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Fractal Zeta Functions: To Ahlfors Spaces and Beyond

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ABSTRACT OF THE DISSERTATION

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by

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While classical analysis dealt primarily with smooth spaces, much research has been done in the last half century on expanding the theory to the nonsmooth case. Metric Measure spaces are the natural setting for such analysis, and it is thus important to understand the geometry of subsets of these spaces. In this dissertation we will focus on the geometry of Ahlfors regular spaces, Metric Measure spaces with an additional regularity condition. Historically, fractals have been studied using different ideas of dimension which have all proven to be unsatisfactory to some degree. The theory of complex dimensions, developed by M.L. Lapidus and a number of collaborators, was developed in part to better understand fractality in the Euclidean case and seeks to overcome these problems. Of particular interest is the recent theory of complex dimensions in higher-dimensional Euclidean spaces, as studied by M.L.Lapidus, G. Radunović, and D. Žubrinić, who introduced and studied the properties of the distance zeta function ζ_A. We will show that this theory of complex dimensions naturally generalizes to the case of Ahlfors regular spaces, as the distance zeta function can be modified to these spaces and all of its main properties carry over. In particular, we will show that we can reconstruct information about the geometry of a subset from their associated distance zeta function through fractal tube formulas. We also provide a selection of examples in Ahlfors spaces, as well as hints that the theory can be expanded to a more general setting.

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Chapter 1

Background

While analysis has classically studied smooth spaces, much work has been done in the past half century in expanding the theory to the non-smooth cases. In particular, in 1971 Coiffman and Weiss introduced in [4] the notion of *spaces of homogeneous type*, a space with a quasimetric and a measure that satisfies a doubling condition, as the natural spaces for which the theory of Calderón-Zygmund singular integrals naturally extends. This suddenly presented a method to perform harmonic analysis on these much more general sets. If we make the further restriction that the quasimetric be instead a metric, in which case the space will be called a *Metric Measure space* (or MM space for short). These spaces have since given rise to activity in many other analytical areas of mathematics, such as the study of function spaces, partial differential equations, probability theory, and analysis on fractals. See e.g. [5],[10].

Definition 1.1. A *Metric Measure Space* (or MM space) is a set X equipped with a metric d and a positive Borel measure μ that is doubling; $\exists C$ positive such that

$$\mu(B_d(x,2r)) \leq C\mu(B_d(x,r)).$$

Ideally, we wish to understand the geometry of subsets of such spaces, particularly through the methods of fractal geometry.

Historically, fractals have been studied under an approach to measure their "size" and complexity (or "roughness"), either through the Minkowski or Hausdorff dimensions. However, these dimensions leave many sets we would like to call fractal indistinguishable from standard Euclidean space. The theory of *complex dimensions*, developed by Lapidus and a number of collaborators since 1990, seeks to alleviate this problem and establish further geometric information intrinsic to a fractal. In particular, both the real and imaginary parts of the complex dimensions encode information about the geometric, spectral and dynamical oscillations of a fractal space and its associated fractal drum (bounded open subset with fractal boundary).

In particular, we wish to generalize the recent theory introduced by Lapidus, Radunović, and Žubrinić in [19] (See also [20]-[27]). Here, the distance zeta function (and related zeta functions) are introduced, wherein bounded subsets A of $\mathbb{R}^N$ are analyzed by studying the tubular neighborhood around the set. Given set A, we define the *t-tubular neighborhood* to be

$$A_\delta = \{x \in \mathbb{R}^N : d(x, A) < \delta\}.$$

The distance zeta function is then defined as

$$\zeta_A(s) = \int_{A_\delta} d(x, A)^{s-N} \mathrm{d}x.$$

This zeta function has many interesting properties that make it worthwhile to study. It is holomorphic in the half-plane to the right of the upper Minkowski dimension of set A, and if the function can be meromorphically continued, the poles of the zeta function give geometric information. We call the collection of poles the *complex dimensions* of set A.

In order to define such an analogous theory in more general settings, we first restrict ourselves to subsets of MM spaces that satisfy an extra regularity condition; they are called *Ahlfors regular spaces*. This added regularity condition roughly translates to requiring the measure of a ball to change consistently with any changes to the radius, regardless of where it is centered (see definition below). The way the measure changes as a power of the radius allows us to define an ambient dimension of the space in which we are working, called the *Ahlfors regularity dimension.* This dimension allows us to justify the geometric information obtained by any zeta function. In Chapter 5, we will study weakening this condition, and the problems that occur in doing so.

Definition 1.2. A MM space (X, d, μ) is *Ahlfors regular of dimension Q* (here on, *regular*) if (X, d) is locally compact and μ locally finite with dense support satisfying, for some $K \geq 1$,

$$K^{-1} r^Q \leq \mu(B(x, r)) \leq K r^Q$$

$\forall x \in X, 0 < r \leq \operatorname{diam} X$.

If only the upper (resp. lower) bounds are satisfied, we call the space *upper (resp. lower) Ahlfors regular of dimension Q.*

Despite requiring this additional regularity condition, many spaces that arise naturally in various areas of mathematics are Ahlfors regular. Important examples are the symbolic Cantor set, Laakso space and finite dimensional Riemannian and sub-Riemannian manifolds; in particular, the Heisenberg group and Heisenberg type groups. Many self-similar fractals, such as the Sierpiński gasket, are Ahlfors regular for the Hausdorff metric induced by the embedding in Euclidean space. However, in the theory of analysis on fractals, there are identifications with other spaces, such as the *harmonic Sierpiński gasket*, which is Ahlfors regular for the Hausdorff metric induced by a natural geodesic metric (see [13], [14]).

The Hausdorff measure has an important role in the study of Ahlfors spaces. This measure is defined on any metric space in the following manner:

Definition 1.3. For any s nonnegative, $A \subset X$, define the *s-dimensional Hausdorff outer measure*:

$$H^s(A) = \lim_{\delta \to 0} H^s_\delta = \lim_{\delta \to 0} \inf \left\{ \sum_{j=1}^{\infty} (\operatorname{diam} E_j)^s : A \subset \bigcup_{j=1}^{\infty} E_j, \operatorname{diam} E_j < \delta \right\}.$$

The *Hausdorff dimension* D_H of a set A is defined by

$$D_H = \inf\{s \geq 0 : H^s(A) = 0\}$$
$$= \sup\{s \geq 0 : H^s(A) = \infty\}.$$

We call H^D the *D-dimensional Hausdorff measure* when restricted to Borel sets.

It turns out that the measure μ of a regular Q dimensional space and H^Q are equivalent in that there is a constant C depending only on K such that

$$C^{-1}\mu(E) \leq H^Q(E) \leq C\mu(E) \quad \forall \text{ Borel } E \subseteq X.$$

In particular, if the MM space triple (X, d, μ) is regular of dimension Q, then so is (X, d, H^D).

Thus, we can always work under the assumption that we are using the Q-dimensional Hausdorff measure, and for ease of notation we will denote the measure of a set A by $|A|$.

In Ahlfors spaces, we will also find that we need to subtly change the definition of the tubular neighborhood of a set:

Definition 1.4. Let $\mathcal{E}$ be an Ahlfors regular space. Given $A \subset \mathcal{E}$ bounded, define the *t-neighborhood of A* (or *t-tubular neighborhood of A*) by $A_t := \{x \in \mathcal{E} : d(x, A) \leq t\}$. Note that in the setting of Ahlfors regular spaces, we take the closed neighborhood.

Finally, we will need a proper definition of Minkowski content and Minkowksi dimension. Given a Q-regular Ahlfors space, we propose using the ambient dimension to generalize the Euclidean definition as follows:

Definition 1.5. The *r-dimensional Minkowski upper content* is defined as

$$\mathcal{M}^{*r} = \limsup_{t \to 0} \frac{|A_t|}{t^{Q-r}}.$$

We then define the *upper Minkowski dimension* by

$$\overline{\dim}_B A = \inf\{r \geq 0 : \mathcal{M}^{*r}(A) = 0\}$$
$$= \sup\{r \geq 0 : \mathcal{M}^{*r}(A) = \infty\}.$$

Similarly, we can define the *r-dimensional Minkowski lower content* as

$$\mathcal{M}_*^r = \liminf \limsup_{t \to 0} \frac{|A_t|}{t^{Q-r}}$$

and the *lower Minkowski dimension* by

$$\underline{\dim}_B A = \inf\{r \geq 0 : \mathcal{M}_*^r(A) = 0\}.$$

If $\overline{\dim}_B A = \underline{\dim}_B A$, we say that $D = \dim_B A := \overline{\dim}_B A$ is the *Minkowski dimension.* If further $\mathcal{M}^{*D} = \mathcal{M}_*^D$, we call this common value the *D-dimensional Minkowski content* and denote it simply by $\mathcal{M}^D$.

If there exists a $D \geq 0$ such that

$$0 < \mathcal{M}_*^D(A) \leq \mathcal{M}^{*D} < \infty,$$

we say that A is *Minkowski nondegenerate.* Finally, if $\mathcal{M}_*^D(A) = \mathcal{M}^{*D}$, we call this common value the *Minkowski content of* A, denoted by $\mathcal{M}^D(A)$. If the content is nondegenerate, i.e. $0 < \mathcal{M}^D(A) < \infty$, then A is said to be *Minkowski measurable.*

With these definitions in hand, we can start generalizing the distance zeta function, as well as related zeta functions. We proceed as follows. Chapter 2 generalizes the major part of the theory, establishing that the distance zeta function is holomorphic in a right half plane and the poles of such give complex dimensions. Chapter 3 studies examples in the Heisenberg group and Laakso space, showing some degree of geometric information that is established by the distance zeta function. Chapter 4 establishes the theory of Fractal Tube Formulas, both point-wise and distributional. We end with Chapter 5, which hints at fractal zeta functions applied to weaker spaces while noting the inherent difficulties in establishing a full generalization.

Chapter 2

Fractal Zeta Functions

In this chapter we will study the basic properties of the distance zeta function in Ahlfors spaces. In particular, our main result is that in Ahlfors spaces, the distance zeta function is holomorphic in the right half-plane $\{\text{Re } s > \overline{\dim}_B A\}$, along with mild assumptions under which this bound cannot be improved. Thus the abscissa of convergence of our distance zeta function on the right-hand side is $\overline{\dim}_B A$, the Minkowski dimension of the set A.

The proof will require the following lemma stated, without proof, by Harvey and Polking in [9]. The following proof is due to Žubrinić in [30], which uses dyadic decomposition. It is given in detail and extended for the needs of the theory in [19]. While originally written for Euclidean spaces, the proof still holds under the appropriate modifications for Ahlfors spaces.

Lemma 2.1 (**Harvey-Polking Lemma**). *Assume that A is an arbitrary bounded subset of an Ahlfors regular space $\mathcal{E}$ and let δ be any fixed positive number. If $\gamma \in (-\infty, Q - \overline{\dim}_B A)$ real, then*

$$\int_{A_\delta} d(x, A)^{-\gamma}\, dx < \infty.$$

Proof. If $\gamma \in (-\infty, 0]$, then $x \mapsto d(x, A)^{-\gamma}$ is continuous and bounded on A_δ and the claim follows immediately. We will therefore focus on the case where $\overline{\dim}_B A < Q$.

Let $s \in (\overline{\dim}_B A, Q-\gamma)$ be arbitrary. Note that since $\gamma < Q - \overline{\dim}_B A$, this interval is nonempty. For $t \in (0,\delta]$, the function $t \mapsto |A_t|/t^{Q-s}$ is continuous by continuity of measures. Since $\mathcal{M}^{*s}(A) = 0$ by assumption, the supremum of this function must be finite. Denoting the supremum by $C(\delta)$, then $|A_t| \leq C(\delta)t^{Q-s}$, for all $t \in (0,\delta)$.

We will now use a type of dyadic decomposition of the set $A_\delta \setminus \overline{A}$:

$$A_\delta = \overline{A} \cup \Big(\bigcup_{i=1}^{\infty} B_i\Big), \quad B_i := A_{2^{-i}\delta} \setminus A_{2^{-i-1}\delta}.$$

Since we have assumed $\overline{\dim}_B A < Q$, we have that $|\overline{A}| = 0$. Thus

$$I(A) := \int_{\overline{A}} d(x,A)^{-\gamma} \mathrm{d}x = 0.$$

If $\delta > 1$, then $d(x,A)^{-\gamma}$ is bounded on $A_\delta \setminus A_1$. Thus, we can assume $\delta \in (0,1]$. Using our dyadic decomposition and the assumption $0 < \gamma < Q - s$, we have:

$$\begin{aligned}
\int_{A_\delta} d(x,A)^{-\gamma}\mathrm{d}x &= I(A) + \sum_{i=0}^{\infty} \int_{B_i} d(x,A)^{-\gamma}\mathrm{d}x \\
&\leq I(A) + \sum_{i=0}^{\infty} \int_{A_{2^{-i}\delta}} d(x,A)^{-\gamma}\mathrm{d}x \\
&\leq I(A) + \sum_{i=0}^{\infty} (2^{-i-1}\delta)^{-\gamma} |A_{2^{-i}\delta}| \\
&\leq I(A) + C(\delta) \sum_{i=0}^{\infty} (2^{-i-1}\delta)^{-\gamma} (2^{-i}\delta)^{Q-s} \\
&\leq I(A) + \frac{2^{\gamma} C(\delta)}{1 - 2^{\gamma - Q + s}} \delta^{Q-s-\gamma} < \infty.
\end{aligned}$$

This completes the proof. □

We can think of this lemma as an extension of the fact that, in $\mathbb{R}^N$, if $\gamma < N$ then the integral of $|x|^{-\gamma}$ over the unit ball is finite valued.

The left-hand integral of the Harvey-Polking lemma can be viewed as a prototype of the distance zeta function, where the exponent is only real-valued. In order to prove a useful identity, we will need the following theorem, stated in Folland's classic real analysis text [8].

Lemma 2.2 (Folland pg. 198)**.** *Let $f \in (X, \mathcal{M}, \mu)$ be a nonnegative measurable function in a measure space X, i.e. $f : X \to [0, +\infty]$, and let $0 < \alpha < \infty$. Then*

$$\int_X f(x)^\alpha \, dx = \alpha \int_0^\infty t^{\alpha-1} |\{f > t\}| \, dt,$$

where $\{f > t\} := \{x \in X : f(x) > t\}$.

This next lemma, which is an exact counterpart of [19, Lemma 2.1.4], establishes the identity that changes the integral over the distance function to the integral of the tubular volume of A_t. We can view this, again, as a prototype of the *tubular zeta function* defined in Def. 2.2.

Lemma 2.3. *Let A be any bounded set in an Ahlfors regular space $\mathcal{E}$. Then, for every value of the parameter γ in the open interval $(-\infty, Q - \overline{\dim}_B A)$, the following identity holds:*

$$\int_{A_\delta} d(x, A)^{-\gamma} \, dx = \delta^{-\gamma} |A_\delta| + \gamma \int_0^\delta t^{-\gamma-1} |A_t| \, dt.$$

Furthermore, both integrals in the above are finite.

Proof. We consider the following three cases:

(a) Case where $\gamma > 0$: Since $0 < \gamma < Q - \overline{\dim}_B A$, we will proceed similarly to [32]. We use Lemma 2.2 with $\alpha = \gamma$ and the Borel measurable function $f : \mathcal{E} \to [0, +\infty]$ given by

$$f(x) := \begin{cases} d(x, A)^{-1} & \text{for } x \in A_\delta \\ 0 & \text{for } x \in \mathcal{E} \setminus A_\delta \end{cases}.$$

By definition, $f(x) = +\infty$ for $x \in \overline{A}$; further, since $\overline{\dim}_B A < Q$, we have $|\overline{A}| = 0$. Note that the set $\{x \in \mathcal{E} : f(x) > t\}$ is equal to $A_{1/t}$ for $t > \delta^{-1}$ and to the constant set A_δ for $t \in (0, \delta^{-1})$. Therefore,

$$\begin{aligned} \int_{A_\delta} d(x, A)^{-\gamma} \mathrm{d}x &= \gamma \Big(\int_0^{1/\delta} + \int_{1/\delta}^{+\infty} \Big) t^{\gamma-1} |\{f > t\}| \mathrm{d}t \\ &= \gamma |A_\delta| \int_0^{1/\delta} t^{\gamma-1} + \gamma \int_{1/\delta}^{+\infty} t^{\gamma-1} |A_{1/t}| \mathrm{d}t. \end{aligned}$$

The result follows by a change of variables $\tau = 1/t$ in the last integral. In order to show this integral is finite, let $\varepsilon > 0$ be small enough so that $\gamma \in (0, Q - D - \varepsilon)$, where $D := \overline{\dim}_B A$. Then

$\mathcal{M}^{*(D+\varepsilon)} = 0$, and so there exists a positive constant $C = C(\delta)$ such that $|A_t| \leq Ct^{Q-d-\varepsilon}$ for all $t \in (0, \delta]$. Hence,

$$\int_0^\delta t^{-\gamma-1}|A_t|\mathrm{d}t \leq C \int_0^\delta t^{Q-D-\varepsilon-\gamma-1}\mathrm{d}t < \infty.$$

(b) Case where $\gamma = 0$: If we assume that $\gamma = 0 < Q - \overline{\dim}_B A$, then it suffices to show that $I := \int_0^\delta t^{-1}|A_t|\mathrm{d}t < \infty$. Letting $D := \overline{\dim}_B A$, we then have $D + \varepsilon < Q$ for $\varepsilon > 0$ small enough; hence, since $\mathcal{M}^{*(D+\varepsilon)}(A) = 0$, there exists a positive constant C such that $|A_t| \leq Ct^{Q-D-\varepsilon}$ for all $t \in (0, \delta)$. This immediately implies that $I \leq C \int_0^\delta t^{Q-D-\varepsilon-1}\mathrm{d}t < \infty$.

(c) Case where $\gamma < 0$: In this case, the left hand side of our equation is clearly finite. We shall use Lemma 2.2 with $\alpha = -\gamma$ and the Borel measurable function $f : \mathcal{E} \to [0, +\infty]$ given by

$$f(x) := \begin{cases} d(x, A) & \text{for } x \in A_\delta \\ 0 & \text{for } x \in \mathcal{E} \setminus A_\delta \end{cases}.$$

Note that $\{f > t\} = \emptyset$ for $t \geq \delta$, and $\{f > t\} = A_\delta \setminus A_t$ for $0 < t < \delta$. Thus for $t < \delta$, we have $|\{f > t\}| = |A_\delta| - |A_t|$. We obtain

$$\int_{A_\delta} d(x, A)^\alpha \mathrm{d}x = \alpha \int_0^\delta t^{\alpha-1}(|A_\delta| - |A_t|)\mathrm{d}t = d^\alpha|A_\delta| - \alpha \int_0^\delta t^{\alpha-1}|A_t|\mathrm{d}t,$$

which, after replacing α by $-\gamma$, completes the proof.

□

The proof of this lemma bears a small but important difference to that in the Euclidean case. In particular, part (c) requires that the tubular neighborhood A_t be closed in the Ahlfors regular setting, which is not the case in $\mathbb{R}^N$. A small but surprising change, given that in this part of the theorem the left-hand integral is clearly finite.

