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A sufficient condition for stochastic stability of an Internet congestion control model in terms of fluid model stability

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A sufficient condition for stochastic stability of an Internet congestion control model in terms of fluid model stability

A dissertation submitted in partial satisfaction of the requirements for the degree Doctor of Philosophy

in

Mathematics

by

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Professor Paul Siegel

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The dissertation of Nam H. Lee is approved, and it is acceptable in quality and form for publication on microfilm:

Chair

University of California, San Diego

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DEDICATION

I dedicate this thesis to my beloved mother.

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ACKNOWLEDGEMENTS

I have been fortunate enough to be around some good people who have helped steer me back in the right direction whenever I got off track, and I would like to take a moment here to thank them.

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VITA

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ABSTRACT OF THE DISSERTATION

A sufficient condition for stochastic stability of an Internet congestion control model in terms of fluid model stability

by

Nam H. Lee
Doctor of Philosophy in Mathematics

University of California San Diego, 2008

Professor Ruth Williams, Chair

We consider a model of Internet congestion control, introduced by Massoulié and Roberts, as an example of a stochastic network with resource sharing and a non-head-of-the-line service discipline. To describe the evolution of this system, we use a stochastic process that tracks the amount of service that has been given to each document that is still in the system and the time since the last arrival to each route. This is a Borel right process with a locally compact with countable base state space. It is shown that under mild assumptions, stability of a related fluid model for residual document sizes is sufficient for stability (positive Harris recurrence) of the Borel right process.

Chapter 1

Introduction

1.1 Overview

Stochastic networks are used as models for complex dynamic systems subject to uncertainty. Applications arise in manufacturing, telecommunications, computer systems, and the service industry. Two major questions of interest for such models are their stability and performance when heavily loaded. For a certain class of stochastic networks known as multiclass queueing networks operating under head-of-the-line (HL) service disciplines, there is now a fairly well developed theory of fluid and diffusion approximations for studying the stability and performance of these networks. For more general stochastic networks operating under non-HL service disciplines, there is currently no analogous theory of fluid and diffusion approximations.

In this thesis, a model of Internet congestion control is considered as an example of a stochastic network with resource sharing and a non-HL service discipline. With generally distributed document sizes, it is an open question whether this model is stable under a nominal load condition. The answer to this question is known to be yes for α-fair sharing disciplines and exponential document sizes (see Bonald and Massoulié [4] and De Veciana et al. [11]) and also for a proportional fair sharing discipline with phase type distributions for the document sizes (see Massoulié [20]). Here, under mild conditions, it is shown that if a related fluid model is stable, then a process describing the Markov dynamics of the stochastic Internet congestion control model is positive Harris recurrent, i.e., is stable. Some components of this process are measure-valued (to keep track of the partial processing of documents). Our result provides an analogue, for this stochastic network model, of a well known theorem for multiclass HL queueing networks due to Dai [8]. Our result proved in this thesis reduces the problem of establishing sufficient conditions for stability for the stochastic network model to proving stability for a simpler deterministic fluid model.

The model of Internet congestion control considered here was introduced by Massoulié and Roberts [21]. It aims to capture connection level dynamics in data networks like the Internet. The bandwidth sharing policies considered with this model are generalizations of the processor sharing discipline. Assuming Poisson arrivals and exponentially distributed document sizes, it is known that the Massoulié-Roberts model is stable when operating under various utility based bandwidth sharing policies, provided a nominal load condition is satisfied. The exponential assumptions permit one to use queue-length as a Markovian state descriptor in this context.

It is of considerable interest to understand the behavior of models like that of Massoulié and Roberts when the document sizes are more generally distributed than exponential. To describe the dynamics of such a model, one needs a higher dimensional state descriptor than queue-length. In particular, we use an age process that has measure-valued components that track the amount of service that has been given to each document that is still in the system and components that track the time since the last arrival to each route. This process is a Borel right process. We associate with this age process a residual document size process. Gromoll and Williams [14] developed a fluid model approximation for such residual processes.

The main result of this thesis is that stability of Gromoll and Williams' fluid model [13] is sufficient for stability (positive Harris recurrence) of our age process.

The thesis is organized as follows. In Section 1.2, we explain elementary terminology and notation that is used in this work, and in Section 1.3, relevant material from the theory of Borel right processes is reviewed. In Chapter 2, we introduce the age process as a Borel right process. We introduce the notion of a fluid model solution in Section 2.6 and state our main theorem in Section 2.7 under two assumptions, Assumptions 2.2.1 and 2.2.2 that are specified in Section 2.2. While the age process is the main object in our stability analysis, the residual document size process associated with the age process plays an important role in connecting the age process to the fluid model developed in Gromoll and Williams [13]. In Chapter 3, we will carefully verify that the hypotheses of the main theorem of Gromoll and Williams [13] are satisfied by the residual document size process associated with the age process. In fact, Assumptions 2.2.1 and 2.2.2 are formulated chiefly to overcome various difficulties that arise when doing this. In Chapter 4, we give our proof of the main theorem. The overall proof strategy used there is similar to that in Dai [8] and Bramson [5]. However, there are differences because of the measure-valued aspects of the age process and the resource sharing aspect of our model.

1.2 Notation and terminology

In this section, we suppose that $\mathbb{X}$ is a locally compact space with countable space, and that $(\Omega, \mathcal{F}, \mathbf{P})$ is a probability space. We list some elementary notation and terminology that will be used throughout our discussion.

We denote by $\mathcal{B}(\mathbb{X})$ the Borel subsets of $\mathbb{X}$ and denote by $\mathbf{M}(\mathbb{X})$ the space of finite non-negative Borel measures on $\mathbb{X}$. For each $x \in \mathbb{X}$, we denote by δ_x, the Dirac delta measure at x. We let χ be the function from $\mathbb{X}$ to $\mathbb{X}$ such that $\chi(x) = x$ for any $x \in \mathbb{X}$. For any measurable function $f : \mathbb{X} \to (-\infty, \infty)$ and $B \in \mathcal{B}(\mathbb{X})$, we let $\|f\|_B = \sup\{|f(x)| : x \in B\}$, where this may take the value ∞. Let $\mu \in \mathbf{M}(\mathbb{X})$. For a Borel measurable function $f : \mathbb{X} \to [0, \infty]$, we denote by $\langle f, \mu \rangle$, the integral $\int_{\mathbb{X}} f(x)\mu(dx)$, which may take the value ∞. Moreover, given another Borel measurable function $g : \mathbb{X} \to [0, \infty)$ such that $\min\{\langle f, \mu\rangle, \langle g, \mu\rangle\} < \infty$, we let $\langle f - g, \mu\rangle = \langle f, \mu\rangle - \langle g, \mu\rangle$ and $\langle g - f, \mu\rangle = \langle g, \mu\rangle - \langle f, \mu\rangle$.

We denote by $\mathcal{C}_b(\mathbb{X})$ the set of bounded continuous real-valued functions on $\mathbb{X}$, by $\mathcal{C}_u(\mathbb{X})$ the set of uniformly continuous real-valued functions on $\mathbb{X}$, and by $\mathcal{C}_c(\mathbb{X})$ the set of continuous real-valued functions on $\mathbb{X}$ having compact support. Also, let $\mathcal{C}^+(\mathbb{X})$ be the space of non-negative continuous functions on $\mathbb{X}$. Then, we let $\mathcal{C}_c^+(\mathbb{X}) = \mathcal{C}_c(\mathbb{X}) \cap \mathcal{C}^+(\mathbb{X})$, $\mathcal{C}_b^+(\mathbb{X}) = \mathcal{C}_b(\mathbb{X}) \cap \mathcal{C}^+(\mathbb{X})$ and $\mathcal{C}_{b,u}^+(\mathbb{X}) = \mathcal{C}_b(\mathbb{X}) \cap \mathcal{C}_u(\mathbb{X}) \cap \mathcal{C}^+(\mathbb{X})$.

