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A finite dimensional approximation to pinned Wiener measure on symmetric spaces

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A Finite Dimensional Approximation to Pinned Wiener Measure on Symmetric Spaces

A dissertation submitted in partial satisfaction of the
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in

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by

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2016

The dissertation of Zhehua Li is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

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Chair

University of California, San Diego

2016

DEDICATION

To my parents.

EPIGRAPH

I... a universe of atoms, an atom in the universe. —Richard Feynman

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ACKNOWLEDGEMENTS

Five years sounds like a long time, but when you are hearing the countdown clock ticking, all the memories frame in minds like a series of flashbacks. My five–year Ph.D. life in San Diego is filled with every kind of emotions, among all of them is my appreciation to all people who have influenced me one way or another.

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VITA

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ABSTRACT OF THE DISSERTATION

A Finite Dimensional Approximation to Pinned Wiener Measure on Symmetric Spaces

by

Zhehua Li

Doctor of Philosophy in Mathematics

University of California, San Diego, 2016

Professor Bruce K. Driver, Chair

Let M be a Riemannian manifold, $o \in M$ be a fixed base point, $W_o(M)$ be the space of continuous paths from $[0,1]$ to M starting at $o \in M$, and let ν_x denote Wiener measure on $W_o(M)$ conditioned to end at $x \in M$. The goal of this thesis is to give a rigorous interpretation of the informal path integral expression for ν_x;

$$d\nu_x(\sigma) \text{ ``}=\text{''} \delta_x(\sigma(1)) \frac{1}{Z} e^{-\frac{1}{2}E(\sigma)} \mathcal{D}\sigma \ , \ \sigma \in W_o(M).$$

In this expression $E(\sigma)$ is the "energy" of the path σ, δ_x is the δ – function based at x, $\mathcal{D}\sigma$ is interpreted as an infinite dimensional volume "measure" and Z is a certain "normalization" constant. We will interpret the above path integral expression as a limit of measures, $\nu^1_{\mathcal{P},x}$, indexed by partitions, $\mathcal{P}$ of $[0,1]$. The measures $\nu^1_{\mathcal{P},x}$ are constructed by restricting the above path integral expression to the finite dimensional manifolds, $H_{\mathcal{P},x}(M)$, of piecewise geodesics in $W_o(M)$ which are allowed to have jumps in their derivatives at the partition points and end at x. The informal volume measure, $\mathcal{D}\sigma$, is then taken to be a certain Riemannian volume measure on $H_{\mathcal{P},x}(M)$. When M is a symmetric space of non–compact type, we show how to naturally interpret the pinning condition, i.e. the δ – function term, in such a way that $\nu^1_{\mathcal{P},x}$, are in fact well defined finite measures on $H_{\mathcal{P},x}(M)$. The main theorem of the this thesis then asserts that $\nu^1_{\mathcal{P},x} \to \nu_x$ (in a weak sense) as the mesh size of $\mathcal{P}$ tends to zero. Along the way we develop a number of integration–by–parts arguments for the approximate measures, $\nu^1_{\mathcal{P},x}$, which are analogous to those known for the measures, ν_x.

Chapter 1

Overview

Throughout this dissertation, we fix $\left(M^d, g, \nabla, o\right)$ to be a pointed complete Riemannian manifold of dimension d with Riemannian metric g, Levi-Civita covariant derivative (∇), and base point $o \in M$. We further let

$$W_o(M) := \{\sigma \in C([0,1] \mapsto M) \mid \sigma(0) = o\}$$

be the **Wiener space** on M and let ν be the **Wiener measure** on $W_o(M)$—i.e. the law of the M–valued Brownian motion which starts at $o \in M$.

Richard Feynman, in his groundbreaking 1942 dissertation, offered a path integral representation of the quantum particle state based on the *principle of least action*. In quantum physics, the state of a quantum particle is described by a wave function ϕ which satisfies the Schrödinger equation,

$$i\frac{\partial}{\partial t}\phi = H\phi$$

where $H = -\frac{1}{2}\Delta_g + V$ is the Schrödinger operator, Δ_g is the Laplace-Beltrami operator on (M, g, o), $V : M \to \mathbb{R}$ is an external potential and i is the imaginary unit. For our purpose, a slight modification is considered: after an analytic continuation (roughly change $t \to it$), the Schrödinger's equation becomes the heat equation;

$$\frac{\partial}{\partial t}\phi = -H\phi \ , \ \phi(x,0) = f(x) \,. \tag{1.1}$$

Let e^{-tH} be the solution operator to (1.1), meaning $e^{-tH}f$ solves heat equation (1.1) when such a solution exists. Under modest regularity conditions, this operator admits an integrable kernel $p_t^H(\cdot,\cdot)$. In the physics literature one frequently finds Feynman type informal identities of the form,

$$p_1^H(o,x) = \text{“}\frac{1}{Z}\int_{W_o(M)} \delta_x(\sigma(1))\, e^{-\int_0^1\left[\frac{1}{2}|\dot{\sigma}(\tau)|^2 + V(\sigma(\tau))\right]d\tau}\mathcal{D}\sigma\text{”} \tag{1.2}$$

and

$$\left(e^{-H}f\right)(o) = \text{“}\frac{1}{Z}\int_{W_o(M)} f(\sigma(1))\, e^{-\int_0^1\left[\frac{1}{2}|\dot{\sigma}(\tau)|^2 + V(\sigma(\tau))\right]d\tau}\mathcal{D}\sigma\text{”} \tag{1.3}$$

Variants of these informal path integrals are often used as the basis for “defining” and making computations in quantum-field theories. From a mathematical perspective, making sense of such path integrals is thought to be a necessary step to developing a rigorous definition of interacting quantum field theories, (see for example; Glimm and Jaffe [17], Barry Simon [31], the Clay Mathematics Institute's Millennium problem involving Yang-Mills and Mass Gap). In general, path integrals like those appearing in (1.2) suffer from at least five distinct flaws;

1. The normalizing constant Z should typically be interpreted as either 0 or ∞ depending on the context.

2. The energy function
$$E(\sigma) := \frac{1}{2}\int_0^1 |\dot{\sigma}(\tau)|^2\, d\tau$$
appearing in the exponent in (1.2) requires σ to be appropriately differentiable; this is at odds with the fact that sample paths of Wiener measure ν are almost surely nowhere differentiable.

3. There is no Lebesgue measure $\mathcal{D}\sigma$ on infinite dimensional path spaces.

4. δ_x is a distribution so pointwise evaluation does not make sense.

5. It is generally not permissible to multiply a distribution δ_x with a measure $\frac{1}{Z}\exp\left(-\frac{1}{2}\int_0^1 |\dot{\sigma}(\tau)|^2 d\tau\right)\mathcal{D}\sigma$.

Various attempts to use path integrals to rigorously construct solutions to the Schrödinger (heat) equation have been made, out of which we highlight two routes. One is to approximate the path integral through piecewise "linear" paths or polygonal paths, which evolves as a finite dimensional approximation scheme that will be discussed more in Section 1.1. Another route, pioneered by Kac, is the realization of taking Wiener measure as the framework of integration over path spaces. Roughly speaking, when $V = 0$, Kac suggests the formal identities;

$$\text{``}\frac{1}{Z}e^{-\frac{1}{2}\int_0^1|\dot{\sigma}(\tau)|^2 d\tau}\mathcal{D}\sigma\text{''} := d\nu(\sigma) \tag{1.4}$$

and

$$\text{``}\int_{W_o(M)} \delta_x(\sigma(1))\frac{1}{Z}e^{-\frac{1}{2}\int_0^1|\dot{\sigma}(\tau)|^2 d\tau}\mathcal{D}\sigma\text{''} := p_1(o,x) \tag{1.5}$$

where $p_t(x,y)$ is the heat kernel on M. For example, if $M = \mathbb{R}$, the heat kernel is given by

$$p_t(0,x) = \frac{1}{\sqrt{2\pi t}}e^{\frac{-x^2}{2t}},$$

which is the well known density function of a normal random variable with mean 0 and variance t. In general, if the potential V is sufficiently regular, one can prove rigorously the following results;

$$p^H(o,x) = p_1(o,x)\int_{W_o(M)} e^{-\int_0^1 V(\Sigma_s)ds} d\nu_x$$

and

$$e^{-H} f(o) = \int_M \left[p_1(o,x) f(x) \int_{W_o(M)} e^{-\int_0^1 V(\Sigma_s) ds} d\nu_x \right] dx$$

where $\Sigma_s : \sigma \ni W_o(M) \to \sigma(s) \in M$ is the coordinate function. The above expressions are typically refered to as Feynman–Kac formulae. Interested readers may refer to [29] and references therein for a thorough summary of this field in Euclidean space with a flavor of rigorous quantum field theory and may refer to [4] for a survey of results in general Riemannian manifolds.

1.1 Finite Dimensional Approximation Scheme for Path Integrals

The central idea behind finite dimensional approximation scheme is to define a path integral as a limit of the same integrands restricted to "natural" approximate path spaces, for example, piecewise linear paths, broken lines, polygonal paths and so on. The ill–defined expression under these finite dimensional approximations usually becomes well–defined or has better interpretations, see ([16], [23]). For example, when $M = \mathbb{R}^d$, it is known that Wiener measure on $W(\mathbb{R}^d)$ may be approximated by Gaussian measures on piecewise linear path spaces. More specifically, Eq. (1.4) restricted to a finite dimensional subspace of piecewise linear paths based on a partition of $[0,1]$ has a natural interpretation as Gaussian probability measure resulting from the canonical isometry between the piecewise linear path space and $\mathbb{R}^{dn}$, where n is the number of partition points. By combining Wiener's theorem on the existence of Wiener measure with the dominated convergence theorem, one can see that these Gaussian measures converge weakly to ν as the mesh of partition tends to zero, (see for example [13, Proposition 6.17] for details). An analogous theory on general manifolds was also developed, see for example [30], Atiyah [3], Bismut [5], Andersson and Driver [2] and references therein. In [2], followed by [28] and [27], the finite dimensional approximation problem is viewed in its full geometric form by restricting the expression in Eq. (1.4) to finite dimensional sub-manifolds of piecewise geodesic paths on M. Unlike the flat case ($M = \mathbb{R}^d$) where the choice of translation invariant Riemannian metric on path spaces is irrelevant, various Riemannian metrics on approximate path spaces are explored. Based on these metrics, different approximate measures are constructed which lead to different limiting measures on $W_o(M)$, see [2], [27], and [28]. In this dissertation we adopt a so–called $G^1_{\mathcal{P}}$ metric on the piecewise geodesic space.

In the remainder of this section, we briefly summarize some results in [2] to give reader a better understanding of how the finite dimensional approximation scheme goes as well as establishing some necessary notations used in this dissertation.

Definition 1.1 (Cameron-Martin space on (M, o)) *Let*

$$H(M) := \left\{ \sigma \in C([0,1] \mapsto M) : \sigma(0) = o \text{ , } \sigma \textit{ is a.c. and } \int_0^1 |\sigma'(s)|^2 \, ds < \infty \right\}$$

*be the **Cameron-Martin space** on (M, o). (Here a.c. means absolutely continuous.)*

Notation 1.2 *Let $\Gamma(TM)$ denote the differentiable sections of TM and $\Gamma_\sigma(TM)$ be the differentiable sections of TM along $\sigma \in H(M)$.*

The space, $H(M)$, is an infinite dimensional Hilbert manifold which is a central object in problems related to the calculus of variation on M. Klingenberg [24] contains a good exposition of the manifold of paths. In particular, Theorem 1.2.9 in [24] presents its differentiable structure in terms of atlases. We will be interested in certain Riemannian metrics on $H(M)$ and on certain finite dimensional submanifolds.

Definition 1.3 *For any* $\sigma \in H(M)$ *and* $X, Y \in \Gamma_\sigma^{a.c.}(TM)$*,*

$$G^1(X,Y) = \int_0^1 \left\langle \frac{\nabla X}{ds}(s), \frac{\nabla Y}{ds}(s) \right\rangle_g ds$$

where $\Gamma_\sigma^{a.c.}(TM)$ *is the set of absolutely continuous vector fields along* σ *with finite energy, i.e.* $\int_0^1 \left\langle \frac{\nabla X}{ds}(s), \frac{\nabla X}{ds}(s) \right\rangle_g ds < \infty$*.*

Remark 1.4 *To see that* G^1 *is a metric on* $H(M)$*, we identify the tangent space* $T_\sigma H(M)$ *with* $\Gamma_\sigma^{a.c.}(TM)$*. To motivate this identification, consider a differentiable one-parameter family of curves* σ_t *in* $H(M)$ *such that* $\sigma_0 = \sigma$*. By definition of tangent vector,* $\frac{d}{dt}|_0 \sigma_t(s)$ *should be viewed as a tangent vector at* σ*. This is actually the case, for detailed proof, see Theorem 1.3.1 in [24].*

Definition 1.5 (Piecewise geodesic space) *Given a partition*

$$\mathcal{P} := \{0 = s_0 < \cdots < s_n = 1\} \text{ of } [0,1],$$

define:

$$H_{\mathcal{P}}(M) := \left\{\sigma \in H(M) \cap C^2([0,1] \setminus \mathcal{P}) : \nabla\sigma'(s)/ds = 0 \text{ for } s \notin \mathcal{P}\right\}. \tag{1.6}$$

The piecewise geodesic space $H_{\mathcal{P}}(M)$ is a finite dimensional embedded submanifold of $H(M)$. As for its tangent space, following the argument of Theorem 1.3.1 in [24], for any $\sigma \in H_{\mathcal{P}}(M)$, the tangent space $T_\sigma H_{\mathcal{P}}(M)$ may be identified with vector-fields along σ of the form $X(s) \in T_{\sigma(s)}M$ where $s \to X(s)$ is piecewise C^1 and satisfies Jacobi equation for $s \notin \mathcal{P}$, i.e.

$$\frac{\nabla^2 X}{ds^2}(s) = R(\dot\sigma(s), X(s))\dot\sigma(s),$$

where R is the curvature tensor. (See Theorem 2.39 below for a more detailed description of $TH_{\mathcal{P}}(M)$). After specifying the tangent space of $H_{\mathcal{P}}(M)$, we can define the $G^1_{\mathcal{P}}$ metric as follows.

Definition 1.6 *For any $\sigma \in H_{\mathcal{P}}(M)$ and $X, Y \in T_\sigma H_{\mathcal{P}}(M)$, let*

$$G^1_{\mathcal{P}} \langle X, Y \rangle := \sum_{j=1}^{n} \left\langle \frac{\nabla X}{ds}(s_{j-1}+), \frac{\nabla Y}{ds}(s_{j-1}+) \right\rangle_g \Delta_j \tag{1.7}$$

where $\Delta_j = s_j - s_{j-1}$ and $\frac{\nabla Y}{ds}(s_{j-1}+) = \lim_{s \downarrow s_{j-1}} \frac{\nabla Y}{ds}(s)$.

Endowed with the Riemannian metric $G^1_{\mathcal{P}}$, $H_{\mathcal{P}}(M)$ becomes a finite dimensional Riemannian manifold and the left hand side of (1.4) is now well–defined on $H_{\mathcal{P}}(M)$ if $\mathcal{D}\sigma$ is interpreted as the volume measure induced from this Riemannian metric. This motivates the following approximate measure definition.

Definition 1.7 (Approximate measure on $H_{\mathcal{P}}(M)$) *Let $\nu^1_{\mathcal{P}}$ be the probability measure on $H_{\mathcal{P}}(M)$ defined by;*

$$d\nu^1_{\mathcal{P}}(\sigma) = \frac{1}{Z^1_{\mathcal{P}}} e^{-\frac{1}{2}\int_0^1 \langle \sigma'(s), \sigma'(s) \rangle ds} dvol_{G^1_{\mathcal{P}}}(\sigma), \tag{1.8}$$

where $dvol_{G^1_{\mathcal{P}}}$ is the volume measure on $H_{\mathcal{P}}(M)$ induced from the metric $G^1_{\mathcal{P}}$ and $Z^1_{\mathcal{P}}$ is the normalization constant.

We now summarize the main theorem in Andersson and Driver [2].

Theorem 1.8 (Andersson-Driver, Theorem 1.8. [2]) *Suppose $f : W_o(M) \to \mathbb{R}$ is bounded and continuous, then*

$$\lim_{|\mathcal{P}| \to 0} \int_{H_{\mathcal{P}}(M)} f(\sigma)\, d\nu^1_{\mathcal{P}}(\sigma) = \int_{W_o(M)} f(\sigma)\, d\nu(\sigma).$$

1.2 Main Theorems

In this section we state the main results of this dissertation while avoiding many technical details.

Definition 1.9 (Pinned piecewise geodesic space) *For any* $x \in M$,

$$H_{\mathcal{P},x}(M) := \{\sigma \in H_{\mathcal{P}}(M) : \sigma(1) = x\}.$$

We prove below in Proposition 3.11 that when M has non–positive sectional curvature, $H_{\mathcal{P},x}(M)$ is an embedded submanifold of $H_{\mathcal{P}}(M)$.

Theorem 1.10 *If M is a Hadamard manifold with bounded sectional curvature and $\mathcal{P} = \{k/n\}_{k=0}^{n}$ are equally-spaced partitions, then there exists a finite measure $\nu^1_{\mathcal{P},x}$ supported on $H_{\mathcal{P},x}(M)$, such that for any bounded continuous function f on $H_{\mathcal{P}}(M)$,*

$$\lim_{m\to\infty}\int_{H_{\mathcal{P}}(M)} \delta_x^{(m)}(\sigma(1))\, f(\sigma)\, d\nu^1_{\mathcal{P}}(\sigma) = \int_{H_{\mathcal{P}}(M)} f(\sigma)\, d\nu^1_{\mathcal{P},x}(\sigma).$$

where $\delta_x^{(m)}$ is an approximating sequence of δ_x in $C_0^\infty(M)$.

Recall that a Hadamard manifold is a simply connected complete Riemannian manifold with non-positive sectional curvature.

Remark 1.11 *The formula for $d\nu^1_{\mathcal{P},x}$ is explicitly given, see Definition 3.13.*

The next theorem asserts, under additional geometric restrictions, that the measure $\nu^1_{\mathcal{P},x}$ we obtained from Theorem 1.10 serves as a good approximation to pinned Wiener measure ν_x.

Theorem 1.12 *If M is a symmetric space of non–compact type, i.e. it is a Hadamard manifold with parallel curvature tensor, then for any restricted cylinder function $f \in \mathcal{RFC}_b^1$, see Definition 2.31,*

$$\lim_{|\mathcal{P}|\to 0}\int_{H_{\mathcal{P}}(M)} f(\sigma)\, d\nu^1_{\mathcal{P},x}(\sigma) = \int_{W_o(M)} f(\sigma)\, d\nu_x(\sigma)$$

where ν_x is pinned Wiener measure, see Theorem 2.17 below.

1.3 Structure of the Dissertation

For the guidance to the reader, we give a brief summary of the contents of this dissertation.

In Chapter 2 we set up some notations and preliminaries in probability and geometry. In particular we present the Eells-Elworthy-Malliavin construction of Brownian motion on manifolds.

In Chapter 3 we define explicitly the pinned approximate meausre $\nu^1_{\mathcal{P},x}$ and study its properties. In Theorem 3.15, we prove that $\nu^1_{\mathcal{P},x}$ is a finite measure and that $x \to \int_{H_{\mathcal{P},x}(M)} f d\nu^1_{\mathcal{P},x}$ is a continuous function on M provided f is bounded and continuous. This property is the key ingredient in proving Theorem 1.10, which is given in Chapter 3.

In Chapter 4 we develop the so–called orthogonal lift of a vector field X on M to a vector field $\tilde{X}(\cdot)$ on $W_o(M)$. We define $\tilde{X}(\cdot)$ first on $H(M)$ by minimizing a norm of $\tilde{X}(\cdot)$ which is induced from a "damped"metric related to the Ricci curvature of M (see Definition 4.6). This lift is then "stochastically"extended to $W_o(M)$. Some tools from Malliavin calculus are reviewed as needed in order to define $\tilde{X}(\cdot)$ as an anticipating differential opearator on $W_o(M)$. We then establish integration–by–parts formula for $\tilde{X}(\cdot)$.

In Chapter 5 we focus on the finite dimensional manifold $H_{\mathcal{P}}(M)$. In Section 5.1 a parametrization of the tangent space of $H_{\mathcal{P}}(M)$ is given. Using this parametrization and some linear algebra we obtain a formula for the orthogonal lift $\tilde{X}_{\mathcal{P}}$ of $X \in \Gamma(TM)$ relative to the norm induced from the $G^1_{\mathcal{P}}$ metrc on $H_{\mathcal{P}}(M)$.

In Chapter 6, (using the development maps introduced in Chapter 2), we view $\tilde{X}_{\mathcal{P}}$ as defined on all of $W_o(M)$ and show that for any bounded cylinder function f (also introduced in Chapter 2), $\tilde{X}_{\mathcal{P}} f \to \tilde{X} f$ in $L^{\infty-}(W_o(M))$ and more challengingly, we show $\tilde{X}^{tr,\nu} f - \tilde{X}^{tr,\nu^1_{\mathcal{P}}}_{\mathcal{P}} f \to 0$, where $\tilde{X}^{tr,\nu}$ is the adjoint of $\tilde{X}$ with respect to ν and $\tilde{X}^{tr,\nu^1_{\mathcal{P}}}_{\mathcal{P}}$ is the adjoint of $\tilde{X}_{\mathcal{P}}$ with respect to $\nu^1_{\mathcal{P}}$.

In Chapter 7, we combine all the tools that are developed from previous chapters to prove the main Theorem 1.12 of this dissertation.

Chapter 2

Background and Notation

For the remainder of the dissertation, let $u_0 : \mathbb{R}^d \to T_oM$ be a fixed linear isometry which we add to the standard setup (M, g, o, u_0, ∇). Let $\Gamma(TM)$ be differentiable sections of the tangent bundle TM. We will first introduce the orthonormal frame bundle $\mathcal{O}(M)$ which is crucial in the Eells-Elworthy-Malliavin construction of Brownian motion. A connection is then defined on $\mathcal{O}(M)$. The reader may refer to Appendix A.2 for a more detailed exposition of principal bundles ($\mathcal{O}(M)$ is a special case of a principal bundle) and connections on them.

Definition 2.1 (Orthonormal Frame Bundle $(\mathcal{O}(M), \pi)$) *For any $x \in M$, denote by $\mathcal{O}(M)_x$ the space of orthonormal frames on T_xM, i.e. the space of linear isometries from $\mathbb{R}^d$ to T_xM. Denote $\mathcal{O}(M) := \cup_{x\in M}\mathcal{O}(M)_x$ and let $\pi : \mathcal{O}(M) \to M$ be the (fiber) projection map, i.e. for each $u \in \mathcal{O}(M)_x$, $\pi(u) = x$. The pair $(\mathcal{O}(M), \pi)$ is the orthonormal frame bundle over M whose structure group is the orthogonal group $O(d)$–the $d \times d$ real orthogonal matrices.*

Definition 2.2 (Connection on $\mathcal{O}(M)$) *A connection on $\mathcal{O}(M)$ is uniquely specified by the $\mathfrak{so}(d)$–valued connection form ω^∇ on $\mathcal{O}(M)$ determined by ∇;*

for any $u \in \mathcal{O}(M)$ *and* $X \in T_u\mathcal{O}(M)$,

$$\omega_u^{\nabla}(X) := u^{-1}\frac{\nabla u(s)}{ds}\mid_{s=0}$$

where $u(\cdot)$ *is a differentiable curve on* $\mathcal{O}(M)$ *such that* $u(0) = u$ *and* $\frac{du(s)}{ds}\mid_{s=0} = X$*. For any* $\xi \in \mathbb{R}^d$, $\frac{\nabla u(s)}{ds}\mid_{s=0}\xi := \frac{\nabla u(s)\xi}{ds}\mid_{s=0}$ *is the covariant derivative of* $u(\cdot)\xi$ *along* $\pi(u(\cdot))$ *at* $\pi(u)$.

ω^{∇} *determines a decomposition of* $T\mathcal{O}(M)$*. We will call the kernel of* ω^{∇} *the horizontal vector space (denoted by* $HT\mathcal{O}(M)$*) and call the compliment space the vertical vector space (denoted by* $VT\mathcal{O}(M)$*).*

Definition 2.3 *For any* $a \in \mathbb{R}^d$*, define the horizontal lift* $B_a \in \Gamma(T\mathcal{O}(M))$ *of* a *in the following way: for any* $u \in \mathcal{O}(M)$,

- $\omega_u^{\nabla}(B_a(u)) = 0$
- $\pi_*(B_a(u)) = ua$

Remark 2.4 *By the rank-nullity theorem, it is easy to see that the above conditions determine uniquely the horizontal lift.*

Recall that we have defined the Cameron-Martin space on M:

$$H(M) := \left\{\sigma \in C([0,1], M) : \sigma(0) = o, \sigma \text{ is a.c. and } \int_0^1 |\sigma'(s)|_g^2\, ds < \infty\right\}$$

Similarly we define $H_0(\mathbb{R}^d)$ and $H_{u_0}(\mathcal{O}(M))$ by changing the state spaces to be $\mathbb{R}^d$, $\mathcal{O}(M)$, reference points to be 0, u_0 and using the usual metric for g on the Euclidean spaces $\mathbb{R}^d$, $\mathbb{R}^{d\times d}$.

Definition 2.5 (Horizontal lift of a path) *For any* $\sigma \in H(M)$*, a curve* $u : [0,1] \to \mathcal{O}(M)$ *is said to be a horizontal lift of* σ *if* $\pi \circ u = \sigma$ *and the tangent vector to* $u(s)$ *always belongs to* $HT_{u(s)}\mathcal{O}(M)$*.*

Theorem 2.6 *Given $\sigma \in H(M)$ and $u_0 \in \pi^{-1}(\sigma(0))$, there exists a unique horizontal lift $u(s)$ such that $u(0) = u_0$. We denote u by $\psi(\sigma)$, so ψ is the horizontal lifting map.*

Proof. The condition of existence of horizontal lift u of σ is equivalent to:

$$\begin{aligned} \pi(u(s)) &= \sigma(s) \\ \omega^{\nabla}(u'(s)) &= 0 \end{aligned} \quad \text{for } s \in [0,1]$$

For any $s \in [0,1]$, there exists U_α in the open cover of M and $\epsilon > 0$ such that $\sigma(\tau) \in U_\alpha$ for $\tau \in (s-\epsilon, s+\epsilon) \cap [0,1]$. Denote by ω_α the restriction of the connection one-form ω on $\pi^{-1}(U_\alpha)$ and $\phi_\alpha \circ u(\tau) = (\sigma(\tau), g(\tau)) \in U_\alpha \times G$, where $\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times G$ is the local trivialization. Then after identifying $T(U_\alpha \times G)$ with $TU_\alpha \times TG$, the condition $\omega^{\nabla}(u'(\tau)) = 0$ is equivalent to $A_{\sigma(\tau)}\sigma'(\tau) + C_{\sigma(\tau)}g'(\tau) = 0$, where A and C are two $\mathfrak{g}-$valued one forms on U_α and G. Since $\sigma(\tau)$ is fixed, this gives rise to a linear system of ODEs of $g(\tau)$, since the initial condition is specified, there is a unique solution $g(\tau)$ and hence the unique $u(\tau)$.

■

Notation 2.7 *A path $u \in H_{u_0}(\mathcal{O}(M))$ is said to be horizontal if the tangent vector to $u(s)$ always belongs to $HT_{u(s)}\mathcal{O}(M)$. We denote the set of horizontal paths by $HH_{u_0}(\mathcal{O}(M))$.*

Fact 2.8 *If $u(\sigma, s) = \psi(\sigma)(s)$, then $u(\sigma, s)\, u_0^{-1}$ is the parallel translation $//_s(\sigma)$ along σ.*

Remark 2.9 *Theorem 2.6 asserts that $\psi : H(M) \to HH_{u_0}(\mathcal{O}(M))$ is a bijection. It is in fact known to be a diffeomorphism with $\psi^{-1}(u) = \pi \circ u$.*

Definition 2.10 (Development map) *Given $w \in H_0\left(\mathbb{R}^d\right)$, the solution to the ordinary differential equation*

$$du(s) = \sum_{i=1}^{d} B_{e_i}(u(s))\, dw^i(s), u(0) = u_0$$

is defined to be the ***development*** *of w to $H_{u_0}(\mathcal{O}(M))$ and we will denote this map $w \to u$ by η, i.e. $\eta(w) = u$. Here $\{e_i\}_{i=1}^d$ is the standard basis of $\mathbb{R}^d$.*

Remark 2.11 *From Definition 2.10 and the smooth dependence of driving path in ODE systems we can see that η is a diffeomorphism from $H_0\left(\mathbb{R}^d\right)$ to $HH_{u_0}(\mathcal{O}(M))$.*

Definition 2.12 (Rolling map) *$\phi = \pi \circ \eta : H_0\left(\mathbb{R}^d\right) \to H(M)$ is said to be the rolling map to $H(M)$.*

Remark 2.13 *From Remark 2.9 and 2.11 one can see that ϕ has a smooth inverse ϕ^{-1}, which can be defined explicitly as follows:*

Definition 2.14 (Anti-rolling map) *Given $\sigma \in H(M)$ with $u = \psi(\sigma)$. The anti-development of σ is a curve $w \in H_0\left(\mathbb{R}^d\right)$ defined by:*

$$w_t = \int_0^t u_s^{-1} \sigma_s' ds$$

It is not hard to see $w = \phi^{-1}(\sigma)$.

The Eells-Elworthy-Malliavin construction of Brownian motion depends in essence on a stochastic version of the maps defined above. Since the development maps on the smooth category are defined through ordinary differential equations, a natural way to introduce probability is to replace ODEs by (Stratonovich) stochastic diffrerential equations.

First we set up some measure theoretic notation and conventions. Suppose that $(\Omega, \{\mathcal{G}_s\}, \mathcal{G}, P)$ is a filtered measurable space with a finite measure P. For any

$\mathcal{G}$—measurable function f, we use $P(f)$ and $\mathbb{E}_P[f]$ (if P is a probability measure) to denote the integral $\int_\Omega f dP$. Given two filtered measurable spaces $(\Omega, \{\mathcal{G}_s\}, \mathcal{G}, P)$ and $(\Omega', \{\mathcal{G}'_s\}, \mathcal{G}', P')$ and a $\mathcal{G}/\mathcal{G}'$ measurable map $f : \Omega \to \Omega'$, the law of f under P is the push-forward measure $f_* P(\cdot) := P(f^{-1}(\cdot))$. We are mostly interested in the path spaces $W_o(M)$, $W_0(\mathbb{R}^d)$ and $W_{u_0}(\mathcal{O}(M))$, where the following notation is being used.

Notation 2.15 *If (Y, y) is a pointed manifold, let $W(Y) := C([0,1], Y)$ be the space of all continuous paths in Y equipped with the uniform topology, $W_y(Y) := \{w \in W(Y) \mid w(0) = y\}$ be the subset of continuous paths that start at y.*

Definition 2.16 *For any $s \in [0,1]$ let $\Sigma_s : W_y(Y) \to Y$ be the* ***coordinate functions*** *given by $\Sigma_s(\sigma) = \sigma(s)$.*

We will often view Σ as a map from $W_y(Y)$ to $W_y(Y)$ in the following way: for any $\sigma \in W_y(Y)$ and $s \in [0,1]$, $\Sigma(\sigma)(s) = \Sigma_s(\sigma)$. Let $\mathcal{F}_s^o$ be the $\sigma-$algebra generated by $\{\Sigma_\tau : \tau \le s\}$. We use $\mathcal{F}_1^o$ as the raw $\sigma-$algebra and $\{\mathcal{F}_s^o\}_{0 \le s \le 1}$ as the filtration on $W_y(Y)$. The next theorem defines the Wiener measure ν and pinned Wiener measure ν_x on $(W_y(Y), \mathcal{F}_1^o)$.

Theorem 2.17 *Assume Y is a geometrically complete Riemannian manifold, then there exist two finite measures ν and ν_x on $(W_y(Y), \mathcal{F}_1^o)$ which are uniquely determined by their finite dimensional distributions as follows. For any partition $0 = s_0 < s_1 < \cdots < s_{n-1} < s_n = 1$ of $[0,1]$ and bounded functions $f : Y^n \to \mathbb{R}$;*

$$\nu(f(\Sigma_{s_1}, \ldots, \Sigma_{s_n})) = \int_{Y^n} f(x_1, \ldots, x_n) \Pi_{i=1}^n p_{\Delta s_i}(x_{i-1}, x_i) \, dx_1 \cdots dx_n \tag{2.1}$$

and

$$\nu_x(f(\Sigma_{s_1}, \ldots, \Sigma_{s_n})) = \int_{Y^{n-1}} f(x_1, \ldots, x_n) \Pi_{i=1}^n p_{\Delta s_i}(x_{i-1}, x_i) \, dx_1 \cdots dx_{n-1} \tag{2.2}$$

where $p_t(\cdot,\cdot)$ is the heat kernel on Y associated to the Riemannian metric, $\Delta_i = s_i - s_{i-1}$, $x_0 \equiv y$ and $x_n \equiv x$ in (2.2).

Definition 2.18 (Brownian motion) *A stochastic process $X : (\Omega, \mathcal{G}_s, \{\mathcal{G}\}, P) \to (W_y(Y), \nu)$ is said to be a Brownian motion on Y if the law of X is ν i.e. $X_* P := P \circ X^{-1} = \nu$.*

Remark 2.19 *From Theorem 2.17 it is clear that the law of the adapted process $\Sigma : W_y(Y) \to W_y(Y)$ is ν and Σ is a Brownian motion. We will call Σ the **canonical Brownian motion** on Y.*

Remark 2.20 *Using Theorem 2.17, we can construct Wiener measure and pinnned Wiener measure on $W_0(\mathbb{R}^d)$, $W_o(M)$ and $W_{u_0}(\mathcal{O}(M))$ respectively. In order to avoid ambiguity from moving between $W_0(\mathbb{R}^d)$ and $W_o(M)$, we fix the symbol $\mu\,(\mu_x)$ as the Wiener (pinned Wiener) measure on $W_0(\mathbb{R}^d)$ and reserve the symbol $\nu\,(\nu_x)$ as the Wiener (pinned Wiener) measure on $W_o(M)$. Meanwhile we reserve Σ as the canonical Brownian motion on M.*

Theorem 2.21 (Horizontal Lift of Brownian Motion) *If Σ is the canonical Brownian motion on M, then there exists a unique (up to $\nu-$equivalence) $\tilde{u} \in W_{u_0}(\mathcal{O}(M))$ such that*

$$\pi(\tilde{u}_s) = \Sigma_s. \tag{2.3}$$

Proof. See Theorem 2.3.5 in [21] ∎

Definition 2.22 (Stochastic Anti–rolling map) *If Σ is the canonical Brownian motion on M, the (stochastic) anti–rolling β of Σ is defined by,*

$$\delta\beta_s = \tilde{u}_s^{-1}\delta\Sigma_s \ , \ \beta_0 = 0 \tag{2.4}$$

$\tilde{u}$ and β defined above are linked through the (stochastic) development map.

Definition 2.23 (Stochastic development map) *Let $\tilde{u}$ and β be as defined in Theorem 2.21 and Definition 2.22, then $\tilde{u}$ satisfies the following SDE driven by β,*

$$\delta \tilde{u}_s = \sum_{i=1}^{d} B_{e_i}(\tilde{u}_s)\, \delta\beta_s \ , \ \tilde{u}(0) = u_0$$

and $\tilde{u}$ is said to be the development of β.

Fact 2.24 *The following facts are frequently used in this dissertation. (The proof can be found in Appendix A.)*

- *ϕ is a diffeomorphism from $H_0(\mathbb{R}^d)$ to $H(M)$,*
- *$\phi\,|_{H_{\mathcal{P}}(\mathbb{R}^d)}$ is a diffeomorphism from $H_{\mathcal{P}}(\mathbb{R}^d)$ to $H_{\mathcal{P}}(M)$,*
- *β is a Brownian motion on $\left(W_0(\mathbb{R}^d), \mu\right)$.*

From now on some notations are fixed for the conveniance of consistency.

Notation 2.25 *For any $\sigma \in H(M)$, $u_{(\cdot)}(\sigma) \in H_{u_0}(\mathcal{O}(M))$ is its horizontal lift and $b_{(\cdot)}(\sigma) \in H_0(\mathbb{R}^d)$ is its anti-rolling. Recall that $\{\Sigma_s\}_{0\leq s\leq 1}$ is fixed to be the canonical Brownian motion on $(W_o(M), \nu)$. We also fix $\beta(\cdot)$ to be the anti-rolling of Σ, (which is a Brownian motion on $\mathbb{R}^d$) and $\tilde{u}(\cdot)$ to be the (stochastic) horizontal lift of Σ.*

Notation 2.26 *Given a partition $\mathcal{P}$, $\beta_{\mathcal{P}}$ is the piecewise linear approximation to the Brownian motion β on $\mathbb{R}^d$ given by:*

$$\beta_{\mathcal{P}}(s) := \beta(s_{i-1}) + \frac{\Delta_i \beta}{\Delta_i}(s - s_{i-1}) \ \text{ if } s \in [s_{i-1}, s_i]$$

where $\Delta_i\beta = \beta(s_i) - \beta(s_{i-1})$and $\Delta_i = s_i - s_{i-1}$.

Notation 2.27 (Geometric Notation)

- ***curvature tensor*** *For any* $X, Y, Z \in \Gamma(TM)$, *define the (Riemann) curvature tensor* $R : \Gamma(TM) \times \Gamma(TM) \to \Gamma(End(TM))$ *to be:*

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

- ***sectional curvature*** *For any* $p \in M$ *and* T_p *a two dimensional subspace of* T_pM, *the sectional curvature* $K(p, T_p)$ *with respect to* T_p *is defined to be:*

$$K(p, T_p) := \langle R(X_p, Y_p) Y_p, X_p \rangle_g$$

where (X_p, Y_p) *is an orthonormal basis of* $K(p, T_p)$.

- *For any* $\sigma \in H(M)$, *define* $R_{u(\sigma,s)}(\cdot,\cdot)\cdot$ *to be a map from* $\mathbb{R}^d \otimes \mathbb{R}^d$ *to* $End(\mathbb{R}^d)$ *given by;*

$$R_{u(\sigma,s)}(a,b)\cdot = u(\sigma,s)^{-1} R(u(\sigma,s)a, u(\sigma,s)b)\, u(\sigma,s) \ \ \forall a,b \in \mathbb{R}^d \tag{2.5}$$

where R *is the curvature tensor of* M. *Similarly define* $R_{\tilde{u}(\sigma,s)}(\cdot,\cdot)\cdot$ *to be a random map (up to* ν*-equivalence) from* $\mathbb{R}^d \otimes \mathbb{R}^d$ *to* $\mathbb{R}^d$ *as follows:*

$$R_{\tilde{u}(\sigma,s)}(\cdot,\cdot)\cdot = \tilde{u}(\sigma,s)^{-1} R(\tilde{u}(\sigma,s)\cdot, \tilde{u}(\sigma,s)\cdot)\, \tilde{u}(\sigma,s) \tag{2.6}$$

- $Ric(\cdot) := \sum_{i=1}^d R(v_i, \cdot) v_i$ *is the Ricci curvature tensor on* M. *Here* $\{v_i\}_{i=1}^d$ *is an orthonormal basis of proper tangent space. Using* $u(\sigma,s)$ *or* $\tilde{u}(\sigma,s)$ *to pull back* R, *we can define* $Ric_{u(\sigma,s)}$ *and* $Ric_{\tilde{u}(\sigma,s)}$ *to be maps (Random maps) from* $\mathbb{R}^d$ *to* $\mathbb{R}^d$.

- *For any* $p \in M$, $exp_p : T_pM \to M$ *is the Riemannian exponential map, i.e.*

for any $\xi \in$ *domain of* exp_p,

$$exp_p(\xi) = \gamma\left(|\xi|, \frac{\xi}{|\xi|}\right)$$

where $\gamma(t,v)$ *is the unique geodesic of* M *with* $\gamma(0) = p$ *and* $\gamma'(0) = v$

Remark 2.28 *The existence of unique local geodesic* $\gamma(t,v)$ *is a standard result in differential geometry, see Proposition 2.17 in [8].*

Remark 2.29 *Sometimes in the dissertation we will suppress* σ, *sometimes even* s *in* $u(\sigma, s)$ *when there is no confusion.*

Remark 2.30 *In this dissertation the partition* $\mathcal{P}$ *is always equally spaced, so* $|\mathcal{P}| \equiv \Delta_i \equiv \frac{1}{n}$ *for* $i = 1, ..., n$.

We introduce two commonly used test function spaces on $W_o(M)$.

Definition 2.31 $f : W_o(M) \mapsto \mathbb{R}$ *is a* ***restricted cylinder function*** *if there exists a partition*

$$\mathcal{P} := \{0 < s_1 < \cdots < s_n \leq 1\}$$

of $[0,1]$ *and a function* $F : C^m(M^n, \mathbb{R})$ *such that*

$$f = F(\Sigma_{s_1}, \Sigma_{s_2}, \ldots, \Sigma_{s_n})$$

Denote this space by $\mathcal{RFC}^m$.

Definition 2.32 $f : W_o(M) \mapsto \mathbb{R}$ *is a* ***cylinder function*** *iff there exists a partition*

$$\mathcal{P} := \{0 < s_1 < \cdots < s_n \leq 1\}$$

of $[0,1]$ *and a function* $F \in C^m(\mathcal{O}(M)^n, \mathbb{R})$ *such that:*

$$f = F(\tilde{u}_{s_1}, \tilde{u}_{s_2}, \ldots, \tilde{u}_{s_n})$$

Denote this space by $\mathcal{FC}^m$.

Notation 2.33 *Denote*

$$\mathcal{FC}_b^1 := \left\{ f := F(u) \in \mathcal{FC}^1, F \text{ and all its partial differentials } F_i' \text{ are bounded} \right\}.$$

Notation 2.34 *Denote*

$$\mathcal{RFC}_b^1 := \left\{ f := F(\sigma) \in \mathcal{FC}^1, F \text{ and all its partial differentials } F_i' \text{ are bounded} \right\}.$$

Remark 2.35 *In Notation 2.34, for each* $i \in \{1, \ldots, n\}$, $F_i' : TM \to \mathbb{R}^d$, *so* F_i' *is bounded iff* $\|F_i'\|_g < \infty$.

Remark 2.36 *In Notation 2.33, for each* $i \in \{1, \ldots, n\}$, $F_i' : T\mathcal{O}(M) \to \mathbb{R}^d$. F_i' *is bounded iff for any* $a \in \mathbb{R}^d$, $A \in \mathfrak{so}(d)$, $\left|A^\dagger F_i\right| \leq C\|A\| < \infty$ *and* $|B_a F_i| \leq C\|a\| < \infty$, *where the vectors fields* $A^\dagger$ *and* B_a *are defined in Definitions A.11 and 2.3.*

Remark 2.37 *Since* $\pi(\tilde{u}_s) = \Sigma_s$ *and* $\pi : \mathcal{O}(M)$ *to* M *is smooth,* $\mathcal{RFC}^m \subset \mathcal{FC}^m$ *for each* $m \in \mathbb{N}$.

Definition 2.38 (Jacobi equation) *For* $\sigma \in H(M)$, $Y \in \Gamma_\sigma(TM)$, *we say* $Y(s) \in T_{\sigma(s)}M$ *satisfies Jacobi equation if:*

$$\frac{\nabla^2}{ds^2} Y(s) = R(\sigma'(s), Y(s))\, \sigma'(s).$$

Further if the horizontal lift $u(s)$ *of* σ *is used, we let* $y(s) := u^{-1}(s) Y(s)$. *It then follows that* $y(s)$ *satisfies the pulled back Jacobi equation,*

$$y''(s) = R_{u(s)}(b'(s), y(s))\, b'(s), \tag{2.7}$$

where $b'(s) = u(s)^{-1}\sigma'(s)$. *Once we have Jacobi equation, we can describe the tangent space* $TH_{\mathcal{P}}(M)$ *of* $H_{\mathcal{P}}(M)$*:*

We formalize the tangent space of $H_{\mathcal{P}}(M)$ mentioned in Definition 1.5.

Theorem 2.39 (Tangent space to $H_{\mathcal{P}}(M)$) *For all* $\sigma \in H_{\mathcal{P}}(M)$,

$$T_\sigma H_{\mathcal{P}}(M) = \left\{ s \to u(s)\, J(s) \mid J \in C\left([0,1], \mathbb{R}^d\right),\ J \in H_{\mathcal{P},\sigma} \text{ with } J(0) = 0 \right\}. \tag{2.8}$$

where $J \in H_{\mathcal{P},\sigma}$ *iff*

$$J''(s) = R_{u(s)}\left(b'(s_{i-1}+), J(s)\right) b'(s_{i-1}+) \text{ for } s \in [s_{i-1}, s_i)\ i = 1, ..., n.$$

Proof. See Theorem 1.3.1 in [24]. ■

Notation 2.40 *Given* $h(\cdot) \in H_0\left(\mathbb{R}^d\right)$, *denote*

$$X^h(\sigma, s) := u(\sigma, s)\, h(s).$$

Notation 2.41 ($\{C_{\mathcal{P},i}(\sigma, s)\}_{i=1}^n$ and $\{S_{\mathcal{P},i}(\sigma, s)\}_{i=1}^n$) *Let*

$$\mathcal{P} := \{0 = s_0 < s_1 < \cdots < s_n = 1\}$$

be a partition of $[0,1]$, $K_i := [s_{i-1}, s_i]$ *and* $\Delta_i := s_i - s_{i-1}$ *for* $1 \leq i \leq n$, *and say that* $f(s)$ *satisfies the* i *–Jacobi's equation if*

$$f''(s) = R_{u_s}\left(u^{-1}\sigma'(s_{i-1}+), f(s)\right) u^{-1}\sigma'(s_{i-1}+) \text{ for } s \in K_i. \tag{2.9}$$

where $u^{-1}\sigma'(s) := u(\sigma, s)^{-1}\sigma'(s) \in \mathbb{R}^d$.

We now let $C_{\mathcal{P},i}(\sigma, s)$ *and* $S_{\mathcal{P},i}(\sigma, s) \in \mathrm{End}(\mathbb{R}^d)$ *denote the solution to Eq.*

(2.9) with initial conditions,

$$C_{\mathcal{P},i}\left(s_{i-1}\right)=I,\ C'_{\mathcal{P},i}\left(s_{i-1}\right)=0,\ S_{\mathcal{P},i}\left(s_{i-1}\right)=0 \text{ and } S'_{\mathcal{P},i}\left(s_{i-1}\right)=I$$

and we further let

$$C_{\mathcal{P},i}\left(\sigma\right):=C_{\mathcal{P},i}\left(\sigma,s_i\right) \text{ and } S_{\mathcal{P},i}\left(\sigma\right):=S_{\mathcal{P},i}\left(\sigma,s_i\right).$$

Here we view $C_{\mathcal{P},i}\left(s\right)$ *and* $S_{\mathcal{P},i}\left(s\right)$ *as maps from* $H_{\mathcal{P}}\left(M\right)$ *to* $\mathrm{End}(\mathbb{R}^d)$.

Definition 2.42 *Define for all* $i=1,\cdots,n$,

$$f_{\mathcal{P},i}\left(\sigma,s\right)=\begin{cases}0 & s\in\left[0,s_{i-1}\right]\\ \frac{S_{\mathcal{P},i}(\sigma,s)}{\Delta_i} & s\in\left[s_{i-1},s_i\right]\\ \frac{C_{\mathcal{P},j}(\sigma,s)C_{\mathcal{P},j-1}(\sigma)\cdots\cdot C_{\mathcal{P},i+1}(\sigma)S_{\mathcal{P},i}(\sigma)}{\Delta_i} & s\in\left[s_{j-1},s_j\right] \text{ for } j=i+1,\cdots,n\end{cases}$$

with the convention that $S_{\mathcal{P},0}\equiv\left|\mathcal{P}\right|I$ *and* $f_{\mathcal{P},0}\equiv I$.