The following lemma (analog of [19, Lemma 2.1.6]) complements the Harvey-Polking theorem, establishing the domain of integrability in the real setting:

Lemma 2.4. *Let A be a bounded set in $\mathcal{E}$ and $\delta > 0$. If $\gamma > Q - \overline{dim}_B A$, then $\int_{A_\delta} d(x, A)^{-\gamma} dx = +\infty$.*

Proof. For all $\gamma > 0$,

$$I_\delta := \int_{A_\delta} d(x,A)^{-\gamma}\mathrm{d}x = \delta^{-\gamma}|A_\delta| + \gamma\int_0^\delta s^{-\gamma-1}|A_s|\mathrm{d}s \geq \delta^{-\gamma}|A_\delta|,$$

where the second equality holds from the same proof as in Lemma 2.3 case (a). Let $D := \overline{\dim}_B A$ and choose $\sigma < D$ sufficiently close to D so that $\gamma > Q - \sigma$. Then $\mathcal{M}^{*\sigma}(A) = +\infty$, which implies that there exists a sequence of positive numbers s_k converging to zero such that

$$C_k := \frac{|A_{s_k}|}{s_k^{Q-\sigma}} \to +\infty \quad \text{as} \quad k \to +\infty.$$

Since $\delta \mapsto I_\delta$ is nondecreasing, we have for all k large enough

$$I_\delta \geq I_{s_k} \geq (s_k)^{-\gamma}|A_{s_k}| = C_k \cdot s_k^{Q-\sigma-\gamma} \to +\infty$$

as $k \to \infty$. Hence, $I_\delta = +\infty$. $\square$

Given these lemmas, which deal with the soon defined distance zeta function restricted to real numbers, we can state our main theorem, which is an exact counterpart of [19, Theorem 2.1.11]. The proof is included for completeness, but is nearly identical to the Euclidean case.

Theorem 2.5. *Let A be an arbitrary bounded subset of $\mathcal{E}$ and let $\delta > 0$. Then:*

(a) *The zeta function ζ_A defined by*

$$\zeta_A(s) = \int_{A_\delta} d(x,A)^{s-Q}\mathrm{d}x,$$

is holomorphic in the half-plane $\{\mathrm{Re} s > \overline{\dim}_B A\}$, and for all complex numbers s in that region,

$$\zeta'_{A_\delta}(s) = \int_{A_\delta} d(x,A)^{s-Q}\log d(x,A)\mathrm{d}x.$$

(b) *The lower bound of absolute convergence is optimal, i.e.*

$$\overline{\dim}_B A = D(\zeta_A),$$

where $D(\zeta_A)$ is the abscissa of absolute convergence of ζ_A.

(c) *If the Minkowski dimension* $D := \dim_B A$ *exists,* $D < Q$*, and* $\mathcal{M}_*^D(A) > 0$*, then* $\zeta_A(s) \to +\infty$ *as* $s \in \mathbb{R}$ *converges to* D *from the right. In this case, the abscissa of holomorphic convergence coincides with the abscissa of absolute convergence.*

Proof. (a) Let $I(s) := \int_{A_\delta} d(x,A)^{s-Q} \log d(x,A) \mathrm{d}x$. To prove the holomorphicity of ζ_A, let us fix any s such that $\mathrm{Re} s > \overline{\dim}_B A$. We thus have to show that

$$\begin{aligned} R(h) &:= \frac{\zeta_A(s+h) - \zeta_A(s)}{h} - I(s) \\ &= \int_{A_\delta} \Big(\frac{d(x,A)^h - 1}{h} - \log d(x,A) \Big) d(x,A)^{s-Q} \mathrm{d}x \end{aligned}$$

converges to zero as $h \to 0$ in $\mathbb{C}$, with $h \neq 0$.

Letting $d := d(x,A) \in (0,\delta)$, we first consider

$$f(h) := \frac{d^h - 1}{h} - \log d = \frac{1}{h}(e^{h \log d} - 1) - \log d.$$

Using the MacLaurin series $e^z = \sum_{j \geq 0} \frac{z^j}{j!}$, which converges for all $\mathbb{C}$, we obtain

$$f(h) = h(\log d)^2 \sum_{k=0}^{\infty} \frac{1}{(k+2)(k+1)} \cdot \frac{(\log d)^k h^k}{k!},$$

for all $h \in \mathbb{C}$. Further, assuming without loss of generality that $\delta \in (0,1]$, and hence $\log d \leq 0$, we have

$$\begin{aligned} |f(h)| &\leq \frac{1}{2}|h|(\log d)^2 \sum_{k=0}^{\infty} \frac{(|\log d||h|)^k}{k!} \\ &= \frac{1}{2}|h|(\log d)^2 e^{-|h| \log d} \\ &= \frac{1}{2}|h|(\log d)^2 d^{-|h|} \end{aligned}$$

Thus,

$$|R(h)| \leq \frac{1}{2}|h| \int_{A_\delta} |\log d(x,A)|^2 d(x,A)^{\mathrm{Re} s - Q - |h|} \mathrm{d}x.$$

Let $\varepsilon > 0$ be sufficiently small, to be specified. Taking $h \in \mathbb{C}$ such that $|h| < \varepsilon$, since $\delta \leq 1$, and hence $d(x,A) \leq 1$, we have

$$|R(h)| \leq \frac{1}{2}|h| \int_{A_\delta} |\log d(x,A)|^2 d(x,A)^{\varepsilon} d(x,A)^{\mathrm{Re} s - Q - 2\varepsilon} \mathrm{d}x.$$

By a simple application of L'Hôpital's Rule, it is clear there exists a positive constant $C = C(\delta, \varepsilon)$ such that $|\log \rho|^2 \rho^\varepsilon \leq C$ for all $\rho \in (0, \delta)$. This implies that

$$|R(h)| \leq \frac{1}{2} C |h| \int_{A_\delta} d(x, A)^{\mathrm{Re} s - Q - 2\varepsilon} \mathrm{d}x.$$

Letting $\gamma := 2\varepsilon + Q - \mathrm{Re} s$, we see that the integrability condition $\gamma < Q - \overline{\dim}_B A$ is equivalent to $\mathrm{Re} s > \overline{\dim}_B A + 2\varepsilon$. Since s is fixed with $\mathrm{Re} s > \overline{\dim}_B A$, we only need specify that ε be small enough to fulfill this inequality. Then $R(h) \to 0$ as $h \to 0$ in $\mathbb{C}$, with $h \neq 0$. Therefore we conclude that $\zeta_A(s)$ is holomorphic for $\mathrm{Re} s > \overline{\dim}_B A$, with derivative $\zeta_A'(s)$ given as desired.

(b) This follows immediately from (a) and Lemma 2.4.

(c) Since $\mathcal{M}_*^D(A) > 0$, then for any $\delta > 0$ there exists $C > 0$ such that for all $t \in (0, \delta)$, we have $|A_t| \geq C t^{Q-D}$. Using Lemma 2.1 and 2.2, we see that for any $\gamma \in (0, Q - D)$,

$$\begin{aligned} \infty > I(\gamma) &= \int_{A_\delta} d(x, A)^{-\gamma} \mathrm{d}x = \delta^{-\gamma} |A_\delta| + \gamma \int_0^\delta t^{-\gamma - 1} |A_t| \mathrm{d}t \\ &\geq \gamma C \int_0^\delta t^{Q - D - \gamma - 1} \mathrm{d}t = \gamma C \frac{\delta^{Q-D-\gamma}}{Q - D - \gamma} \end{aligned}$$

Therefore, if $\gamma \to Q - D$ from the left, then $I(\gamma) \to +\infty$. Equivalently, if $s \in \mathbb{R}$ is such that $s \to D$ from the right, then $\zeta_A(s) \to +\infty$.

$\square$

As noted in the lemmas and clear from the definition, if s is real then ζ_A is also real-valued. Also, using the principle of reflection, we have that for any complex number such that Re $s > \overline{\dim}_B A$, $\overline{\zeta_A(s)} = \zeta_A(\overline{s})$. We will be mostly concerned with meromorphically extending the distance zeta function, when possible, and below we will define the *visible complex dimensions of the set* A as the poles in such extensions. Thus, if the distance zeta function can be meromorphically extended to a region symmetric with respect to the real axis, the non-real (visible) complex dimensions come in conjugate pairs.

Given that our set A is not of "full dimension" in the sense that $\overline{\dim}_B A < Q$, we have the following proposition (analog of [19, Proposition 2.1.13]):

Proposition 2.6. Assuming that $|\overline{A}| = 0$ (which is always the case if $\overline{\dim}_B A < Q$), and given any $\delta > 0$, we can compute the distance zeta function ζ_A as follows for $s \in \mathbb{C}$ with $\mathrm{Re}s > \overline{\dim}_B A$:

$$\zeta_A(s) = \lim_{\varepsilon \to 0^+} \int_{A_\delta \setminus A_\varepsilon} d(x, A)^{s-Q} \mathrm{d}x.$$

Proof. The characteristic function $\chi_{A_\delta \setminus A_\varepsilon}$ converges to χ_{A_δ} a.e. in A_δ as $\varepsilon \to 0^+$. To show that our limit converges uniformly, we take a complex number s and real number ξ such that $\mathrm{Re}s > \xi > \overline{\dim}_B A$, and (without loss of generality) $\varepsilon \in (0,1)$, noting that

$$\left| \int_{A_\varepsilon} d(x, A)^{s-Q} \mathrm{d}x \right| \le \int_{A_\varepsilon} d(x, A)^{\mathrm{Re}s - Q} \mathrm{d}x \le \int_{A_\varepsilon} d(x, A)^{\xi - Q} \mathrm{d}x.$$

Let us choose any $d \in (\overline{\dim}_B A, \xi)$. Since $d > \overline{\dim}_B A$, we have that $\mathcal{M}^{*d}(A) = 0$, which means there exists a positive constant $C = C(d, Q, A)$ such that $|A_t| \le C t^{Q-d}$ for all $t \in (0, \varepsilon]$. Using the Harvey-Polking Lemma with $\gamma := Q - \xi$, it follows that

$$\begin{aligned} \int_{A_\delta} d(x, A)^{\xi - Q} \mathrm{d}x &= \varepsilon^{-\gamma} |A_\varepsilon| + \gamma \int_0^\varepsilon t^{\gamma - 1} |A_t| \mathrm{d}t \\ &\le \varepsilon^{\gamma} C \varepsilon^{Q-d} + \gamma \int_0^\varepsilon t^{\gamma - 1} C t^{Q-d} \mathrm{d}t = C_1 \varepsilon^{\xi - d}, \end{aligned}$$

where $C_1 := C(n-d)/(\xi - d)$. Hence, using $d < \xi$, we conclude that

$$\sup_{\mathrm{Re}s > \xi} \left| \int_{A_\varepsilon} d(x, A)^{s-Q} \mathrm{d}x \right| \le C_1 e^{\xi - d} \to 0^+ \quad \text{as } \varepsilon \to 0^+,$$

and our result follows. $\square$

From this proof, we can deduce that the distance zeta function satisfies the following asymptotic property (analog of [19, Proposition 2.1.14]).

Proposition 2.7. Assume that A is a bounded set in $\mathcal{E}$, $\varepsilon \in (0,1)$, and let us define the corresponding distance zeta function $\zeta_A(\cdot, A_\varepsilon)$ by

$$\zeta_A(s, A_\varepsilon) := \int_{A_\varepsilon} d(x, A)^{s-Q} \mathrm{d}x.$$

Then, for any $\xi > \overline{\dim}_B A$ and $d \in (\overline{\dim}_B A, \xi)$, there exists a positive constant $C_1 = C_1(\xi, d, Q, A)$ such that

$$\sup_{\mathrm{Re}s>\xi} |\zeta_A(s, A_\varepsilon)| \le C_1 \varepsilon^{\xi-d}, \quad \text{for all } \varepsilon \in (0,1).$$

In other words, $\sup_{\mathrm{Re}s>\xi} |\zeta_A(s, A+\varepsilon)| = O(\varepsilon^{\xi-d})$ as $\varepsilon \to 0^+$.

The following theorem, counterpart of [19, Lemma 2.1.15], establishes that the distance zeta function is an entire function away from the set A. Note that since the zeta function is associated with the ordered pair (A, U), where A and U are suitable subsets of $\mathcal{E}$, we could establish this theorem in the manner of *relative fractal drums*. These are studied further at the end of the chapter.

Theorem 2.8. *Let A and U be bounded sets in $\mathcal{E}$ which have disjoint closures such that $\overline{A} \cap \overline{U} = \emptyset$. Further assume that U is Hausdorff measurable. Then*

$$F : \mathbb{C} \to \mathbb{C}, \quad F(s) := \int_U d(x, A)^{s-Q} dx.$$

is an entire function and we have

$$F'(s) = \int_U d(x, A)^{s-Q} \log d(x, A) dx$$

for all $s \in \mathbb{C}$.

Proof. Let s be a fixed complex number and set $R(h) = \frac{1}{h}(F(s+h) - F(s)) - I_1(h)$, for $h \in \mathbb{C},\ h \neq \emptyset$, where $I_1 := \int_U d(x, A)^{s-Q} \log d(x, A) \mathrm{d}x$. We follow the same procedure as in the proof of Theorem 2.5 part (a), where it follows that

$$|R(h)| \le \frac{1}{2}|h| \int_U |\log d(x, A)|^2 \exp(|\log d(x, A)||h|) d(x, A)^{\mathrm{Re}s-Q} \mathrm{d}x.$$

Since A and U are disjoint and bounded, there exists positive constants d_1 and d_2 such that $d_1 \le d(x, A) \le d_2$ for all $x \in U$. Therefore, the function under the integral sign above is bounded from above by a constant C, uniformly for all $h \in \mathbb{C}$ such that $|h| \le \varepsilon$, where $\varepsilon > 0$ is fixed:

$$\begin{aligned} C = &\max\{(\log d_1)^2, (\log d_2)^2\} \exp(\max\{|\log d_1|, |\log d_2|\}\varepsilon) \\ &\cdot \max\{d_1^{\mathrm{Re}s-Q}, d_2^{\mathrm{Re}s-Q}\}. \end{aligned}$$

Hence, $|R(h)| \le \frac{1}{2}|h|C|U|$, and therefore $R(h) \to 0$ as $h \to 0$ in $\mathbb{C}$, with $h \neq 0$. $\square$

We also wish to generalize Lemma 2.3 for the distance zeta function in full. The established identity is used below to define the *tube zeta function of* A, which presents an alternate zeta function for analysis. This result is the counterpart to [19, Theorem 2.2.1]. We will see in Chapter 3 and 5 that, in certain examples, the tube zeta function is much easier to use compared to the distance zeta function.

Theorem 2.9. *Let A be any bounded set in an Ahlfors regular space $\mathcal{E}$. Then, for every value of the parameter $s \in \mathbb{C}$ such that* $\operatorname{Re} s > \overline{dim}_B A$*, the following identity holds:*

$$\int_{A_\delta} d(x,A)^{s-Q}\, dx = \delta^{s-Q}|A_\delta| + (Q-s)\int_0^\delta t^{s-Q-1}|A_t|\, dt. \tag{2.1}$$

Proof. Let $D := \overline{\dim}_B A$. By Lemma 2.3, we know that this identity already holds for all $s \in \mathbb{R}$ such that $s > D$. Let us denote the left hand side of Equation 2.1 by $f(s)$ and the right hand side by $g(s)$. Since $f(s) = g(s)$ on the subset $(D, +\infty) \subset \mathbb{C}$, to prove the theorem, it suffices to show that $f(s)$ and $g(s)$ are both holomorphic in the region $\{\operatorname{Re} s > D\}$; the result will then follow by the principle of analytic continuation. Note that the holomorphicity of $f(s)$ in this region was already proven in Theorem 2.5.

To prove the holomorphicitiy of $g(s)$ on $\{\operatorname{Re} s > D\}$, it suffices to consider $g_1(s) = \int_0^\delta t^{s-Q-1}|A_t|\mathrm{d}t$. Note that $g_1(s)$ has the form of a Dirichlet-type integral: $g_1(s) = \int_E \phi(t)^s \mathrm{d}\mu(x)$, where $E := (0,\delta)$, $\phi(t) := t$, and the positive measure $\mathrm{d}\mu(x) := t^{-Q-1}|A_t|\mathrm{d}t$. Thus it suffices to show that on this region, $g_1(s)$ is well defined. To prove this, let $\varepsilon > 0$ be small enough so that $\operatorname{Re} s > D + \varepsilon$. Since $\mathcal{M}^{*(D+\varepsilon)}(A) = 0$, there exists $C_\delta > 0$ such that $|A_t| \le C_\delta t^{Q-D-\varepsilon}$ for all $t \in (0, \delta]$. Then

$$\begin{aligned}|g_1(s)| &\le \int_0^\delta t^{\operatorname{Re} s-Q-1}|A_t|\mathrm{d}t \\ &\le C_\delta \int_0^\delta t^{\operatorname{Re} s-D-\varepsilon-1}\mathrm{d}t = C_d e\frac{\delta^{\operatorname{Re} s-D-\varepsilon}}{\operatorname{Re} s-D-\varepsilon} < \infty.\end{aligned}$$

$\square$

Definition 2.2. Let δ be a fixed positive number, and let A be a bounded set in $\mathcal{E}$. Then the *tube zeta function of* A, denoted by $\widetilde{\zeta}_A$, is defined by

$$\widetilde{\zeta}_A(s) = \int_0^\delta t^{s-Q-1}|A_t|\mathrm{d}t,$$

for $s \in \mathbb{C}$ with Re s sufficiently large.

We also denote the *abscissa of holomorphic convergence of function* f as $D(f)$ for the sequel. Importantly, we know that if a bounded set A is such that $D := \dim_B A$ exists, $D < Q$, and $\mathcal{M}_*^D(A) > 0$, then $D(\zeta_A) = \overline{\dim}_B A$, see Theorem 2.5 part (c). The following lemma (analog of [19, Lemma 2.1.52]) proves that $D(\zeta_A) \geq 0$ for any bounded set $A \in \mathcal{E}$, which is not the case for relative fractal drums. In particular, there are examples of relative fractal drums in $\mathbb{R}^N$ that have negative abscissa of convergence, see [19], Section 4.1.2.

Lemma 2.10. *For any bounded set A in $\mathcal{E}$, we have $D(\zeta_A) \geq 0$.*

Proof. Assume to the contrary that $D(\zeta_A) < 0$. Then $\zeta_A(s)$ is well defined and continuous for $s \in (D(\zeta_A), +\infty)$, and in particular, it is continuous at $s = 0$.

Let us take any $a \in A$. Since $A_\delta \supset B_\delta(a)$, and $d(x, A) \leq d(x, a)$, we have that for every real number $s \in (0, Q)$,

$$\begin{aligned}\widetilde{\zeta}_A(s) &= \int_0^\delta x^{s-Q-1}|A_x|\mathrm{d}x \geq \int_0^\delta x^{s-Q-1}|B_x(a)|\mathrm{d}x \\ &\geq \int_0^\delta x^{s-Q-1}K^{-1}x^Q\mathrm{d}x = K^{-1}\frac{\delta^s}{s},\end{aligned}$$

where $K^{-1}x^Q$ comes from the bound on the measure of the ball $B_x(a)$ given by the Ahlfors regularity condition. Thus, $\widetilde{\zeta}_A(s)$ (and as a consequence, $\zeta_A(s)$) $\to +\infty$ as $s \to 0^+$, $s \in \mathbb{R}$. This clearly contradicts the continuity of ζ_A at $s = 0$. $\square$

We can now define the *complex dimensions of a set* A, given that the distance or tube zeta functions can be meromorphically extended to an *admissible* domain:

Definition 2.3. A set A such that ζ_A can be meromorphically extended to an open domain G containing the closed half-plane $\{\mathrm{Re}\ s \geq D(\zeta_A)\}$ is called *admissible.*

Definition 2.4. Given an admissible set A, we consider the *set of poles of* ζ_A *located on the critical line* $\{\mathrm{Re}\ s = D(\zeta_A)\}$:

$$\mathscr{P}_c(\zeta_A) = \{\omega \in W : \omega \text{ is a pole of of } \zeta_A \text{ and } \mathrm{Re}\ \omega = D(\zeta_A)\},$$

called the *set of principal complex dimensions of* A, also denoted by $\dim_{\mathbb{C}} A$.