A random element taking values in $\mathbb{X}$ is a measurable function $X : \Omega \to \mathbb{X}$. For each such random element X, the distribution of X is the probability measure μ on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ such that $\mu(B) = \mathbf{P}[X \in B]$ for each $B \in \mathcal{B}(\mathbb{X})$. Suppose that $\{X_n\}_{n=1}^\infty$ is a sequence of random elements (possibly defined on different probability spaces) taking values in $\mathbb{X}$, and for each integer $n \geq 1$, let μ_n be the distribution of X_n. Also, suppose that X_0 is a random element that takes values in $\mathbb{X}$, and let μ_0 denote the distribution of X_0. The sequence $\{X_n\}_{n=1}^\infty$ converges in distribution to X_0 if and only if for any $f \in \mathcal{C}_b^+(\mathbb{X})$, $\langle f, \mu_n\rangle \to \langle f, \mu\rangle$ as $k \to \infty$; in this case, we write $X_n \Rightarrow X_0$ as $n \to \infty$. The random element X_0 is said to be a weak limit point of the sequence $\{X_n\}_{n=1}^\infty$ if there exists a subsequence $\{X_{n_k}\}_{k=1}^\infty$ of $\{X_n\}_{n=1}^\infty$ such that $X_{n_k} \Rightarrow X_0$ as $k \to \infty$. The sequence $\{X_n\}_{n=1}^\infty$ is said to be tight if for each $\varepsilon \in (0, 1)$, there exists a compact subset K_ε of $\mathbb{X}$ such that $\liminf_{n\to\infty} \mu_n(K_\varepsilon) \geq 1 - \varepsilon$.

We denote by $\mathbb{D}(\mathbb{X})$ the set of all functions from $[0, \infty)$ into $\mathbb{X}$ that are right continuous on $[0, \infty)$ and that have finite left limits on $(0, \infty)$ and denote by $\mathbb{C}(\mathbb{X})$ the set of all continuous functions from $[0, \infty)$ into $\mathbb{X}$. Unless it is stated otherwise, the space $\mathbb{D}(\mathbb{X})$ will be equipped with the Skorohod J_1 topology and the space $\mathbb{C}(\mathbb{X})$ will be equipped with the topology of uniform convergence on compact time intervals. With these topologies, the spaces $\mathbb{D}(\mathbb{X})$ and $\mathbb{C}(\mathbb{X})$ are separable metric spaces and we consider them with the associated Borel σ-algebras. A sequence $\{\xi_n\}_{n=1}^\infty$ of random elements taking values in $\mathbb{D}(\mathbb{X})$ is said to be C-tight if $\{\xi_n\}_{n=1}^\infty$ is tight and each weak limit point of $\{\xi_n\}_{n=1}^{\infty}$ take values almost surely in $\mathbb{C}(\mathbb{X})$.

The set $\mathbf{M}(\mathbb{X})$ can be equipped either with the topology of weak convergence or with the topology of vague convergence, and in either case, $\mathbf{M}(\mathbb{X})$ is known to be a separable metric space (cf. Kallenberg [19]). In particular, given a sequence $\{\nu_k\}_{k=1}^{\infty}$ in $\mathbf{M}(\mathbb{X})$ and a measure $\nu_0 \in \mathbf{M}(\mathbb{X})$, the sequence $\{\nu_k\}_{k=1}^{\infty}$ converges weakly (i.e., in the topology of weak convergence) to $\nu_0 \in \mathbf{M}(\mathbb{X})$ if and only if for any $f \in \mathcal{C}_b^+(\mathbb{X})$, $\langle f, \nu_k \rangle \to \langle f, \nu_0 \rangle$ as $k \to \infty$, but the sequence $\{\nu_k\}_{k=1}^{\infty}$ converges vaguely (i.e., in the topology of vague convergence) to ν_0 if and only if for any $f \in \mathcal{C}_c^+(\mathbb{X})$, $\langle f, \nu_k \rangle \to \langle f, \nu_0 \rangle$ as $k \to \infty$. Unless it is stated otherwise, the (default) topology on $\mathbf{M}(\mathbb{X})$ will be the topology of weak convergence.

A filtration is a collection $\{\mathcal{F}_t : t \in [0, \infty)\}$ of sub-σ-algebras of $\mathcal{F}$ such that $\mathcal{F}_s \subset \mathcal{F}_t$ for all s and $t \in [0, \infty)$. A random variable η taking values in $[0, \infty]$ is a stopping time for a filtration $\{\mathcal{F}_t : t \in [0, \infty)\}$ if $\{\eta \leq t\} \in \mathcal{F}_t$ for each $t \in [0, \infty)$.

We let $\mathbb{R}$ denote the set of all real numbers, i.e., $(-\infty, \infty)$, we let $\mathbb{R}_+$ denote $[0, \infty)$, and we let $\mathbb{Q}$ denote the set of all rational numbers. We will simply write $\mathbf{M}$ for $\mathbf{M}(\mathbb{R}_+)$, and for each integer $n \geq 1$, $\mathbf{M}^n$ denotes the product space $\mathbf{M} \times \ldots \times \mathbf{M}$, which involves exactly n copies of $\mathbf{M}$. If x and y are in $[-\infty, \infty)$, then $x \wedge y$ is the minimum of x and y, and $x \vee y$ is the maximum of x and y. For any $x \in [-\infty, \infty]$, $x^+ = x \vee 0$. For each $\mathbb{R}$, let $\lfloor x \rfloor$ be the largest integer that is less than or equal to x. For each $B \subset \mathbb{R}_+$ and $r \in \mathbb{R}_+$, we let $B + r = \{x + r : x \in B\}$. For each integer $n \geq 1$ and $x = (x_1, \ldots, x_n) \in (-\infty, \infty)^n$, $\|x\| = \sum_{k=1}^n |x_k|$. For each function f defined on $\mathbb{R}_+$, we let $\|f\|_\infty = \|f\|_{\mathbb{R}_+}$. Let $\mathcal{C}_b^1(\mathbb{R}_+)$ denote the set of once continuously differentiable functions on $\mathbb{R}_+$ that together with their first derivatives are bounded on $\mathbb{R}_+$. For non-negative finite Borel measures ξ and $\widetilde{\xi}$ on $\mathbb{R}_+$, we denote by $\xi * \widetilde{\xi}$, the convolution of ξ and $\widetilde{\xi}$, and for each integer $n \geq 1$, we denote by $\xi^{(*n)}$ the convolution of n copies of ξ. For any integer $n \geq 1$ and $\mu \in \mathbf{M}(\mathbb{R}_+^n)$, we will always identify the measure μ with the measure $\overline{\mu} \in \mathbf{M}(\mathbb{R}^n)$ such that $\overline{\mu}(B) = \mu(B \cap \mathbb{R}_+^n)$ for each $B \in \mathcal{B}(\mathbb{R}^n)$.

The phrase "independent and identically distributed" will be abbreviated as i.i.d., and the phrase "right continuous with finite left limits" will be abbreviated as r.c.l.l.

1.3 Borel right process and Harris recurrence

Our definition of a Borel right process will follow that in [12], and we begin with some basic notation and terminology used in the theory of Borel right processes.

Throughout this section, suppose that $\mathbb{X}$ is a locally compact space with countable base and let $\Omega = \mathbb{D}(\mathbb{X})$. For each $t \in [0,\infty)$, let X_t be the mapping from Ω to $\mathbb{X}$ given by $X_t(\omega) = \omega(t)$, where $\omega(t)$ is the value of the function ω at t. Next, let $\mathcal{F} = \sigma(X_t : t \in [0,\infty))$. For each $t \in [0,\infty)$, let $\mathcal{F}_t = \sigma\left(X_s : s \in [0,t]\right)$, and let $\mathcal{F}_{t+} = \cap_{s\in(t,\infty)}\mathcal{F}_s$. Then, let $\mathfrak{F} = \{\mathcal{F}_t : t \in [0,\infty)\}$ and $X = \{X_t : t \in [0,\infty)\}$.

