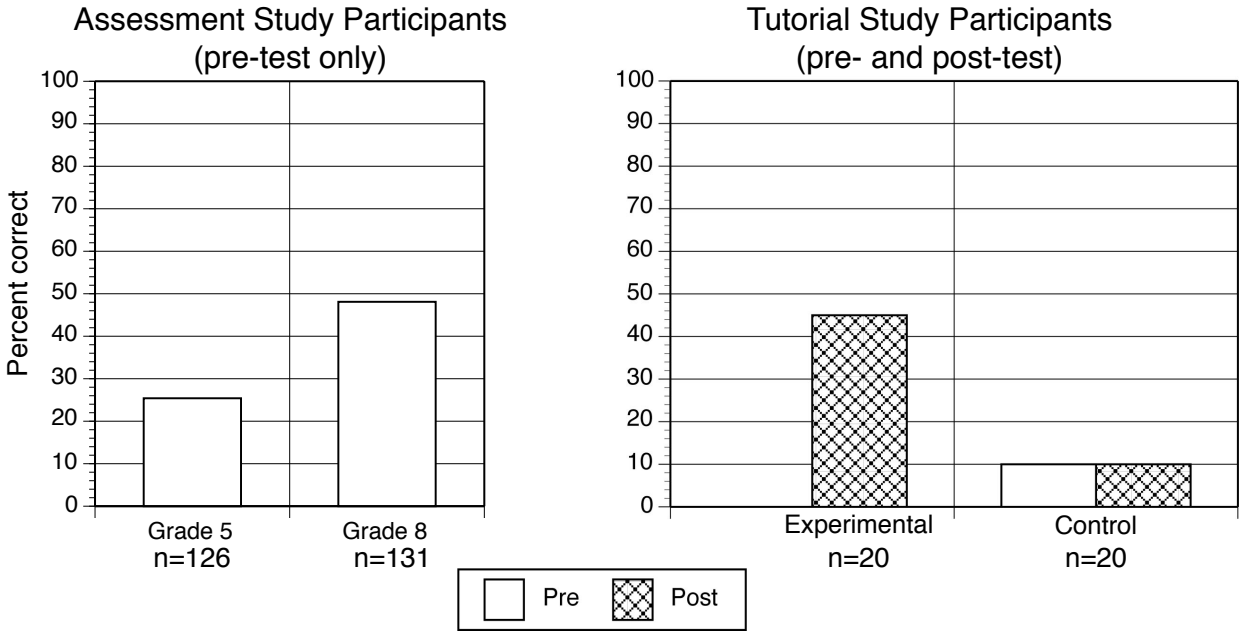


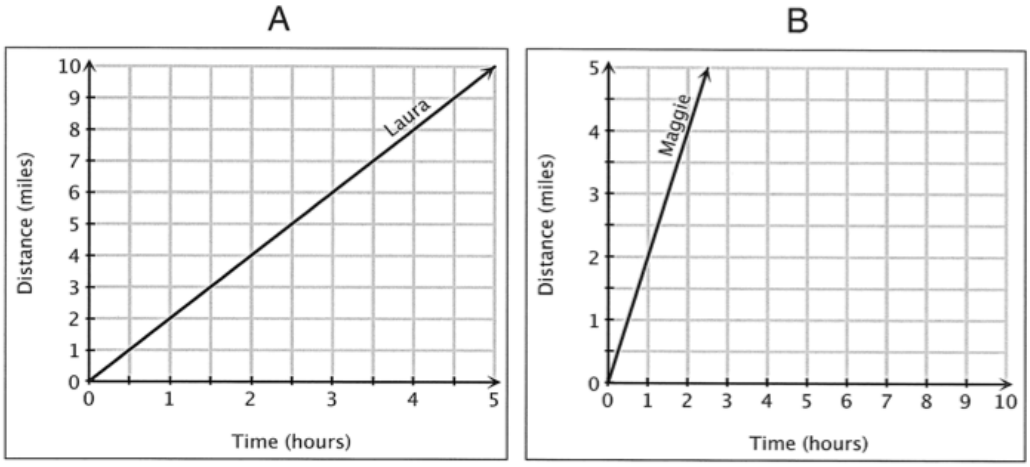
Figure 79: Percentage correct on LRA item *Rate of change* among assessment study participants (left) and tutorial study participants

Jana's performance, like almost half of the tutorial group, showed change on Item *Rate of change* from pre- to post-test. On the pre-assessment, she chose *Graph B: Maggie* as showing more miles walked each hour (Jana's pre and post-test responses are featured in Figure 80). In her written justification, she wrote, "because she runs 5 miles in 2 ½ hours, but Laura runs to 10 miles in 5 hours but still it's still Maggie because she ran half the miles but still in half the time." While Jana was not interviewed on the task, her strategy most closely resembles the strategy of coordinating the function line with values along one axis only, though the data do not make clear exactly how she thought about this. On the post-test with the same problem, Jana responded accurately, choosing *They walk the same rate*. In this case, she stated in her written justification, "I know because the graph looks different but the numbers are in different places." After the tutorial intervention, Jana successfully employed a strategy in which she coordinated linear units and numerical units across graphs.

Pre-test

20. Graph A shows how many miles Laura walks each hour.

Graph B shows how many miles Maggie walks each hour.



Who walks more miles each hour, Laura or Maggie?

Circle one: Laura Maggie They walk the same rate

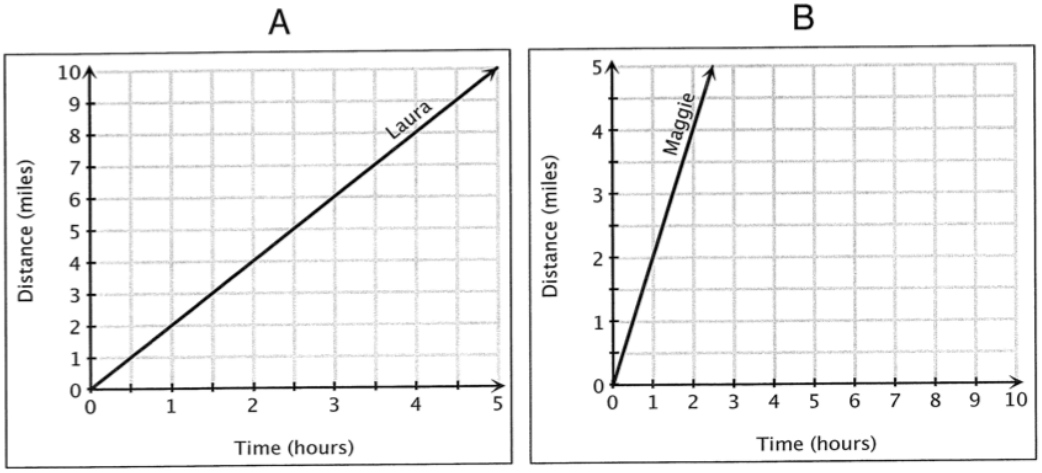
How do you know? because she runs 5 miles in 2½ hours but laura runs 10 miles in 5 hours but still its still Maggie because she ran half the miles but still in half the time

10

Post-test

20. Graph A shows how many miles Laura walks each hour.

Graph B shows how many miles Maggie walks each hour.



Who walks more miles each hour, Laura or Maggie?

Circle one: Laura Maggie They walk the same rate

How do you know? I know because the graph looks different but the numbers are in different places

Figure 80: Jana's performance on LRA item *Rate of change* on the pre (top) and post (bottom) test

Figure 81 contains a profile of Jana's trajectory across Problem Set XIII and XIV, with rows for No Pass corresponding to the six categories presented in a prior section: (a) joint variation established, no inscriptions that contradict the function; (b) joint variation established, some contradicting inscriptions; (c) joint variation NOT established, different function; (d) joint variation NOT established, change in one variable is applied to both variables; (e) joint variation NOT established, change is accurate in one variable with no pattern in the other; and (f) other/idiosyncratic. For both problem sets, Jana passed at the third iteration of the problem set, and was coded as No Pass for versions A and B for both problem sets. Like most tutorial students, Jana ultimately achieved a pass on both problems, passing by the third iteration on each. She displayed various efforts at coordinating representational features for each problem version.

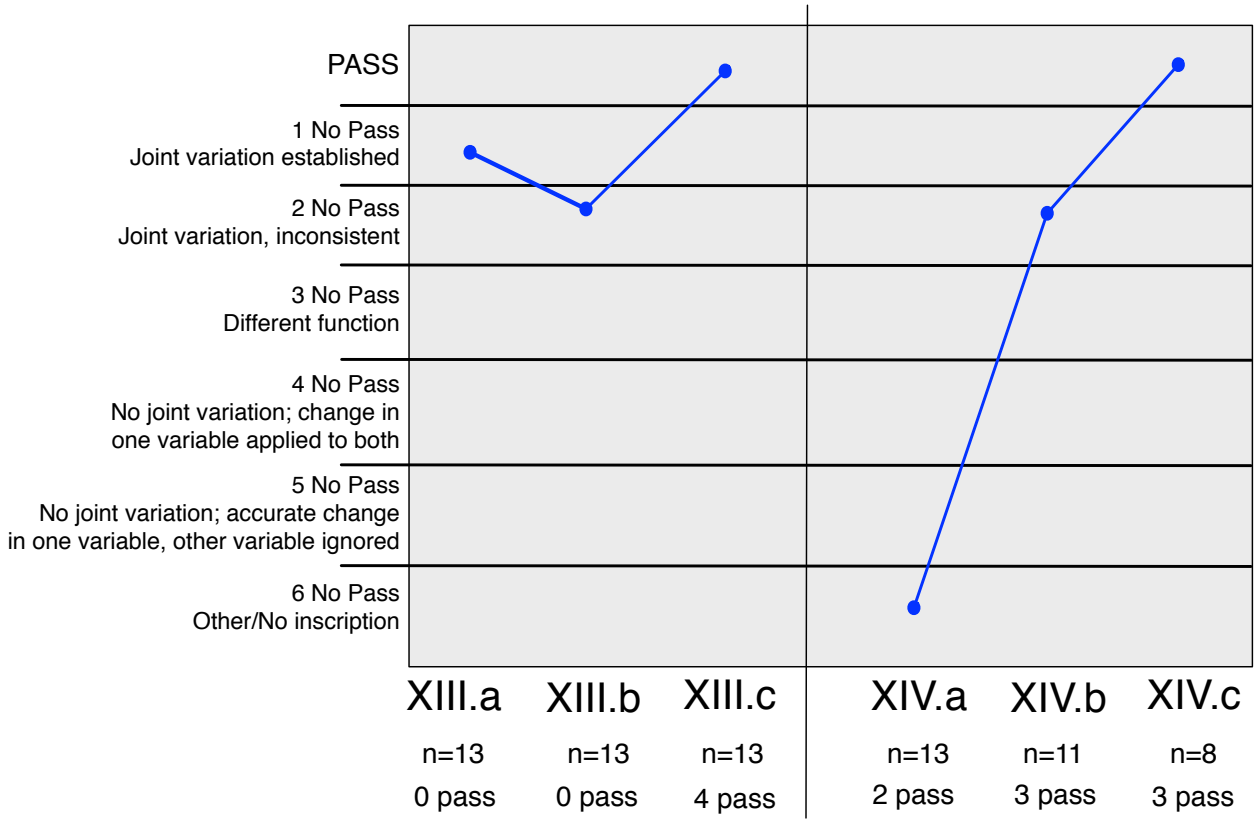


Figure 81: Jana's passing trajectory over Problem Sets XIII and XIV

Problem Set XIII. Figure 82 displays Jana's performances on the three iterations of Problem Set XIII. As described above, Problem Set XIII.A featured the playing card, "Beetle the Cat starts with no water. Mark that he earns 2 inches of water for each yard that he runs. He gets water even if he walks part of a yard." The tutor, together with Jana, first used props to enact the joint variation. Jana moved the cat (a photograph glued onto a popsicle stick) along a yard line, while the tutor poured water into a container marked in inches. The screen was placed between game players and each engaged in construction of their own graph. Jana paused before inscribing anything. She first plotted a point at (1, 4), and then picked up the playing card while sighing, indicating her displeasure at her point placement. She marked an additional point at (1, 2), an accurate point placement for the functional relation. When the screen is removed, she offered right away, "I'm done, but I didn't mean to make that one there," indicating the point (1,

4). The tutor indicated she could cross it out if she preferred. The tutor then overlaid the transparency over Jana's paper and stated, "How did we do? They look different! So tell me how you did yours." In the ensuing interaction, the tutor asked Jana what she did. Jana indicated the playing card and stated, "It says 2 inches of water, and it says he gets 2 inches of water for each distance he runs. So, 1. And I put 2 inches." In turn, the tutor explained his approach, drawing upon the modeled functional relation and drawing upon the agreement introduced in the context of the Cartesian plane, *One point shows two amounts*. He referred to the playing card and stated, "I saw that he started with no water, so I put a point there [at 0, 0] to show those two amounts. And then I did exactly what you did. I know if he goes 1 yard, he gets 2 inches of water." The tutor then draws upon the functional relation: "But then it doesn't say how far he goes, and also when we were doing it, we were doing it so he would keep on going, right? So if he goes 2 full yards, he gets 4 inches of water, because he gets 2 inches for the first yard and 2 inches for the second yard." The tutor went on to make reference to non-integer points along the horizontal axis and the yard line, reminding Jana that Beetle earned water even if he doesn't walk a full yard. In the end, the tutor expressed the *We Can Connect Points to Show Many Points* agreement as a means to (a) resolve the tutor and students' lack of coordination, and (b) support inscriptions on the subsequent problem version. The tutor asked, "What could we do next time to have our graphs look as identical as possible, to which Jana responded, "We should do it all the way," gesturing over the function line the tutor had inscribed on his work.

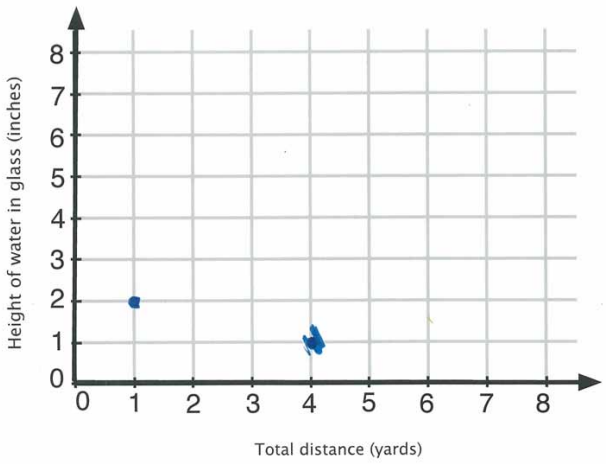
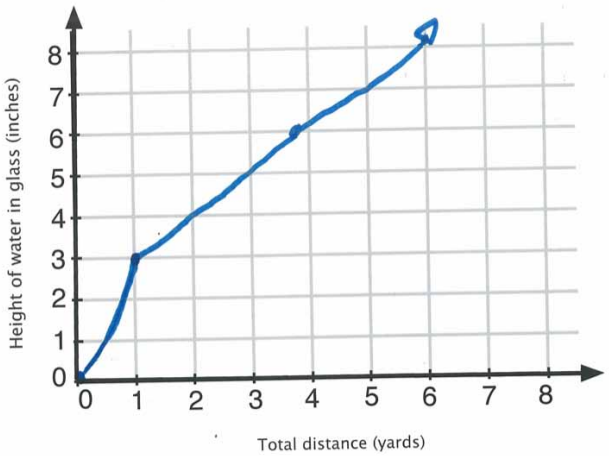
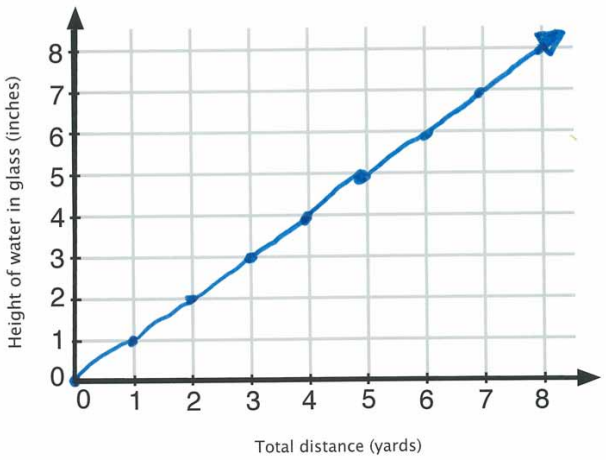
Problem Card	Jana's Construction	Agreement Usage
13a Beetle starts with no water. Mark that he earns 2 inches of water in his glass for each yard he runs. He gets water even if he walks part of a yard.		Tutor introduces the agreement *We can connect points to show many points.* Tutor references the agreement *One point shows two amounts.*
13b Beetle starts with no water. Mark that he earns 3 inches of water in his glass for each yard he runs. He gets water even if he walks part of a yard.		Tutor references the agreement, *One point shows two amounts.*
13c Beetle starts with no water. Mark that he earns 1 inch of water in his glass for each yard he runs. He gets water even if he walks part of a yard.		Jana references the agreement, *One point shows two amounts.* Tutor references the agreement, *We can connect points to show many points.*

Figure 82: Jana's constructions on Problem Set XIII

Because of Jana's No Pass performance on Problem XIII.A, the tutor drew the XIII.B problem card, which stated, "Beetle the cat starts with no water. Mark that he earns 3 inches of water for each yard he walks. He gets water even if he walks part of a yard." Jana's construction for Problem XIII.B was coded as No Pass, joint variation established but inconsistent. She began

by plotting a point at (0, 0), followed by drawing a line from 0, 0 up to 1, 3 while carefully coordinating her inscription with values along the axes. She then traced over it without touching the marker to paper to check her work. To consider subsequent inscriptions, Jana traced from the point 1, 3 and appeared to be counting linear grid units before inscribing the line. She marked a point at 4, 6, applying the change of 3 of the dependent variable to both the distance and height of water variables. She continued inscribing a line to intersect with subsequent gridline intersection points (e.g., one over, one up). Once the screen was removed and the tutor's transparency overlaid on her work, she explained, "Well, I did it regular," indicating the line from 0, 0 to 1, 3, "but then I got mixed up, like, what way I went." (Figure 83). When discussing how to achieve identical constructions on subsequent problems, Jana indicated the instructions on the playing card: "I could just remember, like, how it says he only earns 3 inches for each yard." The tutor then referred to the agreement, *One point shows two amounts*, emphasizing that each of those two amounts need to correspond to the story on the playing card.

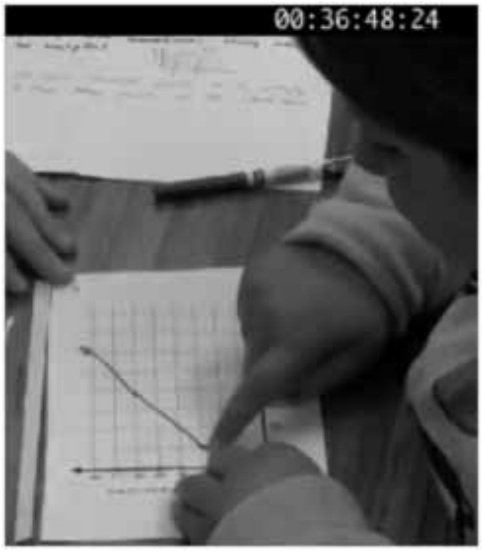


Figure 83: Video still shot of Jana explaining her construction for Problem XIII.B

The tutor drew problem card XIII.C, which stated, "Beetle the cat starts with no water. Mark that he earns 1 inch of water in his glass for each yard he runs. He gets water even if he walks part of a yard." On this, Jana was coded as *repeated point-then-line construction*. Jana began similarly to the prior problem version; she placed a point at 0, 0, then drew a line towards 1, 1, where she made a point. She continued to draw a line from 1, 1 to 2, 2, and once again made a point at 2, 2. She continued in this fashion finally plotting a point at 8, 8. When the tutor overlaid the transparency, he exclaimed, "Looks like we got it! Nice!" In her justification, Jana made reference to the agreement *One Point Shows Two Amounts*, stating, "the graph point shows two amounts at the same time." The tutor affirmed this and added to it a reference to *We Can Connect Points to Show Many Points*.

Problem Set XIV. With Problem Set XIII successfully completed, the tutor drew the playing card for the first version of Problem Set XIV. Recall that the grids for Problem Set XIV featured multiunit intervals along both the horizontal and vertical axes. Figure 84 displays Jana's constructions across the three versions. Problem XIV.A stated, "Rudy the cat starts with no water. Mark that he earns 4 inches of water in his glass for each yard that he runs." On problem XIV.A, Jana was coded as No Pass, idiosyncratic/other, with her work displayed in Figure 84.

Jana paused at the beginning of the Construction Phase (with the screen already down). She first held her finger over "2" on the horizontal axis. Beginning at 0, 0, she drew a line moving diagonally up to the middle of box formed by gridline inscriptions (at approximately 1, 1). She then started again at 0, 0, and once again inscribed a line diagonally up to the right (so that inscriptions depicted two lines from the origin); she placed a point at the coordinate point of 4 along the vertical axis, and approximately 1.5 along the horizontal (see Figure 84). She continued to extend the line, followed by placing a point at 2, 6; she extended the line further then placed another point that corresponded to 8 along the vertical axis. Extending the line further, she places two additional points, each of which correspond to jumps of 4 along the vertical axis. Jana explained her response by first acknowledging the playing card stated "each yard" yet the horizontal axis skip counted by 2s. The tutor made reference to the skip counting along the axes, and pointed out, "So I thought, well here's 1. So I made a point right there [at (1, 2)], to show those two amounts," making reference to the *One Point Shows Two Amounts* agreement. When discussing what to do next time for identical constructions, Jana makes passing reference to the *Every Number Has a Place* agreement by indicating she could inscribe values along the horizontal axis: "I could make the numbers so then it could be more exact."