Remark 2.43 $S_{\mathcal{P},j}\left(s\right)$, $C_{\mathcal{P},j}\left(s\right)$ *may be expressed in terms of* $\left\{f_{\mathcal{P},i}\right\}_{i=0}^{n}$ *by*

$$S_{\mathcal{P},j}\left(s\right)=\Delta_j f_{\mathcal{P},j}\left(s\right)$$

$$C_{\mathcal{P},j}\left(s\right)=f_{\mathcal{P},j-1}\left(s\right)f_{\mathcal{P},j-1}^{-1}\left(s_j\right).$$

Chapter 3

Approximate Pinned Measures

3.1 Representation of $\delta-$ function

Let X be a smooth manifold (for example, M as mentioned in the dissertation, $\mathbb{R}^d$ or open subset of the first two). We will denote the distribution on X by $\mathcal{D}'(X)$ and, compactly supported distribution by $\mathcal{E}'(X)$. For a matrix A, let $eig(A)$ is denote the set of eigenvalues of A. To each $x \in X$, let $\delta_x \in \mathcal{E}'(X)$ be the δ–function at x defined by

$$\delta_x(f) = f(x) \;\; \forall f \in C_0^\infty(X).$$

Lemma 3.1 (Representation of δ_0 on flat space) *There exist functions $\{g_i\}_{i=0}^d$ with $g_0 \in C_0^\infty(\mathbb{R}^d)$, $\{g_j\}_{j=1}^d \subset C^\infty(\mathbb{R}^d/\{0\})$ with supports contained in a compact subset $K \subset \mathbb{R}^d$ and satisfies*

$$|g_j(x)| \leq c|x|^{1-d} \;\; \text{for } j = 1, \cdots, d, \tag{3.1}$$

such that

$$\delta_0 = g_0 + \sum_{j=1}^d \frac{\partial g_j}{\partial x_j} \;\; \text{in } \mathcal{E}'(\mathbb{R}^d). \tag{3.2}$$

In more detail, for any $f \in C_0^\infty\left(\mathbb{R}^d\right)$,

$$f(0) = \int_{\mathbb{R}^d} \left(g_0 + \sum_{j=1}^{d} \frac{\partial g_j}{\partial x_j} \right) f dx = \int_{\mathbb{R}^d} \left(g_0 f - \sum_{j=1}^{d} \frac{\partial f}{\partial x_j} g_j \right) dx. \tag{3.3}$$

This lemma can be derived from Lemma 10.10 in [32]. Here we provide another proof using the fundamental solution to the Laplace's equation.

Proof of Lemma 3.1. Define the Newtonian kernel $\Gamma(x)$ on $\mathbb{R}^d$ $(d > 2)$:

$$\Gamma(x) = \frac{|x|^{2-d}}{d(2-d) w_d}$$

where w_d is the volume of unit ball on $\mathbb{R}^d$. Then it is well-known $\Gamma(x)$ is a fundamental solution of Laplace's equation, i.e. for any $y \in \mathbb{R}^d$, denote by Δ the Laplacian on $\mathbb{R}^d$:

$$\Delta\Gamma(\cdot - y) = \delta_y(\cdot) \text{ in } \mathcal{E}'\left(\mathbb{R}^d\right).$$

where δ_y is the delta function at y and the equality is interpreted in the distributional sense. In particular if $y = 0$, we get:

$$\Delta\Gamma(\cdot) = \delta_0(\cdot).$$

If $Z := \nabla\Gamma \in C^\infty\left(\mathbb{R}^d / \{0\} \to \mathbb{R}^d\right)$, then

$$|Z| = \left| \frac{x|x|^{-d}}{dw_d} \right| \leq C_d |x|^{1-d}$$

where C_d is a constant depending only on d and

$$\nabla \cdot Z = \delta_0 \text{ in } \mathcal{E}'\left(\mathbb{R}^d\right).$$

In order to get compact support, let $g_j = \phi Z_j$, where $\phi \in C_0^\infty\left(\mathbb{R}^d\right)$ such that $\phi \equiv 1$

on $B(0,1)$ and $\phi \equiv 0$ on $\mathbb{R}^d/B(0,2)$, $B(x,r)$ is the ball on $\mathbb{R}^d$ centered at x with radius r. Then we have

$$\nabla \cdot (\phi Z) = \nabla \phi \cdot Z + \phi \nabla \cdot Z \text{ in } \mathcal{E}'\left(\mathbb{R}^d\right).$$

Since the support of δ_0 is $\{0\}$, we get

$$\delta_0 = \nabla \cdot Z = \phi \nabla \cdot Z = \nabla \cdot (\phi Z) - \nabla \phi \cdot Z,$$

where $-\nabla \phi \cdot Z \in C_0^\infty\left(\mathbb{R}^d\right)$ and $\{\phi Z_i\}_{i=1}^d \subset C^\infty\left(\mathbb{R}^d/\{0\}\right)$ with compact support and $|\phi Z_i| \leq c\,|x|^{1-d}$ for some $c > 0$. ■

Based on this representation we can get a representation of δ_p for any $p \in M$. Before we get to the representation of δ_p we state a smooth Urysohn lemma.

Lemma 3.2 (Smooth Urysohn Lemma) *If M is a smooth manifold, then for any two disjoint closed sets V_1 and V_2, there exists a function $f \in C^\infty(M, [0,1])$ such that $f^{-1}(\{0\}) = V_1$ and $f^{-1}(\{1\}) = V_2$.*

This is a standard result in elementary topology, so the proof is skipped here.

Theorem 3.3 (Representation of δ − function on manifold) *For any $p \in M$, there exist functions $\{g_j\}_{j=0}^d \subset C^\infty(M/\{p\}) \cap L^{\frac{d}{d-1}}(M)$ with supports in a compact subsets K of M and smooth vector fields $\{X_j\}_{j=1}^d \subset \Gamma^\infty(TM)$ with compact support such that*

$$\delta_p = g_0 + \sum_{j=1}^d X_j g_j \text{ in } \mathcal{E}'(M). \tag{3.4}$$

Proof. Pick a chart $\{U, x\}$ near $p \in M$ such that $x(p) = 0$. Since $x(U) = \mathbb{R}^d$, one can apply Lemma 3.1 on $x(U) \simeq \mathbb{R}^d$ and get:

$$\delta_0 = \tilde{g}_0 - \sum_{j=1}^{d} \frac{\partial}{\partial x_j} \tilde{g}_j$$

where δ_0 is the delta mass on $x(U)$ supported at the origin. So for any $h \in C^{\infty}(U)$

$$\begin{aligned} h(p) &= h \circ x^{-1}(0) \\ &= \int_{\mathbb{R}^d} \left(\tilde{g}_0 - \sum_{j=1}^{d} \frac{\partial}{\partial x_j} \tilde{g}_j \right) h \circ x^{-1} d\lambda \\ &= \int_{\mathbb{R}^d} \left(\tilde{g}_0 + \sum_{j=1}^{d} \tilde{g}_j \frac{\partial}{\partial x_j} \right) h \circ x^{-1} d\lambda \end{aligned}$$

where $d\lambda$ is the Lebesque measure on $\mathbb{R}^d$. Consider $\left\{ \frac{\tilde{g}_j}{\sqrt{\det g}} \circ x \right\}_{j=0}^{d}$ where $g = (g_{ij})_{1 \le i,j \le d}$ is the metric matrix, i.e. $g_{ij} = \left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_i} \right\rangle_g$. From Lemma 3.1 we know that $\frac{\tilde{g}_j}{\sqrt{\det g}} \circ x$ has compact support in U and therefore $K := \cup_{j=1}^{d} supp \left(\frac{\tilde{g}_j}{\sqrt{\det g}} \circ x \right)$ is compact in U. Using Lemma 3.2 we can construct a smooth function $\phi \in C^{\infty}(M \to [0,1])$ such that $\phi^{-1}(\{0\}) = M/U$ and $\phi^{-1}(\{1\}) = K$. Define

$$\hat{g}_0 = \phi \frac{\tilde{g}_0}{\sqrt{\det g}} \circ x$$

and

$$\hat{g}_j = \phi \frac{\tilde{g}_j}{\sqrt{\det g}} \circ x, \;\; X_j = \phi \cdot \left(x^{-1} \right)_* \frac{\partial}{\partial x_j} \text{ for } j = 1, \ldots, d$$

Then for any $f \in C^\infty(M)$,

$$\begin{aligned}
\int_M \left(\hat{g}_0 + \sum_{j=1}^d \hat{g}_j X_j\right) f dvol \\
&= \int_U \left(\hat{g}_0 + \sum_{j=1}^d \hat{g}_j X_j\right) f dvol \\
&= \int_U \frac{\tilde{g}_0}{\sqrt{\det g}} \circ x \cdot \phi f dvol \\
&+ \sum_{j=1}^d \int_U \phi^2 \frac{\tilde{g}_j}{\sqrt{\det g}} \circ x \left(\left(x^{-1}\right)_* \frac{\partial \phi f}{\partial x_j} - \left(x^{-1}\right)_* \frac{\partial \phi}{\partial x_j} f\right) dvol
\end{aligned}$$

Here $dvol$ is the volume measure on M.

Since $\phi \cdot (x^{-1})_* \frac{\partial \phi}{\partial x_j} \equiv 0$ and $\phi \equiv 1$ on K, we have:

$$\begin{aligned}
\int_M \left(\hat{g}_0 + \sum_{j=1}^d \hat{g}_j X_j\right) f dvol &= \int_U \left(\frac{\tilde{g}_0}{\sqrt{\det g}} \circ x + \sum_{j=1}^d \frac{\tilde{g}_j}{\sqrt{\det g}} \circ x \left(x^{-1}\right)_* \frac{\partial}{\partial x_j}\right) f dvol \\
&= \int_{\mathbb{R}^d} \left(\frac{\tilde{g}_0}{\sqrt{\det g}} + \sum_{j=1}^d \frac{\tilde{g}_j}{\sqrt{\det g}} \frac{\partial}{\partial x_j}\right) f \circ x^{-1} \sqrt{\det g} d\lambda \\
&= \int_{\mathbb{R}^d} \left(\tilde{g}_0 + \sum_{j=1}^d \tilde{g}_j \frac{\partial}{\partial x_j}\right) f \circ x^{-1} d\lambda \\
&= f \circ x^{-1}(0) \\
&= f(p)
\end{aligned}$$

Therefore, by the Divergence Theorem, we can write down δ_p in distributional sense as

$$\delta_p = g_0 + \sum_{j=1}^d X_j g_j$$

where

$$g_0 = \hat{g}_0 - \sum_{j=1}^d \hat{g}_j \cdot div X_j$$

and for $j = 1, \ldots, n$,

$$g_j = -\hat{g}_j.$$

From the construction one can see that $X_j \in \Gamma^\infty(TM)$ and $\{g_j\}_{j=0}^d \subset C^\infty(M/\{p\}) \cap L^{\frac{d}{d-1}}(M)$ with compact support. ■

Lemma 3.4 *$C_0^\infty(M)$ is dense in $L^p(M)$ for any $1 \le p < \infty$.*

Proof. Recall that simple functions on M are finite linear combinations of indicator functions 1_E where $vol(E) < \infty$. Since simple functions are dense in $L^p(M)$. It suffices to show that $C_0^\infty(M)$ is dense in the space of simple functions with respect to L^p-norm. Given a simple function 1_E,

$$\int_M 1_E dvol = vol(E)$$

Since the volume measure is regular, there exists a compact set K and open set U such that

$$K \subset E \subset U$$

and

$$vol(K) \ge vol(U) - \epsilon.$$

Now apply Lemma 3.2 we can find a cutoff function $f \in C_0^\infty(M)$ such that $f^{-1}(\{0\}) = M/U$ and $f^{-1}(\{1\}) = K$. It follows that

$$\|f - 1_E\|_{L^p(M)}^p = \int_M |f - 1_E|^p \, dvol \le vol(U - K) \le \epsilon,$$

which proves the denseness of $C_0^\infty(M)$ in the space of simple functions and thus in $L^p(M)$.

■

Remark 3.5 *Using Lemma 3.4 and Theorem 3.3, for any $g_j, j = 1, \cdots, d$, we can* *find a sequence* $\left\{g_j^{(m)}\right\}_m \subset C_0^\infty(M)$ *such that*

$$g_j^{(m)} \to g_j \ \text{in} \ L^{\frac{d}{d-1}}(M)$$

In particular, since g_j *has compact support, we can make* $\cup_m suppg_j^{(m)}$ *to be compact.*

Corollary 3.6 *Define*

$$\delta_x^{(m)} := g_0^{(m)} + \sum_{j=1}^d X_j g_j^{(m)} \in C_0^\infty(M).$$

Then $\left\{\delta_x^{(m)}\right\}_m$ *is an approximating sequence of delta mass* δ_x*, i.e.*

$$\delta_x^{(m)} \to \delta_x \ \text{in} \ \mathcal{D}'(M).$$

Proof. Using integration by parts, we have for any $f \in C(M)$,

$$\int_M f\delta_x^{(m)} d\lambda = \int_M \left(g_0^{(m)} + \sum_{j=1}^d X_j g_j^{(m)}\right) f d\lambda \tag{3.5}$$

$$= \int_M \left(g_0^{(m)} f + \sum_{j=1}^d g_j^{(m)} X_j^* f\right) d\lambda \tag{3.6}$$

Since $K := \cup_m suppg_j^{(m)}$ is compact, $f \cdot 1_K$ and $X_j^* f \cdot 1_K \in L^{\infty-}(M)$, then 3.6 easily follows by Holder's inequality. ■

3.2 Definition of $\nu_{\mathcal{P},x}^1$

In this section we will give the explicit definition of $\nu_{\mathcal{P},x}^1$ proposed in Theorem 1.10.

Definition 3.7 (End point map) *Define* $E_1 : H(M) \to M$ *to be* $E_1(\sigma) = \sigma(1)$ *and let* $E_1^{\mathcal{P}}$ *denote* $E_1|_{H_{\mathcal{P}}(M)}$ *.*

Recall from Definition 3.16 that

$$H_{\mathcal{P},x}(M) := \{\sigma \in H_{\mathcal{P}}(M) \mid \sigma(1) = x\} = \left(E_1^{\mathcal{P}}\right)^{-1}(\{x\}).$$

In general, it is not guaranteed that $E_1^{\mathcal{P}}$ is a submersion, which would guarantee that $H_{\mathcal{P},x}(M)$ is an embedded submanifold of $H_{\mathcal{P}}(M)$. The following is an easy, yet illuminating, example showing what can go wrong:

Example 3.8 *If* $M = \mathbb{S}^2$ *and* $\mathcal{P} := \{0, 1\}$ *with starting point being the North pole, then* $\dim H_{\mathcal{P}}(M) = 2$. *Consider*

$$X(\sigma, s) := (0, \pi \sin s\pi, 0) \in T_\sigma H_{\mathcal{P}}(M)$$

where

$$\sigma(s) = (\sin s\pi, 0, \cos s\pi).$$

An one parameter family realizing $X(\sigma, s)$ *would be*

$$\sigma_t(s) = (\sin s\pi \cos t\pi, \sin s\pi \sin t\pi, \cos s\pi)$$

From which one can easily see that:

$$E_{1*\sigma}^{\mathcal{P}}(X) = \frac{d}{dt}|_0 E_1^{\mathcal{P}}(\sigma_t) = \frac{d}{dt}|_0 \sigma_t(1) = X(\sigma, 1) = 0.$$

So by Rank-Nullity theorem, $E_{1*\sigma}^{\mathcal{P}}$ *is not surjective.*

The problem comes from the conjugate points on M. Two points p and q are conjugate points along a geodesic σ if there exists non-zero Jacobi field (smooth vector field along σ satisfying Jacobi equation) that vanishes at p and q. This fact will allow the kernel of $E_{1*}^{\mathcal{P}}$ to be “overly large ”(more accurately dimension exceeds $(n-1)\,d$), so by Rank-nullity theorem, E_{1*} can not be surjective. In this dissertation we consider manifolds with non–positive sectional curvature. These manifolds do not have conjugate points. From the next proposition we will see that $E_1^{\mathcal{P}}$ is a submersion on these manifolds.

Notation 3.9 *We construct a $G_{\mathcal{P}}^1$–orthonormal frame*

$$\left\{X^{h_{\alpha,i}} : 1 \leq \alpha \leq d, 1 \leq i \leq n\right\}$$

of $H_{\mathcal{P}}(M)$ as follows,

$$h_{\alpha,i} \in H_{\mathcal{P},\sigma} \text{ and } h'_{\alpha,i}(s_j+) = \frac{\delta_{i-1,j} e_\alpha}{\sqrt{\Delta_{j+1}}} \text{ for } j = 0, ..., n-1 \tag{3.7}$$

where the definition of $H_{\mathcal{P},\sigma}$ can be found in Definition 2.8.

Remark 3.10 *It is not hard to see using Proposition 5.1 that*

$$h_{\alpha,i}(s) = \frac{1}{\sqrt{n}} f_{\mathcal{P},i}(s) e_\alpha \tag{3.8}$$

where $\{f_{\mathcal{P},i}(s)\}$ is given in Definition 2.42.

Proposition 3.11 *If M is complete with non-positive sectional curvature, then for any $x \in M$, $H_{\mathcal{P},x}(M) := (E_1^{\mathcal{P}})^{-1}(\{x\})$ is an embedded submanifold of $H_{\mathcal{P}}(M)$.*

Proof. It suffices to show $E_1^{\mathcal{P}}$ is a submersion. Since M is complete, for any $y \in M$, there exists a geodesic σ parametrized on $[0,1]$ and connecting o and y. So $E_1^{\mathcal{P}}$ is surjective. To show $E_{1*}^{\mathcal{P}}$ is surjective, we use a class of vector fields $\left\{X^{h_{\alpha,n}}\right\}_{\alpha=1}^d$ in Notation 3.9. Since

$$E_{1*}^{\mathcal{P}}\left(X^{h_{\alpha,n}}\right) = X_1^{h_{\alpha,n}} = \sqrt{n} u(1) S_{\mathcal{P},n} e_\alpha$$

where $u(\cdot) = u(\sigma, \cdot)$ is the horizontal lift of $\sigma \in H_{\mathcal{P}}(M)$. From Proposition B.2

we know $S_{\mathcal{P},n}$ is invertible, therefore $\left\{E_{1\,*}^{\mathcal{P}}\left(X^{h_{\alpha,n}}\right)\right\}_{\alpha=1}^{d}$ spans $T_{E_1^{\mathcal{P}}(\sigma)}M$. So $E_{1\,*}^{\mathcal{P}}$ is surjective. ■

Since $H_{\mathcal{P},x}(M)$ is an embedded submanifold of $H_{\mathcal{P}}(M)$, we can restrict the Riemannian metric $G_{\mathcal{P}}^{1}$ on $TH_{\mathcal{P}}(M)$ in Eq. (1.7) to a Riemannian metric on $TH_{\mathcal{P},x}(M)$.

Definition 3.12 *Assuming M has non-positive sectional curvature, for any $x \in M$, let $G_{\mathcal{P},x}^{1}$ be the restriction of $G_{\mathcal{P}}^{1}$ to $T_{\sigma}H_{\mathcal{P},x}(M) \subset T_{\sigma}H_{\mathcal{P}}(M)$. Further, let $\mathrm{vol}_{G_{\mathcal{P},x}^{1}}$ be the associated volume measure on $H_{\mathcal{P},x}(M)$.*

Based on the Volume measure $vol_{G_{\mathcal{P},x}^{1}}$ on $H_{\mathcal{P},x}(M)$, we can construct the pinned approximate measure $\nu_{\mathcal{P},x}^{1}$:

Definition 3.13 *Let $\nu_{\mathcal{P},x}^{1}$ be the measure on $H_{\mathcal{P},x}(M)$ defined by*

$$d\nu_{\mathcal{P},x}^{1}(\sigma) = \frac{1}{J_{\mathcal{P}}(\sigma)}\frac{1}{Z_{\mathcal{P}}^{1}}e^{\frac{-E(\sigma)}{2}}dvol_{G_{\mathcal{P},x}^{1}}(\sigma) \tag{3.9}$$

where $J_{\mathcal{P}}(\sigma) := \sqrt{\det\left(E_{1\,*\sigma}^{\mathcal{P}}E_{1\,*\sigma}^{\mathcal{P}\,tr}\right)}$ and $Z_{\mathcal{P}}^{1} := (2\pi)^{\frac{dn}{2}}$.

3.3 Continuous Dependence of $\nu_{\mathcal{P},x}^{1}$ on x

Recall that a Hadamard manifold is a simply connected complete manifold with non–positive sectional curvature. Throughout this section we assume M is a Hadamard manifold whose sectional curvature is bounded below by $-N$. The following theorem illustrates that the measures, $\nu_{\mathcal{P},x}^{1}$, are finite and "continuously varying"with respect to x.

Notation 3.14 *We will denote by $C_b(Y)$ bounded continuous functions on a topological space Y.*

Theorem 3.15 *For any $x \in M$, $\nu^1_{\mathcal{P},x}$ is a finite measure. Morover, for any $f \in C_b(H_{\mathcal{P},x}(M))$, define*

$$h_{\mathcal{P}}(x) := \int_{H_{\mathcal{P},x}(M)} f(\sigma)\, d\nu^1_{\mathcal{P},x}(\sigma). \tag{3.10}$$

If the mesh size of $|\mathcal{P}| := \frac{1}{n}$ of the partition $\mathcal{P}$ is small enough, i.e. $n \geq 3dN$, then $h_{\mathcal{P}}(x) \in C(M)$.

Before proving this theorem, we need to set up some notations and auxiliary results.

Notation 3.16 *We fix $n \in \mathbb{N}$ and let $s_i := \frac{i}{n}$ and $\tau := 1 - \frac{1}{n} = s_{n-1}$. We further define $\mathcal{K} := H_{\mathcal{P}}([0,\tau], M)$ be the space of piecewise geodesic paths, $\sigma : [0,\tau] \to M$ such that $\sigma(0) = o \in M$.*

Lemma 3.17 *For $x, y \in M$, we can choose an unique element $\log_x(y) \in T_xM$ so that*

$$\gamma_{y,x}(t) := \exp_x\left((t-\tau)\frac{1}{n}\log_x(y)\right),$$

is the unique minimal-lengh-geodesic connecting x to y such that $\gamma_{y,x}(\tau) = x$ and $\gamma_{y,x}(1) = y$.

Proof. Since M is a Hadamard manifold, by the Theorem of Hadamard (See Theorem A.2 in Appendix A), $\exp_x : T_xM \to M$ is a diffeomorphism. Therefore we can see that $\log_x(y) = \exp_x^{-1}(y)$ is unique and it follows that the geodesic $\gamma_{y,x}$ is unique. ■

Definition 3.18 *For any given $y \in M$, let $\psi_y : \mathcal{K} \to H_{\mathcal{P},y}(M) := \left(E_1^{\mathcal{P}}\right)^{-1}(\{y\})$ be defined by*

$$\psi_y(\sigma) := \gamma_{y,\sigma(\tau)} * \sigma$$

where

$$\left(\gamma_{y,\sigma(\tau)} * \sigma\right)(t) = \begin{cases} \sigma(t) & \text{if } 0 \leq t \leq \tau \\ \gamma_{y,\sigma(\tau)}(t) & \text{if } \tau \leq t \leq 1 \end{cases}.$$

Notation 3.19 *For any* $\sigma \in H_{\mathcal{P},y}(M)$*, let* $\xi_{y,\sigma} := u(\sigma,\tau)^{-1} \log_{\sigma(\tau)}(y) \in \mathbb{R}^d$ *and also let* $G(\sigma,s) := (C_y(\sigma,s), S_y(\sigma,s))^t \in \mathbb{R}^{2d\times d}$ *be the fundamental solution to the ODE:*

$$G'(\sigma,s) = \begin{pmatrix} 0 & I_{d\times d} \\ A_{\xi_y}(\sigma,s) & 0 \end{pmatrix} G(\sigma,s)$$

where $A_{\xi_y}(\sigma,s) = R_{u(\sigma,1-s)}(\xi_{y,\sigma},\cdot)\,\xi_{y,\sigma}$ *and* $0 \le s \le 1$*.*

The next lemma characterizes the differential of ψ_y:

Lemma 3.20 *Let* $\sigma \in \mathcal{K}$*, recall from Theorem 2.39 and Notation 2.40 that* $X^h(\sigma,\cdot) = u(\sigma,\cdot)\,h(\sigma,\cdot) \in T_\sigma\mathcal{K}$ *iff* $h(\sigma,\cdot)$ *satisfies the piecewise Jacobi equation as in (2.41). Then*

$$\psi_{y*}\left(X^h(\sigma,\cdot)\right) = X^{\hat{h}}(\psi_y(\sigma),\cdot) := u(\psi_y(\sigma),\cdot)\,\hat{h}(\psi_y(\sigma),\cdot)$$

where

$$\hat{h}(\psi_y(\sigma),s) = \begin{cases} h(\psi_y(\sigma),s) & s \in [0,\tau] \\ S_y(\psi_y(\sigma),1-s)\,S_y\left(\psi_y(\sigma),\frac{1}{n}\right)^{-1} h(\sigma,\tau) & s \in [\tau,1] \end{cases}. \tag{3.11}$$

Proof. From now on we will suppress the path argument $\psi_y(\sigma)$ in $\hat{h}$. Suppose that $t \to \sigma_t \in \mathcal{K}$ is an one-parameter family of curves in $\mathcal{K}$ such that $\sigma_0 = \sigma$ and $\frac{d}{dt}|_0\sigma_t = X^h(\sigma)$. Then we have

$$\psi_{y*}\left(X^h(\sigma)\right) = \frac{d}{dt}|_0\psi_y(\sigma_t) = \frac{d}{dt}|_0\gamma_{y,\sigma_t(\tau)} * \sigma_t.$$

If $s \in [0,\tau]$, then

$$\frac{d}{dt}|_0\left(\gamma_{y,\sigma_t(\tau)} * \sigma_t\right)(s) = \frac{d}{dt}|_0\sigma_t(s) = X^h_s(\sigma).$$

While if $s \in [\tau, 1]$ we have

$$\frac{d}{dt}|_0 \left(\gamma_{y,\sigma_t(\tau)} * \sigma_t\right)(s) = \frac{d}{dt}|_0 \gamma_{y,\sigma_t(\tau)}(t) =: X_s^{\hat{h}}\left(\psi_y(\sigma)\right)$$

We know that $X_s^{\hat{h}}$ is determined by,

1. $\hat{h}$ satisfies Jacobi's equation,

2. $\hat{h}(\tau) = h(\tau)$ and $\hat{h}(1) = 0$.

Denote $\hat{h}(s)$ by $g(1-s)$ for $s \in [\tau, 1]$, the above conditions are equivalent to g being the solution to the following boundary value problem:

$$\begin{cases} g''(s) = A_{\xi_y}(s)\, g(s) \\ g(0) = 0 \\ g\left(\frac{1}{n}\right) = h(\tau) \end{cases} .$$

Then we use $S_y(\cdot)$ to express the solution. Here we have used Proposition B.2 to see that $S_y(s)$ is invertible for $s \in \left[0, \frac{1}{n}\right]$, therefore

$$g(s) = S_y(s)\, S_y\left(\frac{1}{n}\right)^{-1} h(\tau) \ \text{ for } s \in [0, \tau]$$

and thus

$$\hat{h}(s) = g(1-s) = S_y(1-s)\, S_y\left(\frac{1}{n}\right)^{-1} h(\tau) \ \text{ for } s \in [\tau, 1] .$$

■

Corollary 3.21 *For any $y \in M$, ψ_y is a diffeomorphism.*

Proof. From Lemma 3.20 it is easy to see that the push forward $(\psi_y)_*$ of ψ_y is one to one and thus an isomorphism since $\dim(\mathcal{K}) = \dim\left(H_{\mathcal{P},y}(M)\right)$. Therefore the inverse function theorem implies that ψ_y is a local diffeomorphism. Furthermore, M being a Hadamard manifold implies that ψ_y is bijective, so ψ_y is actually a diffeomorphism. ■

Remark 3.22 *An orthonormal frame* $\left\{X^{h_{\alpha,i}} : 1 \leq \alpha \leq d, 1 \leq i \leq n-1\right\}$ *of* $\mathcal{K}$ *can be constructed similarly to Notation* 3.9*,*

$$h_{\alpha,i} \in H_{\mathcal{P},\sigma} \text{ and } h'_{\alpha,i}(s_j+) = \frac{\delta_{i-1,j}e_\alpha}{\sqrt{\Delta_{j+1}}} \text{ for } j = 0, ..., n-2.$$

In this chapter we will use the same notation for both these two sets of orthonormal frames as the meaning should be clear from the context.

Definition 3.23 $f : M \to N$ *is a differentiable map between two Riemannian manifolds* M, N*. The* ***Normal Jacobian*** *of* f *is defined to be* $\sqrt{\det\left(f_* f_*^{tr}\right)}$*.*

We will use the orthonormal basis $\left\{X^{h_{\alpha,i}} : 1 \leq \alpha \leq d, 1 \leq i \leq n-1\right\}$ of $\mathcal{K}$ to estimate the Normal Jacobian $J_{\mathcal{P}}$ of E_1 in Lemma 3.24 and the "volume change "V_x (See precise definition in Lemma 3.26) brought by the diffeomorphism ψ_x in Lemma 3.26 and 3.27.

Lemma 3.24 *If* $J_{\mathcal{P}} := \sqrt{\det E^{\mathcal{P}}_{1\,*}\left(E^{\mathcal{P}}_{1\,*}\right)^{tr}}$ *is the Normal Jacobian of* $E_1^{\mathcal{P}}$*, then*

$$J_{\mathcal{P}}(\sigma) = \sqrt{\det\left(\frac{1}{n}\sum_{i=1}^{n} f_{\mathcal{P},i}(\sigma,1)\, f_{\mathcal{P},i}^{tr}(\sigma,1)\right)}\ \forall \sigma \in H_{\mathcal{P}}(M).$$

Proof. Using

$$E^{\mathcal{P}}_{1\,*\sigma} X^h(\sigma) = X^h(\sigma, 1),$$

if $v \in T_{E_1^{\mathcal{P}}(\sigma)}M$, then

$$\left\langle \left(E^{\mathcal{P}}_{1\,*}\right)^{tr} v, X^h\right\rangle_{G^1_{\mathcal{P}}} = \left\langle v, E^{\mathcal{P}}_{1\,*} X^h\right\rangle_{T_{E_1(\sigma)}M} = \left\langle u(1)^{-1} v, h(1)\right\rangle_{\mathbb{R}^d}.$$

Therefore, using the orthonormal frame of $TH_{\mathcal{P}}(M)$ given by

$$\left\{X^{h_{\alpha,i}} : 1 \le \alpha \le d, 1 \le i \le n\right\},$$

we find

$$\left(E^{\mathcal{P}}_{1\,*}\right)^{tr} v = \sum_{i,\alpha} \left\langle \left(E^{\mathcal{P}}_{1\,*}\right)^{tr} v, X^{h_{\alpha,i}} \right\rangle_{G^1_{\mathcal{P}}} X^{h_{\alpha,i}} = \sum_{i,\alpha} \left\langle u(1)^{-1} v, h_{\alpha,i}(1) \right\rangle_{\mathbb{R}^d} X^{h_{\alpha,i}}.$$

Let $\{e_\alpha\}_{\alpha=1}^d$ be the standard basis of $\mathbb{R}^d$, since $u(1)$ is an isometry, $\{u(1)\, e_\alpha\}_{\alpha=1}^d$ is an O.N. basis of $T_{E^{\mathcal{P}}_1(\sigma)}M$. Using

$$h_{k,i}(1) = \frac{1}{\sqrt{n}} f_{\mathcal{P},i}(1)\, e_k \text{ for } 1 \le k \le d,$$

we can compute:

$$\begin{aligned}
\det\left(E^{\mathcal{P}}_{1\,*}\left(E^{\mathcal{P}}_{1\,*}\right)^{tr}\right) &= \det\left\{\left\langle \left(E^{\mathcal{P}}_{1\,*}\right)^{tr} u(1)\, e_\alpha, \left(E^{\mathcal{P}}_{1\,*}\right)^{tr} u(1)\, e_\beta \right\rangle_{T_{E_1(\sigma)}M}\right\}_{\alpha,\beta} \\
&= \det\left\{\sum_{i=1}^n \sum_{\gamma=1}^d \left\langle h_{\gamma,i}(1), e_\alpha\right\rangle \left\langle h_{\gamma,i}(1), e_\beta\right\rangle\right\}_{\alpha,\beta} \\
&= \det\left\{\sum_{i=1}^n \sum_{\gamma=1}^d \frac{1}{n} \left\langle e_\gamma, f^{tr}_{\mathcal{P},i}(1)\, e_\alpha\right\rangle \left\langle e_\gamma, f^{tr}_{\mathcal{P},i}(1)\, e_\beta\right\rangle\right\}_{\alpha,\beta} \\
&= \det\left\{\sum_{i=1}^n \frac{1}{n} \left\langle f^{tr}_{\mathcal{P},i}(1)\, e_\alpha, f^{tr}_{\mathcal{P},i}(1)\, e_\beta\right\rangle\right\}_{\alpha,\beta} \\
&= \det\left(\frac{1}{n}\sum_{i=1}^n f_{\mathcal{P},i}(1)\, f^{tr}_{\mathcal{P},i}(1)\right).
\end{aligned}$$

■

Using the expression of $J_{\mathcal{P}}$ in Lemma 3.24, we can easily derive the following estimate.

Corollary 3.25 *Let $J_{\mathcal{P}}$ be defined as above, then for any $\sigma \in H_{\mathcal{P}}(M)$, $J_{\mathcal{P}}(\sigma) \ge 1$.*

Proof. For any $v \in \mathbb{C}^d$, using Proposition B.2, we have:

$$\begin{aligned}\left\langle \frac{1}{n}\sum_{i=1}^{n} f_{\mathcal{P},i}(\sigma,1)\, f_{\mathcal{P},i}^{tr}(\sigma,1)\, v, v\right\rangle &= \frac{1}{n}\sum_{i=1}^{n}\left\| f_{\mathcal{P},i}^{tr}(\sigma,1)\, v\right\|^2\\ &\geq \frac{1}{n}\sum_{i=1}^{n}\|v\|^2\\ &= \|v\|^2.\end{aligned}$$

So by Min-max theorem, $eig\left(\frac{1}{n}\sum_{i=1}^{n} f_{\mathcal{P},i}(\sigma,1)\, f_{\mathcal{P},i}^{tr}(\sigma,1)\right) \subset [1,+\infty)$ and therefore:

$$J_{\mathcal{P}}(\sigma) = \sqrt{\det\left(\frac{1}{n}\sum_{i=1}^{n} f_{\mathcal{P},i}(\sigma,1)\, f_{\mathcal{P},i}^{tr}(\sigma,1)\right)} \geq 1.$$

■

Lemma 3.26 *For any $\sigma \in \mathcal{K}$, let $V_x : \mathcal{K} \to \mathbb{R}_+$ be the normal Jacobian of $\psi_x : \mathcal{K} \to H_{\mathcal{P},x}(M)$, i.e. $V_x := \sqrt{\det\left((\psi_{x*})^{tr}\, \psi_{x*}\right)}$, then*

$$V_x(\sigma) = \sqrt{\det\left(I + L_x(\sigma)\, F_{\mathcal{P}}(\sigma)\, L_x(\sigma)^{tr}\right)}\ \forall \sigma \in \mathcal{K}, \tag{3.12}$$

where

$$L_x(\sigma) := C_x\left(\sigma, \frac{1}{n}\right) S_x\left(\sigma, \frac{1}{n}\right)^{-1}$$

and

$$F_{\mathcal{P}}(\sigma) := \frac{1}{n^2}\sum_{i=0}^{n-2} f_{\mathcal{P},i}(\sigma,\tau)\, f_{\mathcal{P},i}(\sigma,\tau)^{tr}$$

Proof. Using (3.11) and differentiating $\hat{h}$ with respect to s, we get:

$$\hat{h}'(\sigma,\tau+) = -C_x\left(\sigma,\frac{1}{n}\right) S_x\left(\sigma,\frac{1}{n}\right)^{-1} h(\sigma,\tau) := -L_x(\sigma)\, h(\sigma,\tau) \tag{3.13}$$

Also since from Proposition 5.1,

$$h(\sigma,\tau)=\frac{1}{n}\sum_{i=0}^{n-1}f_{\mathcal{P},i+1}(\sigma,\tau)\,h'(\sigma,s_i+),$$

so we have

$$\hat{h}'(\sigma,\tau+)=-L_x(\sigma)\frac{1}{n}\sum_{i=0}^{n-1}f_{\mathcal{P},i+1}(\sigma,\tau)\,h'(\sigma,s_i+).\tag{3.14}$$

For any $\alpha,\beta\in\{1,...,d\}$ and $i,j\in\{1,...,n-1\}$:

$$\left\langle\psi_{x*}\left(X^{h_{\alpha,i}}(\sigma)\right),\psi_{x*}\left(X^{h_{\beta,j}}(\sigma)\right)\right\rangle_{T_{\psi_x(\sigma)}H_{\mathcal{P},x}(M)}\tag{3.15}$$

$$=\frac{1}{n}\sum_{k=0}^{n-2}\left\langle h'_{\alpha,i}(s_{k+}),h'_{\beta,j}(s_{k+})\right\rangle+\frac{1}{n}\left\langle\hat{h}'_{\alpha,i}(\tau+),\hat{h}'_{\beta,j}(\tau+)\right\rangle\tag{3.16}$$

$$=\delta_{(\alpha,i)}^{(\beta,j)}+\frac{1}{n}\left\langle L_x(\sigma)\frac{1}{n}\frac{f_{\mathcal{P},i}(\tau)\,e_\alpha}{\sqrt{\frac{1}{n}}},L_x(\sigma)\frac{1}{n}\frac{f_{\mathcal{P},j}(\tau)\,e_\beta}{\sqrt{\frac{1}{n}}}\right\rangle\tag{3.17}$$

$$=\delta_{(\alpha,i)}^{(\beta,j)}+\left\langle L_x(\sigma)\frac{1}{n}f_{\mathcal{P},i}(\tau)\,e_\alpha,L_x(\sigma)\frac{1}{n}f_{\mathcal{P},j}(\tau)\,e_\beta\right\rangle,\tag{3.18}$$

where

$$\delta_{(\alpha,i)}^{(\beta,j)}=\begin{cases}1 & \alpha=\beta,i=j\\ 0 & \text{otherwise.}\end{cases}$$

It follows that the volume change

$$V_x(\sigma)=\sqrt{\det\left(I_{\left(\mathbb{R}^d\right)^{n-1}}+\hat{T}_x(\sigma)\right)}\tag{3.19}$$

where $\hat{T}_x(\sigma)\in End\left(\left(\mathbb{R}^d\right)^{n-1}\right)$ is defined by

$$\left(\hat{T}_x(\sigma)\right)_{d(i-1)+\alpha,d(j-1)+\beta}=\left\langle L_x(\sigma)\frac{1}{n}f_{\mathcal{P},i}(\sigma,\tau)\,e_\alpha,L_x(\sigma)\frac{1}{n}f_{\mathcal{P},j}(\sigma,\tau)\,e_\beta\right\rangle.$$

If

$$S_\sigma = \begin{pmatrix} I_{\left(\mathbb{R}^d\right)^{n-1}} \\ A_x(\sigma) \end{pmatrix} \in M_{nd\times(n-1)d}$$

and

$$A_x(\sigma) = \left(\frac{1}{n}L_x(\sigma) f_{\mathcal{P},0}(\sigma,\tau) e_1, \cdots, \frac{1}{n}L_x(\sigma) f_{\mathcal{P},n-2}(\sigma,\tau) e_d\right) \in M_{d\times(n-1)d},$$

then

$$I_{\left(\mathbb{R}^d\right)^{n-1}} + \hat{T}_x(\sigma) = S_\sigma^{tr} S_\sigma.$$

Apply Lemma D.1 we get:

$$\begin{aligned} \det\left(I_{\left(\mathbb{R}^d\right)^{n-1}} + \hat{T}_x(\sigma)\right) &= \det\left(I_{\left(\mathbb{R}^d\right)} + A_x(\sigma) A_x(\sigma)^{tr}\right) \\ &= \det\left(I + \frac{1}{n^2}\sum_{i=0}^{n-2}\sum_{\alpha=1}^{d} L_x f_{\mathcal{P},i}(\tau) e_\alpha e_\alpha^{tr} f_{\mathcal{P},i}(\tau)^{tr} L_x^{tr}\right) \\ &= \det\left(I + L_x F_{\mathcal{P}} L_x^{tr}\right) \end{aligned}$$

where $F_{\mathcal{P}}(\sigma)$ is as in Eq. (3.26). ■

Lemma 3.27 *For any $\sigma \in \mathcal{K}$,*

$$V_x(\sigma) \le \sum_{k=0}^{d} \binom{d}{k} n^{\frac{k}{2}} e^{\frac{Nk}{2} d^2(\sigma(\tau),x)} \Pi_{j=0}^{n-2} e^{kNd^2(\sigma(s_j),\sigma(s_{j+1}))} \tag{3.20}$$

Proof. From Lemma 3.26 and Appendix D, one can see, after suppressing σ,

$$\begin{aligned} \det\left(I_{\left(\mathbb{R}^d\right)^{n-1}} + \hat{T}_x\right) &= \det\left(I + L_x F_{\mathcal{P}} L_x^{tr}\right) \\ &= \Pi_{i=1}^{d}\left(1 + \lambda_{i,x}\right) \\ &\le \left(1 + \max_{1\le i\le d} \lambda_{i,x}\right)^d \end{aligned}$$

where $\{\lambda_{i,x}\} = eig\left(L_x F_{\mathcal{P}} L_x^{tr}\right)$.

Notice that

$$\max_{1\le i\le d} \lambda_{i,x} = \left\| L_x(\sigma) F_{\mathcal{P}} L_x(\sigma)^{tr} \right\| \le \left\| L_x(\sigma) \right\|^2 \left\| F_{\mathcal{P}} \right\| \\ \le \frac{1}{n} \left\| L_x(\sigma) \right\|^2 \sup_{0\le i\le n-2} \left\| f_{\mathcal{P},i}(\tau) \right\|^2.$$

Using Proposition B.4, we get:

$$\left\| C_x\left(\sigma, \frac{1}{n}\right) \right\| \le e^{\frac{N}{2} d^2(\sigma(\tau),x)},$$

where for any $x, y \in M$, $d(x,y)$ is the geodesic distance between x and y, and

$$\left\| S_x^{-1}\left(\sigma, \frac{1}{n}\right) \right\| \le n,$$

and so

$$\left\| L_x(\sigma) \right\|^2 \le n^2 e^{N d^2(\sigma(\tau),x)}$$

and

$$\max_{1\le i\le d} \lambda_{i,x} \le n e^{N d^2(\sigma(\tau),x)} \sup_{0\le i\le n-2} \left\| f_{\mathcal{P},i}(\sigma,\tau) \right\|^2.$$

Therefore

$$V_x(\sigma) = \left(1 + \max_{1\le i\le d} \lambda_{i,x}\right)^{\frac{d}{2}} \le \left(1 + n e^{N d^2(\sigma(\tau),x)} \sup_{0\le i\le n-2} \left\| f_{\mathcal{P},i}(\sigma,\tau) \right\|^2\right)^{\frac{d}{2}} \\ \le \left(1 + n^{\frac{1}{2}} e^{\frac{N}{2} d^2(\sigma(\tau),x)} \sup_{0\le i\le n-2} \left\| f_{\mathcal{P},i}(\sigma,\tau) \right\|\right)^{d} \\ = \sum_{k=0}^{d} \binom{d}{k} n^{\frac{k}{2}} e^{\frac{Nk}{2} d^2(\sigma(\tau),x)} \sup_{0\le i\le n-2} \left\| f_{\mathcal{P},i}(\sigma,\tau) \right\|^k. \quad (3.21)$$

Applying Proposition B.4 to $f_{\mathcal{P},i}(\sigma,\tau)$ shows

$$\begin{aligned}
&\|f_{\mathcal{P},i}(\tau)\| \leq \|C_{\mathcal{P},n-1}\|\cdots\|C_{\mathcal{P},i+1}\|\left\|\frac{S_i}{\Delta_i}\right\| \\
&\leq e^{\frac{1}{2}Nd^2(\sigma(s_{n-2}),\sigma(s_{n-1}))}\cdot\dots\cdot e^{\frac{1}{2}Nd^2(\sigma(s_{i-1}),\sigma(s_i))}\left(1+\frac{Nd^2(\sigma(s_{i-1}),\sigma(s_i))}{6}\right) \\
&\leq \Pi_{j=i-1}^{n-2} e^{\frac{1}{2}Nd^2(\sigma(s_j),\sigma(s_{j+1}))}\cdot e^{\frac{Nd^2(\sigma(s_{i-1}),\sigma(s_i))}{6}} \\
&\leq \Pi_{j=i-1}^{n-2} e^{Nd^2(\sigma(s_j),\sigma(s_{j+1}))} \\
&\leq \Pi_{j=0}^{n-2} e^{Nd^2(\sigma(s_j),\sigma(s_{j+1}))}
\end{aligned}$$

Taking supremum over i, we get:

$$\sup_{0\leq i\leq n-2}\|f_{\mathcal{P},i}(\sigma,\tau)\| \leq \Pi_{j=0}^{n-2} e^{Nd^2(\sigma(s_j),\sigma(s_{j+1}))}. \tag{3.22}$$

and (3.20) follows. ■

Definition 3.28 *For any X, $Y \in T\mathcal{K}$(the tangent bundle of $\mathcal{K}$), define $G^0_{\mathcal{P},\tau}$, $G^1_{\mathcal{P},\tau}$ to be:*

$$G^0_{\mathcal{P},\tau}(X,Y) = \sum_{i=1}^{n-1}\langle X(s_i), Y(s_i)\rangle\Delta_i$$

and

$$G^1_{\mathcal{P},\tau}(X,Y) = \sum_{i=1}^{n-1}\left\langle\frac{\nabla X}{ds}(s_{i-1}), \frac{\nabla Y}{ds}(s_{i-1})\right\rangle\Delta_i.$$

Lemma 3.29 *$G^0_{\mathcal{P},\tau}$ is a metric on $\mathcal{K}$.*

Proof. The only non–trivial part is to check $G^1_{\mathcal{P},\tau}(X,X) = 0 \implies X = 0$. Since M has non–positive curvature, there are no conjugate points. For each $0 \leq i \leq n-1$, there is a unique Jacobi field X connecting $\sigma(s_i)$ and $\sigma(s_{i+1})$ with specified $X(s_i)$ and $\frac{\nabla Y}{ds}(s_i)$. $G^1_{\mathcal{P},\tau}(X,X) = 0 \implies \frac{\nabla Y}{ds}(s_i) = 0$ for any $1 \leq i \leq n$. Notice that $X(0) = 0$, so by the uniqueness of Jacobi field, $X \equiv 0$. ■

Remark 3.30 *Since M has non-positive curvatures, $G^0_{\mathcal{P},\tau}$ is indeed a metric on $\mathcal{K}$ since the only one-paramater family of geodesics with fixed end points is a constant* *family consisting of the unique geodesic connecting the starting point and the ending point.*

Based on the metric $G^0_{\mathcal{P},\tau}$ and $G^1_{\mathcal{P},\tau}$, we define measures $\nu^0_{\mathcal{P},\tau}$ and $\nu^1_{\mathcal{P},\tau}$ on $\mathcal{K}$ as follows.

Definition 3.31 *Let*

$$d\nu^0_{\mathcal{P},\tau} := \frac{n^{(n-1)d}}{(2\pi)^{(n-1)\frac{d}{2}}} e^{-\frac{1}{2}E} dvol_{G^0_{\mathcal{P},\tau}}$$

and

$$d\nu^1_{\mathcal{P},\tau} = \frac{1}{(2\pi)^{(n-1)\frac{d}{2}}} e^{-\frac{1}{2}E} dvol_{G^1_{\mathcal{P},\tau}}$$

Lemma 3.32 *If*

$$\rho_{\mathcal{P}}(\sigma) := \Pi_{i=1}^{n-1} \det\left(\frac{S_{\mathcal{P},i}(\sigma)}{n}\right) \ \forall \sigma \in \mathcal{K},$$

then $d\nu^0_{\mathcal{P},\tau} = \rho_{\mathcal{P}} d\nu^1_{\mathcal{P},\tau}$ *and moreover,* $\rho_{\mathcal{P}}(\sigma) \geq 1 \ \forall \sigma \in \mathcal{K}$.

