# SPARSE UNIVARIATE POLYNOMIALS WITH MANY ROOTS OVER FINITE FIELDS

QI CHENG, SHUHONG GAO, J. MAURICE ROJAS, AND DAQING WAN

ABSTRACT. Suppose $q$ is a prime power and $f \in \mathbb{F}_q[x]$ is a univariate polynomial with exactly $t$ monomial terms and degree $< q-1$. To establish a finite field analogue of Descartes' Rule, Bi, Cheng, and Rojas (2013) proved an upper bound of $2(q-1)^{\frac{t-2}{t-1}}$ on the number of cosets in $\mathbb{F}_q^*$ needed to cover the roots of $f$ in $\mathbb{F}_q^*$. Here, we give explicit $f$ with root structure approaching this bound: When $q$ is a perfect $(t-1)$-st power we give an explicit $t$-nomial vanishing on $q^{\frac{t-2}{t-1}}$ distinct cosets of $\mathbb{F}_q^*$. Over prime fields $\mathbb{F}_p$, computational data we provide suggests that it is harder to construct explicit sparse polynomials with many roots. Nevertheless, assuming the Generalized Riemann Hypothesis, we find explicit trinomials having $\Omega\left(\frac{\log p}{\log\log p}\right)$ distinct roots in $\mathbb{F}_p$.

## 1. INTRODUCTION

How can one best bound the complexity of an algebraic set in terms of the complexity of its defining polynomials? Over the complex numbers (or any algebraically closed field), Bézout's Theorem [Béz06] bounds the number of roots, for a system of multivariate polynomials, in terms of the degrees of the polynomials. Over finite fields, Weil's famous mid-$20^{\underline{\text{th}}}$ century result [Wei49] bounds the number of points on a curve in terms of the genus of the curve (which can also be bounded in terms of degree). These bounds are optimal for dense polynomials. For sparse polynomials, over fields that are not algebraically closed, these bounds can be much larger than necessary. For example, Descartes' Rule [SL54] tells us that a univariate real polynomial with exactly $t$ monomial terms always has less than $2t$ real roots, even though the terms may have arbitrarily large degree. It has been generalized to number fields and $p$-adic extensions of $\mathbb{Q}$ [Lenstra98], and local fields of positive characteristics [Poonen98].

Is there an analogue of Descartes' Rule over finite fields? Despite the wealth of beautiful and deep $20^{\underline{\text{th}}}$-century results on point-counting for curves and higher-dimensional varieties over finite fields, varieties defined by sparse *univariate* polynomials were all but ignored until [CFKLLS00] (see Lemma 7 there, in particular). Aside from their own intrinsic interest, refined univariate root counts over finite fields are useful in applications such as cryptography (see, e.g., [CFKLLS00]), the efficient generation of pseudo-random sequences (see, e.g., [BBK09]), and refined estimates for certain exponential sums over finite fields [Bou05, Proof of Theorem 4]. For instance, estimates on the number of roots of univariate tetranomials over a finite field were a key step in establishing the uniformity of the *Diffie-Helman distribution* [CFKLLS00, Proof of Thm. 8, Sec. 4] — a quantitative statement relevant to justifying the security of cryptosystems based on the Discrete Logarithm Problem.

We are thus interested in the number of roots of sparse univariate polynomials over finite fields. The polynomial $x^q - x$ having two terms and exactly $q$ roots in $\mathbb{F}_q$ might suggest that there is no finite field analogue of Descartes' rule. However, the roots of $x^q - x$ consist of 0 and the roots of $x^{q-1} - 1$, and the latter roots form the unit group $\mathbb{F}_q^* := \mathbb{F}_q \setminus \{0\}$. For an

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arbitrary binomial $ax^n + bx^m \in \mathbb{F}_q[x]$ with $n > m$ and $a$ and $b$ nonzero, the roots consist of 0 (if $m > 0$) and the roots of $x^{n-m} + b/a$. Note that the number of roots of $x^{n-m} + b/a$ in $\mathbb{F}_q$ is either 0 or $\gcd(n-m, q-1)$. In the latter case, the roots form a coset of a subgroup of $\mathbb{F}_q^*$. For polynomials with three or more terms, the number of roots quickly becomes mysterious and difficult, and, as we shall demonstrate in this paper, may exhibit very different behaviors in the two extreme cases where (a) $q$ is a large power of a prime, and (b) $q$ is a large prime.