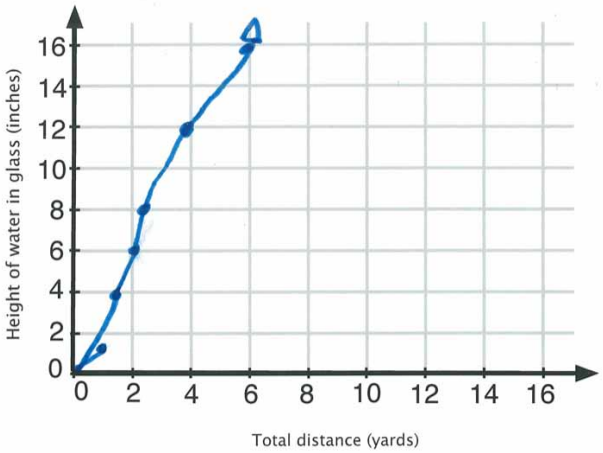
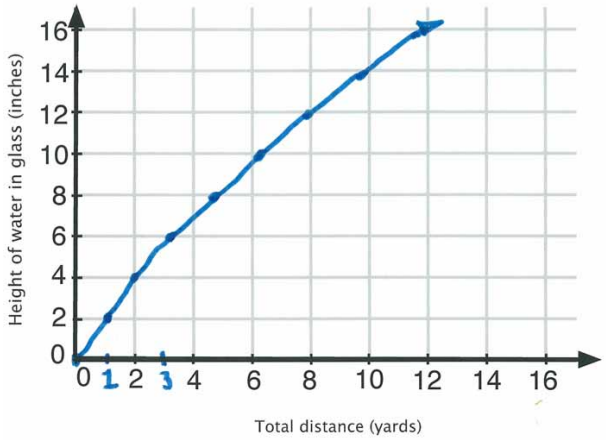
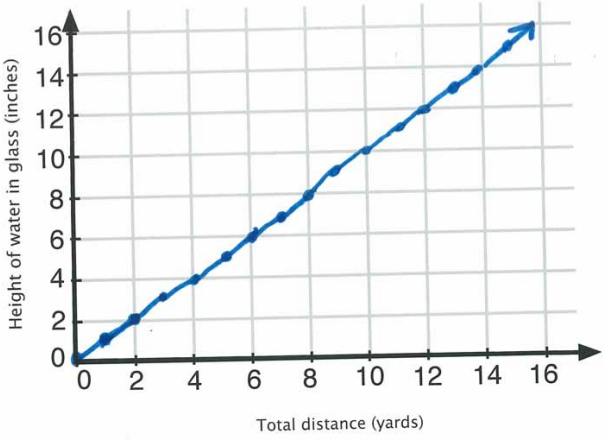
Problem Card	Jana's Construction	Agreement Usage
14a Rudy the cat starts with no water. Mark that he earns 4 inches of water in his glass for each yard that he runs. He gets water even if he walks part of a yard.	14a  	Tutor references the agreements, *One point shows two amounts,* and to Skip Counting Numbers. Jana makes passing reference to Every Number Has a Place agreement.
14b Rudy the cat starts with no water. Mark that he earns 2 inches of water in his glass for each yard that he runs. He gets water even if he walks part of a yard.	14b  	Tutor references the agreement, *One point shows two amounts.*
14c Rudy the cat starts with no water. Mark that he earns 1 inch of water in his glass for each yard that he runs. He gets water even if he walks part of a yard.	14c  	

Figure 84: Jana's constructions on Problem Set XIV

With the failure to achieve identical constructions on Problem XIV.A, the tutor asked Jana to read the playing card for XIV.B: "Rudy the cat starts with no water. Mark that he earns 2 inches of water in his glass for each yard that he runs. He gets water even if he walks part of a yard." On problem XIV.B, Jana was coded as No Pass: Joint variation established but not

consistent. Jana began by inscribing "1" and "3" along the horizontal axis. She then made a point at 0, 0, and drew a line up to 1, 2, where she inscribed a point, coordinating that point with values along each axis. She extended the line to 2, 4, where she inscribed another point. Her subsequent line extension and point placement become less coordinated with the values along the axes. She extended the line up to 6 along the vertical axis and at approximately 3 along the horizontal and inscribed a point; her next extension and point placement is at 8 along the horizontal axis—indicating a change of 2—but also changes by 2 along the horizontal axis, plotting the point at approximately 5, 8. Her subsequent extensions and points corresponded with gridline units. When the tutor overlaid the transparency on her work, she stated, "Oh, wait, I did it wrong, oh wait." She then appeared to analyze the comparison of work, stating, "Oh wait, no wait. Yeah. And then… Okay. I kept on going like that. He kept on getting 2 inches and a yard, 2 inches and a yard." The tutor referenced the One Point Shows Two Amounts, agreement, emphasizing that those two amounts had to correspond to the story on the playing card.

The tutor flipped the card for Problem XIV.C, the final version in the problem set, which stated, "Rudy the cat starts with no water. Mark that he earns 1 inch of water in his glass for each yard he runs. He gets water even if he walks part of a yard." Jana was coded as *repeated point then line construction*. On this final problem, Jana did not inscribe any additional values along either axis. She began by plotting a point at 0, 0, and then drew a line up to 2, 2, where she made another point. She then inscribed an additional point at 1, 1 over the line. She then drew a line up to 3, 3, where she inscribed a point, and continued this pattern. When the transparency was then overlaid on the paper, Jana and the tutor both acknowledged that their constructions were identical.

Discussion

Analyses showed that the tutorial intervention was effective for supporting students' development of target mathematical ideas related to linear representations of quantities. In this discussion, I consider the efficacy of the tutorial as well as sources of student learning.

Efficacy of the tutorial. In this section, I consider two features of the tutorial design that support the assertion of efficacy as a result of the intervention: (a) matched experimental and control groups, and (b) multiple versions of each problem set.

Matched experimental and control groups. Evidence of efficacy includes shifts in the experimental group's overall performance score from pre-test to post-test as well as the subset of matched tasks, each with a large effect size, with no shift for the control group. To be eligible for participation in this study, students were required to have scored in the lower two-thirds of the entire Grade 5 sample from the assessment study. This eligibility requirement resulted in a potential threat to validity given that the LRA was administered as both pre-test and post-test. Because of repeated testing, an increased performance on the post-test may indicate falsely that students developed new understandings across the two administrations, when the shift could be a result of initial measurement error or repeated practice with the same items.

To address this potential threat to validity, eligible participants were paired with a classmate based on identical pre-test score. One partner was randomly assigned to the experimental group, and the other to the control group. Contrasting the gains among experimental students with a large effect size, analyses revealed that students in the control group showed no detectable difference in performance form pre to post-test. This indicates that

gains shown by the experimental group were a function of the intervention, and not due to regression to the mean or repeated testing.

Multiple versions of each problem set. Another design feature included three versions of each problem set. Recall that the purpose of the three versions (A, B, and C) was to provide additional tutoring support to students that may need it, as well as to document pathways through the tutorial sessions. Because each version featured small shifts in the provided values or in the representational design, improvements across versions within a problem set are not a result of students' practice with a type of problem or memorization of the tutor's solution approach. Furthermore, problem sets were designed in pairs, so that the first set introduced a new content element and the second allowed for practice with a more complex design. Analysis revealed that students often solved the problems in the second set more quickly (or, with fewer versions administered) than the first set, again corroborating a learning effect of the tutorial procedure.

Sources of student learning. The tutorial intervention was designed to afford the emergence of a supportive learning environment. Tutorial features included: a progressive development of planned agreements that afforded students' generative use of representational principles to mediate their constructions and adjudicate discrepant solutions to tutorial problems; a sequence of linear representations of quantities that afforded students' understanding and generative use of principles and definitions; communication supported through a repeating game structure; wrap-up sessions to support and consolidate learning; three versions within problem sets to provide students with multiple opportunities to work through and receive differentiated support with the targeted mathematical ideas; modeling by recording physical magnitudes through the use of C-rods on an open number line; and modeling of a functional relation through enacting two jointly varying quantities that were to be represented on a graph. While the variety of tutorial features makes difficult the task of locating any one as a source of learning, I focus on three particular sources and point to evidence to support these as sources of learning: agreement usage, the sequence of representations, and modeling.

Agreement usage. Analysis focused on mindful agreement use as an important indicator of the ways students were learning to solve problems and build understandings of the target mathematical ideas. For this analysis, students' use of the agreements in the wrap-up sessions served as an estimate of students' mindful use of agreements across tutorial sessions. Students varied in the extent that they used the agreements to organize their own problem solving, and this variation enabled analysis of correlation between take-up of agreements and shifts in score from pre-test to post-test (a measure of learning in the tutorial). Analysis revealed that mindful use of agreements differentiated students who gained more and less from the tutorial. Agreement usage, therefore, served as one source of student learning.

Sequence of representations. Recall that the tutorial design reflected a sequence of linear representations of quantities, beginning with number lines, moving to the Cartesian plane, and then introducing linear functions on the plane. Interwoven at specified points within this sequence were agreements to support students' generative understandings of the mathematical properties. The design supported the coordination of linear units and numerical units across the three linear representations of quantities. I argue that this sequence served as an additional source of student learning.

Consider once again the case of Jana as she engaged in Problem Sets XIII and XIV, the point at which linear functions are introduced in the plane. Prior to these problem sets, Jana (and all other tutorial students) was first introduced to the number line and the Cartesian plane. As Jana and the tutor solved each version of the two problem sets, they each relied on agreements

introduced in the context of number lines or the Cartesian plane, an implication of which is that these agreements continued to support tutor-tutee interaction as the representational system gained in complexity. Using the case study, I provide an example in which the tutee spontaneously referred to a prior agreement, and an example in which the tutor referred to a prior agreement that the tutee eventually used herself to justify her solution.

Recall that for Problem XIV.A Jana was coded as No Pass. When the tutor asked Jana what they could do on subsequent problems in order to achieve identical solutions, Jana referred to an agreement inscribed at Problem V.A, *Every Number Has a Place but Not All Have to be Shown*. Originally introduced nine problem sets prior and in a different tutorial session, *Every Number Has a Place* was introduced to support the coordination of linear units and numerical units on a number line. On the subsequent problem, XIV.B, on which the horizontal axis featured multiunit intervals of 2, Jana began by partitioning the first two intervals in half in order to locate the position of 1 and 3 along that axis. I argue that Jana applied this agreement in coordination with another number line agreement (though unstated), unit interval, to regulate her problem solving on the linear function graph task. I take this as corroboration that the prior work with number line tasks and principles supported later work with functions in the coordinate plane

In addition to invoking an agreement from the number line problem sets, the case study also provides an example of an agreement originally introduced in the context of the Cartesian plane invoked to support understanding of functions in the plane. On Problem XIII.A, the introduction of a linear function, Jana was coded as No Pass. To support generative understanding, the tutor directed her attention to the various points he made in his representation, referring to the agreement, *One Point Shows Two Amounts*, which was originally introduced in the first of the Cartesian plane problem sets. He stated, "I saw that he [Beetle the cat] started with no water, so I put a point there [at 0, 0] to show those two amounts." In his use of the agreement, the tutor refers to each point as the two quantities on the problem card, distance walked and height of the water in the glass. On a subsequent problem set, Jana referred to this same agreement to justify her accurate solution to plot a linear function in the plane. I argue that such use of prior agreements, both for coordinating linear and numerical units on a single number line and for a point in the plane as representing two quantities on two independent axes, supports the assertion that the sequence of linear representations of quantities was a source of student learning in the tutorial.

Modeling. Recall that mathematical modeling of the three representations supported a particular conceptual treatment of linear representations of quantities. I argue that the model of a functional relation anchored in a real-world phenomenon supported children's treatment of points in the plane as representing two quantities. I point to two pieces of evidence to support the claim that modeling served as a source of student learning: comparison of Grade 5 experimental students to Grade 8 students, and the use of the modeled functional relation in the case study.

Analysis focused on a comparison of performance between Grade 5 experimental students and Grade 8 students, with each group performing in the lower two-thirds of the original sample for each grade. Since the modeling from the tutorial sequence is different in character from existing curricula, I make an assumption that Grade 8 students, while currently enrolled in algebra, were instructed in ways that did not use such modeling. Based on grade differences, one may expect Grade 8 students currently enrolled in algebra would outperform Grade 5 students on LRA items. For the set of 18 matched problems, nine routine and nine non-routine, analysis revealed that there was no detectable difference in performance on routine problems between the

two groups, and that Grade 5 students outperformed Grade 8 students on non-routine problems. The lack of a detectable difference in performance on routine problems indicates that the intervention supported learning of the target mathematical ideas, with modeling serving as a central design feature that differentiates the intervention from typical instruction on the same targeted ideas.

In addition, the case study highlighted the ways in which modeling supported tutor-tutee interaction. As game players discussed their solutions for problem XIII.A, the tutor referred back to the mathematical model of a cat walking and earning water in order to support a conceptualization of points in the plane as representing different instantiations of the modeled situation. Addressing the fact that no one point represents all possible values of the function, the tutor referred back to the context of a cat earning water, stating, "when we were doing it, we were doing it so he [Beetle] would keep going," gesturing towards the cat on the yard line in order to refer back to the shared discussion. On the subsequent problem, XIII.B, the tutor coordinated modeling with the agreement *One Point Shows Two Amounts* by emphasizing that those two amounts need to correspond to the Beetle the Cat's water earned as a function of walking. The modeling, therefore, anchored discussion during the tutorial sequence, and supported children's generative understanding of inscriptions in the plane.

Chapter 4: Discussion

The two studies presented in this dissertation contribute to the research literature on students' understanding of linear representations of quantities. In this chapter, I present a review of this dissertation's key findings. I also discuss limitations of the study, and provide commentary on implications and next steps.

Summary and Discussion of Findings

In this section, I present a review of this dissertation's key findings, beginning with the assessment study followed by the tutorial study. In particular, I focus on the coordination of linear units and numerical units across the two studies.

Assessment study. In the assessment study featured in Chapter 2, I examined the performance of students in Grades 5 and 8 on the Linear Representations Assessment (LRA). I documented performance on items featuring three types of representations with linear dimensions: number lines, the Cartesian plane, and graphs of simple mathematical functions. I examined the understandings both of students who had not had formal instruction involving graphs of functions (Grade 5) and, then to understand whether the patterns of errors in fifth graders persisted at higher grade levels when students were enrolled in algebra, of students who were in the middle of coursework involving functions (Grade 8). Using an experimental design embedded in the assessment, I compared students' performances on matched routine and non-routine problems. Findings indicated that students at each grade scored significantly better on routine problems as compared to non-routine. These results suggested that performance on canonical routine problems may hide the ways in which students coordinate linear and numerical units with these representations. To understand this, I analyzed performance on target non-routine tasks for each grade across representations, and also between each grade for each representation. There was no detectable difference in performance between Grade 5 and Grade 8 students on non-routine number line tasks, suggesting that, despite the grade difference, a similar proportion of students in each grade did not coordinate linear units and numerical units on number line items.

A qualitative analysis of students' efforts to solve number line, Cartesian plane, and function graphs problems using interview data revealed the various ways students coordinated linear and numerical units to solve LRA problems. Successful approaches were almost always associated with an appropriate coordination of linear and numerical units (see Item *1, 3; Place 0* in Figure 85a). Other resources emerged across students' incorrect responses, including (a) attention to the order of numerical units for positive integers (e.g., placing 0 to the left of 1 in Figure 85a-d); (b) treatment of 0 as a unique numerical unit (e.g., placing 0 at the 'start point', Figure 85b); (c) efforts to extend numerical units (e.g., Figure 85c); (d) efforts to use congruent linear units on the line (e.g., Figure 85d); and (e) naming a point in the plane based on projections from each axis (e.g., in the task *Identify 5, 5,* naming the point (4.5, 4.5) as in Figure 85e).

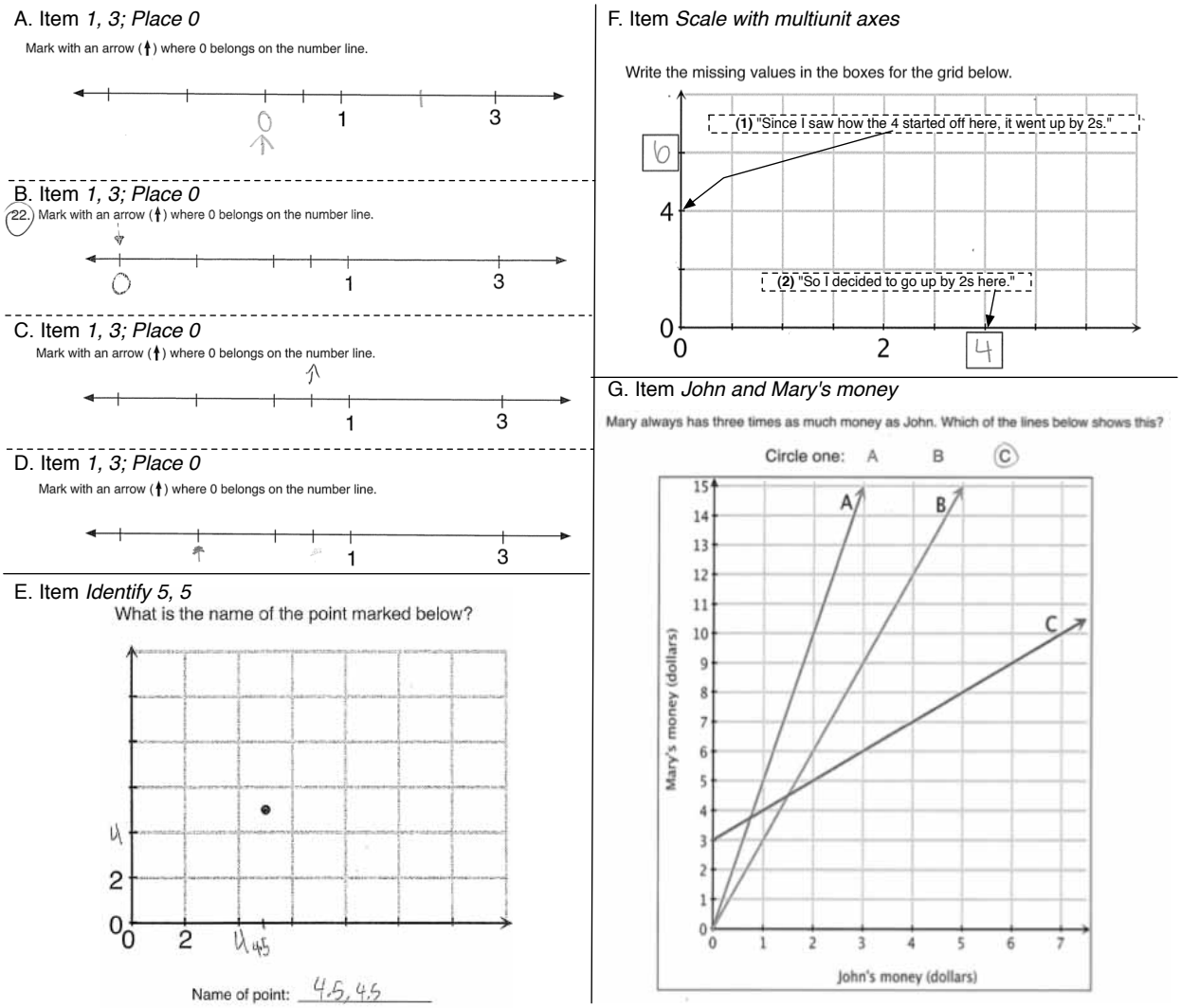


Figure 85: Examples of students' solutions on various items from the LRA

Interviews also provided data on similarities and differences between the two Grades participating in the assessment study. In both grades, students displayed a need for further support in order to coordinate linear units and numerical units on number lines and axes. In both grades, students did not first coordinate linear and numerical units along axes before naming a point in the plane (e.g., in the task *Identify 5, 5* in Figure 85e, treating each interval of 2 as a unit interval to name the point (4.5, 4.5)). Trends that emerged among the Grade 5 population alone included (a) the position of 0 as the leftmost point on a line (e.g., item *1, 3; Place 0* in Figure 85b), and (b) 'matching axes,' or, coordinating linear and numerical units across axes (e.g., item *Scale with multiunit axes* in Figure 85f). A trend more common to the Grade 8 population included ascribing meaning to the orientation of the function line in the plane independent of numerical units or linear units in the plane (e.g., choosing Line B in item *John and Mary's money* because of its orientation in the plane independent of the underlying mathematical function).

Clinical interviews further clarified instructional goals for task design. In particular, the following three areas emerged as ones for which Grade 5 students need further support: (a)

coordinating linear units and numerical units in principled ways (e.g., to locate 0 at the appropriate location, as in Figure 85a); (b) treating two axes as each representing different quantities (e.g., treating the two axes of the item *Scale with multiunit intervals* as unrelated in Figure 85f); and (c) working with story contexts that depict joint variation (e.g., coordinating the story context of item *John and Mary's money* Figure 85g with linear and numerical units on a grid).

Tutorial study. I examined the efficacy of a tutorial intervention designed to support students' generative understanding of linear representations of quantities. Recall that the tutorial intervention was designed to support the coordination of linear units and numerical units across the three representations. Students were selected from the lower performing two-thirds of Grade 5 students on the LRA from the assessment study. After intervention, I compared pre- to post-test gains on the LRA for the experimental and control groups both for overall performance and for the subscale of routine and non-routine items. I also analyzed students' successes and hurdles in solving tutorial problems. This included three analyses. First, I used video data to examine correlation between students' uptake of agreements and learning gains. Second, I analyzed pathways through the tutorial problem sets and patterns in coordination efforts across a subset of tasks for both successful and unsuccessful responses. I concluded with a case study of a typical tutorial student to illustrate one student's pathway through a part of the tutorial and provide further insight into the dynamics of learning in this instructional context.

Quantitative results comparing performances of the experimental and control groups revealed that experimental students demonstrated gains on the overall scale, and also on the routine and non-routine problem subscales. Control students, meanwhile, demonstrated no detectable pre to posttest gains, indicating that gains within the experimental population were not due to regular classroom instruction or due to regression to the mean. These results suggest that the intervention supported students' coordination of linear units and numerical units as measured by overall performance on the LRA, performance on canonical routine problems, and performance on the matched non-routine problems.

I compared performance of experimental Grade 5 students with Grade 8 students on the LRA. Experimental Grade 5 students performed statistically better on non-routine items after intervention when compared to Grade 8 students at the same cut point. Given that lower-performing Grade 5 students outperformed older students enrolled in algebra, this suggests that the intervention well supported Grade 5 students' principled coordination of linear and numerical units. This is further corroborated by performance on the task *Rate of change* for Grade 5 experimental students and Grade 8 students. This challenging problem, which featured two grids of different scales with the same function, required coordination of linear and numerical units to determine that the two functions displayed the same rate of change. The performance of Grade 5 experimental students after intervention was comparable to that of Grade 8 students in the assessment study, indicating that the intervention supported in productive ways the coordination of linear units and numerical units.

Further analyses for the experimental group focused on the character of performances through the tutorial sessions. Analyses revealed a correlation between students who used agreements appropriately in the wrap-up sessions and higher gain scores. Across tutorial problem sets, analyses revealed particular areas of success or struggle across students. Some areas, such as those that introduced the Cartesian plane with unit axes (Problem Set VIII), were met with success across all tutorial students, with all but two students passing on the first attempt

(Problem VIII.A). Yet on other areas, such as the introduction of linear functions (Problem Set XIII), no students passed at on the first version (Problem XIII.A), and 35% of students did not pass the final version (Problem XIII.C). In order to understand students' coordination of linear and numerical units as a large number of these students struggled, I analyzed the two problem sets that introduced linear functions (problem sets XIII and XIV). These analyses revealed patterns within students' approaches that were coded as Pass and No Pass as students made efforts to represent the joint variation indicated on playing cards.

Limitations

In this project, I have made efforts to gather systematic data on students' understanding of linear representations of quantities. Nonetheless, the design has limitations. I focus on two limitations that have implications for the interpretations of the findings and directions for future research: (a) the role of comparison groups, and (b) the lack of additional content in the task sequence.