A *Borel Markov kernel* on $\mathbb{X}$ is a function $K : \mathbb{X} \times \mathcal{B}(\mathbb{X}) \to [0,1]$ such that for each $x \in \mathbb{X}$, the function $K(x,\cdot) : \mathcal{B}(\mathbb{X}) \to [0,1]$ defines a probability measure on $\mathbb{X}$, and for each $B \in \mathcal{B}(\mathbb{X})$, the function $K(\cdot,B) : \mathbb{X} \to [0,1]$ is Borel measurable. For each bounded real-valued Borel measurable function f on $\mathbb{X}$ and Borel Markov kernel K on $\mathbb{X}$, we define a real-valued function Kf on $\mathbb{X}$ by letting, for each $x \in \mathbb{X}$,

$$(Kf)(x) = \int_{\mathbb{X}} K(x,dy)f(y).$$

A (homogeneous) *Borel Markov semigroup* on $\mathbb{X}$ is a collection $\{P_t : t \in [0,\infty)\}$ of Borel Markov kernels on $\mathbb{X}$ such that for any $x \in \mathbb{X}$, $B \in \mathcal{B}(\mathbb{X})$ and $[s,t] \subset [0,\infty)$,

$$P_t(x,B) = \int_{\mathbb{X}} P_s(x,dy)P_{t-s}(y,B). \tag{1.3.1}$$

A *stopping time* τ *for the filtration* $\{\mathcal{F}_{t+} : t \in [0,\infty)\}$ is a non-negative random variables taking values in $[0,\infty]$ such that $\{\tau \le t\} \in \mathcal{F}_{t+}$ for each $t \in [0,\infty)$.

Definition 1.3.1. A Borel Markov semigroup $\{P_t : t \in [0,\infty)\}$ on $(\mathbb{X},\mathcal{B}(\mathbb{X}))$ is said to be *right* if for each probability measure μ on $(\mathbb{X},\mathcal{B}(\mathbb{X}))$, there exists a probability

$\mathbf{P}^{\mu}$ on $(\Omega, \mathcal{F})$ such that for each $B \in \mathcal{B}(\mathbb{X})$,

$$\mathbf{P}^{\mu}[X_0 \in B] = \mu(B), \tag{1.3.2}$$

$$\mathbf{P}^{\mu}[X_{t+s} \in B \,|\mathcal{F}_s] = P_t(X_s, B), \text{ for each } s \text{ and } t \in [0, \infty), \tag{1.3.3}$$

and for each stopping time τ for the filtration $\{\mathcal{F}_{t+} : t \in [0, \infty)\}$ and $f \in \mathcal{C}^+_{b,u}(\mathbb{X})$,

$$\mathbf{E}^{\mu}[f(X_{\tau+t})1_{\{\tau<\infty\}} \,|\mathcal{F}_{\tau+}] = P_t f(X_\tau)1_{\{\tau<\infty\}}, \text{ for each } t \in [0, \infty), \tag{1.3.4}$$

where $\mathbf{E}^{\mu}$ denotes the expectation operator for $\mathbf{P}^{\mu}$.

For the rest of this subsection, we fix a collection $\mathcal{P} = \{P_t : t \in [0, \infty)\}$ of Borel Markov kernels, and we will say that $\mathcal{P}$ is a *Borel right semigroup* if $\mathcal{P}$ is a Borel Markov semigroup that is right.

Definition 1.3.2. The collection $(\Omega, \mathcal{F}, \mathfrak{F}, X, \mathcal{P})$ is said to be a *Borel right process* if the collection $\mathcal{P}$ is a Borel right semigroup.

One of the most frequently used concepts of stability for a Borel right process involves a notion that can be traced back to T. E. Harris [15]. To introduce this, for the rest of this section, we assume that $(\Omega, \mathcal{F}, \mathfrak{F}, X, \mathcal{P})$ is a Borel right process, and for each probability measure μ on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$, let $\mathbf{P}^{\mu}$ be as described in Definition 1.3.1. Then, for simplicity, for each $x \in \mathbb{X}$, let $\mathbf{P}^x$ denote the probability measure $\mathbf{P}^{\delta_x}$.

Definition 1.3.3. The Borel right process $(\Omega, \mathcal{F}, \mathfrak{F}, X, \mathcal{P})$ is *Harris recurrent* if there exists a σ-finite non-trivial measure ψ on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ such that if $B \in \mathcal{B}(\mathbb{X})$ and $\psi(B) > 0$, then $\mathbf{P}^x\,[\eta_B < \infty] = 1$ for each $x \in \mathbb{X}$, where $\eta_B = \inf\{t \in [0, \infty) : X_t \in B\}$.

An *invariant measure* for the Borel right process $(\Omega, \mathcal{F}, \mathfrak{F}, X, \mathcal{P})$ is a non-trivial σ-finite measure π on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ that satisfies

$$\pi(B) = \int_{\mathbb{X}} \pi(dx) P_t(x, B), \tag{1.3.5}$$

for each $t \in [0, \infty)$ and $B \in \mathcal{B}(\mathbb{X})$. Every Harris recurrent Borel right process has an invariant measure that is unique up to multiplication by a constant factor. This was shown initially in Azéma et al. [17] but also later with some simplification in Meyn and Tweedie [22].

Definition 1.3.4. The Borel right process $(\Omega, \mathcal{F}, \mathfrak{F}, X, \mathcal{P})$ is *positive Harris recurrent* if $(\Omega, \mathcal{F}, \mathfrak{F}, X, \mathcal{P})$ is Harris recurrent and there exists a *probability* measure π on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ satisfying (1.3.5) for each $t \in [0, \infty)$ and $B \in \mathcal{B}(\mathbb{X})$.

Chapter 2

Internet congestion control model

2.1 Network structure, bandwidth sharing policy and model description

We consider a flow-level model of Internet congestion control introduced by Massoulié and Roberts [21]. The network has finitely many resources labeled by $j = 1, \ldots, \mathbb{J}$ and a finite set of routes labeled by $i = 1, \ldots, \mathbb{I}$. A route i is a non-empty subset of the resources, interpreted as the set of resources used by the route. Let M be a $\mathbb{J} \times \mathbb{I}$ matrix such that $M_{ji} = 1$ if resource j is used by route i, and $M_{ji} = 0$ otherwise. Since each route is a non-empty subset of $\{1, \ldots, \mathbb{J}\}$, no column of M is identically zero.

A flow on route i corresponds to the continuous transfer of a document through the resources used by the route. While the transfer is in progress, the flow takes simultaneous possession of each resource on its route. The processing rate allocated to a flow is the rate at which the document associated with the flow is being transferred. The bandwidth allocated to route i is the sum of the processing rates allocated to all flows on route i and it is shared equally amongst the flows on route i. The bandwidth allocated through resource j is the sum of the processing rates allocated to all flows on the routes using resource j. Each resource j has finite capacity $C_j > 0$, interpreted as the maximum bandwidth that can be allocated through it.

Our work will focus on the network operating under a certain bandwidth sharing policy Λ, where the policy Λ dynamically determines the bandwidth allocation to the routes as a function of the number of flows on all routes. In other words, for a vector $n = (n_1, \ldots, n_{\mathbb{I}})$ of non-negative integers, when each coordinate n_i represents the number of flows on route i, then the bandwidth sharing policy Λ associates with the vector n another vector $\Lambda(n) \in \mathbb{R}_+^{\mathbb{I}}$, where the i-th component $\Lambda_i(n)$ of $\Lambda(n)$ represents the bandwidth allocated to route i, and the allocation respects the capacity constraints:

$$\sum_{i=1}^{\mathbb{I}} M_{ji}\Lambda_i(n) \leq C_j, \text{ for each resource } j. \tag{2.1.1}$$

The bandwidth sharing policy Λ can be regarded as a measurable function from the set of vectors of non-negative integers to the set of vectors of non-negative real numbers. In fact, we shall define Λ so that we can also apply it to scaled flow counts. We assume that Λ is a Borel measurable function from $\mathbb{R}_+^{\mathbb{I}}$ to $\mathbb{R}_+^{\mathbb{I}}$ such that

(i) for any $r \in (0, \infty)$ and $n \in \mathbb{R}_+^{\mathbb{I}}$, $\Lambda(n) = \Lambda(rn)$,

(ii) for each route i and $n \in \mathbb{R}_+^{\mathbb{I}}$, $\Lambda_i(n) > 0$ if and only if $n_i > 0$,

(iii) for each route i and $n \in \mathbb{R}_+^{\mathbb{I}}$, Λ_i is continuous at n whenever $n_i > 0$,

(iv) $\sum_{i=1}^{\mathbb{I}} M_{ji}\Lambda_i(n) \leq C_j$ for each $n \in \mathbb{R}_+^{\mathbb{I}}$ and resource j.