We call the set of all poles in the region G of meromorphic extension of ζ_A the *set of visible complex dimensions of A (with respect to G)*, and we denote it by $\mathscr{P}(\zeta_A)$ or $\mathscr{P}(\zeta_A, G)$:

$$\mathscr{P}(A) = \{\omega \in G : \omega \text{ is a pole of } \zeta_A\}$$

It may not be convenient or even possible to meromorphically extend ζ_A to all of $\mathbb{C}$. These complex dimensions give important geometric information about the set A, and are the main point of study explored in Chapters 3 and 5. It is important that the dimensions are given as a *set* (or even as a multiset), as each dimension can be interpreted to represent a fuller picture into the geometry. For example, given a simple simplicial 3-complex composed of vertices, edges, and faces, the complex dimensions given by the distance zeta function would be $\{0, 1, 2\}$.

While complex dimensions can be useful for standard geometric subsets, subsets that exhibit *geometric oscillations*, such as Cantor sets, produce non-real complex dimensions. As conjectured by Lapidus and collaborators, a fractal set is a set that has non-real complex dimensions (or more complex behavior, such as essential singularities along the vertical line $\{\mathrm{Re}\ s = D(\zeta_A)\}$).

As stated above, depending on the set we wish to study, it may be preferential to use the tube zeta function. Before we can do so, we first establish their equivalence in terms of complex dimensions as defined in [19]:

Definition 2.5. Two meromorphic functions f and g on a domain $G \subset \mathbb{C}$ are said to be *equivalent* if $D(f) = D(g)$, and their sets of poles contained on the common vertical line $\{\mathrm{Re}\ s = D(f) = D(g)\}$

coincide:

$$f \sim g \iff D(f) = D(g) \text{ and } \mathscr{P}_c(f) = \mathscr{P}_c(g).$$

Now, the relationship between the distance and tube zeta functions is given by

$$\int_{A_\delta} d(x, A)^{s-Q} \mathrm{d}x = \delta^{s-Q}|A_\delta| + (Q - s) \int_0^\delta t^{s-Q-1}|A_t| \mathrm{d}t.$$

As long as $\overline{\dim}_B A < Q$, the two zeta functions are equivalent. However, if $\overline{\dim}_B A = Q$, it is possible that the two zeta functions differ by a pole at $s = Q$. Thus the two functions are almost equivalent, as long as this caveat is kept in mind.

In the following theorems (analogs of [19, Theorem 2.2.3] and [19, Theorem 2.2.14], resp.), we establish the relationship between the residues of the distance and tube zeta functions with the Minkowski content of the set A. We use the notation $\zeta_A(\cdot, A_\delta)$ to emphasize the dependence of the value of δ. In particular, we prove δ has no significance on the value of the residues, just as it does not impact the complex dimensions.

Theorem 2.11. *Assume that the bounded set $A \subset \mathcal{E}$ is Minkowski nondegenerate, i.e., $0 < \mathcal{M}_*^D(A) \leq \mathcal{M}^{*D}(A) < \infty$ for $D := \dim_B A$ and $D < Q$. If $\zeta_A(s, A_\delta)$ can be extended meromorphically to a neighborhood of $s = D$, then D is necessarily a simple pole of $\zeta(s, A_\delta)$, and*

$$(Q - D)\mathcal{M}_*^D(A) \leq \mathit{res}(\zeta_A(\cdot, A_\delta), D) \leq (Q - D)\mathcal{M}^{*D}(A).$$

Furthermore, the value of $\mathit{res}(\zeta_A(\cdot, A_\delta), D)$ does not depend on $\delta > 0$. In particular, if A is Minkowski measurable, then

$$\mathit{res}(\zeta_A(\cdot, A_\delta), D) = (Q - D)\mathcal{M}^D(A).$$

Proof. Since $\mathcal{M}_*^D(A) > 0$, using Theorem 2.5(c) we conclude that $s = D$ is a pole. Therefore, it suffices to show that the order of the pole at $S = D$ is not larger than 1. Let us take any fixed $\delta > 0$, and let

$$C_\delta := \sup_{t \in (0, \delta]} \frac{|A_t|}{t^{Q-D}}.$$

Note that $C_\delta < \infty$ because $M^{*D}(A) < \infty$. Then, for $s \in \mathbb{R}$ with $D < s < Q$, we have

$$\begin{aligned} \zeta_A(s, A_\delta) &= \int_{A_\delta} d(x,A)^{s-Q}\mathrm{d}x = \delta^{s-Q}|A_\delta| + (Q-s)\int_0^\delta t^{s-Q-a}|A_t|\mathrm{d}t \\ &\leq C_\delta \delta^{s-D} + C_\delta(Q-s)\frac{\delta^{s-D}}{s-D} = C_\delta(Q-D)\delta^{s-D}\frac{1}{s-D}. \end{aligned} \tag{2.6}$$

Therefore, $0 < \zeta_A(s, A_\delta) \leq C_1(s-D)^{-1}$ for all $s \in (D, Q)$. This shows that $s = D$ is a pole of $\zeta_A(s, A_\delta)$ which is at most of order 1, and the first claim is established. Namely, D is a simple pole.

It is easy to see that for any positive real numbers δ and δ_1, with $\delta < \delta_1$, the difference

$$\zeta_A(s, A_{\delta_1}) - \zeta_A(s, A_\delta) = \int_{A_{\delta_1}\setminus A_\delta} d(x,A)^{s-Q}\mathrm{d}x$$

is an entire function of s, since $\delta \leq d(x,A) \leq \delta_1$ for any $x \in A_{\delta_1} \setminus A_\delta$. Therefore, the residue of $\zeta_A(s, A_\delta)$ at D does not depend on δ.

In order to prove the second inequality, it suffices to multiply 2.6 by $s - D$, with real s, and take the limit as $s \to D^+$:

$$\operatorname{res}(\zeta_A(\cdot, A_\delta), D) \leq (Q-D)\lim_{s\to D^+} C_\delta \delta^{s-D} = (Q-D)C_\delta.$$

Since the residue at D does not depend on δ, we establish the second inequality by recalling the definition of C_δ and passing to the limit as $\delta \to 0^+$. The first inequality is proved analogously. $\square$

The following is the result stated for the tube zeta functions, and is a direct consequence of Lemma 2.3 and the above proof.

Theorem 2.12. *Assume that A is a nondegenerate bounded set in $\mathcal{E}$ such that $D := dim_B A < Q$, and there exists a meromorphic extension of $\widetilde{\zeta}_A$. Then D is a simple pole, and for any positive δ,* $\operatorname{res}(\widetilde{\zeta}_A, D)$ *is independent of δ. Furthermore, we have*

$$\mathcal{M}_*^D(A) \leq \operatorname{res}(\widetilde{\zeta}_A, D) \leq \mathcal{M}^{*D}(A).$$

In particular, if A is Minkowski measurable, then the residue of the tube zeta function of A at $s = D$ is equal to the D-dimensional Minkowski content of A; that is

$$\operatorname{res}(\widetilde{\zeta}_A, D) = \mathcal{M}^D(A).$$

We end this chapter by defining the relative zeta functions for Ahlfors spaces, along with the main theorem adapted to this setting. While we do not currently study any examples, it is important to note that the theory extends. Relative fractal drums provide several interesting examples in Euclidean spaces, such as a spaces that have negative values or even $-\infty$ as the abscissa of holomorphic convergence. See [19], Chapter 4, for an exploration of the Euclidean case.

In Chapter 4, we present fractal tube formulas, which allow us to reconstruct information about the geometry of a relative fractal drum, given information about their associated fractal zeta functions.

Definition 2.7. Let Ω be an open subset of $\mathcal{E}$, not necessarily bounded, but of finite Hausdorff measure. Let $A \subset \mathcal{E}$, also possibly unbounded, such that Ω is contained in A_δ for some $\delta > 0$. The *distance zeta function* $\zeta_A(\cdot, \Omega)$ *of* A *relative to* Ω (or *the relative distance zeta function*) is defined by

$$\zeta_A(s, \Omega) = \int_\Omega d(x, A)^{s-Q} \mathrm{d}x,$$

for Re s sufficiently large. We will call the ordered pair (A, Ω) a *relative fractal drum.*

Note that, in the case of the relative fractal drum (A, A), we recover the standard setting of the distance zeta function. In this manner, relative fractal drums and their relative zeta functions are direct generalizations of ζ_A.

In this way, we can define the relative Minkowski content and dimension:

Definition 2.8. The *upper* r*-dimensional Minkowski content of* A *relative to* Ω is defined as

$$\mathcal{M}^{*r}(A, \Omega) = \limsup_{t \to 0} \frac{|A_t \cap \Omega|}{t^{Q-r}}.$$

We then define the *upper Minkowski dimension* by

$$\begin{aligned} \overline{\dim}_{BB}(A, \Omega) &= \inf\{r \in \mathcal{E} : \mathcal{M}^{*r}(A, \Omega) = 0\} \\ &= \sup\{r \in \mathcal{E} : \mathcal{M}^{*r}(A, \Omega) = \infty\}. \end{aligned}$$

We can thus define the relative distance zeta function and prove that its main properties still hold:

Theorem 2.13. *Let Ω be an open subset of $\mathcal{E}$ of finite Hausdorff measure, and let $A \subset \mathcal{E}$ be such that $\Omega \subset A_\delta$ for some $\delta > 0$. Then:*

(a) The relative distance zeta function $\zeta_A(s, \Omega)$ is holomorphic in the half-plane $\{\mathrm{Re} s > \overline{dim}_B(A, \Omega)\}$, and for all complex numbers s in that region,

$$\zeta'_A(s, \Omega) = \int_\Omega d(x, A)^{s-Q} \log d(x, A) \mathrm{d}x.$$

(b) The lower bound of absolute convergence is optimal, i.e.

$$\overline{dim}_B(A, \Omega) = D(\zeta_A(\cdot, \Omega)),$$

where $D(\zeta_A(\cdot, \Omega))$ is the abscissa of absolute convergence of $\zeta_A(s, \Omega)$.

(c) If the Minkowski dimension $D := dim_B(A, \Omega)$ exists, $D < Q$, and $\mathcal{M}_*^D(A, \Omega) > 0$, then $\zeta_A(s, \Omega) \to +\infty$ as $s \in \mathbb{R}$ converges to D from the right. In this case, the abscissa of holomorphic convergence coincides with the abscissa of absolute convergence.

Proof. The proof is similar to that of Theorem 2.5, as the Harvey-Polking Lemma generalizes to the case of relative fractal drums ([31, 19]). □

If the difference in two domains Ω_1 and Ω_2 is "nice enough" (such as both sets including the entirety of A_ϵ for ϵ small), then we define the relative fractal drums to be equivalent. Here again, the proof is similar to the Euclidean case and is thus omitted.

Proposition 2.14. Assume that (A, Ω_1) and (A, Ω_2) are relative fractal drums in $\mathcal{E}$ such that $f_j(s) := \int_{\Omega_j \setminus (\Omega_1 \cap \Omega_2)} d(x, A)^{s-Q} \mathrm{d}x$ are entire functions, for $j = 1, 2$. Then the corresponding distance zeta functions are equivalent,

$$\zeta_A(\cdot, \Omega_1) \sim \zeta_A(\cdot, \Omega_2).$$

Finally, we define the complex dimensions of a relative fractal drum, in an analogous manner to the standard distance zeta function.

Definition 2.9. Assume that (A, Ω) is a relative fractal drum in $\mathcal{E}$ such that its distance zeta function possesses a meromorphic extension to a domain which contains the critical line $\{\mathrm{Re}\, s = D(\zeta_A(\cdot, \Omega))\}$ in its interior. The set of poles of $\zeta_A(\cdot, \Omega)$ located on the critical line is called *the set of principal complex dimensions of the relative fractal drum* (A, Ω), or *the set of relative principal complex dimensions of* (A, Ω), and is denoted by $\dim_{\mathbb{C}}(A, \Omega)$, or equivalently, $\mathscr{P}_c(\zeta_A(\cdot, \Omega))$.

Chapter 3

Examples

Ahlfors spaces occur throughout mathematics in a variety of forms, such as Riemannian and sub-Riemannian manifolds, self-similar fractals in $\mathbb{R}^N$, and harmonic embeddings of fractals. In this chapter, we will look at two specific examples, the Heisenberg groups and Laakso spaces, both to calculate the complex dimensions of certain subsets as well as to illustrate the difficulties that arise.

3.1 The Heisenberg Groups

Definition 3.1. The *n-dimensional Heisenberg group* $\mathbb{H}^n$ in its algebra representation is thought of as $\mathbb{H}^n = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ with group multiplication given by

$$h \cdot h' = (x, y, t) \cdot (x', y', t') = (x + x', y + y', t + t' + 2\sum_{i=1}^{n}(x'_i y_i - x_i y'_i)).$$

There is a natural dilation action $\partial_r(x, y, t) = (rx, ry, r^2 t)$, $r > 0$, that gives rise to the homogeneous norm called the *Heisenberg gauge*,

$$\|(x, y, t)\| = \left(\sum_{i=1}^{n}(x_i^2 + y_i^2)^2 + t^2\right)^{\frac{1}{4}}.$$

This norm has the properties that $\|\partial_r(x,y,t)\| = r\|(x,y,t)\|$ and $\|h^{-1}\| = \|h\|$, where $h^{-1} = (-x,-y,-z)$.

This norm defines a metric on $\mathbb{H}^n$, $d(x,y) = \|x^{-1}y\|$.

The Haar measure on $\mathbb{H}^n$ given by Lebesgue measure under the exponential map gives

$$|B(x,r)| = r^{2n+2}|B(x,1)|.$$

The Lebesgue measure is invariant under left multiplication.

Thus the Heisenberg group is Ahlfors regular of dimension $Q = 2n+2$, while its topological dimension is $T = 2n+1$. More information on these spaces, and the more general Carnot-Carathéodory spaces, can be found in [2, 5, 6].

In what follows, we will be working with the space $\mathbb{H}^1 = \mathbb{H}$, and denoting points by their ordered triples. Note that $\mathbb{H}$ is Alhfors 4-regular. Also note there will be a few notational differences due to the standard usage of t as the last coordinate in the Heisenberg setting.

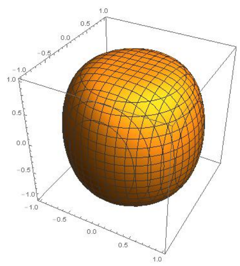
Example 3.2. If we fix any point $(x_1, y_1, t_1) = p$ in $\mathbb{H}$, then we know that $|A_r| = |(B(p,r))| = r^4|(B(p,1))|$, and thus we can explicitly calculate the complex dimension of the point using the tube zeta function:

$$\begin{aligned}\widetilde{\zeta}(s) &= \int_0^r w^{s-4-1}|A_w|\mathrm{d}w = |(B(p,1))|\int_0^r w^{s-4-1}w^4\mathrm{d}w \\ &= |(B(p,1))|\frac{r^s}{s}\end{aligned}$$

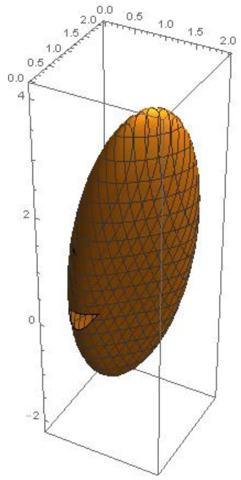
which has a simple pole at $s = 0$. Thus the set of complex dimensions of the point p is $\{0\}$.

Example 3.3. Let us take as our set A the line segment starting at the origin and going to (0,0,1) (or more generally, any line segment that lies on the t-axis). Explicitly, the distance function is given by

$$d((x,y,t),(x_1,y_1,t_1)) = \sqrt[4]{((x-x_1)^2+(y-y_1)^2)^2+(t-t_1+2yx_1-2xy_1)^2}. \tag{3.4}$$



(a) Unit ball centered at the origin



(b) Unit ball centered at (1,1,1)

Since we are along the t-axis, this equation simplifies to the much neater form:

$$d((x, y, t), (x_1, y_1, t_1)) = ((x^2 + y^2)^2 + (t - t_1)^2)^{1/4},$$

and we see that at any given point on our line segment, the horizontal cross-section is a circle. Thus A_r will be a standard cylinder with the hemispheres of the Heisenberg unit ball as the caps.

Again using the tube zeta function, we see that

$$\begin{aligned}\widetilde{\zeta}(s) &= \int_0^r w^{s-4-1}|A_w|\mathrm{d}w = \int_0^r w^{s-4-1}(\pi w^2 * 1 + w^4|B(p,1)|) \\ &= \pi\frac{r^{s-2}}{s-2} + |B(p,1)|\frac{r^s}{s},\end{aligned}$$

which has simple poles at $s = 2$ and $s = 0$, from which we conclude that our set of complex dimensions is $\{0, 2\}$.

The pole at $s = 0$ can be interpreted as the dimension of the endpoints of the line segment. This tells us that a line segment along the t-axis effectively is of dimension 2.

Line segments elsewhere in the plane are much harder to determine the complex dimensions of, due to the twisting nature of the Heisenberg group. However, we conjecture that a line segment entirely along the x or y axes should be of dimension 1, which would then imply a dimensional shift as line segments become "more vertical".

It is also possible that a one-dimensional "line segment" in the Heisenberg group is a curve that accounts for this twist, so that the δ-neighborhood around the curve is easily visualized as a curved cylinder.

3.2 Laakso Spaces

In 2000, Tomi Laakso published a paper describing a construction of spaces now known as Laakso spaces ([16]). These are Q-regular path spaces for any $Q > 1$ that also satisfy a weak (1,1)-Poincaré inequality. This means that, given any ball B in our space,

$$\int_B |u - u_B| d\mu \le C(\mathrm{diam}(B)) \left(\int_{CB} \rho d\mu \right),$$

for u any bounded continuous function on ball CB, and ρ its upper gradient there. Here the upper gradient is defined in the sense of [11] (see also [10]). In particular, ρ is an *upper gradient* of u if ρ is a non-negative Borel measurable function on X, u a real-valued measurable function on X, and

$$|u(x) - u(y)| \le \int_p \rho \, ds$$

whenever p is a rectifiable path connecting x and y. We construct here a simple Laakso space based off of the 1/4-th Cantor set.

We first construct the 1/4-th Cantor set, called K henceforth, viewed as an iterated function system. Our IFS is described by the two maps:

$$\phi_1(x) = \frac{x}{4}, \quad \phi_2(x) = \frac{x}{4} + \frac{3}{4}.$$

K is then the unique nonempty compact set in $\mathbb{R}$ such that $\phi_1(K) \cup \phi_2(K) = K$, with all the usual properties of the Cantor set with a scaling factor of 1/4 (instead of 1/3 for the classic ternary Cantor set). Further, K is an Ahlfors space of dimension $Q_K = \dfrac{\log 4}{\log 2} = \frac{1}{2}$.

The IFS construction is important for us as it gives a unique addressing scheme for any point in K. Specifically, any point can be identified to an infinite binary string $(i_j)_{j=1}^{\infty}$, where

$i_j \in \{1,2\}$. This string represents the composition of maps ϕ_1 and ϕ_2 such that our given point is the limit of compositions, $\lim_{n\to\infty} \phi_{i_n} \circ \cdots \circ \phi_{i_2} \circ \phi_{i_1}([0,1])$. We use the notation $\circ_{j=1}^{\infty}\phi_{i_j}([0,1])$ to represent this limit of compositions.