Alternative comparison groups. A limitation of the tutorial study design was that the intervention was not compared to an alternative intervention. Recall that the tutorial study presented in Chapter 3 showed that the experimental group had significant gains while a matched control group revealed no detectable difference in performance. While such a comparison allowed me to rule out the effect of regular classroom instruction and regression to the mean, such a comparison did not allow me to discern whether another intervention on linear representations of quantities would have supported students in similar or different ways. Alternative options might have included a similar intervention with additional control groups; for example, administering to a control group the same task sequence but without inscribing agreements on an agreement sheet. Such a design would have provided a comparison to understand the role of inscribed definitions on students' coordination of linear and numerical units. Additional comparison groups would have allowed me to understand how particular features of the intervention supported student understanding, and if alternative options would have supported understanding similarly or differently.

Additional content: Fractions and negative numbers. A second limitation of the tutorial study design is that the progression of linear representations of quantities did not include fractions or negative numbers. Research has documented the utility of linear representations of quantities in supporting negative integers and fractions (Bass, 1998; Saxe et al., 2007, 2010; Wu, 2005, 2009). I conjecture that might have allowed students to refine and extend their understandings of the definitions and, therefore, of the inner workings of linear representations of quantities.

I use an example of a simple linear function in the plane to consider this limitation. Figure 86 presents a problem (XIII.A) from the tutorial sequence. The additional points at various positions on the function line represent non-integer values for one or both quantities (e.g., (0.5, 1)). While analyses of the intervention indicate that tutorial interactions supported gains as measured by pre to post-test, I conjecture that work that explicitly focuses on extending linear and numerical units to fractions (e.g., the unit interval from 0 to 1 may be partitioned in half to create two congruent sub-unit intervals) would further support students' interpretation of non-integer points along a function line. The design of the tutorial intervention, however, did not provide direct instruction on non-integer points.

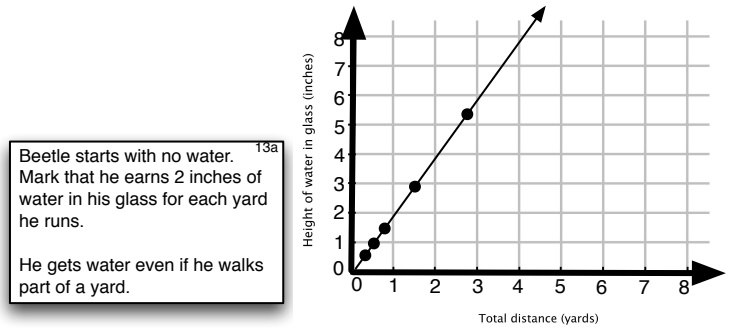


Figure 86: Problem XIII.A from the tutorial intervention with additional points at non-integer positions

Implications and Next Steps

I consider here implications of the two studies presented in this dissertation, as well as next steps. Results of the tutorial study indicated that the intervention led to deep understandings of the mathematics underlying linear representations of quantities. In this section, I draw upon a research framework developed by the design research project *Learning Mathematics through Representations* (Saxe et al., 2009), in which three coordinated sets of studies supported the development of curriculum. I then discuss a recently produced curriculum found to be effective in supporting generative understandings with integers and fractions on number lines. I finish by proposing subsequent curriculum development focusing on the Cartesian plane and function graphs.

A next step for research is to use findings and design of the tutorial study as blueprints for curriculum development. I consider the two studies of this dissertation—assessment study and tutorial study—to be two of three research phases key to developing research-based curriculum. I argue that to develop rich curricula to support generative thinking, further research is necessary. The final type of research study, classroom studies, builds on results of both assessment and tutorial studies. As indicated in Figure 87, the research phases include: (a) *written assessments and interviews* to document students' conceptual coordinations (Saxe et al., in press); (b) *tutorial studies* to explore a particular instructional approach and to investigate variations in learning trajectories of students (Saxe et al., 2010); and (c) *classroom studies* to explore an instructional approach and sequence in a whole class setting (Saxe et al., in press). The bidirectional arrows indicate that any particular phase may lead to further studies in each of the other two phases.

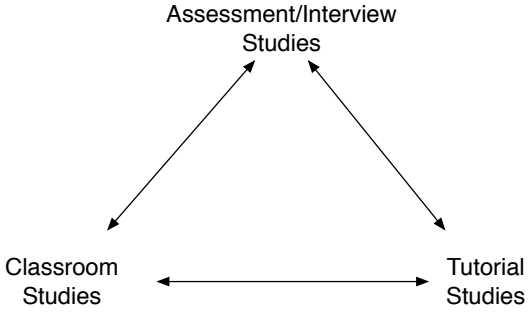


Figure 87: Three phases of research study key to research-based curriculum (adapted from Saxe et al., 2009)

I propose that classroom studies may be well coordinated with the efforts of LMR project. LMR has been engaged with the design of a two-unit, nineteen lesson sequence involving integers and fractions on the number line (Gearhart et al., 2011; Saxe et al., 2010b). Content begins with positive integers and then moves to negative integers and fractions. LMR instruction is designed to incorporate the three design features of the tutorial intervention from Chapter 3: modeling, communication, and the use of definitions. I conjecture that this two-unit sequence would set up well a new unit on the Cartesian plane and graphing functions (Figure 88). Specifically, I see potential in supporting students' generative understandings of negative numbers and fractions as points on the line prior to introducing the plane in order to support their understanding of points along a function line.

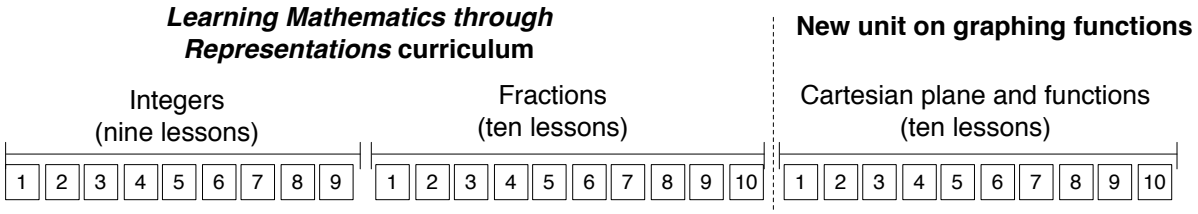


Figure 88: LMR curriculum on integers and fractions with a proposed next step for a new unit on graphing functions

In next steps, I intend to build on this lesson sequence with a unit on the Cartesian plane and graphing functions. Such content is consistent with recent state standards that have been widely adopted in the United States (Common Core State Standards Initiative, 2010). Once lessons have been iteratively refined and piloted, I will recruit classrooms for an experimental group and a control group in order to test the efficacy of the curriculum and also understand opportunities for children to learn. In collaboration with classroom teachers, I anticipate iterative design cycles in which lessons are designed and then piloted, followed by a re-design and a new pilot period to respond to results of the first pilot. Experimental classrooms would teach the two LMR units followed by the unit on graphing functions. The control classrooms would not provide this intervention, but instead provide the standard classroom instruction. In order to investigate the interplay of cognition and instruction, empirical techniques will include video of group work and whole class discussions, as well as after-class interviews with students in order to document individual students' reasoning. This classroom-based research will also explore the utility of the definitional approach in supporting rich and generative understandings.

Concluding Remarks

The research presented in this dissertation consisted of a systematic study of student reasoning and instruction involving linear representations of quantities—number lines, the Cartesian plane, and function graphs. This research provided evidence that a progression of linear representations of quantities focusing on coordinating linear and numerical units well supported student performance on routine and non-routine assessment tasks. Findings from written assessment and interview studies as well as tutorial studies provide rich insights into the coordinations of linear and numerical units students make when working with linear representations of quantities. While much additional research needs to be conducted to further understand both student reasoning and productive instructional approaches, the findings presented in this dissertation represent important next steps in order to ultimately reach this goal.

REFERENCES

Bass, H. (1998). Algebra with integrity and reality. Keynote address. In Proceedings of the National Research Council Symposium on the Nature and Role of Algebra in K-14 curriculum (pp. 9-15). Washington DC: National Academy Press.

Blanton, M.L. (2008). Early Algebra. Proceedings of "Future Curricular Trends in School Algebra and Gemoetry" Conference. University of Chicago.

Brizuela, B.M., & Earnest, D. (2008). Multiple notational systems and algebraic understandings: The case of the best deal problem. In J.J. Kaput, D.W. Carraher, & M.L. Blanton (Eds.), Algebra in the early grades, (pp. 273-301). Mahwah, NJ: Lawrence Erlbaum Associates.

Caddle, M.C., & Earnest, D. (2009). Slope as a procedure: The impact of axis scale. Research Report at the 2009 meeting for the International Group for the Psychology in Mathematics Education. Greece.

Cai, J., & Knuth, E. (2011). Early algrebraization: A global dialogue from multiple perspectives. Heidelberg: Springer.

Carraher, D.W., & Schliemann, A.D. (2007). Early algebra and algebraic reasoning. In F.K. Lester, Second handbook of research on mathematics teaching and learning (pp. 669-705). Greenwich: Information Age Publishing.

Carraher, D., Schliemann, A.D., Brizuela, B.M., & Earnest, D. (2006). Arithmetic and algebra in early mathematics education. Journal for research in mathematics education, 37(2), 87-115.

Carraher, D.W., Schliemann, A.D. & Schwartz, J. (2008). Early algebra is not the same as algebra early. In J. Kaput. D. Carraher, & M. Blanton (Eds.), Algebra in the Early Grades. Mahwah, NJ, Erlbaum, pp. 235-272

Corwin, R. B., Russell, S. J., & Tierney, C. C. (1990). Seeing fractions: A unit for the upper elementary grades. Sacramento, CA: California Department of Education.

Dugdale, S., & Kibbey, D. (1986). Green globs and graphing equations (computer software). Pleasantville, NY: Sunburst Communications, Inc.

Gearhart, M., Saxe, G.B., Haldar, L.C., Earnest, D., Sitabkhan, Y., Lewis, K., et al. (2011, June). Learning Mathematics through Representations: Integers, fractions and the number line. Paper presented at the 41st annual meeting of the Jean Piaget Society, Berkeley, CA.

Gravemeijer, K. & Stephan, M. (2002). Emergent models as an instructional design heuristic. In K. Gravejeijer, R. Lehrer, B. Van Oers, & L. Verschaffel (Eds.), Symbolizing, modeling and tool use in mathematics education (pp. 145-170). Dordrecht, The Netherlands: Kluwer.

Kaput, J. (2008). What is algebra? What is algebraic reasoning? In J. Kaput, D. W. Carraher, & M. Blanton (Eds.) Algebra in the early grades. Mahwah, NJ: Erlbaum. 5-17.

Kaput, J.J., & Blanton, M.L. (2005). Characterizing a classroom practice that promotes algebraic reasoning. Journal for Research in Mathematics Education, 36(5), 412-446.

Kaput, J.J., Carraher, D.W., & Blanton, M.L. (2008). Algebra in the early grades. New York, NY: Erlbaum.

Lehrer, R., Strom, D., & Confrey, J. (2002). Grounding metaphors and inscriptional resonance: Children's emerging understanding of mathematical similarity. Cognition and Instruction, 20, 359-398.

Kerslake, D. (1977). The understanding of graphs. Mathematics in School, 6(2), 56-63.

Leinhardt, G., Zaslavsky, O., & Stein, M.K. (1990). Functions, graphs and graphing: Tasks, learning and teaching. Review of Educational Research, 60, 1-64

Leont'ev, A. A. (1981). "The Problem of Activity in Psychology." In J.V. Wertsch (Ed.) The Concept of Activity in Soviet Psychology (p. 36-71). NY: Sharpe.

Lobato, J., Ellis, A.B., & Muñoz, R. (2003). How "focusing phenomena" in the instructional environment support individual students' generalizations. Mathematical Thinking and Learning, 5(1), 1-36.

Moschkovich, J.N. (2008). "I Went by Twos, He Went by One": Multiple Interpretations of Inscriptions as Resources for Mathematical Discussions. The Journal of the Learning Sciences, 17(4), 551-587.

Moschkovich, J., Schoenfeld, A.H., & Arcavi, A. (1993). Aspects of understanding: Multiple perspectives and representations of linear relations and connections among them. In T.A. Romberg, E. Fennema, and T.P> Carpenter (Eds.), Integrating research on the graphical representations of functions. Hillsdale, NJ: Erlbaum, pp. 69-100.

Moses, R. P., & Cobb, C. (2001). Radical equations: Math literacy and civil rights. Boston: Beacon.

National Governors Association Center for Best Practices, Council of Chief State School Officers. (2010). Common Core State Standards for Mathematics. National Governors Association Center for Best Practices, Council of Chief State School Officers: Washington DC.

Saxe, G.B., Shaughnessy, M.M., Shannon, A., Langer-Osuna, J.M., Chinn, R., & Gearhart, M. (2007). Learning about fractions as points on a number line. In W.G. Martin, M.E. Strutchens, & P.C. Elliott (Eds.), The Learning of Mathematics: Sixty-ninth Yearbook (pp. 221-237). Reston, VA: NCTM.

Saxe, G.B., Shaughnessy, M., Gearhart, M., & Haldar, L.C. (in press). Coordinating numerical and linear units: Elementary students' strategies for locating whole numbers on the number line. Mathematical Thinking and Learning.

Saxe, G.B., Gearhart, M., Shaughnessy, M.M., Haldar, L.C., Earnest, D., & Sitabkhan, Y. (2009, April). Integers on the number line: Students' understanding of linear unit. Poster presented at the research pre-session of the annual meeting of the National Council of Teachers of Mathematics, Washington, D.C.

Saxe, G.B., Earnest, D., Sitabkhan, Y., Haldar, L.C., Lewis, K.E., & Zheng, Y. (2010a). Supporting generative thinking about integers on number Lines in elementary mathematics. Cognition and instruction, 28(4), 433-474.

Saxe, G.B., Earnest, D., Sitabkhan, Y., Haldar, L.C., & Zheng, Y. (2010, May). Integers on number lines: Supporting generative thinking through a communication game tutorial. Poster presented at the annual meeting of the American Educational Research Association, Denver, CO.

Schliemann, A.D., Carraher, D.W., & Caddle, M.C. (2008, March). From number lines to intervals in the Cartesian space. In D. Abrahamson (Chair), D. Earnest (Organizer), and H. Bass (Discussant), The many values of the number line – An interdisciplinary forum. Symposium conducted at the annual meeting of the American Educational Research Association, New York City, NY.

Sfard, A. (2008). Thinking as communicating: Human development, the growth of discourses, and mathematizing. Cambridge, UK: Cambridge University Press.

Stevens, R., & Hall, R. (1998). Disciplined perception: Learning to see in technoscience. In M. Lampert & M.L. Blunk (Eds.), Talking mathematics in school: Studies of teaching and learning (pp. 107-149). New York, NY: Cambridge University Press.

Treffers, A. (1991). Meeting Innumeracy at Primary School. Educational studies in mathematics, 22, 333-352.

Vygotsky, L.S. (1986). Thought and language (A. Kozulin, Trans.). Cambridge, MA: The MIT Press.

Wu, H.-H. (2002). What is so difficult about the preparation of mathematics teachers? http://math.berkeley.edu/~wu.

Wu, H.-H. (2005). Key mathematical ideas in grades 5-8. http://math.berkeley.edu/~wu.

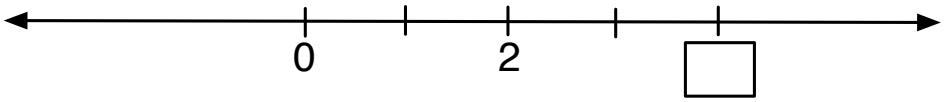
Wu, H-H. (2009). What's sophisticated about elementary mathematics? American Educator, 33(3), 4-14.

Zaslavsky, O., Sela, H., & Leron, U. (2002). Being sloppy about slope: The effect of changing the scale. Educational studies in mathematics, 49, 119-140.

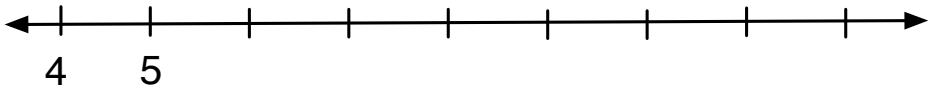
APPENDIX A: LINEAR REPRESENTATIONS ASSESSMENT (LRA)

Name: _____

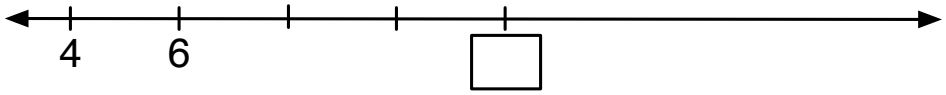
1. Write the number that belongs in the box:



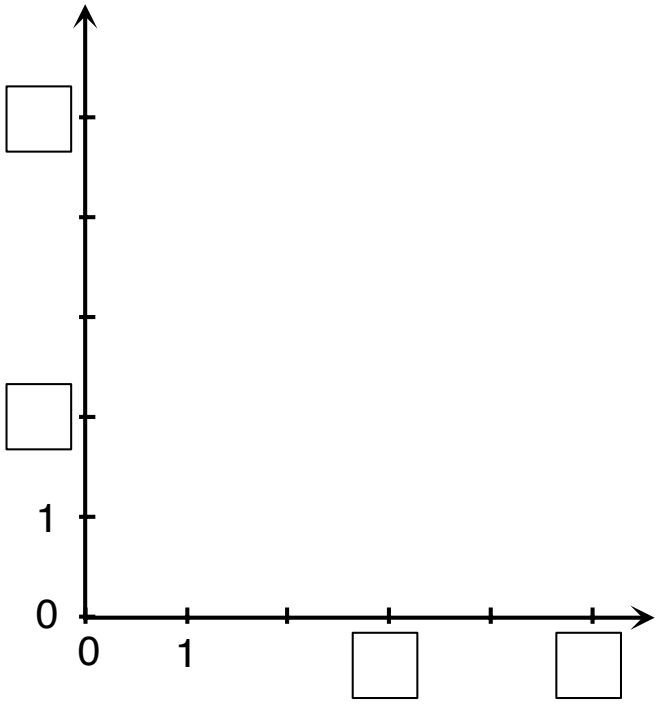
2. Mark where 8 belongs on this number line



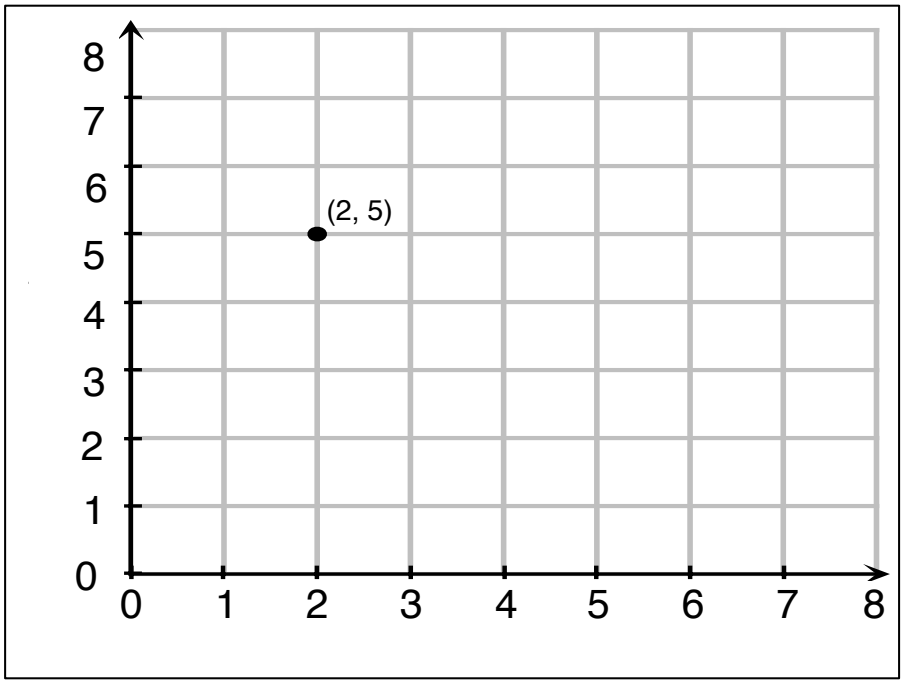
3. Write the number that belongs in the box.

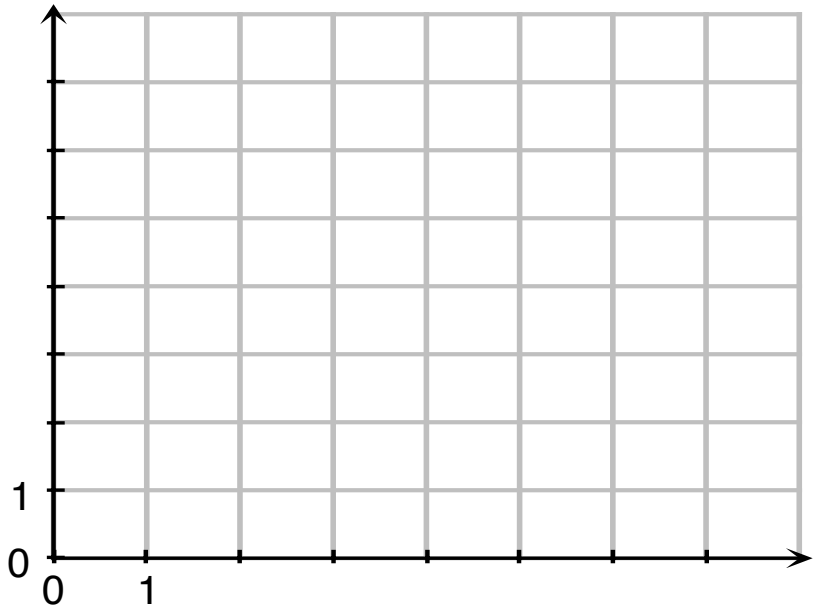


4. Write the missing values in the boxes for the grid below.

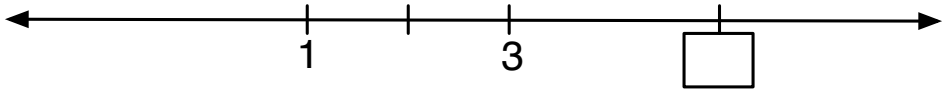


5. The point (2, 5) is shown on the top grid. Mark the point (5, 2) on the bottom grid.

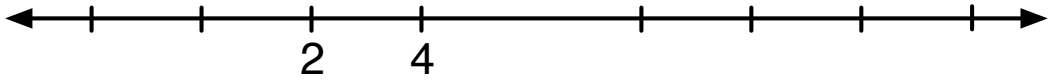




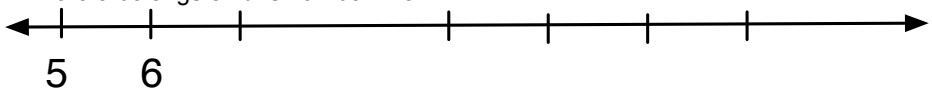
6. Write the number that belongs in the box:



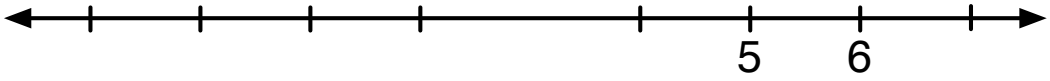
7. Mark with an arrow (↑) where 8 belongs on the number line.