To fix notation, we call a polynomial $f(x) = c_1 x^{e_1} + c_2 x^{e_2} + \cdots + c_t x^{e_t} \in \mathbb{F}_q[x]$ with $e_1 < e_2 < \cdots < e_t < q-1$ and $c_i \neq 0$ for all $i$ a *(univariate) $t$-nomial*. The best current upper bounds on the number of roots of $f$ in $\mathbb{F}_q$, as a function of $q$, $t$, and the coset structure of the roots of $f$, can be summarized as follows, using $|\cdot|$ for set cardinality:

**Theorem 1.1.** *Let $f \in \mathbb{F}_q[x]$ be any univariate $t$-nomial with degree $< q-1$ and exponent set $\{e_1, \ldots, e_t\}$ containing 0. Set $\delta(f) := \gcd(e_1, \ldots, e_t, q-1)$, $Z(f) := \{x \in \mathbb{F}_q \mid f(x) = 0\}$, $R(f) := |Z(f)|$, and let $C(f)$ denote the maximum cardinality of any coset (of any subgroup of $\mathbb{F}_q^*$) contained in $Z(f)$. Then:*

*0. (Special case of* [KS96, Thm. 1]*)* $R(f) \leq \frac{t-1}{t}(q-1)$.

*1.* [BCR16, Thm. 1.1] *$Z(f)$ is a union of no more than $2\left(\frac{q-1}{\delta(f)}\right)^{\frac{t-2}{t-1}}$ cosets, each associated to one of two subgroups $H_1 \subseteq H_2$ of $\mathbb{F}_q^*$, where $|H_1| = \delta(f)$, $|H_2| \geq \delta(f)\left(\frac{q-1}{\delta(f)}\right)^{1/(t-1)}$, and $|H_2|$ can be determined within $2^{O(t)}(\log q)^{O(1)}$ bit operations.*

*2.* [KO16, Thm. 1.2] *For $t = 3$ we have $R(f) \leq \delta(f)\left\lfloor \frac{1}{2} + \sqrt{\frac{q-1}{\delta(f)}} \right\rfloor$ and, if we have in addition that $q$ is a perfect square and $\delta(f) = 1$, then $R(f) \leq \sqrt{q}$.*

*3. (See* [Kel16, Thms. 2.2 & 2.3]*) For any $t \geq 2$ we have $R(f) \leq 2(q-1)^{\frac{t-2}{t-1}} C(f)^{1/(t-1)}$. Furthermore, $C(f) \leq \max\{k \in \mathbb{N} : k \mid (q-1)$ and, for all $i$, there is a $j \neq i$ with $k \mid (e_i - e_j)\}$.* ■

For any fixed $t \geq 2$, *Dirichlet's Theorem* (see, e.g., [BS96, Thm. 8.4.1, Pg. 215]) implies that there are infinitely many prime $p$ with $t \mid (p-1)$, and thus infinitely many prime powers $q$ with $t \mid (q-1)$. For such pairs $(q, t)$ the bound from Assertion (0) is tight: The roots of

$$f(x) = \frac{x^{q-1} - 1}{x^{(q-1)/t} - 1} = 1 + x^{\frac{1}{t}(q-1)} + \cdots + x^{\frac{t-1}{t}(q-1)},$$

are the disjoint union of $t-1$ cosets of size $\delta(f) = \frac{q-1}{t}$. (There are no $H_2$-cosets for this $t$-nomial.) However, Assertions (1) and (3) tell us that we can get even sharper bounds by making use of the structure of the cosets inside $Z(f)$. For instance, when $t = 3$ and $\delta(f) = 1$, Assertion (2) yields the upper bound $\sqrt{q}$, which is smaller than $\frac{2}{3}(q-1)$ for $q \geq 5$.

While Assertion (3) might sometimes not improve on the upper bound $\frac{t-1}{t}(q-1)$, it is often the case that $C(f)$ is provably small enough for a significant improvement. For instance, when $e_1 = 0$ and $\gcd(e_i, q-1) = 1$ for all $i \geq 2$, we have $C(f) = 1$ and then $R(f) \leq 2(q-1)^{\frac{t-2}{t-1}}$.

Our first main result is two explicit families of $t$-nomials revealing that Assertions (1)–(3) are close to optimal for *non*-prime $q$.