Using finite binary strings, we can define subsets of K in terms of the different level approximations under the IFS. We define the subset K_a, where a is a finite string $(i_j)_{j=1}^n$, as the set of points in $\circ_{j=1}^{n}\phi_{i_j}(K) = \phi_{i_n} \circ \cdots \circ \phi_{i_2} \circ \phi_{i_1}(K)$. For example, letting $i_1 = 2$ and $i_2 = 1$, then K_{12} is the subset formed by first applying ϕ_2 followed by ϕ_1:

$$K_{12} = \phi_1(\phi_2(K)) = \frac{1}{4}\left(\frac{1}{4}K + \frac{3}{4}\right).$$

The base space for our Laakso space will then be $K \times I$ in $\mathbb{R}^2$, where I is the unit interval $[0,1]$, with the metric induced from $\mathbb{R}^2$. For our eventual quotient space, we restrict ourselves to a fairly simple choice of identification pairs, otherwise known as *wormholes*. Our construction will place wormholes at heights of 1/4-th powers, where 1/4 is equal to the scaling ratio that defines the Cantor set K.

Starting with the first identifications, we define the first level wormhole function

$$\omega(i) = 4^{-1}i, \quad 0 \le i < 4,$$

so that $\omega(0) = 0$, $\omega(1) = 1/4$, and so on. We continue this construction for the second level function defined as

$$\omega(m_1, m_2) = 4^{-1}(m_1 + 4^{-1}m_2), \quad 0 \le m_i < 4,$$

giving 16 points from 0 to 15/16. In general,

$$\omega(m_1, \ldots, m_k) = \sum_{i=1}^{k} m_i \prod_{h=1}^{i} 4^{-1}.$$

Provided $m_k > 0$ to avoid overlap, we call the sets $K \times \omega(m_1, \ldots, m_k)$ in $K \times I$ the *wormhole levels of order* k. Given any finite binary string of length $k-1$, called a, identify $K_{a1} \times \omega(m_1, \ldots, m_k)$ with $K_{a2} \times \omega(m_1, \ldots, m_k)$. Taking all such identifications, the resulting quotient space is our Laakso space F. Denote the natural projection from $K \times I \to F$ by the map s.

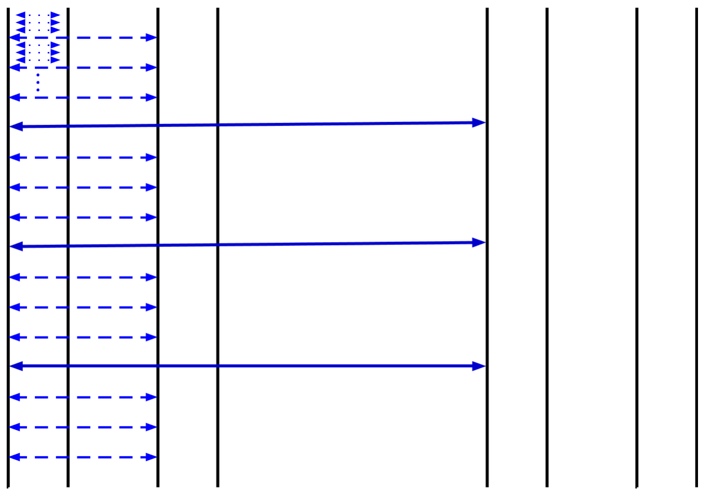


Figure 3.2: A representation of the third-level iterate of our Laakso space, with the left-hand wormholes identified.

The Ahlfors measure on F will be the same as the probability measure on $K \times I$, since only pairs of points are identified at a time. Define a metric on F as

$$d(x, y) := \inf\{H^1(p)|s(p) \text{is a continuous path joining } x \text{ and } y\},$$

where $p \subset K \times I$ and $H^1(p)$ is the one-dimensional Hausdorff measure (or simply, the length) of the path p. As s does not change the vertical distance, define the height of $x \in F$ as $h(x)$. Note that if x, y can be connected by an only upward or only downward path, then

$$d(x, y) = |h(x) - h(y)|.$$

If p is an s-image of a countable number or line segments, then the length of p will simply be the sum of the lengths of the line segments. The following theorem importantly guarantees and classifies the existence of geodesics in F (see [16, Proposition 1.1]).

Theorem 3.1. *Let $[a, b] \subset I$ be a smallest possible interval containing $h(x)$ and $h(y)$ as well as all wormhole levels needed to connect x and y. Assume that $h(x) \leq h(y)$. Let p be any path starting from x, going downward to height a, up to height b, then back down to point y. Then p is a geodesic, and all are of this form.*

The Laakso space can also be defined as a limit of graphs, with identifications made at each step (see [1]).

With this metric and the stated Ahlfors measure, F is thus an Ahlfors space of dimension $Q = 1/2$. With the space defined, we now look at subsets of F and compute their complex dimensions.

Example 3.5. Let us first look at a single point. Let $A = \{(0, 1/4)\}$ and let $\delta = 1/4$ (for an explicit calculuation). Using the tube zeta function, we have

$$\widetilde{\zeta}_A(s) = \int_0^\delta t^{s-1/2}|B_t(A)|\mathrm{d}t,$$

where $B_t(A)$ is the ball of radius t centered at point A, and $|\cdot|$ is the Ahlfors measure.

Notice that for $2/16 \leq r \leq 1/4$, $|B_r(A)| = 2r$, whereas for $1/16 \leq r \leq 2/16$, $|B_r(A)| = \frac{1}{2}2r + \frac{1}{2}2(2(r - \frac{1}{16}))$, due to the interference of the wormholes. Similarly we have for $2/64 \leq r \leq 1/16$, $|B_r(A)| = \frac{1}{2}2r$, and for $1/64 \leq r \leq 2/64$, $|B_r(A)| = \frac{1}{2}(\frac{1}{2}2r) + \frac{1}{2}(\frac{1}{2}2(r - \frac{1}{64}))$, with a similar pattern continuing as r goes to 0.

Thus we can solve for $\widetilde{\zeta}_A$ as follows:

$$\begin{aligned}
\int_0^\delta t^{s-1/2}|B_t(A)|\mathrm{d}t &= \sum_{k=1}^{\infty} \int_{2/4^{k+1}}^{1/4^k} t^{s-1/2}2^{-k+1}2t\mathrm{d}t + \sum_{k=1}^{\infty} \int_{1/4^{k+1}}^{2/4^{k+1}} t^{s-1/2}2^{-k}(6t - 4^{-k})\mathrm{d}t \\
&= \sum_{k=1}^{\infty} \frac{2^{-k+2}}{s-1/2}\left[\left(\frac{1}{4^k}\right)^{s-1/2} - \left(\frac{2}{4^{k+1}}\right)^{s-1/2}\right] \\
&\quad + 2^{-k}\frac{6}{s-1/2}\left[\left(\frac{2}{4^{k+1}}\right)^{s-1/2} - \left(\frac{1}{4^{k+1}}\right)^{s-1/2}\right] \\
&\quad - 2^{-3k}\frac{1}{s-3/2}\left[\left(\frac{2}{4^{k+1}}\right)^{s-3/2} - \left(\frac{1}{4^{k+1}}\right)^{s-3/2}\right],
\end{aligned}$$

where in the second steps we merged the series into one as they were absolutely convergent. After some expansion and factoring, we can rewrite this entire series as:

$$\left(\frac{4(1-2^{-s+1/2})}{s-1/2} + \frac{6(2^{-s+1/2})(1-2^{-s+1/2})}{s-1/2} - \frac{2^{-s+3/2}(1-2^{-s+3/2})}{s-3/2}\right)\sum_{k=1}^{\infty}(2^{-2s})^k.$$

There is a removable singularity that occurs as $s = 1/2$ in the first two fractions, and a removable singularity at $s = 3/2$ is the third. The geometric series we meromorphically extend to $\mathbb{C}$, using its

Figure 3.3: A representation of the $t = \frac{1}{12}$ neighborhood around the point $(0, \frac{1}{4})$.

formula of convergence:

$$\sum_{k=1}^{\infty} (2^{-2s})^k = \frac{2^{-2s}}{1-2^{-2s}}.$$

The poles thus occur when $1 - 2^{-2s} = 0$, which gives as a set the complex dimensions

$$\mathcal{D} = \{0 + \frac{\pi i n}{\log 2} : n \in \mathbb{Z}\}.$$

This tells us that wormhole points, while zero dimensional, have the same complex dimensions as those of the 1/4th Cantor set. These complex dimensions capture the geometric oscillations occurring in the neighborhood around the point, which can be pictured as "branching paths" across the different wormholes. These can be viewed as the splitting point in the integrals above, where the neighborhood A_t represents a collection full line segments for t ranging from $1/4^k$ to $2/4^{k+1}$, and then as t gets smaller, certain segments of our neighborhood "split from the center" and disappear. See Figure 3.3 for an example of such a neighborhood.

Since the set of wormholes are dense in our space, we expect any point to have the same complex dimensions.

Example 3.6. We now let $A = \{0\} \times K$, or the 1/4th-Cantor set that lies on the vertical interval I, and $\delta = 1/4$. Note that for δ between $1/4^k$ and $1/4^{k+1}$, we can view the neighborhood around A as if we were at the k-th step of construction of the 1/4-Cantor set. For example, if $k = 1$ and δ in the above range, we are essentially viewing the neighborhood for the set $\{(0, y) : y \in [0, 1/4] \cup [3/4, 1]\}$, since δ is large enough to make this set and $\{0\} \times K$ indistinguishable in terms of our zeta function. In other words, at the kth level of the 1/4 Cantor set construction, we have 2^k "full intervals" of length 4^{-k}, plus $2^k - 1$ replicas of example 3.5, since every two endpoints of the k-level approximation together give one side of a point neighborhood. The tube zeta function is thus

$$
\begin{aligned}
\tilde{\zeta_A}(s) &= \int_0^{1/4} t^{s-3/2-1}|A_t|\mathrm{d}t \\
&= \sum_{k=1}^{\infty} \int_{2/4^{k+1}}^{1/4^k} t^{s-3/2-1} 2^{-k+1} \left((2^k-1)2t + 2^k(4)^{-k}\right)\ \mathrm{d}t \\
&+ \sum_{k=1}^{\infty} \int_{1/4^{k+1}}^{2/4^{k+1}} t^{s-3/2-1} 2^{-k+1} \left((2^k-1)(2t+4t-4^{-k}) + 2^k(4)^{-k}\right)\ dt \\
&= 2\sum_{k=1}^{\infty} \left\{ \frac{2}{s-1/2}\left[\left(\frac{1}{4^k}\right)^{s-1/2} - \left(\frac{2}{4^{k+1}}\right)^{s-1/2}\right]\right. \\
&+ \frac{6}{s-1/2}\left[\left(\frac{2}{4^{k+1}}\right)^{s-1/2} - \left(\frac{1}{4^{k+1}}\right)^{s-1/2}\right] \\
&\left. - \frac{4^k}{s-3/2}\left[\left(\frac{2}{4^{k+1}}\right)^{s-3/2} - \left(\frac{1}{4^{k+1}}\right)^{s-3/2}\right]\right\} \\
&+ \sum_{k=1}^{\infty} \frac{2^{-2k+1}}{s-3/2}\left[\left(\frac{1}{4^k}\right)^{s-3/2} - \left(\frac{1}{4^{k+1}}\right)^{s-3/2}\right] \\
&- \int_{(0,1/4)_\delta} t^{s-3/2-1}|(0,1/4)|\mathrm{d}t.
\end{aligned}
$$

The first series is example 3.5, multiplied by 2^k. Thus we can write this first series as

$$
2\left(\frac{4(1-2^{-s+1/2})}{s-1/2} + \frac{6(2^{-s+1/2})(1-2^{-s+1/2})}{s-1/2} - \frac{2^{-s+3/2}(1-2^{-s+3/2})}{s-3/2}\right)\sum_{k=1}^{\infty}(2^{-2s+1})^k.
$$

As before, we have removable singularities at $s = 1/2$ and $s = 3/2$, but now our geometric series meromorphically extends as

$$
\sum_{k=1}^{\infty}(2^{-2s+1})^k = \frac{2^{-2s}}{1-2^{-2s+1}},
$$

which has poles at $s = 1/2 + \pi i n/\log 2$, for any integer n. The second series (separated here only for emphasis) can be written as

$$
\sum_{k=1}^{\infty} \frac{2^{-2k+1}}{s-3/2}\left[\left(\frac{1}{4^k}\right)^{s-3/2} - \left(\frac{1}{4^{k+1}}\right)^{s-3/2}\right] = \frac{1-2^{2s-3}}{s-3/2}\sum_{k=1}^{\infty}(2^{-2s+1})^k.
$$

Again, we have a removable singularity at $s = 3/2$, and the same series as above shows itself. Finally, we subtract off exactly Example 3.5, and so the same poles will occur. Thus we see that the complex dimensions of A are

$$
\mathcal{D} = \left\{0 + \frac{\pi i n}{\log 2}, \frac{1}{2} + \frac{\pi i n}{\log 2} : n \in \mathbb{Z}\right\}.
$$

Thus we recapture the dimensions of the 1/4th Cantor set itself, as well as the complex dimensions at each of the wormhole points.

Chapter 4

Fractal Tube Formulas

In this chapter, we reconstruct information about the geometry of relative fractal drums from their associated fractal zeta functions. The ideas and proofs are largely similar to the Euclidean case, first obtained for fractal strings (see [17], chapters 5 and 8), and then developed for general higher dimensional sets in [19], Chapter 5 (see also [24]).

The rough idea is that we can obtain an asymptotic formula for the relative tube function $t \mapsto |A_t \cap \Omega|$ as $t \to 0^+$, which is expressed as a sum of the residues of the (suitably modified) fractal zeta functions, taken over the complex dimensions of A. In this chapter we present the pointwise formulas, although conjecture that the distributional formulas should still hold true in Ahlfors regular spaces, as is the case in Euclidean spaces [19, Chapter 5].

We begin by defining *screens* and *windows*, which we use to define a more rigid definition of *admissible* relative fractal drums than was used in Chapter 2. In particular, it is necessary to meromorphically extend our distance or tube zeta function into a domain left of $\{\mathrm{Re}\ s = \overline{\dim}_B A\}$, with the extra condition that we can define a screen as follows:

Definition 4.1. The *screen* $\mathbf{S}$ is the graph of a bounded, real-valued Lipshitz continuous function $S(\tau)$, with the horizontal and vertical axes interchanged:

$$\mathbf{S} := \{S(\tau) + \mathrm{i}\tau : \tau \in \mathbb{R}\}.$$

The Lipschitz constant of S is denoted by $\|S\|_{\mathrm{Lip}}$: so that

$$|S(x) - S(y)| \leq \|S\|_{\mathrm{Lip}}|x - y|, \quad \text{for all } x, y \in \mathbb{R}.$$

Furthermore, the following quantities are associated to the screen:

$$\inf S := \inf_{\tau\in\mathbb{R}} S(\tau) \quad \text{and} \quad \sup S := \sup_{\tau\in\mathbb{R}} S(\tau).$$

We will always assume that the screen S lies to the left of the *critical line* $\{\mathrm{Re}\ s = \overline{\dim}_B(A, \Omega)\}$ and that no poles lie on the screen. We shall also assume that $\inf S > -\infty$.

The *window* $\mathbf{W}$ is defined as the part of the complex plane to the right of S; that is,

$$\mathbf{W} := \{s \in \mathbb{C} : \mathrm{Re}\ s \geq S(\mathrm{Im}\ s)\}.$$

We say that the relative fractal drum (A, Ω) is *admissible* if its relative tube (or distance) zeta function can be meromorphically extended (necessarily uniquely) to an open connected neighborhood of some window W.

The screen thus ensures that the boundary of the region of meromorphic extension we view is at least Lipschitz continuous. We will also need the following technical growth conditions on the tube zeta function:

Definition 4.2. An admissible relative fractal drum (A, Ω) in $\mathcal{E}$ is said to be *languid* if for some fixed $\delta > 0$, its tube zeta function $\widetilde{\zeta}_A(\cdot, \Omega; \delta)$ satisfies the following growth conditions:

There exists a real constant κ and a two-sided sequence $(T_n)_{n\in\mathbb{Z}}$ of real numbers such that $T_{-n} < 0 < T_n$ for all $n \geq 1$ and

$$\lim_{n\to\infty} T_n = +\infty, \quad \lim_{n\to\infty} T_{-n} = -\infty$$

satisfying the following two hyoptheses:

L1 For a fixed real constant $c > \overline{\dim}_B(A, \Omega)$, there exists a positive constant $C > 0$ such that for all $n \in \mathbb{Z}$ and all $\sigma \in (S(T_n), c)$,

$$|\widetilde{\zeta}_A(\sigma + \mathrm{i}T_n, \Omega; \delta)| \leq C(|T_n| + 1)^\kappa.$$

L2 For all $\tau \in \mathbb{R}, |\tau| \geq 1$,

$$|\widetilde{\zeta}_A(S(\tau) + \mathrm{i}\tau, \Omega; \delta)| \leq C|\tau|^\kappa,$$

where C is a positive constant which can be chosen to be the same one as in condition **L1**.

We can view **L1** as a polynomial growth condition along horizontal segments (which do not pass through singularities), while **L2** is polynomial growth condition along the vertical direction of the screen. These hypotheses will be necessary to establish the pointwise tube formulas with error term.

In order to find the exact pointwise tube formula, without error term, we will need a stronger languidity condition:

Definition 4.3. We say that an admissible relative fractal drum (A, Ω) in $\mathcal{E}$ is *strongly languid* if for some $\delta > 0$, its tube zeta function satisfies condtion **L1** with $S(T_n)$ replaced by $-\infty$ (i.e. for every $\sigma < c$), with the additional assumption that there exists a sequence of screens $\mathbf{S_m} : \tau \mapsto S_m(\tau) + \mathrm{i}\tau$ for $m \geq 1, \tau \in \mathbb{R}$ with $\sup S_m \to -\infty$ as $m \to \infty$ and with a uniform Lipshitz bound, $\sup_{m\geq 1} \|S_m\|_{\mathrm{Lip}} < \infty$, such that the following condition holds:

L2' There exists constants $B, C > 0$ such that for all $\tau \in \mathbb{R}$ and $m \geq 1$,

$$|\widetilde{\zeta}_A(S_m(\tau) + \mathrm{i}\tau, \Omega; \delta)| \leq CB^{|S_m(\tau)|}(|\tau| + 1)^\kappa.$$

As before, δ is non essential for the languidity conditions, although changing δ may change the languidity exponent κ_δ. The next proposition covers the details, whose proof is similar to the Euclidean case ([19, Proposition 5.1.5]).

Proposition 4.1. Let (A, Ω) be a relative fractal drum in $\mathcal{E}$. If the relative tube zeta function $\widetilde{\zeta}_A(\cdot, \Omega; \delta)$ satisfies the languidity conditions **L1** and **L2** for some $\delta > 0$ and $\kappa \in \mathbb{R}$, then so does $\widetilde{\zeta}_A(\cdot, \Omega; \delta_1)$ for any $\delta_1 > 0$ and with $\kappa_{de_1} := \max\{\kappa, 0\}$.

Futhermore, the analogous statement is also true in the case when $\widetilde{\zeta}_A(\cdot, \Omega, \delta)$ is strongly languid, under the additional assumption that $\delta \geq 1$ and $\delta_1 \geq 1$.

Proof. Without loss of generality, we may assume that $\delta < \delta_1$. Then the conclusion follows from the fact that $\widetilde{\zeta}_A(\cdot, \Omega; \delta_1) = \widetilde{\zeta}_A(\cdot, \Omega; \delta) + f(s)$, where f is entire and

$$|f(s)| \leq \int_\delta^{\delta_1} t^{\mathrm{Re}\ s-Q-1} |A_t \cap \Omega| \mathrm{d}t \leq \begin{cases} |\Omega| \dfrac{\delta_1^{\mathrm{Re}\ s-Q} - \delta^{\mathrm{Re}\ s-Q}}{\mathrm{Re}\ s - Q} & \mathrm{Re}\ s \neq Q \\ |\Omega|(\log \delta_1 - \log \delta) & \mathrm{Re}\ s = Q \end{cases}.$$

Since the upper bound on $|f(s)|$ does not depend on Im s, we conclude that f satisfies the languidity conditions **L1** and **L2** with the languidity exponent $\kappa_f = 0$ and for any given window W. This observation implies that $\widetilde{\zeta}_A(\cdot, \Omega; \delta_1)$ is languid for $\kappa_{\delta_1} = \max\{\kappa, 0\}$ with the same window as $\widetilde{\zeta}_A(\cdot, \Omega; \delta)$.