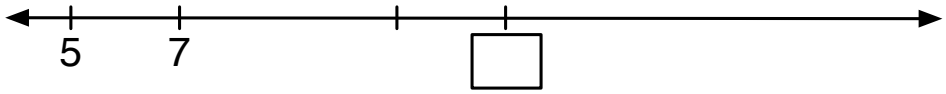
8. Mark where 9 belongs on this number line



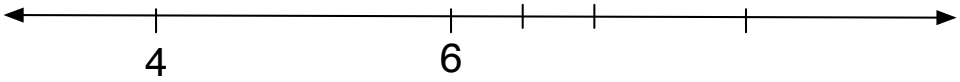
9. Mark with an arrow (↑) where 1 belongs on the number line.



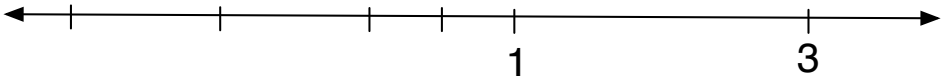
10. Write the number that belongs in the box.



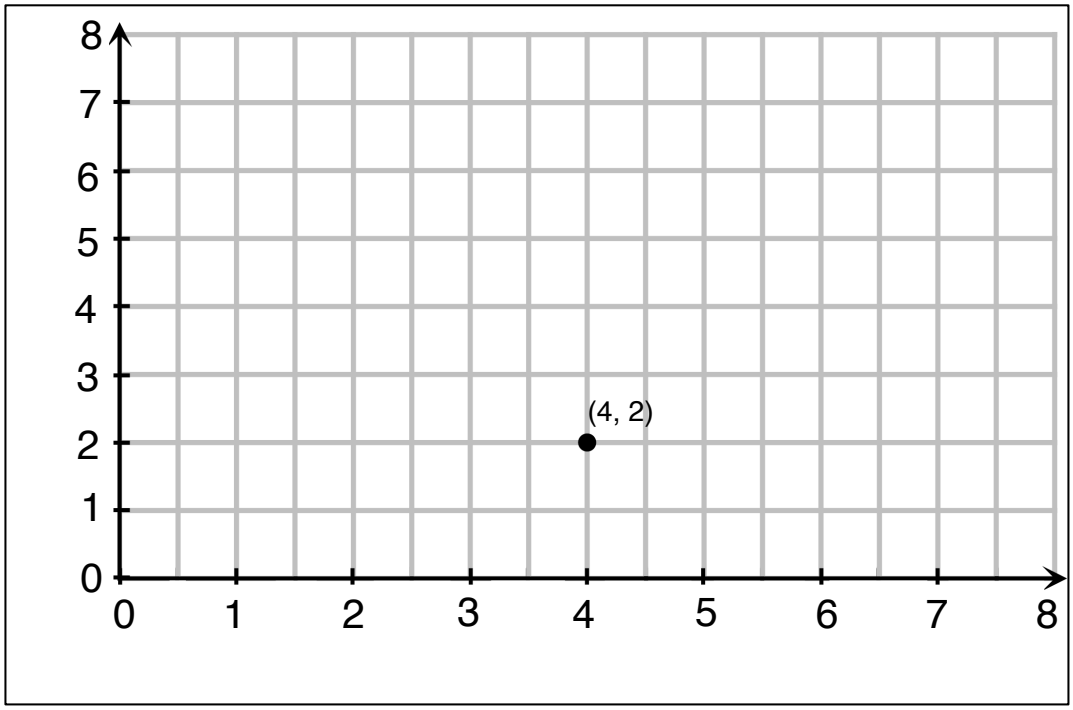
11. Mark with an arrow (↑) where 7 belongs on the number line.

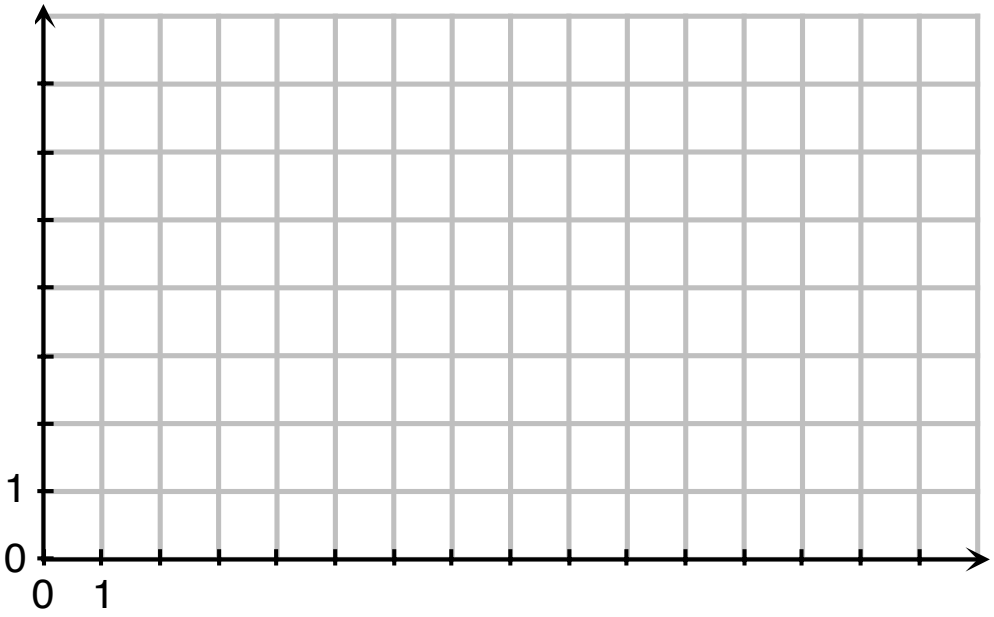


12. Mark with an arrow (↑) where 0 belongs on the number line.



13. The point (4, 2) is shown on the top grid. Mark the point (4, 2) on the bottom grid.

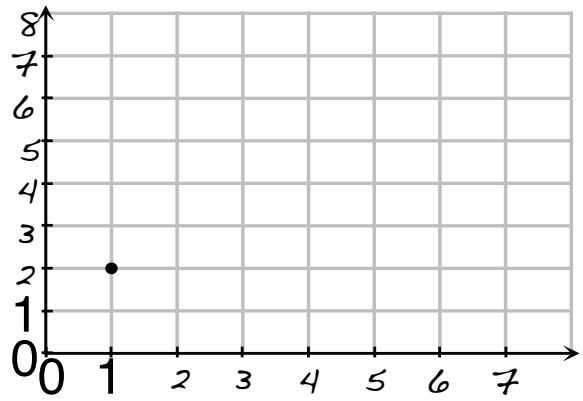




4

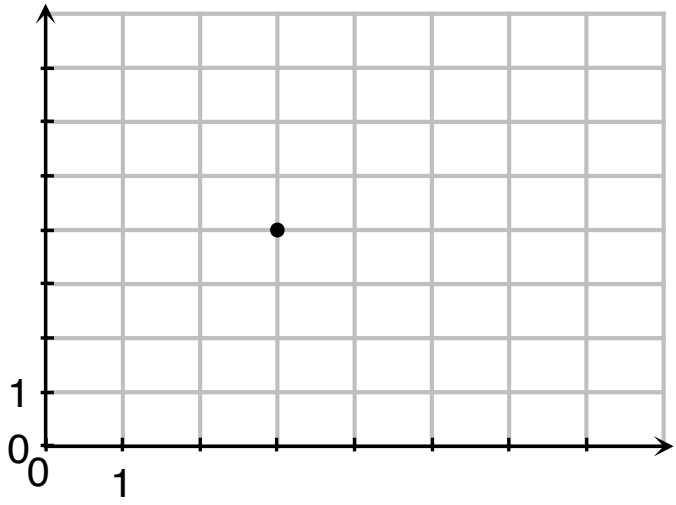
Example

What is the name of the point marked below?



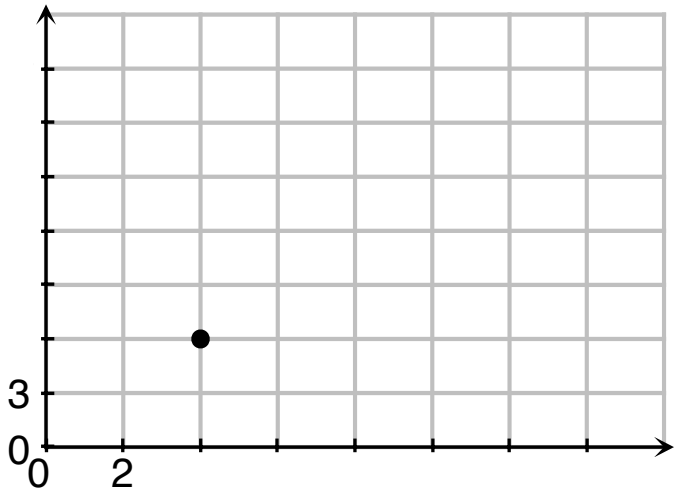
Name of point: (1, 2)

14. What is the name of the point marked below?



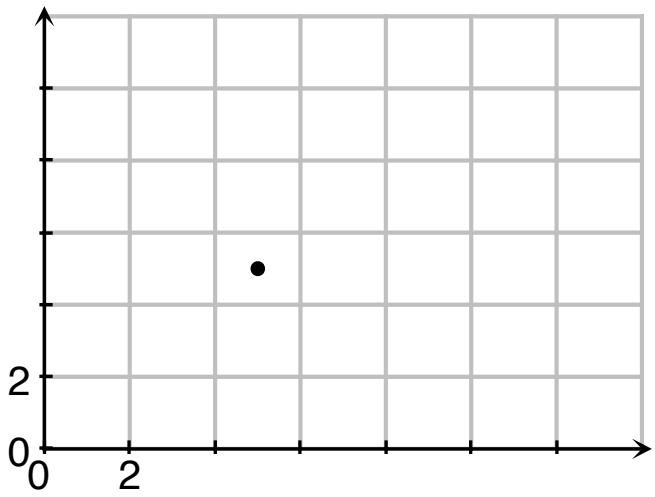
Name of point: _____

15. What is the name of the point marked below?



Name of point: _____

16. What is the name of the point marked below?

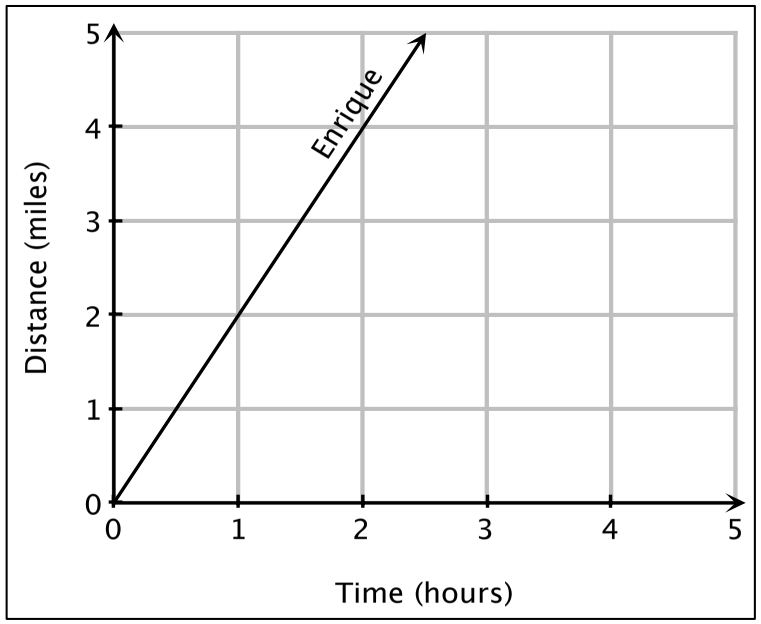


Name of point: _____

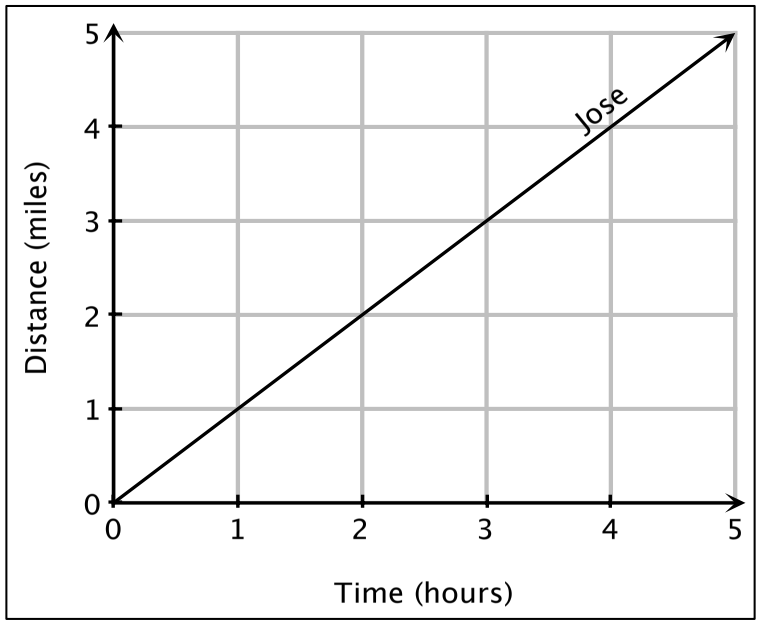
17. Graph A shows how many miles Enrique walks each hour.

Graph B shows how many miles Jose walks each hour.

A



B



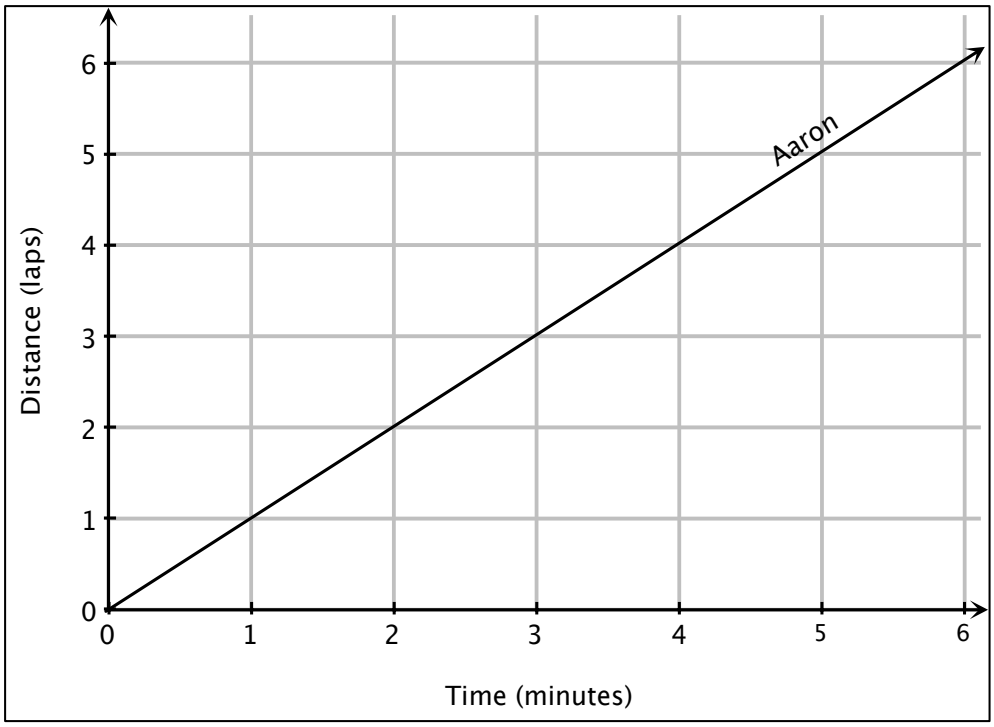
Who walks more miles each hour, Enrique or Jose?

Circle one: Enrique Jose They walk the same rate

How do you know? _____

_____

18. The graph below shows how far Aaron runs over time.



a. How many laps does Aaron run in 3 minutes? _____

b. How many laps does Aaron run between minute 2 and minute 4? _____

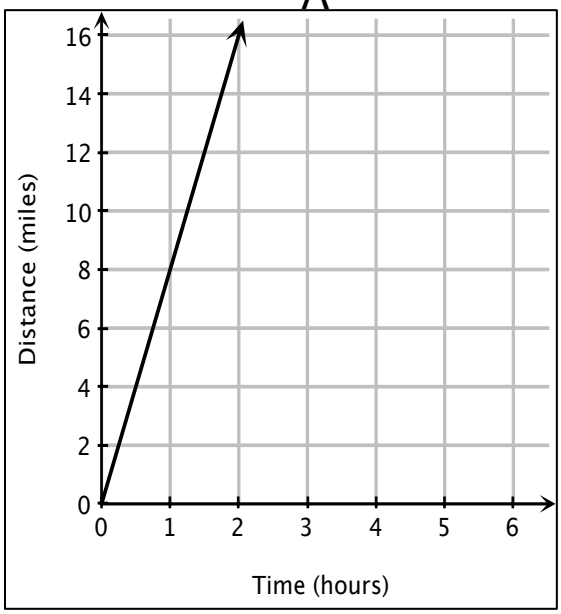
c. How long does it take Aaron to run 4 laps? _____

d. How many laps does Aaron run each minute? _____

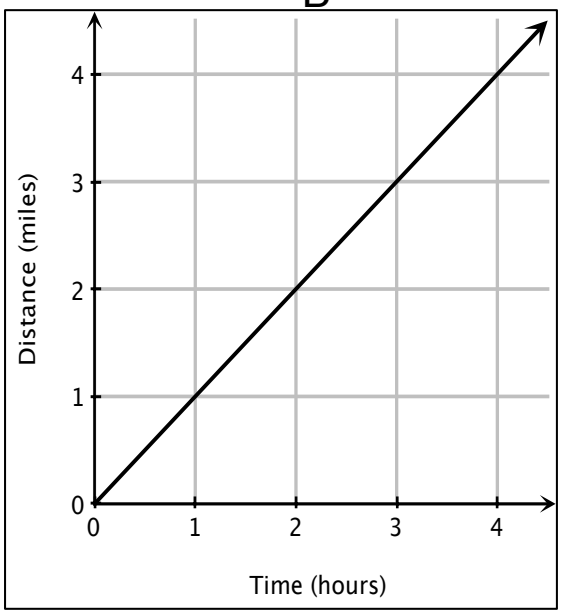
19. Ben and his dad walk 4 miles each hour.

Which graph below best shows this (circle one): A B C D

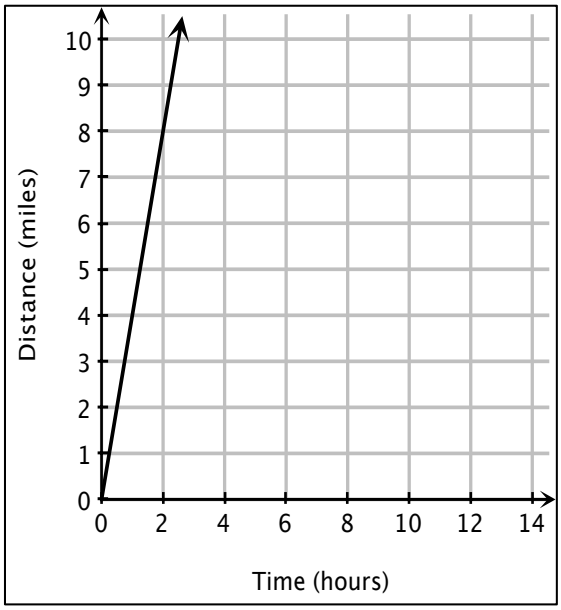
A



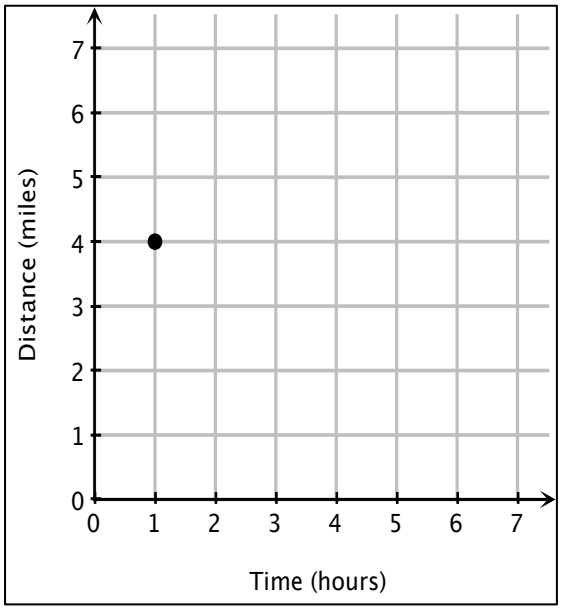
B



C



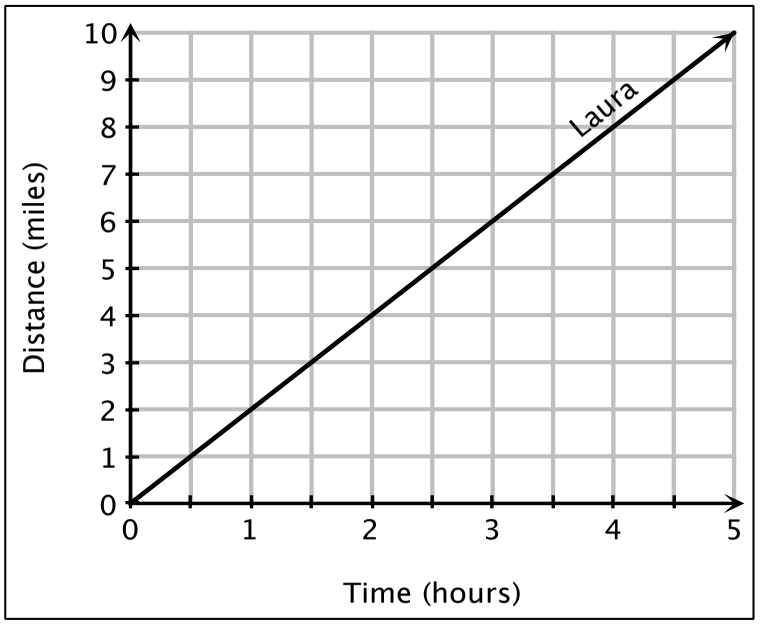
D



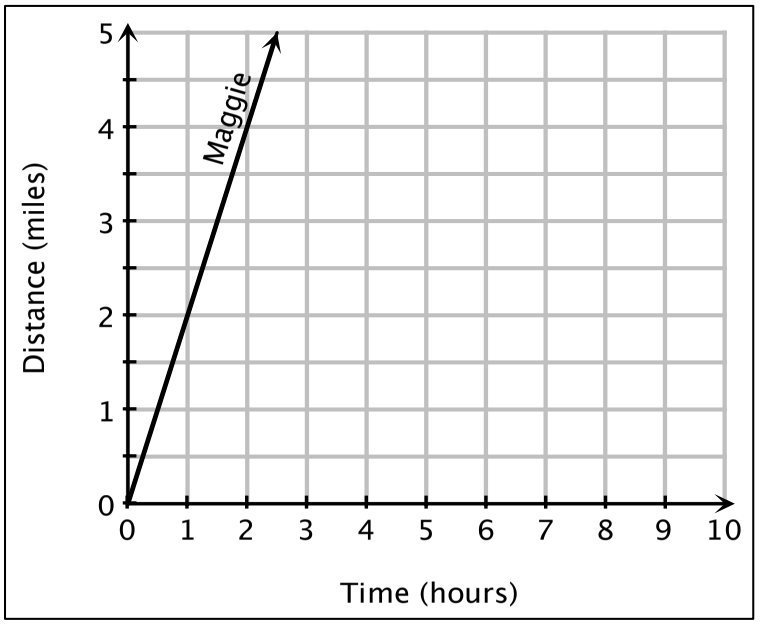
20. Graph A shows how many miles Laura walks each hour.