An important class of bandwidth sharing policies is described below. This family of policies was introduced by Mo and Warland [24].

Example 2.1.1. Fix $\alpha \in (0, \infty)$ and $w = (w_1, \ldots, w_{\mathbb{I}}) \in (0, \infty)^{\mathbb{I}}$. For each $n =$

$(n_1, \ldots, n_{\mathbb{I}}) \in \mathbb{R}_+^{\mathbb{I}}$, let

$$\mathcal{I}_0(n) = \{i \le \mathbb{I} : n_i = 0\},$$
$$\mathcal{I}_+(n) = \{i \le \mathbb{I} : n_i > 0\},$$
$$\mathbb{O}(n) = \{\lambda \in \mathbb{R}_+^{\mathbb{I}} : \lambda_i = 0 \text{ for all } i \in \mathcal{I}_0(n)\},$$

and define a function $F_n : \mathbb{R}_+^{\mathbb{I}} \to [-\infty, \infty)$ by letting, for each $\lambda = (\lambda_1, \ldots, \lambda_{\mathbb{I}}) \in \mathbb{R}_+^{\mathbb{I}}$,

$$F_n(\lambda) = \begin{cases} \sum_{i \in \mathcal{I}_+(n)} w_i n_i^\alpha \frac{\lambda_i^{1-\alpha}}{1-\alpha}, & \alpha \in (0,\infty) \setminus \{1\}, \\ \sum_{i \in \mathcal{I}_+(n)} w_i n_i \log(\lambda_i), & \alpha = 1, \end{cases} \tag{2.1.2}$$

where $F_n(\lambda) = 0$ if $\mathcal{I}_+(n) = \varnothing$, but $F_n(\lambda) = -\infty$ if $\alpha \in [1, \infty)$ and there exists $i \in \mathcal{I}_+(n)$ such that $\lambda_i = 0$.

Define a function $\Lambda : \mathbb{R}_+^{\mathbb{I}} \to \mathbb{R}_+^{\mathbb{I}}$ by letting, for each $n = (n_1, \ldots, n_{\mathbb{I}}) \in \mathbb{R}_+^{\mathbb{I}}$, $\Lambda(n)$ to be the unique vector $\lambda = (\lambda_1, \ldots, \lambda_{\mathbb{I}}) \in \mathbb{R}_+^{\mathbb{I}}$ that solves the following optimization problem:

$$\begin{aligned} &\text{maximize} && F_n(\lambda) && (2.1.3)\\ &\text{subject to} && \textstyle\sum_{i=1}^{\mathbb{I}} M_{ji}\lambda_i \le C_j \text{ for each resource } j, && (2.1.4)\\ &\text{over} && \mathbb{O}(n). && (2.1.5) \end{aligned}$$

The function $\Lambda : \mathbb{R}_+^{\mathbb{I}} \to \mathbb{R}_+^{\mathbb{I}}$ is called a *weighted α-fair bandwidth sharing policy.* Note that by (2.1.4), Λ satisfies the condition (iv) in our definition of a bandwidth sharing policy, and it can also be seen that the other conditions (i)–(iii) in our definition of a bandwidth sharing policy are also satisfied by Λ (cf. Gromoll and Williams [13]).

When $w_i = 1$ for each route i, the case $\alpha = 1$ and the limiting cases $\alpha \to 0$ and $\alpha \to \infty$ correspond respectively to a bandwidth sharing policy that is *proportionally fair*, achieves *maximum throughput*, or is *max-min fair* (cf. Bonald and Massoulié [4], and Mo and Walrand [24]). □

We now return to the case where n is a vector of non-negative integers so that each $\Lambda_i(n)$ represents the bandwidth allocated to route i. If there is only one flow on route i, or equivalently, if $n_i = 1$, then the allocated bandwidth $\Lambda_i(n)$ is used in full to process that one flow. On the other hand, if there are multiple flows on route i, or equivalently, if $n_i \geq 2$, then the bandwidth $\Lambda_i(n)$ allocated to route i is shared equally amongst all of the flows on route i so that $\Lambda_i(n)/n_i$ is the processing rate for each flow on route i. If $n_i = 0$ for route i, then $\Lambda_i(n) = 0$. Once the document associated with a flow is fully transmitted, the flow is assumed to disappear from the network.

Every flow is assumed to arrive to the network at some point, and we distinguish the flows that have arrived prior to or at time zero from the flows arriving to the network after time zero. In particular, an *exogenous flow* is any flow that enters the network after time 0, and an *initial flow* is any flow already on some route at time zero.

The exogenous flows are specified by $2\mathbb{I}$ independent sequences of random variables: $\{u_{1k}\}_{k=1}^{\infty}, \ldots, \{u_{\mathbb{I}k}\}_{k=1}^{\infty}$ and $\{v_{1k}\}_{k=1}^{\infty}, \ldots, \{v_{\mathbb{I}k}\}_{k=1}^{\infty}$ describing respectively the interarrival times and document sizes for the exogenous arrivals to the network after time zero. For each route i, $\{u_{ik}\}_{k=1}^{\infty}$ are i.i.d. strictly positive random variables with common distribution φ_i satisfying $\langle \chi, \varphi_i \rangle < \infty$, and $\{v_{ik}\}_{k=1}^{\infty}$ are i.i.d. strictly positive random variables with common distribution ϑ_i satisfying $\langle \chi, \vartheta_i \rangle < \infty$. Here, for each route i and integer $k \geq 1$, the random variable u_{ik} denotes the time between the k-th and the $(k+1)$-st arrival to route i after time zero, and v_{ik} denotes the size of the k-th document to arrive to route i after time zero. For future reference, we let, for each route i,

$$\nu_i = \frac{1}{\langle \chi, \varphi_i \rangle} \text{ and } \rho_i = \frac{\langle \chi, \vartheta_i \rangle}{\langle \chi, \varphi_i \rangle}, \tag{2.1.6}$$

and then, let

$$\nu = (\nu_1, \ldots, \nu_{\mathbb{I}}) \text{ and } \rho = (\rho_1, \ldots, \rho_{\mathbb{I}}). \tag{2.1.7}$$

A convenient way to specify an initial state for the network is by identifying a suitable element from the set $\mathbb{A}$:

$$\mathbb{A} = \mathbb{A}_{\alpha_1} \times \ldots \times \mathbb{A}_{\alpha_{\mathbb{I}}} \times [0, \beta_1) \times \ldots \times [0, \beta_{\mathbb{I}}), \tag{2.1.8}$$

where for each route i,

$$\begin{aligned} \alpha_i &= \inf\{\alpha \in [0, \infty) : \vartheta_i((\alpha, \infty)) = 0\}, \\ \beta_i &= \inf\{\beta \in [0, \infty) : \varphi_i((\beta, \infty)) = 0\}, \\ \mathbb{A}_{\alpha_i} &= \{0\} \cup \textstyle\bigcup_{n=1}^{\infty} \left\{\sum_{k=1}^{n} \delta_{c_k} : c_1, \ldots, c_n \in [0, \alpha_i), c_1 < c_2 < \ldots < c_n\right\}; \end{aligned}$$

here, 0 denotes the zero measure on $[0, \alpha_i)$. For instance, suppose that at time zero, for each route i, there are n_i flows on route i and the flow on route i with the k-th smallest completed work has completed work size $c_{ik} \in [0, \alpha_i)$. Also, suppose that the last time that a flow arrived to route i is $a_i \in [0, \beta_i)$ units of time ago for each route i. Then, these can be neatly encoded into an element in $\mathbb{A}$,

$$\left(\sum_{k=1}^{n_1} \delta_{c_{1k}}, \ldots, \sum_{k=1}^{n_{\mathbb{I}}} \delta_{c_{\mathbb{I}k}}, a_1, \ldots, a_{\mathbb{I}} \right), \tag{2.1.9}$$

where an empty sum is defined to be zero. The topology on $\mathbb{A}$ is the topology induced on $\mathbb{A} \subset \mathbf{M}^{\mathbb{I}} \times \mathbb{R}_+^{\mathbb{I}}$ by the product topology on $\mathbf{M}^{\mathbb{I}} \times \mathbb{R}_+^{\mathbb{I}}$, where each $\mathbf{M}$ of $\mathbf{M}^{\mathbb{I}}$ is equipped with the topology of weak convergence. It is shown in Section B.1 that with this topology, $\mathbb{A}$ is locally compact with countable base. We forewarn the reader here that in Chapter 3, we also consider use of the vague topology for $\mathbf{M}$.