The additional assumption for strong languidity is needed since **L1** must then be satisfied for all $\sigma \in (-\infty, c)$, and for this to be achieved we need that $\delta_1 > \delta \geq 1$, since otherwise we do not have an upper bound on $|f(s)|$ when Re $s \to -\infty$. $\square$

4.1 Pointwise Fractal Tube Formulas

In this section, we obtain fractal tube formulas via the tube zeta function which are valid pointwise. The results in this section will be used in the following sections to develop tube formulas via the distance zeta function, which is important since the distance zeta function can be calculated without knowing the tubular volume.

As motivation for the approach used below, we note that the tube zeta function coincides with the Mellin transform of a modification of the tube function $t \mapsto |A_t \cap \Omega|$. Specifically, one has for all $s \in \mathbb{C}$ such that Re $s > \overline{\dim}_B(A, \Omega)$,

$$\widetilde{\zeta}_A(s, \Omega; \delta) = \int_0^{\infty} t^{s-1}(\chi_{(0,\delta)}(t)t^{-Q}|A_t \cap \Omega|)\mathrm{d}t. \tag{4.4}$$

Theorem 4.2. ***(Mellin's inversion theorem)***, *[29, Theorem 28]. Let $f : (0, +\infty) \to \mathbb{R}$ be such that for a given $y > 0$, $f(t)$ is of bounded variation in a neighborhood of the point $t = y$. Furthermore, assume that $t \mapsto t^{c-1}f(t)$ belongs to $L^1(0, +\infty)$, where c is a real number, and define*

$$\{\mathfrak{M}f\}(s) := \int_0^{+\infty} t^{s-1}f(t)\,dt$$

for all $s \in \mathbb{C}$ such that Re $s = c$. Then, for the above value of y, the following inversion formula holds:

$$\frac{1}{2}(f(y+0) + f(y-0)) = \frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} y^{-s}\{\mathfrak{M}f\}(s)\,ds,$$

where $f(y+0$ and $f(y-0)$ denote, respectively, the right and left limits of f at y. Here, on the right-hand side, the contour integral is taken over the vertical line $\{Re\ s = c\}$.

We can now relate the relative tube function of the RFD (A, Ω) and the tube zeta function $\widetilde{\zeta}_{A,\Omega}(\cdot; \delta)$ through the following integral (analog to [19, Theorem 5.1.7]):

Theorem 4.3. *Let (A, Ω) be a relative fractal drum in $\mathcal{E}$ and fix $\delta > 0$. Then, for any fixed $c > \overline{dim}_B(A, \Omega)$ and for every $t \in (0, \delta)$, we have*

$$|A_t \cap \Omega| = \frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} t^{Q-s}\widetilde{\zeta}_A(s, \Omega; \delta)\,ds.$$

Proof. Let$f(t) := \chi_{(0,\delta)}(t)t^{-Q}|A_t \cap \Omega|$ and observe that $t \mapsto |A_t \cap \Omega|$ is nondecreasing, and hence, is locally of bounded variation on $(0, +\infty)$. Since the product of two functions of locally bounded variation is also a function of locally bounded variation, we conclude that f is also locally of bounded variation on $(0, +\infty)$. Furthermore, we deduce from Theorem 2.13 and from the functional equality that the integral defining the tube zeta function $\widetilde{\zeta}(\cdot, \Omega)$ in Equation 4.4 is absolutely convergent for all $s \in \mathbb{C}$ such that Re $s > \overline{\dim}_B(A, \Omega)$. In other words, $t \mapsto t^{\mathrm{Re}\ s-1}f(t)$ belongs to $L^1(0, +\infty)$ for such s. Consequently, the Mellin transform $\{\mathfrak{M}f\}(s)$ of f is well defined and coincides with

$\widetilde{\zeta}_A(s,\Omega;\delta)$ for $c = \operatorname{Re} s > \overline{\dim}_B(A,\Omega)$. Therefore, by Mellin's inversion theorem, we can recover the relative tube function from the relative tube zeta function and for positive $y \neq \delta$, we have

$$\chi_{(0,\delta)}(y)y^{-Q}|A_y \cap \Omega| = \frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} y^{-s}\widetilde{\zeta}_A(s,\Omega;\delta)\mathrm{d}s,$$

where $c > \overline{\dim}_B(A,\Omega)$ is arbitrary. Moving $\chi_{(0,\delta)}(y)y^{-Q}$ to the other side proves our result. $\square$

We wish to represent the integral on the right-hand side in a more useful manner, which will be the goal of the rest of this chapter. In particular, as the stated sum of residues over the complex dimensions of the given RFD.

Before moving forward, we will introduce the definition of the k-th primitive of an RFD, as well as recall the definition of the Pochammer symbol.

Definition 4.5. Given a relative fractal drum (A,Ω), we define the k-th primitive (or k-th anti-derivative) function, $V^{[k]} = V^{[k]}(t)$, of the relative tube function $V = V(t)$ as follows:

$$V(t) = V_{A,\Omega}(t) = V^{[0]}(t) := |A_t \cap \Omega|,$$

$$V^{[k]}(t) = V^{[k]}_{A,\Omega}(t) := \int_0^t V^{[k-1]}(\tau)\mathrm{d}\tau, \quad \text{for each } k \in \mathbb{N}.$$

Definition 4.6. For any $s \in \mathbb{C}$, the *Pochammer symbol* is defined by

$$(s)_0 := 1, \quad (s)_k := s(s+1)\cdots(s+k-1)$$

for any nonnegative integer k. More generally, for any $k \in \mathbb{Z}$ we have

$$(s)_k := \frac{\Gamma(s+k)}{\Gamma(s)},$$

where Γ denotes the gamma function.

With these definitions, we can obtain the following proposition (analog to [19, Proposition 5.1.8]), based on the last theorem:

Proposition 4.4. Let (A, Ω) be a relative fractal drum in $\mathcal{E}$ and let $\delta > 0$ be fixed. Then for every $t \in (0, \delta)$ and $k \in \mathbb{Q}_0$, we have

$$V_{A,\Omega}^{[k]}(t) = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{t^{Q-s+k}}{(Q-s+1)_k} \widetilde{\zeta}_A(s, \Omega; \delta) \mathrm{d}s,$$

where $c \in (\overline{\dim}_B(A, \Omega), Q+1)$ is arbitrary.

Proof. By the previous theorem, we have the following equalities, valid pointwise for all $t \in (0, \delta)$:

$$\begin{aligned} V_{A,\Omega}^{[1]}(t) = \int_0^t V_{A,\Omega}(\tau) \mathrm{d}\tau &= \frac{1}{2\pi \mathrm{i}} \int_0^t \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \tau^{Q-s} \widetilde{\zeta}_A(s, \Omega; \delta) \mathrm{d}s \mathrm{d}\tau \\ &= \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \int_0^t \tau^{Q-s} \widetilde{\zeta}_A(s, \Omega; \delta) \mathrm{d}\tau \mathrm{d}s \\ &= \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{t^{Q-s+1}}{Q-s+1} \widetilde{\zeta}_A(s, \Omega; \delta) \mathrm{d}s, \end{aligned}$$

since $Q - c + 1 > 0$. The change of the order of integration is justified by Lebesgue's dominated convergence (since $V_{A,\Omega}(\tau)$ is non-decreasing and finite at $\tau = t$, we can dominate by the constant $V_{A,\Omega}(t)$) and the Fubini-Tonelli theorem. Iterating $k - 1$ more times proves the statement. $\square$

The following definition of the truncated screen and window follows directly from the Euclidean case:

Definition 4.7. Given an integer $n \geq 1$ and a languid relative fractal drum in $\mathcal{E}$, the *truncated screen* $\mathbf{S}_{|n}$ is the part of the screen $\mathbf{S}$ restricted to the interval $[T_{-n}, T_n]$, and the *truncated window* $\mathbf{W}_{|n}$ is the window $\mathbf{W}$ intersected with the horizontal strip between T_{-n} and T_n;

$$\mathbf{W}_{|n} := \mathbf{W} \cap \{s \in \mathbb{C} : T_{-n} \leq \text{Im } s \leq T_n\}.$$

We call $\mathscr{P}(\widetilde{\zeta}_A(\cdot, \Omega), W_{|n})$ the set of *truncated visible complex dimensions*, i.e., it is the set of visible complex dimensions of (A, Ω) relative to the window W and with imaginary parts between T_{-n} and T_n.

We use the truncated screen and window to first prove a truncated pointwise formula (the analog of [19, Lemma 5.1.10]), from which the full general pointwise formula will follow.

Lemma 4.5. ***(Truncated pointwise tube formula).*** *Let $k \geq 0$ be an integer and (A, Ω) a languid relative fractal drum in $\mathcal{E}$ for a fixed $\delta > 0$. Furthermore, fix a constant $c \in (\overline{dim}_B(A,\Omega), Q+1)$. Then, for all $t \in (0,\delta)$ and all $n \geq 1$, we have*

$$\begin{aligned} I_n &:= \frac{1}{2\pi\mathrm{i}} \int_{c+\mathrm{i}T_{-n}}^{c+\mathrm{i}T_n} \frac{t^{Q-s+k}}{(Q-s+1)_k} \widetilde{\zeta}_A(s,\Omega;\delta)\, ds \\ &= \sum_{\omega \in \mathscr{P}(\widetilde{\zeta}_A(\cdot,\Omega,W_{|n})} \operatorname{res}\left(\frac{t^{Q-s+k}}{(Q-s+1)_k} \widetilde{\zeta}_A(s,\Omega;\delta), \omega \right) \\ &+ \frac{1}{2\pi\mathrm{i}} \int_{S_{|n}} \frac{t^{Q-s+k}}{(Q-s+1)_k} \widetilde{\zeta}_A(s,\Omega;\delta)\, ds + E_n(t). \end{aligned}$$

Moreover, assuming that hypothesis L1 *is fulfilled, we have the following pointwise remainder estimate, valid for all $t \in (0,\delta)$:*

$$|E_n(t)| \leq t^{Q+k} K_\kappa \max\{T_n^{\kappa-k}, |T_{-n}|^{\kappa-k}\}(c - \inf S) \max\{t^{-c}, t^{\inf S}\},$$

where K_κ is a positive constant depending only on κ.

Finally, for each point $s = S(\tau + \mathrm{i}\tau$, where $\tau \in \mathbb{R}$ is such that $|\tau| > 1$, and for all $t \in (0,\delta)$, the integrand over the truncated screen appearing in I_n above is bounded in absolute value by

$$Ct^{Q+k} \max\{t^{-\sup S}, t^{-\inf S}\}|\tau|^{\kappa-k},$$

when hypothesis L2 *holds, and by*

$$C_\kappa t^{Q+k} \max\{B^{|\inf S|}, B^{|\sup S|}\} \max\{t^{-\sup S}, t^{-\inf S}\}|\tau|^{\kappa-k},$$

when hypothesis L2' *holds, with the constant C_κ depending only on κ.*

Proof. Let $\overline{D} := \overline{\dim}_B(A,\Omega)$ and $\widetilde{\zeta}_{A,\Omega}(s) := \widetilde{\zeta}_A(s,\Omega;\delta)$ throughout the proof. Qow we replace the integral over the segment $[c + \mathrm{i}T_{-n}, c + \mathrm{i}T_n]$ with the integral over the contour Γ consisting of this segment, the truncated screen $S_{|n}$ and the two horizontal segments joining $S(T_{\pm n}) + \mathrm{i}T_{\pm n}$ and

$c + \mathrm{i}T_{\pm n}$. In other words, we have:

$$\begin{aligned} I_n &= \frac{1}{2\pi \mathrm{i}} \int_{c+\mathrm{i}T_{-n}}^{c+\mathrm{i}T_n} \frac{t^{Q-s+k}}{(Q-s+1)_k} \widetilde{\zeta}_{A,\Omega} \mathrm{d}s \\ &= \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma} \frac{t^{Q-s+k}}{(Q-s+1)_k} \widetilde{\zeta}_{A,\Omega} \mathrm{d}s \\ &+ \frac{1}{2\pi \mathrm{i}} \int_{S_{|n}} \frac{t^{Q-s+k}}{(Q-s+1)_k} \widetilde{\zeta}_{A,\Omega} \mathrm{d}s + E_n(t), \end{aligned}$$

where

$$E_n(t) := \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_L \cup \Gamma_U} \frac{t^{Q-s+k}}{(Q-s+1)_k} \widetilde{\zeta}_{A,\Omega} \mathrm{d}s.$$

Furthermore, the integrand appearing above is meromorphic on the bounded domain having Γ as its boundary and its poles are exactly the poles of the relative tube zeta function since $c \in (\overline{D}, Q+1)$ ensures that there are no zeroes of $(Q-s+1)_k$ inside of Γ. Consequently, we deduce from the residue theorem that

$$\begin{aligned} I_n = & \sum_{\omega \in \mathscr{P}(\widetilde{\zeta}_A(\cdot, \Omega, W_{|n})} \operatorname{res}\left(\frac{t^{Q-s+k}}{(Q-s+1)_k} \widetilde{\zeta}_A(s, \Omega; \delta), \omega \right) \\ &+ \frac{1}{2\pi \mathrm{i}} \int_{S_{|n}} \frac{t^{Q-s+k}}{(Q-s+1)_k} \widetilde{\zeta}_{A,\Omega} \mathrm{d}s + E_n(t). \end{aligned}$$

To obtain the upper bound on$|E_n(t)|$, we first observe that for $s = \sigma + \mathrm{i}T_n$ we have $|(Q-s+1)_k| \geq T_n^k$ and we estimate the integrals over the upper segment Γ_U and the lower segment Γ_L under hypothesis **L1**:

$$\begin{aligned} \left| \int_{\Gamma_U} \frac{t^{Q-s+k}}{(Q-s+1)_k} \widetilde{\zeta}_{A,\Omega}(s) \mathrm{d}s \right| &= \left| \int_{S(T_n)}^{c} \frac{t^{Q+k-\sigma-\mathrm{i}T_n}}{(Q+1-(\sigma+\mathrm{i}T_n))_k} \widetilde{\zeta}_{A,\Omega}(\sigma + \mathrm{i}T_n) \mathrm{d}\sigma \right| \\ &\leq t^{Q+k} C(T_n+1)^{\kappa} T_n^{-k} \int_{S(T_n)}^{c} t^{-\sigma} \mathrm{d}\sigma \\ &\leq T^{Q+k} K_{\kappa} T_n^{\kappa-k} (c - S(T_n)) \max\{t^{-c}, t^{-S(T_n)}\}, \end{aligned}$$

where K_κ is a positive constant such that $C(|T_n|+1)^\kappa \leq K_\kappa |T_n|^\kappa$ for all $n \in \mathbb{Z}$. Furthermore, since $\inf S \leq S(\tau)$ for all $\tau \in \mathbb{R}$, we have

$$\left| \int_{\Gamma_U} \frac{T^{Q-s+k} \widetilde{\zeta}_{A,\Omega}(s) \mathrm{d}s}{(Q-s+1)_k} \right| \leq t^{Q+k} K_{\kappa} T_n^{\kappa-k} (c - \inf S) \max\{t^{-c}, t^{-\inf S}\}.$$

A similar calculation for the integral over the lower line segment yields

$$\left| \int_{\Gamma_L} \frac{T^{Q-s+k}\widetilde{\zeta}_{A,\Omega}(s)\mathrm{d}s}{(Q-s+1)_k} \right| \leq t^{Q+k} K_\kappa |T_{-n}|^{\kappa-k}(c - \inf S)\max\{t^{-c}, t^{-\inf S}\}.$$

Putting these estimates together gives us our upper bound on $|E_n|$.

In order to estimate the integrand over the truncated screen $S_{|n}$, we observe that for $s = S(\tau) + \mathrm{i}\tau$ with $|\tau| > 1$, we have

$$\begin{aligned} \left| \frac{t^{Q-s+k}}{(Q-s+1)_k} \widetilde{\zeta}_{A,\Omega}(s) \right| &\leq C t^{Q-S(\tau)+k} |\tau|^{\kappa-k} \\ &\leq C t^{Q+k} \max\{t^{-\sup S}, t^{-\inf S}\} |\tau|^{\kappa-k}, \end{aligned}$$

under hypothesis **L2** or **L2'** (with C_κ a constant such that $C(|\tau|+1)^\kappa \leq C_\kappa |\tau|^\kappa$ for all τ such that $|\tau| > 1$. This completes the proof of the lemma.

□

We can now state the pointwise tube formula with error. It is the precise counterpart of [19, Theorem 5.1.11]; see also [24]. Note that the sum is understood to be a limit over the truncated windows, $\mathbf{W}_{|n}$. The proof establishes that the series converges pointwise and conditionally, but does not establish anything about absolute convergence.