Graph B shows how many miles Maggie walks each hour.

A



B



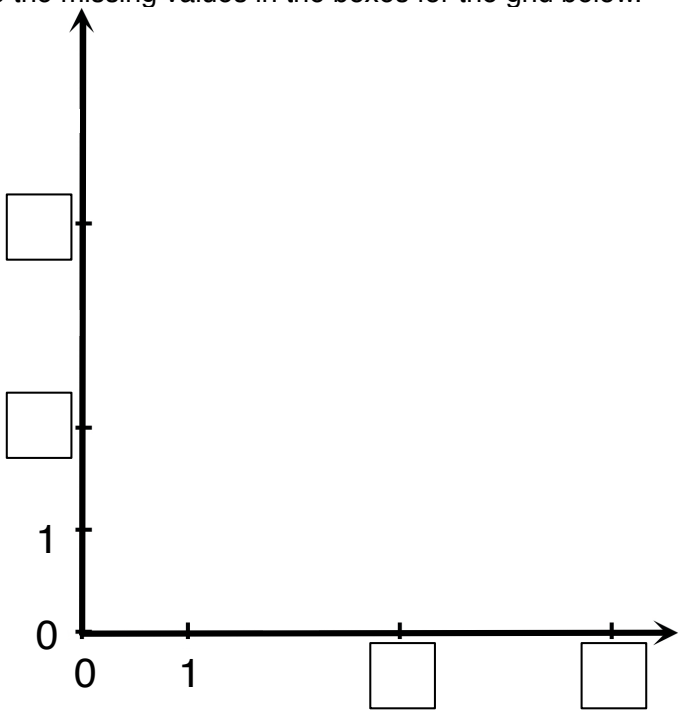
Who walks more miles each hour, Laura or Maggie?

Circle one: Laura Maggie They walk the same rate

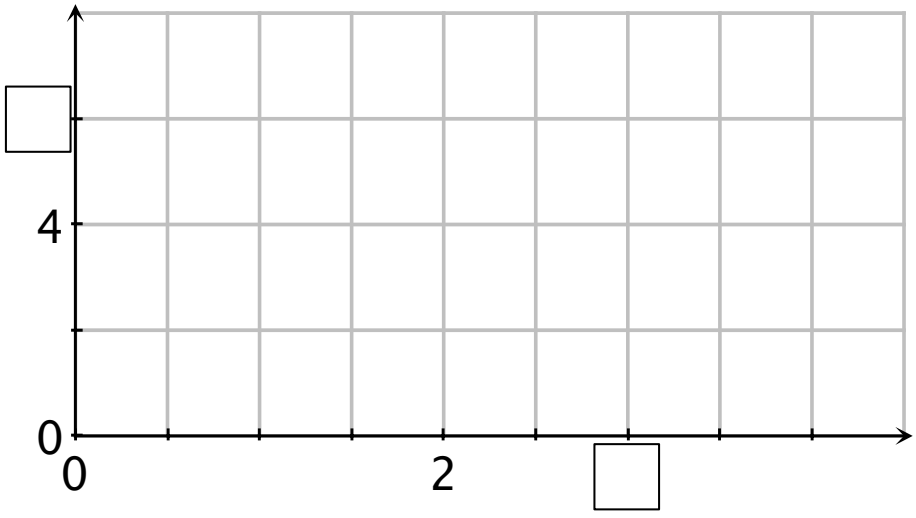
How do you know? _____

_____

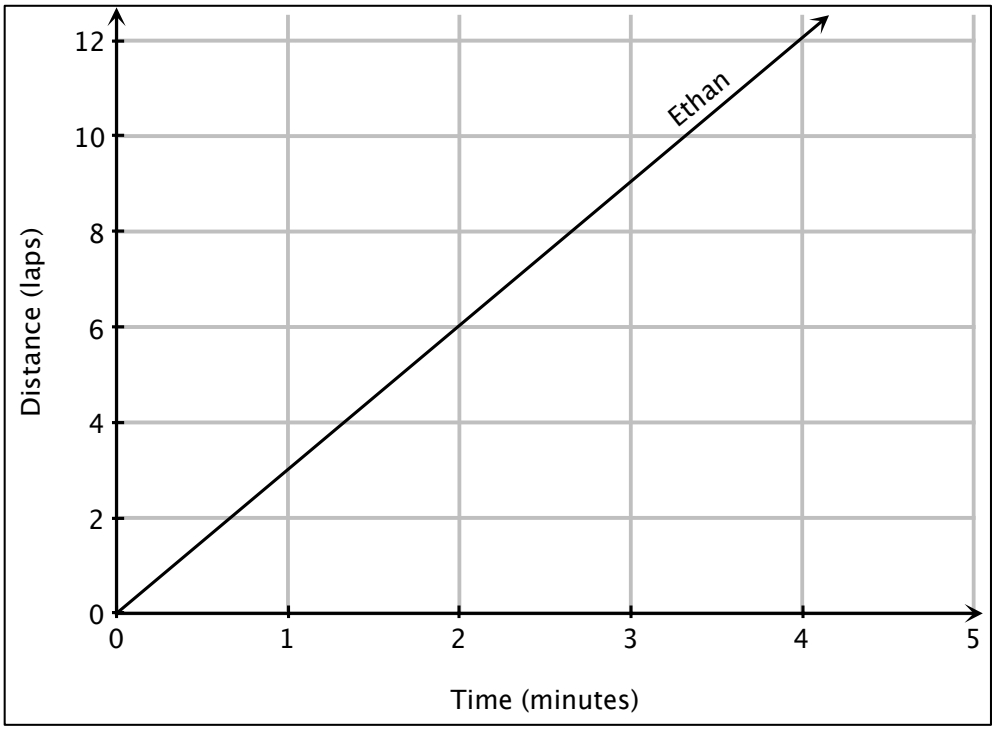
21. Write the missing values in the boxes for the grid below.



22. Write the missing values in the boxes for the grid below.



23. The graph below shows how far Ethan runs over time.



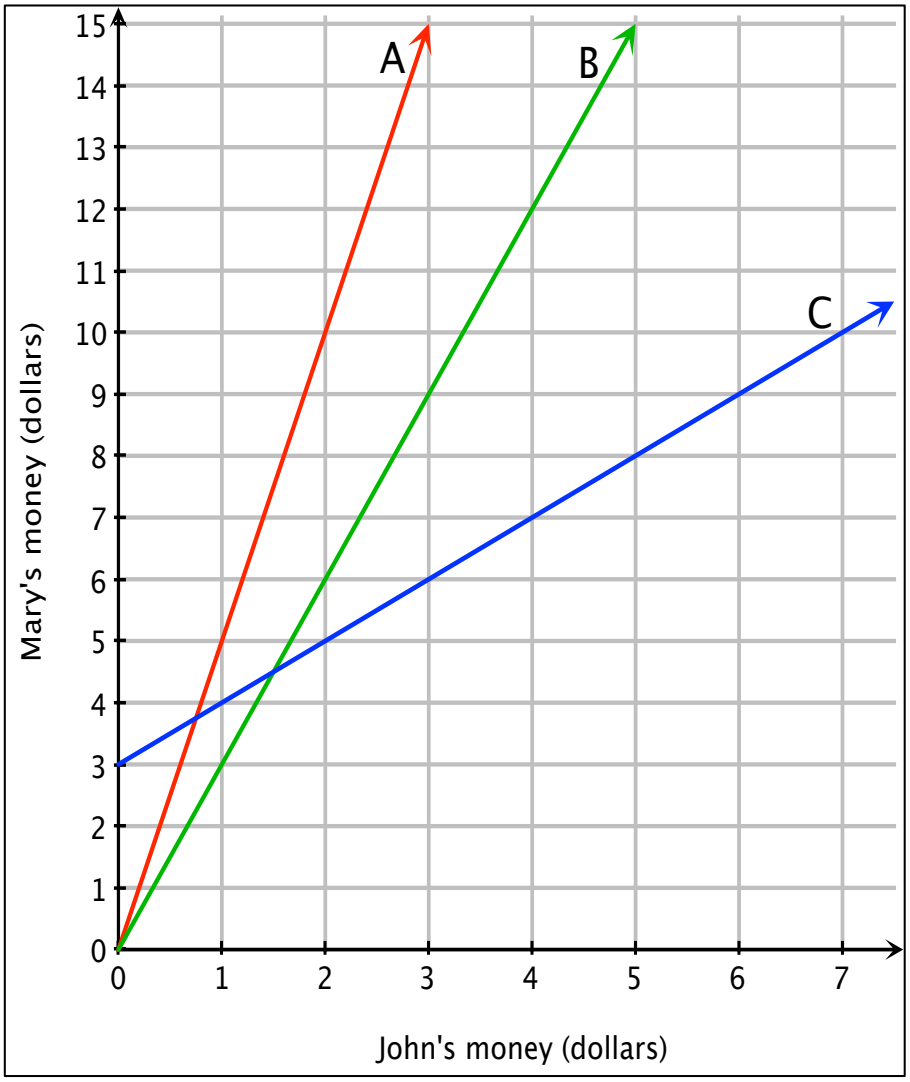
a. How many laps does Ethan run in 3 minutes? _____

b. How many laps does Ethan run between minute 2 and minute 4? _____

c. How many laps does Ethan run each minute? _____

24. Mary always has three times as much money as John. Which of the lines below shows this?

Circle one: A B C

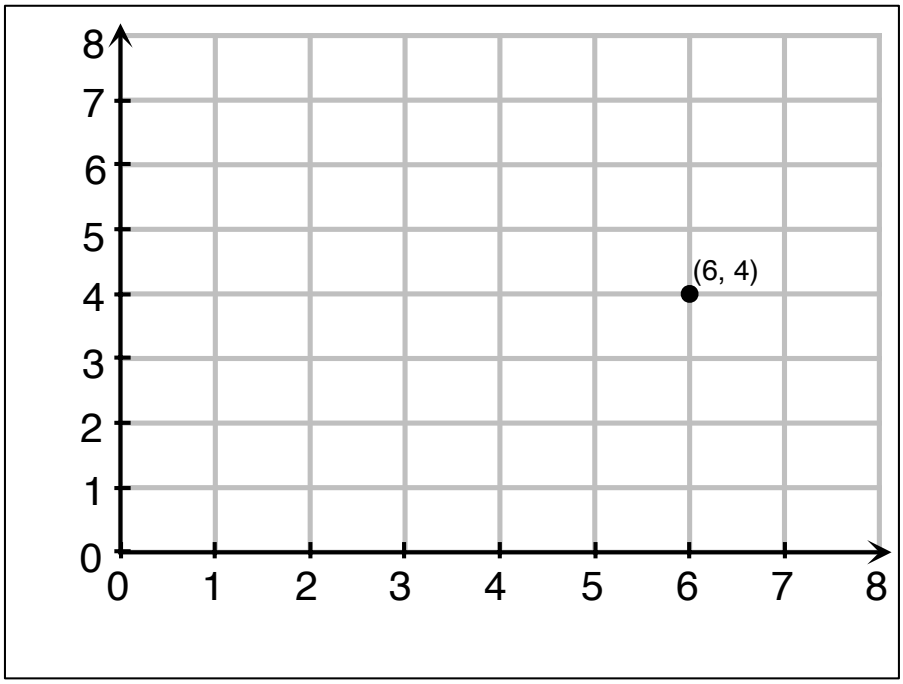


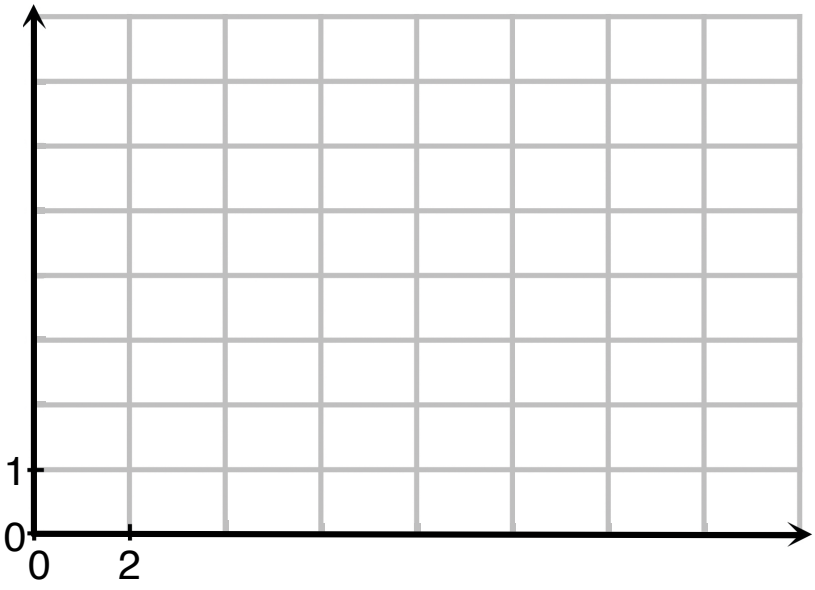
How do you know?

_____

_____

25. The point (6, 4) is shown on the top grid. Mark the point (6, 4) on the bottom grid.

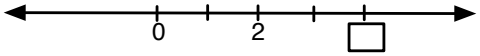




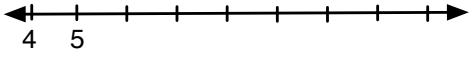
APPENDIX B: MATCHED ROUTINE AND NON-ROUTINE ITEMS ON THE LRA

ROUTINE

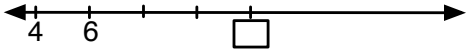
Write the number that belongs in the box:



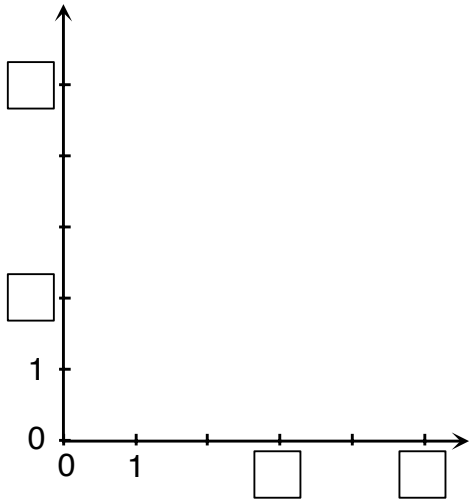
Mark where 8 belongs on this number line



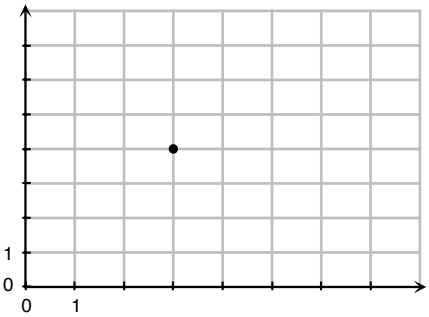
Write the number that belongs in the box.



Write the missing values in the boxes for the grid below.



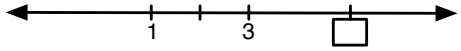
What is the name of the point marked below?



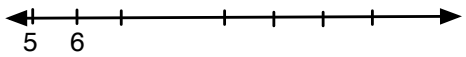
Name of point: _____

NON-ROUTINE

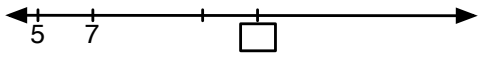
Write the number that belongs in the box:



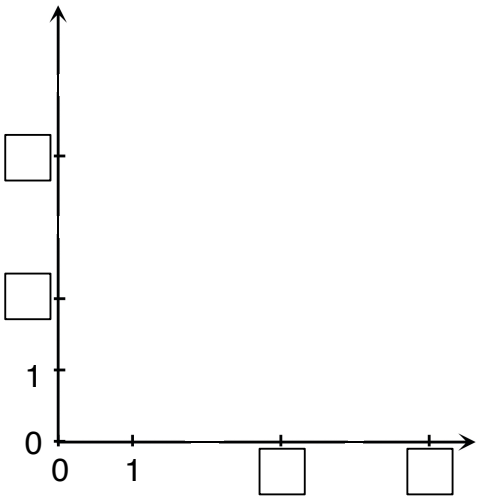
Mark where 9 belongs on this number line



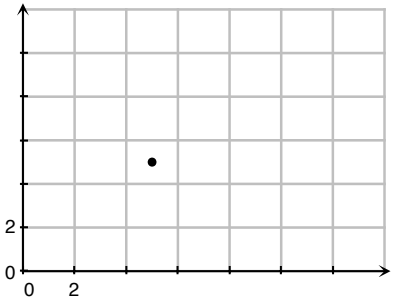
Write the number that belongs in the box.



Write the missing values in the boxes for the grid below.



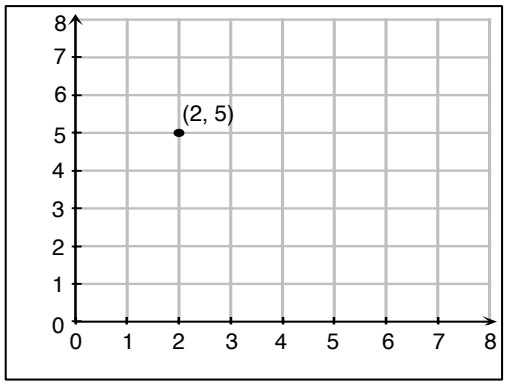
What is the name of the point marked below?

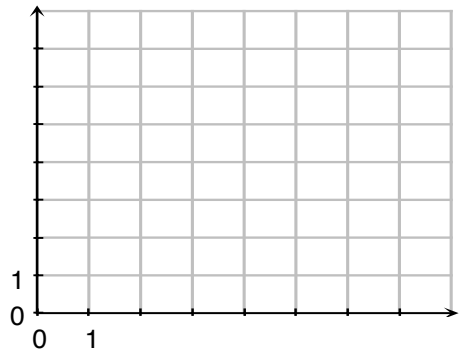


Name of point: _____

ROUTINE

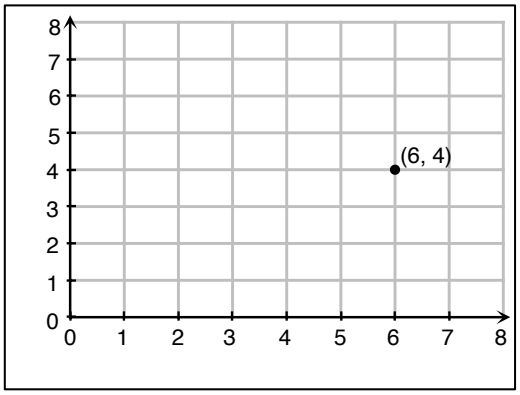
The point (2, 5) is shown on the grid below.
Mark the point (5, 2) on the bottom grid.

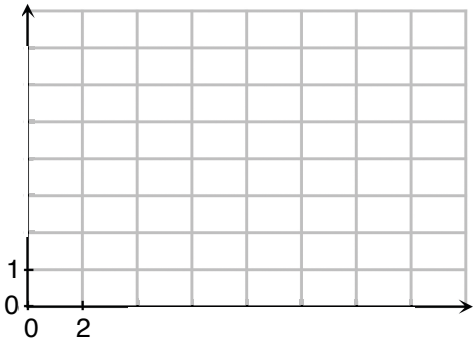




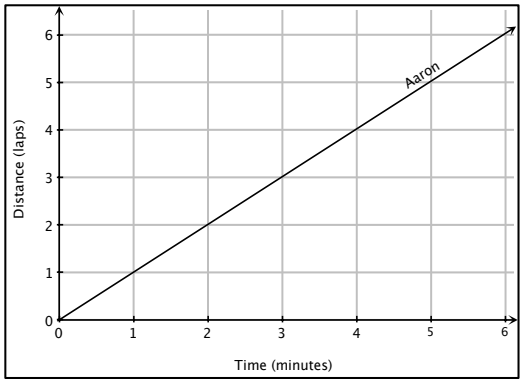
NON-ROUTINE

The point (6, 4) is shown on the grid below.
Mark the point (6, 4) on the bottom grid.





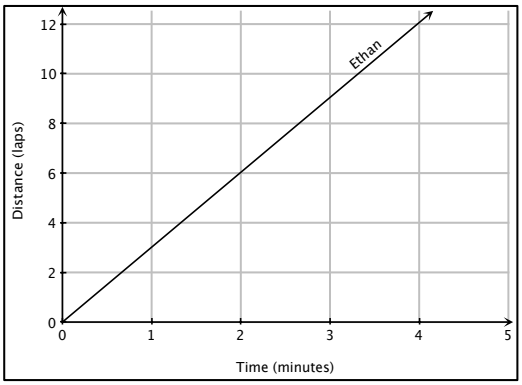
The graph below shows how far Aaron runs over time.



a. How many laps does Aaron run in 3 minutes? _____

b. How many laps does Aaron run between minute 2 and minute 4? _____

The graph below shows how far Ethan runs over time.



a. How many laps does Ethan run in 3 minutes? _____

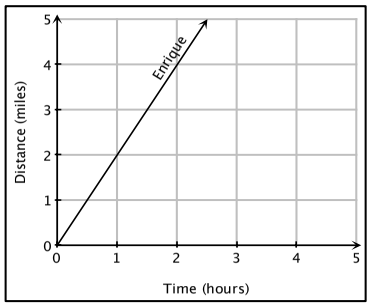
b. How many laps does Ethan run between minute 2 and minute 4? _____

ROUTINE

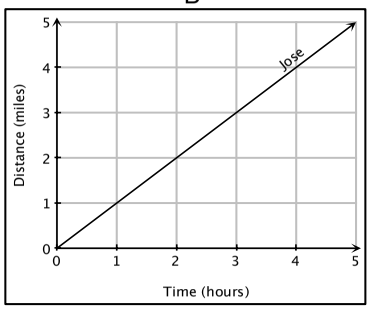
Graph A shows how many miles Enrique walks each hour.

Graph B shows how many miles Jose walks each hour.