**Theorem 1.2.** *Let $t, u, p \in \mathbb{N}$ with $t \geq 2$ and $p$ prime. If $q = p^{(t-1)u}$ then the polynomial*

$$r_{t,u,p}(x) := 1 + x + x^{p^u} + \cdots + x^{p^{(t-2)u}}$$

*has $\delta(r_{t,u,p}) = C(r_{t,u,p}) = 1$ and exactly $q^{(t-2)/(t-1)}$ roots in $\mathbb{F}_q$. Furthermore, if $q = p^{tu}$, then the polynomial*

$$g_{t,u,p}(x) := 1 + x + x^{1+p^u} + \cdots + x^{1+p^u+\cdots+p^{(t-2)u}}$$

*has* $\delta(g_{t,u,p})=1$, $C(g_{t,u,p})\le\lfloor t/2\rfloor$, *and exactly* $q^{(t-2)/t}+\cdots+q^{1/t}+1$ *roots in* $\mathbb{F}_q$.

Theorem 1.2 is proved in Section 2 below. The polynomials $r_{t,u,p}$ show that the bounds from Assertions (1) and (3) of Theorem 1.1 *are within a factor of* 2 *of being optimal*, at least for $\delta(f)=C(f)=1$ and a positive density of prime powers $q$. Note in particular that $r_{3,u,p}$ shows that Assertion (2) of Theorem 1.1 is *optimal* for $q$ a perfect square and $\delta(f)=1$. (See [KO16, Thm. 1.3] for a different set of extremal trinomials when $q$ is an odd perfect square.) The second family $g_{t,u,p}$ reveals similar behavior for a different family of prime powers.

Optimally bounding the maximal number of roots (or cosets of roots) for the case of *prime* $q$ is more subtle already in the trinomial case. One reason is that explicit families of trinomials with many roots over prime fields appear much harder to generate. For instance, as we'll see below, the smallest prime $p$ for which there is a trinomial $f$ with $\delta(f)=1$ and 15 roots in $\mathbb{F}_p$ is $p=71237$. (Note that $15<71237^{1/4}$.) Also, it wasn't until the 1970s that Cohen proved that, given any $n\in\mathbb{N}$ and any prime $p=e^{\Omega(n\log n)}$, one can always find $b\in\mathbb{F}_p$ with $x^n+x+b$ having $n$ distinct roots in $\mathbb{F}_p$ [Coh70, Coh72], i.e., he found a family of trinomials with $\Omega\left(\frac{\log p}{\log\log p}\right)$ roots in $\mathbb{F}_p$ (albeit with unknown constant terms).

Using conditional results on the splitting of primes in number fields, we are able to give a completely explicit family of trinomials with a similar number of roots in certain $\mathbb{F}_p$.

**Theorem 1.3.** *Fix any* $n\ge2$ *and set* $h_n(x):=x^n-x-1$. *Then* $\delta(h_n)=C(h_n)=1$ *and there is a prime* $p\ge n+2$ *satisfying*

(a) $p=e^{O(n^n\sqrt{n}\log n)}$ *unconditionally, and*

(b) $p=O(n^{2n+1}\log^2 n)$ *if the Generalized Riemann Hypothesis (GRH) is true,*

*with* $h_n$ *having* $n$ *distinct roots in* $\mathbb{F}_p$.

Theorem 1.3 is proved in Section 3. In particular, we use a classic estimate of Lagarias, Montgomery, and Odlyzko [LMO79] (reviewed in Section 3 below) on the least prime ideal possessing a Frobenius element exhibiting a particular Galois conjugacy class. The latter result is in turn heavily based on the effective Chebotarev Density Theorem of Lagarias and Odlyzko [LO77].

Could the existence of trinomials $f$ with $\delta(f)=1$ and, say, $\Omega\left(\sqrt{p}\right)$ roots in $\mathbb{F}_p$ (as one might conjecture from Theorem 1.2) be obstructed by $p$ not being a perfect square? We feel that $p$ being prime is such an obstruction and, based on experimental results below, we conjecture the following upper bound:

**Conjecture 1.4.** *There is an absolute constant* $c\ge1$ *such that, for any prime* $p\ge3$ *and* $\gamma,e_2,e_3\in\{1,\dots,p-2\}$, *with* $e_3>e_2>0$ *and* $\gcd(e_2,e_3,p-1)=1$, *the trinomial* $\gamma+x^{e_2}+x^{e_3}$ *has no more than* $(\log p)^c$ *roots in* $\mathbb{F}_p$.