Theorem 4.6. ***(Pointwise fractal tube formula with error term, via $\widetilde{\zeta}_{A,\Omega}$).*** *Let (A, Ω) be a relative fractal drum in $\mathcal{E}$ which is languid for some fixed $\delta > 0$ and some fixed exponent $\kappa \in \mathbb{R}$. Furthermore, let $k > \kappa + 1$ be a nonnegative integer. Then, the following pointwise fractal tube formula with error term, expressed in terms of the tube zeta function $\widetilde{\zeta}_{A,\Omega} := \widetilde{\zeta}_A(\cdot, \Omega; \delta)$, is valid for every $t \in (0, \delta)$:*

$$V_{A,\Omega}^{[k]}(t) = \sum_{\omega \in \mathscr{P}(\widetilde{\zeta}_{A,\Omega}, W)} \operatorname{res}\left(\frac{t^{Q-s+k}}{(Q-s+1)_k} \widetilde{\zeta}_{A,\Omega}(s), \omega \right) + \tilde{R}_{A,\Omega}^{[k]}(t).$$

Here, for every $t \in (0, \delta)$, the (pointwise) error term $\tilde{R}_{A,\Omega}^{[k]}$ is given by the absolutely convergent integral

$$\tilde{R}_{A,\Omega}^{[k]}(t) = \frac{1}{2\pi\mathrm{i}} \int_S \frac{t^{Q-s+k}}{(Q-s+1)_k} \widetilde{\zeta}_{A,\Omega}(s)\mathrm{d}s. \tag{4.8}$$

Furthermore, we have the following pointwise error estimate, valid for all $t \in (0, \delta)$*:*

$$|\tilde{R}^{[k]}_{A,\Omega}(t)| \leq \frac{C}{2\pi}(1 + \|S\|_{\mathrm{Lip}}) \frac{t^{Q+k} \max\{t^{-supS}, t^{-\inf S}\}}{k - \kappa - 1} + C',$$

where C *is the positive constant appearing in* ***L1*** *and* ***L2*** *and* C' *is some suitable positive constant. These constants depend only on the relative fractal drum* (A, Ω) *and the screen, but not on the value of the nonnnegative integer* k*.*

In particular, we have the following pointwise error estimate:

$$\tilde{R}^{[k]}_{A,\Omega}(t) = O(t^{Q - \sup s + k}) \quad \text{as} \quad t \to 0^+.$$

Moreover, if $S(\tau) < \sup S$ *for all* $\tau \in \mathbb{R}$*, then we have the following stronger pointwise estimate:*

$$\tilde{R}^{[k]}_{A,\Omega}(t) = o(t^{Q - \sup S + k}) \quad \text{as} \quad t \to 0^+.$$

Proof. Without loss of generality, let $c \in (\overline{\dim}_B(A, \Omega), Q + 1)$ be the constant from the languidity condition **L1**. We will prove the theorem by using the representation obtained in the lemma above and then letting $n \to \infty$. We note that $E_n(t)$ tends to zero for $k > \kappa$ at the rate of some negative power of $\min\{T_n, |T_{-n}|\}$. Futhermore, for $k > \kappa + 1$, the error term $\tilde{R}^{[k]}(t)_{A,\Omega}$ is absolutely convergent. Indeed, note that, since $\tau \mapsto S(\tau)$ is Lipschitz continuous, it is differentiable almost everywhere and, consequently, the derivative of $\tau \mapsto S(\tau) + \mathrm{i}\tau$ is bounded by $(1 + \|S\|_{\mathrm{Lip}})$ for almost all $\tau \in \mathbb{R}$. Moreover, since

$$\int_1^\infty \tau^{\kappa - k} \mathrm{d}\tau = \frac{1}{k - \kappa - 1}$$

for $k > \kappa + 1$, the upper bound on the error term $\tilde{R}^{[k]}_{A,\Omega}(t)$ now follows from the lemma. The positive constant C' in the upper bound is the constant which corresponds to the integral over the part of the screen for which $|\tau| < 1$;i.e.,

$$C' := \frac{1}{2\pi} \left| \int_{S \cap \{\mathrm{Im}\ S| < 1\}} \frac{t^{Q-s+k}}{(Q - s + 1)_k} \widetilde{\zeta}_{A,\Omega}(s) \mathrm{d}s \right|.$$

In the case when the screen stays strictly to the left of the line $\{\mathrm{Re}\ s = \sup S\}$, we can obtain the better estimate using a well-known estimate [12]. Namely, for any given $\varepsilon > 0$, we have to show that the integral defining $\tilde{R}^{[k]}_{A,\Omega}(t)$ is bounded by $\varepsilon t^{Q-\sup S+k}$. For a given $T>0$, we can split this integral into the following two parts. The first one, is the integral over the part of the screen for which $|\mathrm{Im}\ S|>T$ and the second one is the integral over the part of the screen for which $|\mathrm{Im}\ S|\leq T$. Since the first integral is absolutely convergent, we can choose T sufficiently large so that it is bounded by $\frac{1}{2}\varepsilon t^{Q-\sup S+k}$. For the second integral, we observe that the maximum of $S(\tau)$ for $\tau\in[-T,T]$ is strictly less than $\sup S$; i.e., we can choose $\alpha>0$ such that $S(\tau)<\sup S-\alpha$ for $\tau\in[-T,T]$. This implies that the integral over the part of the screen for which $|\mathrm{Im}\ S|\leq T$ is of order $O(t^{Q-\sup S+k+\alpha})$ as $t\to 0^+$. Hence, for all sufficiently small $t>0$ it is bounded by $\frac{1}{2}\varepsilon t^{Q-\sup S+k}$. This proves that $\tilde{R}^{[k]}_{A,\Omega}(t)=o(t^{Q-\sup S+k})$ as $t\to 0^+$, as desired, and therefore completes the proof of the theorem.

□

If the given RFD is strongly languid, then we can establish a pointwise tube formula without error term. The following theorem, which is the complement of the result in [19, Theorem 5.1.13] and [24], establishes the exact pointwise formula:

Theorem 4.7. ***Exact pointwise fractal tube formula via*** $\widetilde{\zeta}_{A,\Omega}$***.*** *Let (A,Ω) be a relative fractal drum in $\mathcal{E}$ which is strongly languid for some fixed $\delta>0$ and some fixed exponenet $\kappa\in\mathbb{R}$. Furthermore, let $k>\kappa$ be a nonnegative integer. Then, the following exact pointwise fractal tube formula, expressed in terms of the tube zeta function $\widetilde{\zeta}_{A,\Omega}:=\widetilde{\zeta}_a(\cdot,\Omega;\delta)$, holds for all $t\in(0,\min\{1,\delta,B^{-1}\})$:*

$$V^{[k]}_{A,\Omega}(t)=\sum_{\omega\in\mathscr{P}(\widetilde{\zeta}_{A,\Omega},\mathbb{C})} res\left(\frac{t^{Q-s+k}}{(Q-s+1)_k}\widetilde{\zeta}_{A,\Omega}(s),\omega\right).$$

Here, B is the positive constant appearing in hypothesis ***L2'****.*

Proof. For a fixed integer $n\geq 1$, we apply Lemma 4.5 with the screen S_m given by hypothesis **L2'**. We first let $m\to\infty$ while keeping n fixed. Since the screens S_m have a uniform Lipschitz bound, if we take $t<\min\{1,B^{-1}\}$, then the sequence of integrals over the truncated screens $(S_m)_{|n}$ converges to 0 as $m \to \infty$. Indeed, let us take m_0 large enough so that $\sup S_m < 0$ for every $m \geq m_0$. (This is possible since $\sup S_m \to -\infty$ as $m \to \infty$.)

Furthermore, for every $m \geq 1$ and $n \geq 1$, the integral over the truncated screen $(S_m)_{|n}$ is given by

$$I_{n,m} := \frac{1}{2\pi\mathrm{i}} \int_{(S_m)_{|n}} \frac{t^{Q-s+k}}{(Q-s+1)_k} \widetilde{\zeta}_{A,\Omega}(s)\mathrm{d}s$$

and, similarly as in the proof of Lemma 4.5, we have that the integrand is bounded in absolute value by

$$C_\kappa |\sup S_m|^{\kappa-k} \max\{B^{|\inf(S_m)_{|n}}, B^{|\sup(S_m)_{|n}}\} t^{Q+|\sup(S_m)_{|n}|+k},$$

where C_κ is a suitable constant depending only on κ, Here, we use the notation

$$\inf(S_m)_{|n} := \inf_{\tau \in [T_{-n}, T_n]} S_m(\tau) \quad \text{and} \quad \sup(S_m)_{|n} := \sup_{\tau \in [T_{-n}, T_n]} S_m(\tau).$$

We now let $L := \sup_{m \geq 1} \|S_m\|$ be the uniform Lipschitz bound for the sequence of screens S_m. Then, the derivative of $\tau \mapsto S_m(\tau) + \mathrm{i}\tau$ is bounded for almost every $\tau \in [T_{-n}, T_n]$ by $(1 + L)$.

We must next consider the following two cases: firstly, if $B < 1$, we then have that

$$|I_{n,m}| \leq \frac{C_\kappa (1+L) B^{|\sup(S_m)_{|n}|}}{2\pi |\sup(S_m)_{|n}|^{k-\kappa}} (T_n - T_{-n}) t^{Q+|\sup(S_m)_{|n}|+k},$$

and, since $t < 1$, we have that $I_{n,m} \to 0$ as $m \to \infty$. Secondly, if $B \geq 1$, we deduce from the Lipschitz condition on S_m that we have

$$\sup(S_m)_{|n} - \inf(S_m)|n \leq L(T_n - T_{-n});$$

i.e.,

$$|\inf(S_m)_{|n}| \leq |\sup(S_m)_{|n}| + L(T_n - T_{-n}),$$

from which we deduce the estimate

$$|I_{n,m}| \leq \frac{C_\kappa (1+L) B^{L(T_n - T_{-n})}}{2\pi |\sup(S_m)_{|n}|^{k-\kappa}} (T_n - T_{-n}) (Bt)^{|\sup(S_m)_{|n}|} t^{Q+k}.$$

Therefore, $I_{n,m} \to 0$ as $m \to \infty$ since $Bt < 1$.

We now let $E_{n,m}(t)$ be the error function for the truncated screen $(S_m)_{|n}$ and we will complete the proof by showing that its iterated limit converges to zero pointwise. For $c \in (\overline{\dim}_B(A,\Omega), Q+1)$ and since $0 < t < 1$, we have, much as in the proof of Lemma 4.5, that

$$\begin{aligned}\left| \int_{\Gamma_{U_m}} \frac{t^{Q-s+k}\widetilde{\zeta}_{A,\Omega}(s)\mathrm{d}s}{(Q-s+1)_k} \right| &\leq t^{Q+k} C(T_n+1)^{\kappa} T_n^{-k} \int_{-\infty}^{c} t^{-\sigma}\mathrm{d}\sigma \\ &\leq t^{Q+k} K_\kappa T_n^{\kappa-k} \frac{t^{-c}}{\log(1/t)}.\end{aligned}$$

Here, Γ_{U_m} is the segment connecting $S_m(T_n)+\mathrm{i}T_n$ and $c+\mathrm{i}T_n$. A similar reasoning for the corresponding integral over the lower segment gives us the following upper bound on $|E_{n,m}(t)|$, independent of m:

$$|E_{n,m}(t)| \leq \frac{t^{Q-c+k}}{\pi \log(1/t)} K_\kappa \max\{T_n^{\kappa-k}, |T_{-n}^{\kappa-k}\}.$$

Finally, this inequality, which is valid for all $m \geq 1$ and all $n \geq 1$, implies that for a fixed $k > \kappa$, the iterated limit of $E_{n,m}(t)$ tends to 0 when $m \to \infty$ and $n \to \infty$; i.e., we have

$$\lim_{n\to\infty} (\lim_{m\to\infty} E_{n,m}(t)) = 0.$$

This concludes the proof of the theorem.

$\square$

We can now state both previous theorems for the case that $k = 0$; in other words, when we obtain a pointwise formula for the volume $|A_t \cap \Omega|$ in terms of the complex dimensions of (A,Ω).

Theorem 4.8. ***(Pointwise fractal tube formula via $\widetilde{\zeta}_{A,\Omega}$; level $k = 0$).*** *Under the same hypothesis as in Theorem 4.6, with $\kappa < -1$ (resp., under the same hypotheses as in Theorem 4.7, with $\kappa < 0$), with $k := 0$, we have the following pointwise formula for the tube function of the relative fractal drum (A,Ω) in $\mathcal{E}$:*

$$|A_t \cap \Omega| = \sum_{\omega \in \mathscr{P}(\widetilde{\zeta}_{A,\Omega}, W)} \operatorname{res}\left(t^{Q-s}\widetilde{\zeta}_{A,\Omega}(s), \omega\right) + \tilde{R}^{[0]}_{A,\Omega}(t),$$

valid pointwise for all $t \in (0, \delta)$ *and where* $\tilde{R}^{[0]}_{A,\Omega}(t)$ *is the error term given by formula 4.8 with* $k = 0$. *Furthermore, we have the following pointwise estimate:*

$$\tilde{R}^{[0]}_{A,\Omega}(t) = O(t^{Q-\sup S}) \quad as \quad t \to 0^+.$$

Moreover, if $S(\tau) < \sup S$ *for every* $\tau \in \mathbb{R}$ *(i.e. if the screen* S *lies strictly to the left of the vertical line* $\{Re\ s = \sup S\}$*), we then have*

$$\tilde{R}^{[0]}_{A,\Omega}(t) = o(t^{Q-\sup S}) \quad as \quad t \to 0^+.$$

Finally, in the special case of Theorem 4.7 where (A, Ω) *is assumed to be strongly languid, then* $\tilde{R}^{[0]}_{A,\Omega} \equiv 0$ *and* $W := \mathbb{C}$ *above, so that the pointwise fractal tube formula becomes exact.*

In a similar way, the distributional tube formulas from Euclidean space should carry over to setting of Ahlfors spaces.

The tube formulas can also, importantly, be written in terms of the distance zeta function. In order to do so, we first introduce the concept of the Relative Shell Zeta Function.

4.2 Relative Shell Zeta Function

We begin by introducing the concept of the *relative shell zeta function*, which will be useful since it satisfies a more direct functional equation than the one between the distance and tube zeta functions. For $A \subset \mathcal{E}$ and $t, \delta > 0$ with $t \leq \delta$, we define the (t, δ)-*shell of* A as

$$A_{t,\delta} := A_\delta \setminus A_t.$$

Let $\widetilde{\zeta}_{A,\Omega}(\,\cdot\,; \underset{\cdot}{)}$ be the tube zeta function of the relative fractal drum (A, Ω) in $\mathcal{E}$ and assume that $\mathrm{Re}\ s > Q$, then we have

$$\begin{aligned}\widetilde{\zeta}_{A,\Omega}(s; \underset{\cdot}{)} &= \int_0^\delta t^{s-Q-1}|A_t \cap \Omega| \ \mathrm{d}t = \int_0^\delta t^{s-Q-1}(|A_\delta \cap \Omega| - |A_{t,\delta} \cap \Omega|) \ \mathrm{d}t \\ &= \frac{\delta^{s-Q}|A_\delta \cap \Omega|}{s-Q} - \int_0^\delta t^{s-Q-1}|A_{t,\delta} \cap \Omega| \ \mathrm{d}t.\end{aligned}$$

Definition 4.9. Let (A, Ω) be an RFD in $\mathcal{E}$ and fix $\delta > 0$. We define the *shell zeta function* $\breve{\zeta}_{A,\Omega} := \breve{\zeta}_{A,\Omega}(\,\cdot\,;\delta)$ *of* A *relative to* Ω (or the *relative shell zeta function* of (A, Ω)) by

$$\breve{\zeta}_{A,\Omega}(s;\delta) := -\int_0^\delta t^{s-Q-1}|A_{t,\delta} \cap \Omega| \; \mathrm{d}t, \tag{4.10}$$

for all $s \in \mathbb{C}$ with Re s sufficiently large.

This definition, in relation to the tube zeta function, gives us the following theorem (analog of [19, Theorem 5.3.2]):

Theorem 4.9. *Let* (A, Ω) *be an RFD in* $\mathcal{E}$ *and fix* $\delta > 0$*. Then the shell zeta function* $\breve{\zeta}_{A,\Omega}(\,\cdot\,;\delta)$ *of* (A, Ω) *is holomorphic on the open right half-plane* $\{Re\ s > Q\}$ *and*

$$\frac{\mathrm{d}}{\mathrm{d}s}\breve{\zeta}_{A,\Omega}(s;\delta) = -\int_0^\delta t^{s-Q-1}|A_{t,\delta} \cap \Omega| \log t \;\, dt, \tag{4.11}$$

for all $s \in \mathbb{C}$ *such that* $Re\ s > Q$*.*

Furthermore, for all $s \in \mathbb{C}$ *such that* $Re\ s > Q$*,* $\breve{\zeta}_{A,\Omega}(\,\cdot\,;\delta)$ *satisfies the following functional equations, connecting it to the tube and distance zeta functions of* (A, Ω)*, respectively:*

$$\widetilde{\zeta}_{A,\Omega}(s;\delta) = \frac{\delta^{s-Q}|A_\delta \cap \Omega|}{s-Q} + \breve{\zeta}_{A,\Omega}(s;\delta) \tag{4.12}$$

and

$$\zeta_{A,\Omega}(s;\delta) = (Q-s)\breve{\zeta}_{A,\Omega}(s;\delta). \tag{4.13}$$

Proof. To prove the holomorphicity of $\breve{\zeta}_{A,\Omega}(\,\cdot\,;\delta)$, observe that for every real number $\sigma > Q$, we have $|\breve{\zeta}_{A,\Omega}(\sigma;\delta)| \leq |A_\delta \cap \Omega| \int_0^\delta t^{\sigma-Q-1}\ \mathrm{d}t < \infty$, and uses a well-known theorem about differentiation of an integral depending analytically on a parameter (see, e.g., [29, Theorem 31]) which also gives the formula (4.11) for the derivative. Formula (4.12) is a rewriting of the tube decomposition above, and by combining it with the functional equation (2.1), which connects the relative distance and tube zeta functions, we obtain (4.13). □

In light of Theorem 2.13, the principle of analytic continuation combined with Equation (4.12) (or (4.13)) now immediately yields the following properties of the relative shell zeta function (complement of [19, Theorem 5.3.3]).

Theorem 4.10. *Let (A, Ω) be a relative fractal drum in $\mathcal{E}$ such that $\overline{\dim}_B(A, \Omega) < Q$ and fix $\delta > 0$. Then the following properties hold:*

(a) *The relative shell zeta function $\breve{\zeta}_{A,\Omega}(s; \delta)$ is meromorphic in the half-plane $\{\mathrm{Re}\, s > \overline{\dim}_B(A, \Omega)\}$, with a single simple pole at $s = Q$. Furthermore,*

$$\operatorname{res}(\breve{\zeta}_{A,\Omega}(\,\cdot\,; \delta), Q) = -|A_\delta \cap \Omega|. \tag{4.14}$$

(b) *If the relative box (or Minkowski) dimension $D := \dim_B(A, \Omega)$ exists, $D < Q$ and $\mathcal{M}_*^D(A, \Omega) > 0$, then $\breve{\zeta}_{A,\Omega}(s) \to +\infty$ as $s \in \mathbb{R}$ converges to D from the right.*

Proof. We deduce from the principle of analytic continuation that the functional equations (4.12) and (4.13) continue to hold on any open connected set $U \supseteq \{\mathrm{Re}\, s > Q\}$ to which any of the three relative zeta functions, $\breve{\zeta}_{A,\Omega}$, $\widetilde{\zeta}_{A,\Omega}$ or $\zeta_{A,\Omega}$, has a holomorphic continuation. In light of this, part (a) follows from the counterpart of Theorem 2.13 for the relative tube zeta function and (4.12), while part (b) follows from Theorem 2.13 and (4.13). ☐

The following corollary (analog of [19, Corollary 5.3.4]) is an immediate consequence of the above theorem. More precisely, of the functional equation and the fact that, given a relative fractal drum (A, Ω) in $\mathcal{E}$ with fixed $\delta_1, \delta_2 > 0$, the difference $\widetilde{\zeta}_{A,\Omega}(s; \delta_1) - \widetilde{\zeta}_{A,\Omega}(s; \delta_2)$ is an entire function.

Corollary 4.15. *Let (A, Ω) be an RFD in $\mathcal{E}$ such that $\overline{\dim}_B(A, \Omega) < Q$ and fix $\delta_1, \delta_2 > 0$ such that $\delta_1 < \delta_2$. Then, the difference $\breve{\zeta}_{A,\Omega}(s; \delta_1) - \breve{\zeta}_{A,\Omega}(s; \delta_2)$ is meromorphic on all of $\mathbb{C}$ with a single simple pole at $s = Q$ of residue $|A_{\delta_1,\delta_2} \cap \Omega|$.*

The next corollary follows at once from the first part of the proof of Theorem 4.10 (analog of [19, Corollary 5.3.5].

Corollary 4.16. *Let (A,Ω) be an RFD in $\mathcal{E}$. The functional equations* (4.12) *and* (4.13) *continue to hold on any connected open neighborhood $U \subseteq \mathbb{C}$ of the critical line $\{Re\ s = \overline{\dim}_B(A,\Omega)\}$ to which any of the three relative zeta functions $\breve{\zeta}_{A,\Omega}$, $\widetilde{\zeta}_{A,\Omega}$ or $\zeta_{A,\Omega}$ can be meromorphically continued. More specifically, if either $\breve{\zeta}_{A,\Omega}$, $\widetilde{\zeta}_{A,\Omega}$ or $\zeta_{A,\Omega}$ has a (necessarily unique) meromorphic continuation on the domain $U \subseteq \mathbb{C}$, then so do the other two fractal zeta functions and the functional equations* (4.12) *and* (4.13) *continue to hold for all $s \in U$ between the resulting meromorphic extensions of $\breve{\zeta}_{A,\Omega}$, $\widetilde{\zeta}_{A,\Omega}$ and $\zeta_{A,\Omega}$.*

In light of the obvious counterpart of 2.11 for relative fractal drums, as well as the functional equation (4.12), we have the following result (analog of [19, Theorem 5.3.6]).