A



B



Who walks more miles each hour, Enrique or Jose?

Circle one: Enrique Jose They walk the same rate

How do you know? _____

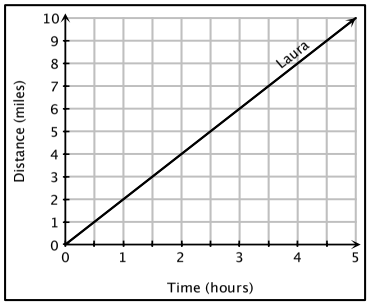
_____

NON-ROUTINE

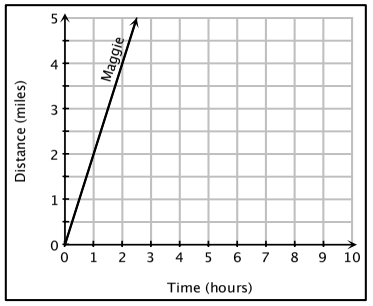
Graph A shows how many miles Laura walks each hour.

Graph B shows how many miles Maggie walks each hour.

A



B



Who walks more miles each hour, Laura or Maggie?

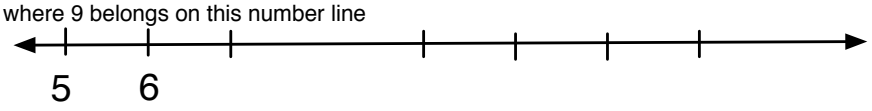
Circle one: Laura Maggie They walk the same rate

How do you know? _____

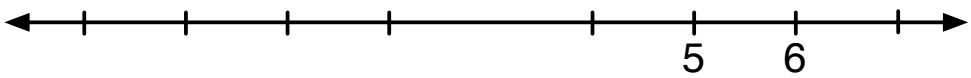
_____

APPENDIX C: TARGET INTERVIEW ITEMS FROM THE LRA

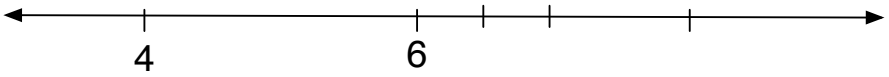
Mark where 9 belongs on this number line



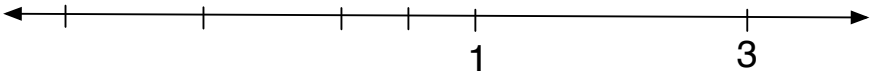
Mark with an arrow (↑) where 1 belongs on the number line.



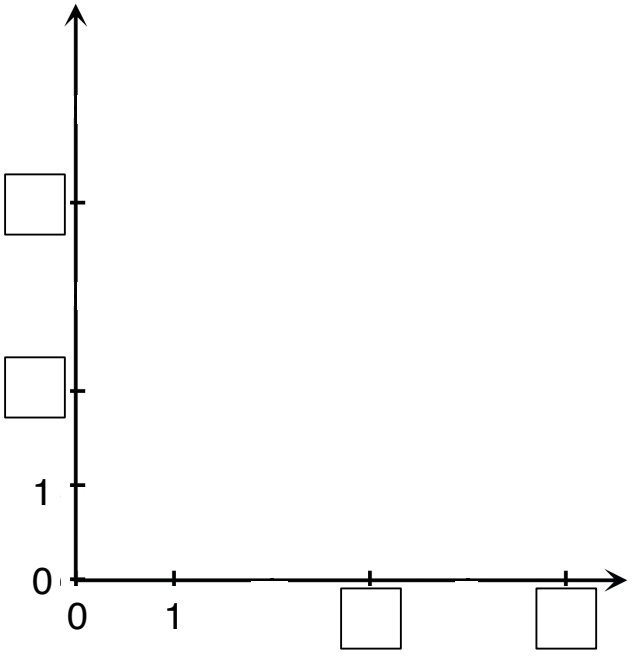
Mark with an arrow (↑) where 7 belongs on the number line.



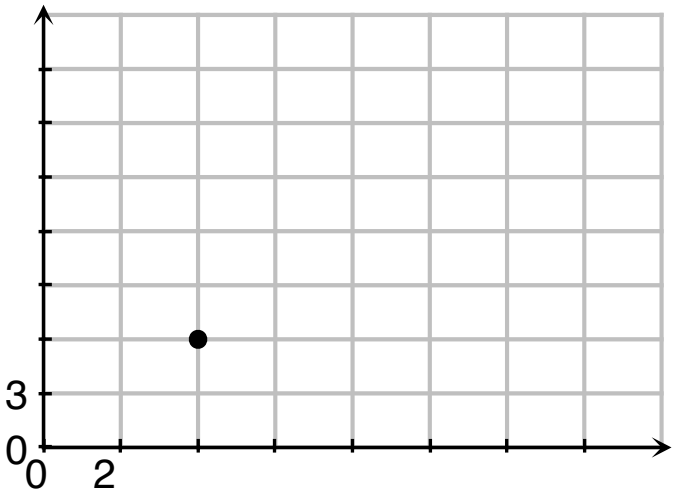
Mark with an arrow (↑) where 0 belongs on the number line.



Write the missing values in the boxes for the grid below.

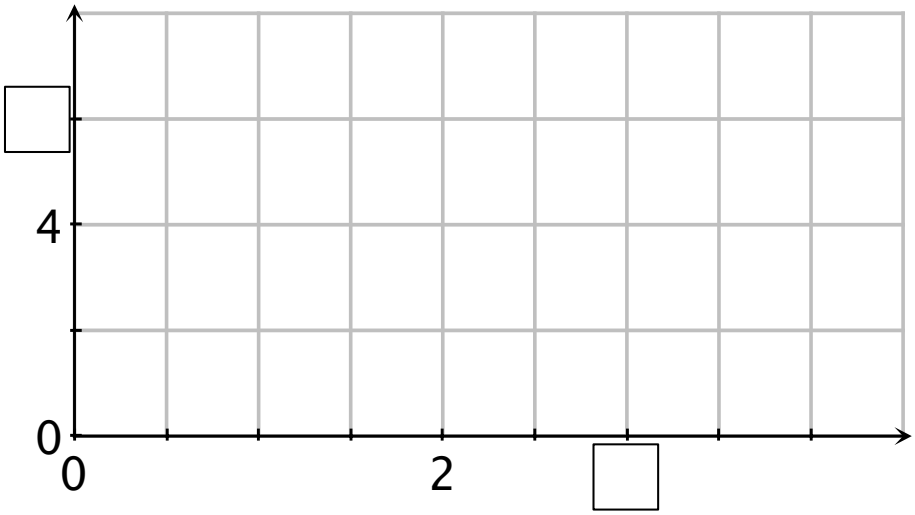


What is the name of the point marked below?

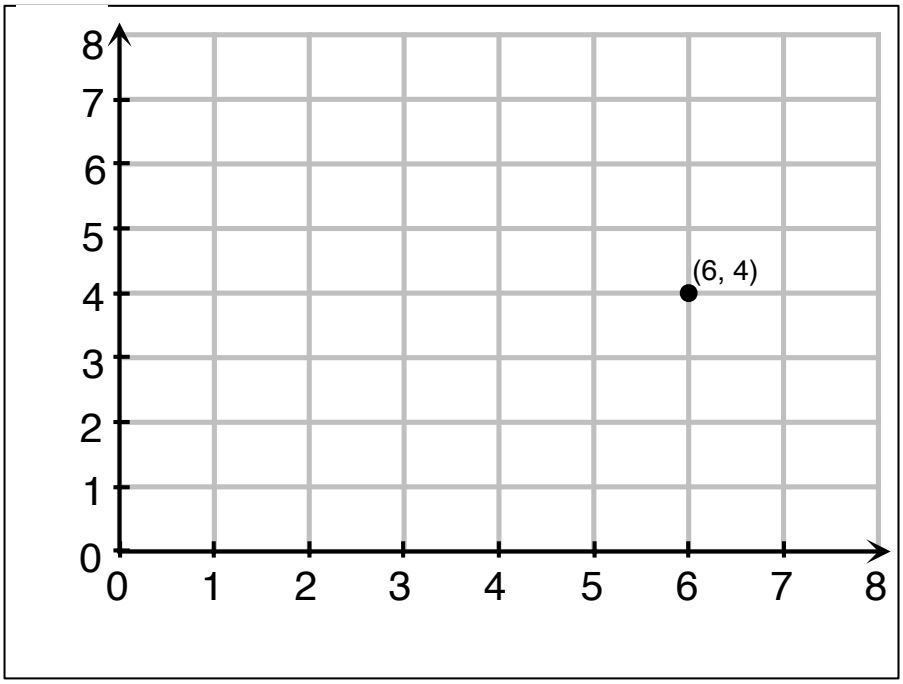


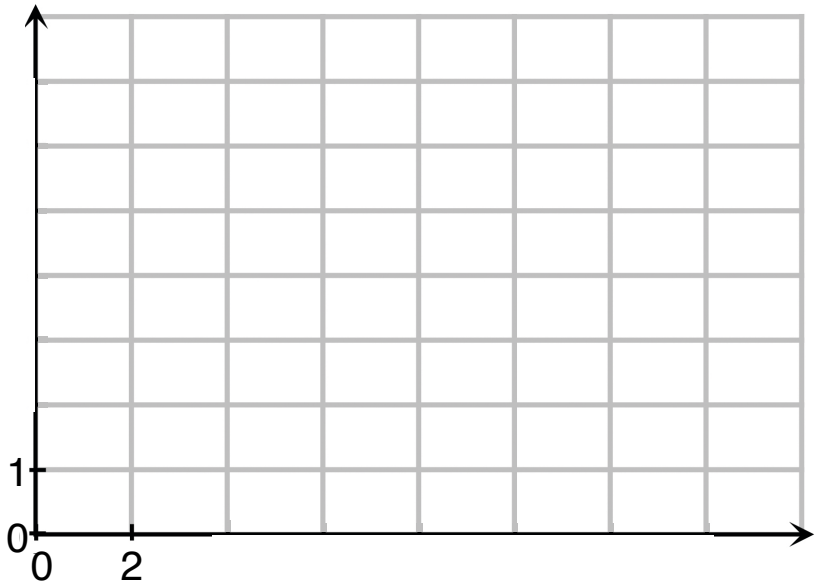
Name of point: _____

Write the missing values in the boxes for the grid below.

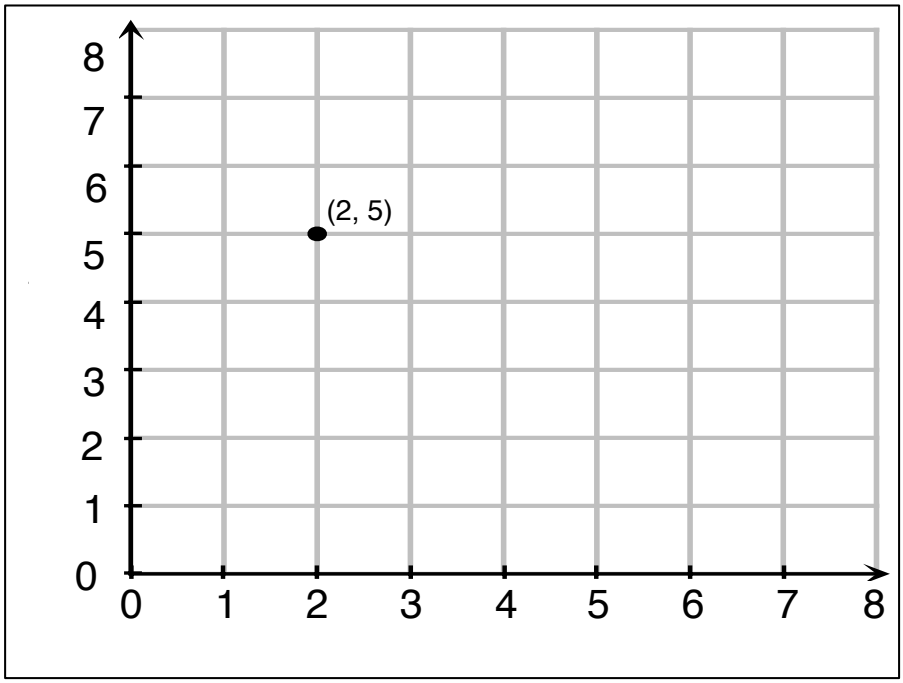


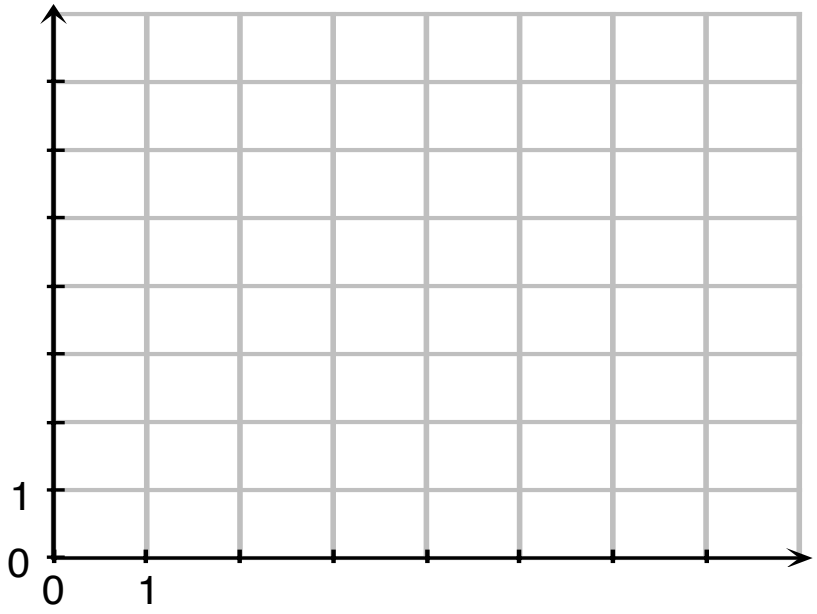
The point (6, 4) is shown on the grid below. Mark the point (6, 4) on the bottom grid.





The point (2, 5) is shown on the grid below. Mark the point (5, 2) on the bottom grid.





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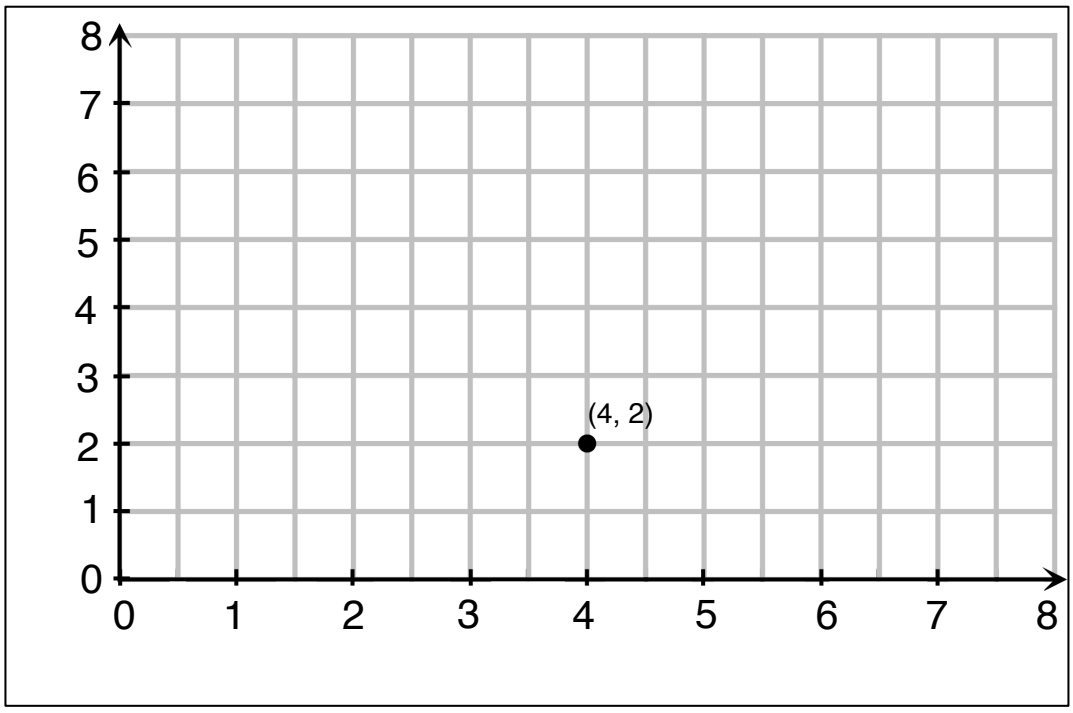
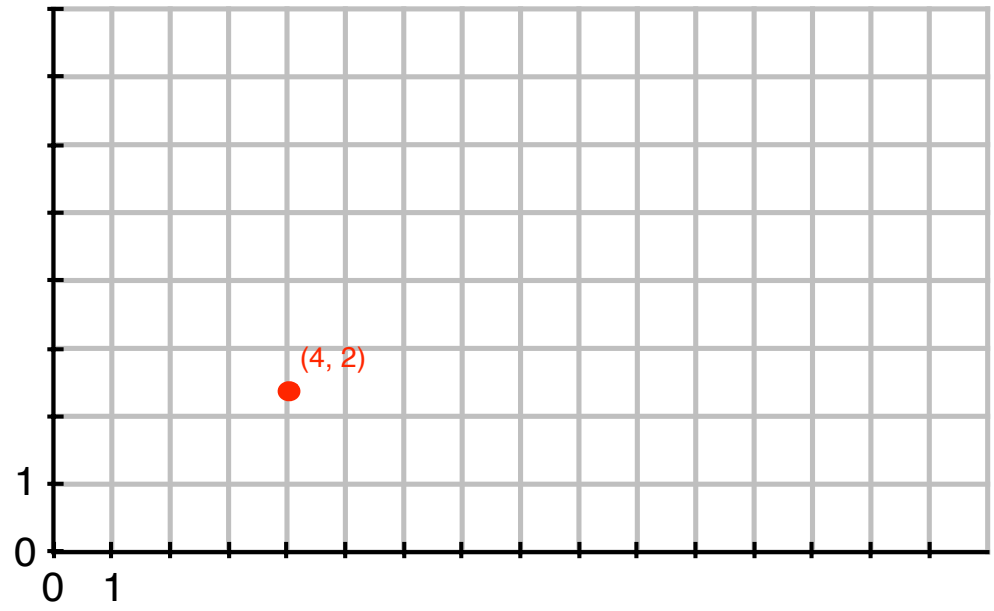
The point (4, 2) is shown on the grid below. Mark the point (4, 2) on the bottom grid.

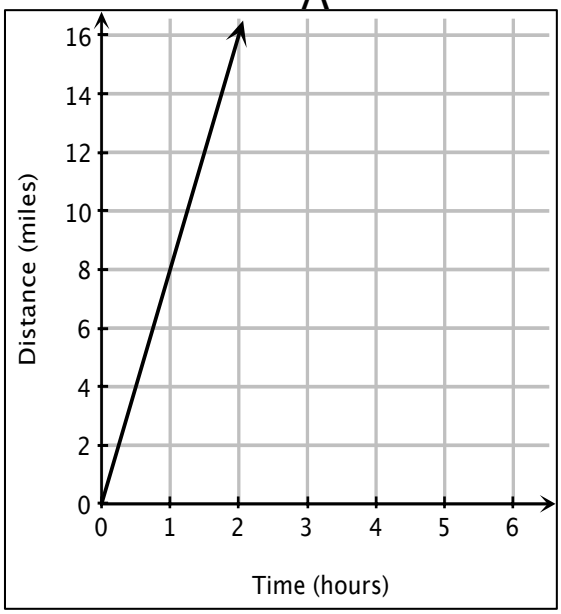




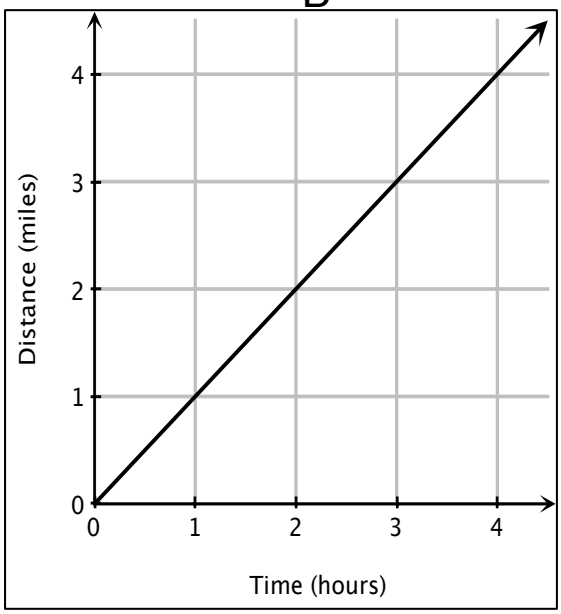
Ben and his dad walk 4 miles each hour.