(See also [KO16, Conj. 1.5 & Sec. 4] for other refined conjectures and heuristics in this direction.) It is a simple exercise to show that, to compute the maximal number of roots in $\mathbb{F}_p$ of trinomials with $\delta(f)=1$, one can restrict to the family of trinomials in the conjecture.

For any $n\in\mathbb{N}$, let $p_n$ be the least prime for which there exists a univariate trinomial $f_n$ with $\delta(f_n)=1$ and exactly $n$ distinct roots in $\mathbb{F}_{p_n}$. Note that $p_n$ is well-defined according to [Coh70, Coh72]. We did a computer search to find the values of $p_n$ for $1\le n\le16$. They are...

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $p_n$ | 3 | 5 | 11 | 23 | 47 | 151 | 173 | 349 | 619 | 1201 | 2753 | 4801 | 10867 | 16633 | 71237 | 8581 |

For example, $p_{16} = 8581$ because $-364 + 363x + x^{2729}$ has exactly 16 roots in $\mathbb{F}_{8581}$, and *no* trinomial $f \in \mathbb{F}_p[x]$ with $p < 8581$ and $\delta(f) = 1$ has more than 15 roots in $\mathbb{F}_p$. In the appendix, we give representative trinomials for each $p_n$ above.

To get a feel for how the maximal number of roots of a trinomial grows with the field size, let us compare the graphs (drawn darker) of the functions $0.91 \log x$ and $1.77 \log x$ with the piecewise linear curve (drawn lighter) going through the sequence of points $((p_1, 1), \ldots, (p_{12}, 12), (p_{16}, 16), (p_{13}, 13), (p_{14}, 14), (p_{15}, 15))$ as shown in Figure 1 below. We used some simple `Maple`

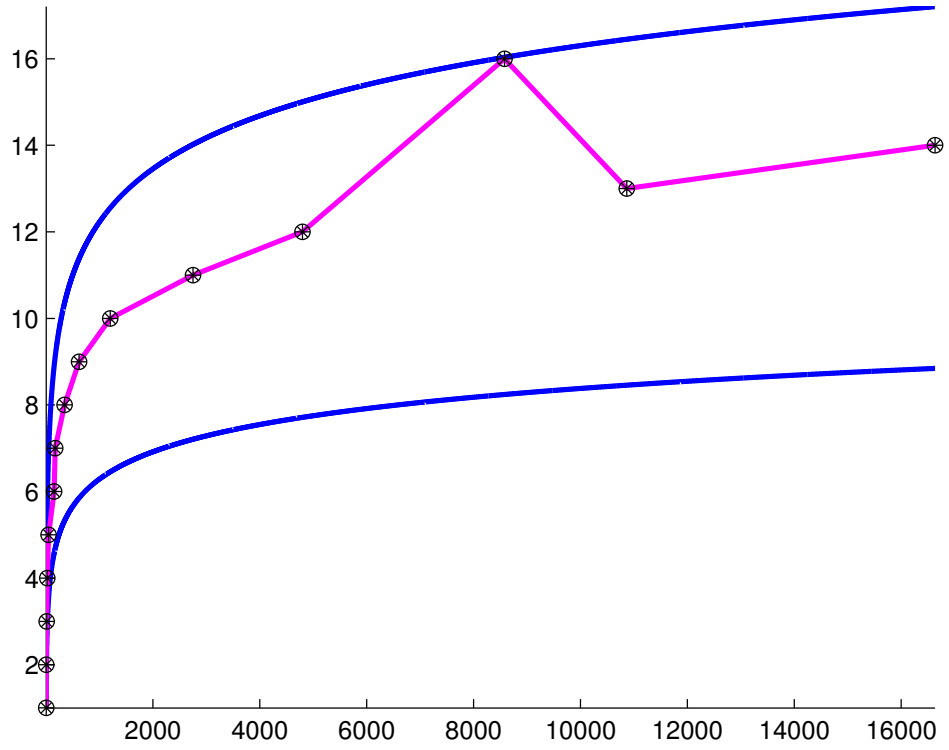


FIGURE 1

and `Sage` programs that took a few hours to find most of the data above. The value of $p_{15}$ took `C` code (written by Zander Kelley) running on a supercomputer for 2 days.