Theorem 4.11. *Assume that (A,Ω) is a Minkowski nondegenerate RFD in $\mathcal{E}$, that is, $0 < \mathcal{M}_*^D(A,\Omega) \leq \mathcal{M}^{*D}(A,\Omega) < \infty$ (in particular, $\dim_B(A,\Omega) = D$), and $D < Q$. If $\breve{\zeta}_{A,\Omega}(s)$ can be meromorphically extended to a connected open neighborhood of $s = D$, then D is necessarily a simple pole of $\breve{\zeta}_{A,\Omega}(s)$ and*

$$\mathcal{M}_*^D(A,\Omega) \leq \operatorname{res}(\breve{\zeta}_{A,\Omega}, D) \leq \mathcal{M}^{*D}(A,\Omega). \tag{4.17}$$

Furthermore, if (A,Ω) is Minkowski measurable, then

$$\operatorname{res}(\breve{\zeta}_{A,\Omega}, D) = \mathcal{M}^D(A,\Omega). \tag{4.18}$$

The most useful fact about the relative shell zeta function is that the residues of its meromorphic extension at any of its (simple) poles belonging to the open left half-plane $\{\text{Re } s < Q\}$ have a simple connection to the residues of the relative tube or distance zeta functions. This follows from the functional equations relating each of the zeta functions, and is the complement to [19, Lemma 5.3.7].

Lemma 4.12. *Assume that (A,Ω) is an RFD in $\mathcal{E}$ such that its tube or distance or shell zeta function is meromorphic on some connected open neighborhood $U \subseteq \mathbb{C}$ of the critical line $\{Re\ s = \overline{\dim}_B(A,\Omega)\}$. Then, the multisets of poles located in $U \setminus \{Q\}$ of each of the three zeta functions,*

$\breve{\zeta}_{A,\Omega}$, $\widetilde{\zeta}_{A,\Omega}$ *and* $\zeta_{A,\Omega}$, *coincide*:

$$\mathcal{D}\big(\breve{\zeta}_{A,\Omega}, U \setminus \{Q\}\big) = \mathcal{D}\big(\widetilde{\zeta}_{A,\Omega}, U \setminus \{Q\}\big) = \mathcal{D}\big(\zeta_{A,\Omega}, U \setminus \{Q\}\big). \tag{4.19}$$

Moreover, if $\omega \in U \setminus \{Q\}$ *is a simple pole of one of the three fractal zeta functions* $\breve{\zeta}_{A,\Omega}$, $\widetilde{\zeta}_{A,\Omega}$ *or* $\zeta_{A,\Omega}$, *then it is also a simple pole of the other two fractal zeta functions and we have*

$$\operatorname{res}(\breve{\zeta}_{A,\Omega}, \omega) = \operatorname{res}(\widetilde{\zeta}_{A,\Omega}, \omega) = \frac{\operatorname{res}(\zeta_{A,\Omega}, \omega)}{Q - \omega}. \tag{4.20}$$

The shell zeta function will be useful in the upcoming section, where it will act as a "translation tool" for deriving the tube formulas in terms of the distance zeta function.

4.3 Pointwise Tube Formula via the Distance Zeta Function

Analogously as in the case of the relative tube zeta function of (A, Ω), we observe that $\breve{\zeta}_{A,\Omega}(s) = \{\mathfrak{M}f\}(s)$, where $f(s) := -t^{-Q}\chi_{(0,\delta)}(t)|A_{t,\delta} \cap \Omega|$. We also note that f is continuous and of bounded variation on $(0, +\infty)$; so that we can apply the Mellin inversion theorem, much as in the proof of Theorem 4.3, and conclude that

$$|A_{t,\delta} \cap \Omega| = -\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} t^{Q-s} \breve{\zeta}_{A,\Omega}(s;\delta)\, \mathrm{d}s, \tag{4.21}$$

where $c > Q$ is arbitrary and $t \in (0, \delta)$. Since $|A_{t,\delta} \cap \Omega| = |A_\delta \cap \Omega| - |A_t \cap \Omega|$, the following theorem is an immediate consequence of the above identity, and is the analog of to [19, Theorem 5.3.8]

Theorem 4.13. *Let* (A, Ω) *be a relative fractal drum in* $\mathcal{E}$ *and fix* $\delta > 0$*. Then, for every* $t \in (0, \delta)$ *and any real number* $c > Q$*, we have*

$$|A_t \cap \Omega| = |A_\delta \cap \Omega| + \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} t^{Q-s} \breve{\zeta}_{A,\Omega}(s;\delta)\, ds. \tag{4.22}$$

It is now clear that if the shell zeta function of (A, Ω) satisfies the languidity conditions of Definition 4.2, with the constant $c > Q$ in the condition **L1**, or the strong languidity conditions of Definition 4.3, we can rewrite the results of Section 4.1 verbatim in terms of the shell zeta function.

Note that for this to work, it was crucial that in the truncated pointwise formula of Lemma 4.5, we had the freedom to choose any $c \in (\overline{\dim}_B(A,\Omega), Q+1)$. Furthermore, observe that the additional pole of the shell zeta function at $s = Q$ will cancel out the term $|A_\delta \cap \Omega|$ in (4.22) above. More specifically, in the analog for the relative shell zeta function of the pointwise formula stated in Theorem 4.6, we obtain the following pointwise fractal tube formula with error term, expressed in terms of the shell zeta function $\breve{\zeta}_{A,\Omega} := \breve{\zeta}_{A,\Omega}(\,\cdot\,;\delta)$:

$$V_{A,\Omega}^{[k]}(t) = \sum_{\omega \in \mathcal{D}(\breve{\zeta}_{A,\Omega},\boldsymbol{W})} \operatorname{res}\left(\frac{t^{Q-s+k}}{(Q-s+1)_k}\breve{\zeta}_{A,\Omega}(s),\omega\right) + |A_\delta \cap \Omega|\frac{t^k}{(1)_k} + \breve{R}_{A,\Omega}^{[k]}(t), \tag{4.23}$$

valid pointwise for all $t \in (0,\delta)$. Here, just as in the statement of Theorem 4.6, the shell zeta function $\breve{\zeta}_{A,\Omega)}$ of the RFD (A,Ω) is assumed to be languid for some fixed $\delta > 0$ and some fixed constant $\kappa \in \mathbb{R}$, as well as with the constant c satisfying $c > Q$. Furthermore, the nonnegative integer k is assumed to be such that $k > \kappa + 1$ and for every $t \in (0,\delta)$, the error term $\breve{R}_{A,\Omega}^{[k]}$ is given by the absolutely convergent (and hence, convergent) integral

$$\breve{R}_{A,\Omega}^{[k]}(t) = \frac{1}{2\pi i}\int_S \frac{t^{Q-s+k}}{(Q-s+1)_k}\breve{\zeta}_{A,\Omega}(s;\delta)\,\mathrm{d}s. \tag{4.24}$$

In addition, by singling out the residue at $s = Q$ from the above sum and using Lemma 4.12 and Theorem 4.10(a), along with the functional equation (4.13), we can rewrite the above equation (in (4.23)) as follows:

$$V_{A,\Omega}^{[k]}(t) = \sum_{\omega \in \mathcal{D}(\zeta_{A,\Omega},\boldsymbol{W})} \operatorname{res}\left(\frac{t^{Q-s+k}}{(Q-s)_{k+1}}\zeta_{A,\Omega}(s;\delta),\omega\right) + R_{A,\Omega}^{[k]}(t), \tag{4.25}$$

where the pointwise error term $R_{A,\Omega}^{[k]}$ is now given by the absolutely convergent (and hence, convergent) integral

$$R_{A,\Omega}^{[k]}(t) = \frac{1}{2\pi i}\int_S \frac{t^{Q-s+k}}{(Q-s)_{k+1}}\zeta_{A,\Omega}(s;\delta)\,\mathrm{d}s. \tag{4.26}$$

We next introduce the analogs of the languidity conditions for a relative fractal drum, now formulated in terms of its relative distance zeta function. We call them *d-languidity conditions* in order to stress that they are related to the distance zeta function.

Definition 4.27. (*d-languidity and strong d-languidity*). We say that a relative fractal drum (A,Ω) in $\mathcal{E}$ is *d-languid* (resp., *strongly d-languid*) if it is languid in the sense of Definition 4.2 (resp., Definition 4.3), but with the relative tube zeta function $\widetilde{\zeta}_{A,\Omega} = \widetilde{\zeta}_{A,\Omega}(\,\cdot\,;\delta)$ replaced by the relative distance zeta function $\zeta_{A,\Omega} = \zeta_{A,\Omega}(\,\cdot\,;\delta)$ and with the constant c appearing in **L1** satisfying $c > Q$.

The following lemma (analog of [19, Lemma 5.3.10]) is an immediate consequence of the functional equation (4.13). It is crucial in the sense that it allows us to deduce the languidity exponent κ of the shell zeta function directly from the d-languidity exponent κ_d of the distance zeta function. This cannot be done for the tube zeta function, due to the presence of the term $\delta^{s-Q}|A_\delta \cap \Omega|$ in the functional equation (2.9); this is in fact the technical reason for introducing the shell zeta function in the first place.

Lemma 4.14. *Let (A,Ω) be a relative fractal drum in $\mathcal{E}$ such that $\overline{\dim}_B(A,\Omega) < Q$ and which is d-languid for some value $\delta > 0$ and with some exponent $\kappa_d \in \mathbb{R}$. Then the shell zeta function $\check{\zeta}_{A,\Omega}$ of (A,Ω) satisfies the languidity conditions of Definition 4.2 for the same value of δ and with the exponent $\kappa := \kappa_d - 1$.*

Furthermore, if (A,Ω) is strongly d-languid with the corresponding constant $B > 0$ and for some exponent $\kappa_d \in \mathbb{R}$ and some $\delta > 0$, then the shell zeta function $\check{\zeta}_{A,\Omega}$ of (A,Ω) satisfies the strong languidity conditions of Definition 4.3 with the exponent $\kappa := \kappa_d - 1$ and with the same constant B as well as the same value of δ.

We are now able to state the main theorem of this section, which is the analog for $\zeta_{A,\Omega}$ of Theorem 4.6 stated in terms of $\widetilde{\zeta}_{A,\Omega}$, and is the complement of the [19, Theorem 5.3.11].

Theorem 4.15 (Pointwise fractal tube formula with error term, via $\zeta_{A,\Omega}$)**.** *Let (A,Ω) be a relative fractal drum in $\mathcal{E}$ which is d-languid for some $\delta > 0$ and with exponent $\kappa_d \in \mathbb{R}$. Furthermore, assume that $\overline{\dim}_B(A,\Omega) < Q$ and let $k > \kappa_d$ be a nonnegative integer. Then, the following pointwise fractal tube formula, expressed in terms of the distance zeta function $\zeta_{A,\Omega} := \zeta_{A,\Omega}(\,\cdot\,;\delta)$, is valid for every*

$t \in (0, \delta)$:

$$V^{[k]}_{A,\Omega}(t) = \sum_{\omega \in \mathcal{D}(\zeta_(A,\Omega),\boldsymbol{W})} \operatorname{res}\left(\frac{t^{Q-s+k}}{(Q-s)_{k+1}}\zeta_{A,\Omega}(s), \omega\right) + R^{[k]}_{A,\Omega}(t). \tag{4.28}$$

Here, for every $t \in (0, \delta)$, the error term $R^{[k]}_{A,\Omega}$ is given by the absolutely convergent (and hence, convergent) integral

$$R^{[k]}_{A,\Omega}(t) = \frac{1}{2\pi i}\int_S \frac{t^{Q-s+k}}{(Q-s)_{k+1}}\zeta_{A,\Omega}(s)\ ds. \tag{4.29}$$

Furthermore, for every $t \in (0, \delta)$, we have

$$|R^{[k]}_{A,\Omega}(t)| \leq t^{Q+k}\max\{t^{-\sup S}, t^{-\inf S}\}\left(\frac{C(1 + \|S\|_{\text{Lip}})}{2\pi(k - \kappa_d)} + C'\right), \tag{4.30}$$

where C is the constant appearing in **L1** *and* **L2** *and C' is some suitable positive constant. These constants depend only on the relative fractal drum (A, Ω) and the screen, but not on k.*

In particular, we have the following pointwise error estimate:

$$R^{[k]}_{A,\Omega}(t) = \Omega(t^{Q-\sup S+k}) \quad \text{as} \quad t \to 0^+. \tag{4.31}$$

Moreover, if $S(\tau) < \sup S$ (i.e., if the screen $\boldsymbol{S}$ lies strictly left of the vertical line $\{Re\ s =$ $\sup S\}$), then we have the following stronger pointwise error estimate:

$$R^{[k]}_{A,\Omega}(t) = o(t^{Q-\sup S+k}) \quad \text{as} \quad t \to 0^+. \tag{4.32}$$

Proof. In light of Lemma 4.14, we have that $\breve{\zeta}_{A,\Omega}$, the shell zeta function of (A, Ω), also satisfies the appropriate languidity conditions with $\kappa := \kappa_d - 1$ and for the same value of δ. The theorem now follows much as in the case of the relative tube zeta function $\widetilde{\zeta}_{A,\Omega}$; see the proof of Theorem 4.6 and the discussion following Theorem 4.13. □

The next result is the counterpart for $\zeta_{A,\Omega}$ of Theorem 4.7, which is stated in terms of $\widetilde{\zeta}_{A,\Omega}$, and complements [19, Theorem 5.3.13]

Theorem 4.16 (Exact pointwise fractal tube formula via $\zeta_{A,\Omega}$)**.** *Let (A, Ω) be a relative fractal drum in $\mathcal{E}$ which is strongly d-languid for some $\delta > 0$ and with exponent $\kappa_d \in \mathbb{R}$. Furthermore, let*

$k > \kappa_d - 1$ be a nonnegative integer and assume that $\overline{\dim}_B(A,\Omega) < Q$. Then, the following exact pointwise fractal tube formula, expressed in terms of the distance zeta function $\zeta_{A,\Omega} := \zeta_{A,\Omega}(\,\cdot\,;\delta)$, holds for every $t \in (0, \min\{1, \delta, B^{-1}\})$:

$$V_{A,\Omega}^{[k]}(t) = \sum_{\omega \in \mathcal{D}(\zeta_{A,\Omega}, \mathbb{C})} \operatorname{res}\left(\frac{t^{Q-s+k}}{(Q-s)_{k+1}} \zeta_{A,\Omega}(s), \omega\right). \tag{4.33}$$

Here, B is the constant appearing in **L2'** *and κ_d is the exponent occurring in the statement of hypotheses* **L1** *and* **L2'**.

Proof. In light of Lemma 4.14 and the functional equation (4.13), the proof of the theorem parallels that of Theorem 4.15 and of Theorem 4.7, except (in the latter case) for the tube zeta function $\widetilde{\zeta}_{A,\Omega}(\,\cdot\,;\delta)$ now being replaced by the shell zeta function $\check{\zeta}_{A,\Omega}(\,\cdot\,;\delta)$. ☐

The most interesting case of the above theorems is when $k = 0$, and so we state the above two theorems for this specific case. This is an analog of the first part of [19, 5.3.16], although the second part does not translate as Ahlfors spaces in general do not satisfy the scaling property of Euclidean distance zeta functions.

Theorem 4.17 (Pointwise fractal tube formula via $\zeta_{A,\Omega}$; level $k = 0$)**.**

Under the same hypotheses as in Theorem 4.15, with $k := 0$, and using the same notation as in that theorem, with $\kappa_d < 0$, the following pointwise fractal tube formula with error term, expressed in terms of the distance zeta function $\zeta_{A,\Omega} := \zeta_{A,\Omega}(\,\cdot\,;\delta)$, holds for all $t \in (0,\delta)$:

$$|A_t \cap \Omega| = \sum_{\omega \in \mathcal{D}(\zeta_{A,\Omega}, \boldsymbol{W})} \operatorname{res}\left(\frac{t^{Q-s}}{Q-s} \zeta_{A,\Omega}(s), \omega\right) + R_{A,\Omega}^{[0]}(t), \tag{4.34}$$

where $R_{A,\Omega}^{[0]}(t)$ is the error term given by formula (4.29) with $k := 0$. Furthermore, we have the following pointwise error estimate:

$$R_{A,\Omega}^{[0]}(t) = \Omega(t^{Q-\sup S}) \quad \text{as} \quad t \to 0^+. \tag{4.35}$$

Moreover, if $S(\tau) < \sup S$ for every $\tau \in \mathbb{R}$ (i.e., if the screen $\boldsymbol{S}$ lies strictly to the left of the vertical line $\{Re\ s = \sup S\}$), we then have the following stronger pointwise error estimate:

$$R^{[0]}_{A,\Omega}(t) = o(t^{Q-\sup S}) \quad \text{as} \quad t \to 0^+. \tag{4.36}$$

We also present the case when the poles are simple, and complements [19, Theorem 5.3.17].

Theorem 4.18 (Pointwise fractal tube formula via $\zeta_{A,\Omega}$; level $k = 0$ and the case of simple poles)**.** *Assume that the hypotheses of Theorem 4.17 hold. Suppose, in addition, that all of the visible complex dimensions of the RFD (A, Ω) are simple (i.e., all of the poles of $\zeta_{A,\Omega}$ or, equivalently, since $\overline{D} := \overline{\dim}_B(A, \Omega) < Q$ here, of $\widetilde{\zeta}_{A,\Omega}$, belonging to the window $\boldsymbol{W}$ are simple). Then, the pointwise fractal tube formula (4.34), expressed in terms of $\zeta_{A,\Omega}$, takes the following simpler form, valid for all $t \in (0, \delta)$:*

$$|A_t \cap \Omega| = \sum_{\omega \in \mathcal{D}(\zeta_{A,\Omega}, \boldsymbol{W})} \frac{t^{Q-\omega}}{Q-\omega} \operatorname{res}(\zeta_{A,\Omega}(s), \omega) + R^{[0]}_{A,\Omega}(t), \tag{4.37}$$

where the (pointwise) error term $R^{[0]}_{A,\Omega}$ is the same as in Theorem 4.15 at level $k = 0$ and hence, satisfies the same (pointwise) error estimates [(4.35) or (4.36), depending on the hypotheses] as in Theorem 4.17.

As with the tube formulas via the tube zeta function, much of the distributional tube formulas in terms of the distance zeta function should transfer immediately over to the case of Ahlfors spaces.

Chapter 5

To Beyond

5.1 Patchwork Spaces

The distance zeta function does not need the space to necessarily be an Ahlfors space. All that is needed is a metric measure space with an appropriate idea of "ambient dimension", or alternatively of the co-dimension, as well as a generalization of Minkowski dimension. In this chapter we attempt to loosen the restriction on the space in order to understand how the theory should be generalized in order to still give relevant geometric information.

To begin, we define a new type of space, more general than Ahlfors spaces:

Definition 5.1. A space X is of **Ahlfors-type between dimensions** D_1 **and** D_2 (here on, $A(D_1, D_2)$**-type**) if it has both a metric d and a measure μ and $\exists K > 0$ such that

$$K^{-1} r^{D_1} \leq \mu(B(x, r)) \leq K r^{D_2}$$

$\forall x \in X, 0 < r \leq \operatorname{diam} X$.

Given a metric measure space (with finite, normalized diameter), we can deform the metric in order to obtain a new metric space. We will use these deformations in order to construct simple examples of Ahlfors-type spaces.

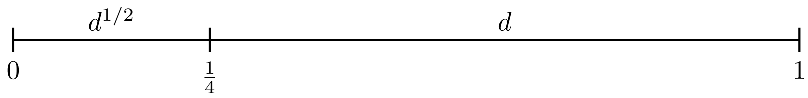


Figure 5.1: A 1-dimensional patchwork space, the line segment $[0,1]$ with the metric on $[0, \frac{1}{4}]$ the standard Euclidean metric raised to the $1/2$ power.

Definition 5.2. Let $(M, d(x,y))$ be a metric space with diameter ≤ 1. Given a real number $0 < s < 1$ consider the space $(M, d(x,y)^s)$. This is a metric space, and the transformation is called the *snowflake functor* in [7].

Further, if $(M, d(x,y), \mu)$ is regular of dimension D, then $(M, d(x,y)^s, \mu)$ is regular of dimension D/s.