Proof. The argument to show $\rho_{\mathcal{P}}$ is the density of $\nu^0_{\mathcal{P},\tau}$ with respect to $\nu^1_{\mathcal{P},\tau}$ is almost exactly the same as Theorem 5.9 in [2] with a slight change of ending point from 1 to τ. Here we focus on the lower bound estimate of $\rho_{\mathcal{P}}(\sigma)$. Since for any $v \in \mathbb{C}^d$,

$$\left\| \frac{S_{\mathcal{P},i}}{n} v \right\| \geq \|v\|,$$

we know from propsition B.2 that for any $\lambda \in eig\left(\frac{S_{\mathcal{P},i}}{n}\right)$,

$$|\lambda| \geq 1$$

And from which we know:

$$\rho_{\mathcal{P}}(\sigma) = \Pi_{i=1}^{n-1} \det\left(\frac{S_{\mathcal{P},i}(\sigma)}{n}\right) \geq 1.$$

■

Proof of Theorem 3.15. Since ψ_x is a diffeomorphism, apply Theorem C.1 and we have:

$$h_{\mathcal{P}}(x) = \int_{H_{\mathcal{P},x}(M)} \frac{1}{Z^1_{\mathcal{P}}} \frac{f}{J_{\mathcal{P}}}(\sigma) e^{-\frac{1}{2}E(\sigma)} dvol_{G^1_{\mathcal{P},x}}(\sigma) \tag{3.23}$$

$$= \int_{\mathcal{K}} \frac{1}{Z^1_{\mathcal{P}}} \frac{f}{J_{\mathcal{P}}} \circ \psi_x(\sigma) e^{-\frac{1}{2}E\circ\psi_x(\sigma)} V_x(\sigma) dvol_{G^1_{\mathcal{P},\tau}}(\sigma) \tag{3.24}$$

Notice that

$$\frac{1}{Z^1_{\mathcal{P}}} e^{-\frac{1}{2}E\circ\psi_x(\sigma)} = \frac{1}{(2\pi)^{\frac{d}{2}}} \frac{1}{(2\pi)^{(n-1)\frac{d}{2}}} e^{-\frac{1}{2}E(\sigma)} e^{-\frac{n}{2}d^2(\sigma(\tau),x)}, \tag{3.25}$$

So

$$h_{\mathcal{P}}(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathcal{K}} \frac{f}{J_{\mathcal{P}}} \circ \psi_x(\sigma) e^{-\frac{n}{2}d^2(\sigma(\tau),x)} V_x(\sigma) d\nu_{G^1_{\mathcal{P},\tau}}(\sigma) \tag{3.26}$$

Combine (3.21), (3.22) we know that:

$$e^{-\frac{n}{2}d^2(\sigma(\tau),x)} V_x(\sigma) \le \sum_{k=0}^{d} \binom{d}{k} n^{\frac{k}{2}} e^{\frac{Nk-n}{2}d^2(\sigma(\tau),x)} \Pi_{j=0}^{n-2} e^{Nd^2(\sigma(s_j),\sigma(s_{j+1}))} \tag{3.27}$$

So

$$\sup_{x\in M} e^{-\frac{n}{2}d^2(\sigma(\tau),x)} V_x(\sigma) \le \sup_{x\in M} e^{-\frac{n-Nk}{2}d^2(\sigma(\tau),x)} \sum_{k=0}^{d} \binom{d}{k} n^{\frac{k}{2}} \Pi_{j=0}^{n-2} e^{Nkd^2(\sigma(s_j),\sigma(s_{j+1}))} \tag{3.28}$$

When n is large enough, $n - Nk > 0$. Therefore $e^{-\frac{n-Nk}{2}d^2(\sigma(\tau),x)} \le 1$ and it suffices to show

$$\mathbb{E}_{\nu_{G^1_{\mathcal{P},\tau}}} \left[\sum_{k=0}^{d} \binom{d}{k} n^{\frac{k}{2}} \Pi_{j=0}^{n-2} e^{Nkd^2(\sigma(s_j),\sigma(s_{j+1}))} \right] < \infty. \tag{3.29}$$

For each $k \leq d$ we have:

$$\mathbb{E}_{\nu_{G^1_{\mathcal{P},\tau}}}\left[\binom{d}{k} n^{\frac{k}{2}} \Pi_{j=0}^{n-2} e^{Nkd^2(\sigma(s_j),\sigma(s_{j+1}))}\right] = C_n \mathbb{E}_\mu\left[\Pi_{j=0}^{n-2} e^{Nk|\Delta_{j+1}\beta|^2}\right] \tag{3.30}$$

where C_n is a generic constant.

Since for each j, $|\Delta_j\beta|^2 = \sum_{l=1}^d \left|(\Delta_j\beta)_l\right|^2$, where $\left\{(\Delta_j\beta)_l\right\}_{l=1}^d$ are coordinates of $\Delta_j\beta$, i.e. $\Delta_j\beta = \left((\Delta_j\beta)_1, \ldots, (\Delta_j\beta)_d\right)$. Since β is a Brownian motion on $\mathbb{R}^d$, $\left\{(\Delta_j\beta)_l\right\}_{l=1}^d$ are i.i.d with Gaussian distribution of mean 0 and variance $\frac{1}{n}$. Using Lemma B.1 in Appendix B, notice that $Nk < \frac{n}{2}$, we have

$$\mathbb{E}\left[e^{Nk|\Delta_j\beta|^2}\right] = \Pi_{l=1}^d \mathbb{E}\left[e^{Nk\left|(\Delta_j\beta)_l\right|^2}\right] = \left(1 - \frac{2kN}{n}\right)^{-\frac{d}{2}}$$

and thus the right–hand side of Eq. (3.30) is bounded (the bound here depends on n).

Since for any $\sigma \in \mathcal{K}$, $\frac{f}{J_\mathcal{P}} \circ \psi_x(\sigma) e^{-\frac{n}{2}d^2(\sigma(\tau),x)} V_x(\sigma)$ is continuous with respect to $x \in M$, so by dominated convergence theorem, $h_\mathcal{P}(x) \in C(M)$. ■

Not only can we show that $h_\mathcal{P}(x)$ is a continuous function, it is bounded uniformly in $x \in M$ and partition $\mathcal{P}$, as is shown in the following proposition.

Proposition 3.33 $\sup_\mathcal{P} \sup_{x\in M} \nu^1_{\mathcal{P},x}(H_{\mathcal{P},x}(M)) < \infty$.

Proof. Based on Eq. (3.26),

$$\nu^1_{\mathcal{P},x}(H_{\mathcal{P},x}(M)) \leq C_d \int_\mathcal{K} e^{-\frac{n}{2}d^2(\sigma(\tau),x)} V_x(\sigma)\, d\nu_{G^1_{\mathcal{P},\tau}}(\sigma) \tag{3.31}$$

Combine Eq.(3.21) and Eq.(3.22) we know that:

$$e^{-\frac{n}{2}d^2(\sigma(\tau),x)} V_x(\sigma) \leq \sum_{k=0}^d \binom{d}{k} n^{\frac{k}{2}} e^{\frac{Nk-n}{2}d^2(\sigma(\tau),x)} \Pi_{j=0}^{n-2} e^{Nd^2(\sigma(s_j),\sigma(s_{j+1}))} \tag{3.32}$$

For each $k \leq d$, apply Lemma 3.32, we have:

$$\mathbb{E}_{\nu_{G^1_{\mathcal{P},\tau}}}\left[e^{-\frac{n-Nk}{2}d^2(\sigma(\tau),x)}\binom{d}{k}n^{\frac{k}{2}}\Pi_{j=0}^{n-2}e^{Nkd^2(\sigma(s_j),\sigma(s_{j+1}))}\right] \tag{3.33}$$

$$= \binom{d}{k}n^{\frac{k}{2}}\int_{\mathcal{K}} e^{-\frac{n-Nk}{2}d^2(\sigma(\tau),x)}\Pi_{j=0}^{n-2}e^{Nkd^2(\sigma(s_j),\sigma(s_{j+1}))}d\nu_{G^1_{\mathcal{P},\tau}}(\sigma) \tag{3.34}$$

$$= \binom{d}{k}n^{\frac{k}{2}}\int_{\mathcal{K}} e^{-\frac{n-Nk}{2}d^2(\sigma(\tau),x)}\Pi_{j=0}^{n-2}e^{Nkd^2(\sigma(s_j),\sigma(s_{j+1}))}\frac{1}{\rho_{\mathcal{P}}(\sigma)}d\nu^0_{\mathcal{P},\tau}(\sigma) \tag{3.35}$$

$$\leq \binom{d}{k}n^{\frac{k}{2}}\int_{\mathcal{K}} e^{-\frac{n-Nk}{2}d^2(\sigma(\tau),x)}\Pi_{j=0}^{n-2}e^{Nkd^2(\sigma(s_j),\sigma(s_{j+1}))}d\nu^0_{\mathcal{P},\tau}(\sigma) \tag{3.36}$$

Now define the projection map $\pi_{\mathcal{P}} : \mathcal{K} \to M^{n-1}$, for any $\sigma \in \mathcal{K}$,

$$\pi_{\mathcal{P}}(\sigma) := (\sigma(s_1), \ldots, \sigma(s_{n-1})).$$

Since M is a Hadamard manifold, $\pi_{\mathcal{P}}$ is a diffeomorphism. From there one can get:

$$\binom{d}{k}n^{\frac{k}{2}}\int_{\mathcal{K}} e^{-\frac{n-Nk}{2}d^2(\sigma(\tau),x)}\Pi_{j=0}^{n-2}e^{Nkd^2(\sigma(s_j),\sigma(s_{j+1}))}d\nu^0_{\mathcal{P},\tau}(\sigma) \tag{3.37}$$

$$= \frac{\binom{d}{k}n^{\frac{k+(n-1)d}{2}}}{(2\pi)^{\frac{(n-1)d}{2}}}\int_{M^{n-1}} e^{-\frac{n-Nk}{2}d^2(x_{n-1},x)}\Pi_{j=0}^{n-2}e^{-\frac{1}{2}(n-2Nk)d^2(x_j,x_{j+1})}dx_1\cdots dx_{n-1} \tag{3.38}$$

Corollary 4.2 in [33] gives a lower bound of heat kernels of manifold M such that $Ric \geq (1-d)N$:

$$p_t(x,y) \geq (2\pi t)^{-\frac{d}{2}}e^{-\frac{\rho^2}{2t}}\left(\frac{\sinh\sqrt{N}\rho}{\sqrt{N}\rho}\right)^{\frac{1-d}{2}}e^{-Ct}$$

where N is the curvature bound and C is some constant depending only on d and N and $\rho = d(x,y)$. Using the fact that:

$$\frac{\sinh\sqrt{N}\rho}{\sqrt{N}\rho} \leq e^{\frac{N\rho^2}{2}}$$

It follows that

$$p_t(x,y) \geq (2\pi t)^{-\frac{d}{2}} e^{-\frac{1}{2}\left(\frac{1}{t}+\frac{N(d-1)}{2}\right)\rho^2} e^{-Ct}$$

let

$$t = \frac{1}{n-N_1}$$

where $N_1 = 2Nd + \frac{N(d-1)}{2}$.

We have, for any $j \in \{0, \ldots, n-1\}$:

$$e^{-\frac{1}{2}(n-2Nd)d^2(x_j,x_{j+1})} \leq e^{Ct} p_t(x_j, x_{j+1})(2\pi t)^{\frac{d}{2}}.$$

So

$$\begin{aligned}
&\frac{\binom{d}{k} n^{\frac{k+(n-1)d}{2}}}{(2\pi)^{\frac{(n-1)d}{2}}} \int_{M^{n-1}} \sup_{x\in M} e^{-\frac{n-2Nk}{2}d^2(x_{n-1},x)} \Pi_{j=0}^{n-2} e^{-\frac{1}{2}(n-2Nd)d^2(x_j,x_{j+1})} dx_1 \cdots dx_{n-1} \\
&\leq \frac{\binom{d}{k} n^{\frac{k+(n-1)d}{2}}}{(n-N_1)^{\frac{nd}{2}}} e^{C\frac{n}{n-N_1}} \int_{M^{n-1}} p_{\frac{1}{n-N_1}}(x_{n-1},x) \Pi_{j=0}^{n-2} p_{\frac{1}{n-N_1}}(x_j,x_{j+1}) dx_1 \cdots dx_{n-1} \\
&= \frac{\binom{d}{k} e^{\frac{Cn}{n-N_1}}}{n^{\frac{d-k}{2}}\left(1-\frac{N_1}{n}\right)^{\frac{nd}{2}}} \int_M p_{\frac{1}{n-N_1}}(x_{n-1},x)\, p_{\frac{n-1}{n-N_1}}(o,x_{n-1})\, dx_{n-1} \qquad (3.39) \\
&= \frac{\binom{d}{k} e^{\frac{Cn}{n-N_1}}}{n^{\frac{d-k}{2}}\left(1-\frac{N_1}{n}\right)^{\frac{nd}{2}}} p_{\frac{n}{n-N_1}}(o,x) \qquad (3.40)
\end{aligned}$$

Since the heat kernel is continuous w.r.t. to time, combine (3.36) ,(3.38) and (3.40), we get

$$\frac{\binom{d}{k} e^{\frac{Cn}{n-N_1}}}{n^{\frac{d-k}{2}}\left(1-\frac{N_1}{n}\right)^{\frac{nd}{2}}} p_{\frac{n}{n-N_1}}(0,x) \leq C.$$

and hence

$$\nu^1_{\mathcal{P},x}\left(H_{\mathcal{P},x}(M)\right) \leq C.$$

where C is a constant depending only on d and N. ■

Theorem 3.15 shows that the class of approximate pinned measures $\left\{\nu^1_{\mathcal{P},x}\right\}$

are finite measures and using the continuity result for $h_{\mathcal{P}}(x)$, one can see that $\nu^1_{\mathcal{P},x}$ is deserved to be formally expressed as $\delta_x(\sigma(1))\,\nu^1_{\mathcal{P}}$ and it should be interpreted in the sense of Corollary 3.35. First we state a co–area formula.

Theorem 3.34 (Theorem 2.3 in [15]) *Let H and M be two Riemannian manifolds with volume measures $dvol_H$ and $dvol_M$ respectively. If $p: H \to M$ is a smooth submersion, $g: H \to [0,\infty)$ is a density function, for each $x \in M$, let $dvol_{H_x}$ be the volume measure on $H_x := p^{-1}(\{x\})$ and $J(y) := \sqrt{\det\left(p_{*y}p_{*y}^{tr}\right)}$ on $y \in H_x$, then for any non–negative measurable function $f: H \to [0,\infty)$,*

$$\int_H (f \circ p)\, g dvol_H = \int_M dvol_M(x)\, f(x) \int_{H_x} \frac{1}{J(y)} g(y)\, dvol_{H_x}(y). \tag{3.41}$$

Corollary 3.35 *Denote by $\delta_x \in \mathcal{E}'(M)$ the delta–function at $x \in M$, then for any $\left\{\delta_x^{(m)}\right\} \subset C_0^\infty(M)$ such that*

$$\delta_x^{(m)} \to \delta_x \text{ in } \mathcal{E}'(M)$$

i.e. for any $h \in C^\infty(M)$, we have:

$$\lim_{m\to\infty} \int_M h(y)\, \delta_x^{(m)}(y)\, dy = \int_M h(y)\, \delta_x(y)\, dy =: h(x)$$

where dy is the volume measure on M. Then for any $f \in C_b^\infty(H_{\mathcal{P}}(M))$,

$$\lim_{m\to\infty} \int_{H_{\mathcal{P}}(M)} \delta_x^{(m)}(\sigma(1))\, f(\sigma)\, d\nu^1_{\mathcal{P}}(\sigma) = \int_{H_{\mathcal{P},x}(M)} f(\sigma)\, d\nu^1_{\mathcal{P},x}(\sigma).$$

Proof. Using the co-area formula in (3.41) with

$$(H, M, p, g, f) = \left(H_{\mathcal{P}}(M), M, E_1^{\mathcal{P}}, \frac{1}{Z^1_{\mathcal{P}}} e^{-\frac{E}{2}}, \delta_x^{(m)}(\sigma(1))\, f(\sigma)\right),$$

we have

$$\int_{H_{\mathcal{P}}(M)} \delta_x^{(m)}(\sigma(1)) f(\sigma) d\nu_{\mathcal{P}}^1(\sigma) = \int_M dy \delta_x^{(m)}(y) \int_{H_{\mathcal{P},y}(M)} f(\sigma) d\nu_{\mathcal{P},y}^1(\sigma)$$
$$= \int_M h_{\mathcal{P}}(y) \delta_x^{(m)}(y) dy$$

From Theorem 3.15 we know $h_{\mathcal{P}}(x) \in C(M)$, therefore:

$$\lim_{m\to\infty} \int_{H_{\mathcal{P}}(M)} \delta_x^{(m)}(\sigma(1)) f(\sigma) d\nu_{\mathcal{P}}^1(\sigma) = \lim_{m\to\infty} \int_M h_{\mathcal{P}}(y) \delta_x^{(m)}(y) dy$$
$$= h_{\mathcal{P}}(x)$$
$$= \int_{H_{\mathcal{P},x}(M)} f(\sigma) d\nu_{\mathcal{P},x}^1(\sigma).$$

■

Chapter 4

The Orthogonal Lift $\tilde{X}$ of X on $H(M)$ and Its Stochastic Extension

4.1 Damped Metrics and Adjoints

Definition 4.1 (α–inner product) *Let $\alpha(t) \in \operatorname{End}\left(\mathbb{R}^d\right)$ be a continuously varying matrix valued function. For $h, k \in H_0\left(\mathbb{R}^d\right)$ let*

$$\langle h, k \rangle_\alpha := \int_0^1 \left(\frac{d}{dt} h(t) + \alpha(t) h(t)\right) \cdot \left(\frac{d}{dt} k(t) + \alpha(t) k(t)\right) dt.$$

Remark 4.2 *We denote the norm induced by α–inner product by $\|\cdot\|_\alpha$, differenting from the notation $\|\cdot\|_{H_0\left(\mathbb{R}^d\right)}$ for the norm induced by the H^1– inner product: $\langle h, l \rangle_{H^1} = \int_0^1 h'(s) \cdot l'(s)\, ds$.*

For the moment, let $E_1 : H_0\left(\mathbb{R}^d\right) \to \mathbb{R}^d$ be the end point evaluation map in the case where $M = \mathbb{R}^d$. Let ${E_1}^* : \mathbb{R}^d \to H_0\left(\mathbb{R}^d\right)$ be the adjoint of E_1 with respect to the α–inner product, i.e. for any $a \in \mathbb{R}^d$ and $h \in H_0\left(\mathbb{R}^d\right)$,

$$\langle E_1 h, a\rangle_{\mathbb{R}^d} = \langle h, \left(E_1{}^*\right) a\rangle_{\alpha}.$$

The next theorem computes E_1^* which is crucial in constructing the orthogonal lift in Section 4.2.

Theorem 4.3 *Let $a \in \mathbb{R}^d$ and $\alpha(t)$ be as in Definition 4.1, then $E_1^* a \in H_0\left(\mathbb{R}^d\right)$ is given by*

$$\left(E_1^* a\right)(t) = \left(S(t)\int_0^t \left[S(s)^* S(s)\right]^{-1} S(1)^* \, ds\right) a. \tag{4.1}$$

where $S(t) \in \operatorname{Aut}\left(\mathbb{R}^d\right)$ solves

$$\frac{d}{dt} S(t) + \alpha(t) S(t) = 0 \text{ with } S(0) = I$$

and

$$v(t) = \left(\int_0^t \left[S(s)^* S(s)\right]^{-1} S(1)^* \, ds\right) a.$$

Proof. Notice that if $h(t) = S(t) w(t)$ with $w(\cdot) \in H_0\left(\mathbb{R}^d\right)$, then

$$\begin{aligned}
\left(\frac{d}{dt} + \alpha(t)\right) h(t) &= \left(\frac{d}{dt} + \alpha(t)\right)\left[S(t) w(t)\right] \\
&= \left[\left(\frac{d}{dt} + \alpha(t)\right) S(t)\right] w(t) + S(t)\dot{w}(t) \\
&= S(t)\dot{w}(t).
\end{aligned}$$

And in particular,

$$\langle Sv, Sw\rangle_{\alpha} = \int_0^1 S(t)\dot{v}(t) \cdot S(t)\dot{w}(t) \, dt.$$

Given $a \in \mathbb{R}^d$, let $w(t) = E_1^* a$ and define $v(t) := S(t)^{-1} w(t)$ so that $E_1^* a =$

$S(t)\,v(t)$. Then by the definition of the adjoint we find,

$$\int_0^1 S(t)\,\dot{v}(t)\cdot S(t)\,\dot{w}(t)\,dt = \langle Sv, Sw\rangle_\alpha = \langle E_1^* a, Sw\rangle_\alpha = a\cdot E_1(Sw)$$
$$= a\cdot S(1)\,w(1) = \int_0^1 S(1)^* a\cdot \dot{w}(t)\,dt$$

As $w\in H_0\left(\mathbb{R}^d\right)$ is arbitrary we may conclude that

$$S(t)^* S(t)\,\dot{v}(t) = S(1)^* a \implies v(t) = \int_0^t \left[S(s)^* S(s)\right]^{-1} S(1)^* a ds$$

which proves (4.1). ■

Theorem 4.4 *If $a\in\mathbb{R}^d$, then $h(\cdot)\in H_0\left(\mathbb{R}^d\right)$ defined by*

$$h(t) := S(t)\left(\int_0^t \left[S(s)^* S(s)\right]^{-1} ds\right)\left(\int_0^1 \left[S(s)^* S(s)\right]^{-1} ds\right)^{-1} S(1)^{-1} a, \quad (4.2)$$

is the minimal length element of $H_0\left(\mathbb{R}^d\right)$ such that $E_1 h = a$.

i.e.

$$\|h\|_\alpha = \inf\left\{\|k\|_\alpha \mid k(\cdot)\in H_0\left(\mathbb{R}^d\right),\ E_1 k = a\right\}.$$

Proof. Since $H_0\left(\mathbb{R}^d\right) = \mathrm{Nul}(E_1)^\perp \oplus \mathrm{Nul}(E_1)$, we have $E_1 h = a \implies E_1 h_k = a$ and $\|h\|_\alpha \geq \|h_k\|_\alpha$ where h_k is the orthogonal projection of h onto $\mathrm{Nul}(E_1)^\perp$. So we are looking for the element, $h\in H_0\left(\mathbb{R}^d\right)$, such that $E_1 h = a$ and $h\in \mathrm{Nul}(E_1)^\perp = \mathrm{Ran}\left({E_1}^*\right)$. In other words we should have $h = E_1^* v$ for some $v\in\mathbb{R}^d$. Thus, using (4.1), we need to demand that

$$a = E_1 {E_1}^* v = \left({E_1}^* v\right)(1) = \left(S(1)\int_0^1 \left[S(s)^* S(s)\right]^{-1} S(1)^* ds\right) v,$$

i.e.

$$v = \left(S(1)\int_0^1 \left[S(s)^* S(s)\right]^{-1} S(1)^* ds\right)^{-1} a.$$

It then follows that

$$h(t) = E_1{}^* \left(S(1) \int_0^1 [S(s)^* S(s)]^{-1} S(1)^* \, ds \right)^{-1} a$$
$$= \left(S(t) \int_0^t [S(s)^* S(s)]^{-1} S(1)^* \, ds \right) \left(S(1) \int_0^1 [S(s)^* S(s)]^{-1} S(1)^* \, ds \right)^{-1} a$$

which is equivalent to (4.2).

Alternative proof: Let $h := E_1{}^* a \in H_0(\mathbb{R}^d)$ and $k \in H_0(\mathbb{R}^d)$, then

$$a \cdot k(1) = a \cdot E_1(k) = \langle E_1{}^* a, k \rangle_\alpha = \langle h, k \rangle_\alpha$$
$$= \int_0^1 \left(\frac{d}{dt} h(t) + \alpha(t) h(t) \right) \cdot z(t) \, dt \tag{4.3}$$

where

$$\frac{d}{dt} k(t) + \alpha(t) k(t) =: z(t).$$

Solving the previous equation for k in terms of z gives,

$$k(t) = S(t) \int_0^t S(s)^{-1} z(s) \, ds.$$

Using this result with $t = 1$ back in (4.3) shows

$$\int_0^1 \left(\frac{d}{dt} h(t) + \alpha(t) h(t) \right) \cdot z(t) \, dt = a \cdot S(1) \int_0^1 S(s)^{-1} z(s) \, ds$$
$$= \int_0^1 S^*(s)^{-1} S(1)^* a \cdot z(s) \, ds.$$

As $z(s)$ is arbitrary in $L^2([0,1], \mathbb{R}^d)$ we may conclude that

$$\frac{d}{dt} h(t) + \alpha(t) h(t) = S^*(t)^{-1} S(1)^* a.$$

Solving this equation for h then shows,

$$\begin{aligned}(E_1{}^*a)(t) = h(t) &= S(t)\int_0^t S(s)^{-1} S^*(s)^{-1} S(1)^* a ds \\ &= \left(S(t)\left[\int_0^t S(s)^{-1} S^*(s)^{-1} ds\right] S(1)^*\right) a\end{aligned}$$

and so we again recover (4.1). ■

Remark 4.5 *The expression in (4.2) matches the well known result for damped metrics where* $\alpha = \frac{1}{2}\operatorname{Ric}_u$. *Further observe that if* $\alpha(t) = 0$ *(i.e. we are in the flat case) then* $S(t) = I$ *and the above expression reduces to* $h(t) = ta$ *as we know to be the correct result.*

Definition 4.6 *Let* $\langle \cdot, \cdot \rangle_{Ric_u}$ *be the* ***damped metric on*** $TH(M)$ *defined by*

$$\langle X, Y\rangle_{Ric_u} := \int_0^1 \left\langle \left[\frac{\nabla}{ds} + \frac{1}{2}Ric\right] X(s), \left[\frac{\nabla}{ds} + \frac{1}{2}Ric\right] Y(s)\right\rangle ds \tag{4.4}$$

for all $X, Y \in \Gamma_\sigma(TM) = T_\sigma H(M)$ *and* $\sigma \in H(M)$.

If $X = X^{J_1}$ and $Y = X^{J_2}$ with that $J_1, J_2 \in H_0(\mathbb{R}^d)$, then we have

$$\left\langle X^{J_1}, X^{J_2}\right\rangle_{Ric_u} = \int_0^1 \left\langle \left[\frac{d}{ds} + \frac{1}{2}Ric_{u_s}\right] J_1(s), \left[\frac{d}{ds} + \frac{1}{2}Ric_{u_s}\right] J_2(s)\right\rangle ds. \tag{4.5}$$

4.2 The Orthogonal Lift $\tilde{X}$ on $H(M)$

In this section we construct the orthogonal lift $\tilde{X} \in \Gamma(TH(M))$ of $X \in \Gamma(TM)$ which is defined to be the minimal length element in $\Gamma(TH(M))$ relative to the damped metric introduced in Definition 4.6.

Definition 4.7 *For each* $\sigma \in H(M)$, *recall that* $u_s(\sigma)$ *is the horizontal lift of* σ. *Denote by* $T_{(\cdot)} : H(M) \to End(\mathbb{R}^d)$ *the solution to the following initial value* *problem:*

$$\begin{cases} \frac{d}{ds}T_s + \frac{1}{2}Ric_{u_s}T_s = 0 \\ T_0 = I \end{cases} \tag{4.6}$$

Lemma 4.8 *For all* $s \in [0,1]$*,* T_s *is invertible. Further both* $\sup_{0\le s\le 1} \|T_s\|$ *and* $\sup_{0\le s\le 1} \|T_s^{-1}\|$ *are bounded by* $e^{\frac{1}{2}(d-1)N}$*, where* $(d-1)\,N$ *is a bound of* $\|\mathrm{Ric}\|$*.*

Proof. Let U_s solve the ODE,

$$\begin{cases} \frac{d}{ds}U_s = \frac{1}{2}U_s\,\mathrm{Ric}_{u_s} \\ U_0 = I. \end{cases} \tag{4.7}$$

Then one easily shows that

$$\frac{d}{ds}\left(U_sT_s\right) = 0 \implies U_sT_s = U_0T_0 = I$$

and this shows that U_s is a left inverse to T_s. As we are in finite dimensions it follows that T_s^{-1} exists and is equal to U_s. The stated bounds now follow by Gronwall's inequality. ∎

Definition 4.9 *Let* $\mathbf{K} : [0,1] \times H(M) \to End\left(\mathbb{R}^d\right)$ *be defined by*

$$\mathbf{K}_s := T_s\left[\int_0^s T_r^{-1}\left(T_r^{-1}\right)^* dr\right]T_1^*. \tag{4.8}$$

Remark 4.10 *A simple computation shows that* $\mathbf{K}_s$ *satisfies the following initial value problem:*

$$\begin{cases} \mathbf{K}_s' = & -\frac{1}{2}\,\mathrm{Ric}_{u_s}\mathbf{K}_s + \left(T_1T_s^{-1}\right)^* \\ \mathbf{K}_0 = & 0. \end{cases} \tag{4.9}$$

Conversely, from Duhamel's principle and (4.6) it is easy to deduce the formula in Definition 4.9.

Lemma 4.11 *With $\mathbf{K}_s$ as in Definition 4.9, $\mathbf{K}_1$ is invertible and $\left\|\mathbf{K}_1^{-1}\right\| \leq e^{(d-1)N}$, provided $\|\mathrm{Ric}\| \leq (d-1)\,N$.*

Proof. Since

$$\mathbf{K}_1 := \int_0^1 \left(T_1 T_r^{-1}\right)\left(T_1 T_r^{-1}\right)^* dr$$

is a symmetric positive semi-definite operator such that

$$\langle \mathbf{K}_1 v, v\rangle = \int_0^1 \left\|\left(T_1 T_r^{-1}\right)^* v\right\|^2 dr \ \forall v \in \mathbb{C}^d.$$

Apply Lemma 4.8 to the expression given;

$$\begin{aligned}\langle \mathbf{K}_1 v, v\rangle &\geq \int_0^1 e^{-(d-1)N} \left\|\left(T_r^{-1}\right)^* v\right\|^2 dr \\ &\geq \int_0^1 e^{-2(d-1)N} \|v\|^2 dr \\ &= e^{-2(d-1)N} \|v\|^2\end{aligned}$$

From which it follows that $eig\,(\mathbf{K}_1) \subset [e^{-(d-1)N}, \infty)$. ■

Definition 4.12 *Let $X \in \Gamma(TM)$, define two maps $H : H(M) \to \mathbb{R}^d$ and $J : [0,1] \times H(M) \to \mathbb{R}^d$ as follows,*

$$\tilde{H} = u_1^{-1}(\sigma)\, X \circ E_1(\sigma) \tag{4.10}$$

and

$$J(\sigma, s) := J_s(\sigma) := \mathbf{K}_s(\sigma)\, \mathbf{K}_1^{-1}(\sigma)\, H(\sigma)\,. \tag{4.11}$$

Theorem 4.13 *Given $X \in \Gamma(TM)$ the minimal length lift, $\tilde{X}$, relative to the damped metric in Definition 4.6 of X to $\Gamma(TH(M))$ is given by $\tilde{X} = X^J$. Further we know that J_s is the solution to the following ODE:*

$$J_s' = -\frac{1}{2} Ric_{u_s} J_s + \phi_s,\ \ J_0 = 0$$

where $\phi_s = (T_1 T_s^{-1})^* \mathbf{K}_1^{-1} H = (T_s^{-1})^* \left[\int_0^1 T_r^{-1} (T_r^{-1})^* dr\right]^{-1} T_1^{-1} H.$

Proof. Apply Theorem 4.4 with $\alpha_s = \frac{1}{2} Ric_{u_s}$. ■

Following the construction above, one can define an similar object (still denoted by $\tilde{X}$) on $W_o(M)$. Recall from Notation 2.25 that $\tilde{u}$ is the stochastic horizontal lift of the canonical Brownian motion Σ on M.

Definition 4.14 *Define* $\tilde{T}_{(\cdot)} : [0,1] \times W_o(M) \to End(\mathbb{R}^d)$ *to be the solution to the following initial value problem:*

$$\begin{cases} \frac{d}{ds}\tilde{T}_s + \frac{1}{2} Ric_{\tilde{u}_s} \tilde{T}_s = 0 \\ \tilde{T}_0 = I \end{cases} \tag{4.12}$$

Remark 4.15 *Following the same arguments used in Lemma 4.8 and 4.11, one can see the bounds obtained there still hold for* $\tilde{T}$ *and* $\tilde{\mathbf{K}}$*.*

Definition 4.16 *Using* $\tilde{T}_s$*, we define* $\tilde{\mathbf{K}} : [0,1] \times W_o(M) \to End(\mathbb{R}^d)$*:*

$$\tilde{\mathbf{K}}_s := \tilde{T}_s \left[\int_0^s \tilde{T}_r^{-1} \left(\tilde{T}_r^{-1}\right) dr\right] \tilde{T}_1^*. \tag{4.13}$$

Definition 4.17 *For each* $X \in \Gamma(TM)$ *define two maps* $\tilde{H} : W_o(M) \to \mathbb{R}^d$ *and* $\tilde{J} : W_o(M) \to H_0(\mathbb{R}^d)$ *by*

$$\tilde{H} = \tilde{u}_1^{-1} X \circ E_1 \tag{4.14}$$

and

$$\tilde{J}_s := \tilde{\mathbf{K}}_s \tilde{\mathbf{K}}_1^{-1} \tilde{H} \text{ for } s \in [0,1]. \tag{4.15}$$

Notation 4.18 *Given a measurable function* $h : W_o(M) \to H_0(\mathbb{R}^d)$*, let* $Z_h : W_o(M) \to H_0(\mathbb{R}^d)$ *be the solution to the following initial value problem:*

$$\begin{cases} Z_h{}'(s) = -\frac{1}{2} Ric_{\tilde{u}_s} Z_h(s) + h_s' \\ Z_h(0) = 0. \end{cases}$$

Definition 4.19 *For any* $X \in \Gamma(TM)$, *define*

$$\tilde{X}_s = X_s^{Z_\Phi} := \tilde{u}_s Z_\Phi(s) \text{ for } 0 \le s \le 1$$

where

$$\Phi_s = \int_0^s \left(\tilde{T}_\tau^{-1}\right)^* \left[\int_0^1 \left(\tilde{T}_r^* \tilde{T}_r\right)^{-1} dr\right]^{-1} \tilde{T}_1^{-1} \tilde{H} d\tau.$$

4.3 Review of Calculus on Wiener Space

In this section we interpret X^{Z_Φ} as a first order differential operator on some geometric Wiener functionals (see Definition 4.36). The main difficulty there is the non-adaptedness of Φ. To overcome this difficulty, we express X^{Z_Φ} in terms of geometric vector field (see Definition 4.27) with non–adapted coefficients. However, these coefficients are differentiable Wiener functionals in "Malliavin calculus "sense. Based on this observation we derive an integration–by–parts formula for X^{Z_Φ} which naturally shows X^{Z_Φ} is a closable first order differential operator on $L^2(W_o(M))$. The integration–by–parts formula will also be one of our main tool of dealing with δ—function pinning in this dissertation. We begin with a brief review of the classical theory of calculus on Wiener space that is needed in our work.

The first order differential geometry on path spaces that we will use can be traced back to the famous Cameron-Martin Theorem (see [6]).

Theorem 4.20 (Cameron-Martin) *For any* $h \in H_0(\mathbb{R}^d)$, *consider the flow* ϕ_t^h *generated by* h, *i.e. for any* $w \in W_0(\mathbb{R}^d)$, $\phi_t^h(w) = w + th$. *Notice that* ϕ_t^h *is the flow of the vector field* $D_h := \frac{\partial}{\partial h}$. *Then the pull–back measure* $\mu^h(\cdot) := \left(\phi_1^h\right)_* \mu(\cdot) = \mu(\cdot - h)$ *and Wiener measure* μ *are mutually absolutely continuous.*

The map ϕ_t^h is usually called Cameron-Martin shift and the phenomenon described in Theorem 4.20 is called quasi-invariance of μ under the Cameron-Martin shift.

The generalization of Cameron-Martin Theorem to path spaces on a manifold came quite a while later in 1990s. Driver initiated the geometric Cameron-Martin theory in [10] and [11] where he considered the "vector field "X^h(or more precisely an equivalence class of vector fields) on $W_o(M)$ defined as follows,

$$X_s^h(\sigma) = \tilde{u}_s(\sigma) h_s$$

where $h \in \{f \in C^1([0,1]) : f(0) = 0\} \subset H_0(\mathbb{R}^d)$.

Theorem 4.21 *Let (M, g, o, ∇) be a compact manifold and h be as above, then for any $\sigma \in W_o(M)$, there exists a unique flow ϕ_t^h of X^h, i.e. $\phi_t^h : W_o(M) \mapsto W_o(M)$ satisfying:*

$$\frac{d}{dt}\phi_t^h(\sigma) = X^h\left(\phi_t^h(\sigma)\right) \text{ with } \phi_0^h = I$$

and $\nu_t^h(\cdot) := \left(\phi_t^h\right)_* \nu$ is equivalent to ν.

The existence of the flow and the quasi-invariance of the Wiener measure were later extended to all Cameron-Martin vector field X^h, $h \in H_0(\mathbb{R}^d)$ in [19] and [14] and then to a geometrically and stochastically complete Riemannian manifold in [20] and [22]. Owing to the facts that Cameron-Martin vector fields do not form a Lie Algrbra and more general vector fields naturally appreared in practice, it is useful to introduce a broader class of so called "adapted vector fields ", see [12] and [7].

Definition 4.22 (Vector Valued Brownian Semimartingales) *V is a finite dimensional vector space. A function $f : W_o(M) \times [0,1] \to V$ is called a Brownian semimartingale if f has the following representation:*

$$f(s) = \int_0^s Q_\tau d\beta_\tau + \int_0^s r_\tau d\tau$$

where (Q_s, r_s) is a predictable process with values in $Hom\left(\mathbb{R}^d, V\right) \times V$, V is a vector space. We will call (Q_s, r_s) the kernels of f.

Definition 4.23 ($\mathcal{H}^q$ Space) *For each $q \geq 1$, $f : W_o(M) \times [0,1] \to V$ jointly measurable, we define the root mean square norm in $L^q(W_o(M), \nu)$ to be:*

$$\|f\|_{R^q(V)} \equiv \left\| \left(\int_0^1 |f(\cdot, s)|_V^2 \, ds \right)^{\frac{1}{2}} \right\|_{L^q(W_o(M),\nu)}$$

Let $\mathcal{H}^q$ be the space of all Brownian semimartingales such that

$$\|f\|_{\mathcal{H}^q} := \left\|Q^f\right\|_{R^q} + \left\|r^f\right\|_{R^q} < \infty$$

Definition 4.24 ($\mathcal{B}^q$ Space) *For each $q \geq 1$, $f : W_o(M) \times [0,1] \to V$ jointly measurable, we define the supremum norm in $L^q(W_o(M), \nu)$ to be:*

$$\|f\|_{S^q(V)} \equiv \|f^*\|_{L^q(W_o(M),\nu)}$$

where f^* is the essential supremum of $s \to f(\cdot, s)$ relative to Lebesque measure on $[0,1]$. Let $\mathcal{B}^q$ be the space of all Brownian semimartingales such that

$$\|f\|_{\mathcal{B}^q} := \left\|Q^f\right\|_{S^q} + \left\|r^f\right\|_{S^q} < \infty$$

Definition 4.25 (Adapted Vector Field) *An adapted vector field on $W_0(\mathbb{R}^d)$ is an $\mathbb{R}^d$–valued Brownian semimartingale with predictable kernels $Q_. \in \mathfrak{so}(d)$ and $r_. \in L^2[0,1]$ $\nu - a.s.$ We denote the space of* ***adapted vector fields*** *by $\mathcal{V}$ and $\mathcal{V}^q := \mathcal{V} \cap \mathcal{H}^q$.*

Notation 4.26 *We will use the following notation in this dissertation: $\mathcal{H}^{\infty-} := \cap_{q\geq 1}\mathcal{H}^q$, $\mathcal{B}^{\infty-} = \cap_{q\geq 1}\mathcal{B}^q$ and $\mathcal{V}^{\infty-} = \mathcal{V} \cap \mathcal{H}^{\infty-}$.*

A class of vector field called geometric vector field can be constructed using adapted vector fields.

Definition 4.27 (Geometric Vector Field) *For any $h \in \mathcal{V}$,*

$$X_s^h := \tilde{u}_s h_s \ 0 \le s \le 1$$

is said to be a ***geometric vector field****.*

Theorem 4.28 (Approximate Flow of Geometric Vector Field) *Let X^h be a geometric vector field as above with $h \in \mathcal{V} \cap \mathcal{S}^\infty \cap \mathcal{B}^\infty$, $t \in \mathbb{R}$, there exists a funcion $E\left(tX^h\right) : W_o(M) \to W_o(M)$ such that*

$$\frac{d}{dt}|_0 \, E\left(tX^h\right) = X^h \text{ in } \mathcal{B}^{\infty-}.$$

Proof. See Corollary 4.6 in [9]. ■

For a geometric vector field, one can not construct a real flow as is constructed for Cameron–Martin vector field in Theorem 4.21. However the theorem above gurantees we can view them as vector fields from a natural tangent vector point of view. In the next definition we specify a domain of these operators.

Notation 4.29 *In this chapter, we fix $\mathcal{D}(L)$ to be the domain of an operator L.*

Definition 4.30 *Given a geometric vector field X^h, let $\mathcal{D}\left(X^h\right)$ be*

$$\mathcal{D}\left(X^h\right) := \left\{ f : W_o(M) \to \mathbb{R} \mid X^h f := \frac{d}{dt}|_0 \, f\left(E\left(tX^h\right)\right) \in L^{\infty-}\left(W_o(M)\right) \right\}.$$

Notation 4.31 *Recall from Notation 4.18 that Z_h satisfies the following ODE,*

$$Z_h'(s) = -\frac{1}{2} Ric_{\tilde{u}_s} Z_h(s) + h_s' \text{ with } Z_h(0) = 0. \tag{4.16}$$

We will use Z_α as the shorthand of Z_h where $h_s = \int_0^s \left(\tilde{T}_r^{-1}\right)^* e_\alpha dr$, $1 \le \alpha \le d$.

Lemma 4.32 *Let X^{Z_α} be given above, then X^{Z_α} is a geometric vector field with $Z_\alpha \in \mathcal{V}^\infty \cap \mathcal{B}^\infty$.*

Proof. Recall that Z_α satisfies the following ODE:

$$Z'_\alpha(s) = -\frac{1}{2} Ric_{\tilde{u}_s} Z_\alpha(s) + \left(\tilde{T}_s^{-1}\right)^* e_\alpha \text{ with } Z_\alpha(0) = 0. \tag{4.17}$$

Since $\left(\tilde{T}_s^{-1}\right)^* e_\alpha$ is adapted, Z'_α is adapted. So Z_α is a Brownian semimartingale with $Q \equiv 0$ and $r = Z'_\alpha$. Since $\tilde{T}_s$ is bounded, from Gronwall's inequality we have Z_α is bounded, and the bound is independent of $\sigma \in W_o(M)$ and $s \in [0,1]$. Therefore $Z_\alpha \in \mathcal{V}^\infty \cap \mathcal{B}^\infty$. ■

The next theorem shows how to differentiate a cylinder function $f \in \mathcal{FC}$ along a geometric vector field.

Notation 4.33 *Given $k : W_o(M) \to H_0(\mathbb{R}^d)$, denote $\int_0^s R_{\tilde{u}_r}(k_r, \delta\beta_r)$ by $A_s\langle k\rangle$, where δ is the stratonovich differential.*

Notation 4.34 *Suppose $F \in C(\mathcal{O}(M)^n)$ and $\mathcal{P} = \{0 < s_1 < \cdots < s_n \le 1\}$ is a partition of $[0,1]$, set*

$$F(u) = F(u_{s_1}, \ldots, u_{s_n}),$$

then for $A : [0,1] \to \mathfrak{so}(d)$ and $h : [0,1] \to \mathbb{R}^d$, set

$$F'(u)\langle A + h\rangle := \frac{d}{dt}|_0 F\left(ue^{tA}\right) + \frac{d}{dt}|_0 F\left(e^{tB_h}(u)\right)$$

where $ue^{tA}(s) = u_s e^{tA_s} \in \mathcal{O}(M)$ and $e^{tB_h}(u)(s) = e^{tB_{h_s}}(u_s) \in \mathcal{O}(M)$.

Theorem 4.35 *Following Notation 2.33, if $h \in \mathcal{V}^\infty \cap \mathcal{B}^\infty$, then $\mathcal{FC}_b^1 \subset \mathcal{D}\left(X^h\right)$. In more detail, if $f = F(\tilde{u}) \in \mathcal{FC}_b^1$, then*

$$X^{Z_h} f = F'(\tilde{u})\langle -A\langle Z_h\rangle + Z_h\rangle \tag{4.18}$$

Moreover, if $g \in \mathcal{D}\left(X^h\right)$, then

$$\mathbb{E}\left[X^{Z_h} f \cdot g\right] = \mathbb{E}\left[f \cdot \left(X^{Z_h}\right)^{tr,\nu} g\right] \tag{4.19}$$

where $\left(X^{Z_h}\right)^{tr,\nu} := -X^{Z_h} + \int_0^1 \langle h_s', d\beta_s \rangle$.

Proof. See Proposition 4.10 in [9] . ■

The following lemma gives an anticipating expansion of $\tilde{X}$ in terms of $\{X^{Z_h}\}_{h \in H(M)}$.

Definition 4.36 (Orthogonal lift on $W_o(M)$) *For any* $f \in \mathcal{FC}^\infty$, *define*

$$\tilde{X} f := \sum_{\alpha=1}^{d} \left\langle \tilde{C}\tilde{H}, e_\alpha \right\rangle X^{Z_\alpha} f \tag{4.20}$$

where $\tilde{C} = \left[\int_0^1 \left(\tilde{T}_r^* \tilde{T}_r\right)^{-1} dr\right]^{-1} \tilde{T}_1^{-1}$ *and by the previous notation (Notation 4.18),*

$$X_s^{Z_\alpha} = \tilde{u}_s Z_\alpha(s)$$

Remark 4.37 *To motivate this definition, recall that we have obtained a lift* $\tilde{X} = X^{Z_\Phi} := \tilde{u}_s Z_\Phi(s)$ *of* $X \in \Gamma(TM)$, *where*

$$\Phi_s = \int_0^s \left(\tilde{T}_\tau^{-1}\right)^* \left[\int_0^1 \left(\tilde{T}_r^* \tilde{T}_r\right)^{-1} dr\right]^{-1} \tilde{T}_1^{-1} \tilde{H} d\tau.$$

It is clear that $\Phi \in H_0\left(\mathbb{R}^d\right)$ *is not adapted. Therefore we cannot apply the theory for geometric vector field. Alternatively we can expand* Φ *in terms of adapted vector fields,*

$$\Phi_s = \sum_{\alpha=1}^{d} \left\langle \tilde{C}\tilde{H}, e_\alpha \right\rangle \int_0^s \left(\tilde{T}_r^{-1}\right)^* e_\alpha dr. \tag{4.21}$$

By superposition principle,

$$Z_\Phi(s) = \sum_{\alpha=1}^{d} \left\langle \tilde{C}\tilde{H}, e_\alpha \right\rangle Z_\alpha(s)$$

and further

$$X^{Z_\Phi} = \sum_{\alpha=1}^{d} \left\langle \tilde{C}\tilde{H}, e_\alpha \right\rangle X^{Z_\alpha}. \tag{4.22}$$

Definition 4.38 *Let $\tilde{X}$ be given above, then define*

$$\mathcal{D}\left(\tilde{X}\right) := \cap_{\alpha=1}^{d} \mathcal{D}\left(X^{Z_\alpha}\right).$$

Remark 4.39 *From the multiplicative system theorem, one can see that $\mathcal{FC}_b^1$ is dense in $L^2(W_o(M))$, therefore $\tilde{X}$ is a densely defined operator on $L^2(W_o(M))$. The integration–by–parts formula for $\tilde{X}$ in the next section will show that it is a closable operator.*

4.4 Computing $\tilde{X}^{tr,\nu}$

This section is devoted to studying of the existence of $\tilde{X}^{tr,\nu}$ (The adjoint operator of $\tilde{X}$ with respect to ν restricted to $\mathcal{D}\left(\tilde{X}\right)$). The crucial step to show existence is checking the anticipating coefficients in (4.20) are differentiable in the Malliavin sense reviewed in Section 4.3. Moreover, an explicit formula which has clearer structure as indicated in Corollary C.3 is given under the condition that the covariant derivative of the curvature tensor is bounded, which includes manifold with non–positive constant sectional curvature.