Notice the strict inequalities: "$c_1 < c_2 < \ldots < c_{n-1} < c_n$" in the definition of each $\mathbb{A}_{\alpha_i}$. These strict inequalities reflect the fact that for any two distinct flows that are being transmitted on the same route at the same time, their completed works must be different if their arrival times are different. This is because any flow must be transmitted at a positive rate at any moment while it is in the network and at any instant, the rate is the same for all flows currently on the route. Hence, in order for two distinct flows on the same route that are active at the same time to have the same completed work size, they must have arrived to their common route at the same time. However, for each route i, interarrival times of flows to the route i are strictly positive and so, this coincidence does not occur.

We now return to our discussion of how elements in $\mathbb{A}$ can be used as a way to describe initial states of the network. First, for each route i, $\alpha \in [0, \alpha_i)$ and $\beta \in [0, \beta_i)$, let ϑ_i^α and φ_i^β be the probability measures on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ such that

$$\vartheta_i^\alpha(B) = \vartheta_i(B + \alpha)/\vartheta_i((\alpha, \infty)), \tag{2.1.10}$$

$$\varphi_i^\beta(B) = \varphi_i(B + \beta)/\varphi_i((\beta, \infty)), \tag{2.1.11}$$

for $B \in \mathcal{B}((0, \infty))$ and $\vartheta_i^\alpha(\{0\}) = 0 = \varphi_i^\beta(\{0\})$. Then, for each route i, $\alpha \in [0, \alpha_i)$ and $\beta \in [0, \beta_i)$, ϑ_i^α is the probability measure describing the residual work size for initial flows on route i whose completed work size at time zero is α, and φ_i^β is the probability measure describing the time of the first exogenous route i arrival after time zero given that at time zero, the last arrival to route i occurred β units of time ago. Also, note here that $\vartheta_i^0 = \vartheta_i$ and $\varphi_i^0 = \varphi_i$ for each route i.

Now, let x be the element that is described in (2.1.9). In our interpretation of the element x given earlier, while the completed work of the initial flows has been specified, nothing about the residual work of the initial flows has been said. Similarly, nothing has been said about the residual time till the first exogenous arrival. These quantities unspecified by x are specified by families of independent random variables $\{\widetilde{u}_i^x\}_{i=1}^{\mathbb{I}}$ and $\{\widetilde{v}_{1k}^x\}_{k=1}^{n_1}, \dots, \{\widetilde{v}_{\mathbb{I}k}^x\}_{k=1}^{n_\mathbb{I}}$ that are independent of $\{u_{1k}\}_{k=1}^\infty, \dots, \{u_{\mathbb{I}k}\}_{k=1}^\infty$ and $\{v_{1k}\}_{k=1}^\infty, \dots, \{v_{\mathbb{I}k}\}_{k=1}^\infty$. Here, for each route i, $\widetilde{u}_i^x$ is a random variable with distribution $\varphi_i^{a_i}$ describing the initial residual time till the first exogenous arrival to route i and for each $k \in \{1, \dots, n_i\}$, $\widetilde{v}_{ik}^x$ is a random variable with distribution $\vartheta_i^{c_{ik}}$ describing the initial residual work for the initial flow whose completed work is c_{ik}. Given the sequences

$$\{\widetilde{u}_i^x\}_{i=1}^{\mathbb{I}}, \{\widetilde{v}_{1k}^x\}_{k=1}^{n_1}, \dots, \{\widetilde{v}_{\mathbb{I}k}^x\}_{k=1}^{n_\mathbb{I}}, \{u_{1k}\}_{k=1}^\infty, \dots, \{u_{\mathbb{I}k}\}_{k=1}^\infty, \{v_{1k}\}_{k=1}^\infty, \dots, \{v_{\mathbb{I}k}\}_{k=1}^\infty$$

and a bandwidth sharing policy Λ, the dynamics of the Internet congestion control model starting from x and operating under the bandwidth sharing policy can be described in a discrete event manner.

2.2 Assumptions 2.2.1 and 2.2.2

The main theorem of this thesis shows that under mild assumptions, stability of the deterministic fluid model implies stability of the stochastic network model. Assumptions 2.2.1 and 2.2.2 are the two mild assumptions as defined below. These are assumed to hold for the rest of our discussion.

Assumption 2.2.1. *For each route* i*,*

(i) *the document size distribution* ϑ_i *has no atoms in* $\mathbb{R}_+$*,*

(ii) *there exists a subprobability measure* Θ_i *on* $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ *such that for each* $f \in \mathcal{C}_c^+(\mathbb{R}_+)$*,*

$$\lim_{\alpha \uparrow \alpha_i} \langle f, \vartheta_i^\alpha \rangle = \langle f, \Theta_i \rangle, \tag{2.2.1}$$

(iii)

$$\lim_{r\to\infty} \sup_{\alpha \in [0,\alpha_i)} \langle \chi 1_{[r,\infty)}, \vartheta_i^\alpha \rangle = 0. \tag{2.2.2}$$

Assumption 2.2.2. *For each route* i*,*

(i) *for each* $r \in \mathbb{R}_+$*,* $\varphi_i((r,\infty)) > 0$*; in other words,* $\beta_i = \infty$*,*

(ii) *there exist an integer* $\kappa_i \geq 1$ *and a Borel measurable function* $\upsilon_i : \mathbb{R}_+ \to \mathbb{R}_+$ *such that* $\int_0^\infty \upsilon_i(x)dx > 0$ *and for each* $B \in \mathcal{B}(\mathbb{R}_+)$*,*

$$\varphi_i^{(*\kappa_i)}(B) \geq \int_0^\infty 1_B(x)\upsilon_i(x)dx, \tag{2.2.3}$$

(iii)

$$\limsup_{r\to\infty} \frac{1}{r} \sup_{\beta\in[0,r]} \langle \chi, \varphi_i^\beta \rangle = 0. \tag{2.2.4}$$

Remarks about Assumptions 2.2.1 and 2.2.2

(a) For each route i, since Assumption 2.2.1 requires that ϑ_i has no atoms, we have $\vartheta_i(\{\alpha_i\}) = 0$ and hence, without loss of generality, we may further assume that for each $x \in \mathbb{A}$ taking the form in (2.1.9), each random variable in $\{\widetilde{v}_{ik}^x\}_{k=1}^{n_1} \cup \{v_{ik}\}_{k=1}^\infty$ only takes values in $(0, \alpha_i)$ rather than $(0, \alpha_i]$; in fact, for route i such that $n_i > 0$, each $\widetilde{v}_{ik}^x$ is a random variable taking values almost surely in the smaller interval $(0, \alpha_i - c_{ik})$.

(b) The condition (iii) in Assumption 2.2.1 relates to tail behavior of ϑ_i. Roughly speaking, it says that no ϑ_i should have a "heavy" tail. For example, on one extreme, if ϑ_i is a Pareto distribution or a log-normal distribution, then one can easily compute that

$$\lim_{\alpha\to\infty} \langle \chi, \vartheta_i^\alpha \rangle = \infty,$$

which is inconsistent with (2.2.2). On the other extreme, if ϑ_i is a probability distribution with bounded support i.e., $\alpha_i < \infty$, then (2.2.2) is clearly satisfied. Another notable class of distributions satisfying (2.2.2) is the class of finite mixtures of Erlang distributions with the same intensity; the density of such a distribution has the form

$$\sum_{\ell=1}^m q_\ell \lambda^{n_\ell+1} \frac{x^{n_\ell}}{(n_\ell)!} e^{-\lambda x}, \quad x > 0, \tag{2.2.5}$$

where $m \geq 1$, $n_1, \ldots, n_m \geq 0$ are integers, $\lambda \in (0, \infty)$, and $\sum_{\ell=1}^m q_\ell = 1$ with each $q_\ell > 0$. If the distribution ϑ_i in fact has a density of the form (2.2.5), one can check that

$$\sup_{\alpha\in[0,\infty)} \langle \chi^2, \vartheta_i^\alpha \rangle < \infty,$$

which implies (2.2.2) holds. It is well-known that as well as being a subclass of the phase type distributions, this class of distributions, approximates any probability distribution on $\mathbb{R}_+$ arbitrarily well in the sense of Theorem 4.2 in Asmussen [1].