Quantitative results on sparse polynomials over finite fields sometimes admit stronger analogues involving complex roots of unity. For instance, [BCR13, BCR16] and [Che07] deal with the complexity of deciding whether a sparse polynomial vanishes on a (finite) subgroup of, respectively, $\mathbb{F}_q^*$ or $\mathbb{C}^*$. It is thus interesting to observe a recent complex analogue to our trinomial root counts over $\mathbb{F}_q$: Theobald and de Wolff [TdW14] proved that, if $\gcd(e_2, e_3) = 1$, a trinomial $c_1 + c_2 x^{e_2} + c_3 x^{e_3} \in \mathbb{C}[x]$ can have at most 2 complex roots of the same absolute value. So any such trinomial has at most 2 roots in the *torsion subgroup* $\{\zeta \in \mathbb{C} \mid \zeta^n = 1 \text{ for some } n \in \mathbb{N}\}$ of $\mathbb{C}^*$. This upper bound is sharp: Consider $(x-1)(x-\zeta)$ for any $\zeta \notin \{\pm 1\}$ satisfying $\zeta^n = 1$ for some $n \geq 3$.

## 2. Main Lower Bounds in the Prime Power Case

**Proof of Theorem 1.2**: To establish the root count for $r_{t,u,p}$ it clearly suffices to prove that $r_{t,u,p}$ divides $x^q - x$ or, equivalently, $x^{p^{(t-1)u}} = x$ in the ring $R := \mathbb{F}_p[x]/\langle r_{t,u,p}(x) \rangle$.

Toward this end, observe that $x^{p^{(t-1)u}} = \left(x^{p^{(t-2)u}}\right)^{p^u} = \left(-1 - x^{p^0} - \cdots - x^{p^{(t-3)u}}\right)^{p^u}$ in $R$. Since $(a+b)^{p^u} = a^{p^u} + b^{p^u}$ in any ring of characteristic $p$, we thus obtain $x^{p^{(t-1)u}} = (-1)^{p^u}\left(1 + x^{p^u} + \cdots + p^{(t-2)u}\right)$ in $R$. The last factor is merely $r_{t,u,p}(x) - x$, so we obtain $x^{p^{(t-1)u}} = (-1)^{p^u}(-x) = (-1)^{1+p^u}x$ in $R$. Since $(-1)^{1+p^k} = 1$ in $\mathbb{F}_p$ for all primes $p$, we have thus established the root count for $r_{t,u,p}$.

That $\delta(r_{t,u,p}) = 1$ is clear since $r_{t,u,p}$ has a nonzero constant term and 1 as one of its exponents. Likewise, $\delta(g_{t,u,p}) = 1$. That $C(r_{t,u,p}) = 1$ is clear because the lowest exponents of $r_{t,u,p}$ are 0 and 1, the rest are powers of $p$, and $\gcd(p^k, q-1) = 1$ for all $k \in \mathbb{N}$. We postpone proving our upper bound on $C(g_{t,u,p})$ until after we prove our stated root count for $g_{t,u,p}$.

Consider now the set $S$ of elements in $\mathbb{F}_q$ whose trace to $\mathbb{F}_{p^u}$ is zero, that is,

$$S := \left\{ a \in \mathbb{F}_q \mid a + a^{p^u} + a^{p^{2u}} + \cdots + a^{p^{(t-1)u}} = 0 \right\}.$$

Then $S$ has $q/p^u = p^{(t-1)u}$ elements and is a vector space of dimension $t-1$ over $\mathbb{F}_{p^u}$. Let $a \in S$ be nonzero. We show that $a^{p^u-1}$ is a root of $g_{t,u,p}$:

$$g_{t,u,p}(a^{p^u-1}) = 1 + \sum_{i=1}^{t-2} a^{(p^u-1)(p^{iu}+\cdots+p^u+1)} = 1 + \sum_{i=1}^{t-2} a^{p^{(i+1)u}-1} = \frac{1}{a}\sum_{i=0}^{t-1} a^{p^{iu}} = 0.$$

Now note that, for any $a \in S$ and any nonzero $w \in \mathbb{F}_{p^u}$, the element $aw$ is also in $S$. Also, for $a, b \in \mathbb{F}_q$, $a^{p^u-1} = b^{p^u-1}$ if and only if $b = aw$ for some nonzero $w \in \mathbb{F}_{p^u}$. Therefore, when $a$ runs through $S \setminus \{0\}$, the element $a^{p^u-1}$ yields $(p^{(t-1)u}-1)/(p^u-1) = 1 + p^u + \cdots + p^{(t-2)u}$ roots for $g_{t,u,p}$.