Proposition 4.40 *Our standard assumption of bounded curvature tensor implies that Ric is bounded. If we further assume ∇Ric is bounded, then for any $h \in \mathcal{V}^\infty$ and $s \in [0,1]$, we have $Ric_{\tilde{u}_s} \in \mathcal{D}\left(X^h\right)$. Moreover,*

$$\sup_{s\in[0,1]} \left|X^h Ric_{\tilde{u}_s}\right| \in L^{\infty-}\left(W_o\left(M\right)\right). \tag{4.23}$$

Proof. Since $Ric_{\tilde{u}_s} \in \mathcal{FC}^\infty$, from Theorem 4.35 we know $Ric_{\tilde{u}_s} \in \mathcal{D}\left(X^h\right)$ and

$$X^h Ric_{\tilde{u}_s} = \left(\nabla_{X_s^h} Ric\right)_{\tilde{u}_s} + \left[A_s \langle h \rangle, Ric_{\tilde{u}_s}\right],$$

where $[\cdot,\cdot]$ is the Lie bracket of matrices and $\left(\nabla_{X_s^h} Ric\right)_{\tilde{u}_s} : \mathbb{R}^d \to \mathbb{R}^d$ is defined to be

$$\left(\nabla_{X_s^h} Ric\right)_{\tilde{u}_s} = \tilde{u}_s^{-1} \nabla_{X_s^h} Ric \cdot \tilde{u}_s.$$

Since ∇Ric is bounded,

$$\left|\left(\nabla_{X_s^h} Ric\right)_{\tilde{u}_s}\right| \leq C h^* \tag{4.24}$$

where C is a constant and h^* is the essential supremum of $s \to h_s$. Since $h \in \mathcal{B}^\infty$, we know

$$\sup_{s\in[0,1]} \left|\left(\nabla_{X_s^h} Ric\right)_{\tilde{u}_s}\right| \in L^\infty\left(W_o\left(M\right)\right). \tag{4.25}$$

Then by Burkholder's inequality, for any $q \in [1,\infty)$,

$$\mathbb{E}\left[\sup_{s\in[0,1]} \left|A_s \langle h \rangle\right|^q\right] \leq C \left\|h\right\|^{\frac{q}{2}}_{L^{\frac{q}{2}}(W_o(M))} < \infty.$$

Since Ric is bounded, we have

$$\sup_{s\in[0,1]} \left|\left[A_s \langle h \rangle, Ric_{\tilde{u}_s}\right]\right| \in L^{\infty-}\left(W_o\left(M\right)\right). \tag{4.26}$$

Combining (4.26) and (4.25) gives (4.23). ■

Theorem 4.41 *Let $\tilde{T}_s$ be as defined in Definition 4.14, then*

$$\tilde{T}_s \in \mathcal{D}\left(X^{Z_\alpha}\right) \text{ for } 1 \leq \alpha \leq d.$$

Proof. For each X^{Z_α}, since Lemma 4.32 shows $Z_\alpha \in \mathcal{V}^{\infty-}$, so we can construct the approximate flow $E\left(tX^{Z_\alpha}\right)$ of X^{Z_α}. Define $\tilde{T}_s\left(t\right) := \tilde{T}_s \circ E\left(tX^{Z_\alpha}\right)$ and $G_s\left(t\right) :=$

$\frac{\tilde{T}_s(t)-\tilde{T}_s}{t}$, it is easy to see that $G_s(t)$ satisfies the following ODE:

$$G_s'(t) = -\frac{1}{2}Ric_{\tilde{u}_s}G_s(t) - \frac{1}{2t}\left(Ric_{\tilde{u}_s(t)} - Ric_{\tilde{u}_s}\right)\tilde{T}_s \text{ with } G_0(t) = 0. \tag{4.27}$$

Then denote by G_s the solution to the following ODE

$$G_s' = -\frac{1}{2}Ric_{\tilde{u}_s}G_s - \frac{1}{2}\left(X^{Z_\alpha}Ric_{\tilde{u}_s}\right)\tilde{T}_s \text{ with } G_0 = 0 \tag{4.28}$$

and let $H_s(t)$ be $H_s(t) := G_s(t) - G_s$, we know $H_s(t)$ satisfies

$$H_s'(t) = -\frac{1}{2}Ric_{\tilde{u}_s}H_s(t) - \frac{1}{2}\left(\frac{Ric_{\tilde{u}_s(t)} - Ric_{\tilde{u}_s}}{t}\tilde{T}_s(t) + \left(X^{Z_\alpha}Ric_{\tilde{u}_s}\right)\tilde{T}_s\right),\ H_0(t) = 0. \tag{4.29}$$

According to Definition 4.30,

$$\tilde{T}_s \in \mathcal{D}\left(X^{Z_\alpha}\right) \iff H_s(t) \to 0 \text{ in } L^{\infty-}(W_o(M)).$$

By Gronwall's inequality, we have

$$|H_s(t)| \leq \int_0^s \left|\frac{Ric_{\tilde{u}_r(t)} - Ric_{\tilde{u}_r}}{t}\tilde{T}_r(t) + X^{Z_\alpha}Ric_{\tilde{u}_r}\tilde{T}_r\right| dr e^{\frac{d(N-1)}{2}} \tag{4.30}$$

Following Theorem 4.4 in [9], we know

$$\frac{Ric_{\tilde{u}_r(t)} - Ric_{\tilde{u}_r}}{t} \to X^{Z_\alpha}Ric_{\tilde{u}_r}$$

and

$$\tilde{T}_r(t) \to \tilde{T}_r \to 0$$

uniformly on $r \in [0,1]$ in $L^{\infty-}(W_o(M))$ as $t \to 0$. So we have $H_s(t) \to 0$ in $L^{\infty-}(W_o(M))$ as $t \to 0$. ■

Corollary 4.42 *Recall that we have defined* $\tilde{C} = \left[\int_0^1 \left(\tilde{T}_r^*\tilde{T}_r\right)^{-1} dr\right]^{-1} \tilde{T}_1^{-1}$ *in Def-*

inition 4.36, then

$$\tilde{C} \in \mathcal{D}\left(X^{Z_\alpha}\right) \text{ for } 1 \leq \alpha \leq d.$$

Proof. By the product rule, for any $s \in [0,1]$,

$$X^{Z_\alpha}\left(\tilde{T}_s^{-1}\right) = -\tilde{T}_s\left(X^{Z_\alpha}\tilde{T}_s\right)\tilde{T}_s \in L^{\infty-}\left(W_o\left(M\right)\right),$$

so $\tilde{T}_s^{-1} \in \mathcal{D}\left(X^{Z_\alpha}\right)$ and thus $\int_0^1 \left(\tilde{T}_r^*\tilde{T}_r\right)^{-1} dr \in \mathcal{D}\left(X^{Z_\alpha}\right)$. Then apply the product rule again we get $\tilde{C} \in \mathcal{D}\left(X^{Z_\alpha}\right)$. ■

Lemma 4.43 *Given $X \in \Gamma(TM)$ with compact support, recall from Definition* 4.36 *that $\tilde{X}$ is its orthogonal lift on $W_o(M)$, then*

$$\tilde{X}^{\mathrm{tr},\nu} = -\tilde{X} + \sum_{\alpha=1}^{d}\left\langle \tilde{C}\tilde{H}, e_\alpha\right\rangle \int_0^1 \left\langle \left(\tilde{T}_s^{-1}\right)^* e_\alpha, d\beta_s\right\rangle + \sum_{\alpha=1}^{d}\left\langle -X^{Z_\alpha}\left(\tilde{C}\tilde{H}\right), e_\alpha\right\rangle.$$

In other words we are claiming that

$$\mathbb{E}\left[\tilde{X}f \cdot g\right] = \mathbb{E}\left[f \cdot \tilde{X}^{tr,\nu}g\right]$$

for all $f, g \in \mathcal{D}\left(\tilde{X}\right)$.

Proof. First of all, $f \in \mathcal{D}\left(\tilde{X}\right) \iff f \in \mathcal{D}\left(X^{Z_\alpha}\right)\ \forall 1 \leq \alpha \leq d$. Then since $\tilde{H} \in \mathcal{FC}_b^\infty$, $\tilde{H} \in \mathcal{D}\left(X^{Z_\alpha}\right)\ \forall 1 \leq \alpha \leq d$. Based on the above observation and Corollary 4.42, we obtain

$$\mathbb{E}\left[\tilde{X}f \cdot g\right] = \mathbb{E}\left[\sum_{\alpha=1}^{d}\left\langle \tilde{C}\tilde{H}, e_\alpha\right\rangle X^{Z_\alpha}f \cdot g\right] \tag{4.31}$$

$$= \sum_{\alpha=1}^{d}\mathbb{E}\left[X^{Z_\alpha}f \cdot \left(g \cdot \left\langle \tilde{C}\tilde{H}, e_\alpha\right\rangle\right)\right] \tag{4.32}$$

$$= I + II + III \tag{4.33}$$

where

$$I = \mathbb{E}\left[f \cdot \left(-\tilde{X}\right) g\right]$$
$$II = \mathbb{E}\left[f \cdot g \cdot \sum_{\alpha=1}^{d} \left\langle \tilde{C}\tilde{H}, e_\alpha \right\rangle \int_0^1 \left\langle \left(\tilde{T}_s^{-1}\right)^* e_\alpha, d\beta_s \right\rangle\right]$$
$$III = \mathbb{E}\left[f \cdot g \cdot \sum_{\alpha=1}^{d} \left\langle -X^{Z_\alpha}\left(\tilde{C}\tilde{H}\right), e_\alpha \right\rangle\right].$$

■

The following lemma gives a more explicit expression of the last term in $\tilde{X}^{tr,\nu}$

$$\sum_{\alpha=1}^{d} \left\langle -X^{Z_\alpha}\left(\tilde{C}\tilde{H}\right), e_\alpha \right\rangle$$

under an extra condition that $\nabla R \equiv 0$.

Lemma 4.44 *If further the covariant differential of the curvature tensor is 0, i.e.* $\nabla R \equiv 0$, *then*

$$-\sum_{\alpha=1}^{d} \left\langle X^{Z_\alpha}\left(\tilde{C}\tilde{H}\right), e_\alpha \right\rangle = divX \circ E_1 - \sum_{\alpha=1}^{d} \left\langle \tilde{C}A_1 \langle Z_\alpha \rangle \tilde{H}, e_\alpha \right\rangle. \tag{4.34}$$

Proof. Since for tensors, contraction commutes with covariant differentiation, and Ric is the contraction of curvature tensor R, so $\nabla Ric \equiv 0$. For any $\sigma \in H_0\left(\mathbb{R}^d\right)$, using its horizontal lift u we find that

$$\frac{d}{ds} Ric_{u_s} = (\nabla_{\sigma'_s} Ric)_{u_s} = 0.$$

It follows that $\tilde{T}_s = e^{-\frac{1}{2}N' sI}$ is deterministic and thus

$$\tilde{C} = \left(\int_0^1 \left[\tilde{T}_r^* \tilde{T}_r\right]^{-1} dr\right)^{-1} \tilde{T}_1^{-1} \text{ is deterministic.}$$

Since $\tilde{H} = \tilde{u}_1^{-1} X(\pi \circ \tilde{u}_1) \in \mathcal{FC}_b^\infty$, we can apply Theorem 4.35 to $\tilde{H}$ to find

$$\sum_{\alpha=1}^{d} \left\langle X^{Z_\alpha}\left(\tilde{C}\tilde{H}\right), e_\alpha \right\rangle = \sum_{\alpha=1}^{d} \left\langle \tilde{C} X^{Z_\alpha} \tilde{H}, e_\alpha \right\rangle$$
$$= I + II$$

where

$$I = -\sum_{\alpha=1}^{d} \left\langle \tilde{C}\tilde{u}_1^{-1} \nabla_{Z_\alpha(1)} X, e_\alpha \right\rangle$$

and

$$II = \sum_{\alpha=1}^{d} \left\langle \tilde{C} A_1 \langle Z_\alpha \rangle \tilde{H}, e_\alpha \right\rangle.$$

Claim: $I = -divX \circ E_1$.

Proof of Claim:

$$I = -\sum_{\alpha=1}^{d} \left\langle \tilde{u}_1 \tilde{C}\tilde{u}_1^{-1} \nabla_{\tilde{u}_1 \tilde{C}^{-1} \tilde{u}_1^{-1} \tilde{u}_1 e_\alpha} X, \tilde{u}_1 e_\alpha \right\rangle$$
$$= -\sum_{\alpha=1}^{d} \left\langle A^{-1} \nabla_{A f_\alpha} X, f_\alpha \right\rangle$$
$$= -\sum_{\alpha=1}^{d} \left\langle \nabla_{A f_\alpha} X, \left(A^{-1}\right)^* f_\alpha \right\rangle$$

where $A = \tilde{u}_1 \tilde{C}^{-1} \tilde{u}_1^{-1} \in End\left(T_{E_1(\sigma)} M\right)$ and $\{f_\alpha\} = \{\tilde{u}_1 e_\alpha\}$ is an orthonormal basis of $T_{E_1(\sigma)} M$. Since $\langle \nabla . X, \cdot \rangle$ is bilinear on $T_{E_1(\sigma)} M$, by the Universal property of tensor product we know there exists a linear map $l : T_{E_1(\sigma)} M \otimes T_{E_1(\sigma)} M \mapsto \mathbb{R}$ such that

$$\left\langle \nabla_{A f_\alpha} X, \left(A^{-1}\right)^* f_\alpha \right\rangle = l\left(A f_\alpha \otimes \left(A^{-1}\right)^* f_\alpha\right)$$

and therefore:

$$\sum_{\alpha=1}^{d} \left\langle \nabla_{Af_\alpha} X, \left(A^{-1}\right)^* f_\alpha \right\rangle = l\left(\sum_{\alpha=1}^{d} Af_\alpha \otimes \left(A^{-1}\right)^* f_\alpha\right) \tag{4.35}$$

Using the isomorphism between $T_1^1(V) \mapsto End(V)$:$(a \otimes b)\, v = a \cdot \langle b, v\rangle$ one can easily see:

$$\sum_{\alpha=1}^{d} Af_\alpha \otimes \left(A^{-1}\right)^* f_\alpha = \sum_{\alpha=1}^{d} f_\alpha \otimes f_\alpha \tag{4.36}$$

Combine (4.35) and (4.36) we have

$$I = -\sum_{\alpha=1}^{d} \langle \nabla_{f_\alpha} X, f_\alpha\rangle = -divX \circ E_1$$

and (4.34). ■

Chapter 5

The Orthogonal Lift $\tilde{X}_{\mathcal{P}}$ on $H_{\mathcal{P}}(M)$

As a remainder, unless mentioned seperately, M is a complete Riemannian manifold with non–positive sectional curvature bounded below by $-N$. In this chapter we focus on the unpinned piecewise geodesic space $H_{\mathcal{P}}(M)$.

5.1 A Parametrization of $T_\sigma H_{\mathcal{P}}(M)$

Recall from Theorem 2.39 that for $Y \in \Gamma(TH_{\mathcal{P}}(M))$ iff for each $\sigma \in H_{\mathcal{P}}(M)$, $J(\sigma, s) := u(\sigma, s)^{-1} Y(\sigma, s)$ satisfies (in the following equation we suppress σ)

$$J''(s) = R_{u(s)}(b'(s_{i-1}+), J(s))\, b'(s_{i-1}+) \text{ for } s \in [s_{i-1}, s_i) \; i = 1, ..., n. \tag{5.1}$$

where $b = \phi(\sigma) \in H_0(\mathbb{R}^d)$ is the anti–rolling of σ.

From above we observe that J can be parametrized by

$$\{J'(s_i+) = k_i\}_{i=0}^{n-1} \tag{5.2}$$

where $(k_0, k_1, \ldots, k_{n-1})$ is an arbitrary element of $\left(\mathbb{R}^d\right)^n$. Proposition 5.1 explains this parametrization in more detail.

Proposition 5.1 *If $(k_0, k_1, \ldots, k_{n-1}) \in \left(\mathbb{R}^d\right)^n$, then the unique $J(\cdot) \in C\left([0,1], \mathbb{R}^d\right)$ satisfying (5.1) and (5.2) above is given by*

$$J(s) = \frac{1}{n} \sum_{i=0}^{l-1} f_{\mathcal{P},i+1}(s)\, k_i \text{ for } s \in [s_{l-1}, s_l]\ ,\ 1 \le l \le n. \tag{5.3}$$

Proof. From the definition of $f_{\mathcal{P},i+1}$ (see Definition 2.42), J in Eq. (5.3) may be written as

$$J(s) = C_{\mathcal{P},l}(s) \left[\sum_{i=0}^{l-2} C_{\mathcal{P},l-1} \ldots C_{\mathcal{P},i+2} S_{\mathcal{P},i+1} k_i \right] + S_{\mathcal{P},l}(s)\, k_{l-1} \text{ when } s \in [s_{l-1}, s_l].$$

To finish the proof, we need only verify that J is continuous, $J'(s_i+) = k_i$ for $0 \le i \le n-1$ and J solves the Jacobi equation (5.1). Since $C_{\mathcal{P},l}(s)$ and $S_{\mathcal{P},l}(s)$ satisfies Jacobi equation for $s \in [s_{l-1}, s_l)$, J satisfies (5.1) and is continuous at $s \notin \mathcal{P}$. For each s_l, $1 \le l \le n-1$, since $C_{\mathcal{P},l+1}(s_l) = I$, $S_{\mathcal{P},l+1}(s_l) = 0$ and J is right continuous on $[0,1]$,

$$\begin{aligned} J(s_l-) = \lim_{s \uparrow s_l} J(s) &= C_{\mathcal{P},l} \left[\sum_{i=0}^{l-2} C_{\mathcal{P},l-1} \ldots C_{\mathcal{P},i+2} S_{\mathcal{P},i+1} k_i \right] + S_{\mathcal{P},l} k_{l-1} \\ &= C_{\mathcal{P},l+1}(s_l) \left[\sum_{i=0}^{l-1} C_{\mathcal{P},l} \ldots C_{\mathcal{P},i+2} S_{\mathcal{P},i+1} k_i \right] + S_{\mathcal{P},l+1}(s_l)\, k_l \\ &= J(s_l) = J(s_l+). \end{aligned}$$

So J is also continuous at partition points. Then since

$$\begin{aligned} J'(s_{l-1}+) = C'_{\mathcal{P},l}(s_{l-1}+) \left[\sum_{i=0}^{l-2} C_{\mathcal{P},l-1} \ldots C_{\mathcal{P},i+2} S_{\mathcal{P},i+1} k_i \right] + S'_{\mathcal{P},l}(s_{l-1}+)\, k_{l-1} \\ = 0 + I \cdot k_{l-1} = k_{l-1}, \end{aligned}$$

J satisfies (5.2). The uniqueness of J is easily seen from the uniqueness of solutions to ODE with initial values.

■

Definition 5.2 *For each* $s \in [0,1]$, *define* $\mathbf{L}_s : \left(\mathbb{R}^d\right)^n \to \mathbb{R}^d$ *as follows: for* $s \in [s_{l-1}, s_l]$,

$$\mathbf{L}_s \left(k_0, \ldots, k_{n-1}\right) = \frac{1}{n} \sum_{i=0}^{l-1} f_{\mathcal{P},i+1}\left(s\right) k_i \tag{5.4}$$

and in particular

$$\mathbf{L}_1 \left(k_0, \ldots, k_{n-1}\right) = \frac{1}{n} \sum_{i=0}^{n-1} f_{\mathcal{P},i+1}\left(1\right) k_i \tag{5.5}$$

We now compute the adjoint of $\mathbf{L}_1$.

Lemma 5.3 *For any* $v \in \mathbb{R}^d$, *let* $\mathbf{L}_1^* : \mathbb{R}^d \to \left(\mathbb{R}^d\right)^n$ *be the adjoint of* $\mathbf{L}_1$, *then*

$$\mathbf{L}_1^* v = \frac{1}{n} \left(f_{\mathcal{P},1}^*\left(1\right) v, f_{\mathcal{P},2}^*\left(1\right) v, \ldots, f_{\mathcal{P},n}^*\left(1\right) v\right), \tag{5.6}$$

where $f_{\mathcal{P},i}^*\left(1\right)$ *is the matrix adjoint of* $f_{\mathcal{P},i}\left(1\right)$.

Proof. Equation (5.6) immediately follows from the identity;

$$\left\langle \mathbf{L}_1 \left(k_0, \ldots, k_{n-1}\right), v\right\rangle = \sum_{i=0}^{n-1} \left\langle \frac{1}{n} f_{\mathcal{P},i+1}\left(1\right) k_i, v \right\rangle = \sum_{i=0}^{n-1} \left\langle k_i, \frac{1}{n} f_{\mathcal{P},i+1}^*\left(1\right) v \right\rangle. \tag{5.7}$$

■

Definition 5.4 *We now define*

$$\mathbf{K}_{\mathcal{P}}\left(s\right) v := n\mathbf{L}_s \left(\mathbf{L}_1^* v\right). \tag{5.8}$$

In particular,

$$\mathbf{K}_{\mathcal{P}}\left(1\right) v = \frac{1}{n} \sum_{i=0}^{n-1} f_{\mathcal{P},i+1}\left(1\right) f_{\mathcal{P},i+1}^*\left(1\right) v. \tag{5.9}$$

Recall that given a matrix A, $eig\left(A\right)$ denotes the eigenvalues of A.

Lemma 5.5 (Invertibility of $\mathbf{K}_{\mathcal{P}}(1)$) *If M has non-positive sectional curvature, then*

$$eig\left(\mathbf{K}_{\mathcal{P}}(1)\right) \subset [1, \infty) \tag{5.10}$$

and thus $\mathbf{K}_{\mathcal{P}}(1)$ is invertible.

Proof. Denote $R_{u_s}\left(b'\left(s_{i-1}+\right), \cdot\right) b'\left(s_{i-1}+\right)$ by $A_{\mathcal{P},i}(s) : H_{\mathcal{P}}(M) \to End\left(\mathbb{R}^d\right)$. Notice that M having non-positive sectional curvature guarantees $A_{\mathcal{P},i}(s)$ is non-negative. Then apply Proposition B.2 to find, for any $i = 1, \cdots, n$ and $v \in \mathbb{C}^d$,

$$\|C_{\mathcal{P},i} v\| \geq \|v\| \text{ and } \|S_{\mathcal{P},i} v\| \geq \frac{1}{n}\|v\|.$$

From these inequalities it follows that

$$\begin{aligned}\|f_{\mathcal{P},i}(1) v\| &= n \|C_{\mathcal{P},n} C_{\mathcal{P},n-1} \cdots C_{\mathcal{P},i+1} S_{\mathcal{P},i} v\| \\ &\geq n \cdot \frac{1}{n}\|v\| = \|v\|.\end{aligned}$$

So $f_{\mathcal{P},i}(1)$ is invertible and $\left\|f_{\mathcal{P},i}(1)^{-1}\right\| \leq 1$. Therefore for any $v \in \mathbb{C}^d$,

$$\left\|f *_{\mathcal{P},i}(1)^{-1} v\right\| = \left\|f_{\mathcal{P},i}(1)^{-1} v\right\| \leq \|v\|,$$

now replace v by $f^*_{\mathcal{P},i}(1) v$, we get $\left\|f^*_{\mathcal{P},i}(1) v\right\| \geq \|v\|$ and thus

$$\begin{aligned}\langle \mathbf{K}_{\mathcal{P}}(1) v, v\rangle &= \frac{1}{n}\sum_{i=0}^{n-1}\left\langle f_{\mathcal{P},i+1}(1) f^*_{\mathcal{P},i+1}(1) v, v\right\rangle \\ &= \frac{1}{n}\sum_{i=0}^{n-1}\left\|f^*_{\mathcal{P},i+1}(1) v\right\|^2 \\ &\geq \frac{1}{n}\cdot n\|v\|^2 = \|v\|^2 \ \ \forall v \in \mathbb{C}^d.\end{aligned}$$

This implies that

$$eig\left(\mathbf{K}_{\mathcal{P}}(1)\right) \subset [1, \infty)$$

In particular, $\mathbf{K}_{\mathcal{P}}(1)$ is invertible. ■

5.2 Orthogonal Lifts on $H_{\mathcal{P}}(M)$

In this section we use the least square method to lift a vector field $X \in \Gamma(TM)$ to a vector field $\tilde{X}_{\mathcal{P}} \in \Gamma(TH_{\mathcal{P}}(M))$.

Theorem 5.6 (Orthogonal lift) *For all $X \in \Gamma(TM)$, there exists a unique orthogonal lift $\tilde{X}_{\mathcal{P}} \in \Gamma(TH_{\mathcal{P}}(M))$. In more detail, $\tilde{X}_{\mathcal{P}}$ is uniquely determined by;*

1. *For all $h \in C^1(M)$,*

$$\tilde{X}_{\mathcal{P}}(h \circ E_1)(\sigma) = (Xh)(E_1(\sigma)), \text{ i.e. } E_{1*}\tilde{X}_{\mathcal{P}}(\sigma) = X(\sigma(1)). \tag{5.11}$$

2. *For all $\sigma \in H_{\mathcal{P}}(M)$,*

$$\left\|\tilde{X}_{\mathcal{P}}(\sigma)\right\|_{G^1_{\mathcal{P}}} = \inf\{\|Y(\sigma)\|_{G^1_{\mathcal{P}}} : Y \in \Gamma(TH_{\mathcal{P}}(M)), Y \text{ satisfies } (5.11)\}. \tag{5.12}$$

First we use the parametrization in Section 5.1 to characterize $\left\{\mathrm{Nul}\left(E_{1*,\sigma}\right)\right\}^{\perp}$.

Lemma 5.7 *Suppose $Y \in \Gamma(TH_{\mathcal{P}}(M))$ with $k(\cdot) := u(\cdot)^{-1}Y(\cdot) : H_{\mathcal{P}}(M) \to H_0(\mathbb{R}^d)$. Then $Y \in \{\mathrm{Nul}(E_{1*})\}^{\perp}$ iff*

$$(k'(s_0+), ..., k'(s_{n-1}+)) \in (\mathrm{Nul}\,\mathbf{L}_1)^{\perp} = \mathrm{Ran}(\mathbf{L}_1^*).$$

Proof. Given $Y(\cdot) := u(\cdot)k(\cdot)$ and $Z(\cdot) := u(\cdot)J(\cdot) \in \Gamma(TH_{\mathcal{P}}(M))$, then

$$\begin{aligned}\langle Y(\sigma), Z(\sigma)\rangle_{G^1_{\mathcal{P}}} = 0 &\iff \sum_{i=0}^{n-1} \langle J'(\sigma, s_i+), k'(\sigma, s_i+)\rangle \Delta_{i+1} = 0 \\ &\iff \sum_{i=0}^{n-1} \langle J'(\sigma, s_i+), k'(\sigma, s_i+)\rangle = 0,\end{aligned}$$

and

$$Z(\sigma) \in \mathrm{Nul}\left(E_{1*,\sigma}\right) \iff E_{1*,\sigma}\left(X^{J_1}\right) = u_1(\sigma)\, J(\sigma, 1) = 0 \iff J_1(\sigma) = 0.$$

Recall that $J_1 = \mathbf{L}_1\left(J'(s_0+), ..., J'(s_{n-1}+)\right)$, so

$$J_1 = 0 \iff \left(J'(s_0+), ..., J'(s_{n-1}+)\right) \in \mathrm{Nul}\left(\mathbf{L}_1\right) \tag{5.13}$$

Since

$$\sum_{i=0}^{n-1} \left\langle J'(s_i+), k'(s_i+)\right\rangle = \left\langle \left(J'(s_0+), ..., J'(s_{n-1}+)\right), \left(k'(s_0+), ..., k'(s_{n-1}+)\right)\right\rangle,$$

so $Y \in \{\mathrm{Nul}(E_{1*})\}^{\perp}$ iff

$$\left(k'(s_0+), ..., k'(s_{n-1}+)\right) \in \{\mathrm{Nul}(\mathbf{L}_1)\}^{\perp} = \mathrm{Ran}\left(\mathbf{L}_1^*\right).$$

■

Remark 5.8 *According to (5.6) and (5.13), it is immediate that*

$$\mathrm{Ran}\left(\mathbf{L}_1^*\right) = \left\{ \left(\frac{1}{n} f_{\mathcal{P},1}^*(1)\, v, \frac{1}{n} f_{\mathcal{P},2}^*(1)\, v, \ldots, \frac{1}{n} f_{\mathcal{P},n}^*(1)\, v \right), \quad \forall\, v \in \mathbb{R}^d \right\},$$

Definition 5.9 *Given $X \in \Gamma(TM)$, define $\tilde{X}_{\mathcal{P}} \in \Gamma(TH_{\mathcal{P}}(M))$ to be $\tilde{X}_{\mathcal{P}}(\cdot) = u.J_{\mathcal{P}}(\cdot)$ where*

$$J_{\mathcal{P}}(s) := \mathbf{K}_{\mathcal{P}}(s)\, \mathbf{K}_{\mathcal{P}}(1)^{-1}\, {u_1}^{-1} X \circ E_1.$$

Proof of Theorem 5.6. We will show $\tilde{X}_{\mathcal{P}}$ is the unique orthogonal lift of X. Since $T_\sigma H_{\mathcal{P}}(M) = \mathrm{Nul}\left(E_{1*,\sigma}\right) \oplus_{G^1_{\mathcal{P}}} \left\{\mathrm{Nul}\left(E_{1*,\sigma}\right)\right\}^{\perp}$, given a lift $Z \in \Gamma(TH_{\mathcal{P}}(M))$ of $X \in \Gamma(TM)$, its orthogonal projection to $\left\{\mathrm{Nul}\left(E_{1*,\sigma}\right)\right\}^{\perp}$ is also a lift but with a smaller $G^1_{\mathcal{P}}$ norm. So if Z is an orthogonal lift, then $Z \in \{\mathrm{Nul}(E_{1*})\}^{\perp}$. From

Lemma 5.7 and 5.8 it follows that if $k(\cdot) := u^{-1}(\cdot) Z(\cdot)$, then

$$(k'(s_0), \ldots, k'(s_{n-1})) = \left(\frac{1}{n} f^*_{\mathcal{P},1}(1) v, \frac{1}{n} f^*_{\mathcal{P},2}(1) v, \ldots, \frac{1}{n} f^*_{\mathcal{P},n}(1) v\right)$$

for some $v \in \mathbb{R}^d$. Then using Definition 5.4 and Proposition 5.1, k must have the following form,

$$k_s = \mathbf{K}_{\mathcal{P}}(s) v$$

for some $v \in \mathbb{R}^d$ to be determined. To specify v, we use condition (5.11)

$$\tilde{X}_{\mathcal{P}}(\sigma, 1) = X(\sigma(1)).$$

This implies $\mathbf{K}_{\mathcal{P}}(1) v = u_1^{-1} X \circ E_1$. Since $\mathbf{K}_{\mathcal{P}}(1)$ is invertible, we can just pick v to be $\mathbf{K}_{\mathcal{P}}(1)^{-1} u_1^{-1} X \circ E_1$.

Definition 5.10 *We will view $\tilde{X}_{\mathcal{P}}$ as a differential operator with domain,*

$$\mathcal{D}\left(\tilde{X}_{\mathcal{P}}\right) := C_b^1(H_{\mathcal{P}}(M)).$$

Since $C_b^1(H_{\mathcal{P}}(M))$ is dense in $L^2(H_{\mathcal{P}}(M), \nu^1_{\mathcal{P}})$, we can view $\tilde{X}_{\mathcal{P}}$ as a densely defined operator on $L^2(H_{\mathcal{P}}(M), \nu^1_{\mathcal{P}})$. Using Lemma 7.5 we know its range is in $L^{\infty-}(H_{\mathcal{P}}(M), \nu^1_{\mathcal{P}}) \subset L^2(H_{\mathcal{P}}(M), \nu^1_{\mathcal{P}})$.

We will explore the limit of the orthogonal lift $\tilde{X}_{\mathcal{P}}$ as the mesh of $|P| \to 0$ in Chapter 6. ■

5.3 Restricted Adjoint $\tilde{X}_{\mathcal{P}}^{tr,\nu^1_{\mathcal{P}}}$

In this section we study $\tilde{X}_{\mathcal{P}}^{tr,\nu^1_{\mathcal{P}}}$—the adjoint of $\tilde{X}_{\mathcal{P}}$ with respect to $\nu^1_{\mathcal{P}}$ restricted to $\mathcal{D}\left(\tilde{X}_{\mathcal{P}}\right)$, i.e. we require $\mathcal{D}\left(\tilde{X}_{\mathcal{P}}^{tr,\nu^1_{\mathcal{P}}}\right) = \mathcal{D}\left(\tilde{X}_{\mathcal{P}}\right)$.

Lemma 5.11 *Given $X \in \Gamma(TM)$, if $\tilde{X}_{\mathcal{P}}$ is the orthogonal lift of X, then*

$$\tilde{X}_{\mathcal{P}}^{tr,\nu_{\mathcal{P}}^1} = -\tilde{X}_{\mathcal{P}} + M_{\int_0^1 \langle J_{\mathcal{P}}'(s), b'(s)\rangle ds} - M_{div\tilde{X}_{\mathcal{P}}} \tag{5.14}$$

where $M_{\cdot}$ is the multiplication operator, b is the anti-rolling of σ and $div\tilde{X}_{\mathcal{P}}$ is the divergence of $\tilde{X}_{\mathcal{P}}$ with respect to $vol_{G_{\mathcal{P}}^1}$.

Proof. In this proof we identify the measure $\nu_{\mathcal{P}}^1$ with the associated nd—form. So by "Cartan's magic formula", first assume $f \in C_b^1(H_{\mathcal{P}}(M))$ with compact support,

$$\mathcal{L}_{\tilde{X}_{\mathcal{P}}}\left(f\nu_{\mathcal{P}}^1\right) = d\left(i_{\tilde{X}_{\mathcal{P}}}\left(f\nu_{\mathcal{P}}^1\right)\right) + i_{\tilde{X}_{\mathcal{P}}}\left(d\left(f\nu_{\mathcal{P}}^1\right)\right).$$

Since $f\nu_{\mathcal{P}}^1$ is a top degree form, $d\left(f\nu_{\mathcal{P}}^1\right) = 0$. By Stokes' theorem,

$$\int_{H_{\mathcal{P}}M} d\left(i_{\tilde{X}_{\mathcal{P}}}\left(f\nu_{\mathcal{P}}^1\right)\right) = 0.$$

Therefore we have:

$$\int_{H_{\mathcal{P}}(M)} \mathcal{L}_{\tilde{X}_{\mathcal{P}}}\left(f\nu_{\mathcal{P}}^1\right) = 0$$

and

$$\begin{aligned} \int_{H_{\mathcal{P}}(M)} \left(\tilde{X}_{\mathcal{P}}f\right) d\nu_{\mathcal{P}}^1 &= \int_{H_{\mathcal{P}}(M)} \mathcal{L}_{\tilde{X}_{\mathcal{P}}}\left(f\nu_{\mathcal{P}}^1\right) - \int_{H_{\mathcal{P}}(M)} f\mathcal{L}_{\tilde{X}_{\mathcal{P}}}\left(\nu_{\mathcal{P}}^1\right) \\ &= -\int_{H_{\mathcal{P}}(M)} f\mathcal{L}_{\tilde{X}_{\mathcal{P}}}\left(\nu_{\mathcal{P}}^1\right). \end{aligned} \tag{5.15}$$

Recall that $\nu_{\mathcal{P}}^1 = \frac{1}{Z_{\mathcal{P}}^1} e^{-\frac{1}{2}E} vol_{G_{\mathcal{P}}^1}$, so

$$\mathcal{L}_{\tilde{X}_{\mathcal{P}}}\left(\nu_{\mathcal{P}}^1\right) = \left[\tilde{X}_{\mathcal{P}}\left(\frac{1}{Z_{\mathcal{P}}^1} e^{-\frac{1}{2}E}\right)\right] vol_{G_{\mathcal{P}}^1} + \left(div\tilde{X}_{\mathcal{P}}\right)\nu_{\mathcal{P}}^1. \tag{5.16}$$

In (5.16)

$$\begin{aligned}
\tilde{X}_{\mathcal{P}}\left(\frac{1}{Z_{\mathcal{P}}^{1}}e^{-\frac{1}{2}E}\right) &= -\frac{1}{2}\tilde{X}_{\mathcal{P}}\left(E\right)\frac{1}{Z_{\mathcal{P}}^{1}}e^{-\frac{1}{2}E} \\
&= -\int_{0}^{1}\left\langle \sigma'\left(s+\right), \frac{\nabla \tilde{X}_{\mathcal{P}}}{ds}\left(s+\right)\right\rangle ds\frac{1}{Z_{\mathcal{P}}^{1}}e^{-\frac{1}{2}E} \\
&= -\int_{0}^{1}\left\langle b'\left(s+\right), J_{\mathcal{P}}'\left(s+\right)\right\rangle ds\frac{1}{Z_{\mathcal{P}}^{1}}e^{-\frac{1}{2}E}. \qquad (5.17)
\end{aligned}$$

Combining (5.15), (5.16) and (5.17) we get, if $f \in C_b^1\left(H_{\mathcal{P}}\left(M\right)\right)$ with compact support and bounded differential df, then

$$\int_{H_{\mathcal{P}}(M)} \tilde{X}_{\mathcal{P}} f d\nu_{\mathcal{P}}^{1} = \int_{H_{\mathcal{P}}(M)} f \cdot \left(\tilde{X}_{\mathcal{P}}^{tr,\nu_{\mathcal{P}}^{1}} 1\right) d\nu_{\mathcal{P}}^{1}, \qquad (5.18)$$

where $\tilde{X}_{\mathcal{P}}^{tr,\nu_{\mathcal{P}}^{1}}$ is defined in Eq. (5.14). For the general case choose a cut–off function $\phi \in C_0^\infty\left(\mathbb{R}^d,[0,1]\right)$ such that $\phi \equiv 1$ on $B\left(0,1\right)$ and $\phi \equiv 0$ on $\mathbb{R}^d/B\left(0,2\right)$, $B\left(x,r\right)$ is the ball on $\mathbb{R}^d$ centered at x with radius r. Let $f_n := f \cdot \phi\left(\frac{E}{n}\right)$, observe, using product rule, that

$$\tilde{X}_{\mathcal{P}} f_n = \phi\left(\frac{E}{n}\right) \cdot \tilde{X}_{\mathcal{P}} f + \frac{1}{n} f \cdot \phi'\left(\frac{E}{n}\right)\int_0^1 \left\langle J_{\mathcal{P}}'\left(s\right), b'\left(s\right)\right\rangle ds, \qquad (5.19)$$

so $\tilde{X}_{\mathcal{P}} f_n \to \tilde{X}_{\mathcal{P}} f$ as $n \to \infty$ $\nu_{\mathcal{P}}^1$ a.s.

Using Proposition 6.26 and Lemma 4.43 we have

$$\int_0^1 \left\langle J_{\mathcal{P}}'\left(s\right), b'\left(s\right)\right\rangle ds \in L^{\infty-}\left(H_{\mathcal{P}}\left(M\right), d\nu_{\mathcal{P}}^1\right)$$

where $L^{\infty-} := \cap_{q\geq 1} L^q$.

Since f has bounded derivative, from Definition 5.9 and Lemma 6.12 we have

$$\left|\tilde{X}_{\mathcal{P}} f\right| \leq C\left\langle \tilde{X}_{\mathcal{P}}, \tilde{X}_{\mathcal{P}}\right\rangle_{G_{\mathcal{P}}^1} \in L^{\infty-}\left(H_{\mathcal{P}}\left(M\right), d\nu_{\mathcal{P}}^1\right).$$

So using bounded convergence theorem, Eq.(5.18) with f_n, and Eq.(5.19) we obtain Eq. (5.18) for f. ■

5.4 Computing $div\tilde{X}_{\mathcal{P}}$

Recall from Definition 3.9 that

$$X^{h_{\alpha,i}}(\sigma,s)=u(\sigma,s)\frac{1}{\sqrt{n}}f_{\mathcal{P},i}(s)\,e_{\alpha}\ ,\ 1\leq\alpha\leq d\ ,\ 1\leq i\leq n$$

is an orthonormal frame on $(TH_{\mathcal{P}}(M),G^1_{\mathcal{P}})$. Using this orthonormal frame, one can get an expression of $div\tilde{X}_{\mathcal{P}}$.

Proposition 5.12 *Let $\tilde{X}_{\mathcal{P}}$ be the orthogonal lift of $X\in\Gamma(TM)$, then*

$$div\tilde{X}_{\mathcal{P}}=\sum_{\alpha=1}^{d}\sum_{j=1}^{n}\left\langle X^{h_{\alpha,j}}J'_{\mathcal{P}}(s_{j-1}+),e_{\alpha}\right\rangle\sqrt{\Delta_j} \tag{5.20}$$

Proof. We know

$$div\tilde{X}_{\mathcal{P}}=\sum_{\alpha=1}^{d}\sum_{j=1}^{n}G^1_{\mathcal{P}}\left\langle\left[X^{h_{\alpha,j}},\tilde{X}_{\mathcal{P}}\right],X^{h_{\alpha,j}}\right\rangle, \tag{5.21}$$

where $[\cdot,\cdot]$ is the Lie bracket of vector fields.

Now fix j and α, notice that $\tilde{X}_{\mathcal{P}}=X^{J_{\mathcal{P}}}$, apply Theorem 3.5 in [2] to find

$$\left[X^{h_{\alpha,j}},\tilde{X}_{\mathcal{P}}\right]=X^{f(h_{\alpha,j},J_{\mathcal{P}})},$$

where

$$\begin{aligned}f_s(h_{\alpha,j},J_{\mathcal{P}})=&\left(X^{h_{\alpha,j}}J_{\mathcal{P}}\right)(s)-\left(X^{J_{\mathcal{P}}}h_{\alpha,j}\right)(s)+\\&q_s\left(X^{h_{\alpha,j}}\right)J_{\mathcal{P}}(s)-q_s\left(X^{J_{\mathcal{P}}}\right)h_{\alpha,j}(s)\end{aligned}$$

and

$$q_s\left(X^f\right) = \int_0^s R_{u_r}\left(b'\left(r+\right), f\left(r\right)\right) dr.$$

Therefore

$$G^1_{\mathcal{P}}\left\langle\left[X^{h_{\alpha,j}}, \tilde{X}_{\mathcal{P}}\right], X^{h_{\alpha,j}}\right\rangle = \sum_{i=1}^{n}\left\langle f', h'_{\alpha,j}\right\rangle_{s_{i-1}+}\Delta_i \tag{5.22}$$

$$= \sum_{i=1}^{n}\left\langle\left(X^{h_{\alpha,j}}J_{\mathcal{P}}\right)' - \left(X^{J_{\mathcal{P}}}h_{\alpha,j}\right)', h'_{\alpha,j}\right\rangle_{s_{i-1}+}\Delta_i$$

$$+ \sum_{i=1}^{n}\left\langle\left(q_s\left(X^{h_{\alpha,j}}\right)J_{\mathcal{P}}\left(s\right)\right)' - \left(q_s\left(X^{J_{\mathcal{P}}}\right)h_{\alpha,j}\left(s\right)\right)', h'_{\alpha,j}\right\rangle_{s_{i-1}+}\Delta_i$$

Here $'$ is the derivative with respect to (time) s.

Since $h'_{\alpha,j}\left(s_{i-1}+\right)$ is independent of σ, so

$$\left(X^{J_{\mathcal{P}}}h_{\alpha,j}\right)'\left(s_{i-1}+\right) = X^{J_{\mathcal{P}}}\left(\sigma \to h'_{\alpha,j}\left(\sigma, s_{i-1}+\right)\right) = 0.$$

and thus

$$\sum_{i=1}^{n}\left\langle\left(X^{J_{\mathcal{P}}}h_{\alpha,j}\right)', h'_{\alpha,j}\right\rangle_{s_{i-1}+}\Delta_i = 0. \tag{5.23}$$

We now claim that

$$\left(q_s\left(X^{h_{\alpha,j}}\right)J_{\mathcal{P}}\left(s\right)\right)' = q'_s\left(X^{h_{\alpha,j}}\right)J_{\mathcal{P}}\left(s\right) + q_s\left(X^{h_{\alpha,j}}\right)J'_{\mathcal{P}}\left(s\right) = 0 \text{ for } s \in \mathcal{P}.$$

Since

$$h'_{\alpha,j}\left(s_{i-1}+\right) \neq 0 \text{ iff } i = j,$$

and when $i = j$,

$$h_{\alpha,j}\left(s\right) = 0 \text{ for } s \leq s_{i-1},$$

so both $q'_{s_{i-1}}\left(X^{h_{\alpha,j}}\right) = 0$ and $q_{s_{i-1}}\left(X^{h_{\alpha,j}}\right) = 0$. It then follows that the claim is true and

$$\sum_{i=1}^{n}\left\langle\left(q_s\left(X^{h_{\alpha,j}}\right)J_{\mathcal{P}}\left(s\right)\right)',h'_{\alpha,j}\right\rangle_{s_{i-1}+}\Delta_i=0. \tag{5.24}$$

and

$$\sum_{i=1}^{n}\left\langle q'_s\left(X^{J_{\mathcal{P}}}\right)h_{\alpha,j}\left(s\right),h'_{\alpha,j}\right\rangle_{s_{i-1}+}\Delta_i=0 \tag{5.25}$$

Lastly because $q_s\left(X^{J_{\mathcal{P}}}\right)$ is skew-symmetric,

$$\sum_{i=1}^{n}\left\langle q_s\left(X^{J_{\mathcal{P}}}\right)h'_{\alpha,j},h'_{\alpha,j}\right\rangle_{s_{i-1}+}\Delta_i=0 \tag{5.26}$$

Combining Eq.(5.23), (5.24), (5.25) and (5.26) shows,

$$G^1_{\mathcal{P}}\left\langle\left[X^{h_{\alpha,j}},\tilde{X}_{\mathcal{P}}\right],X^{h_{\alpha,j}}\right\rangle=\sum_{i=1}^{n}\left\langle X^{h_{\alpha,j}}J'_{\mathcal{P}},h'_{\alpha,j}\right\rangle_{s_{i-1}+}\Delta_i \tag{5.27}$$

$$=\left\langle X^{h_{\alpha,j}}J'_{\mathcal{P}}\left(s_{j-1}+\right),e_\alpha\right\rangle\sqrt{\Delta_j}. \tag{5.28}$$

Summing Eq.(5.28) on α and j while making use of (5.21) gives (5.20). ∎

Chapter 6

Convergence Result

In this chapter M is a complete Riemannian manifold with non–positive and bounded sectional curvature. We fix N to be a bound of the sectional curvature. Other conditions will be mentioned specificly in theorems if needed. First we modify and abuse a few notations we have defined before in order to avoid messy arguments.