(c) By Alaoglu's theorem, the set of all subprobability measures on $\mathbb{R}_+$ is *compact in the vague topology*. Hence, even without the condition (ii) of Assumption 2.2.1, for each route i, the collection $\{\vartheta_i^\alpha : \alpha \in [0, \alpha_i)\}$ is relatively compact in the vague topology. So, the restriction posed by the condition (ii) of Assumption 2.2.1 is that for each route i, the collection $\{\vartheta_i^\alpha : \alpha \in [0, \alpha_i)\}$ of probability measures has a *unique* vague limit point while the limit point can be a subprobability measure. This uniqueness requirement combined with the no atoms property of ϑ_i implies that Θ_i has no atoms in $(0, \infty)$ as shown in Lemma B.2.1.

(d) For examples satisfying the condition (ii) of Assumption 2.2.1, note that if ϑ_i is a probability distribution with $\alpha_i < \infty$, then $\Theta_i = \delta_0$, and on the other hand, if ϑ_i is a finite mixture of Erlang distributions with intensity λ_i, then Θ_i is the exponential distribution with mean $1/\lambda_i$.

(e) The condition (ii) of Assumption 2.2.2 requires that each φ_i is a *spread-out* distribution. To be precise, a distribution F on $\mathbb{R}_+$ is *spread-out* if there exists an integer $n \geq 1$ and a non-negative non-trivial subprobability measure G on $\mathbb{R}_+$ such that $F^{(*n)} \geq G$ and G is absolutely continuous with respect to Lebegue measure. Use of spread-out distributions is not uncommon. Indeed, this condition is used in assumptions for ergodicity of renewal processes (cf. Asmussen [1]) and stability of multi-class queueing networks (cf. Dai [8]).

2.3 Uniform framework for random variables

To provide a solid footing for studying our network model with differing initial states, for the rest of this work, we fix a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and then, fix the random variables

$$\{U_{1k}\}_{k=1}^{\infty}, \dots, \{U_{\mathbb{I}k}\}_{k=1}^{\infty}, \tag{2.3.1}$$

$$\{V_{1k}\}_{k=1}^{\infty}, \dots, \{V_{\mathbb{I}k}\}_{k=1}^{\infty}, \tag{2.3.2}$$

$$\{\widetilde{U}_1, \dots, \widetilde{U}_{\mathbb{I}}\}, \tag{2.3.3}$$

$$\{\widetilde{V}_{1k}\}_{k=1}^{\infty}, \dots, \{\widetilde{V}_{\mathbb{I}k}\}_{k=1}^{\infty}, \tag{2.3.4}$$

that are independent, uniformly distributed on $(0,1)$ and are all defined on $(\Omega, \mathcal{F}, \mathbf{P})$. Then, for each route i and $x \in \mathbb{A}$ that takes the form in (2.1.9), we assume that

$$u_{ik} = \inf\{s \in (0,\infty) : \varphi_i([0,s]) \geq U_{ik}\}, \text{ for each } k \geq 1, \tag{2.3.5}$$

$$v_{ik} = \inf\{s \in (0,\infty) : \vartheta_i([0,s]) \geq V_{ik}\}, \text{ for each } k \geq 1, \tag{2.3.6}$$

$$\widetilde{u}_i^x = \inf\{s \in (0,\infty) : \varphi_i^{a_i}([0,s]) \geq \widetilde{U}_i\}, \tag{2.3.7}$$

$$\widetilde{v}_{ik}^x = \inf\{s \in (0,\infty) : \vartheta_i^{c_{ik}}([0,s]) \geq \widetilde{V}_{ik}\}, \text{ if } k \leq n_i. \tag{2.3.8}$$

2.4 Descriptive processes

For this section, we fix $x \in \mathbb{A}$, and we assume that x takes the form in (2.1.9). Here, we will define various stochastic processes each of which describes some aspect of the network with initial state specified by the element x. We shall use the superscript x to indicate a quantity associated with the network starting at x.

2.4.1 Finite dimensional state descriptors

For each route i and $t \in [0,\infty)$, let $Q_i^x(t)$ be the number of flows on route i at time t and let $W_i^x(t)$ be the sum of the residual document sizes at time t of the flows on route i at time t. For the rest of this subsection, we fix route i.

At time t, the bandwidth allocated to route i is $\Lambda_i(Q^x(t))$, and this bandwidth is shared equally by all $Q_i^x(t)$ flows on route i. In other words, each such flow receives an instantaneous processing rate of $\Lambda_i(Q^x(t))/Q_i^x(t)$, which equals zero by convention if $Q_i^x(t) = 0$. Thus, a flow that is on route i during a time interval $[t_1, t_2] \subset [0, \infty)$ receives cumulative service during that interval that is equal to

$$S_i^x(t_1, t_2) = \int_{t_1}^{t_2} \frac{\Lambda_i(Q^x(s))}{Q_i^x(s)} ds. \tag{2.4.1}$$

Let $\tau_i^x(1) = \widetilde{u}_i^x$, and for each integer $k \geq 1$, define inductively $\tau_i^x(k+1) = u_{ik} + \tau_i^x(k)$. In words, each random variable $\tau_i^x(k)$ specifies the time when the k-th exogenous arrival to route i occurs. For each $t \in [0, \infty)$, let $E_i^x(t)$ be the number of exogenous arrivals to route i by time t:

$$E_i^x(t) = \sup\{k \geq 1 : \tau_i^x(k) \leq t\}, \tag{2.4.2}$$

where the supremum of an empty set is zero.

For $k \leq n_i$, consider the k-th initial flow (the k-th largest initial flow) on route i which arrived to route i at some time at or before time 0; this flow has received total processing in the amount of c_{ik} by time 0. At time 0, there still remains $\widetilde{v}_{ik}^x$ amount of work to be completed for the flow, and at time $t > 0$, the quantity $S_i^x(0,t) \wedge \widetilde{v}_{ik}^x$ describes the cumulative bandwidth allocated to this flow during $[0, t]$. For each $t \in [0, \infty)$, let

$$c_{ik}^{o,x}(t) = c_{ik} + \left(S_i^x(0,t) \wedge \widetilde{v}_{ik}^x\right), \tag{2.4.3}$$

$$r_{ik}^{o,x}(t) = \left(\widetilde{v}_{ik}^x - S_i^x(0,t)\right)^+. \tag{2.4.4}$$

describing respectively the completed work at time t and the residual work at time t of the k-th initial flow for route i.

Next, consider the k-th exogenous flow to have arrived to route i; this flow arrives at time $\tau_i^x(k)$ and has initial document size v_{ik}. At a time $t \geq \tau_i^x(k)$, the cumulative service received by this flow during $[\tau_i^x(k), t]$ equals $S_i^x(\tau_i^x(k), t) \wedge v_{ik}$. The amount of work still to be done on the flow at time t equals $(v_{ik} - S_i^x(\tau_i^x(k), t))^+$. For each $t \in [0, \infty)$, define the completed work and residual work at time t of the k-th exogenous flow for route i by letting

$$c_{ik}^{e,x}(t) = S_i(\tau_i^x(k), t) \wedge v_{ik}, \tag{2.4.5}$$

$$r_{ik}^{e,x}(t) = (v_{ik} - S_i^x(\tau_i^x(k), t))^+, \tag{2.4.6}$$

respectively.