Using the snowflake functor, we can create new Ahlfors-type spaces by "patching together" known spaces with each patch snowflaked in a different manner. For example, the patchwork unit interval, with segment $[0, \frac{1}{4}]$ the Euclidean metric raised to the $1/2$ power, is an $A(1,2)$-type space (see Figure 5.2 below).

This example of space is of interest in determining the proper generalization of the distance zeta function. In a Q-Ahlfors space, Q appears in both the distance and tube zeta functions:

$$\zeta_A(s) = \int_{A_\delta} d(x,A)^{s-Q} \mathrm{d}x,$$

$$\widetilde{\zeta}_A(s) = \int_0^\delta t^{s-Q-1} |A_t| \mathrm{d}t.$$

How should this exponent be changed if the dimension is no longer constant throughout our space?

We propose that, in order to obtain adequate geometrical data, the "local dimension" should be used, and the integral split accordingly.

Example 5.3. Take $A = [\frac{1}{8}, \frac{1}{2}]$, with $\delta = \frac{1}{4}$, then the tube zeta function (for explicit calculations since A is a full subset of the space) would be realized as:

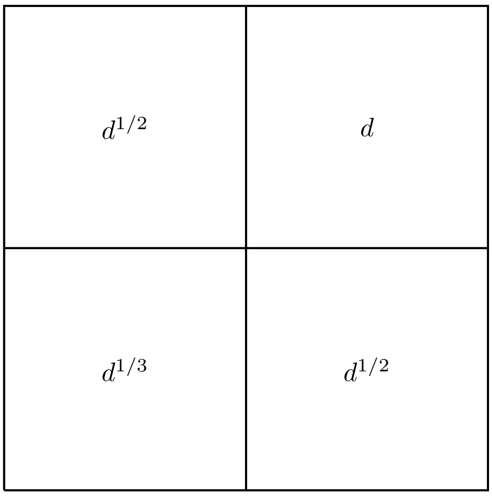


Figure 5.2: A square patchwork space

$$
\begin{aligned}
\widetilde{\zeta}_A(s) &= \int_0^\delta t^{s-2-1}(t^2 + \frac{1}{4})\mathrm{d}t + \int_0^\delta t^{s-1-1}(t + \frac{1}{4})\mathrm{d}t \\
&= \frac{\delta^s}{s} + \frac{\delta^{s-2}}{4(s-2)} + \frac{\delta^s}{s} + \frac{\delta^{s-1}}{4(s-1)}.
\end{aligned}
$$

This would lead to the set of complex dimensions of A being

$$\mathcal{D} = \{0, 1, 2\},$$

accounting for the endpoints, the one-dimensional section existing in $[\frac{1}{4}, \frac{1}{2}]$, and the two-dimensional section existing in $[\frac{1}{8}, \frac{1}{4}]$.

Conversely, the use of a global dimension provides counter-intuitive results. For example, letting $Q = 2$ throughout our space would lead any point in $[\frac{1}{4}, 1]$ to be of dimension 1, and any line segment to be of dimension 2. Not only would the geometric information obtained be muddled, it would not conform to dimensions of line segments and points in standard $\mathbb{R}$. Similar problems occur if $Q = 1$ is chosen to be the global dimension, with a line segment in $[0, \frac{1}{4}]$ attaining dimension 0 with endpoints of dimension -1.

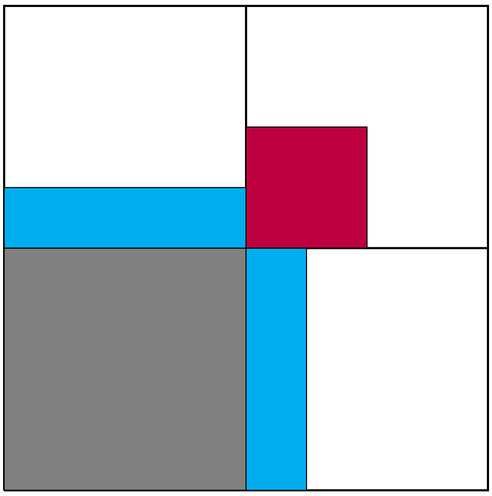


Figure 5.3: The $\delta = 1/4$ neighborhood of A with the ℓ_∞ metric

Just as we did with the line segment, we can create new patchwork spaces out of the unit square, or any other subset of $\mathbb{R}^N$ with normalized, finite diameter. Let us take the unit square and split it into fourths, giving each fourth a different metric, with the condition that the shared edge belongs to the minimizing distance. In particular, let us use the distances in the figure below, where d is the ℓ_∞ metric, we will have an $A(1,6)$-type space with the new metric given by the minimizing path.

Example 5.4. Let $A = [0, 1/2] \times [0, 1/2]$ and $\delta \in (0, 1/2)$. Then using the tube zeta function, we find that

$$\begin{aligned}\tilde{\zeta_A}(s) &= \int_0^\delta t^{s-6-1}\Big(\frac{1}{4}\Big)\mathrm{d}t + \int_0^\delta t^{s-4-1}\Big(\frac{1}{2}t^2 * 2\Big)\mathrm{d}t + \int_0^\delta t^{s-2-1}t^2\mathrm{d}t \\ &= \frac{1}{4}\frac{\delta^{s-6}}{s-6} + \frac{\delta^{s-2}}{s-2} + 2\frac{\delta^{s-1}}{s-1} - \frac{t^s}{s}\end{aligned}$$

which has as complex dimensions the set

$$D = \{0, 1, 2, 6\}.$$

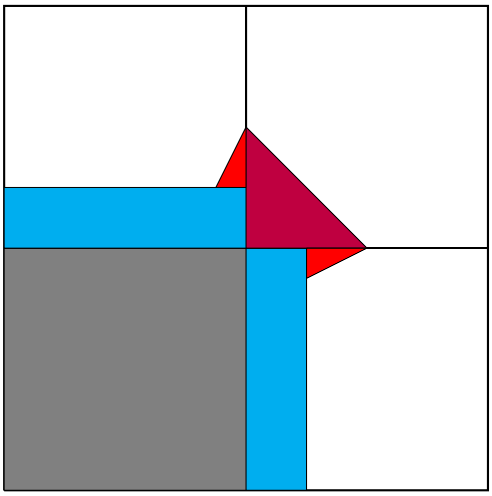


Figure 5.4: The $\delta = 1/4$ neighborhood of A with the ℓ_1 metric

If instead we started with d as the ℓ_1 (taxi-cab) metric, we find new, initially surprising dimensions:

Example 5.5.

$$\begin{aligned}\tilde{\zeta_A}(s) &= \int_0^\delta t^{s-6-1}\left(\frac{1}{4}\right) + t^{s-4-1}(t^2 + (t-t^2)(t-t^2)^2) + \frac{1}{2}t^{s-2-1}t^2 \mathrm{d}t \\ &= \frac{1}{4}\frac{\delta^{s-6}}{s-6} + \frac{\delta^{s-2}}{s-2} + \frac{\delta^{s-1}}{s-1} - \frac{t^s}{s} + 3\frac{t^s+1}{s+1} - \frac{t^{s+2}}{s+2},\end{aligned}$$

which has as complex dimensions the set

$$\mathcal{D} = \{-2, -1, 0, 1, 2, 6\}.$$

These negative dimensions appear due to geodesics that utilize the lower-dimensional boundary connecting the patches, where in this instance the negative dimensions seem to indicate a "faster" travel time than should be possible in standard Ahlfors spaces.

5.2 Laakso Graph

While studying spaces of Ahlfors-type is instructive in understanding the geometry and generalizations of the distance zeta function, they do not encompass all types of spaces that one would like to study. In particular, the Laakso graph described below is an upper Ahlfors space, but not a space of Ahlfors type. While no lower bound exists, all neighborhoods are scaled similarities and act nearly as a 2 dimensional Ahlfors space. Recall that the fact that the Laakso graph is an *upper Ahlfors space* means that there exists a positive constant K such that $\mu(B(x,r)) \leq Kr^2$, for any $x \in X$ and $0 < r < \text{diam}X$, where the diameter of the Laakso graph is 1.

To construct the Laakso graph rigorously, we follow the same procedure as in [3], starting with $X_0 = [0,1]$. For $i > 0$, define X_i by replacing each edge of X_{i-1} by a $4^{-(i-1)}$ scaled copy of Γ (below). Then $\{X_i\}_{i=0}^{\infty}$ forms an inverse system

$$X_0 \xleftarrow{\pi_0} \ldots \xleftarrow{\pi_{i-1}} X_i \xleftarrow{\pi_i} \ldots,$$

where $\pi_{i-1} : X_i \to X_{i-1}$ collapses the copies of Γ at the i-th level.

Then the inverse limit X_∞ is the **Laakso graph**, with metric

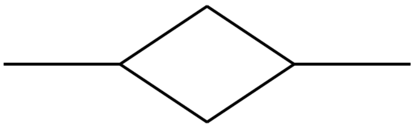
$$d_\infty(x,x') = \lim_{i\to\infty} d_{X_i}(\pi_i^\infty(x), \pi_i^\infty(x')),$$

where X_∞ is the Gromov-Hausdorff limit of $\{X_i\}$ and $\pi_i^\infty : X_\infty \to X_i$ is the canonical projection.

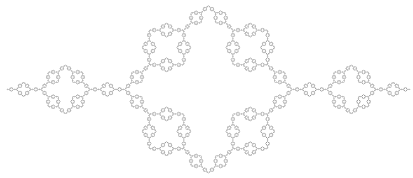
We can also view the Laakso graph as being constructed in a similar manner to the 1/4th-Cantor set, where instead of removing the middle halves, two copies of the middle halves are created at each step in the construction (see Figure 5.5). This relationship to the Cantor set plays an integral role in the geometry we expect to see in the graph.

The Laakso graph is a compact, and hence complete, metric space. If we place a Bernoulli measure on the graph, we get not an Ahlfors regular space, but an upper Ahlfors regular space. Further, the Laakso graph cannot be embedded in a bi-Lipschitz manner into any $\mathbb{R}^N$ (with $N \geq 1$), and the diameter of our graph is equal to 1.

Example 5.6. Let A be the upper geodesic from $\frac{1}{4}$ to $\frac{1}{2}$, and let $\delta = \frac{1}{4}$.Then using the tube zeta function we get



(a) Graph of Γ



(b) Laakso Graph

Figure 5.5

$$\begin{aligned}
\tilde{\zeta_A}(s) &= \sum_{k=1}^{\infty} \int_{4^{-k-1}}^{4^{-k}} t^{s-2}|A_t|\mathrm{d}t \\
&= \sum_{k=1}^{\infty} \int_{4^{-k-1}}^{4^{-k}} t^{s-2} \left(2t + t(2^{k-1} - 1) + 1\right) \mathrm{d}t \\
&= \sum_{k=1}^{\infty} \frac{3}{2s} t^s + \frac{2^{k-2}}{s} t^s + \frac{t^{s-1}}{s-1}\Big|_{4^{-k-1}}^{4^{-k}} \\
&= \frac{3}{2s} 4^{-2s} + \frac{1}{4}\left(\frac{1}{1-2^{-2s+1}} - 1 - 2^{-2s+1}\right) + \frac{1}{s-1} 4^{-2s}.
\end{aligned}$$

From these calculations, we find that the complex dimensions are

$$\mathcal{D} = \left\{0, 1, \text{ and } \frac{1}{2} + \frac{\pi i n}{\log 2} : n \in \mathbb{Z}\right\}.$$

Geometrically, these dimensions give us what intuitively should be the correct dimensions. In particular, we get dimension 0 for the endpoints, and dimension 1 corresponds to the fact that the geodesic is simply a copy of the interval $[\frac{1}{4}, \frac{1}{2}]$. Additionally, the complex dimensions $\left\{\frac{1}{2} + \frac{\pi i n}{\log 2} : n \in \mathbb{Z}\right\}$ are exactly those of the 1/4th Cantor set.

5.3 Concluding Comments

From the above examples, it is clear that the distance (and consequently, tube) zeta function should be extended to more general spaces than simply Ahlfors spaces. Of course, neither the patchwork spaces or the Laakso graph are far removed from Ahlfors spaces, and much is yet unknown about spaces in which the geometry changes throughout the space. While analysis in metric measure spaces are well studied, most examples of such spaces are, in practice, Ahlfors spaces. Of course, the study of such spaces is still recent, and it would be interesting to look at far more general examples than those above.

The above examples show that a notion of "local dimension" is necessary if we wish to capture relevant geometric data about any given set. In our examples the local dimension was obvious, but it is still a question of how to define such a dimension in general. It is easy to imagine a space in which the dimension shifts in a continuous manner, but even in this setting it is difficult to both pinpoint the local dimension at any point, and to further show that the complex dimensions of the distance zeta function can be interpreted as geometric information.

It is also currently unknown how general such spaces that admit a local dimension are. It may be possible that, with the correct definition of dimension, the theory of fractal zeta functions can be extended to any doubling metric space, in the sense of the complex dimensions truly giving geometric data. However it seems likely that some necessary conditions on the regularity of the space will be necessary.

Bibliography

[1] M.T. Barlow and S.N Evans, Markov processes on vermiculated spaces, *Random walks and geometry*, Walter de Gruyter GmbH & Co. KG, Berlin, 2004, pg. 337-348.

[2] A. Björn, J. Björn, *Nonlinear Potential Theory on Metric Spaces*, European Mathematical Society, 2011.

[3] J. Cheeger, B. Kleiner, Realization of Metric Spaces as Inverse Limits and Bilipschitz Embedding in L_1, *Geom. Funct. Anal.*, (23) **1** (2013), 96-133.

[4] R. Coifman, G. Weiss, *Analyse harmonique non-commutative sur certains espaces homogènes*, Lecture Notes in Math., vol. 242, Springer-Verlag, Berlin and New York, 1971.

[5] G. Dafni, R. J. McCann, A. Stancu (eds.), *Analysis and Geometry of Metric Measure Spaces*, Lecture Notes of the 50th Séminaire de Mathématiques Supérieures (SMS), (Montréal 2011), CRM Proceedings & Lecture Notes, **56**, Centre de Recherches Mathématiques (CRM), Montréal and Amer. Math. Soc., Providence, R.I., 2013.

[6] D. Danielli, N Garofalo, D. Nhieu, Non-Doubling Ahlfors Measures, Perimeter Measures, and the Characterization of the Trace Spaces of Sobolev Functions in Carnot-Carathéodory Spaces, *Mem. Am. Math. Soc.*, (857) **182** (2006), 1-119.

[7] G. David, S. Semmes, *Fractured Fractals and Broken Dreams: Self-Similar Geometry through Metric and Measure*, Oxford University Press, Oxford, 1997.

[8] G.B. Folland, *Real Analysis: Modern Techniques and Their Applications*, second edition, John Wiley and Sons, New York, 1999.

[9] R. Harvey, J. Polking, Removable singularities of solutions of linear partial differential equations, *Acta Math.* **125** (1970), 39-56.

[10] J. Heinonen, Nonsmooth calculus, *Bull. Amer. Math. Soc.*, **44** (2007), 163-232.

[11] J. Heinonen, P Koskela, Quasiconformal maps in metric spaces with controlled geometry, *Acta. Math.* **181** (1998), 1-61.

[12] A.E. Ingham, *The Distribution of Prime Numbers*, second edition (reprinted from the 1932 edition), Cambridge Univ. Press, Cambridge, 1992.

[13] N. Kajino, Analysis and geometry of the measurable Riemannian structure on the Sierpiński gasket, *Contemp. Math.*, **600** (2013), 91-134.

[14] J. Kigami, *Volume Doubling Measures and Heat Kernel Estimates of Self-Similar Sets*, *Mem. Amer. Math. Soc.*, (932) **199** (2009), 94 pages.

[15] J. Kigami, Measurable Riemannian structure on the Sierpiński gasket: The Kusuoka measure and the Gaussian heat kernel estimate, *Math. Ann.*, (4) **340** (2008), 781-804.

[16] T.J. Laakso, Ahlfors Q-Regular spaces with arbitrary $Q > 1$ admitting weak Poincaré inequality, *Geom. Func. Anal.* **10** (2000), 111-123.

[17] M.L. Lapidus, M. Frankenhuijsen, *Fractal Geometry, Complex Dimensions and Zeta Functions: Geometry and Spectra of Fractal Strings*, second edition (of the 2006 edition), Springer Monograph in Mathematics, Springer, New York, 2013, 593 pages.

[18] M.L. Lapidus, H. Maier, The Riemann hypothesis and inverse spectral problems for fractal strings, *J. London Math. Soc.* (2) **52** (1995), 15-34.

[19] M.L. Lapidus, G. Radunović, D. Žubrinić *Fractal Zeta Functions and Fractal Drums: Higher-Dimensional Theory of Complex Dimensions* Springer Monographs in Mathematics, Springer, New York, 2017, 684 pages.

[20] M.L. Lapidus, G. Radunović, D. Žubrinić, Distance and Tube zeta functions of fractals and arbitrary compact sets and relative fractal drums, *Adv. in Math.*, **307** (2017), 1215-1267.

[21] M.L. Lapidus, G. Radunović, D. Žubrinić, Complex dimensions of fractals and meromorphic extensions of fractal zeta functions, *J. Math. Anal. Appl.* No. 1, **453** (2017), 458-484.

[22] M.L. Lapidus, G. Radunović, D. Žubrinić, Zeta functions and complex dimensions of relative fractal drums: Theory, examples and applications, *Dissertationes Mathematicae*, in press, 2017, 101 pages.

[23] M.L. Lapidus, G. Radunović, D. Žubrinić, Fractal tube formulas and a Minkowski measurability criterion for compact subsets of Euclidean spaces, *Discrete and Continuous Dynamical Systems Ser. S*, in press, 2017.

[24] M.L. Lapidus, G. Radunović, D. Žubrinić, Fractal tube formulas for compact sets and relative fractal drums: Oscillations, complex dimensions and fractality, *Journal of Fractal Geometry*, in press, 2017, 104 pages.

[25] M.L. Lapidus, G. Radunović, D. Žubrinić, Minkowski measurability criteria for compact sets and relative fractal drums in Euclidean spaces, preprint, 2016.

[26] M.L. Lapidus, G. Radunović, D. Žubrinić, Fractal zeta functions and complex dimensions of relative fractal drums, survey article, *J. Fixed Point Theory Appl.* **15** (2014), 321-378.

[27] M.L. Lapidus, G. Radunović, D. Žubrinić, Fractal zeta functions and complex dimensions: A general higher-dimensional theory, survey article, in: *Fractal Geometry and Stochastics V* (C. Bandt, K. Falconer and M. Zähle, eds.), Proc. Fifth Internat. Conf. (Tabarz, Germany, March 2014), *Progress in Probability*, vol. 70, Birkhäuser/Springer Internat., Basel, Boston and Berlin, 2015, pp. 229-257.

[28] M.L. Lapidus, J. Sarhad, Dirac operators and geodesic metric on the harmonic Sierpiński gasket and other fractal sets, *J. Noncommut. Geom.* **8** (2014), 947-985.

[29] E.C. Titchmarsh, *Introduction to the Theory of Fourier Integrals*, second edition, Oxford University Press, Oxford, 1948.

[30] D. Žubrinić, Analysis of Minkowski contents of fractal sets and applications, *Chaos, Solitons & Fractals* **21** (2004), 1281-1287.

[31] D. Žubrinić, Analysis of Minkowski contents of fractal sets and applications *Real Anal. Exchange* No. 2, **31** (2005/2006), 315-354.

[32] D. Žubrinić, Hausdorff dimension of singular sets of Sobolev functions and applications, in: *More Progress in Analysis,* Proc. of the 5th Internat. ISAAC Congress (H.G.W. Begher and F. Nicolosi, eds.), World Scientific, Singapore, 2009, pp. 793-802.