Notation 6.1 *Recall that $\beta : W_o(M) \to W_0(\mathbb{R}^d)$ is the Brownian motion on $\mathbb{R}^d$ defined in Definition 2.22. We have also defined $\beta_{\mathcal{P}} : W_o(M) \to H_{\mathcal{P}}(\mathbb{R}^d)$ to be the linear approximation to Brownian motion on $\mathbb{R}^d$ as in Notation 2.26. Now denote by $u_{\mathcal{P}} := \eta \circ \beta_{\mathcal{P}}$ the development map of $\beta_{\mathcal{P}}$. Notice that $\phi \circ \beta_{\mathcal{P}} \in H_{\mathcal{P}}(M)$–$\nu$ a.s, here ϕ is the development map onto $H(M)$. So after identifying $C_{\mathcal{P},i}$, $S_{\mathcal{P},i}$ and hence $f_{\mathcal{P},i}$ with $C_{\mathcal{P},i} \circ \phi \circ \beta_{\mathcal{P}}$, $S_{\mathcal{P},i} \circ \phi \circ \beta_{\mathcal{P}}$ and $f_{\mathcal{P},i} \circ \phi \circ \beta_{\mathcal{P}}$, we can view them as maps from $W_o(M)$ to $End(\mathbb{R}^d)$. The point here is to make the notations short and it should not cause confusions after this explanation.*

Remark 6.2 *Let $L^{\infty-}(W_o(M)) := \cap_{q\geq 1} L^q(W_o(M))$. This is a Frechet space and for any $\{f_n\}_n$ and f in $L^{\infty-}(W_o(M))$, $f_n \to f$ as $n \to \infty$ in $L^{\infty-}(W_o(M))$ iff $f_n \to f$ in $L^q(W_o(M))$ $\forall q \geq 1$.*

Convention 6.3 *We use C to denote a generic constant. It can vary from line to* *line. In this chapter it depends only on an upper bound of the mesh size* $|\mathcal{P}| := \frac{1}{n}$ *of the partition* $|\mathcal{P}|$ *(We may allow* C *to depend on some other factors as well, but this is good enough for our purpose of taking the limit as* $|\mathcal{P}| \to 0$. *)*

6.1 Wong-Zakai Approximation Scheme

Wong-Zakai approximation scheme are types of theorems that approximate solutions to stochastic differential equations (SDEs) by solutions to (random) ordinary differential equations driven by "smooth "approximations of the semi-martingale that drives the SDE. Wong and Zakai [34], [35] first studied this problem in the case of one dimensional Brownian motion and there are a lot of generalizations that follow, which are partially listed in here : [1], [18] and so on. We record a Wong-Zakai type theorem in the form that fits our need.

Theorem 6.4 *Let* $f : \mathbb{R}^d \times \mathbb{R}^n \to End(\mathbb{R}^d, \mathbb{R}^n)$ *and* $f_0 : \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}^n$ *be either twice differentiable with bounded continuous derivatives or linear. Let* $\xi_0 \in \mathbb{R}^n$ *and* $\mathcal{P}$ *be a partition of* $[0,1]$. *Further let* β *and* $\beta_{\mathcal{P}}$ *be as in Notation 3.16 and* $\xi_{\mathcal{P}}(s)$ *denote the solution to the ordinary differential equation:*

$$\xi'_{\mathcal{P}}(s) = f(\xi_{\mathcal{P}}(s))\beta'_{\mathcal{P}}(s) + f_0(\xi_{\mathcal{P}}(s)), \qquad \xi_{\mathcal{P}}(0) = \xi_0 \tag{6.1}$$

and ξ *denote the solution to the Stratonovich stochastic differential equation,*

$$d\xi(s) = f(\xi(s))\delta\beta(s) + f_0(\xi(s))ds, \qquad \xi(0) = \xi_0. \tag{6.2}$$

Then, for any $\gamma \in (0, \frac{1}{2})$, $p \in [1, \infty)$, *there is a constant* $C(p,\gamma) < \infty$ *depending only on* f *and* M, *so that*

$$\lim_{|\mathcal{P}| \to 0} \mathbb{E}\left[\sup_{s \le 1} |\xi_{\mathcal{P}}(s) - \xi(s)|^p\right] \le C(p,\gamma)|\mathcal{P}|^{\gamma p}. \tag{6.3}$$

Corollary 6.5 $\sup_{0\leq s\leq 1}\left|u_{\mathcal{P}}(s)-\tilde{u}(s)\right|\to 0$ *as* $|\mathcal{P}|\to 0$ *in* $L^{\infty-}(W_o(M))$.

6.2 Convergence of $\tilde{X}_{\mathcal{P}}$ to $\tilde{X}$

6.2.1 Some Useful Estimates for $\{C_{\mathcal{P},i}\}_{i=1}^n$ and $\{S_{\mathcal{P},i}\}_{i=1}^n$

We apply Proposition B.2 to get the estimates in Lemmas 6.6 to 6.9 below.

Lemma 6.6 *For any* $i\in\{1,...,n\}$ *and* $s\in[s_{i-1},s_i]$, *we have*

$$\left|C_{\mathcal{P},i}(s)\right|\leq\cosh\left(\sqrt{N}\left|\Delta_i\beta\right|\right)\leq e^{\frac{1}{2}N|\Delta_i\beta|^2}.$$

Lemma 6.7 *For any* $i\in\{1,...,n\}$ *and* $s\in[s_{i-1},s_i]$, *we have*

$$\begin{aligned}\left|S_{\mathcal{P},i}(s)\right| &\leq \sqrt{N}\left|\Delta_i\beta\right|\frac{\sinh\left(\sqrt{N}\left|\Delta_i\beta\right|\right)}{\sqrt{N}\left|\Delta_i\beta\right|}\\ &\leq\cosh\left(\sqrt{N}\left|\Delta_i\beta\right|\right)\sqrt{N}\left|\Delta_i\beta\right|\leq\sqrt{N}\left|\Delta_i\beta\right|e^{\frac{1}{2}N|\Delta_i\beta|^2}.\end{aligned}$$

Lemma 6.8 *For any* $i\in\{1,...,n\}$, *we have*

$$\left|S_{\mathcal{P},i}-\Delta_i I\right|\leq\frac{N\left|\Delta_i\beta\right|^2\Delta_i}{6}e^{\frac{1}{2}N|\Delta_i\beta|^2}$$

Lemma 6.9 *For any* $i\in\{1,...,n\}$, *we have*

$$\left|C_{\mathcal{P},i}-I\right|\leq\frac{N\left|\Delta_i\beta\right|^2}{2}e^{\frac{1}{2}N|\Delta_i\beta|^2}$$

Lemma 6.10 *For all* $\gamma\in\left(0,\frac{1}{2}\right)$, *define* $K_\gamma:=\underset{s,t\in[0,1],s\neq t}{sup}\left\{\frac{|\beta_t-\beta_s|}{|t-s|^\gamma}\right\}$, *then there exists an* $\epsilon_\gamma>0$ *such that* $\mathbb{E}\left[e^{\epsilon K_\gamma^2}\right]<\infty$.

Proof. See Fernique's Theorem (Theorem 3.2) in [26]. ■

Remark 6.11 *From Lemma 6.10, it is easy to see any polynomial of* ϵK_γ *has finite moments of all orders.*

6.2.2 Size Estimates of $f_{\mathcal{P},i}(s)$

Recall from Definition 2.42 that $f_{\mathcal{P},i} : W_o(M) \times [0,1] \to End\left(\mathbb{R}^d\right)$ $0 \le i \le n$ is given by

$$f_{\mathcal{P},i}(s) = \begin{cases} 0 & s \in [0, s_{i-1}] \\ \frac{S_{\mathcal{P},i}(s)}{\Delta_i} & s \in [s_{i-1}, s_i] \\ \frac{C_{\mathcal{P},j}(s) C_{\mathcal{P},j-1} \cdots \cdot C_{\mathcal{P},i+1} S_{\mathcal{P},i}}{\Delta_i} & s \in [s_{j-1}, s_j] \text{ for } j = i+1, \cdots, n \end{cases}$$

with the convention that $S_{\mathcal{P},0} \equiv |\mathcal{P}| I$ and $f_{\mathcal{P},0} \equiv I$.

Using the estimates in Subsection 6.2.1, it is easy to get an estimate of $f_{\mathcal{P},i}(s)$.

Lemma 6.12 *Recall from the begining of this chapter that* $n := \frac{1}{|\mathcal{P}|}$ *and* N *is the sectional curvature bound. For each* $q \ge 1$*, we have*

$$\sup_{n \ge 2qN} \mathbb{E}\left[\sup_{i \in \{0, \cdots, n\}} \sup_{s \in \mathcal{P}} |f_{\mathcal{P},i}(s)|^q\right] < \infty.$$

Proof. For all $i, j \in \{0, \cdots, n\}$, if $j < i$, $f_{\mathcal{P},i}(s_j) \equiv 0$. So we only need to consider the case when $j \ge i$. Since

$$f_{\mathcal{P},i}(s_j) = \frac{C_{\mathcal{P},j} C_{\mathcal{P},j-1} \cdot \cdots \cdot C_{\mathcal{P},i+1} S_{\mathcal{P},i}}{\Delta_i},$$

so

$$|f_{\mathcal{P},i}(s_j)|^q \le |C_{\mathcal{P},j}|^q |C_{\mathcal{P},j-1}|^q \cdot \cdots \cdot |C_{\mathcal{P},i+1}|^q \left|\frac{S_{\mathcal{P},i}}{\Delta_i}\right|^q.$$

Apply Lemma 6.6 and 6.8 to find

$$|f_{\mathcal{P},i}(s_j)|^q \le e^{\frac{1}{2}qN\sum_{k=i}^{j}|\Delta_k\beta|^2}\left(e^{-\frac{N}{2}|\Delta_i\beta|^2}+\frac{N|\Delta_i\beta|^2}{6}\right)^q \tag{6.4}$$

$$\le e^{\frac{1}{2}qN\sum_{k=i}^{j}|\Delta_k\beta|^2}\left(1+\frac{N|\Delta_i\beta|^2}{6}\right)^q \tag{6.5}$$

$$\le e^{\frac{1}{2}qN\sum_{k=i}^{j}|\Delta_k\beta|^2}e^{\frac{Nq|\Delta_i\beta|^2}{6}} \tag{6.6}$$

$$\le e^{qN\sum_{k=1}^{n}|\Delta_k\beta|^2}. \tag{6.7}$$

Since $e^{qN\sum_{k=1}^{n}|\Delta_k\beta|^2}$ is independent of i and j, we have

$$\sup_{i\in\{1,\cdots,n\}}\sup_{s\in\mathcal{P}}|f_{\mathcal{P},i}(s)|^q \le e^{qN\sum_{k=1}^{n}|\Delta_k\beta|^2}. \tag{6.8}$$

Since for each k, $|\Delta_k\beta|^2 = \sum_{l=1}^{d}|(\Delta_k\beta)_l|^2$, where $\{(\Delta_k\beta)_l\}_{l=1}^{d}$ are coordinates of $\Delta_k\beta$, i.e. $\Delta_k\beta = ((\Delta_k\beta)_1,\ldots,(\Delta_k\beta)_d)$. Since β is a Brownian motion on $\mathbb{R}^d$, $\{(\Delta_k\beta)_l\}_{l=1}^{d}$ are i.i.d with Gaussian distribution of mean 0 and variance $\frac{1}{n}$. Using Lemma B.1 in Appendix B, notice that $qN < \frac{n}{2}$, we have

$$\mathbb{E}\left[e^{qN|\Delta_k\beta|^2}\right] = \Pi_{l=1}^{d}\mathbb{E}\left[e^{qN\left|(\Delta_k\beta)_l\right|^2}\right] = \left(1-\frac{2qN}{n}\right)^{-\frac{d}{2}}$$

and

$$\mathbb{E}\left[\sup_{i\in\{0,\cdots,n\}}\sup_{s\in\mathcal{P}}|f_{\mathcal{P},i}(s)|^q\right] \le \mathbb{E}\left[e^{qN\sum_{k=1}^{n}|\Delta_k\beta|^2}\right] \tag{6.9}$$

$$= \Pi_{k=1}^{n}\mathbb{E}\left[e^{qN|\Delta_k\beta|^2}\right] = \left(1-\frac{2qN}{n}\right)^{-\frac{nd}{2}}. \tag{6.10}$$

Since $\left(1-\frac{2qN}{n}\right)^{-\frac{nd}{2}} \to e^{-dqN}$ as $n\to\infty$, so $\left\{\left(1-\frac{2qN}{n}\right)^{-\frac{nd}{2}}\right\}_{n>2qN}$ is bounded and thus

$$\sup_{n\geq 2qN}\mathbb{E}\left[\sup_{i\in\{0,\cdots,n\}}\sup_{s\in\mathcal{P}}\left|f_{\mathcal{P},i}\left(s\right)\right|^{q}\right]<\infty.$$

■

Notation 6.13 *Given $n\in\mathbb{N}$ and $s\in[0,1]$, let $\underline{s}=s_{k-1}$ when $s\in[s_{k-1},s_k)$, $|\mathcal{P}|=\frac{1}{n}$ is the mesh size of the partition $\mathcal{P}$ and also let*

$$A_{\mathcal{P},k}\left(s\right):=R_{u_{\mathcal{P}}(s)}\left(\beta_{\mathcal{P}}'\left(s_{k-1}+\right),\cdot\right)\beta_{\mathcal{P}}'\left(s_{k-1}+\right).$$

Lemma 6.14 *For each $q\geq 1$, $\gamma\in\left(0,\frac{1}{2}\right)$ there exists a constant C such that for all $n>5qN$,*

$$\mathbb{E}\left[\sup_{i\in\{0,\cdots,n\},s\in[0,1]}\left|f_{\mathcal{P},i}\left(s\right)-f_{\mathcal{P},i}\left(\underline{s}\right)\right|^{q}\right]\leq C\left|\mathcal{P}\right|^{2q\gamma}. \tag{6.11}$$

Proof. For $s\in[s_{k-1},s_k)$, Taylor's expansion gives

$$f_{\mathcal{P},i}\left(s\right)-f_{\mathcal{P},i}\left(\underline{s}\right)=\int_{\underline{s}}^{s}A_{\mathcal{P},k}\left(r\right)f_{\mathcal{P},i}\left(r\right)\left(s-r\right)dr \tag{6.12}$$

$$=\int_{\underline{s}}^{s}A_{\mathcal{P},k}\left(r\right)\left(f_{\mathcal{P},i}\left(r\right)-f_{\mathcal{P},i}\left(\underline{r}\right)\right)\left(s-r\right)dr+\int_{\underline{s}}^{s}A_{\mathcal{P},k}\left(r\right)f_{\mathcal{P},i}\left(\underline{r}\right)\left(s-r\right)dr. \tag{6.13}$$

Since $\left|A_{\mathcal{P},k}\left(s\right)\right|\leq N\left|\frac{\Delta_k\beta}{\Delta_k}\right|^2$, we have

$$\left|f_{\mathcal{P},i}\left(s\right)-f_{\mathcal{P},i}\left(\underline{s}\right)\right|\leq\frac{N}{\Delta_k}\left|\Delta_k\beta\right|^2\int_{\underline{s}}^{s}\left|f_{\mathcal{P},i}\left(r\right)-f_{\mathcal{P},i}\left(\underline{r}\right)\right|dr+\frac{1}{2}N\left|\Delta_k\beta\right|^2\sup_{s\in\mathcal{P}}\left|f_{\mathcal{P},i}\left(s\right)\right|.$$

By Gronwall's inequality, we have:

$$\begin{aligned}|f_{\mathcal{P},i}(s)-f_{\mathcal{P},i}(\underline{s})| &\leq \frac{1}{2}N\left|\Delta_k\beta\right|^2\sup_{s\in\mathcal{P}}\left|f_{\mathcal{P},i}(s)\right|e^{\frac{N}{\Delta_k}\left|\Delta_k\beta\right|^2(s-\underline{s})}\\ &\leq \frac{1}{2}N\left|\Delta_k\beta\right|^2\sup_{s\in\mathcal{P}}\left|f_{\mathcal{P},i}(s)\right|e^{N\left|\Delta_k\beta\right|^2}\end{aligned}$$

Using estimate (6.8) gives

$$\begin{aligned}|f_{\mathcal{P},i}(s)-f_{\mathcal{P},i}(\underline{s})|^q &\leq \frac{N^q}{2^q}\left|\Delta_k\beta\right|^{2q}e^{qN\left|\Delta_k\beta\right|^2}e^{qN\sum_{j=1}^n\left|\Delta_j\beta\right|^2} &(6.14)\\ &\leq C\left|\mathcal{P}\right|^{2q\gamma}e^{2qN\sum_{k=1}^n\left|\Delta_k\beta\right|^2}K_\gamma^{2q}. &(6.15)\end{aligned}$$

Based on a computation exactly the same as (6.10), we know $\mathbb{E}\left[e^{2qN(1+\epsilon)\sum_{k=1}^n\left|\Delta_k\beta\right|^2}\right]$ is finite for some $\epsilon>0$ and the value is bounded above independently of n. Then using Remark 6.10 we see K_γ has finite moments of all orders. The estimate in (6.11) then follows by Holder's inequality. ■

Theorem 6.15 *Let $\tilde{T}_{(\cdot)}$ be as in Definition 4.14, then for each $q\geq 1$, $\gamma\in\left(0,\frac{1}{2}\right)$, there exists a constant C such that for all $n>5q\gamma$,*

$$\mathbb{E}\left[\sup_{i\in\{0,\cdots,n\}}\sup_{s\in[s_i,1]}\left|f_{\mathcal{P},i}(s)-\tilde{T}_s\tilde{T}_{s_i}^{-1}\right|^q\right]\leq C\left|\mathcal{P}\right|^{\gamma q}. \tag{6.16}$$

In order to prove Theorem 6.15, we need the following result.

Lemma 6.16 *For each $q\geq 1$, $\gamma\in\left(0,\frac{1}{2}\right)$, there exists a constant C such that for all $n>5q\gamma$,*

$$\begin{aligned}&\mathbb{E}\left[\sup_{i\in\{1,\cdots,n\}}\sup_{j\geq i}\left|f_{\mathcal{P},i}(s_j)-\left(f_{\mathcal{P},i}(s_i)-\int_{s_i}^{s_j}Ric_{u_{\mathcal{P}}(\underline{r})}f_{\mathcal{P},i}(\underline{r})\,dr\right)\right|^q\right] &(6.17)\\ &\leq C\left|\mathcal{P}\right|^{\gamma q}. &(6.18)\end{aligned}$$

Proof. For all $s_j \in \mathcal{P}$ with $j \geq i+1$ and for $k = i, \cdots, j-1$, we have

$$f_{\mathcal{P},i}(s_{k+1}) = f_{\mathcal{P},i}(s_k) + \frac{1}{\Delta_{k+1}^2}\int_{s_k}^{s_{k+1}} R_{u_{\mathcal{P}}(r)}(\Delta_{k+1}\beta, f_{\mathcal{P},i}(r))\,\Delta_{k+1}\beta\,(s_{k+1}-r)\,dr \tag{6.19}$$

$$= f_{\mathcal{P},i}(s_k) + \frac{1}{2}R_{u_{\mathcal{P}}(s_k)}(\Delta_{k+1}\beta, f_{\mathcal{P},i}(s_k))\,\Delta_{k+1}\beta + e_{i,k}$$

where

$$\begin{aligned} e_{i,k} = & \frac{1}{\Delta_{k+1}^2}\int_{s_k}^{s_{k+1}} R_{u_{\mathcal{P}}(r)}(\Delta_{k+1}\beta, f_{\mathcal{P},i}(r))\,\Delta_{k+1}\beta\,(s_{k+1}-r)\,dr \\ & - \frac{1}{\Delta_{k+1}^2}\int_{s_k}^{s_{k+1}} R_{u_{\mathcal{P}}(s_k)}(\Delta_{k+1}\beta, f_{\mathcal{P},i}(s_k))\,\Delta_{k+1}\beta\,(s_{k+1}-r)\,dr. \end{aligned}$$

Since $\{f_{\mathcal{P},i}(s_j)\}_j$ is adapted, by Ito's lemma

$$\begin{aligned} \frac{1}{2}R_{u_{\mathcal{P}}(s_k)}(\Delta_{k+1}\beta, f_{\mathcal{P},i}(s_k))\,\Delta_{k+1}\beta = & \frac{1}{2}\int_{s_k}^{s_{k+1}} R_{u_{\mathcal{P}}(s_k)}(\beta_r - \beta_{s_k}, f_{\mathcal{P},i}(s_k))\,d\beta_r \\ & + \frac{1}{2}\int_{s_k}^{s_{k+1}} R_{u_{\mathcal{P}}(s_k)}(d\beta_r, f_{\mathcal{P},i}(s_k))\,(\beta_r - \beta_{s_k}) \\ & - \frac{1}{2}Ric_{u_{\mathcal{P}}(s_k)} f_{\mathcal{P},i}(s_k)\,\Delta_k. \end{aligned}$$

Summing (6.19) over k from i to $j-1$, we have

$$f_{\mathcal{P},i}(s_j) = f_{\mathcal{P},i}(s_i) - \frac{1}{2}\int_{s_i}^{s_j} Ric_{u_{\mathcal{P}}(\underline{r})} f_{\mathcal{P},i}(\underline{r})\,dr + M_{\mathcal{P},s_j} + \sum_{k=i}^{j-1} e_{i,k}$$

where

$$M_{\mathcal{P},s} := \frac{1}{2}\int_{s_i}^{s} R_{u_{\mathcal{P}}(\underline{r})}(\beta_r - \beta_{\underline{r}}, f_{\mathcal{P},i}(\underline{r}))\,d\beta_r + \frac{1}{2}\int_{s_i}^{s} R_{u_{\mathcal{P}}(\underline{r})}(d\beta_r, f_{\mathcal{P},i}(\underline{r}))\,(\beta_r - \beta_{\underline{r}})$$

is a $\mathbb{R}^d$-valued martingale starting from s_i. By the Burkholder-Davis-Gundy inequality, for $q \geq 1$,

$$\mathbb{E}\left[\sup_{s\in[s_i,1]} |M_{\mathcal{P},s}|^q\right] \leq C\mathbb{E}\left[\langle M_{\mathcal{P}}\rangle_1^{\frac{q}{2}}\right] \tag{6.20}$$

where $\langle M_{\mathcal{P}}\rangle$ is the quadratic variation process of $M_{\mathcal{P}}$. An estimate of $\langle M_{\mathcal{P}}\rangle$ gives

$$\langle M_{\mathcal{P}}\rangle_1 \leq dN^2 \int_{s_i}^1 |\beta_r - \beta_{\underline{r}}|^2 |f_{\mathcal{P},i}(\underline{r})|^2 dr \leq dN^2 \int_0^1 |\beta_r - \beta_{\underline{r}}|^2 |f_{\mathcal{P},i}(\underline{r})|^2 dr,$$

and by Jensen's inequality,

$$\langle M_{\mathcal{P}}\rangle_1^{\frac{q}{2}} \leq d^{\frac{q}{2}} N^q \int_0^1 |\beta_r - \beta_{\underline{r}}|^q |f_{\mathcal{P},i}(\underline{r})|^q dr.$$

Since $\{f_{\mathcal{P},i}(\underline{r})\}_{r\in[0,1]}$ is adapted to the filtration generated by β, using the independence of $|\beta_r - \beta_{\underline{r}}|^q$ and $f_{\mathcal{P},i}(\underline{r})$ we have:

$$\begin{aligned}\mathbb{E}\left[\langle M_{\mathcal{P}}\rangle_1^{\frac{q}{2}}\right] &\leq d^{\frac{q}{2}} N^q \int_0^1 \mathbb{E}\left[|\beta_r - \beta_{\underline{r}}|^q\right] \mathbb{E}\left[|f_{\mathcal{P},i}(\underline{r})|^q\right] dr \\ &= C\sup_{s\in\mathcal{P}}\mathbb{E}\left[|f_{\mathcal{P},i}(s)|^q\right] |\mathcal{P}|^{\frac{q}{2}}.\end{aligned}$$

By Lemma 6.12, we know

$$\mathbb{E}\left[\langle M_{\mathcal{P}}\rangle_1^{\frac{q}{2}}\right] \leq C|\mathcal{P}|^{\frac{q}{2}} \tag{6.21}$$

So to finish the proof of Lemma 6.16, it suffices to show:

$$\mathbb{E}\left[\sup_{i\in\{0,\cdots,n\},j\in\{i+1,\cdots,n\}} \left|\sum_{k=i}^{j-1} e_{i,k}\right|^q\right] \leq C|\mathcal{P}|^{\gamma q}. \tag{6.22}$$

Since $|e_{i,k}| \leq I_{\mathcal{P}} + II_{\mathcal{P}}$, where

$$I_{\mathcal{P}} = \frac{1}{\Delta_{k+1}^2}\left|\int_{s_k}^{s_{k+1}} R_{u_{\mathcal{P}}(r)}\left(\Delta_{k+1}\beta, f_{\mathcal{P},i}(r) - f_{\mathcal{P},i}(s_k)\right)\Delta_{k+1}\beta\,(s_{k+1}-r)\,dr\right|$$

and

$$II_{\mathcal{P}} = \frac{1}{\Delta_{k+1}^2} \left| \int_{s_k}^{s_{k+1}} \left(R_{u_{\mathcal{P}}(s_k)} - R_{u_{\mathcal{P}}(r)} \right) \left(\Delta_{k+1}\beta, f_{\mathcal{P},i}(s_k) \right) \Delta_{k+1}\beta \left(s_{k+1} - r \right) dr \right|,$$

using (6.15) we know

$$I_{\mathcal{P}} \leq \frac{N}{2} \sup_{i \in \{1,\cdots,n\}, r \in [0,1]} \left| f_{\mathcal{P},i}(r) - f_{\mathcal{P},i}(\underline{r}) \right| \left| \Delta_{k+1}\beta \right|^2 \leq C K_\gamma^4 \left| \mathcal{P} \right|^{4\gamma} e^{2N \sum_{k=1}^n |\Delta_k \beta|^2}.$$

Since

$$\left| R_{u_{\mathcal{P}}(s_k)} - R_{u_{\mathcal{P}}(r)} \right| \leq C \int_{s_k}^{s_{k+1}} \left| \beta'_{\mathcal{P}}(s) \right| ds = C \left| \Delta_{k+1}\beta \right| \leq C K_\gamma \left| \mathcal{P} \right|^\gamma,$$

using (6.8) we have

$$\begin{aligned} II_{\mathcal{P}} &\leq C \sup_{i \in \{1,\cdots,n\}, r \in \mathcal{P}} \left| f_{\mathcal{P},i}(r) \right| \left| \Delta_{k+1}\beta \right|^2 \sup_{r \in [s_k, s_{k+1}]} \left| R_{u_{\mathcal{P}}(s_k)} - R_{u_{\mathcal{P}}(r)} \right| \\ &\leq C K_\gamma^3 \left| \mathcal{P} \right|^{3\gamma} e^{N \sum_{k=1}^n |\Delta_k \beta|^2}. \end{aligned}$$

So

$$\left| \sum_{k=i}^{j-1} e_{i,k} \right| \leq \frac{1}{|\mathcal{P}|} (I + II) \leq C \left(K_\gamma^4 \left| \mathcal{P} \right|^{4\gamma - 1} + K_\gamma^3 \left| \mathcal{P} \right|^{3\gamma - 1} \right) e^{2N \sum_{k=1}^n |\Delta_k \beta|^2}.$$

Since if γ approaches $\frac{1}{2}$, $3\gamma - 1$ approaches $\frac{1}{2}$, so using Lemma 6.10 we get

$$\mathbb{E}\left[\sup_{i \in \{0,\cdots,n\}, j \in \{i+1,\cdots,n\}} \left| \sum_{k=i}^{j-1} e_{i,k} \right|^q \right] \leq C \left| \mathcal{P} \right|^{\gamma q}.$$

Combining (6.21) and (6.22) we obtain (6.18). ■

Proof of Theorem 6.15. For $s \geq s_i$, define

$$\hat{f}_{\mathcal{P},i}(s) := f_{\mathcal{P},i}(s_i) - \frac{1}{2}\int_{s_i}^{s} Ric_{u_{\mathcal{P}}(r)} f_{\mathcal{P},i}(r)\, dr. \tag{6.23}$$

Then

$$\begin{aligned}\left|\hat{f}_{\mathcal{P},i}(s_j) - f_{\mathcal{P},i}(s_j)\right| \leq &\left|\frac{1}{2}\int_{s_i}^{s_j} \left(Ric_{u_{\mathcal{P}}(r)} - Ric_{u_{\mathcal{P}}(\underline{r})}\right) f_{\mathcal{P},i}(\underline{r})\, dr\right| \\ &+ \left|\frac{1}{2}\int_{s_i}^{s_j} Ric_{u_{\mathcal{P}}(r)} \left(f_{\mathcal{P},i}(r) - f_{\mathcal{P},i}(\underline{r})\right) dr\right|.\end{aligned}$$

Since

$$\left|Ric_{u_{\mathcal{P}}(r)} - Ric_{u_{\mathcal{P}}(\underline{r})}\right| \leq C K_\gamma \left|\mathcal{P}\right|^\gamma,$$

using Lemma 6.12 and Eq.(6.8), we know:

$$\left|\int_{s_i}^{s_j} \left(Ric_{u_{\mathcal{P}}(r)} - Ric_{u_{\mathcal{P}}(\underline{r})}\right) f_{\mathcal{P},i}(\underline{r})\, dr\right|^q \leq C K_\gamma^q \left|\mathcal{P}\right|^{\gamma q} \tag{6.24}$$

and

$$\mathbb{E}\left[\left|\int_{s_i}^{s_j} \left(Ric_{u_{\mathcal{P}}(r)} - Ric_{u_{\mathcal{P}}(\underline{r})}\right) f_{\mathcal{P},i}(\underline{r})\, dr\right|^q\right] \leq C \left|\mathcal{P}\right|^{\gamma q}.$$

Then consider

$$\left|\int_{s_i}^{s_j} Ric_{u_{\mathcal{P}}(r)} \left(f_{\mathcal{P},i}(r) - f_{\mathcal{P},i}(\underline{r})\right) dr\right|.$$

By Lemma 6.14, one can easily see

$$\mathbb{E}\left[\sup_{i\in\{0,\cdots,n\}} \left|\int_{s_i}^{s_j} Ric_{u_{\mathcal{P}}(r)} \left(f_{\mathcal{P},i}(r) - f_{\mathcal{P},i}(\underline{r})\right) dr\right|^q\right] \leq C \left|\mathcal{P}\right|^q. \tag{6.25}$$

Combining (6.24) and (6.25) we get

$$\mathbb{E}\left[\sup_{i\in\{0,\cdots,n\}, j\geq i} \left|\hat{f}_{\mathcal{P},i}(s_j) - f_{\mathcal{P},i}(s_j)\right|^q\right] \leq C \left|\mathcal{P}\right|^{\gamma q}. \tag{6.26}$$

Then for $s \geq s_i$, define $\tilde{f}_{\mathcal{P},i}(s)$ to be the solution to the following ODE

$$\begin{cases} \frac{d}{ds}\tilde{f}_{\mathcal{P},i}(s) + \frac{1}{2}Ric_{u_{\mathcal{P}}(s)}\tilde{f}_{\mathcal{P},i}(s) = 0 \\ \tilde{f}_{\mathcal{P},i}(s_i) = I. \end{cases}$$

Therefore

$$\tilde{f}_{\mathcal{P},i}(s) = I - \frac{1}{2}\int_{s_i}^{s} Ric_{u_{\mathcal{P}}(r)}\tilde{f}_{\mathcal{P},i}(r)\,dr$$

and

$$\left|\tilde{f}_{\mathcal{P},i}(s) - \hat{f}_{\mathcal{P},i}(s)\right| \leq \left|f_{\mathcal{P},i}(s_i) - I\right| + \frac{1}{2}\int_{s_i}^{s} N\left|\tilde{f}_{\mathcal{P},i}(r) - \hat{f}_{\mathcal{P},i}(r)\right| dr.$$

By Gronwall's inequality we have

$$\left|\tilde{f}_{\mathcal{P},i}(s) - \hat{f}_{\mathcal{P},i}(s)\right| \leq \left|f_{\mathcal{P},i}(s_i) - I\right| e^{\frac{1}{2}N}.$$

Thus by Lemma 6.8, it follows that

$$\mathbb{E}\left[\sup_{i\in\{0,\cdots,n\},s\geq s_i}\left|\tilde{f}_{\mathcal{P},i}(s) - \hat{f}_{\mathcal{P},i}(s)\right|^q\right] \leq C\left|\mathcal{P}\right|^q. \tag{6.27}$$

Lastly, we look at $\tilde{f}_{\mathcal{P},i}(s) - \tilde{T}_s\tilde{T}_{s_i}^{-1}$ where $s \geq s_i$. Note that $\tilde{T}_s\tilde{T}_{s_i}^{-1}$ satisfies the following ODE,

$$\begin{cases} \left(\tilde{T}_s\tilde{T}_{s_i}^{-1}\right)' + \frac{1}{2}Ric_{\tilde{u}_s}\left(\tilde{T}_s\tilde{T}_{s_i}^{-1}\right) = 0 \\ \left(\tilde{T}_{s_i}\tilde{T}_{s_i}^{-1}\right) = I. \end{cases}$$

So

$$\tilde{f}_{\mathcal{P},i}(s) - \tilde{T}_s\tilde{T}_{s_i}^{-1} = \frac{1}{2}\int_{s_i}^{s}\left(Ric_{u_{\mathcal{P}}(r)} - Ric_{\tilde{u}_r}\right)\left(\tilde{f}_{\mathcal{P},i}(r) - \tilde{T}_r\tilde{T}_{s_i}^{-1}\right) dr.$$

By Gronwall's inequality again we have

$$\left|\tilde{f}_{\mathcal{P},i}(s)-\tilde{T}_s\tilde{T}_{s_i}^{-1}\right|\le CK_\gamma\left|\mathcal{P}\right|^\gamma e^{\frac{1}{2}N},$$

so

$$\mathbb{E}\left[\sup_{i\in\{0,\cdots,n\},s\ge s_i}\left|\tilde{f}_{\mathcal{P},i}(s)-\tilde{T}_s\tilde{T}_{s_i}^{-1}\right|^q\right]\le C\left|\mathcal{P}\right|^{\gamma q}. \tag{6.28}$$

The proof is complteted by combining Lemma 6.16 and (6.25), (6.26), (6.27) and (6.28). ■

6.2.3 Convergence of $\mathbf{K}_{\mathcal{P}}(s)$ to $\tilde{\mathbf{K}}_s$

Recall from Definition 5.4 that $\mathbf{K}_{\mathcal{P}}(s)$ satisfies the piecewise Jacobi equation:

$$\begin{cases}\mathbf{K}_{\mathcal{P}}''(s)=R_{u_{\mathcal{P}}(s)}\left(\beta_{\mathcal{P}}'(s_{i-1}+),\mathbf{K}_{\mathcal{P}}(s)\right)\beta_{\mathcal{P}}'(s_{i-1}+)\ \text{ for } s\in[s_{i-1},s_i)\\ \mathbf{K}_{\mathcal{P}}'(s_{i-1}+)=f_{\mathcal{P},i}^*(1)\ \text{ and } \mathbf{K}_{\mathcal{P}}(0)=0,\text{for } i=1,...,n\end{cases} \tag{6.29}$$

where $f_{\mathcal{P},i}(1)$ is given in Definition 2.42.

Before we state the main theorem in this section, we need some supplementary lemmas.

Lemma 6.17 *Recall that $n:=\frac{1}{|\mathcal{P}|}$ and N is the curvature bound. For each $q\ge1$,*

$$\sup_{n>2qN}\mathbb{E}\left[\sup_{r\in\mathcal{P}}\left|\mathbf{K}_{\mathcal{P}}(r)\right|^q\right]<\infty. \tag{6.30}$$

Proof. For all $i\in\{1,\cdots,n\}$, recall from (5.6) that

$$\mathbf{K}_{\mathcal{P}}(s_i)=\frac{1}{n}\sum_{j=0}^{i-1}f_{\mathcal{P},j+1}(s)\,f_{\mathcal{P},j+1}^*(1).$$

where $f_{\mathcal{P},i}$ is defined in Definition 2.42.

So for all $q \geq 1$, we have

$$|\mathbf{K}_{\mathcal{P}}(s_i)|^q \leq i^{q-1}\frac{1}{n^q}\sum_{j=0}^{i-1}|f_{\mathcal{P},j+1}(s_i)|^q\,|f_{\mathcal{P},j+1}(1)|^q\,.$$

Using (6.8) we have

$$|\mathbf{K}_{\mathcal{P}}(s_i)|^q \leq e^{2qN\sum_{k=1}^{n}|\Delta_k\beta|^2}. \tag{6.31}$$

Then taking expectations as in Lemma 6.12 gives (6.30). ■

Lemma 6.18 *For each $q \geq 1$ and $\gamma \in \left(0, \frac{1}{2}\right)$, there exists a constant $C > 0$ such that for all $n > 5qN$,*

$$\mathbb{E}\left[\sup_{i\in\{1,\cdots,n\},r\in[0,1]}|\mathbf{K}_{\mathcal{P}}(r) - \mathbf{K}_{\mathcal{P}}(\underline{r})|^q\right] \leq C\,|\mathcal{P}|^{2q\gamma}$$

Proof. For $s \in [s_{i-1}, s_i]$,

$$\mathbf{K}_{\mathcal{P}}(s) = \mathbf{K}_{\mathcal{P}}(s_{i-1}) \tag{6.32}$$

$$+\, f^*_{\mathcal{P},i}(1)(s - s_{i-1}) + \int_{s_{i-1}}^{s} R_{u_{\mathcal{P}}(s)}\left(\beta'_{\mathcal{P}}(s_{i-1}+), \mathbf{K}_{\mathcal{P}}(r)\right)\beta'_{\mathcal{P}}(s_{i-1}+)(s-r)\,dr. \tag{6.33}$$

Therefore

$$|\mathbf{K}_{\mathcal{P}}(s) - \mathbf{K}_{\mathcal{P}}(s_{i-1})| \tag{6.34}$$

$$\leq |f_{\mathcal{P},i}(1)|(s - s_{i-1}) \tag{6.35}$$

$$+\left|\int_{s_{i-1}}^{s} R_{u_{\mathcal{P}}(s)}\left(\beta'_{\mathcal{P}}(s_{i-1}+), \mathbf{K}_{\mathcal{P}}(r) - \mathbf{K}_{\mathcal{P}}(s_{i-1}) + \mathbf{K}_{\mathcal{P}}(s_{i-1})\right)\beta'_{\mathcal{P}}(s_{i-1}+)(s-r)\,dr\right|$$

$$\leq |f_{\mathcal{P},i}(1)|(s - s_{i-1}) \tag{6.36}$$

$$+\, N\frac{|\Delta_i\beta|^2}{\Delta_i^2}\int_{s_{i-1}}^{s}|\mathbf{K}_{\mathcal{P}}(r) - \mathbf{K}_{\mathcal{P}}(s_{i-1})|(s-r)\,dr + \frac{1}{2}N|\Delta_i\beta|^2\,|\mathbf{K}_{\mathcal{P}}(s_{i-1})|$$

We use the shorthand

$$f(s) := |f_{\mathcal{P},i}(1)|(s - s_{i-1}) + N\frac{|\Delta_i\beta|^2}{\Delta_i^2}\int_{s_{i-1}}^{s} |\mathbf{K}_{\mathcal{P}}(r) - \mathbf{K}_{\mathcal{P}}(s_{i-1})|(s-r)\,dr$$
$$+\frac{1}{2}N|\Delta_i\beta|^2|\mathbf{K}_{\mathcal{P}}(s_{i-1})|.$$

Then it is easily seen that

$$f'(s) = |f_{\mathcal{P},i}(1)| + N\frac{|\Delta_i\beta|^2}{\Delta_i^2}\int_{s_{i-1}}^{s} |\mathbf{K}_{\mathcal{P}}(r) - \mathbf{K}_{\mathcal{P}}(s_{i-1})|\,dr,$$

$$f''(s) = N\frac{|\Delta_i\beta|^2}{\Delta_i^2}|\mathbf{K}_{\mathcal{P}}(s) - \mathbf{K}_{\mathcal{P}}(s_{i-1})| \le N\frac{|\Delta_i\beta|^2}{\Delta_i^2}f(s),$$

and $f(s)$ satisfies the following ODE

$$\begin{cases} f''(s) = N\frac{|\Delta_i\beta|^2}{\Delta_i^2}f(s) + \delta(s) \\ f'(s_{i-1}) = |f_{\mathcal{P},i}(1)| \\ f(s_{i-1}) = \frac{1}{2}N|\Delta_i\beta|^2|\mathbf{K}_{\mathcal{P}}(s_{i-1})| \end{cases} \tag{6.37}$$

where

$$\delta(s) = f''(s) - N\frac{|\Delta_i\beta|^2}{\Delta_i^2}f(s) \le 0.$$

This ODE can be solved exactly to obtain

$$f(s) = \mathcal{C}_i(s)\frac{1}{2}N|\Delta_i\beta|^2|\mathbf{K}_{\mathcal{P}}(s_{i-1})| + \mathcal{S}_{s_{i-1}}(s)|f_{\mathcal{P},i}(1)| + \int_{s_{i-1}}^{s}\mathcal{S}_r(s)\,\delta(r)\,dr$$

where

$$\mathcal{C}_i(s) := \cosh\left(\sqrt{N}\,|\beta'_{\mathcal{P}}(s_{i-1}+)|(s - s_{i-1})\right)$$

and

$$\mathcal{S}_r(s) := \frac{\sinh\left(\sqrt{N}\,|\beta'_{\mathcal{P}}(s_{i-1}+)|(s-r)\right)}{\sqrt{N}\,|\beta'_{\mathcal{P}}(s_{i-1}+)|}.$$

Since $\delta(r) \leq 0$ and $\mathcal{S}_r(s) \geq 0$, we have

$$f(s) \leq \mathcal{C}_i(s) \frac{1}{2} N |\Delta_i \beta|^2 |\mathbf{K}_{\mathcal{P}}(s_{i-1})| + \mathcal{S}_i(s) |f_{\mathcal{P},i}(1)| .$$

Then using the following estimate

$$\frac{\mathcal{S}_{s_{i-1}}(s)}{\Delta_i} \leq \mathcal{C}_i(s) \frac{s - s_{i-1}}{\Delta_i} \leq e^{N|\Delta_i \beta|^2},$$

we obtain

$$\begin{aligned} f(s) &\leq e^{N|\Delta_i \beta|^2} \left(\frac{1}{2} N |\Delta_i \beta|^2 |\mathbf{K}_{\mathcal{P}}(s_{i-1})| + |\mathcal{P}| |f_{\mathcal{P},i}(1)| \right) \qquad (6.38) \\ &\leq e^{NK_\gamma^2 |\mathcal{P}|^{2\gamma}} \left(\frac{1}{2} N K_\gamma^2 |\mathcal{P}|^{2\gamma} \sup_{i \in \{1,\cdots,n\}} |\mathbf{K}_{\mathcal{P}}(s_{i-1})| + |\mathcal{P}| \sup_{i \in \{1,\cdots,n\}, s \in [0,1]} |f_{\mathcal{P},i}(s)| \right). \end{aligned}$$

Note that $f \geq 0$, using (6.8) and (6.31) we have for $q \geq 1$,

$$f^q(s) \leq U_q |P|^{2q\gamma},$$

where

$$U_q = e^{qNK_\gamma^2 |\mathcal{P}|^{2\gamma}} \left(\frac{1}{2} N K_\gamma^2 + |\mathcal{P}|^{1-2\gamma} \right)^q e^{qN \sum_{k=1}^n |\Delta_k \beta|^2}$$

is a random variable with finite first moment which can be bounded uniformly for $n > 5qN$. Therefore,

$$\mathbb{E}\left[\sup_{i \in \{1,\cdots,n\}, r \in [0,1]} |\mathbf{K}_{\mathcal{P}}(r) - \mathbf{K}_{\mathcal{P}}(\underline{r})|^q \right] \leq C |\mathcal{P}|^{2q\gamma} \qquad (6.39)$$

■

Lemma 6.19 *Let* $\mathbf{K}_{\mathcal{P}}$ *and* $\tilde{\mathbf{K}}$ *be defined as in Definition 5.6 and 4.16. Then for* *each* $q \geq 1$ *and* $\gamma \in \left(0, \frac{1}{2}\right)$*, there exists a constant* $C > 0$ *such that for all* $n > 5qN$*,*

$$\mathbb{E}\left[\sup_{s\in\mathcal{P}}\left|\mathbf{K}_{\mathcal{P}}(s) - \tilde{\mathbf{K}}_s\right|^q\right] \leq C_{q,\gamma}\left|\mathcal{P}\right|^q. \tag{6.40}$$

Proof. For all $i \in \{1, \cdots, n\}$, $\mathbf{K}_{\mathcal{P}}(s_i)$ and $\tilde{\mathbf{K}}_{s_i}$ can be rewritten as

$$\mathbf{K}_{\mathcal{P}}(s_i) = f_{\mathcal{P},i-1}(s_i)\, f_{\mathcal{P},i-1}(1)^{-1}\left(\sum_{j=0}^{i-1} f_{\mathcal{P},j+1}(1)\, f^*_{\mathcal{P},j+1}(1)\right)\left|\mathcal{P}\right| \tag{6.41}$$

and

$$\tilde{\mathbf{K}}_{s_i} = \tilde{T}_{s_i}\tilde{T}_1^{-1}\int_0^{s_i}\left(\tilde{T}_1\tilde{T}_r^{-1}\right)\left(\tilde{T}_1\tilde{T}_r^{-1}\right)^* dr.$$

First define

$$\bar{\mathbf{K}}_{\mathcal{P}}(s_i) := \tilde{T}_{s_i}\tilde{T}_1^{-1}\int_0^{s_i}\left(\tilde{T}_1\tilde{T}_{\overline{r}}^{-1}\right)\left(\tilde{T}_1\tilde{T}_{\overline{r}}^{-1}\right)^* dr,$$

where $\overline{r} = s_i$ if $s \in [s_{i-1}, s_i)$. We will show, for each $q \geq 1$,

$$\sup_{s\in\mathcal{P}}\left|\tilde{\mathbf{K}}_s - \bar{\mathbf{K}}_{\mathcal{P}}(s)\right|^q \leq C\left|\mathcal{P}\right|^q. \tag{6.42}$$

Recall from (4.7) that $\tilde{T}_1\tilde{T}_r^{-1}$ satisfies the following ODE,

$$\frac{d}{dr}\left(\tilde{T}_1\tilde{T}_r^{-1}\right) = \frac{1}{2}\left(\tilde{T}_1\tilde{T}_r^{-1}\right)Ric_{\tilde{u}_r}.$$

So by Lemma 4.8,

$$\left|\frac{d}{dr}\left(\tilde{T}_1\tilde{T}_r^{-1}\right)\right| \leq N\left|\tilde{T}_1\tilde{T}_r^{-1}\right| \leq N.$$

Therefore

$$\begin{aligned}\left|\left(\tilde{T}_1\tilde{T}_r^{-1}\right)\left(\tilde{T}_1\tilde{T}_r^{-1}\right)^* - \left(\tilde{T}_1\tilde{T}_{\overline{r}}^{-1}\right)\left(\tilde{T}_1\tilde{T}_{\overline{r}}^{-1}\right)^*\right| &\leq \int_r^{\overline{r}}\left|\frac{d}{ds}\left[\left(\tilde{T}_1\tilde{T}_s^{-1}\right)\left(\tilde{T}_1\tilde{T}_s^{-1}\right)^*\right]\right| ds \\ &\leq 2\int_r^{\overline{r}}\left|\frac{d}{ds}\left(\tilde{T}_1\tilde{T}_s^{-1}\right)\right|\left|\left(\tilde{T}_1\tilde{T}_s^{-1}\right)^*\right| ds \\ &\leq C\left(\overline{r}-r\right) \\ &\leq C\left|\mathcal{P}\right|,\end{aligned}$$

and

$$\begin{aligned}\left|\tilde{\mathbf{K}}_{s_i} - \bar{\mathbf{K}}_{\mathcal{P}}\left(s_i\right)\right| &\leq \left|\tilde{T}_{s_i}\tilde{T}_1^{-1}\right|\int_0^{s_i}\left|\left(\tilde{T}_1\tilde{T}_r^{-1}\right)\left(\tilde{T}_1\tilde{T}_r^{-1}\right)^* - \left(\tilde{T}_1\tilde{T}_{\overline{r}}^{-1}\right)\left(\tilde{T}_1\tilde{T}_{\overline{r}}^{-1}\right)^*\right| dr \\ &\leq C\left|\mathcal{P}\right|.\end{aligned}$$

Since the right–hand side is independent of i, we proved (6.42). Secondly, define

$$\hat{\mathbf{K}}_{\mathcal{P}}\left(s_i\right) := \tilde{T}_{s_i}\tilde{T}_1^{-1}\left(\sum_{j=0}^{i-1} f_{\mathcal{P},j+1}\left(1\right) f_{\mathcal{P},j+1}^*\left(1\right)\right)\left|\mathcal{P}\right|.$$

We will show, for each $q \geq 1, \gamma \in \left(0, \frac{1}{2}\right)$, there exists a constant $C > 0$ such that for all $n > 5qN$, we have

$$\mathbb{E}\left[\sup_{s\in\mathcal{P}}\left|\hat{\mathbf{K}}_{\mathcal{P}}\left(s\right) - \bar{\mathbf{K}}_{\mathcal{P}}\left(s\right)\right|^q\right] \leq C\left|\mathcal{P}\right|^{q\gamma}. \tag{6.43}$$