For future reference, we split the processes Q^x and W^x into parts corresponding to documents initially in the system and documents that arrived after time zero. We let $Q_i^{o,x}(t)$ be the number of flows that were on route i at time zero and are still on route i at time t, and let $W_i^{o,x}(t)$ be the sum of the residual document sizes at time t for these flows. Then, for each route i and $t \in [0, \infty)$,

$$Q_i^{o,x}(t) = \textstyle\sum_{k=1}^{n_i} 1_{(0,\infty)}(r_{ik}^{o,x}(t)), \tag{2.4.7}$$

$$W_i^{o,x}(t) = \textstyle\sum_{k=1}^{n_i} r_{ik}^{o,x}(t) 1_{(0,\infty)}(r_{ik}^{o,x}(t)). \tag{2.4.8}$$

Let $Q_i^{e,x}(t)$ be the number of exogenous flows on route i at time t, and let $W_i^{e,x}(t)$ be the sum of the residual document sizes at time t for these flows. Then,

$$Q_i^{e,x}(t) = \textstyle\sum_{k=1}^{E_i^x(t)} 1_{(0,\infty)}(r_{ik}^{e,x}(t)), \tag{2.4.9}$$

$$W_i^{e,x}(t) = \textstyle\sum_{k=1}^{E_i^x(t)} r_{ik}^{e,x}(t) 1_{(0,\infty)}(r_{ik}^{e,x}(t)). \tag{2.4.10}$$

Finally, for each route i and $t \in [0, \infty)$, we have $Q_i^x(t) = Q_i^{o,x}(t) + Q_i^{e,x}(t)$ and $W_i^x(t) = W_i^{o,x}(t) + W_i^{e,x}(t)$.

2.4.2 Infinite dimensional state descriptors

In this subsection, we introduce two processes $\mathbf{A}^x$ and $\mathbf{R}^x$. The letters $\mathbf{A}$ and $\mathbf{R}$ are mnemonics for “age” and “residual”, respectively.

First, we let $\mathbf{A}^x$ be the random element taking values in $\mathbb{D}(\mathbb{A})$ such that for each $t \in [0, \infty)$,

$$\mathbf{A}^x(t) = \left(\mathcal{A}_1^x(t), \ldots, \mathcal{A}_{\mathbb{I}}^x(t), A_1^x(t), \ldots, A_{\mathbb{I}}^x(t)\right), \tag{2.4.11}$$

where

$$\mathcal{A}_i^x(t) = \sum_{k=1}^{n_i} \delta_{c_{ik}^{o,x}(t)} 1_{(0,\infty)}(r_{ik}^{o,x}(t)) + \sum_{k=1}^{E_i^x(t)} \delta_{c_{ik}^{e,x}(t)} 1_{(0,\infty)}(r_{ik}^{e,x}(t)), \tag{2.4.12}$$

$$A_i^x(t) = \begin{cases} t - \tau_i^x(E_i^x(t)), & \text{if } E_i^x(t) \geq 1, \\ a_i^x + t, & \text{if } E_i^x(t) = 0. \end{cases} \tag{2.4.13}$$

In words, $\mathcal{A}^x(t)$ describes the flows in the network at time t in terms of their completed work and $A^x(t)$ describes for each route, the time since the last arrival.

Next, we let $\mathbf{R}^x$ be the random element taking values in $\mathbb{D}(\mathbf{M}^{\mathbb{I}} \times \mathbb{R}_+^{\mathbb{I}})$ such that for each $t \in [0, \infty)$,

$$\mathbf{R}^x(t) = \left(\mathcal{R}_1^x(t), \ldots, \mathcal{R}_{\mathbb{I}}^x(t), R_1^x(t), \ldots, R_{\mathbb{I}}^x(t)\right), \tag{2.4.14}$$

for each route i, where

$$\mathcal{R}_i^x(t) = \sum_{k=1}^{n_i} \delta_{r_{ik}^{o,x}(t)} 1_{(0,\infty)}(r_{ik}^{o,x}(t)) + \sum_{k=1}^{E_i^x(t)} \delta_{r_{ik}^{e,x}(t)} 1_{(0,\infty)}(r_{ik}^{e,x}(t)), \tag{2.4.15}$$

$$R_i^x(t) = \tau_i^x(E_i^x(t) + 1) - t. \tag{2.4.16}$$

In words, $\mathcal{R}^x(t)$ describes the flows in the network at time t in terms of their residual work, and $R^x(t)$ describes for each route the time until the next arrival to that route after time t.

For future reference, we split the processes $\mathcal{A}^x$ and $\mathcal{R}^x$ into parts corresponding to documents initially in the system and documents that arrived after time zero: for each route i and $t \in [0,\infty)$,

$$\mathcal{A}_i^{o,x}(t) = \sum_{k=1}^{n_i} \delta_{c_{ik}^{o,x}(t)} 1_{(0,\infty)}(r_{ik}^{o,x}(t)), \tag{2.4.17}$$

$$\mathcal{A}_i^{e,x}(t) = \sum_{k=1}^{E_i^x(t)} \delta_{c_{ik}^{e,x}(t)} 1_{(0,\infty)}(r_{ik}^{e,x}(t)), \tag{2.4.18}$$

$$\mathcal{R}_i^{o,x}(t) = \sum_{k=1}^{n_i} \delta_{r_{ik}^{o,x}(t)} 1_{(0,\infty)}(r_{ik}^{o,x}(t)), \tag{2.4.19}$$

$$\mathcal{R}_i^{e,x}(t) = \sum_{k=1}^{E_i^x(t)} \delta_{r_{ik}^{e,x}(t)} 1_{(0,\infty)}(r_{ik}^{e,x}(t)). \tag{2.4.20}$$

2.5 Age process

In this work, the centerpiece of our analysis is a Borel right process which we will name *the age process.* The age process will be a canonical representation for $\{\mathbf{A}^x : x \in \mathbb{A}\}$. Stability of our network model will be formulated in terms of this process.

Let $\Omega^\dagger = \mathbb{D}(\mathbb{A})$. For each $t \in [0,\infty)$, let $\mathbf{A}_t^\dagger$ be the mapping from $\Omega^\dagger$ to $\mathbb{A}$ given by $\mathbf{A}_t^\dagger(\omega) = \omega(t)$, where for each $\omega \in \Omega^\dagger$, $\omega(t)$ is the value of the function ω at t. Next, let $\mathcal{F}^\dagger = \sigma(\mathbf{A}_t^\dagger : t \in [0,\infty))$. For each $t \in [0,\infty)$, let $\mathcal{F}_t^\dagger = \sigma\left(\mathbf{A}_s^\dagger : s \in [0,t]\right)$ and $\mathcal{F}_{t+}^\dagger = \cap_{s\in(t,\infty)}\mathcal{F}_s^\dagger$. Then, let $\mathfrak{F}^\dagger = \{\mathcal{F}_t^\dagger : t \in [0,\infty)\}$ and $\mathbf{A}^\dagger = \{\mathbf{A}_t^\dagger : t \in [0,\infty)\}$. For each $t \in [0,\infty)$, define $P_t : \mathbb{A} \times \mathcal{B}(\mathbb{A}) \to [0,1]$ by letting, for each $x \in \mathbb{A}$ and $B \in \mathcal{B}(\mathbb{A})$,

$$P_t(x,B) = \mathbf{P}[\mathbf{A}^x(t) \in B]. \tag{2.5.1}$$

Then, let $\mathcal{P}^\dagger = \{P_t : t \in [0,\infty)\}$. For convenience, for each $t \in [0,\infty)$, we will also write $\mathbf{A}^\dagger(t)$ for $\mathbf{A}_t^\dagger$.

Theorem 2.5.1. *The collection $\mathcal{P}^\dagger$ is a Borel right semigroup,* *Proof.* See Appendix A. □

Henceforth, for each probability measure μ on $(\mathbb{A}, \mathcal{B}(\mathbb{A}))$, we fix a probability measure $\mathbf{P}^\mu$ on $(\Omega^\dagger, \mathcal{F}^\dagger)$ that has the properties described in Definition 1.3.1 for the Borel right semigroup $\mathcal{P}^\dagger$. In fact, it follows from the proof of Lemma A.3.1 that for each $B \in \mathcal{F}^\dagger$, we have

$$\mathbf{P}^\mu(B) = \int_{\mathbb{A}} \mathbf{P}[\mathbf{A}^x(\cdot) \in B]\mu(dx).$$

We denote by $\mathbf{E}^\mu$, the expectation operator for $\mathbf{P}^\mu$.