Which graph below best shows this (circle one): A B C D

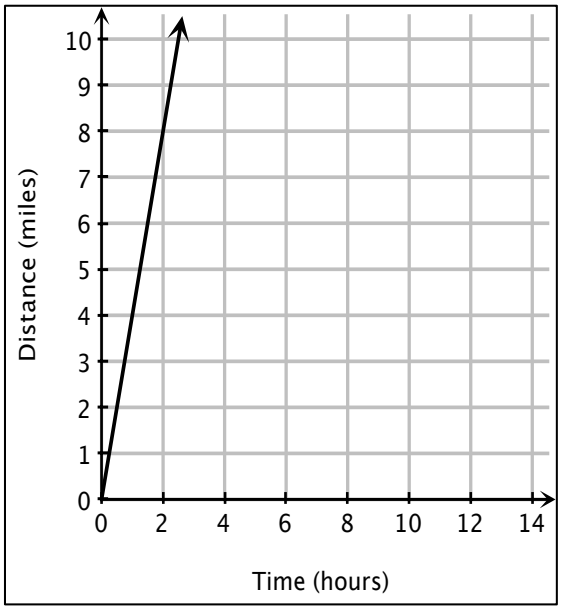
A



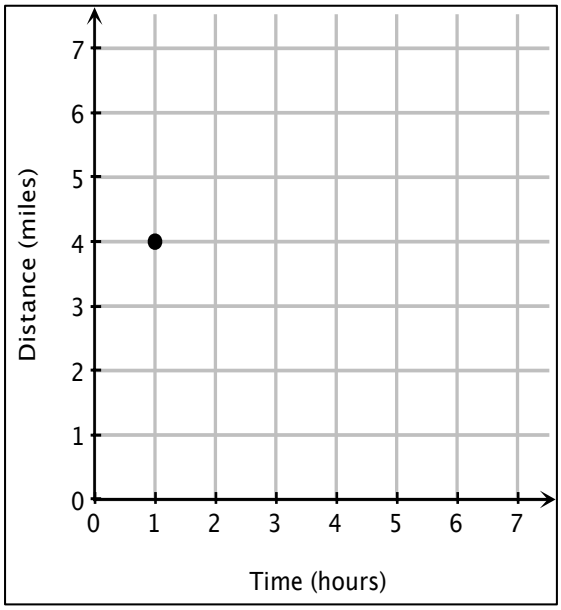
B



C



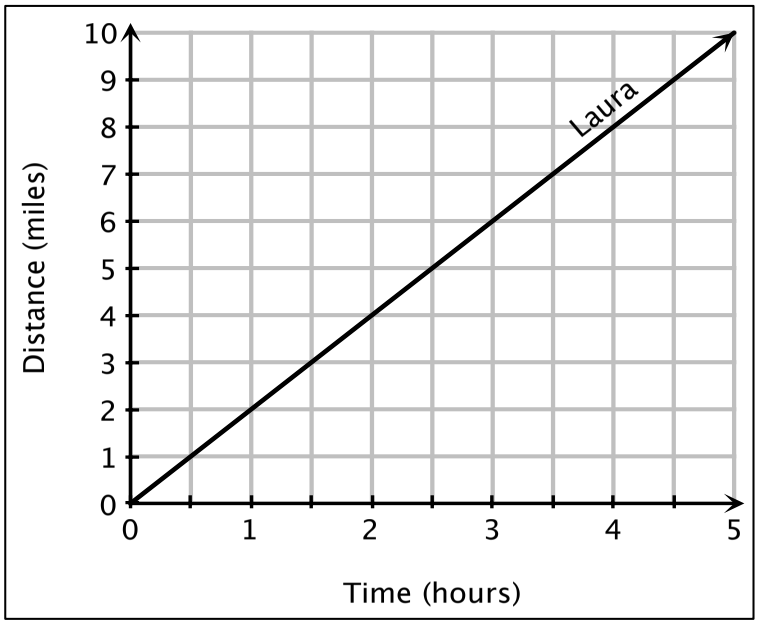
D



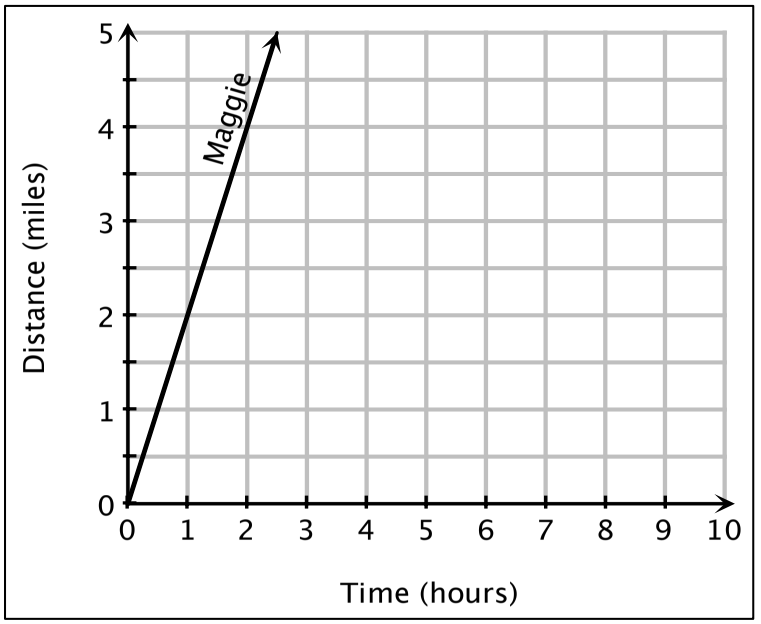
Graph A shows how many miles Laura walks each hour.

Graph B shows how many miles Maggie walks each hour.

A



B



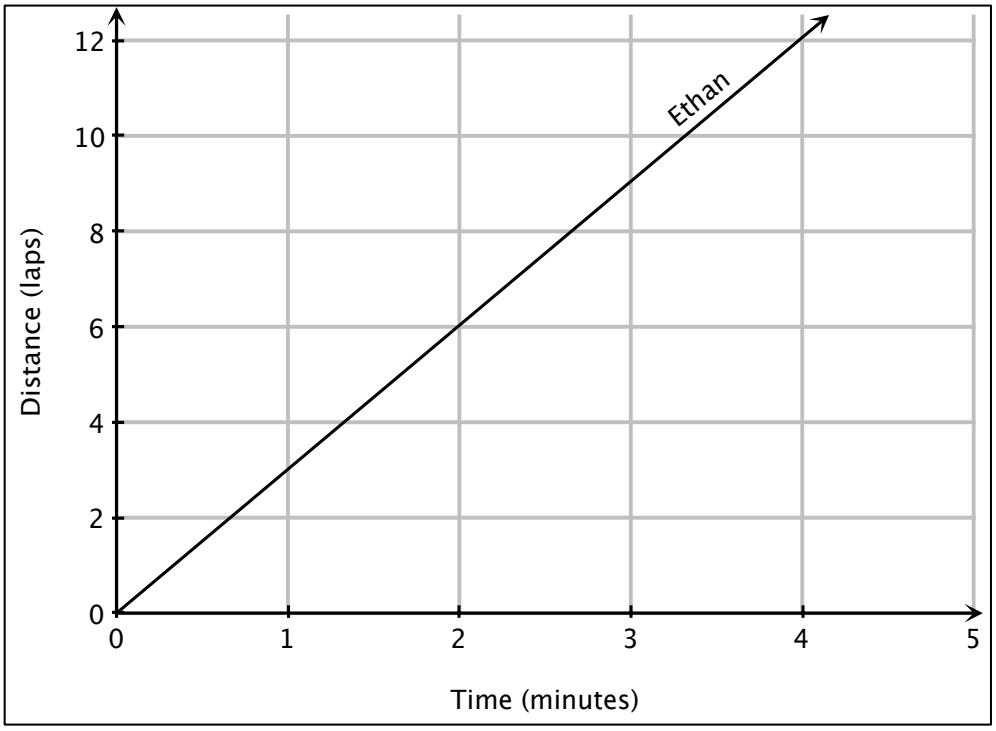
Who walks more miles each hour, Laura or Maggie?

Circle one: Laura Maggie They walk the same rate

How do you know? _____

_____

The graph below shows how far Ethan runs over time.



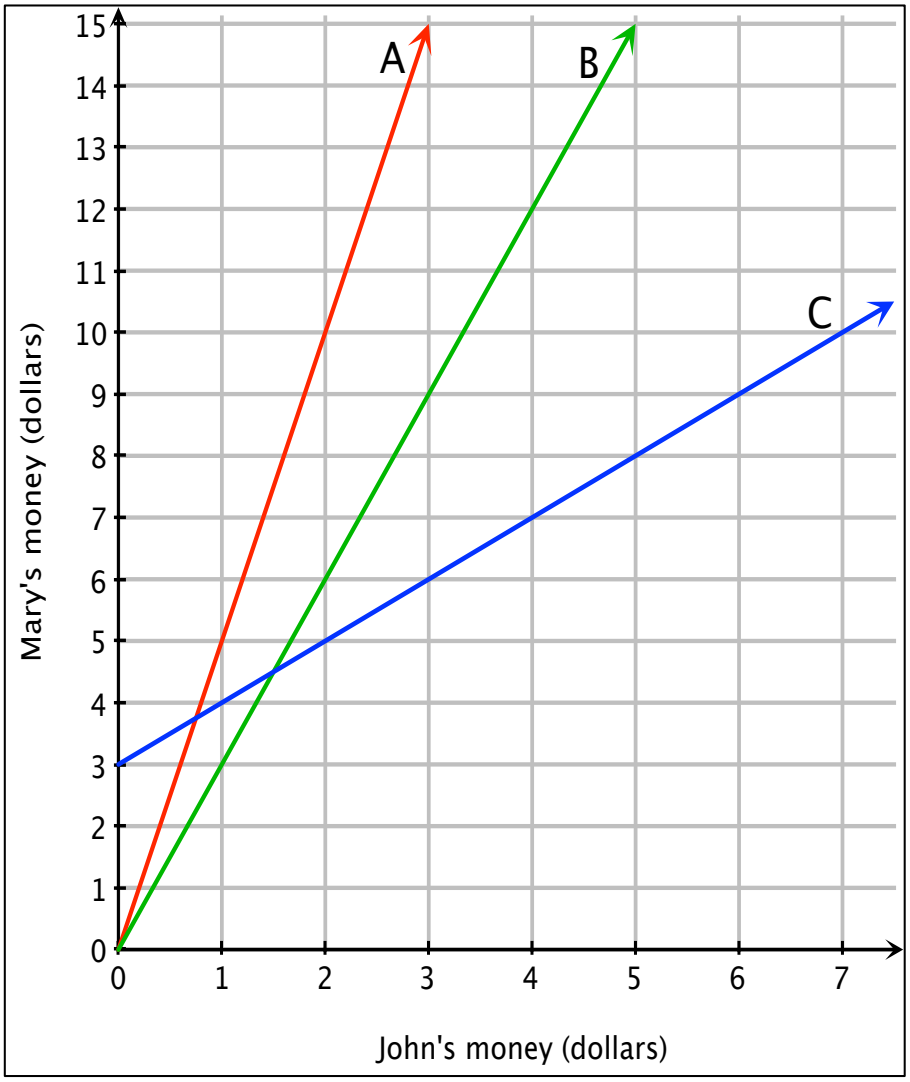
a. How many laps does Ethan run in 3 minutes? _____

b. How many laps does Ethan run between minute 2 and minute 4? _____

c. How many laps does Ethan run each minute? _____

Mary always has three times as much money as John. Which of the lines below shows this?

Circle one: A B C

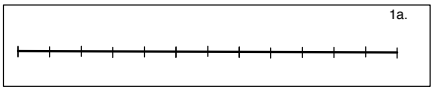
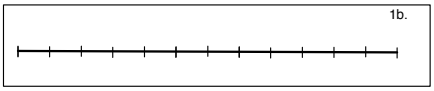
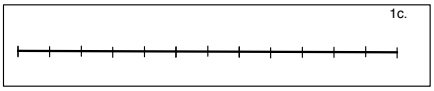
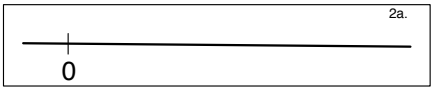
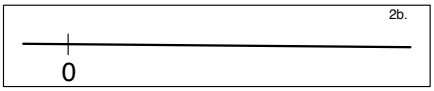
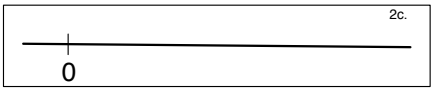
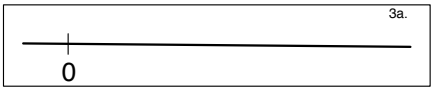
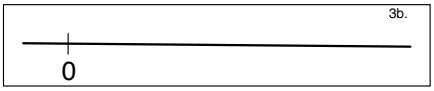
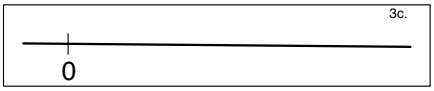
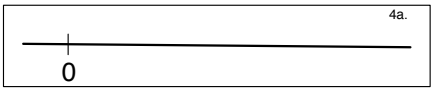


How do you know?

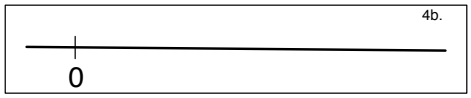
_____

_____

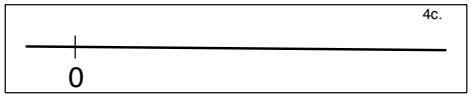
APPENDIX D: TUTORIAL INTERVENTION PROBLEM SETS

1a Mark where 3 is.	1a. 
1b Mark where 4 is.	1b. 
1c Mark where 5 is.	1c. 
2a Mark where 4 is.	2a. 
2b Mark where 4 is.	2b. 
2c Mark where 4 is.	2c. 
3a Mark the length of 6 reds using the purple rods.	3a. 
3b Mark the length of 4 reds using the purple rods.	3b. 
3c Mark the length of 8 reds using the purple rods.	3c. 
4a Mark the length of 3 light greens. Use BOTH the light green and dark green rods.	4a. 

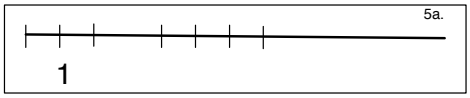
4b
Mark the length of 5 light greens.
Use BOTH the light green and dark green rods.



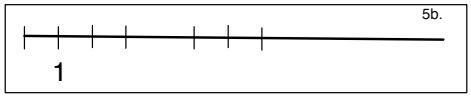
4c
Mark the length of 7 light greens.
Use BOTH the light green and dark green rods.



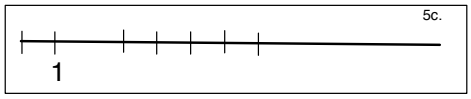
5a
Mark the length of 5 reds. You can use the red and purple rods.



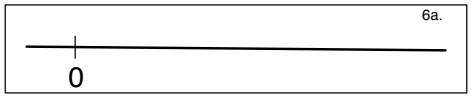
5b
Mark the length of 6 reds. You can use the red and purple rods.



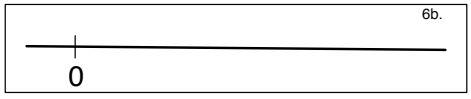
5c
Mark the length of 4 reds. You can use the red and purple rods.



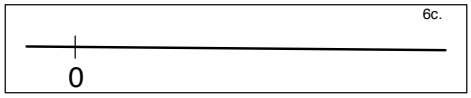
6a
Mark the length of 5 light greens.
Use BOTH the light green and dark green rods.



6b
Mark the length of 7 light greens.
Use BOTH the light green and dark green rods.



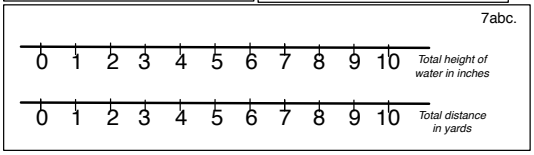
6c
Mark the length of 5 reds. You can use the red and purple rods.



7a
Beetle earns water in his glass when he runs. His glass is measured in inches so you can see exactly how much water he has.
Mark that he walks 6 yards and earns 3 inches of water in his glass.

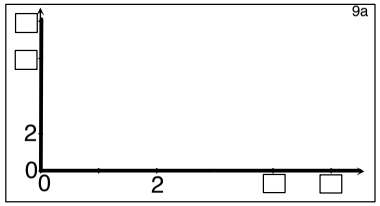
7b
Use the same lines to mark that Rudy the cat walks 3 yards and earns 4 inches of water in his glass

7c
Use the same lines to mark that Viggo the cat walks 2 yards and earns 3 inches of water in his glass.



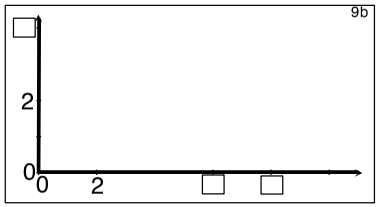
9a

Someone forgot to finish writing in the values on each axis. Fill them in!



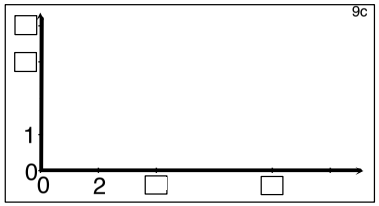
9b

Someone forgot to finish writing in the values on each axis. Fill them in!



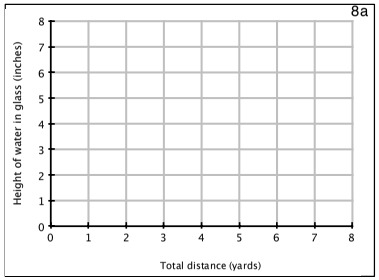
9c

Someone forgot to finish writing in the values on each axis. Fill them in!



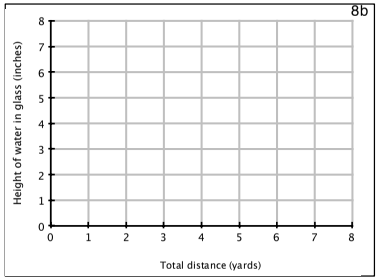
8a

Mark that:
Beetle walks 2 yards and earns 6 inches of water, that **Rudy** walks 2 yards and earns 4 inches of water, and that **Viggo** walks 1 yard and earns 6 inches of water.



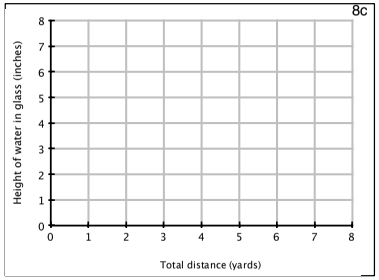
8b

Mark that:
Beetle walks 4 yards and earns 6 inches of water, that **Rudy** walks 4 yards and earns 5 inches of water, and that **Viggo** walks 3 yards and earns 5 inches of water.



8c

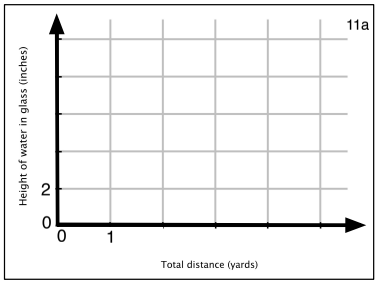
Mark that:
Beetle walks 3 yards and earns 6 inches of water, that **Rudy** walks 3 yards and earns 4 inches of water, and that **Viggo** walks 1 yard and earns 2 inches of water.



11a

Mark that Beetle walks 3 yards and earns 5 inches of water in his glass.

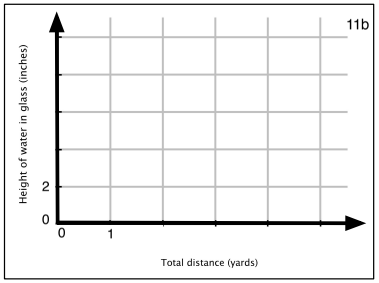
You may use the rods.



11b

Mark that Beetle walks 4 yards and earns 3 inches of water in his glass.

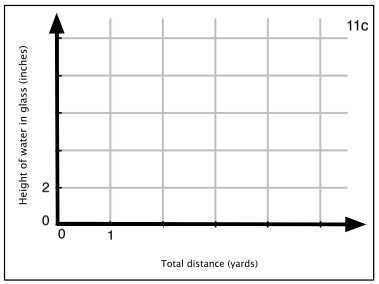
You may use the rods.



11c

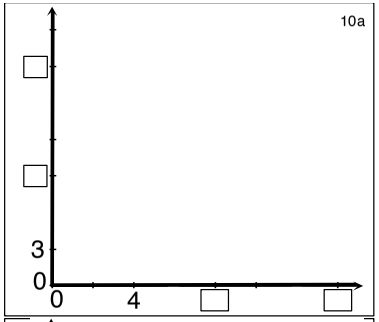
Mark that Beetle walks 3 yards and earns 1 inches of water in his glass.

You may use the rods.



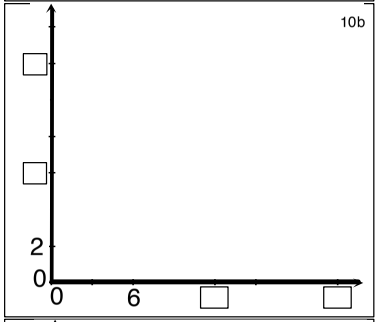
10a

Someone forgot to finish writing in the values on each axis. Fill them in!



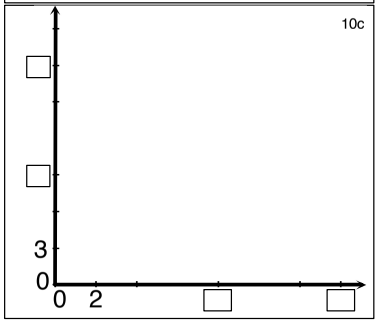
10b

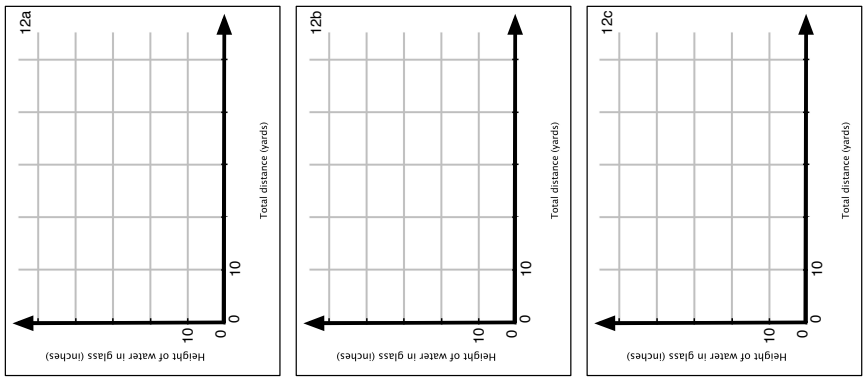
Someone forgot to finish writing in the values on each axis. Fill them in!



10c

Someone forgot to finish writing in the values on each axis. Fill them in!





12a

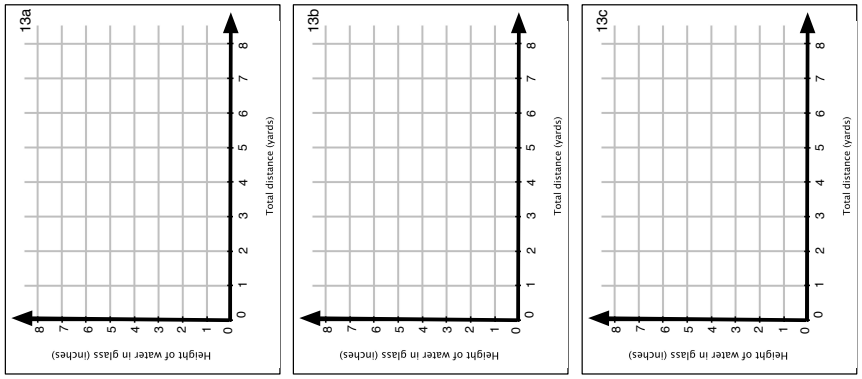
Mark that Rudy walks 11 yards and earns 15 inches of water in his glass.

12b

Mark that Rudy walks 25 yards and earns 29 inches of water in his glass.

12c

Mark that Rudy walks 9 yards and earns 15 inches of water in his glass.



13a

Beetle starts with no water. Mark that he earns 2 inches of water in his glass for each yard he runs.

He gets water even if he walks part of a yard.

13b

Beetle starts with no water. Mark that he earns 3 inches of water in his glass for each yard he runs.

He gets water even if he walks part of a yard.

13c

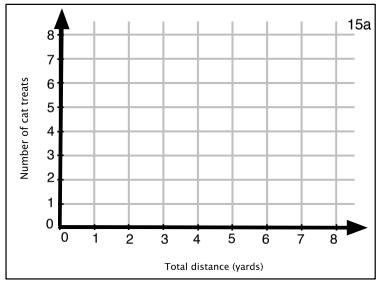
Beetle starts with no water. Mark that he earns 1 inch of water in his glass for each yard he runs.

He gets water even if he walks part of a yard.

15a

Beetle starts with no cat treats. Mark that he earns 2 cat treats for each yard he runs.