For all $j \in \{1, \cdots, n\}$,

$$\begin{aligned}
&\left| f_{\mathcal{P},j+1}(1) f^*_{\mathcal{P},j+1}(1) - \left(\tilde{T}_1 \tilde{T}^{-1}_{s_{j+1}}\right)\left(\tilde{T}_1 \tilde{T}^{-1}_{s_{j+1}}\right)^* \right| \\
&\leq \left| f_{\mathcal{P},j+1}(1) f^*_{\mathcal{P},j+1}(1) - f_{\mathcal{P},j+1}(1)\left(\tilde{T}_1 \tilde{T}^{-1}_{s_{j+1}}\right)^* \right| \\
&+ \left| f_{\mathcal{P},j+1}(1)\left(\tilde{T}_1 \tilde{T}^{-1}_{s_{j+1}}\right)^* - \left(\tilde{T}_1 \tilde{T}^{-1}_{s_{j+1}}\right)\left(\tilde{T}_1 \tilde{T}^{-1}_{s_{j+1}}\right)^* \right| \\
&\leq \left(\left| f_{\mathcal{P},j+1}(1) \right| + \left| \tilde{T}_1 \tilde{T}^{-1}_{s_{j+1}} \right| \right) \left| f_{\mathcal{P},j+1}(1) - \tilde{T}_1 \tilde{T}^{-1}_{s_{j+1}} \right|.
\end{aligned}$$

Since $|f_{\mathcal{P},j+1}(1)| \leq e^{N\sum_{k=1}^n |\Delta_k \beta|^2}$ by (6.8), ans also $\left|\tilde{T}_1 \tilde{T}^{-1}_{s_{j+1}}\right| \leq 1$, we have

$$\begin{aligned}
&\left| f_{\mathcal{P},j+1}(1) f^*_{\mathcal{P},j+1}(1) - \left(\tilde{T}_1 \tilde{T}^{-1}_{s_{j+1}}\right)\left(\tilde{T}_1 \tilde{T}^{-1}_{s_{j+1}}\right)^* \right| \\
&\leq \left(e^{N\sum_{k=1}^n |\Delta_k \beta|^2} + 1 \right) \sup_{j \in \{1, \cdots, n\}} \left| f_{\mathcal{P},j+1}(1) - \tilde{T}_1 \tilde{T}^{-1}_{s_{j+1}} \right|.
\end{aligned}$$

Thus for all $i \in \{1, \cdots, n\}$,

$$\begin{aligned}
\left| \hat{\mathbf{K}}_{\mathcal{P}}(s_i) - \tilde{\mathbf{K}}_{\mathcal{P}}(s_i) \right|^q &\leq |\mathcal{P}|^q \, i^{-q} \sum_{j=0}^{i-1} \left| f_{\mathcal{P},j+1}(1) f^*_{\mathcal{P},j+1}(1) - \left(\tilde{T}_1 \tilde{T}^{-1}_{s_{j+1}}\right)\left(\tilde{T}_1 \tilde{T}^{-1}_{s_{j+1}}\right)^* \right|^q \\
&\leq \left(e^{N\sum_{k=1}^n |\Delta_k \beta|^2} + 1 \right)^q \sup_{j \in \{1, \cdots, n\}} \left| f_{\mathcal{P},j+1}(1) - \tilde{T}_1 \tilde{T}^{-1}_{s_{j+1}} \right|^q.
\end{aligned}$$

Since $\left(e^{N\sum_{k=1}^n |\Delta_k \beta|^2} + 1 \right)^q \leq e^{qN\sum_{k=1}^n |\Delta_k \beta|^2}$, using Holder's inequality and Theorem 6.14 we get

$$\mathbb{E}\left[\sup_{s \in \mathcal{P}} \left| \hat{\mathbf{K}}_{\mathcal{P}}(s) - \tilde{\mathbf{K}}_{\mathcal{P}}(s) \right|^q \right] \leq C \, |\mathcal{P}|^{q\gamma}.$$

Lastly, we estimate $\hat{\mathbf{K}}_{\mathcal{P}}(s_i) - \mathbf{K}_{\mathcal{P}}(s_i)$. Using (6.41) we have

$$\begin{aligned}
&\left|\hat{\mathbf{K}}_{\mathcal{P}}(s_i) - \mathbf{K}_{\mathcal{P}}(s_i)\right| \\
&\leq \left|f_{\mathcal{P},i-1}(s_i)\, f_{\mathcal{P},i-1}(1)^{-1} - \tilde{T}_{s_i}\tilde{T}_1^{-1}\right| \left|\left(\sum_{j=0}^{i-1} f_{\mathcal{P},j+1}(1)\, f^*_{\mathcal{P},j+1}(1)\right)|\mathcal{P}|\right| \\
&\leq \left|f_{\mathcal{P},i-1}(s_i)\, f_{\mathcal{P},i-1}(1)^{-1} - \tilde{T}_{s_i}\tilde{T}_1^{-1}\right| \sup_{j\in\{1,\cdots,n\}} \left|f_{\mathcal{P},j+1}(1)\right|^2
\end{aligned}$$

Since

$$\begin{aligned}
&\left|f_{\mathcal{P},i-1}(s_i)\, f_{\mathcal{P},i-1}(1)^{-1} - \tilde{T}_{s_i}\tilde{T}_1^{-1}\right| \\
&= \left|f_{\mathcal{P},i-1}(s_i) - \tilde{T}_{s_i}\tilde{T}_{s_{i-1}}^{-1}\right| \left|f^{-1}_{\mathcal{P},i-1}(1)\right| + \left|\tilde{T}_{s_i}\tilde{T}_{s_{i-1}}^{-1}\right| \left|\left(\tilde{T}_1\tilde{T}_{s_{i-1}}^{-1}\right)^{-1} - f^{-1}_{\mathcal{P},i-1}(1)\right|,
\end{aligned}$$

and from Lemma 5.5, we know $\left|f_{\mathcal{P},i-1}(1)^{-1}\right| \leq 1$, and

$$\begin{aligned}
&\left|\left(\tilde{T}_1\tilde{T}_{s_{i-1}}^{-1}\right)^{-1} - f_{\mathcal{P},i-1}(1)^{-1}\right| \\
&\leq \left|\left(\tilde{T}_1\tilde{T}_{s_{i-1}}^{-1}\right)^{-1}\right| \left|\tilde{T}_1\tilde{T}_{s_{i-1}}^{-1} - f_{\mathcal{P},i-1}(1)\right| \left|f_{\mathcal{P},i-1}(1)^{-1}\right| \\
&\leq \left|\tilde{T}_1\tilde{T}_{s_{i-1}}^{-1} - f_{\mathcal{P},i-1}(1)\right|.
\end{aligned}$$

So

$$\left|f_{\mathcal{P},i-1}(s_i)\, f_{\mathcal{P},i-1}(1)^{-1} - \tilde{T}_{s_i}\tilde{T}_1^{-1}\right| \leq 2 \sup_{1\leq i,j\leq n} \left|\tilde{T}_{s_j}\tilde{T}_{s_i}^{-1} - f_{\mathcal{P},i}(s_j)\right|.$$

Then using Lemma 6.15, 6.12 and Holder's inequality we have

$$\mathbb{E}\left[\sup_{s\in\mathcal{P}} \left|\hat{\mathbf{K}}_{\mathcal{P}}(s) - \mathbf{K}_{\mathcal{P}}(s)\right|^q\right] \leq C_{q,\gamma}\,|\mathcal{P}|^{q\gamma} \tag{6.44}$$

Finally Lemma 6.19 is proved by combining (6.42),(6.43) and (6.44). ∎

Lemma 6.20 *For each $q \geq 1$, there exists a constant $C > 0$ such that*

$$\sup_{s\in[0,1]} \left|\tilde{\mathbf{K}}_{\underline{s}} - \tilde{\mathbf{K}}_s\right|^q \leq C\left|\mathcal{P}\right|^q$$

Proof. By the fundamental theorem of calculus, we have

$$\tilde{\mathbf{K}}_s = -\frac{1}{2}\int_0^s Ric_{\tilde{u}_r}\tilde{\mathbf{K}}_r dr + \int_0^s \left(\tilde{T}_1\tilde{T}_r^{-1}\right)^* dr.$$

Using Lemma 4.8, note that Ric is bounded by $(d-1)\,N$, we have

$$\left|\tilde{\mathbf{K}}_s\right| \leq (d-1)\,N\int_0^s \left|\tilde{\mathbf{K}}_r\right| dr + C$$

where C and $(d-1)\,N$ are two constants independent of s. Then using Gronwall's inequality we get

$$\left|\tilde{\mathbf{K}}_s\right| \leq Ce^{Ns} \leq Ce^N \tag{6.45}$$

so $\sup_{s\in[0,1]}\left|\tilde{\mathbf{K}}_s\right|$ is bounded. Then using the fundamental theorem of calculus again from $\underline{s}$ to s we have

$$\begin{aligned}\tilde{\mathbf{K}}_s - \tilde{\mathbf{K}}_{\underline{s}} =& -\frac{1}{2}\int_{\underline{s}}^s Ric_{\tilde{u}_r}\tilde{\mathbf{K}}_r dr + \int_{\underline{s}}^s \left(\tilde{T}_1\tilde{T}_r^{-1}\right)^* dr \\ =& -\frac{1}{2}\int_{\underline{s}}^s Ric_{\tilde{u}_r}\left(\tilde{\mathbf{K}}_r - \tilde{\mathbf{K}}_{\underline{r}}\right) dr + \int_{\underline{s}}^s \left(\tilde{T}_1\tilde{T}_r^{-1}\right)^* dr + \frac{1}{2}\int_{\underline{s}}^s Ric_{\tilde{u}_r}\tilde{\mathbf{K}}_{\underline{r}} dr.\end{aligned}$$

Therefore

$$\left|\tilde{\mathbf{K}}_s - \tilde{\mathbf{K}}_{\underline{s}}\right| \leq \frac{N}{2}\int_{\underline{s}}^s \left|\tilde{\mathbf{K}}_r - \tilde{\mathbf{K}}_{\underline{r}}\right| dr + C\left|\mathcal{P}\right|.$$

By Gronwall's inequality again we have

$$\left|\tilde{\mathbf{K}}_s - \tilde{\mathbf{K}}_{\underline{s}}\right| \leq C\left|\mathcal{P}\right| e^{\frac{N}{2}}$$

and thus

$$\sup_{s\in[0,1]} \left|\tilde{\mathbf{K}}_{\underline{s}} - \tilde{\mathbf{K}}_s\right|^q \leq C\left|\mathcal{P}\right|^q$$

■

The next theorem is a generalization to Lemma 6.19 in the sense that s now can be taken to be arbitrary between 0 and 1.

Theorem 6.21 *Let $\mathbf{K}_{\mathcal{P}}$ and $\tilde{\mathbf{K}}$ be defined as in Definition 5.6 and 4.16. Then for each $q \geq 1$ and $\gamma \in \left(0, \frac{1}{2}\right)$, there exists a constant $C > 0$ such that for all $n > 5qN$,*

$$\mathbb{E}\left[\sup_{s\in[0,1]} \left|\tilde{\mathbf{K}}_s - \mathbf{K}_{\mathcal{P}}(s)\right|^q\right] \leq C_{q,\gamma}\left|\mathcal{P}\right|^{\gamma q} \tag{6.46}$$

Proof. For any $s \in [0,1]$, $s \in [s_{i-1}, s_i]$ for some $i \in \{1, \cdots, n\}$. So

$$\begin{aligned}\left|\mathbf{K}_{\mathcal{P}}(s) - \tilde{\mathbf{K}}_s\right| &\leq \left|\mathbf{K}_{\mathcal{P}}(s) - \mathbf{K}_{\mathcal{P}}(s_{i-1})\right| \\ &\quad + \left|\mathbf{K}_{\mathcal{P}}(s_{i-1}) - \tilde{\mathbf{K}}_{s_{i-1}}\right| + \left|\tilde{\mathbf{K}}_{s_{i-1}} - \tilde{\mathbf{K}}_s\right|.\end{aligned}$$

Then using Lemma 6.18, 6.19 and 6.20 we prove this theorem. ■

6.2.4 Convergence of $J_{\mathcal{P}}(s)$ to $\tilde{J}_s$

Recall from Definition 5.9 that

$$J_{\mathcal{P}}(s) := \mathbf{K}_{\mathcal{P}}(s)\,\mathbf{K}_{\mathcal{P}}(1)^{-1} H_{\mathcal{P}} \tag{6.47}$$

where $H_{\mathcal{P}} : W_o(M) \to \mathbb{R}^d$ is given by

$$H_{\mathcal{P}} = u_{\mathcal{P}}(1)^{-1} X\left(\pi \circ u_{\mathcal{P}}(1)\right)$$

and $u_{\mathcal{P}}$ is interpreted in Notation 6.1.

Proposition 6.22 *Let $\tilde{J}_s$ be as in Definition 4.17 and $X \in \Gamma(TM)$ with compact support (here both $J_{\mathcal{P}}$ and $\tilde{J}$ depend on X, see Definition 4.17 and 5.9), then*

$$\sup_{s\in[0,1]}\left|J_{\mathcal{P}}(s)-\tilde{J}_s\right| \to 0 \text{ in } L^{\infty-}(W_o(M)) \text{ as } |\mathcal{P}|\to 0.$$

Proof.

$$\left|J_{\mathcal{P}}(s)-\tilde{J}_s\right| \le I_{\mathcal{P}}(s)+II_{\mathcal{P}}(s)+III_{\mathcal{P}}(s),$$

where

$$\begin{aligned} I_{\mathcal{P}}(s) &= \left|\tilde{\mathbf{K}}_s-\mathbf{K}_{\mathcal{P}}(s)\right|\left|\mathbf{K}_{\mathcal{P}}(1)^{-1}\right||H_{\mathcal{P}}| \\ II_{\mathcal{P}}(s) &= \left|\tilde{\mathbf{K}}_s\right|\left|\mathbf{K}_{\mathcal{P}}(1)^{-1}-\tilde{\mathbf{K}}_1^{-1}\right||H_{\mathcal{P}}| \\ III_{\mathcal{P}}(s) &= \left|\tilde{\mathbf{K}}_s\right|\left|\tilde{\mathbf{K}}_1^{-1}\right|\left|H_{\mathcal{P}}-\tilde{H}\right|. \end{aligned}$$

For $I_{\mathcal{P}}(s)$, since X has compact support, $|H_{\mathcal{P}}(\sigma)|$ is bounded. By Lemma 5.5 $\left|\mathbf{K}_{\mathcal{P}}(1)^{-1}\right| \le 1$. Then using Theorem 6.21 we have

$$\mathbb{E}\left[\sup_{0\le s\le 1} I_{\mathcal{P}}^q(s)\right] \le C|\mathcal{P}|^{q\gamma} \text{ for } n > 5qN. \tag{6.48}$$

For $II_{\mathcal{P}}(s)$: since

$$\mathbf{K}_{\mathcal{P}}(1)^{-1}-\tilde{\mathbf{K}}_1^{-1} = \mathbf{K}_{\mathcal{P}}(1)^{-1}\left(\tilde{\mathbf{K}}_1-\mathbf{K}_{\mathcal{P}}(1)\right)\tilde{\mathbf{K}}_1^{-1}, \tag{6.49}$$

so

$$\begin{aligned} II_{\mathcal{P}}(s) &\le \left|\tilde{\mathbf{K}}_s\right|\left|\mathbf{K}_{\mathcal{P}}(1)^{-1}\right|\left|\tilde{\mathbf{K}}_1-\mathbf{K}_{\mathcal{P}}(1)\right|\left|\tilde{\mathbf{K}}_1^{-1}\right||H_{\mathcal{P}}| \\ &\le C\sup_{s\in[0,1]}\left|\tilde{\mathbf{K}}_s\right|\left|\tilde{\mathbf{K}}_1-\mathbf{K}_{\mathcal{P}}(1)\right|. \end{aligned}$$

Recall from (6.45) that $\sup_{s\in[0,1]}\left|\tilde{\mathbf{K}}_s\right|$ is bounded (the bound is deterministic), using

Theorem 6.21 again we have

$$\mathbb{E}\left[\sup_{0\le s\le 1} II^q_{\mathcal{P}}(s)\right] \le C\,|\mathcal{P}|^{q\gamma} \text{ for } n > 5qN. \tag{6.50}$$

For $III_{\mathcal{P}}(s)$: Since $F : \mathcal{O}(M) \to \mathbb{R}^d$ given by $F(y) = y^{-1}X \circ \pi(y)$ is bounded, and by Wong–Zakai approximation (Corollary 6.5), $u_{\mathcal{P}}(1) \to \tilde{u}_1$ in $L^{\infty-}(W_o(M))$ as $|\mathcal{P}| \to 0$, by DCT,

$$H_{\mathcal{P}} \to \tilde{H} \text{ in } L^{\infty-}(W_o(M)) \text{ as } |\mathcal{P}| \to 0. \tag{6.51}$$

Also since $\sup_{s\in[0,1]}\left|\tilde{\mathbf{K}}_s\right|$ and $\left|\tilde{\mathbf{K}}_1^{-1}\right|$ are bounded, we have

$$\sup_{0\le s\le 1} III_{\mathcal{P}}(s) \to 0 \text{ in } L^{\infty-}(W_o(M)) \text{ as } |\mathcal{P}| \to 0. \tag{6.52}$$

Combining (6.48), (6.50) and (6.52) we prove this proposition. ■

6.3 Convergence of $\tilde{X}^{tr,\nu^1_{\mathcal{P}}}_{\mathcal{P}}$ to $\left(\tilde{X}\right)^{tr,\nu}$

Recall from Lemma 5.11 and 4.43 that

$$\tilde{X}^{tr,\nu^1_{\mathcal{P}}}_{\mathcal{P}} = -\tilde{X}_{\mathcal{P}} + \int_0^1 \langle J'_{\mathcal{P}}(s+), d\beta_{\mathcal{P},s}\rangle + div\tilde{X}_{\mathcal{P}} \tag{6.53}$$

and

$$\left(\tilde{X}\right)^{tr,\nu} = -\tilde{X} + \sum_{\alpha=1}^{d}\left\langle \tilde{C}\tilde{H}, e_\alpha\right\rangle \int_0^1 \left\langle \left(\tilde{T}_s^{-1}\right)^* e_\alpha, d\beta_s\right\rangle - \sum_{\alpha=1}^{d}\left\langle X^{Z_\alpha}\left(\tilde{C}\tilde{H}\right), e_\alpha\right\rangle. \tag{6.54}$$

Theorem 6.23 *If M has parallel curvature tensor, i.e. $\nabla R \equiv 0$, then for any $f \in \mathcal{FC}^1_b$,*

$$\tilde{X}^{tr,\nu^1_{\mathcal{P}}}_{\mathcal{P}} f_{\mathcal{P}} - \tilde{X}^{tr,\nu} f \to 0 \text{ in } L^{\infty-}(W_o(M)) \text{ as } |\mathcal{P}| \to 0$$

where if $f = F(\tilde{u})$*, then* $f_{\mathcal{P}} : H_{\mathcal{P}}(M) \to \mathbb{R}$ *is defined to be* $F(u) \in \mathcal{FC}^1_{\mathcal{P},b}$.

Proof. In correspondence with the three–term formulae (6.53) and (6.54), this theorem is decomposed as three propositions: Proposition 6.25 states that

$$\tilde{X}_{\mathcal{P}} f_{\mathcal{P}} \to \tilde{X} f \text{ in } L^{\infty-}(W_o(M)) \text{ as } |\mathcal{P}| \to 0,$$

Proposition 6.26 states that

$$\int_0^1 \langle J'_{\mathcal{P}}(s+), d\beta_{\mathcal{P},s}\rangle \to \sum_{\alpha=1}^d \left\langle \tilde{C}\tilde{H}, e_\alpha \right\rangle \int_0^1 \left\langle \left(\tilde{T}_s^{-1}\right)^* e_\alpha, d\beta_s \right\rangle \text{ in } L^{\infty-}(W_o(M))$$

and Proposition 6.27 states that

$$div\tilde{X}_{\mathcal{P}} \to \sum_{\alpha=1}^d \left\langle -X^{Z_\alpha}\left(\tilde{C}\tilde{H}\right), e_\alpha \right\rangle \text{ in } L^{\infty-}(W_o(M)) \text{ as } |\mathcal{P}| \to 0.$$

Thus the proof will be complete once the stated propositions are proved. ■

Remark 6.24 *For Proposition 6.25 and 6.26 we assume the assumption of bounded sectional curvature as is mentioned in the begining of this chapter. For Proposition 6.27 we further require the curvature tensor to be covariantly constant.*

Proposition 6.25 *If* $X \in \Gamma(TM)$ *with compact support and* $f \in \mathcal{FC}^1_b$*, then*

$$\tilde{X}_{\mathcal{P}} f_{\mathcal{P}} - \tilde{X} f \to 0 \ \text{in } L^{\infty-}(W_o(M)) \ \text{as } |\mathcal{P}| \to 0.$$

Proposition 6.26 *Keeping the notation above, we have*

$$\int_0^1 \langle J'_{\mathcal{P}}(s+), d\beta_{\mathcal{P},s}\rangle - \sum_{\alpha=1}^d \left\langle \tilde{C}\tilde{H}, e_\alpha \right\rangle \int_0^1 \left\langle \left(\tilde{T}_s^{-1}\right)^* e_\alpha, d\beta_s \right\rangle \to 0 \tag{6.55}$$

in $L^{\infty-}(W_o(M))$ *as* $|\mathcal{P}| \to 0$.

Proposition 6.27 *Continuing the notation above, if we further assume* $\nabla R \equiv 0$*, then*

$$div\tilde{X}_{\mathcal{P}} - \sum_{\alpha=1}^{d} \left\langle -X^{Z_\alpha}\left(\tilde{C}\tilde{H}\right), e_\alpha \right\rangle \to 0 \ \textit{in} \ L^{\infty-}\left(W_o\left(M\right)\right) \ \textit{as} \ |\mathcal{P}| \to 0.$$

Proof of Proposition 6.25. Notice that $\tilde{X}_{\mathcal{P}} = X^{J_{\mathcal{P}}}$ and $\tilde{X} = X^{\tilde{J}}$, since for a general geometric vector field of the form X^z and a cylinder function $f = F\left(u\right)$,

$$\left(X^z f\right)\left(\sigma\right) = \sum_{i=1}^{n} \left\langle \left(\nabla_i f\right)\left(\sigma\right), X^z_{s_i}\left(\sigma\right)\right\rangle,$$

where $\left(\nabla_i f\right)$ denotes the gradient of F in the i–th variable. Therefore, note that $\pi \circ u_{\mathcal{P}} = \phi \circ \beta_{\mathcal{P}}$, we have

$$\begin{aligned}\tilde{X}_{\mathcal{P}} f_{\mathcal{P}}\left(\phi \circ \beta_{\mathcal{P}}\right) &= \sum_{i=1}^{n} \left\langle \left(\nabla_i f\right)\left(\pi \circ u_{\mathcal{P}}\right), u_{\mathcal{P}}\left(s_i\right) J_{\mathcal{P}}\left(s_i\right)\right\rangle \\ &= \sum_{i=1}^{n} \left\langle u_{\mathcal{P}}^{-1}\left(s_i\right)\left(\nabla_i f\right)\left(\pi \circ u_{\mathcal{P}}\right), J_{\mathcal{P}}\left(s_i\right)\right\rangle,\end{aligned}$$

and

$$\tilde{X} f = \sum_{i=1}^{n} \left\langle \left(\nabla_i f\right)\left(\pi \circ \tilde{u}\right), \tilde{u}_{s_i} \tilde{J}_{s_i}\right\rangle = \sum_{i=1}^{n} \left\langle \tilde{u}_{s_i}^{-1}\left(\nabla_i f\right)\left(\pi \circ \tilde{u}\right), \tilde{J}_{s_i}\right\rangle.$$

If $f \in \mathcal{FC}_b^1$, then $u \to u_{s_i}^{-1}\left(\nabla_i f\right)\left(\pi \circ u\right)$ is continuous and bounded. Using Corollary 6.5 and DCT, we know

$$u_{\mathcal{P}}^{-1}\left(s_i\right)\left(\nabla_i f\right)\left(\pi \circ u_{\mathcal{P}}\right) \to \tilde{u}_{s_i}^{-1}\left(\nabla_i f\right)\left(\pi \circ \tilde{u}\right) \ \text{in} \ L^{\infty-}\left(W_o\left(M\right)\right) \ \text{as} \ |\mathcal{P}| \to 0. \tag{6.56}$$

The proof is then completed by making use of (6.56) and Proposition 6.22. ■

Proof of Proposition 6.26.

$$\begin{aligned}\int_0^1 \langle J'_{\mathcal{P}}(s+), d\beta_{\mathcal{P}}(s)\rangle &= \sum_{i=1}^{n} \left\langle \frac{J_{\mathcal{P}}(s_i) - J_{\mathcal{P}}(s_{i-1})}{\Delta_i}, \Delta_i\beta \right\rangle \\ &= \sum_{i=1}^{n} \langle J'_{\mathcal{P}}(s_{i-1}), \Delta_i\beta\rangle + \sum_{i=1}^{n} \left\langle \int_{s_{i-1}}^{s_i} J''_{\mathcal{P}}(s)(s - s_{i-1})\, ds, \Delta_i\beta \right\rangle \\ &= I_{\mathcal{P}} + II_{\mathcal{P}},\end{aligned}$$

where

$$I_{\mathcal{P}} = \sum_{i=1}^{n} \langle J'_{\mathcal{P}}(s_{i-1}), \Delta_i\beta\rangle$$

and

$$II_{\mathcal{P}} = \sum_{i=1}^{n} \left\langle \int_{s_{i-1}}^{s_i} J''_{\mathcal{P}}(s)(s - s_{i-1})\, ds, \Delta_i\beta \right\rangle.$$

Using the fact that $J_{\mathcal{P}}$ satisfies Jacobi equation, we further have

$$\begin{aligned}II_{\mathcal{P}} &= \sum_{i=1}^{n} \left\langle \frac{1}{\Delta_i^2} \int_{s_{i-1}}^{s_i} R_{u_{\mathcal{P}}(s)}(\Delta_i\beta, J_{\mathcal{P}}(s))\, \Delta_i\beta\,(s - s_{i-1})\, ds, \Delta_i\beta \right\rangle \\ &= \sum_{i=1}^{n} \frac{1}{\Delta_i^2} \int_{s_{i-1}}^{s_i} \left\langle R_{u_{\mathcal{P}}(s)}(\Delta_i\beta, J_{\mathcal{P}}(s))\, \Delta_i\beta, \Delta_i\beta \right\rangle (s - s_{i-1})\, ds.\end{aligned}$$

Since the curvature tensor is anti-symmertric,

$$\left\langle R_{u_{\mathcal{P}}(s)}(\Delta_i\beta, J_{\mathcal{P}}(s))\, \Delta_i\beta, \Delta_i\beta \right\rangle \equiv 0 \ \nu\text{–a.s},$$

so $II_{\mathcal{P}} \equiv 0$.

$$\begin{aligned}I_{\mathcal{P}} &= \sum_{i=1}^{n} \left\langle f^*_{\mathcal{P},i}(1)\, \mathbf{K}_{\mathcal{P}}(1)^{-1} H_{\mathcal{P}}, \Delta_i\beta \right\rangle \\ &= \sum_{i=1}^{n} \left\langle \mathbf{K}_{\mathcal{P}}(1)^{-1} H_{\mathcal{P}}, f_{\mathcal{P},i}(1)\, \Delta_i\beta \right\rangle = \left\langle \mathbf{K}_{\mathcal{P}}(1)^{-1} H_{\mathcal{P}}, \sum_{i=1}^{n} f_{\mathcal{P},i}(1)\, \Delta_i\beta \right\rangle.\end{aligned}$$

For each $i \geq 1, s \in [s_{i-1}, s_i]$, define $g_i(s) = S_{\mathcal{P},i}(s) - C_{\mathcal{P},i}(s)\, S_{\mathcal{P},i-1}$. Then Taylor's expansion of g_i at s_{i-1} gives

$$g_i(s) = -S_{\mathcal{P},i-1} + (s - s_{i-1}) I + \int_{s_{i-1}}^{s} R_{u_{\mathcal{P}}(r)} \left(\beta'_{\mathcal{P}}(r), g_i(r)\right) \beta'_{\mathcal{P}}(r)(s-r)\, dr.$$

So

$$|g_i(s)| \leq |S_{\mathcal{P},i-1} - (s - s_{i-1}) I| + N \left|\beta'_{\mathcal{P}}(s_{i-1})\right|^2 \int_{s_{i-1}}^{s} |g_i(r)|(s-r)\, dr.$$

By Gronwall's inequality and Lemma 6.8, we have

$$|g_i(s_i)| \leq \frac{N}{6} K_\gamma^2 |\mathcal{P}|^{2\gamma+1} e^{\frac{1}{2}N|\Delta_i \beta|^2}.$$

Note that $g_i(s_i) = S_{\mathcal{P},i} - C_{\mathcal{P},i} S_{\mathcal{P},i-1}$, so by Lemma 6.6,

$$\begin{aligned} |f_{\mathcal{P},i}(1) - f_{\mathcal{P},i-1}(1)| &\leq \frac{1}{|\mathcal{P}|} |C_{\mathcal{P},n}| \cdot \cdots \cdot |C_{\mathcal{P},i+1}| \cdot |S_{\mathcal{P},i} - C_{\mathcal{P},i} S_{\mathcal{P},i-1}| \\ &\leq \frac{N}{6} K_\gamma^2 |\mathcal{P}|^{2\gamma} e^{\sum_{i=1}^n N|\Delta_i \beta|^2} \end{aligned}$$

and thus

$$\begin{aligned} \left| \sum_{i=1}^n f_{\mathcal{P},i}(1) \Delta_i \beta - \sum_{i=1}^n f_{\mathcal{P},i-1}(1) \Delta_i \beta \right|^q &\leq |\mathcal{P}|^{1-q} \left[\sum_{i=1}^n |f_{\mathcal{P},i}(1) - f_{\mathcal{P},i-1}(1)|^q |\Delta_i \beta|^q \right] \\ &\leq C K_\gamma^{3q} |\mathcal{P}|^{3q\gamma - q} e^{\sum_{i=1}^n qN|\Delta_i \beta|^2}. \end{aligned}$$

Picking $\gamma \in \left(\frac{1}{2}, \frac{1}{3}\right)$ we know for any $q \geq 1$,

$$\mathbb{E}\left[\left| \sum_{i=1}^n f_{\mathcal{P},i}(1) \Delta_i \beta - \sum_{i=1}^n f_{\mathcal{P},i-1}(1) \Delta_i \beta \right|^q \right] \to 0 \text{ as } |\mathcal{P}| \to 0. \tag{6.57}$$

Since $f_{\mathcal{P},i-1}(1) = f_{\mathcal{P},0}(1) f_{\mathcal{P},0}^{-1}(s_{i-1}) \frac{S_{\mathcal{P},i-1}}{\Delta_{i-1}}$, so

$$\left\langle \mathbf{K}_{\mathcal{P}}(1)^{-1} H_{\mathcal{P}}, \sum_{i=1}^{n} \tilde{T}_{s_{i-1}} \Delta_i \beta \right\rangle \tag{6.58}$$

$$= \left\langle f_{\mathcal{P},0}^{*}(1) \mathbf{K}_{\mathcal{P}}(1)^{-1} H_{\mathcal{P}}, \sum_{i=1}^{n} f_{\mathcal{P},0}^{-1}(s_{i-1}) \frac{S_{\mathcal{P},i-1}}{\Delta_{i-1}} \Delta_i \beta \right\rangle. \tag{6.59}$$

Using Lemma 5.5 we have $\left| f_{\mathcal{P},0}^{-1}(s_{i-1}) \right| \leq 1$. Then using Lemma 6.8 we obtain

$$\begin{aligned} \left| f_{\mathcal{P},0}^{-1}(s_{i-1}) \frac{S_{\mathcal{P},i-1}}{\Delta_{i-1}} - f_{\mathcal{P},0}^{-1}(s_{i-1}) \right| |\Delta_i \beta| &\leq \left| \frac{S_{\mathcal{P},i-1}}{\Delta_{i-1}} - I \right| |\Delta_i \beta| \\ &\leq \frac{N K_{\gamma}^{3} |\mathcal{P}|^{3\gamma+1}}{6} e^{\frac{N}{2} |\Delta_{i-1}\beta|^2}. \end{aligned}$$

Therefore for each $q \geq 1$,

$$\left| \sum_{i=1}^{n} f_{\mathcal{P},0}^{-1}(s_{i-1}) \frac{S_{\mathcal{P},i-1}}{\Delta_{i-1}} \Delta_i \beta - \sum_{i=1}^{n} f_{\mathcal{P},0}^{-1}(s_{i-1}) \Delta_i \beta \right|^{q} \tag{6.60}$$

$$\leq |\mathcal{P}|^{1-q} \sum_{i=1}^{n} \frac{N^{q} K_{\gamma}^{3q} |\mathcal{P}|^{3q\gamma+q}}{6^{q}} e^{\frac{Nq}{2} |\Delta_{i-1}\beta|^2} \tag{6.61}$$

$$\leq C |\mathcal{P}|^{3\gamma q} K_{\gamma}^{3q} e^{\sum_{i=1}^{n} \frac{Nq}{2} |\Delta_{i-1}\beta|^2} \tag{6.62}$$

and thus

$$\mathbb{E}\left[\left| \sum_{i=1}^{n} f_{\mathcal{P},0}^{-1}(s_{i-1}) \frac{S_{\mathcal{P},i-1}}{\Delta_{i-1}} \Delta_i \beta - \sum_{i=1}^{n} f_{\mathcal{P},0}^{-1}(s_{i-1}) \Delta_i \beta \right|^{q} \right] \leq C |\mathcal{P}|^{3\gamma q} \overset{|\mathcal{P}| \to 0}{\longrightarrow} 0.$$

Rewrite

$$\sum_{i=1}^{n} f_{\mathcal{P},0}^{-1}(s_{i-1}) \Delta_i \beta \text{ as } \int_0^1 f_{\mathcal{P}}(s) \, d\beta_s,$$

where $f_{\mathcal{P}}(s) := \sum_{i=1}^{n} f_{\mathcal{P},0}^{-1}(s_{i-1}) 1_{[s_{i-1}, s_i)}(s)$. Define

$$M_r := \int_0^r f_{\mathcal{P}}(s) \, d\beta_s - \int_0^r \tilde{T}_s^{-1} d\beta_s,$$

it is easy to see M_r is a martingale, then by Burkholder-Davis-Gundy inequality, for each $q \geq 1$,

$$\mathbb{E}\left[\sup_{r\in[0,1]} |M_r|^q\right] \leq C\mathbb{E}\left[\langle M\rangle_1^{\frac{q}{2}}\right].$$

Since

$$\langle M\rangle_1 \leq \int_0^1 \left|f_{\mathcal{P}}(s) - \tilde{T}_s^{-1}\right|^2 ds \leq 2\int_0^1 \left|f_{\mathcal{P}}(s) - \tilde{T}_{\underline{s}}^{-1}\right|^2 ds + 2\int_0^1 \left|\tilde{T}_{\underline{s}}^{-1} - \tilde{T}_s^{-1}\right|^2 ds,$$

we have

$$\begin{aligned}
\int_0^1 \left|f_{\mathcal{P}}(s) - \tilde{T}_{\underline{s}}^{-1}\right|^2 ds &= \sum_{i=1}^n \left|f_{\mathcal{P},0}^{-1}(s_{i-1}) - \tilde{T}_{s_{i-1}}^{-1}\right|^2 \Delta_i \\
&\leq \sum_{i=1}^n \left|f_{\mathcal{P},0}^{-1}(s_{i-1})\right|^2 \left|f_{\mathcal{P},0}(s_{i-1}) - \tilde{T}_{s_{i-1}}\right|^2 \left|\tilde{T}_{s_{i-1}}^{-1}\right|^2 \Delta_i \\
&\leq \sup_{s\in\mathcal{P}} \left|f_{\mathcal{P},0}(s) - \tilde{T}_s\right|^2 \qquad (6.63)
\end{aligned}$$

and

$$\int_0^1 \left|\tilde{T}_{\underline{s}}^{-1} - \tilde{T}_s^{-1}\right|^2 ds = \int_0^1 \left|\int_s^{\underline{s}} \left(\tilde{T}_r^{-1}\right)' dr\right|^2 ds \leq \int_0^1 N\left|s - \underline{s}\right|^2 ds \leq N\left|\mathcal{P}\right|^2.$$

Therefore,

$$\begin{aligned}
\langle M\rangle_1^{\frac{q}{2}} &\leq C\left(\int_0^1 \left|f_{\mathcal{P}}(s) - \tilde{T}_{\underline{s}}^{-1}\right|^2 ds\right)^{\frac{q}{2}} + C\left(\int_0^1 \left|\tilde{T}_{\underline{s}}^{-1} - \tilde{T}_s^{-1}\right|^2 ds\right)^{\frac{q}{2}} \\
&\leq C\left(\sup_{s\in\mathcal{P}} \left|f_{\mathcal{P},0}(s) - \tilde{T}_s\right|^q + \left|\mathcal{P}\right|^q\right).
\end{aligned}$$

Then using Theorem 6.15 we have

$$\mathbb{E}\left[\langle M\rangle_1^{\frac{q}{2}}\right] \leq C\left|\mathcal{P}\right|^{q\gamma}. \qquad (6.64)$$

From (6.64) it follows that for each $q \geq 1$,

$$\int_0^1 f_{\mathcal{P}}(s)\, d\beta_s - \int_0^1 \tilde{T}_s^{-1} d\beta_s \to 0 \text{ in } L^q(W_o(M)) \text{ as } |\mathcal{P}| \to 0.$$

Then using Eq.(6.49), Eq.(6.51) and Theorem 6.21 we have

$$\mathbf{K}_{\mathcal{P}}(1)^{-1} H_{\mathcal{P}} \to \tilde{\mathbf{K}}_1^{-1} \tilde{H} \text{ in } L^{\infty-}(W_o(M)) \text{ as } |\mathcal{P}| \to 0$$

and

$$f_{\mathcal{P},0}^*(1) \to \tilde{T}_1^* \text{ in } L^{\infty-}(W_o(M)) \text{ as } |\mathcal{P}| \to 0,$$

therefore

$$I_{\mathcal{P}} \to \left\langle T_1^* \tilde{\mathbf{K}}_1^{-1} \tilde{H}, \int_0^1 \tilde{T}_s^{-1} d\beta_s \right\rangle \text{ in } L^{\infty-}(W_o(M)) \text{ as } |\mathcal{P}| \to 0. \tag{6.65}$$

Lastly, notice that

$$\tilde{\mathbf{K}}_1 = \tilde{T}_1 \int_0^1 \left(\tilde{T}_r^* \tilde{T}_r \right)^{-1} dr T_1^*,$$

so

$$\tilde{\mathbf{K}}_1^{-1} = \left(\tilde{T}_1^{-1} \right)^* \tilde{C}$$

where $\tilde{C}$ is defined in Definition 4.36, and

$$\begin{aligned} \left\langle T_1^* \tilde{\mathbf{K}}_1^{-1} \tilde{H}, \int_0^1 \tilde{T}_s^{-1} d\beta_s \right\rangle &= \left\langle \tilde{C} \tilde{H}, \int_0^1 \tilde{T}_s^{-1} d\beta_s \right\rangle & (6.66) \\ &= \sum_{\alpha=1}^d \left\langle \tilde{C} \tilde{H}, e_\alpha \right\rangle \int_0^1 \left\langle \left(\tilde{T}_s^{-1} \right)^* e_\alpha, d\beta_s \right\rangle. & (6.67) \end{aligned}$$

Using (6.65) we get as $|\mathcal{P}| \to 0$,

$$\int_0^1 \langle J'_{\mathcal{P}}(s+), d\beta_{\mathcal{P},s} \rangle - \sum_{\alpha=1}^d \left\langle \tilde{C} \tilde{H}, e_\alpha \right\rangle \int_0^1 \left\langle \left(\tilde{T}_s^{-1} \right)^* e_\alpha, d\beta_s \right\rangle \to 0 \text{ in } L^{\infty-}(W_o(M)).$$

■

Lemma 6.28 *Fix $s \in [0,1]$, consider an one parameter family of paths $\{\sigma_t\} \subset H_{\mathcal{P}}(M)$ and denote by $u_t(\cdot)$: the Horizontal lift of σ_t. For simplicity, we will denote $u_t(1)$ by u_t, σ_0 by σ, the derivative with respect to t by $\cdot$ and the derivative with respect to s by $\prime$. For any $X \in \Gamma(TM)$, define $f_X : \mathcal{O}(M) \mapsto \mathbb{R}^d \simeq T_oM$ by*

$$f_X(u) = u^{-1}(X \circ \pi)(u)$$

Then:

$$\frac{d}{dt}|_0 f_X(u_t) = \left(\frac{d}{dt}|_0 u_t\right) f_X = u_0^{-1}\nabla_{\dot{\sigma}(1)}X \tag{6.68}$$

$$-\int_0^1 R_{u_0(r)}\left(u_0(r)^{-1}\sigma'(r+), u_0(r)^{-1}\dot{\sigma}(r)\right) dr f_X(u_0) \tag{6.69}$$

Proof. Based on the decomposition of $\mathcal{O}(M)$ as in Definition A.12, we have:

$$\dot{u}_0 = B_a(u_0) + \tilde{A}(u_0)$$

where $a = u_0^{-1}\frac{d}{dt}|_0\sigma_t(1) = u_0^{-1}\dot{\sigma}(1) \in T_oM$ and $\tilde{A}(u_0) = \frac{d}{dt}|_0 u_0 e^{tA}$ for some $A = u_0^{-1}\frac{\nabla u_t}{dt}(0) \in \mathfrak{so}(d)$ and $B_a(u_0) = \frac{d}{dt}|_0 //_t(\gamma)\, u_0$ where γ satisfies $\dot{\gamma}(0) = u_0 a$ and $\gamma(0) = \sigma(1)$. In this example, we can choose $\gamma(\cdot)$ to be $\sigma_\cdot(1)$. So

$$B_a(u_0) f_X = \frac{d}{dt}|_0 u_0^{-1} //_t^{-1}(\gamma)(X \circ \pi)(//_t(\gamma)\, u_0) = u_0^{-1}\nabla_{\dot{\sigma(s)}}X$$

and

$$\tilde{A}(u) f_X = \frac{d}{dt}|_0 e^{-tA}u^{-1}(X \circ \pi)\left(ue^{tA}\right) = -Au_0^{-1}X(\sigma(1)) = -Af_x(u_0)$$

Following the computation in Theorem 3.3 in [2], we know that

$$A = \int_0^1 R_{u_0(r)} \left(u_0 (r)^{-1} \sigma' (r+) , u_0 (r)^{-1} \dot{\sigma} (r) \right) dr.$$

■

Proof of Proposition 6.27. Because of Lemma 4.44, it suffices to prove

$$div \tilde{X}_{\mathcal{P}} \to div X \circ E_1 - \sum_{\alpha=1}^{d} \left\langle \tilde{C} A_1 \langle Z_\alpha \rangle \tilde{H}, e_\alpha \right\rangle \text{ as } |\mathcal{P}| \to 0 \text{ in } L^{\infty-} (W_o (M)) .$$

Recall from Definition 5.9 that

$$J_{\mathcal{P}} (s) = \mathbf{K}_{\mathcal{P}} (s) \mathbf{K}_{\mathcal{P}} (1)^{-1} H_{\mathcal{P}}.$$

From there we get, for each $\alpha \in \{1, \ldots, d\}$ and $j \in \{1, \ldots, n\}$, that

$$J'_{\mathcal{P}} (s_{j-1}+) = \mathbf{K}'_{\mathcal{P}} (s_{j-1}+) \mathbf{K}_{\mathcal{P}} (1)^{-1} H_{\mathcal{P}} = f^*_{\mathcal{P},j} (1) \mathbf{K}_{\mathcal{P}} (1)^{-1} H_{\mathcal{P}},$$

and

$$X^{h_{\alpha,j}} J'_{\mathcal{P}} (s_{j-1}+) = I_{\mathcal{P}} (\alpha, j) + II_{\mathcal{P}} (\alpha, j) + III_{\mathcal{P}} (\alpha, j) ,$$

where

$$\begin{aligned} I_{\mathcal{P}} (\alpha, j) &= \left(X^{h_{\alpha,j}} f^*_{\mathcal{P},j} (1) \right) \mathbf{K}_{\mathcal{P}} (1)^{-1} H_{\mathcal{P}} \\ II_{\mathcal{P}} (\alpha, j) &= f^*_{\mathcal{P},j} (1) \left(X^{h_{\alpha,j}} \mathbf{K}_{\mathcal{P}} (1)^{-1} \right) H_{\mathcal{P}} \\ III_{\mathcal{P}} (\alpha, j) &= f^*_{\mathcal{P},j} (1) \mathbf{K}_{\mathcal{P}} (1)^{-1} \left(X^{h_{\alpha,j}} H_{\mathcal{P}} \right) . \end{aligned} \tag{6.70}$$

Using Proposition 5.12, we have

$$div \tilde{X}_{\mathcal{P}} = \sum_{\alpha=1}^{d} \sum_{j=1}^{n} \langle (I_{\mathcal{P}} + II_{\mathcal{P}} + III_{\mathcal{P}}) (\alpha, j) , e_\alpha \rangle \sqrt{\Delta_j}$$

Based on the expression above, Proposition 6.27 will be proved as a corollary of Lemma 6.29 to Lemma 6.32. In Lemma 6.29 and Lemma 6.30 we show that

$$\sum_{\alpha=1}^{d}\sum_{j=1}^{n}\left\langle III_{\mathcal{P}}\left(\alpha,j\right),e_{\alpha}\right\rangle\sqrt{\Delta_j}\rightarrow\sum_{\alpha=1}^{d}\left\langle\tilde{C}A_1\left\langle Z_{\alpha}\right\rangle\tilde{H},e_{\alpha}\right\rangle\text{ as }\left|\mathcal{P}\right|\rightarrow 0.$$

In Lemma 6.31 we show that

$$\sum_{\alpha=1}^{d}\sum_{j=1}^{n}\left\langle II_{\mathcal{P}}\left(\alpha,j\right),e_{\alpha}\right\rangle\sqrt{\Delta_j}\rightarrow 0\text{ as }\left|\mathcal{P}\right|\rightarrow 0.$$

In Lemma 6.32 we show that

$$\sum_{\alpha=1}^{d}\sum_{j=1}^{n}\left\langle I_{\mathcal{P}}\left(\alpha,j\right),e_{\alpha}\right\rangle\sqrt{\Delta_j}\rightarrow 0\text{ as }\left|\mathcal{P}\right|\rightarrow 0.$$

■

Lemma 6.29 *If* $\nabla R\equiv 0$*, then*

$$\sum_{\alpha=1}^{d}\sum_{j=1}^{n}\left\langle III_{\mathcal{P}}\left(\alpha,j\right),e_{\alpha}\right\rangle\sqrt{\Delta_j}\rightarrow\sum_{\alpha=1}^{d}\left\langle\tilde{C}A_1\left\langle Z_{\alpha}\right\rangle\tilde{H},e_{\alpha}\right\rangle\tag{6.71}$$

in $L^{\infty-}\left(W_o\left(M\right)\right)$ *as* $\left|\mathcal{P}\right|\rightarrow 0$.