To finally prove our upper bound on $C(g_{t,u,p})$, note that, for $j > i$,

$$1 + p^u + \cdots + p^{(j-2)u} = \frac{p^{(j-1)u}-1}{p^u-1},$$

and

$$\frac{p^{(j-1)u}-1}{p^u-1} - \frac{p^{(i-1)u}-1}{p^u-1} = p^{(i-1)u}\left(\frac{p^{(j-i)u}-1}{p^u-1}\right).$$

So for $i \geq 2$,

$$\begin{aligned}\max_{j\in\{1,\ldots,t\}\setminus\{i\}} \gcd\left(p^{(i-1)u}\left(\frac{p^{(j-i)u}-1}{p^u-1}\right), p^{tu}-1\right) &= \max_{j\in\{1,\ldots,t\}\setminus\{i\}} \gcd\left(\frac{p^{|j-i|u}-1}{p^u-1}, p^{tu}-1\right) \\ &= \max_{\ell\in\{1,\ldots,\max\{i-1,t-i\}\}} \gcd\left(\frac{p^{\ell u}-1}{p^u-1}, p^{tu}-1\right).\end{aligned}$$

$$\begin{aligned}
\text{Hence,}\quad D(g_{t,u,p}) &:= \min_{i\in\{1,\ldots,t\}} \max_{\ell\in\{1,\ldots,\max\{i-1,t-i\}\}} \gcd\left(\tfrac{p^{\ell u}-1}{p^u-1}, p^{tu}-1\right)\\
&= \max_{\ell\in\{1,\ldots,\lfloor t/2\rfloor\}\}} \gcd\left(\frac{p^{\ell u}-1}{p^u-1}, p^{tu}-1\right)\\
&= \max_{\ell\in\{1,\ldots,\lfloor t/2\rfloor\}\}} \gcd\left(\frac{p^{\ell u}-1}{p^u-1}, \left(\frac{p^{tu}-1}{p^u-1}\right)(p^u-1)\right)\\
&\le \max_{\ell\in\{1,\ldots,\lfloor t/2\rfloor\}\}} \gcd\left(\frac{p^{\ell u}-1}{p^u-1}, \frac{p^{tu}-1}{p^u-1}\right)\gcd\left(\frac{p^{\ell u}-1}{p^u-1}, p^u-1\right),\\
&= \max_{\ell\in\{1,\ldots,\lfloor t/2\rfloor\}\}} \left(\frac{p^{\gcd(\ell,t)u}-1}{p^u-1}\right)\gcd(\ell, p^u-1)\,.
\end{aligned}$$

The last equality follows easily from two elementary facts: (1) $\gcd(x^\ell-1, x^t-1) = x^{\gcd(\ell,t)}-1$, and (2) $(x-1)(x^{\ell-2}+2x^{\ell-3}+\cdots+(\ell-2)x+(\ell-1)) = x^{\ell-1}+\cdots+x^2+x-(\ell-1)$. So $D(g_{t,u,p}) \le \max_{\ell\in\{1,\ldots,\lfloor t/2\rfloor\}} 1\cdot\ell \le \lfloor t/2\rfloor$. By [Kel16, Prop. 2.4] and [Kel16, Thm. 2.2], $D(g_{t,u,p}) \ge C(g_{t,u,p})$, so we are done. ■

## 3. Main Lower Bounds in the Prime Case

We'll need several results from algebraic number theory. First, let $K$ be any number field, i.e., a finite algebraic extension of $\mathbb{Q}$. Let $d_K$ denote the discriminant of $K$ over $\mathbb{Q}$, and $\mathcal{O}_K$ the ring of algebraic integers of $K$, i.e., those elements of $K$ with monic minimal polynomial in $\mathbb{Z}[x]$. We need to know the size of the smallest prime $p \in \mathbb{Z}$ that splits completely in $\mathcal{O}_K$. There are various bounds in the literature that are proved via some effective version of the Chebotarev Density Theorem. For instance:

**Theorem 3.1.** *(See* [LO77, Cor. 1.2 & pp. 461–462] *and* [LMO79, Thm. 1.1]*.) If $f \in \mathbb{Z}[x]$ is any irreducible polynomial of degree $n$ then the least prime $p$ for which the reduction of $f$ mod $p$ has $n$ distinct roots in $\mathbb{F}_p$ is (unconditionally) no greater than $d_K^{O(1)}$, where $K \subset \mathbb{C}$ is the splitting field of $f$. Furthermore, if GRH is true, then $p = O((\log d_K)^2)$.*

The papers [LO77, LMO79] in fact work in much greater generality: Our limited paraphrase above is simply the special case where one is looking for a prime yielding a Frobenius element corresponding to the identity element of the Galois group of $f$ over $\mathbb{Q}$.

The best recent estimates indicate that, in the unconditional case of Theorem 3.1, we can take the $O$-constant to be 40, for sufficiently large $d_K$ [KN14]. Also, for an abelian extension $K$ over $\mathbb{Q}$, Pollack [Pol14] gives a much better bound (in the unconditional case): $p = O_{\varepsilon,K}\left(d_K^{\frac{1}{4}+\varepsilon}\right)$ where $\varepsilon > 0$ is arbitrary, and the implied $O$-constant depends only on $\varepsilon$ and the degree of $K$ over $\mathbb{Q}$.

We will also need good bounds on discriminants of number fields. In the following theorem, the lower bound is due to Minkowski [Min91] and the upper bound follows easily from work of Tôyama [Tôy55] (by induction on the number of composita generated by the distinct roots of $f$).

**Theorem 3.2.** *(See, e.g.,* [BS96, pp. 259–260]*.) For any number field $K$ of degree $n$ over $\mathbb{Q}$, we have $d_K \ge \frac{n^{2n}}{(n!)^2} \ge \frac{(\pi e^2/4)^n}{2\pi n} > \frac{5.8^n}{6.3n}$. Also, if $K$ has minimal polynomial $f \in \mathbb{Q}[x]$ and $L$ is the splitting field of $f$, then $d_L$ divides $d_K^{(n-1)!+(n-2)!n+\cdots+1!n^{n-2}+0!n^{n-1}}$.* ■

**Proof of Theorem 1.3**: Clearly, $\delta(h_n)=1$. Also, since $\gcd(n,n-1)=1$, it is clear that $C(h_n)=1$. Now, for any $n\geq 2$, the trinomial $h_n := x^n-x-1$ is irreducible over $\mathbb{Q}$ [Sel56]. Let $\alpha\in\mathbb{C}$ be any root of $h_n$ and let $K=\mathbb{Q}(\alpha)$, so that $[K:\mathbb{Q}]=n$. Then $d_K$ divides the resultant of $h_n$ and $h_n'$ [BS96, Thm. 8.7.1, pg. 228]. The resultant of $h_n$ and $h_n'$ can then be computed explicitly to be $(-1)^{\frac{(n+2)(n-1)}{2}}\left(n^n+(-1)^n(n-1)^{n-1}\right)$ [Swa62]. Hence $d_K$ divides $n^n+(-1)^n(n-1)^{n-1}$. (We note that an elegant modern development of trinomial discriminants can be found in Chapter 12 of [GKZ94]; see Formula 1.38 on Page 406 in particular).

Let $L$ be the the splitting field of $h_n$. Then $L$ has degree at most $n!$ and, by Theorem 3.2, $d_K>\frac{5.8^n}{6.3n}$ and $d_L$ divides $\left(n^n+(-1)^n(n-1)^{n-1}\right)^{(n-1)!+(n-2)!n+\cdots+1!n^{n-2}+0!n^{n-1}}$. Note that, for $n\geq 3$, we have $n^n+(-1)^n(n-1)^{n-1}\leq n^n+(n-1)^{n-1}\leq e^{n\log n+\frac{4}{27}}$.