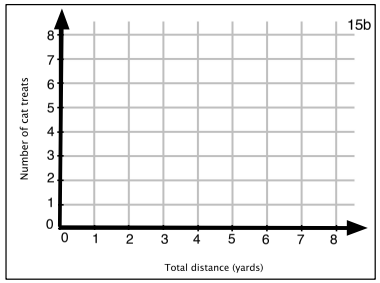
He gets his treats once he finishes each yard.



15b

Beetle starts with no cat treats. Mark that he earns 3 cat treats for each yard he runs.

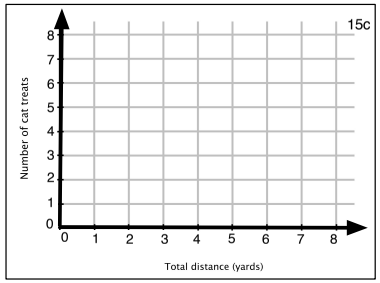
He gets his treats once he finishes each yard.



15c

Beetle starts with no cat treats. Mark that he earns 1 cat treat for each yard he runs.

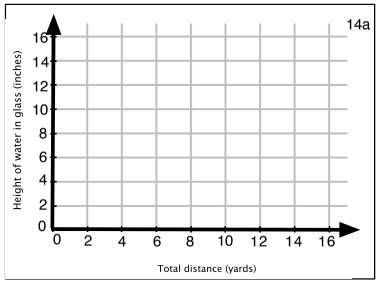
He gets his treats once he finishes each yard.



14a

Rudy starts with no water. Mark that he earns 4 inches of water in his glass for each yard he runs.

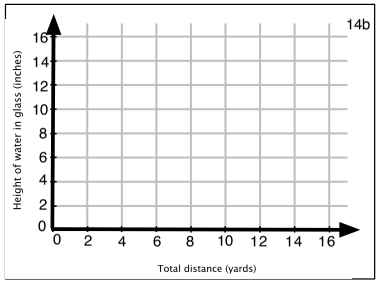
He receives water even if he walks part of a yard.



14b

Rudy starts with no water. Mark that he earns 2 inches of water in his glass for each yard he runs.

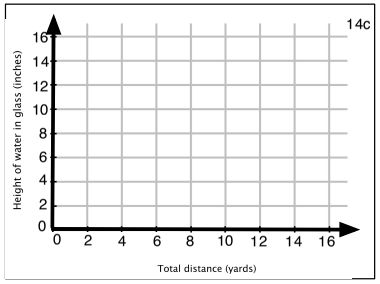
He receives water even if he walks part of a yard.



14c

Rudy starts with no water. Mark that he earns 1 inch of water in his glass for each yard he runs.

He receives water even if he walks part of a yard.



17a

Beetle starts with 2 inches of water in his glass.

Mark that he earns 2 inches of water in his glass for each yard he walks.

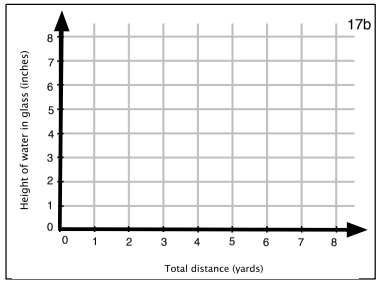
He gets water even if he walks part of a yard.

17b

Beetle starts with 1 inch of water in his glass.

Mark that he earns 2 inches of water in his glass for each yard he walks.

He gets water even if he walks part of a yard.

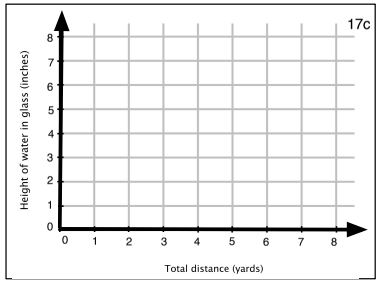


17c

Beetle starts with 3 inches of water in his glass.

Mark that he earns 2 inches of water in his glass for each yard he walks.

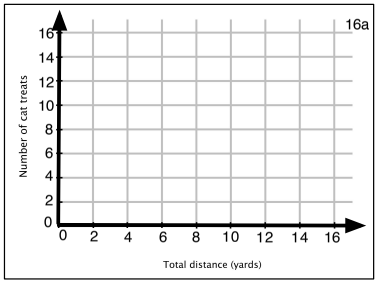
He gets water even if he walks part of a yard.



16a

Rudy starts with no cat treats. Mark that he earns 2 treats for each yard he runs.

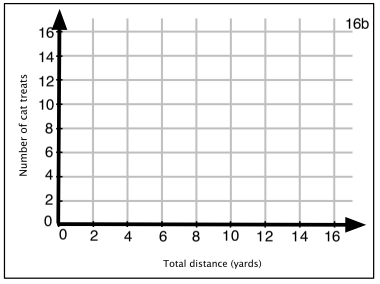
He gets his treats once he finishes each yard.



16b

Rudy starts with no cat treats. Mark that he earns 6 treats for each yard he runs.

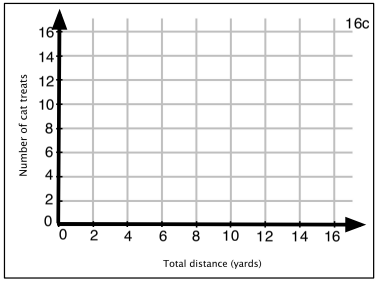
He gets his treats once he finishes each yard.



16c

Rudy starts with no cat treats. Mark that he earns 4 treats for each yard he runs.

He gets his treats once he finishes each yard.

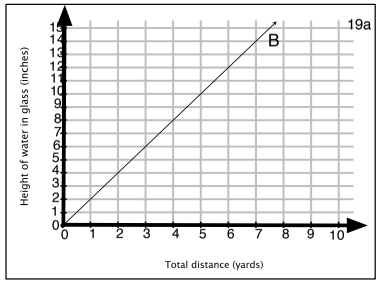


19a

The graph shows how much water Beetle earns.

Show on the same graph that Rudy starts with no water. He earns 6 inches of water in his glass each yard he walks.

Rudy gets water even if he walks part of a yard.

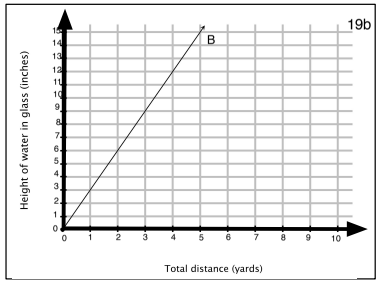


19b

The graph shows how much water Beetle earns.

Show on the same graph that Rudy starts with no water. He earns 2 inches of water in his glass each yard he walks.

Rudy gets water even if he walks part of a yard.

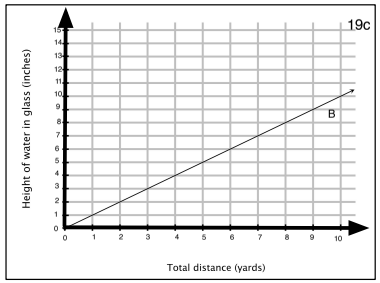


19c

The graph shows how much water Beetle earns.

Show on the same graph that Rudy starts with no water. He earns 4 inches of water in his glass each yard he walks.

Rudy gets water even if he walks part of a yard.

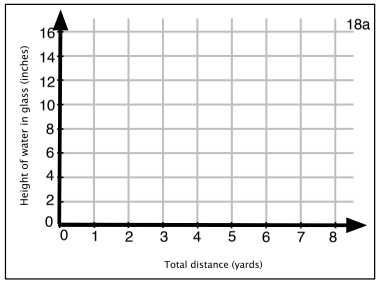


18a

Beetle starts with 2 inches of water in his glass.

Mark that he earns 2 inches of water in his glass for each yard he walks.

He gets water even if he walks part of a yard.

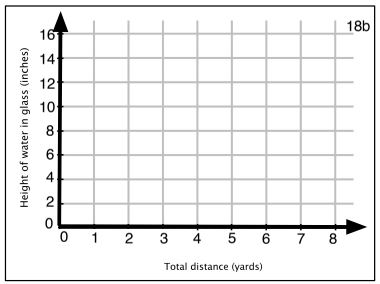


18b

Beetle starts with 6 inches of water in his glass.

Mark that he earns 2 inches of water in his glass for each yard he walks.

He gets water even if he walks part of a yard.

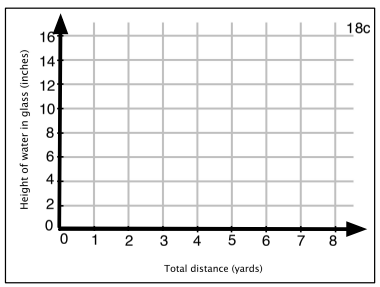


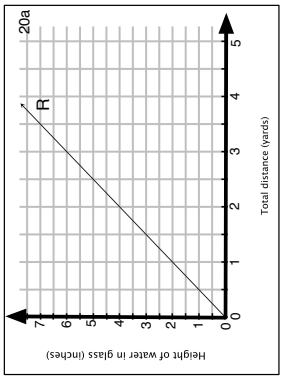
18c

Beetle starts with 4 inches of water in his glass.

Mark that he earns 2 inches of water in his glass for each yard he walks.

He gets water even if he walks part of a yard.



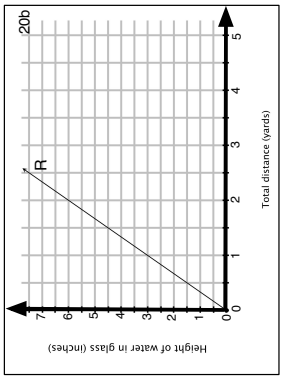


20a

The graph shows how much water Rudy earns.

Show on the same graph that Beetle starts with no water. He earns 1 inch of water in his glass each yard he walks.

Rudy gets water even if he walks part of a yard.

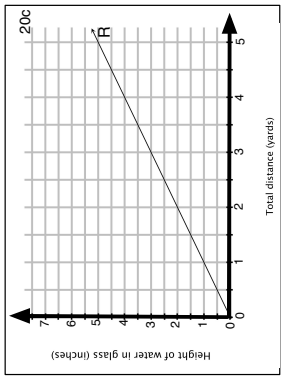


20b

The graph shows how much water Rudy earns.

Show on the same graph that Beetle starts with no water. He earns 2 inches of water in his glass each yard he walks.

Rudy gets water even if he walks part of a yard.



20c

The graph shows how much water Rudy earns.

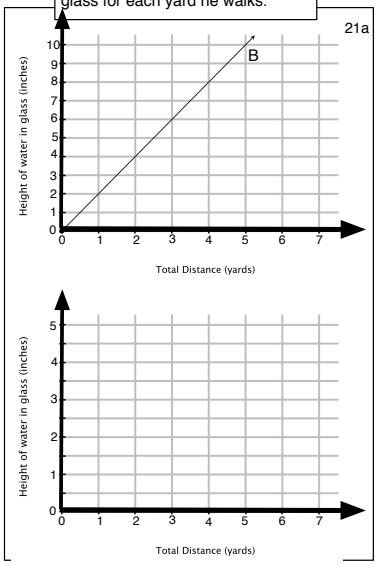
Show on the same graph that Beetle starts with no water. He earns 2 inches of water in his glass each yard he walks.

Rudy gets water even if he walks part of a yard.

21a

The graph shows that Beetle earns 2 inches of water each yard he walks.

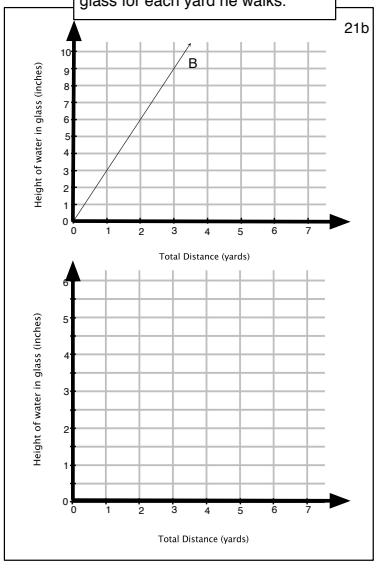
Show on the new graph that Beetle still earns 2 inches of water in his glass for each yard he walks.



21b

The graph shows that Beetle earns 2 inches of water each yard he walks.

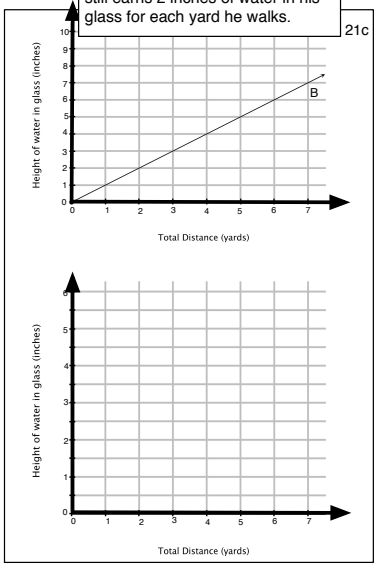
Show on the new graph that Beetle still earns 2 inches of water in his glass for each yard he walks.



21c

The graph shows that Beetle earns 2 inches of water each yard he walks.

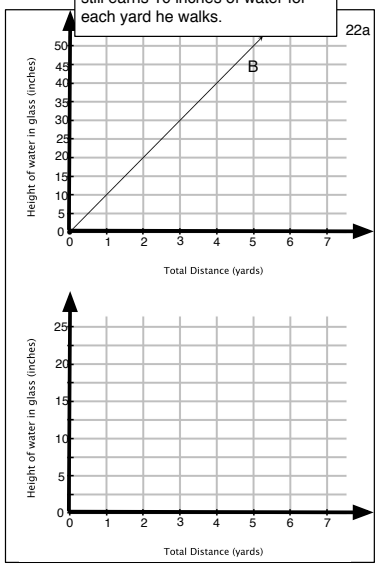
Show on the new graph that Beetle still earns 2 inches of water in his glass for each yard he walks.



22a

The graph shows that Beetle earns 10 inches of water for each yard he walks.

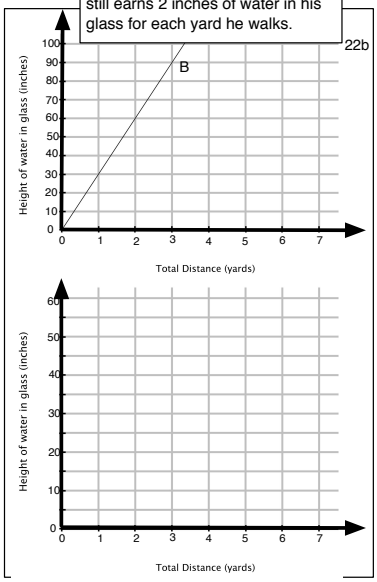
Show on the new graph that Beetle still earns 10 inches of water for each yard he walks.



22b

The graph shows that Beetle earns 2 inches of water each yard he walks.

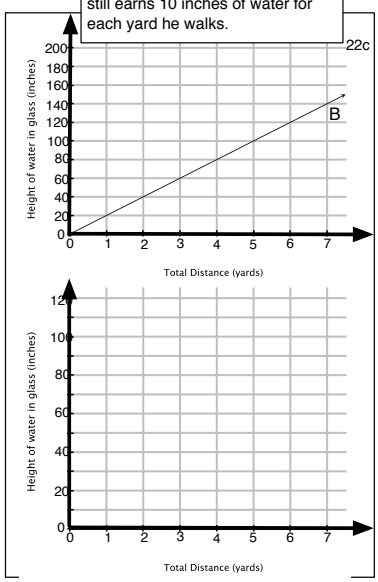
Show on the new graph that Beetle still earns 2 inches of water in his glass for each yard he walks.



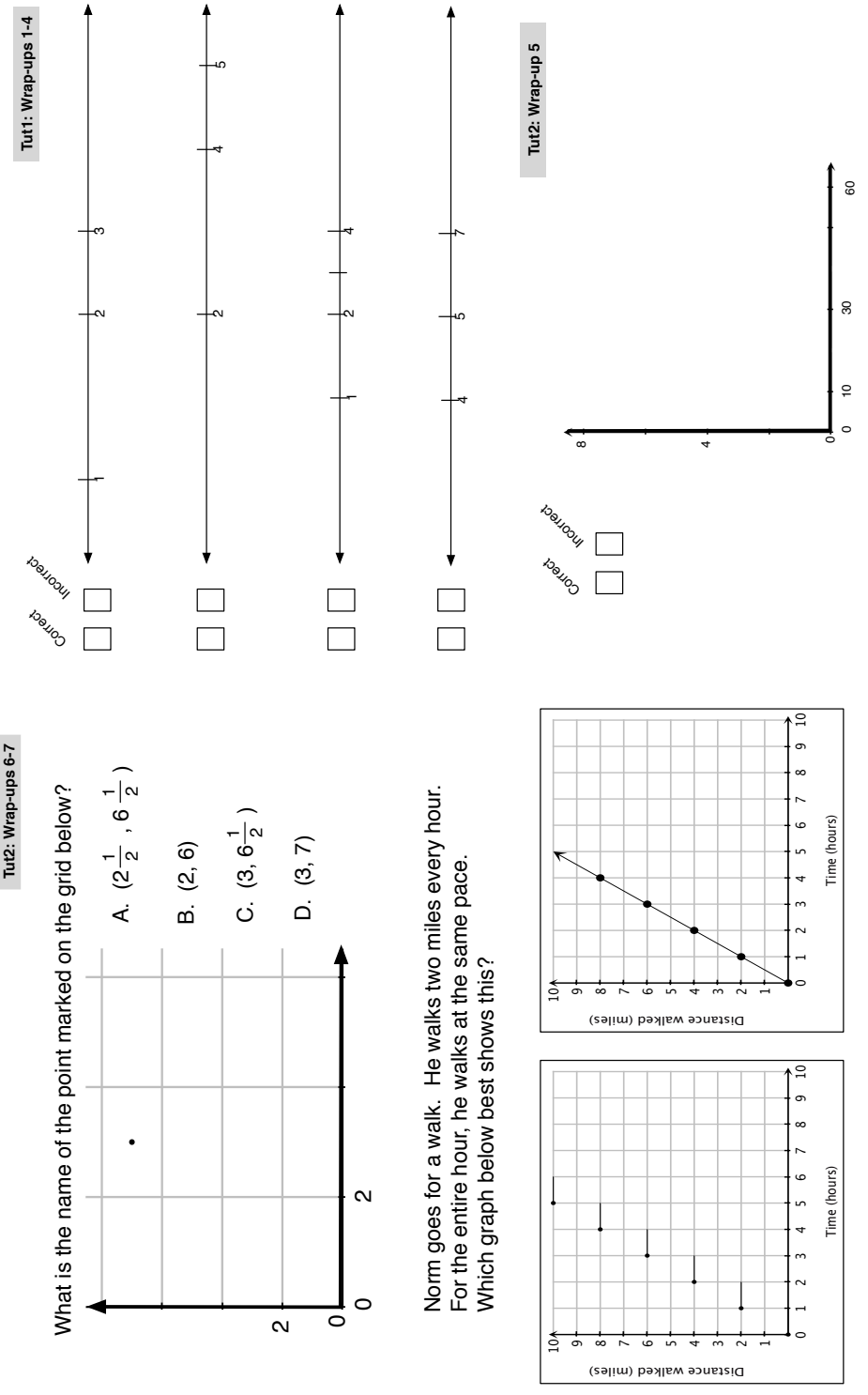
22c

The graph shows that Beetle earns 10 inches of water for each yard he walks.

Show on the new graph that Beetle still earns 10 inches of water for each yard he walks.



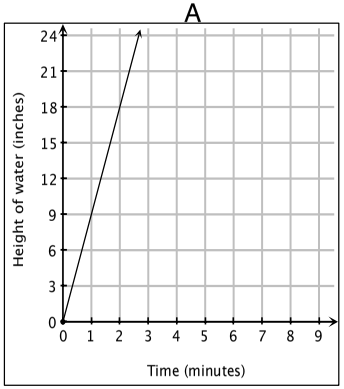
APPENDIX E: TUTORIAL WRAP-UP PROBLEMS



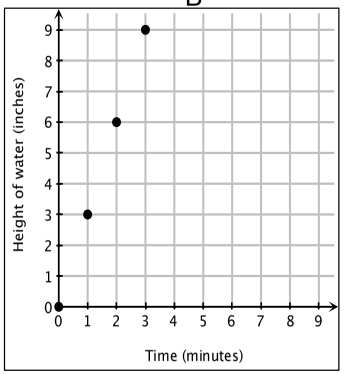
Tut3: Wrap-up 10

Joe turns on the bathtub and lets it fills.
The water rises 3 inches each minute.
Which graph below best shows this (circle one): A B C D

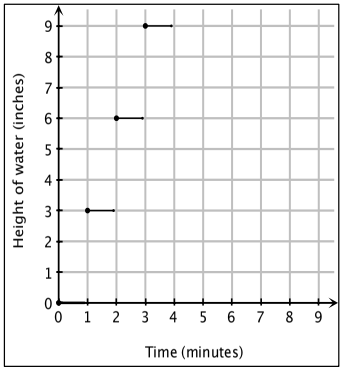
A



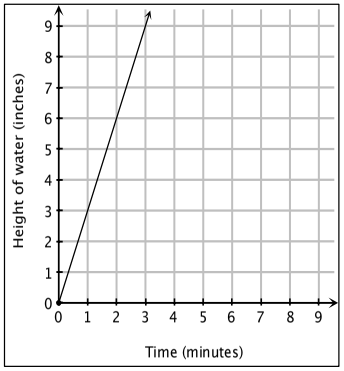
B



C



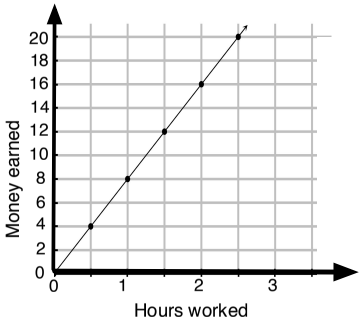
D

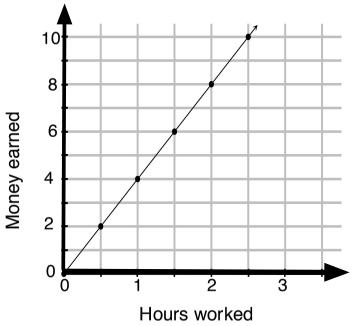


Tut3: Wrap-up 8-9

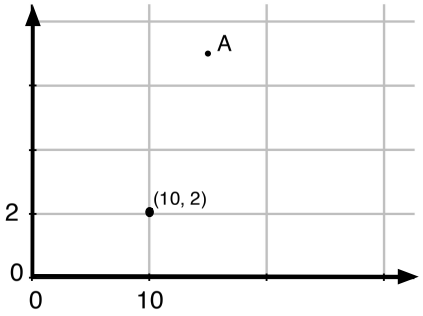
Joe earns $4 each hour he works.
Do both graphs below show this?

Yes ☐ No ☐





What is the name of the point A on the grid below?



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