Proof. Applying Lemma 6.28 to $X^{h_{\alpha,j}}H_{\mathcal{P}}$ gives

$$\sum_{\alpha=1}^{d}\sum_{j=1}^{n}\left\langle III_{\mathcal{P}}\left(\alpha,j\right),e_{\alpha}\right\rangle\sqrt{\Delta_j}=IV_{\mathcal{P}}+V_{\mathcal{P}},$$

where

$$IV_{\mathcal{P}}=\sum_{\alpha=1}^{d}\sum_{j=1}^{n}\left\langle f_{\mathcal{P},j}^{*}\left(1\right)\mathbf{K}_{\mathcal{P}}\left(1\right)^{-1}u_{\mathcal{P}}\left(1\right)^{-1}\nabla_{u_{\mathcal{P}}(1)\sqrt{\Delta_j}f_{\mathcal{P},j}(1)e_{\alpha}}X,e_{\alpha}\right\rangle\sqrt{\Delta_j}$$

and

$$V_{\mathcal{P}} = -\sum_{\alpha=1}^{d}\sum_{j=1}^{n}\left\langle f_{\mathcal{P},j}^{*}\left(1\right)\mathbf{K}_{\mathcal{P}}\left(1\right)^{-1}\int_{0}^{1}R_{u_{\mathcal{P}}(r)}\left(\beta_{\mathcal{P}}'\left(r+\right),h_{\alpha,j}\left(r\right)\right)drH_{\mathcal{P}},e_{\alpha}\right\rangle\sqrt{\Delta_{j}}.$$

We first compute $IV_{\mathcal{P}}$. After viewing $L\left(\cdot\right) = u_{\mathcal{P}}\left(1\right)^{-1}\nabla_{u_{\mathcal{P}}(1)(\cdot)}X$ as a linear functional on $\mathbb{R}^{d}$ we have

$$\begin{aligned}
IV_{\mathcal{P}} &= \sum_{j=1}^{n}\sum_{\alpha=1}^{d}\left\langle f_{\mathcal{P},j}^{*}\left(1\right)\mathbf{K}_{\mathcal{P}}\left(1\right)^{-1}L\left(f_{\mathcal{P},j}\left(1\right)e_{\alpha}\right),e_{\alpha}\right\rangle\Delta_{j}\\
&= \sum_{j=1}^{n}Tr\left(f_{\mathcal{P},j}^{*}\left(1\right)\mathbf{K}_{\mathcal{P}}\left(1\right)^{-1}Lf_{\mathcal{P},j}\left(1\right)\right)\Delta_{j}\\
&= \sum_{j=1}^{n}Tr\left(\Delta_{j}f_{\mathcal{P},j}\left(1\right)f_{\mathcal{P},j}^{*}\left(1\right)\mathbf{K}_{\mathcal{P}}\left(1\right)^{-1}L\right)\\
&= Tr\left(\sum_{j=1}^{n}\Delta_{j}f_{\mathcal{P},j}\left(1\right)f_{\mathcal{P},j}^{*}\left(1\right)\mathbf{K}_{\mathcal{P}}\left(1\right)^{-1}L\right) \qquad (6.72)\\
&= Tr\left(L\right)\\
&= divX\circ E_{1},
\end{aligned}$$

where in Eq. (6.72) we use identity (5.9):

$$\sum_{j=1}^{n}\Delta_{j}f_{\mathcal{P},j}\left(1\right)f_{\mathcal{P},j}^{*}\left(1\right) = \mathbf{K}_{\mathcal{P}}\left(1\right)$$

and given $A\in M_{d\times d}$, $Tr\left(A\right) := \sum_{\alpha=1}^{d}\left\langle Ae_{\alpha},e_{\alpha}\right\rangle$ is the trace of the matrix A.

The proof of the lemma will be completed by Lemma 6.30 below which shows V_{P} term converges to the right side of Eq. (6.71). ■

Lemma 6.30 *Let $V_{\mathcal{P}}$ be defined as in Lemma 6.29 and $\nabla R\equiv 0$, then*

$$V_{\mathcal{P}} - \sum_{\alpha=1}^{d}\left\langle\tilde{C}A_{1}\left\langle Z_{\alpha}\right\rangle\tilde{H},e_{\alpha}\right\rangle \to 0 \text{ in } L^{\infty-}\left(W_{o}\left(M\right)\right) \text{ as } \left|\mathcal{P}\right|\to 0. \qquad (6.73)$$

Proof. Recall that

$$V_{\mathcal{P}} = -\sum_{\alpha=1}^{d}\sum_{j=1}^{n}\left\langle f_{\mathcal{P},j}^{*}(1)\,\mathbf{K}_{\mathcal{P}}(1)^{-1}\int_0^1 R_{u_{\mathcal{P}}(r)}\left(\beta_{\mathcal{P}}'(r+), h_{\alpha,j}(r)\right)dr H_{\mathcal{P}}, e_{\alpha}\right\rangle\sqrt{\Delta_j}.$$

For each $\alpha \in \{1,\ldots,d\}$ and $j \in \{1,\ldots,n\}$, since $h_{\alpha,j}(r) = \sqrt{\Delta_j} f_{\mathcal{P},j}(r)$, we have

$$\begin{aligned}\int_0^1 R_{u_{\mathcal{P}}(r)}\left(\beta_{\mathcal{P}}'(r+), \frac{1}{\sqrt{\Delta_j}}h_{\alpha,j}(r)\right)dr &= \int_0^1 R_{u_{\mathcal{P}}(r)}\left(\beta_{\mathcal{P}}'(r+), f_{\mathcal{P},j}(r)\,e_{\alpha}\right)dr \\ &= \int_0^1 R_{u_{\mathcal{P}}(\underline{r})}\left(\beta_{\mathcal{P}}'(r+), f_{\mathcal{P},j}(\underline{r})\,e_{\alpha}\right)dr + e_0\end{aligned}$$

where $e_0 := e_{0,1} + e_{0,2}$

$$e_{0,1} = \int_0^1 R_{u_{\mathcal{P}}(r)}\left(\beta_{\mathcal{P}}'(r+), f_{\mathcal{P},j}(r)\,e_{\alpha}\right)dr - \int_0^1 R_{u_{\mathcal{P}}(\underline{r})}\left(\beta_{\mathcal{P}}'(r+), f_{\mathcal{P},j}(r)\,e_{\alpha}\right)dr$$

and

$$e_{0,2} = \int_0^1 R_{u_{\mathcal{P}}(\underline{r})}\left(\beta_{\mathcal{P}}'(r+), f_{\mathcal{P},j}(r)\,e_{\alpha}\right)dr - \int_0^1 R_{u_{\mathcal{P}}(\underline{r})}\left(\beta_{\mathcal{P}}'(r+), f_{\mathcal{P},j}(\underline{r})\,e_{\alpha}\right)dr.$$

Since $\nabla R \equiv 0$, we have argued in Lemma 4.44 that R_u is independent of u, therefore $e_{0,1} = 0$.

As for $e_{0,2}$, since

$$|e_{0,2}|^q \leq N \sup_{r\in[0,1]}|\beta_{\mathcal{P}}'(r+)|^q \sup_{r\in[0,1],j\in\{1,\cdots,n\}}|f_{\mathcal{P},j}(r) - f_{\mathcal{P},j}(\underline{r})|^q,$$

using (6.15) we have

$$|e_{0,2}|^q \leq C_{q,\gamma}K_{\gamma}^q\,|\mathcal{P}|^{q\gamma-1}\,|\mathcal{P}|^{2q\gamma}\,e^{qN\sum_{k=1}^{n}|\Delta_k\beta|^2}K_{\gamma}^{2q}\left(1+\frac{NK_{\gamma}\,|\mathcal{P}|^{\gamma}}{6}\right)^q,$$

and from which it follows

$$\mathbb{E}\left[\left|e_{0,2}\right|^q\right] \leq C \left|\mathcal{P}\right|^{3q\gamma-1} \ \forall n \geq 5qN. \tag{6.74}$$

Picking $\gamma > \frac{1}{3}$, so $3q\gamma - 1 > 0$ for any $q \geq 1$, so $\mathbb{E}\left[\left|e_{0,2}\right|^q\right] \to 0$ as $\left|\mathcal{P}\right| \to 0$.

Next we analyze

$$\begin{aligned}\int_0^1 R_{u_{\mathcal{P}}(\underline{r})}\left(\beta'_{\mathcal{P}}(r+), f_{\mathcal{P},j}(\underline{r}) e_\alpha\right) dr &= \sum_{k=1}^n R_{u_{\mathcal{P}}(s_{k-1})}\left(\Delta_k\beta, f_{\mathcal{P},j}(s_{k-1}) e_\alpha\right) \\ &= \int_0^1 g_1(s)\, d\beta_s,\end{aligned}$$

where

$$g_1(s) = \sum_{k=1}^n R_{u_{\mathcal{P}}(s_{k-1})}\left(\cdot, f_{\mathcal{P},j}(s_{k-1}) e_\alpha\right) 1_{[s_{k-1},s_k)}(s).$$

Define

$$\begin{aligned}g_2(s) &= \sum_{k=1}^n R_{\tilde{u}_{s_{k-1}}}\left(\cdot, f_{\mathcal{P},j}(s_{k-1}) e_\alpha\right) 1_{[s_{k-1},s_k)}(s) \\ g_3(s) &= \sum_{k=j+1}^n R_{\tilde{u}_{s_{k-1}}}\left(\cdot, \tilde{T}_{s_{k-1}}\tilde{T}_{s_j}^{-1} e_\alpha\right) 1_{[s_{k-1},s_k)}(s) \\ g_4(s) &= R_{\tilde{u}_{\underline{s}}}\left(\cdot, \tilde{T}_s\tilde{T}_{s_j}^{-1} e_\alpha\right) 1_{[s_j,1]}(s) \\ g_5(s) &= R_{\tilde{u}_s}\left(\cdot, \tilde{T}_s\tilde{T}_{s_j}^{-1} e_\alpha\right) 1_{[s_j,1]}(s).\end{aligned}$$

For each $i = 1,2,3,4$, denote

$$e_{\mathcal{P},i}(r) = \int_0^r g_i(s)\, d\beta_s - \int_0^r g_{i+1}(s)\, d\beta_s,$$

then

$$\int_0^1 R_{u_{\mathcal{P}}(\underline{r})}\left(\beta'_{\mathcal{P}}(r+), f_{\mathcal{P},j}(\underline{r})\, e_\alpha\right) dr - \int_{s_j}^1 R_{\tilde{u}_r}\left(d\beta_r, \tilde{T}_r \tilde{T}_{s_j}^{-1} e_\alpha\right)$$
$$= e_{\mathcal{P},1}(1) + e_{\mathcal{P},2}(1) + e_{\mathcal{P},3}(1) + e_{\mathcal{P},4}(1).$$

We are about to show for each $i \in \{1, 2, 3, 4\}$,

$$e_{\mathcal{P},i}(1) \to 0 \text{ in } L^{\infty-}(W_o(M)) \text{ as } |\mathcal{P}| \to 0. \tag{6.75}$$

For $e_{\mathcal{P},1}(1)$, since

$$\langle e_{\mathcal{P},1}\rangle(r) \le \int_0^r |g_1(s) - g_2(s)|^2\, ds,$$

so for each $q \ge 1$,

$$\begin{aligned}
\mathbb{E}\left[\langle e_{\mathcal{P},1}\rangle^{\frac{q}{2}}(1)\right] &\le \mathbb{E}\left[\int_0^1 |g_1(s) - g_2(s)|^q\, ds\right] \\
&\le \mathbb{E}\left[\sum_{k=1}^n \left|R_{u_{\mathcal{P}}(s_{k-1})} - R_{\tilde{u}_{s_{k-1}}}\right|^q |f_{\mathcal{P},j}(s_{k-1})|^q \Delta_k\right] \\
&\le \mathbb{E}\left[\mathcal{A}_{\mathcal{P}}^q \sup_{j\in\{1,\cdots,n\}, s\in[0,1]} |f_{\mathcal{P},j}(s)|^q\right],
\end{aligned}$$

where

$$A_{\mathcal{P}} := \sup_{s\in[0,1]} \left|R_{u_{\mathcal{P}}(s)} - R_{\tilde{u}_s}\right|.$$

Using Theorem 6.4 we know

$$\mathbb{E}\left[A_{\mathcal{P}}^q\right] \le C(q,\gamma)\, |\mathcal{P}|^{q\gamma} \ \ \forall \gamma \in \left(0, \frac{1}{2}\right), q \ge 1.$$

Then by Holder's inequality and Lemma 6.12,

$$\sup_{n\ge 2qN} \mathbb{E}\left[\mathcal{A}_{\mathcal{P}}^q \sup_{j\in\{1,\cdots,n\}, s\in[0,1]} |f_{\mathcal{P},j}(s)|^q\right] \le C(q,\gamma)\, |\mathcal{P}|^{q\gamma}.$$

Then using Burkholder-Davies-Gundy inequality, we have

$$\mathbb{E}\left[\left|e_{\mathcal{P},1}(1)\right|^q\right] \le C\mathbb{E}\left[\langle e_{\mathcal{P},1}\rangle^{\frac{q}{2}}(1)\right] \le C(q,\gamma)\left|\mathcal{P}\right|^{q\gamma}$$

and thus

$$e_{\mathcal{P},1}(1) \to 0 \text{ in } L^{\infty-}(W_o(M)) \text{ as } |\mathcal{P}| \to 0. \tag{6.76}$$

For $e_{\mathcal{P},2}(1)$, since

$$\langle e_{\mathcal{P},2}\rangle(r) \le \int_0^r \left|g_2(s) - g_3(s)\right|^2 ds,$$

so for each $q \ge 1$,

$$\begin{aligned}
\mathbb{E}\left[\langle e_2\rangle^{\frac{q}{2}}(1)\right] &\le \mathbb{E}\left[\int_0^1 \left|g_2(s) - g_3(s)\right|^q ds\right] \\
&\le \mathbb{E}\left[\sum_{k=1}^n \left|R_{\tilde{u}(s_{k-1})}\right|^q \left|f_{\mathcal{P},j}(s_{k-1}) - \tilde{T}_{s_{k-1}}\tilde{T}_{s_j}^{-1}\right|^q \Delta_k\right] \\
&\le \mathbb{E}\left[N\sup_{j,s}\left|f_{\mathcal{P},j}(s) - \tilde{T}_s\tilde{T}_{s_j}^{-1}\right|^q\right].
\end{aligned}$$

By Holder's inequality and Theorem 6.15,

$$\mathbb{E}\left[N\sup_{j,s}\left|f_{\mathcal{P},j}(s) - \tilde{T}_s\tilde{T}_{s_j}^{-1}\right|^q\right] \le C(q,\gamma)\left|\mathcal{P}\right|^{q\gamma}.$$

Then using Burkholder-Davies-Gundy inequality, we get

$$\mathbb{E}\left[\left|e_{\mathcal{P},2}(1)\right|^q\right] \le C\mathbb{E}\left[\langle e_{\mathcal{P},2}\rangle^{\frac{q}{2}}(1)\right] \le C(q,\gamma)\left|\mathcal{P}\right|^{q\gamma}$$

and thus

$$e_{\mathcal{P},2}(1) \to 0 \text{ in } L^{\infty-}(W_o(M)) \text{ as } |\mathcal{P}| \to 0. \tag{6.77}$$

For $e_{\mathcal{P},3}(1)$, since

$$\langle e_{\mathcal{P},3}\rangle(r) \le \int_0^r \left|g_3(s) - g_4(s)\right|^2 ds,$$

so for each $q \geq 1$,

$$\begin{aligned}\mathbb{E}\left[\langle e_{\mathcal{P},3}\rangle^{\frac{q}{2}}(1)\right] &\leq \mathbb{E}\left[\int_0^1 |g_3(s) - g_4(s)|^q \, ds\right] \\ &\leq \mathbb{E}\left[\sum_{k=1}^n \left|R_{\tilde{u}(s_{k-1})}\right|^q \int_{s_{k-1}}^{s_k} \left|\tilde{T}_s \tilde{T}_{s_j}^{-1} - \tilde{T}_{s_{k-1}} \tilde{T}_{s_j}^{-1}\right|^q ds\right] \\ &\leq C\,|\mathcal{P}|^q .\end{aligned}$$

Then using Burkholder-Davies-Gundy inequality, we get

$$\mathbb{E}\left[|e_{\mathcal{P},3}(1)|^q\right] \leq C\mathbb{E}\left[\langle e_{\mathcal{P},3}\rangle^{\frac{q}{2}}(1)\right] \leq C(q,\gamma)\,|\mathcal{P}|^q$$

and thus

$$e_{\mathcal{P},3}(1) \to 0 \text{ in } L^{\infty-}(W_o(M)) \text{ as } |\mathcal{P}| \to 0. \tag{6.78}$$

For $e_{\mathcal{P},4}(1)$, since

$$\langle e_{\mathcal{P},4}\rangle(r) \leq \int_0^r |g_5(s) - g_4(s)|^2 \, ds,$$

so for each $q \geq 1$,

$$\begin{aligned}\mathbb{E}\left[\langle e_{\mathcal{P},4}\rangle^{\frac{q}{2}}(1)\right] &\leq \mathbb{E}\left[\int_0^1 |g_5(s) - g_4(s)|^q \, ds\right] \\ &\leq \mathbb{E}\left[\sum_{k=1}^n \left|\tilde{T}_s \tilde{T}_{s_j}^{-1}\right|^q \int_{s_{k-1}}^{s_k} \left|R_{\tilde{u}_s} - R_{\tilde{u}_{s_{k-1}}}\right|^q ds\right] \\ &\leq \mathbb{E}\left[K_\gamma^q\right] |\mathcal{P}|^{q\gamma} .\end{aligned}$$

Then using Burkholder-Davies-Gundy inequality, we have

$$\mathbb{E}\left[|e_{\mathcal{P},4}(1)|^q\right] \leq C\mathbb{E}\left[\langle e_{\mathcal{P},4}\rangle^{\frac{q}{2}}(1)\right] \leq C(q,\gamma)\,|\mathcal{P}|^{q\gamma}$$

and thus

$$e_{\mathcal{P},4}(1) \to 0 \text{ in } L^{\infty-}(W_o(M)) \text{ as } |\mathcal{P}| \to 0. \tag{6.79}$$

Combining Eq.(6.76), (6.77), (6.78) and (6.79) gives Eq. (6.75). Then using Eq.(6.75) and Eq.(6.74) we have

$$\left|V_{\mathcal{P}}-\tilde{V}_{\mathcal{P}}\right|\to 0 \text{ in } L^{\infty-}\left(W_o\left(M\right)\right) \text{ as } \left|\mathcal{P}\right|\to 0. \tag{6.80}$$

where

$$\tilde{V}_{\mathcal{P}}=-\sum_{\alpha=1}^{d}\sum_{j=1}^{n}\left\langle \left(\tilde{T}_{s_j}^{-1}\right)^{*}\tilde{T}_1^{*}\tilde{\mathbf{K}}_1^{-1}\int_{s_j}^{1}R_{\tilde{u}_r}\left(d\beta_r,\tilde{T}_r\tilde{T}_{s_j}^{-1}e_\alpha\right)\tilde{H},e_\alpha\right\rangle \Delta_j.$$

We view $\tilde{V}_{\mathcal{P}}:=\tilde{V}_{\mathcal{P}}\left(e_\alpha,e_\alpha\right)$ as a bilinear form on $\mathbb{R}^d$, therefore

$$\begin{aligned}\tilde{V}_{\mathcal{P}}\left(e_\alpha,e_\alpha\right)&=\tilde{V}_{\mathcal{P}}\left(\tilde{T}_{s_j}e_\alpha,\left(\tilde{T}_{s_j}^{-1}\right)^{*}e_\alpha\right)\\&=\sum_{\alpha=1}^{d}\sum_{j=1}^{n}\left\langle \tilde{T}_1^{*}K_1^{-1}\int_{s_j}^{1}R_{\tilde{u}_r}\left(d\beta_r,\tilde{T}_re_\alpha\right)\tilde{H},\tilde{T}_{s_j}^{-1}\left(\tilde{T}_{s_j}^{-1}\right)^{*}e_\alpha\right\rangle \Delta_j\\&=\sum_{\alpha=1}^{d}\int_0^1\left\langle \tilde{T}_1^{*}K_1^{-1}\int_{\underline{s}}^{1}R_{\tilde{u}_r}\left(d\beta_r,\tilde{T}_re_\alpha\right)\tilde{H},\tilde{T}_{\underline{s}}^{-1}\left(\tilde{T}_{\underline{s}}^{-1}\right)^{*}e_\alpha\right\rangle ds.\end{aligned} \tag{6.81}$$

Then we are about to show

$$\tilde{V}_{\mathcal{P}}-\sum_{\alpha=1}^{d}\int_0^1\left\langle \tilde{T}_1^{*}K_1^{-1}\int_{s}^{1}R_{\tilde{u}_r}\left(d\beta_r,\tilde{T}_re_\alpha\right)\tilde{H},\tilde{T}_{s}^{-1}\left(\tilde{T}_{s}^{-1}\right)^{*}e_\alpha\right\rangle ds\to 0 \tag{6.82}$$

in $L^{\infty-}\left(W_o\left(M\right)\right)$ as $\left|\mathcal{P}\right|\to 0$.

Using Eq.(6.81) we know

$$\text{the left–hand side of Eq.(6.82)}\le\sum_{\alpha=1}^{d}\int_0^1\left(I_{\mathcal{P}}\left(s\right)+II_{\mathcal{P}}\left(s\right)\right)ds,$$

where

$$I_{\mathcal{P}}\left(s\right)=\left\langle \tilde{T}_1^{*}K_1^{-1}\int_{\underline{s}}^{s}R_{\tilde{u}_r}\left(d\beta_r,\tilde{T}_re_\alpha\right)\tilde{H},\tilde{T}_{s}^{-1}\left(\tilde{T}_{s}^{-1}\right)^{*}e_\alpha\right\rangle$$

and

$$II_{\mathcal{P}}(s) = \left\langle \tilde{T}_1^* K_1^{-1} \int_s^1 R_{\tilde{u}_r}\left(d\beta_r, \tilde{T}_r e_\alpha\right) \tilde{H}, \left(\tilde{T}_{\underline{s}}^{-1}\left(\tilde{T}_{\underline{s}}^{-1}\right)^* - \tilde{T}_s^{-1}\left(\tilde{T}_s^{-1}\right)^*\right) e_\alpha \right\rangle. \tag{6.83}$$

For $I_{\mathcal{P}}(s)$, since

$$|I_{\mathcal{P}}(s)|^q \le C \left| \int_{\underline{s}}^s R_{\tilde{u}_r}\left(d\beta_r, \tilde{T}_r e_\alpha\right)\right|^q,$$

by Burkholder-Davies-Gundy inequality,

$$\mathbb{E}\left[|I_{\mathcal{P}}(s)|^q\right] \le C |\mathcal{P}|^{\frac{q}{2}}.$$

Notice that

$$\mathbb{E}\left[|II_{\mathcal{P}}(s)|^q\right] \le C |\mathcal{P}|^q \, \mathbb{E}\left[\left|\int_{\underline{s}}^s R_{\tilde{u}_r}\left(d\beta_r, \tilde{T}_r e_\alpha\right)\right|^q\right] \le C |\mathcal{P}|^q,$$

using Holder's inequality, we have

$$\begin{aligned} &\mathbb{E}\left[|\text{the left–hand side of Eq.(6.82)}|^q\right] \\ &\le C \sum_{\alpha=1}^d \sum_{j=1}^n \int_{s_{j-1}}^{s_j} \mathbb{E}\left[|I_{\mathcal{P}}(s)|^q + |II_{\mathcal{P}}(s)|^q\right] \\ &\le C \sum_{\alpha=1}^d \sum_{j=1}^n \int_{s_{j-1}}^{s_j} \left(|\mathcal{P}|^{\frac{q}{2}} + |\mathcal{P}|^q\right) \\ &= C |\mathcal{P}|^{\frac{q}{2}} \end{aligned}$$

and from which Eq. (6.82) follows.

The last step is to show a change of integration order:

$$\sum_{\alpha=1}^{d}\int_0^1\left\langle \tilde{T}_1^* K_1^{-1}\int_s^1 R_{\tilde{u}_r}\left(d\beta_r,\tilde{T}_r e_\alpha\right)\tilde{H},\tilde{T}_s^{-1}\left(\tilde{T}_s^{-1}\right)^* e_\alpha\right\rangle ds$$
$$=\sum_{\alpha=1}^{d}\int_0^r\left\langle \tilde{T}_1^* K_1^{-1}\int_0^1 R_{\tilde{u}_r}\left(d\beta_r,\tilde{T}_r e_\alpha\right)\tilde{H},\tilde{T}_s^{-1}\left(\tilde{T}_s^{-1}\right)^* e_\alpha\right\rangle ds. \qquad (6.84)$$

Define

$$f(s)=\sum_{\alpha=1}^{d}\int_0^t\left\langle \tilde{T}_1^* K_1^{-1}\int_s^t R_{\tilde{u}_r}\left(d\beta_r,\tilde{T}_r e_\alpha\right)\tilde{H},\tilde{T}_s^{-1}\left(\tilde{T}_s^{-1}\right)^* e_\alpha\right\rangle ds$$

and

$$g(s)=\sum_{\alpha=1}^{d}\int_0^r\left\langle \tilde{T}_1^* K_1^{-1}\int_0^t R_{\tilde{u}_r}\left(d\beta_r,\tilde{T}_r e_\alpha\right)\tilde{H},\int_0^r \tilde{T}_s^{-1}\left(\tilde{T}_s^{-1}\right)^* ds e_\alpha\right\rangle .$$

Then

$$df=\sum_{\alpha=1}^{d}\left\langle \tilde{T}_1^* K_1^{-1} R_{\tilde{u}_t}\left(d\beta_t,\tilde{T}_t e_\alpha\right)\tilde{H},\int_0^t \tilde{T}_s^{-1}\left(\tilde{T}_s^{-1}\right)^* ds e_\alpha\right\rangle$$

and $f(0)\equiv 0$. Since

$$dg=\sum_{\alpha=1}^{d}\left\langle \tilde{T}_1^* K_1^{-1} R_{\tilde{u}_t}\left(d\beta_t,\tilde{T}_t e_\alpha\right)\tilde{H},\int_0^t \tilde{T}_s^{-1}\left(\tilde{T}_s^{-1}\right)^* ds e_\alpha\right\rangle = df$$

and $g(0)=0$, Eq. (6.84) is proved by observing that left–hand side$= f_1 = g_1 =$ right–hand side.

Finally, after changing the pair (e_α, e_α) to

$$\left(\int_0^r \tilde{T}_s^{-1}\left(\tilde{T}_s^{-1}\right)^* ds e_\alpha, \left[\left(\int_0^r \tilde{T}_s^{-1}\left(\tilde{T}_s^{-1}\right)^* ds\right)^{-1}\right]^* e_\alpha\right)$$

in the right–hand side of Eq. (6.84) (note that this action does not change its value by exactly the same argument as (6.81)), and recognizing

$$\tilde{T}_r \int_0^r \tilde{T}_s^{-1} \left(\tilde{T}_s^{-1}\right)^* dse_\alpha = Z_\alpha(r),$$

we combine Eq. (6.80), (6.82) and (6.84) to prove Eq.(6.73). ■

Lemma 6.31 *If* $\nabla R \equiv 0$*, then*

$$\sum_{\alpha=1}^{d} \sum_{j=1}^{n} \langle I_{\mathcal{P}}(\alpha, j), e_\alpha \rangle \sqrt{\Delta_j} \to 0$$

as $|\mathcal{P}| \to 0$ *in* $L^{\infty-}(W_o(M))$.

Proof. Define $\tilde{g}_j(s) := X^{h_{\alpha,j}} f_{\mathcal{P},j}(s)$ and $g_j(s) := \tilde{g}_j(s) - \tilde{g}_j(\underline{s})$. Then we know that $g_j(s)$ satisfies the following ODE: for $k = j, \cdots, n$

$$\begin{cases} g_j''(s) = A_{\mathcal{P},k}(s)\, g_j(s) + \dot{A_{\mathcal{P},k}}(s)\left(f_{\mathcal{P},j}(s) - f_{\mathcal{P},j}(\underline{s})\right) & s \in [s_{k-1}, s_k] \\ g_j(\underline{s}) = 0 \\ g_j'(\underline{s}) = 0 \end{cases}$$

where

$$\dot{A_{\mathcal{P},k}}(s) = \frac{d}{dt}|_0 \left(R_{u_{\mathcal{P}}(t,s)}\left(\beta_{\mathcal{P}}'(t,s), \cdot\right) \beta_{\mathcal{P}}'(t,s)\right).$$

For $s \in [s_{k-1}, s_k]$, we know

$$g_j(s) = \int_{s_{k-1}}^{s} S_k(s-r)\, \dot{A_{\mathcal{P}}^k}(r)\left(f_{\mathcal{P},j}(r) - f_{\mathcal{P},j}(s_{k-1})\right) dr.$$

Using Lemma 6.8 and 6.15, we have

$$\begin{aligned} \left|f_{\mathcal{P},i}(s) - f_{\mathcal{P},i}(\underline{s})\right|^q &\le \frac{N^q}{2^q} \left|\Delta_k \beta\right|^{2q} e^{N|\Delta_k\beta|^2} e^{\frac{1}{2}qN\sum_{k=1}^n |\Delta_k\beta|^2} \left(1 + \frac{NK_\gamma |\mathcal{P}|^\gamma}{6}\right)^q \\ &\le C\,|\mathcal{P}|^{2q\gamma} e^{qN\sum_{k=1}^n |\Delta_k\beta|^2} K_\gamma^{2q} \left(1 + \frac{NK_\gamma |\mathcal{P}|^\gamma}{6}\right)^q \end{aligned}$$

and

$$\left|S_k\left(s-r\right)\right| \le \left(s-r\right)\left(1+\frac{N}{6}K_\gamma^2\left|\mathcal{P}\right|^{2\gamma}e^{\frac{1}{2}N\sum_{i=1}^n\left|\Delta_i\beta\right|^2}\right).$$

Therefore

$$\begin{aligned}
&\left|g_j\left(s\right)\right| \\
&\le \int_{s_{k-1}}^{s}\left|S_k\left(s-r\right)\right|\left|\dot{A_{\mathcal{P}}^k}\left(r\right)\right|\left|f_{\mathcal{P},j}\left(r\right)-f_{\mathcal{P},j}\left(s_{k-1}\right)\right|dr \\
&\le C\sup_{k\in\{1,\dots,n\},r\in[0,1]}\left|\dot{A_{\mathcal{P},k}}\left(r\right)\right|\left|\mathcal{P}\right|^{2\gamma}K_\gamma^2\left(1+\frac{NK_\gamma\left|\mathcal{P}\right|^\gamma}{6}\right)e^{N\sum_{i=1}^n\left|\Delta_i\beta\right|^2}\int_{s_{k-1}}^{s}\left(s-r\right)dr \\
&= C\sup_{k\in\{1,\dots,n\},r\in[0,1]}\left|\dot{A_{\mathcal{P}}^k}\left(r\right)\right|\left|\mathcal{P}\right|^{2\gamma+2}K_\gamma^2\left(1+\frac{NK_\gamma\left|\mathcal{P}\right|^\gamma}{6}\right)e^{N\sum_{i=1}^n\left|\Delta_i\beta\right|^2},
\end{aligned}$$

and thus

$$\left|\tilde{g}_j\left(1\right)\right| \le \sum_{k=j}^{n}\left|g_j\left(s_k\right)\right| \tag{6.85}$$

$$\le C\sup_{k\in\{1,\dots,n\},r\in[0,1]}\left|\dot{A_{\mathcal{P}}^k}\left(r\right)\right|\left|\mathcal{P}\right|^{2\gamma+1}K_\gamma^2\left(1+\frac{NK_\gamma\left|\mathcal{P}\right|^\gamma}{6}\right)e^{N\sum_{i=1}^n\left|\Delta_i\beta\right|^2}. \tag{6.86}$$

It remains to analyze $\sup_{k\in\{1,\dots,n\},r\in[0,1]}\left|\dot{A_{\mathcal{P}}^k}\left(r\right)\right|$:

$$\begin{aligned}
\dot{A_{\mathcal{P}}^k}\left(r\right) =& \left(\frac{d}{dt}|_0 R_{u_{\mathcal{P}}(t,s)}\right)\left(\beta_{\mathcal{P}}'\left(s\right),\cdot\right)\beta_{\mathcal{P}}'\left(s\right)+R_{u_{\mathcal{P}}(s)}\left(\frac{d}{dt}|_0\beta_{\mathcal{P}}'\left(t,s\right),\cdot\right)\beta_{\mathcal{P}}'\left(s\right) \\
&+R_{u_{\mathcal{P}}(s)}\left(\beta_{\mathcal{P}}'\left(s\right),\cdot\right)\frac{d}{dt}|_0\beta_{\mathcal{P}}'\left(t,s\right)
\end{aligned}$$

Using $\nabla R\equiv 0$ we find

$$\left(\frac{d}{dt}|_0 R_{u_{\mathcal{P}}(t,s)}\right)\left(\beta_{\mathcal{P}}'\left(s\right),\cdot\right)\beta_{\mathcal{P}}'\left(s\right)=0.$$

Notice that

$$\beta'_{\mathcal{P}}(t,s) = u_s(\sigma_t)^{-1}\sigma'_{\mathcal{P}}(t,s),$$

using Lemma 6.28 and we have

$$X^{h_{\alpha,j}}\beta'_{\mathcal{P}}(s_{k-1}+) = \frac{\delta_k^j e_\alpha}{\sqrt{\Delta_j}} - \int_0^{s_{k-1}} R_{u_{\mathcal{P}}(\tau)}\left(\beta'_{\mathcal{P}}(\tau+), h_{\alpha,j}(\tau)\right) d\tau \beta'_{\mathcal{P}}(s_{k-1}+). \quad (6.87)$$

Therefore

$$\begin{aligned}
\left|\dot{A}^k_{\mathcal{P}}(r)\right| &\leq N\left|X^{h_{\alpha,j}}\beta'_{\mathcal{P}}(s_{k-1}+)\right|\left|\beta'_{\mathcal{P}}(s_{k-1})\right| \\
&\leq N\left(\frac{1}{\sqrt{|\mathcal{P}|}} + N\sup_{j,s}|h_{\alpha,j}(s)|\sup_{s\in[0,1]}\left|\beta'_{\mathcal{P}}(s)\right|^2\right)\left|\beta'_{\mathcal{P}}(s_{k-1})\right| \\
&\leq N\left(\frac{1}{\sqrt{|\mathcal{P}|}} + Nf(K_\gamma)\sqrt{|\mathcal{P}|}\,|\mathcal{P}|^{2(\gamma-1)}\right)K_\gamma|\mathcal{P}|^{\gamma-1} \\
&\leq f(K_\gamma)|\mathcal{P}|^{3\gamma-\frac{5}{2}}
\end{aligned}$$

where $f(K_\gamma)$ is some random variable in $L^1(W_o(M))$, so

$$|\tilde{g}_j(1)| \leq Cf(K_\gamma)|\mathcal{P}|^{5\gamma-\frac{3}{2}}. \quad (6.88)$$

From above one can see

$$\begin{aligned}
\sum_{\alpha,j=1,1}^{d,n}\langle I, e_\alpha\rangle\sqrt{\Delta_j} &= \sum_{\alpha,j=1,1}^{d,n}\left\langle\left(X^{h_{\alpha,j}}T_j^*\right)\mathbf{K}_{\mathcal{P}}^{-1}(1)H_{\mathcal{P}}, e_\alpha\right\rangle\sqrt{\Delta_j} \\
&= \sum_{\alpha=1}^{d}\left\langle\sum_{j=1}^{n}\left(\tilde{g}_j^*(1)\sqrt{|\mathcal{P}|}\right)\mathbf{K}_{\mathcal{P}}(1)^{-1}H_{\mathcal{P}}, e_\alpha\right\rangle.
\end{aligned}$$

From (6.88) we know that $\sum_{j=1}^n\left(\tilde{g}_j^*(1)\sqrt{|\mathcal{P}|}\right) \to 0$ in $L^{\infty-}(W)$, also notice that

$\mathbf{K}_{\mathcal{P}}(1)^{-1} H_{\mathcal{P}} \to \mathbf{K}(1)^{-1} \tilde{H}$ in $L^{\infty-}(W_o(M))$, so:

$$\sum_{\alpha=1}^{d} \left\langle \sum_{j=1}^{n} \left(\tilde{g}_j^*(1) \sqrt{|\mathcal{P}|} \right) \mathbf{K}_{\mathcal{P}}(1)^{-1} H_{\mathcal{P}}, e_\alpha \right\rangle \to 0 \text{ in } L^{\infty-}(W_o(M)).$$

■

Lemma 6.32 *If* $\nabla R \equiv 0$*, then*

$$\sum_{\alpha=1}^{d} \sum_{j=1}^{n} \langle II_{\mathcal{P}}(\alpha, j), e_\alpha \rangle \sqrt{\Delta_j} \to 0$$

as $|\mathcal{P}| \to 0$ *in* $L^{\infty-}(W_o(M))$.

Proof. Since

$$X^{h_{\alpha,j}} \left(\mathbf{K}_{\mathcal{P}}(1)^{-1} \right) = -\mathbf{K}_{\mathcal{P}}(1)^{-1} X^{h_{\alpha,j}} (\mathbf{K}_{\mathcal{P}}(1)) \mathbf{K}_{\mathcal{P}}(1)^{-1},$$

so

$$\left| X^{h_{\alpha,j}} \left(\mathbf{K}_{\mathcal{P}}(1)^{-1} \right) \right| \leq \left| X^{h_{\alpha,j}} (\mathbf{K}_{\mathcal{P}}(1)) \right|.$$

Then using $\tilde{g}_j(s) := X^{h_{\alpha,j}}(\mathbf{K}_{\mathcal{P}}(s))$ and this lemma follows from a Lemma 6.31-type argument. ■

Chapter 7

Proof of Main Theorem

First we restate the main theorem of our paper.

Theorem 7.1 (Theorem 1.12) *If M is a symmetric space of non–compact type, then for any restricted cylinder function $f \in \mathcal{RFC}_b^1$,*

$$\lim_{|\mathcal{P}| \to 0} \int_{H_{\mathcal{P}}(M)} f(\sigma)\, d\nu^1_{\mathcal{P},x}(\sigma) = \int_{W_o(M)} f(\sigma)\, d\nu_x(\sigma)$$

Before proving Theorem 1.12, first we need some supplementary results. Recall that the manifold considered in Theorem 1.12 is a Hadamard manifold with parallel curvature tensor.

Proposition 7.2 *For any $f \in \mathcal{FC}_b^1$, $X \in \Gamma(TM)$ with compact support,*

$$\tilde{X}^{tr,\nu} f \in L^{\infty-}(W_o(M), \nu).$$

The proof comes after Lemma 7.3 and 7.4.

Lemma 7.3 *Following the notations in Lemma 4.43,*

$$\sum_{\alpha=1}^{d} \left\langle \tilde{C}\tilde{H}, e_\alpha \right\rangle \int_0^1 \left\langle \left(\tilde{T}_s^{-1}\right)^* e_\alpha, d\beta_s \right\rangle \in L^{\infty-}(W_o(M), \nu).$$

Proof. For any $v \in \mathbb{C}^d$,

$$\left\langle \left(\int_0^1 \tilde{T}_r^{-1} (\tilde{T}_r^{-1})^* dr \right) v, v \right\rangle = \int_0^1 \left\| (\tilde{T}_r^{-1})^* v \right\|^2 dr \geq C \left\| v \right\|^2 .$$

So

$$\left\| \left(\int_0^1 \tilde{T}_r^{-1} (T_r^{-1})^* dr \right)^{-1} \right\| \leq \frac{1}{C}$$

where C is a generic constant.

Since X has compact support and is smooth, $\|X(\cdot)\| \in C_0(M)$ and

$$\left\| \tilde{H}(\sigma) \right\| = \|X \circ E_1(\sigma)\| \leq \sup \|X\| < C.$$

Also notice that $\tilde{C}$ is deterministic, so we have

$$\left\| \left\langle \tilde{C}\tilde{H}, e_\alpha \right\rangle \right\| \leq \left\| \tilde{C} \right\| \left\| \tilde{H} \right\| < \infty.$$

Since $\left(\tilde{T}_s^{-1} \right)$ is bounded, so $\left(\tilde{T}_s^{-1} \right) \in L^\infty([0,1])$. By Burkholder's inequality, we get

$$\int_0^1 \left\langle \left(\tilde{T}_s^{-1} \right)^* e_\alpha, d\beta_s \right\rangle \in L^{\infty-}(W_o(M)).$$

Therefore,

$$\sum_{\alpha=1}^d \left\langle \tilde{C}\tilde{H}, e_\alpha \right\rangle \int_0^1 \left\langle \left(\tilde{T}_s^{-1} \right)^* e_\alpha, d\beta_s \right\rangle \in L^{\infty-}(W_o(M)).$$

■

Lemma 7.4 *Following the notations in Definition 4.36,*

$$\sum_{\alpha=1}^d \left\langle \tilde{C} X^{Z_\alpha} \tilde{H}, e_\alpha \right\rangle \in L^{\infty-}(W_o(M), \nu).$$

Proof. From Lemma 4.44 we know:

$$-\sum_{\alpha=1}^{d}\left\langle X^{Z_\alpha}\left(\tilde{C}\tilde{H}\right),e_\alpha\right\rangle = divX\circ E_1-\sum_{\alpha=1}^{d}\left\langle \tilde{C}A_1\left\langle Z_\alpha\right\rangle \tilde{H},e_\alpha\right\rangle. \tag{7.1}$$

where

$$A_1\left\langle Z_\alpha\right\rangle = \int_0^1 R_{\tilde{u}(s)}\left(Z_\alpha\left(s\right),\delta\beta_s\right) \tag{7.2}$$

Since $\int_0^{\cdot}\left[\tilde{T}\left(r\right)^{-1}\right]^* e_\alpha dr$ is bounded, by Gronwall's inequality one can see that Z_α is bounded and thus using Burkholder's inequality, we have:

$$A_1\left\langle Z_\alpha\right\rangle \in L^{\infty-}\left(W_o\left(M\right)\right). \tag{7.3}$$

It is easy to see $divX\circ E_1\left(\sigma\right)$ is bounded because $X\in\Gamma\left(TM\right)$ with compact support. Therefore:

$$\sum_{\alpha=1}^{d}\left\langle \tilde{C}X^{Z_\alpha}\tilde{H},e_\alpha\right\rangle \in L^{\infty-}\left(W_o\left(M\right),\nu\right).$$

■

Proof of Proposition 7.2. Recall that from Lemma 4.44 and 4.43, we have:

$$\tilde{X}^{tr,\nu}f=-X^{Z_\Phi}f+\sum_{\alpha=1}^{d}\left\langle \tilde{C}\tilde{H},e_\alpha\right\rangle\int_0^1\left\langle\left(\tilde{T}_s^{-1}\right)^* e_\alpha,d\beta_s\right\rangle\cdot f-\sum_{\alpha=1}^{d}\left\langle \tilde{C}X^{Z_\alpha}\tilde{H},e_\alpha\right\rangle\cdot f$$

A similar argument as in Lemma 4.44 can show that $\tilde{X}f\in L^{\infty-}\left(W_o\left(M\right)\right)$, then combine Lemma 7.3 and 7.4 and we can prove Proposition 7.2. ■

Lemma 7.5 *For any* $f\in\mathcal{RFC}_b^1$, $\tilde{X}_{\mathcal{P}}^{tr,\nu_{\mathcal{P}}^1}f\in L^{\infty-}\left(H_{\mathcal{P}}\left(M\right),\nu_{\mathcal{P}}^1\right).$

Proof. From Theorem 6.23 we know that

$$\tilde{X}_{\mathcal{P}}^{tr,\nu_{\mathcal{P}}^1}f\left(\phi\left(\beta_{\mathcal{P}}\right)\right)-\tilde{X}f\to 0 \text{ in } L^{\infty-}\left(W_o\left(M\right)\right). \tag{7.4}$$

From Proposition 7.2 we know $\tilde{X}\tilde{f} \in L^{\infty-}(W_o(M))$, so $\tilde{X}_{\mathcal{P}}^{tr,\nu_{\mathcal{P}}^1} f(\phi(\beta_{\mathcal{P}})) \in L^{\infty-}(W_o(M))$.

Since the law of $\phi(\beta_{\mathcal{P}})$ under ν is $\nu_{\mathcal{P}}^1$, so

$$\tilde{X}_{\mathcal{P}}^{tr,\nu_{\mathcal{P}}^1} f \in L^{\infty-}\left(H_{\mathcal{P}}(M), \nu_{\mathcal{P}}^1\right) \iff \tilde{X}_{\mathcal{P}}^{tr,\nu_{\mathcal{P}}^1} f(\phi(\beta_{\mathcal{P}})) \in L^{\infty-}(W_o(M)).$$

■

Notation 7.6 *Denote by g any one of $\{g_i\}_{i=0}^d$ as in Theorem 3.3 and $\left\{g^{(m)}\right\}_m \subset C_0^\infty(M)$ be the approximate sequence in $L^{\frac{d}{d-1}}(M)$ as defined in Remark 3.5.*

Lemma 7.7 *Define $\tilde{g}(\sigma) = g(\sigma(1))$ and $\tilde{g}^{(m)}(\sigma) = g^{(m)}(\sigma(1))$, then for any $f \in \mathcal{FC}_b^1$,*

$$\int_{W_o(M)} \left|\tilde{g} \cdot \left(\tilde{X}^{tr,\nu} f\right)\right|(\sigma)\, d\nu(\sigma) < \infty$$

and

$$\lim_{m\to\infty} \int_{W_o(M)} \tilde{g}^{(m)}(\sigma)\left(\tilde{X}^{tr,\nu} f\right)(\sigma)\, d\nu(\sigma) = \int_{W_o(M)} \tilde{g}(\sigma)\left(\tilde{X}^{tr,\nu} f\right)(\sigma)\, d\nu(\sigma).$$

Proof. Since $\nu\{\sigma : \sigma(1) = e\} = 0$, so $\tilde{g}$ is $\nu - a.s.$ well-defined. In particular, for any $p > 0$,

$$\int_{W_o(M)} |\tilde{g}(\sigma)|^p\, d\nu(\sigma) = \int_M |g(x)|^p\, p_1(0, x)\, d\lambda(x), \tag{7.5}$$

where λ is the volume measure on M.