Also, by Stirling's Estimate [Rud76, Pg. 200], $n!<e\sqrt{n}\left(\frac{n}{e}\right)^n$ (for all $n\geq 1$), so we have

$$\begin{aligned}
&(n-1)!+n(n-2)!+\cdots+2!n^{n-3}+1!n^{n-2}+0!n^{n-1}\\
&< e\sqrt{n-1}\left(\frac{n-1}{e}\right)^{n-1}+e\sqrt{n-2}\left(\frac{n-2}{e}\right)^{n-2}n+\cdots+e\sqrt{1}\left(\frac{1}{e}\right)^{1}n^{n-2}+n^{n-1}\\
&< e\sqrt{n}\left(1+\frac{1}{e}+\cdots+\frac{1}{e^{n-1}}\right)n^{n-1}=\left(\frac{e}{1-1/e}\right)n^{n-1/2}<4.31n^{n-1/2},
\end{aligned}$$

and thus $d_L<e^{4.5n^{n+1/2}\log n}$.

Theorem 3.1 then tells us that there is a prime $p\in\mathbb{Z}$ so that $h_n$ splits completely modulo $p$ with no repeated roots where

(a) $p=e^{O(n^{n+1/2}\log n)}=e^{e^{(n+1/2+o(1))\log n}}$ unconditionally, and
(b) $p=O((n^{n+1/2}\log n)^2)=e^{(2n+1+o(1))\log n}$ if GRH is true. ■

We used the family of trinomials $x^n-x-1$ mainly for the sake of simplifying our proof. Many other families would likely exhibit the same behavior, albeit with some additional intricacies in applying prime ideal bounds. However, the deeper question is to understand the structure of *truly* extremal trinomials over prime fields, such as those appearing in the Appendix below.

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## Appendix: Some Extremal Trinomials

We list in Figure 2, for $n\in\{1,\ldots,16\}$, trinomials $f_n$ with $\delta(f_n)=1$ and $f_n$ having exactly $n$ distinct roots in $\mathbb{F}_{p_n}$, with $p_n$ the smallest prime admitting such a trinomial. In particular, for each $n\in\{1,\ldots,16\}$, a full search was done so that the trinomial $f_n$ below has the least degree among all trinomials over $\mathbb{F}_{p_n}$ having exactly $n$ roots in $\mathbb{F}_{p_n}$. (It happens to be the case that, for $n\in\{1,\ldots,16\}$, we can also pick the middle degree monomial of $f_n$ to be $x$.)

| $n$ | $f_n$ | $p_n$ |
|---|---|---|
| 1 | $1+x-2x^2$ | 3 |
| 2 | $1+x-2x^2$ | 5 |
| 3 | $1-3x+2x^3$ | 11 |
| 4 | $-2+x+x^4$ | 23 |
| 5 | $1+4x-5x^8$ | 47 |
| 6 | $1+24x-25x^{33}$ | 151 |
| 7 | $-2+x+x^{34}$ | 173 |
| 8 | $1+23x-24x^{21}$ | 349 |
| 9 | $-71+70x+x^{184}$ | 619 |
| 10 | $1+5x-6x^{152}$ | 1201 |
| 11 | $-797+796x+x^{67}$ | 2753 |
| 12 | $-82+81x+x^{1318}$ | 4801 |
| 13 | $-1226+1225x+x^{225}$ | 10867 |
| 14 | $-39+38x+x^{2264}$ | 16633 |
| 15 | $29574-29573x-x^{27103}$ | 71237 |
| 16 | $-364+363x+x^{2729}$ | 8581 |

FIGURE 2. Trinomials with exactly $n$ distinct roots in $\mathbb{F}_{p_n}$ and $p_n$ minimal

By rescaling the variable as necessary, we have forced 1 to be among the roots of each of the trinomials above. It is easily checked via the last part of Assertion (3) of Theorem 1.1 that $C(f_n)=1$ for each $n\in\{1,\ldots,16\}$.

The least prime $p_{17}$ for which there is a trinomial $f_{17}$ with $\delta(f_{17})=1$ and exactly 17 roots in $\mathbb{F}_{p_{17}}$ is currently unknown (as of July 2016). Better and faster code should hopefully change this situation soon.

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*E-mail address*: `qcheng@cs.ou.edu`

School of Computer Science, University of Oklahoma, Norman, OK 73019

*E-mail address*: `sgao@math.clemson.edu`

Department of Mathematical Sciences, Clemson University, Clemson, SC 29634-0975

*E-mail address*: `rojas@math.tamu.edu`

TAMU 3368, College Station, TX 77843-3368

*E-mail address*: `dwan@math.uci.edu`

Department of Mathematics, University of California, Irvine, CA 92697-3875