Since g has compact support and $p_1(0, \cdot) \in C^\infty(M)$,

$$\int_M |g(x)|^p\, p_1(0, x)\, d\lambda(x) \leq C\, \|g\|_{L^p(M)}^p. \tag{7.6}$$

Since $g \in L^{1+\frac{1}{d-1}}(M)$, we have

$$\tilde{g} \in L^{1+\frac{1}{d-1}}\left(W_o(M)\right). \tag{7.7}$$

Notice that from Proposition 7.2, we have $\tilde{X}^{tr,\nu} f \in L^{\infty-}\left(W_o(M)\right)$, so by Holder's inequality, we get:

$$\int_{W_o(M)} \left|\tilde{g}(\sigma)\, \tilde{X}^{tr,\nu} f(\sigma)\right| d\nu(\sigma) < \infty.$$

To prove (7.7), just notice that the support of $g^{(m)}$ is contained in a compact set for all m, so we have, following the same argument as before

$$\int_{W_o(M)} \left|\tilde{g}^{(m)} - \tilde{g}\right|^p (\sigma)\, d\nu(\sigma) = \int_M \left|g^{(m)}(x) - g(x)\right|^p p_1(0,x)\, d\lambda(x) \tag{7.8}$$

$$\leq C \left\|g^{(m)} - g\right\|_{L^p(M)}^p. \tag{7.9}$$

Using Holder's inequality again we can get (7.7). ■

Lemma 7.8 *Define* $\tilde{g} : H_{\mathcal{P}}(M) \to \mathbb{R}$ *to be* $\tilde{g}(\sigma) = g(\sigma(1))$, *then*

$$\tilde{g} \in L^{\frac{d}{d-1}}\left(H_{\mathcal{P}}(M), \nu_{\mathcal{P}}^1\right).$$

Proof. Apply the co-area formula (3.41) to $|\tilde{g}|^{\frac{d}{d-1}}$, we have:

$$\int_{H_{\mathcal{P}}(M)} |\tilde{g}(\sigma)|^{\frac{d}{d-1}}\, d\nu_{\mathcal{P}}^1(\sigma) = \int_M |g(x)|^{\frac{d}{d-1}}\, h_{\mathcal{P}}(x)\, dx$$

where $h_{\mathcal{P}}(x) \in C(M)$ is defined in Theorem 3.35 with $f \equiv 1$. Since g has compact support, we know:

$$\int_M |g(x)|^{\frac{d}{d-1}}\, h_{\mathcal{P}}(x)\, dx \leq C \int_M |g(x)|^{\frac{d}{d-1}}\, dx. \tag{7.10}$$

Therefore $\tilde{g} \in L^{\frac{d}{d-1}}\left(H_{\mathcal{P}}(M), \nu_{\mathcal{P}}^1\right)$. ■

Lemma 7.9 *Define* $\tilde{g}(\sigma) = g(\sigma(1))$ *and* $\tilde{g}^{(m)}(\sigma) = g^{(m)}(\sigma(1))$*, then for any* $f \in \mathcal{FC}^1_{\mathcal{P},b}$*,*

$$\int_{H_{\mathcal{P}}(M)} \left|\tilde{g} \cdot \left(\tilde{X}^{tr,\nu^1_{\mathcal{P}}} f\right)\right| (\sigma)\, d\nu^1_{\mathcal{P}}(\sigma) < \infty \tag{7.11}$$

and

$$\lim_{m\to\infty} \int_{H_{\mathcal{P}}(M)} \tilde{g}^{(m)}(\sigma) \left(\tilde{X}^{tr,\nu^1_{\mathcal{P}}} f\right)(\sigma)\, d\nu^1_{\mathcal{P}}(\sigma) = \int_{H_{\mathcal{P}}(M)} \tilde{g}(\sigma) \left(\tilde{X}^{tr,\nu^1_{\mathcal{P}}} f\right)(\sigma)\, d\nu^1_{\mathcal{P}}(\sigma).$$

Proof. Using Lemma 7.5, Lemma 7.8 and Holder's inequality, we can easily see Eq.(7.11). Then apply the co-area formula 3.41 with

$$(H, M, p, g, f) = \left(H_{\mathcal{P}}(M), M, E_1^{\mathcal{P}}, \frac{1}{Z^1_{\mathcal{P}}} e^{-\frac{E}{2}}, \left|\left(\tilde{g}^{(m)} - \tilde{g}\right)(\sigma)\right|^{\frac{d}{d-1}}\right),$$

we have:

$$\int_{H_{\mathcal{P}}(M)} \left|\left(\tilde{g}^{(m)} - \tilde{g}\right)(\sigma)\right|^{\frac{d}{d-1}} d\nu^1_{\mathcal{P}}(\sigma) = \int_M |(g^m - g)(x)|^{\frac{d}{d-1}} h_{\mathcal{P}}(x)\, dx.$$

Since $h_{\mathcal{P}}(x) \in C(M)$ as in Theorem 3.35 with $f \equiv 1$, and $\cup_m supp\,(g^m - g)$ is contained in a compact subset of M, so

$$\int_M |(g^m - g)(x)|^{\frac{d}{d-1}} h_{\mathcal{P}}(x)\, dx \to 0 \text{ as } m \to 0$$

and

$$\tilde{g}^{(m)} - \tilde{g} \to 0 \text{ in } L^{\frac{d}{d-1}}\left(d\nu^1_{\mathcal{P}}\right).$$

Using Holder's inequality again we have:

$$\left|\int_{H_{\mathcal{P}}(M)} \left(\tilde{g}^{(m)}(\sigma) - \tilde{g}(\sigma)\right) \tilde{X}^{tr,\nu^1_{\mathcal{P}}}_{\mathcal{P}} f(\sigma)\, d\nu^1_{\mathcal{P}}(\sigma)\right| \tag{7.12}$$

$$\leq \left\|\tilde{g}^{(m)} - \tilde{g}\right\|_{L^{\frac{d}{d-1}}\left(\nu^1_{\mathcal{P}}\right)} \left\|\tilde{X}^{tr,\nu^1_{\mathcal{P}}}_{\mathcal{P}} f\right\|_{L^d\left(\nu^1_{\mathcal{P}}\right)}. \tag{7.13}$$

Therefore

$$\lim_{m\to\infty}\int_{H_{\mathcal{P}}(M)}\tilde{g}^{(m)}(\sigma)\,\tilde{X}_{\mathcal{P}}^{tr,\nu_{\mathcal{P}}^{1}}f(\sigma)\,d\nu_{\mathcal{P}}^{1}(\sigma)=\int_{H_{\mathcal{P}}(M)}\tilde{g}(\sigma)\,\tilde{X}_{\mathcal{P}}^{tr,\nu_{\mathcal{P}}^{1}}f(\sigma)\,d\nu_{\mathcal{P}}^{1}(\sigma).$$

■

Lemma 7.10 *For any* $p\le\frac{d}{d-1}$,

$$\sup_{\mathcal{P}}\mathbb{E}\left[\left|\tilde{g}\left(\phi\circ\beta_{\mathcal{P}}\right)\right|^{p}\right]<\infty. \tag{7.14}$$

Proof. Since the law of $\phi\circ\beta_{\mathcal{P}}$ under ν is $\nu_{\mathcal{P}}^{1}$, we have:

$$\mathbb{E}\left[\left|\tilde{g}\left(\phi\circ\beta_{\mathcal{P}}\right)\right|^{p}\right]=\int_{H_{\mathcal{P}}(M)}|\tilde{g}|^{p}(\sigma)\,d\nu_{\mathcal{P}}^{1}(\sigma). \tag{7.15}$$

Then apply co-area formula (3.41) with

$$(H,M,p,g,f)=\left(H_{\mathcal{P}}(M),M,E_{1}^{\mathcal{P}},\frac{1}{Z_{\mathcal{P}}^{1}}e^{-\frac{E}{2}},|\tilde{g}|^{p}\right),$$

we get:

$$\int_{H_{\mathcal{P}}(M)}|\tilde{g}|^{p}(\sigma)\,d\nu_{\mathcal{P}}^{1}(\sigma)=\int_{M}|g(x)|^{p}\,h_{\mathcal{P}}(x)\,dx \tag{7.16}$$

where $h_{\mathcal{P}}(x)$ is defined as in Theorem 3.15 with $f\equiv 1$.

Apply Proposition 3.33 we know that:

$$\sup_{\mathcal{P}}h_{\mathcal{P}}(x)<\infty \tag{7.17}$$

Since g has compact support, $\sup_{\mathcal{P}}h_{\mathcal{P}}(x)$ is bounded on its support and the bound is independent of $\mathcal{P}$, from there it follows that (using Holder's inequality):

$$\sup_{\mathcal{P}}\int_{M}|g(x)|^{p}\,h_{\mathcal{P}}(x)\,dx<\infty. \tag{7.18}$$

■

Theorem 7.11 *For any $f \in \mathcal{FC}_b^1$,*

$$\lim_{|\mathcal{P}|\to 0}\int_{H_{\mathcal{P}}(M)} \tilde{g}(\sigma)\,\tilde{X}_{\mathcal{P}}^{tr,\nu_{\mathcal{P}}^1} f(\sigma)\,d\nu_{\mathcal{P}}^1(\sigma) = \int_{W_o(M)} \tilde{g}(\sigma)\,\tilde{X}^{tr,\nu} f(\sigma)\,d\nu(\sigma).$$

Proof. Since the law of $\phi\circ\beta_{\mathcal{P}}$ under ν is ${\nu_{\mathcal{P}}}^1$, we have:

$$\int_{H_{\mathcal{P}}(M)} \tilde{g}(\sigma)\left(\tilde{X}_{\mathcal{P}}^{tr,\nu_{\mathcal{P}}^1} f\right)(\sigma)\,d\nu_{\mathcal{P}}^1(\sigma) = \mathbb{E}_\nu\left[\tilde{g}\cdot\left(\tilde{X}_{\mathcal{P}}^{tr,\nu_{\mathcal{P}}^1} f\right)(\phi\circ\beta_{\mathcal{P}})\right]. \tag{7.19}$$

Also

$$\int_{W_o(M)} \tilde{g}(\sigma)\left(\tilde{X}^{tr,\nu} f\right)(\sigma)\,d\nu(\sigma) = \mathbb{E}_\nu\left[\tilde{g}\cdot\tilde{X}^{tr,\nu} f\right]. \tag{7.20}$$

So

$$\left|\int_{H_{\mathcal{P}}(M)} \tilde{g}(\sigma)\,\tilde{X}_{\mathcal{P}}^{tr,\nu_{\mathcal{P}}^1} f(\sigma)\,d\nu_{\mathcal{P}}^1(\sigma) - \int_{W_o(M)} \tilde{g}(\sigma)\,\tilde{X}^{tr,\nu} f(\sigma)\,d\nu(\sigma)\right| \tag{7.21}$$

$$\le \mathbb{E}\left[\left|\tilde{g}\cdot\tilde{X}_{\mathcal{P}}^{tr,\nu_{\mathcal{P}}^1} f(\phi\circ\beta_{\mathcal{P}}) - \tilde{g}\cdot\tilde{X}^{tr,\nu} f\right|\right] \tag{7.22}$$

$$\le \mathbb{E}\left[\left|\tilde{g}(\phi\circ\beta_{\mathcal{P}})\right|\cdot\left|\tilde{X}_{\mathcal{P}}^{tr,\nu_{\mathcal{P}}^1} f(\phi\circ\beta_{\mathcal{P}}) - \tilde{X}^{tr,\nu} f\right|\right] + \mathbb{E}\left[\left|\tilde{g}(\phi\circ\beta_{\mathcal{P}}) - \tilde{g}\right|\cdot\left|\tilde{X}^{tr,\nu} f\right|\right]. \tag{7.23}$$

From Lemma 7.8, we have

$$\tilde{g}(\phi\circ\beta_{\mathcal{P}}) \in L^{\frac{d}{d-1}}(W_o(M)),$$

and from Theorem 6.23 we have

$$\tilde{X}_{\mathcal{P}}^{tr,\nu_{\mathcal{P}}^1} f(\phi\circ\beta_{\mathcal{P}}) - \tilde{X}^{tr,\nu} f \to 0 \text{ in } L^\infty(W_o(M)).$$

So by Holder's inequality,

$$\mathbb{E}\left[\left|\tilde{g}\left(\phi\circ\beta_{\mathcal{P}}\right)\right|\cdot\left|\tilde{X}_{\mathcal{P}}^{tr,\nu_{\mathcal{P}}^{1}}f\left(\phi\circ\beta_{\mathcal{P}}\right)-\tilde{X}^{tr,\nu}f\right|\right]\to 0 \text{ as } |\mathcal{P}|\to 0. \tag{7.24}$$

Then we consider

$$\mathbb{E}\left[\left|\tilde{g}\left(\phi\circ\beta_{\mathcal{P}}\right)-\tilde{g}\right|\cdot\left|\tilde{X}^{tr,\nu}f\right|\right].$$

By Holder's inequality,

$$\mathbb{E}\left[\left|\tilde{g}\left(\phi\circ\beta_{\mathcal{P}}\right)-\tilde{g}\right|\cdot\left|\tilde{X}^{tr,\nu}f\right|\right]\le\mathbb{E}\left[\left|\tilde{g}\left(\phi\circ\beta_{\mathcal{P}}\right)-\tilde{g}\right|^{p}\right]^{\frac{1}{p}}\cdot\mathbb{E}\left[\left|\tilde{X}^{tr,\nu}f\right|^{q}\right]^{\frac{1}{q}} \tag{7.25}$$

where $p>1$ and $q>1$ satisfying $\frac{1}{p}+\frac{1}{q}=1$.

From Proposition 7.2 we know $\tilde{X}^{tr,\nu}f\in L^{\infty-}\left(W_o\left(M\right)\right)$, therefore in order to show

$$\left|\int_{H_{\mathcal{P}}(M)}\tilde{g}\left(\sigma\right)\tilde{X}_{\mathcal{P}}^{tr,\nu_{\mathcal{P}}^{1}}f\left(\sigma\right)d\nu_{\mathcal{P}}^{1}\left(\sigma\right)-\int_{W_o(M)}\tilde{g}\left(\sigma\right)\tilde{X}^{tr,\nu}f\left(\sigma\right)d\nu\left(\sigma\right)\right|\to 0 \text{ as } |\mathcal{P}|\to 0, \tag{7.26}$$

it suffices to find a $p>1$ such that

$$\mathbb{E}_{\nu}\left[\left|\tilde{g}\left(\phi\circ\beta_{\mathcal{P}}\right)-\tilde{g}\right|^{p}\right]\to 0 \text{ as } |\mathcal{P}|\to 0. \tag{7.27}$$

Since for any $\epsilon>0$, there exists a constant $C_{p,\epsilon}$ such that

$$\left|\tilde{g}\left(\phi\circ\beta_{\mathcal{P}}\right)-\tilde{g}\right|^{p(1+\epsilon)}\le C_{p,\epsilon}\left(\left|\tilde{g}\left(\phi\circ\beta_{\mathcal{P}}\right)\right|^{p(1+\epsilon)}+\left|\tilde{g}\right|^{p(1+\epsilon)}\right)$$

We choose p and ϵ such that $p\left(1+\epsilon\right)<\frac{d}{d-1}$. From Eq. (7.7) we know $\mathbb{E}\left[\left|\tilde{g}\right|^{p(1+\epsilon)}\right]<\infty$. Then using Lemma 7.10 we have

$$\sup_{\mathcal{P}}\mathbb{E}_{\nu}\left[\left|\tilde{g}\left(\phi\circ\beta_{\mathcal{P}}\right)\right|^{p(1+\epsilon)}\right]<\infty,$$

and thus

$$\sup_{\mathcal{P}} \mathbb{E}_{\nu}\left[\left|\tilde{g}\left(\phi\circ\beta_{\mathcal{P}}\right)-\tilde{g}\right|^{p(1+\epsilon)}\right]<\infty. \tag{7.28}$$

Therefore

$$\left\{\left|\tilde{g}\left(\phi\circ\beta_{\mathcal{P}}\right)-\tilde{g}\right|^{p}\right\} \text{ is uniformly integrable under } \nu.$$

Then consider

$$U_{\mathcal{P}} := \left\{\sigma\in W_{o}\left(M\right):\pi\circ\Phi^{-1}\circ\beta_{\mathcal{P}}\left(\sigma\right)=x\right\}. \tag{7.29}$$

Since the law of $\Phi^{-1}\circ\beta_{\mathcal{P}}$ under ν is $\nu_{\mathcal{P}}^{1}$, denote

$$V_{\mathcal{P}} := \left\{\sigma\in H_{\mathcal{P}}\left(M\right):E_{1}^{\mathcal{P}}\left(\sigma\right)=x\right\}, \tag{7.30}$$

then $\nu_{\mathcal{P}}^{1}\left(V_{\mathcal{P}}\right)=\nu\left(U_{\mathcal{P}}\right)$.

Apply the co-area formula (3.41) with $f\left(y\right)=1_{\{y=x\}}$, we get:

$$\nu_{\mathcal{P}}^{1}\left(V_{\mathcal{P}}\right)=\int_{H_{\mathcal{P}}(M)} f\left(\sigma\left(1\right)\right)d\nu_{\mathcal{P}}^{1}\left(\sigma\right)=\int_{M} f\left(y\right)h_{\mathcal{P}}\left(y\right)dy=0. \tag{7.31}$$

From there we can construct a $\nu-$Null set;

$$N:=\cup_{\mathcal{P}}U_{\mathcal{P}}\cup\left\{\sigma\in W_{o}\left(M\right):E_{1}\left(\sigma\right)=x\right\}.$$

Recall from Corollary 6.5, we have

$$\mathbb{E}_{\nu}\left[\left|u_{\mathcal{P}}\left(1\right)-\tilde{u}\left(1\right)\right|^{q}\right]\to 0 \text{ as } \left|\mathcal{P}\right|\to 0 \text{ for any } q\geq 1. \tag{7.32}$$

This implies that

$$\left|u_{\mathcal{P}}\left(1\right)-\tilde{u}\left(1\right)\right|\to 0 \text{ in probability.}$$

Notice that $g\in C^{\infty}\left(M/\left\{x\right\}\right)$ and $\pi:\mathcal{O}\left(M\right)\to M$ is smooth, so excluding N, we have

$$\left|\tilde{g}\left(\phi\circ\beta_{\mathcal{P}}\right)-\tilde{g}\right|=\left|g\circ\pi(u_{\mathcal{P}}\left(1\right))-g\circ\pi\left(\tilde{u}\left(1\right)\right)\right|\rightarrow 0 \text{ in probability.} \tag{7.33}$$

Combining 7.27 and 7.33 we know

$$\mathbb{E}\left[\left|\tilde{g}\left(\phi\circ\beta_{\mathcal{P}}\right)-\tilde{g}\right|\cdot\left|\tilde{X}^{tr,\nu}f\right|\right]\rightarrow 0.$$

■

Proposition 7.12 *Let $f\in\mathcal{RFC}_b^1$, then*

$$\lim_{m\rightarrow\infty}\int_{W_o(M)}\delta_x^{(m)}\left(\Sigma_1\right)fd\nu=\int_{W_o(M)}fd\nu_x$$

where $\Sigma_r\left(\sigma\right)=\sigma\left(r\right)$ is the canonical process on $W_o\left(M\right)$.

Proof. Since $f=F\left(\Sigma_{s_1},\ldots,\Sigma_{s_n}\right)$, we have by Markov property,

$$\int_{W_o(M)}\delta_x^{(m)}\left(\Sigma_1\right)fd\nu=\int_{M^n}\delta_x^{(m)}\left(x_n\right)F\left(x_1,\ldots,x_n\right)\Pi_{j=1}^n p_{\frac{1}{n}}\left(x_{j-1},x_j\right)dx_1\cdots dx_n.$$

Viewing $\int_{M^{n-1}}F\left(x_1,\ldots,x_n\right)\Pi_{j=1}^n p_{\frac{1}{n}}\left(x_{j-1},x_j\right)dx_1\cdots dx_{n-1}$ as a function of x_n, observe that it is uniformly integrable with respect to x_n, therefore it is a continuous function of x_n. Thus

$$\begin{aligned}
&\lim_{m\rightarrow\infty}\int_{W_o(M)}\delta_x^{(m)}\left(\Sigma_1\right)fd\nu\\
&\quad=\lim_{m\rightarrow\infty}\int_M\delta_x^{(m)}\left(x_n\right)\int_{M^{n-1}}F\left(x_1,\ldots,x_n\right)\Pi_{j=1}^n p_{\frac{1}{n}}\left(x_{j-1},x_j\right)dx_1\cdots dx_{n-1}\\
&\quad=\int_{M^{n-1}}F\left(x_1,\ldots,x_{n-1},x\right)\Pi_{j=1}^{n-1}p_{\frac{1}{n}}\left(x_{j-1},x_j\right)\cdot p_{\frac{1}{n}}\left(x_{n-1},x\right)dx_1\cdots dx_{n-1}\\
&\quad=\int_{W_o(M)}fd\nu_x.
\end{aligned}$$

■

Proof of Theorem 1.12. Recall from Remark 3.5 that we can approximate the delta mass δ_x on M in the following way:

$$\delta_x^{(m)} := g_0^{(m)} + \sum_{j=1}^{d} X_j g_j^{(m)} \in C_0^{\infty}(M)$$

and

$$\delta_x^{(m)} \to \delta_x \text{ in } \mathcal{D}'(M),$$

where $\left\{g_j^{(m)} : 0 \le j \le d, m \ge 1\right\} \subset C_0^{\infty}(M)$ and $\{X_j : 1 \le j \le d\} \subset \Gamma(TM)$ with compact supports. Using the Orthogonal lift, we get:

$$\delta_x^{\widetilde{(m)}} := g_0^{\widetilde{(m)}} + \sum_{j=1}^{d} X_{\mathcal{P},j} g_j^{\widetilde{(m)}} \in C_0^{\infty}(M)$$

where $\tilde{g}(\sigma) = g \circ E_1(\sigma)$ for any $g \in C(M)$ and $X_{\mathcal{P},i}$ is the Orthogonal lift of X_i into $\Gamma(TH_{\mathcal{P}}(M))$.

For any $0 \le j \le d$ (with the convention that $X_{\mathcal{P},0} = I$), using integration by parts, we get:

$$\int_{H_{\mathcal{P}}(M)} \left(g_0^{\widetilde{(m)}} + \sum_{j=1}^{d} X_{\mathcal{P},j} g_j^{\widetilde{(m)}} \right) f d\nu_{\mathcal{P}}^1 = \int_{H_{\mathcal{P}}(M)} \left(g_0^{\widetilde{(m)}} \cdot f + \sum_{j=1}^{d} X_{\mathcal{P},j}^{tr,\nu_{\mathcal{P}}^1} f \cdot g_j^{\widetilde{(m)}} \right) d\nu_{\mathcal{P}}^1. \tag{7.34}$$

Now let $m \to \infty$, from Corollary 3.35 we have:

$$\text{the left–hand side of (7.34)} = \int_{H_{\mathcal{P},x}(M)} f d\nu_{\mathcal{P},x}^1$$

Apply Lemma 7.9 to each $\left(g_j^{\widetilde{(m)}}, X_{\mathcal{P},j}\right)$, we have:

$$\text{right hand side of 7.34} = \int_{H_{\mathcal{P}}(M)} \left(\tilde{g}_0 \cdot f + \sum_{j=1}^{d} X_{\mathcal{P},j}^{tr,\nu_{\mathcal{P}}^1} f \cdot \tilde{g}_j \right) d\nu_{\mathcal{P}}^1. \tag{7.35}$$

Then let $|\mathcal{P}| \to 0$, from Theorem 7.11 we have:

$$\lim_{|\mathcal{P}|\to 0} \int_{H_{\mathcal{P},x}(M)} f d\nu^1_{\mathcal{P},x} = \int_{W_o(M)} \left(\tilde{g_0} \cdot f + \sum_{j=1}^d \tilde{X_j}^{tr,\nu} f \cdot \tilde{g_j} \right) d\nu. \tag{7.36}$$

According to Lemma 7.7,

$$\int_{W_o(M)} \left(\tilde{g_0} \cdot f + \sum_{j=1}^d \tilde{X_j}^{tr,\nu} f \cdot \tilde{g_j} \right) d\nu \tag{7.37}$$

$$= \lim_{m\to\infty} \int_{W_o(M)} \left(\tilde{g_0^{(m)}} \cdot f + \sum_{j=1}^d \tilde{X_j}^{tr,\nu} f \cdot \tilde{g_j^{(m)}} \right) d\nu. \tag{7.38}$$

Then use integration by parts formula developed in Lemma 4.43 we have:

$$\int_{W_o(M)} \left(\tilde{g_0^{(m)}} \cdot f + \sum_{j=1}^d \tilde{X_j}^{tr,\nu} f \cdot \tilde{g_j^{(m)}} \right) d\nu = \int_{W_o(M)} \left(\tilde{g_0^{(m)}} + \sum_{j=1}^d \tilde{X}_j \tilde{g_j^{(m)}} \right) \cdot f d\nu \tag{7.39}$$

$$= \int_{W_o(M)} \tilde{\delta_x}^{(m)} f d\nu. \tag{7.40}$$

If $f \in \mathcal{RFC}_b^1$, using Proposition 7.12 we have

$$\int_{W_o(M)} \tilde{\delta_x}^{(m)} f d\nu \to \int_{W_o(M)} f d\nu_x.$$

Therefore

$$\lim_{|\mathcal{P}|\to 0} \int_{H_{\mathcal{P},x}(M)} f d\nu^1_{\mathcal{P},x} = \int_{W_o(M)} f d\nu_x. \tag{7.41}$$

■

Appendix A

Riemannian Manifolds

A.1 Hadamard Manifold

Definition A.1 (Hadamard Manifold) *A Hadamard manifold is a complete Riemannian manifold, simply connected and with non–positive sectional curvature.*

Hadamard manifolds share very nice global properties as recorded in the following theorem as the Theorem of Hadamard.

Theorem A.2 *If M is a Hadamard manifold, then M is diffeomorphic to $\mathbb{R}^d$, $d = \dim M$; more precisely for any $x \in M$, $\exp_x : T_x M \to M$ is a diffeomorphism.*

A.2 Connections on Principal Bundle

Notation A.3 *Denote by $\Gamma^\infty(TM)$ the smooth sections of the tangent bundle. You can think of this as the space of smooth vector field.*

Definition A.4 (Affine connection) *An affine connection is a map $\nabla : \Gamma(TM) \times \Gamma(TM) \mapsto \Gamma(TM)$ or $(X, Y) \mapsto \nabla_X Y$ satisfying the following conditions: for*

$X, Y, Z \in \Gamma(TM)$ *and* $f, g \in C^{\infty}(M)$:

$$\nabla_X fY = (Xf)\, Y + f \nabla_X fY$$
$$\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z$$
$$\nabla_{fX+gY} Z = f \nabla_X Z + g \nabla_Y Z$$

Definition A.5 *An affine connection* ∇ *is said to be* ***metric compatible*** *if the following is true for any* $X, Y, Z \in \Gamma(TM)$*:*

$$(\nabla_Z g)(X, Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$$

A metric compatible connection is also called the ***metric connection****.*

Definition A.6 *For any* $X, Y, Z \in \Gamma(TM)$, *define the* ***Riemann curvature tensor*** $R : \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ *and* ***torsion tensor*** T : $\Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ *to be:*

$$R(X, Y, Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$
$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

A connection is said to be symmetric if $T \equiv 0$.

Theorem A.7 (Levi-Civita) *There exists a unique symmetric metric connection, which is called the Levi-Civita connection.*

Throughout this paper we stick with the Levi-Civita connection ∇.

Definition A.8 (Principal bundle) *A principal bundle* $(P, G, \pi, M, \{U_\alpha\}, \phi_\alpha)$ *consists of the following data:*

- P, M *are smooth manifolds.* $\pi : P \to M$ *smooth submersion is called the fibre projection map.*

- *A Lie group G is said to be the structure group of P: i.e. G admits a free and transitive group action on P on the right:*

$$(G, P) \ni (g, u) \to u \cdot g \in P$$

- *(Local trivialization) $\{U_\alpha\}$ is an open covering of M, then $\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times G$ is a diffeomorphism.*

Example A.9 (Frame bundle $L(M)$) *Let G be the general linear group $GL(d, \mathbb{R})$ where $d = \dim M$ and for each $x \in M$, denote by $L(M)_x$ the linear frames of T_xM (Here we will identify a linear frame with a linear isomorphism from $\mathbb{R}^d \to T_xM$). Then $L(M) := \cup_{x\in M} L(M)_x$ can be made a principal bundle with structure group $GL(d, \mathbb{R})$. We will call this principal bundle the frame bundle over M, simply denoted by $L(M)$.*

Example A.10 (Orthonormal frame bundle $(\mathcal{O}(M), \pi)$) *See Definition 2.1*

Definition A.11 (Fundamental vector field) *Given a principal bundle P over M with structure group G, for any $p \in M$, dentote by $G_p := \pi^{-1}(\{p\})$ the fiber at $p = \pi(u)$. Let V_uP be the tangent space of P at u which is tangent to G_p. Since $G_p \cong G$, so*

$$\dim V_uP = \dim G = \dim \mathfrak{g}.$$

One can construct a base of V_uP in the following way: take a basis $\{A_i\}$ of $\mathfrak{g}$, consider

$$u(s) := u \exp(sA_i)$$

then $u(s)$ is a differentiable curve on V_uP with $u(0) = u$. Define:

$$A_i^\dagger := \frac{d}{ds} |_0 \, u(s)$$

This is called the fundamental vector field generated by A_i. Using substitution, one can see that the map $A \to A^\dagger$ is a real vector space isomorphism. (Actually this is a Lie algebra isomorphism.) However, there is no unique way to specify the "orthogonal compliment" of this vector bundle VP unless some more structures are involved, which is called connection on P.

Definition A.12 (Connection on principal bundle) *A (smooth) connection on a principal bundle P is a choice of (smooth) decomposition of the tangent bundle TP over P as follows, for any $u \in P$:*

$$T_u P = V_u P \oplus H_u P$$

and

$$H_{ug} P = R_{g*} H_u P$$

where $R_g : P \ni u \to ug \in P$ is the right action of G on P.

Definition A.13 (Connection one-form) *A connection form is a Lie–algebra valued one form on P, i.e. $\omega \in \mathfrak{g} \otimes T^*P$ satisfying the following requirement:*

$$\begin{array}{lll} (i) & \omega\left(A^\dagger\right) = A & \text{for any } A \in \mathfrak{g} \\ (ii) & R_g^* \omega = Ad_{g^{-1}} \omega & \text{for any } g \in G \end{array} \tag{A.1}$$

here $Ad_{g^{-1}} X = g^{-1} X g$ for any $X \in \mathfrak{g}$.

Remark A.14 *Given a smooth connection on P, we can naturally get a connection one-form ω in the following way: for each $X \in T_u P$, there exists unique $A_X \in \mathfrak{g}$ and $X^H \in H_u P$ such that $X = A_X^\dagger + X^H$. define $\omega(X) = A_X$. It is easy to see that ω satisfies A.1. Conversely, given a smooth connection one-form ω, we can define $H_u P = \ker \omega_u$ and it gives a smooth connection on P.*

Remark A.15 *It is known that a smooth connection on a principal bundle P induces a smooth connection on its associated vector bundles. In particular, it gives* *rise to a connection on M defined as in Definition A.4. There are usually two ways to see that. One is to use the connection on P to derive "horizontal lift" and further parallel translation, then use parallel translation to define covariant derivative and further a connection on M. Interested readers can refer to the Chapter III section 1 in the classical book [25] by Kobayashi and Nomizu for a more detailed exposition. The other way is to use local one-forms of ω in P and the push-forward of the representation of G to derive a compatible local one-forms on M from which one can construct a connection on M.*

Conversely, an affine connection on M gives rise to a connection on the frame bundle $L(M)$ introduced on Example A.9, see Chapter III section 2 in [25] and section 2.1 in [21]. In particular, if the connection ∇ is a metric connection on M, the connection on $L(M)$ reduces to a connection on $\mathcal{O}(M)$. Throughout this paper we will fix ∇ to be the Levi-civita connection and consider only the connection on $\mathcal{O}(M)$ induced by ∇. We also fix a $u_0 \in \mathcal{O}(M)_o$ so that $\mathcal{O}(M)$ becomes a pointed manifold and further we use u_0 to identify T_oM with $\mathbb{R}^d$.

Remark A.16 *π induces an isomorphism $\pi_* : H_u\mathcal{O}(M) \to T_{\pi(u)}M$ following the decomposition specified by ∇. This is a result of the fact that $\pi_* \{V_u\mathcal{O}(M)\}$ and $\dim T_u\mathcal{O}(M) = d + \dim \mathfrak{so}(d) = d + \dim V_u\mathcal{O}(M)$. Therefore for any $x \in M$, $u \in \pi^{-1}(\{x\})$, $X \in T_xM$, there exists a unique tangent vector $X^* \in H_u\mathcal{O}(M)$ such that $\pi_* X^* = X$. X^* is called the* ***horizontal lift*** *of X to u.*

Appendix B

ODE estimates

Lemma B.1 *If X is a normal random variable with mean 0 and variance t, then*

$$\mathbb{E}\left[e^{k|X|^2}\right] = \begin{cases} \infty & \text{if } k \geq \frac{1}{2t} \\ (1-2kt)^{-\frac{1}{2}} & \text{if } k < \frac{1}{2t} \end{cases}$$

Proof. The result follows from the direct computation below.

$$\mathbb{E}\left[e^{k|X|^2}\right] = \int_{-\infty}^{\infty} e^{kx^2} \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{\left(k-\frac{1}{2t}\right)x^2} dx.$$

If $k \geq \frac{1}{2t}$, then

$$\frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{\left(k-\frac{1}{2t}\right)x^2} dx \geq \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} dx = \infty.$$

If $k < \frac{1}{2t}$, then

$$\frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{\left(k-\frac{1}{2t}\right)x^2} dx = \frac{1}{\sqrt{2\pi t\left(\frac{1}{2t}-k\right)}} \int_{-\infty}^{\infty} e^{y^2} dy = (1-2kt)^{-\frac{1}{2}}.$$

■

Proposition B.2 *Consider an ODE:*

$$Y''(s) = A(s) Y(s)$$

where $Y(s), A(s) \in M_{n\times n}(\mathbb{R})$ *are real* $n \times n$ *matrices and* $A(s)$ *is positive semi-definite.*

Denote by $\{C(s), S(s)\}$*the solutions to this ODE with initial values:*

$$C(0) = I, C'(0) = 0 \text{ and } S(0) = 0, S'(0) = I$$

Recall that in this paper we use $eig(X)$ *to denote the set of eigenvalues of matrix* X*. Then*

- *If* $\lambda \in eig(C(s))$, *then* $|\lambda| \geq 1$.
- *If* $\lambda \in eig(S(s))$, *then* $|\lambda| \geq s$.

Proof. For all $v \in \mathbb{C}^d$, define $v(s) := C(s) v$, then:

$$\langle v''(s), v(s)\rangle = \langle A(s) v(s), v(s)\rangle \geq 0.$$

Therefore,

$$\frac{d}{ds}\langle v'(s), v(s)\rangle = \langle v''(s), v(s)\rangle + \|v'(s)\|^2 \geq 0.$$

Since $\langle v'(0), v(0)\rangle = 0$, so $\langle v'(s), v(s)\rangle \geq 0$. Therefore

$$\frac{d}{ds}\|v(s)\|^2 = 2Re\langle v'(s), v(s)\rangle \geq 0.$$

Notice that $\|v(0)\|^2 = \|v\|^2$, so

$$\|v(s)\|^2 \geq \|v\|^2.$$

Therefore if $\lambda \in eig\left(C\left(s\right)\right)$, choose $v \in \mathbb{C}^d$ to be an eigenvector associated to λ, then

$$\left\|\lambda v\right\|^2 = \left\|C\left(s\right)v\right\|^2 \geq \left\|v\right\|^2.$$

So

$$|\lambda| \geq 1.$$

Therefore $C\left(s\right)$ is invertible and

$$\left\|C\left(s\right)\right\| = \max_{\lambda \in eig(C(s))} |\lambda| \geq 1.$$

A lower bound result for $\|S(s)v\|$ can be found in [27, Appendix E]:

$$\|S(s)v\| \geq s\left\|v\right\|.$$

From there it follows

$$\text{If } \lambda \in eig\left(S\left(s\right)\right), \text{ then } |\lambda| \geq s$$

and $S\left(s\right)$ is invertible with

$$\left\|S\left(s\right)\right\| = \max_{\lambda \in eig(S(s))} |\lambda| \geq s.$$

■

Definition B.3 *Denote* $R_{u(s)}\left(\xi,\cdot\right)\xi$ *by* $A_\xi\left(s\right)$, $\left(C_\xi\left(s\right), S_\xi\left(s\right)\right)^t$ *is the fundamental solution to the ODE:*

$$V'\left(s\right) = \begin{pmatrix} 0 & 1 \\ A_{\xi_x} & 0 \end{pmatrix} V\left(s\right)$$

Proposition B.4 *If R is bounded by a constant N, i.e.* $|R(\xi,\cdot)\xi| \leq N|\xi|^2$, *then*

$$|C_\xi(s)| \leq \cosh\left(\sqrt{N}|\xi|s\right) \leq e^{\frac{1}{2}N|\xi|^2 s^2} \tag{B.1}$$

$$\begin{aligned}|S_\xi(s)| &\leq \sqrt{N}|\xi|s\frac{\sinh\left(\sqrt{N}|\xi|s\right)}{\sqrt{N}|\xi|s} \\ &\leq \cosh\left(\sqrt{N}|\xi|s\right)\sqrt{N}|\xi|s \\ &\leq \sqrt{N}|\xi|se^{\frac{1}{2}N|\xi|^2 s^2}\end{aligned} \tag{B.2}$$

$$|S_\xi(s) - sI| \leq \frac{N|\xi|^2 s^3}{6}e^{\frac{1}{2}N|\xi|^2 s^2} \tag{B.3}$$

and

$$|C_\xi(s) - I| \leq \frac{N|\xi|^2 s^2}{2}e^{\frac{1}{2}N|\xi|^2 s^2} \tag{B.4}$$

Proof. B.1 and B.2 are quite elementary, so here we only resent the proof of B.3 and B.4.

By Taylor's expansion,

$$S_\xi(s) = sI + \int_0^s R_{\tilde{u}_r}(\xi, S_\xi(r))\xi(s-r)\,dr.$$

$$\begin{aligned}|S_\xi(s) - sI| &\leq N|\xi|^2\int_0^s |S_\xi(r)|(s-r)\,dr \\ &\leq N|\xi|^2\int_0^s [|S_\xi(r) - rI| + r](s-r)\,dr\end{aligned}$$

Define $f(s) := |S_\xi(s) - sI|$, then we have:

$$f(s) \leq \int_0^s N|\xi|^2(s-r)f(r)\,dr + N|\xi|^2\frac{s^3}{6}$$

By Gronwall's inequality:

$$f(s) \leq N|\xi|^2 \frac{s^3}{6} e^{\frac{1}{2}N|\xi|^2 s^2}$$

Then we consider $C_\xi(s)$:

$$C_\xi(s) = I + \int_0^s R_{\tilde{u}_r}(\xi, C_\xi(r))\, \xi(s-r)\, dr.$$

So

$$\begin{aligned} |C_\xi(s) - I| &\leq N|\xi|^2 \int_0^s |C_\xi(r)|\,(s-r)\,dr \\ &\leq N|\xi|^2 \int_0^s [|C_\xi(r) - I| + 1]\,(s-r)\,dr. \end{aligned}$$

Define $f(s) := |C_\xi(s) - I|$, then we have:

$$f(s) \leq \int_0^s N|\xi|^2 (s-r) f(r)\,dr + N|\xi|^2 \frac{s^2}{2}.$$

By Gronwall's inequality:

$$f(s) \leq N|\xi|^2 \frac{s^2}{2} e^{\frac{1}{2}N|\xi|^2 s^2}.$$

■

Appendix C

Calculus on Differential Forms

Theorem C.1 (change of variable formula on manifold) *If $F : M \to N$ is an orientation preserving diffeomorphism and α is a $d-$form on N with $d = \dim M$. Then $F^*\alpha$ is a $d-$form on M and the following is true:*

$$\int_M F^*\alpha = \int_N \alpha. \tag{C.1}$$

In particular, if M and N are Riemannian manifolds with volume forms vol_M and vol_N, then

$$F^*vol_N = \mathcal{J}_F vol_M. \tag{C.2}$$

where $\mathcal{J}_F = \sqrt{\det (DF)^{tr} DF}$.

Proof. Since the integral of forms are independent of the choice of open coverings, so it suffices to prove for in a chart (U, x) of N,

$$\int_{F^{-1}(U)} F^*\alpha = \int_U \alpha$$

Locally on U, $\alpha = f(x)\, dx_1 \wedge \cdots \wedge dx_d$ and $F^*\alpha = f \circ F d(x_1 \circ F) \wedge \cdots \wedge d(x_d \circ F)$.

Choose a chart map y on $F^{-1}(U) \cong \mathbb{R}^d$, then

$$F^*\alpha = f \circ F \circ y^{-1} d\left(x_1 \circ F \circ y^{-1}\right) \wedge \cdots \wedge d\left(x_d \circ F \circ y^{-1}\right) \tag{C.3}$$

$$= f \circ F \circ y^{-1} \det\left(\frac{\partial\left(x_i \circ F \circ y^{-1}\right)}{\partial y_j}\right) dy_1 \wedge \cdots \wedge dy_d \tag{C.4}$$

Notice that F is orientation preserving, so Equation C.1 is easily follows from the change of variable formula on $\mathbb{R}^d$ applied to $x \circ F \circ y^{-1} : \mathbb{R}^d \to \mathbb{R}^d$. Equation C.1 is thus easily obtained by using orthonormal frames on M and N. ■

C.1 A Structure Theorem for $\operatorname{div}_g\left(\tilde{X}\right)$

This section is devoted to a structure theorem for $\operatorname{div}_g\left(\tilde{X}\right)$ which is t

Let $\pi : (M, g) \to (N, h)$ be a submersion of two smooth Riemannian manifolds. To each $m \in M$ and $v \in T_{\pi(m)}N$, let $\hat{v} := \pi_{*m}^{\mathrm{tr}}\left(\pi_{*m}\pi_{*m}^{\mathrm{tr}}\right)^{-1} v \in T_mM$ so that $\hat{v}$ is the unique shortest vector in T_mM such that $\pi_{*m}\hat{v} = v$. So if $X \in \Gamma(TN)$ is a vector field on N, then $\hat{X} \in \Gamma(TM)$ is defined by $\hat{X}(m) = \pi_{*m}^{\mathrm{tr}}\left(\pi_{*m}\pi_{*m}^{\mathrm{tr}}\right)^{-1} X(\pi(m))$ and we have $\pi_*\hat{X} = X \circ \pi$. Finally, let Vol_g and Vol_h be the volume forms on (M, g) and (N, h) respectively.

Lemma C.2 *If $K := \dim M > k := \dim N$, then there exists a unique $K - k$ – form (γ) on M such that;*

1. $\mathrm{Vol}_g = (\pi^* \mathrm{Vol}_h) \wedge \gamma$
2. $i_{\hat{v}}\gamma = 0$ *for any $v \in T_{\pi(m)}N$ and $m \in M$.*

Proof. Uniqueness. Assuming such a γ exists, choose an orthonormal basis

$\{e_1, \ldots, e_k\}$ for $T_{\pi(m)}N$ such that $\mathrm{Vol}_h(e_1, \ldots, e_k) = 1$. Then it follows that

$$\begin{aligned}\mathrm{Vol}_g(\hat{e}_1, \ldots, \hat{e}_k, \cdot, \ldots, \cdot) &= (\pi^* \mathrm{Vol}_h)(\hat{e}_1, \ldots, \hat{e}_k) \wedge \gamma \\ &= \mathrm{Vol}_h(\pi_* \hat{e}_1, \ldots, \pi_* \hat{e}_k) \wedge \gamma \\ &= \mathrm{Vol}_h(e_1, \ldots, e_k) \wedge \gamma = \gamma\end{aligned}$$

which shows γ is unique if it exists.

Existence. Now suppose that $\{e_1, \ldots, e_k\}$ is a local orthonormal frame on M in a neighborhood of $\pi(m)$ such that $\mathrm{Vol}_h(e_1, \ldots, e_k) = 1$. Then by above we must define

$$\gamma := \mathrm{Vol}_g(\hat{e}_1, \ldots, \hat{e}_k, \cdot, \ldots, \cdot) \text{ in a neighborhood of } m.$$

It is now straightforward to check that this γ has the desired properties and is defined independent of the choice of frame. ■

Corollary C.3 *If $X \in \Gamma(TN)$ and $\hat{X} \in \Gamma(TM)$ is its lift as described above, then*

$$\mathrm{div}_g\left(\hat{X}\right) = \mathrm{div}_h(X) \circ \pi + \rho_{\hat{X}}$$

where $\rho_{\hat{X}}(m)$ is a function on M depending only on $\hat{X}(m)$. {To compute $\rho_{\hat{X}}$ explicitly will require a better understanding of $d\gamma$.]

Proof. From Lemma C.2 we learn,

$$\begin{aligned}\mathrm{div}_g\left(\hat{X}\right)\mathrm{Vol}_g &= d\left[i_{\hat{X}}\mathrm{Vol}_g\right] = d\left[i_{\hat{X}}\left((\pi^*\mathrm{Vol}_h) \wedge \gamma\right)\right] \\ &= d\left[\left(i_{\hat{X}}(\pi^*\mathrm{Vol}_h) \wedge \gamma\right)\right] \\ &= \left[d\left(i_{\hat{X}}(\pi^*\mathrm{Vol}_h)\right)\right] \wedge \gamma + (-1)^k \left(i_{\hat{X}}(\pi^*\mathrm{Vol}_h) \wedge d\gamma\right).\end{aligned}$$

Since

$$i_{\hat{X}}\left(\pi^{*}\operatorname{Vol}_{h}\right)=\left(\pi^{*}\operatorname{Vol}_{h}\right)\left(\hat{X},--\right)=\operatorname{Vol}_{h}\left(\pi_{*}\hat{X},\pi_{*}--\right)$$
$$=\operatorname{Vol}_{h}\left(X\circ\pi,\pi_{*}--\right)=\pi^{*}\left(i_{X}\operatorname{Vol}_{h}\right)$$

it follows that

$$d\left(i_{\hat{X}}\left(\pi^{*}\operatorname{Vol}_{h}\right)\right)=d\left(\pi^{*}\left(i_{X}\operatorname{Vol}_{h}\right)\right)=\pi^{*}\left(d\left(i_{X}\operatorname{Vol}_{h}\right)\right)$$
$$=\pi^{*}\left(\operatorname{div}_{h}\left(X\right)\operatorname{Vol}_{h}\right)=\operatorname{div}_{h}\left(X\right)\circ\pi\cdot\pi^{*}\operatorname{Vol}_{h}.$$

Combining these equations then shows,

$$\operatorname{div}_{g}\left(\hat{X}\right)\operatorname{Vol}_{g}=\operatorname{div}_{h}\left(X\right)\circ\pi\cdot\left(\pi^{*}\operatorname{Vol}_{h}\right)\wedge\gamma+\left(-1\right)^{k}\left(i_{\hat{X}}\left(\pi^{*}\operatorname{Vol}_{h}\right)\wedge d\gamma\right)$$
$$=\left[\operatorname{div}_{h}\left(X\right)\circ\pi+\rho_{\hat{X}}\right]\cdot\operatorname{Vol}_{g}$$

where

$$\rho_{\hat{X}}=\frac{\left(-1\right)^{k}\left(i_{\hat{X}}\left(\pi^{*}\operatorname{Vol}_{h}\right)\wedge d\gamma\right)}{\operatorname{Vol}_{g}}.$$

■

Appendix D

Some matrix analysis

Consider

$$a := \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{R}^n \text{ and } S = \begin{bmatrix} I_{n\times n} \\ a^{\mathrm{tr}} \end{bmatrix}$$

so that

$$S^{\mathrm{tr}} = \begin{bmatrix} I_{n\times n} & a \end{bmatrix}.$$

Notice that S is a $(n+1)\times n$ and S^{tr} is $n \times (n+1)$ matrix. For $x \in \mathbb{R}^n$ and $u \in \mathbb{R}$ we have

$$S^{\mathrm{tr}} \begin{bmatrix} x \\ u \end{bmatrix} = x + ua \text{ and } Sx = \begin{bmatrix} x \\ a \cdot x \end{bmatrix}$$

$$S^{\mathrm{tr}} S x = x + (a \cdot x)\, a = x + a\ a^{\mathrm{tr}} x = \left(I + aa^{\mathrm{tr}}\right) x.$$

Thus choosing an orthonormal basis $\{u_i\}_{i=1}^n$ for $\mathbb{R}^n$ such that $u_1 = \hat{a}$ we learn that

$$S^{\mathrm{tr}} S u_1 = \left(1 + \|a\|^2\right) u_1 \text{ and } S^{\mathrm{tr}} S u_i = u_i \text{ for } i > 1.$$

Thus it follows that $\det\left(S^{\mathrm{tr}}S\right) = 1 + \|a\|^2$. We record the higher dimensional generalization of the result above. It is used in computing some determinants in the dissertation.

Theorem D.1 *Suppose that V is a finite dimensional inner product space, $A : V^n \to V$ is a linear map, and*

$$S := \begin{bmatrix} I_{V^n \times V^n} \\ A \end{bmatrix} : V^n \to V^{n+1}.$$

Then

$$\det\left[S^{\mathrm{tr}}S\right] = \det\left[I_V + AA^{\mathrm{tr}}\right].$$

Proof. First observe that

$$S^{\mathrm{tr}}S = \begin{bmatrix} I & A^{\mathrm{tr}} \end{bmatrix} \begin{bmatrix} I \\ A \end{bmatrix} = I + A^{\mathrm{tr}}A.$$

We let $\{u_j\}_{j=1}^n \subset V$ be an orthonormal basis of eigenvectors for $AA^{\mathrm{tr}} : V \to V$ so that $AA^{\mathrm{tr}}u_j = \lambda_j u_j$ and then let $v_j := A^{\mathrm{tr}}u_j$. Then it follows that

$$A^{\mathrm{tr}}Av_j = A^{\mathrm{tr}}AA^{\mathrm{tr}}u_j = A^{\mathrm{tr}}\lambda_j u_j = \lambda_j A^{\mathrm{tr}}u_j = \lambda_j v_j.$$

Now extend $\{v_j\}_{j=1}^n$ to a basis for all V^n. From this we will find that $S^{\mathrm{tr}}S$ has eigenvalues $\{1\} \cup \{1 + \lambda_j\}_{j=1}^n$ and therefore

$$\det\left(S^{\mathrm{tr}}S\right) = \prod_{j=1}^n (1 + \lambda_j) = \det\left(I + AA^{\mathrm{tr}}\right).$$

■

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