Definition 2.5.1. *The age process* is the Borel right process

$$(\Omega^\dagger, \mathcal{F}^\dagger, \mathfrak{F}^\dagger, \mathbf{A}^\dagger, \mathcal{P}^\dagger). \tag{2.5.2}$$

Definition 2.5.2. *The network model is stable* if the age process is positive Harris recurrent.

2.6 Fluid model solutions

Our main result is that the network model is stable if an associated fluid model is stable. The fluid model solutions we use are formal functional law of large numbers approximations to sequences taken from the family $\{\mathcal{R}^x : x \in \mathbb{A}\}$ associated with the residual processes.

Definition 2.6.1 (Gromoll and Williams [13])**.** Given a continuous function $\zeta : [0, \infty) \to \mathbf{M}^{\mathbb{I}}$, the auxiliary functions for ζ are the functions (z, w, τ, u) such that

$$z_i(t) = \langle 1, \zeta_i(t) \rangle, \tag{2.6.1}$$

$$w_i(t) = \langle \chi, \zeta_i(t) \rangle, \tag{2.6.2}$$

$$\tau_i(t) = \int_0^t \left(\Lambda_i(z(s)) 1_{(0,\infty)}(z_i(s)) + \rho_i 1_{\{0\}}(z_i(s)) \right) ds, \tag{2.6.3}$$

for each $t \in [0, \infty)$ and route i, and

$$u_j(t) = C_j t - \sum_{i=1}^{\mathbb{I}} M_{ji}\tau_i(t), \tag{2.6.4}$$

for each $t \in [0, \infty)$ and resource j.

For our next definition, we let

$$\mathcal{C} = \{f \in \mathcal{C}_b^1(\mathbb{R}_+) : f(0) = f'(0) = 0\}, \tag{2.6.5}$$

where f' denotes the first derivative of f.

Definition 2.6.2 (Gromoll and Williams [13])**.** A *fluid model solution* is a continuous function $\zeta : [0, \infty) \to \mathbf{M}^{\mathbb{I}}$ such that the following three properties are satisfied by ζ and its auxiliary functions (z, w, τ, u):

(i) $\langle 1_{\{0\}}, \zeta_i(t)\rangle = 0$ for each $t \in [0, \infty)$ and route i,

(ii) u_j is non-decreasing for each resource j,

(iii) for each $f \in \mathcal{C}$, route i and $t \in [0, \infty)$,

$$\langle f, \zeta_i(t)\rangle = \langle f, \zeta_i(0)\rangle + \int_0^t \left(\nu_i\langle f, \vartheta_i\rangle - \langle f', \zeta_i(s)\rangle \frac{\Lambda_i(z(s))}{z_i(s)}\right) 1_{(0,\infty)}(z_i(s))ds.$$

When the initial fluid workload is finite, the following is also true.

Proposition 2.6.1 (Gromoll and Williams [13])**.** *Suppose that ζ is a fluid model solution with finite initial workload, i.e., $w_i(0) = \langle \chi, \zeta_i(0)\rangle < \infty$ for each route i. Then, the fluid workload function w associated with ζ satisfies the following for each route i and $t \in [0, \infty)$:*

$$\begin{aligned} w_i(t) &= w_i(0) + \int_0^t (\rho_i - \Lambda_i(z(s)))1_{(0,\infty)}(z_i(s))ds && (2.6.6)\\ &= w_i(0) + \rho_i t - \tau_i(t). && (2.6.7)\end{aligned}$$

In particular, the fluid workload $w_i(t)$ is finite for each $t \in [0, \infty)$ and route i.

Definition 2.6.3. *The fluid model is stable* if for each $r \in (0, \infty)$, there exists $t_r \in (0, \infty)$ such that whenever ζ is a fluid model solution with $\|\langle 1, \zeta(0)\rangle\| \vee \|\langle \chi, \zeta(0)\rangle\| \leq r$, then $\zeta(t) = 0$ for each $t \in [t_r, \infty)$.

For an illustration of fluid model stability, the reader is encouraged to see Gromoll and Williams [14] for two examples. There it is shown that, when the underlying network is either a linear network or a simple tree network, under the nominal condition that the average load placed on each resource is less than its capacity, i.e., $\sum_{i=1}^{\mathbb{I}} M_{ji}\rho_i < C_j$ for each resource j, the associated fluid model is stable.

2.7 Main theorem

Theorem 2.7.1 (Main theorem)**.** *If the fluid model is stable, then the network model is stable.*

Chapter 3

Sequence of states and fluid limits

For each initial state $x \in \mathbb{A}$, we define a numerical quantity that corresponds to a size for x as follows:

$$|x| = \sum_{i=1}^{\mathbb{I}} Q_i^x(0) + \sum_{i=1}^{\mathbb{I}} A_i^x(0). \tag{3.0.1}$$

For instance, for any element x taking the form in (2.1.9), we have

$$|x| = \sum_{i=1}^{\mathbb{I}} (n_i + a_i).$$

Roughly speaking, for each $x \in \mathbb{A}$, the quantity $|x|$ measures a size of the state of the network. We caution the reader that although we use the $|\cdot|$ notation, $|\cdot|$ is not a norm on $\mathbb{A}$.

For the rest of this chapter, we fix a sequence $\{x^\ell\}_{\ell=1}^\infty$ in $\mathbb{A}$ such that as $\ell \to \infty$, $|x^\ell| \to \infty$ and for each integer $\ell \geq 1$, $|x^\ell| > 0$ and

$$x^\ell = \left(\sum_{k=1}^{n_1^\ell} \delta_{c_{1k}^\ell}, \ldots, \sum_{k=1}^{n_{\mathbb{I}}^\ell} \delta_{c_{\mathbb{I}k}^\ell}, a_1^\ell, \ldots, a_{\mathbb{I}}^\ell \right), \tag{3.0.2}$$

where for each route i, n_i^ℓ is a non-negative integer, $a_i^\ell \in \mathbb{R}_+$ and $0 \leq c_{i1}^\ell < \ldots < c_{in_i}^\ell < \alpha_i$. In other words, for the system starting at x^ℓ, on route i, there are exactly

n_i^ℓ initial flows at time zero, where for each $k \in \{1, \ldots, n_i^\ell\}$, the number c_{ik}^ℓ represents the completed work of the initial flow with the k-th largest completed work among the n_i^ℓ initial flows on route i, and a_i^ℓ is the time (measured from time zero) since the last arrival to route i.

For future use, for each route i, let N_i be the random element taking values in $\mathbb{D}(\mathbb{R}_+)$ such that for each $t \in [0, \infty)$,

$$N_i(t) = \max \left\{ n \geq 0 : \sum_{k=1}^{n} u_{ik} \leq t \right\}. \tag{3.0.3}$$

3.1 Sequence of scaled descriptors

For each integer $\ell \geq 1$, there is a network starting at x^ℓ with size $|x^\ell|$ of x^ℓ. We *speed up* time by $|x^\ell|$ and *scale down* the magnitude of the network starting at x^ℓ by $|x^\ell|$ as follows; for each $t \in [0, \infty)$, let

$$\overline{\mathbf{A}}^\ell(t) = \tfrac{1}{|x^\ell|}\mathbf{A}^{x^\ell}(t|x^\ell|), \tag{3.1.1}$$
$$\overline{E}^\ell(t) = \tfrac{1}{|x^\ell|}E^{x^\ell}(t|x^\ell|), \tag{3.1.2}$$
$$\overline{Q}^\ell(t) = \tfrac{1}{|x^\ell|}Q^{x^\ell}(t|x^\ell|), \tag{3.1.3}$$
$$\overline{\mathbf{R}}^\ell(t) = \tfrac{1}{|x^\ell|}\mathbf{R}^{x^\ell}(t|x^\ell|), \tag{3.1.4}$$
$$\overline{W}^\ell(t) = \tfrac{1}{|x^\ell|}W^{x^\ell}(t|x^\ell|). \tag{3.1.5}$$

For each integer $\ell \geq 1$ and route i, we introduce the random element $\overline{N}_i^\ell$ taking values in $\mathbb{D}(\mathbb{R}_+)$ by letting, for each $t \in [0, \infty)$,

$$\overline{N}_i^\ell(t) = \frac{1}{|x^\ell|}N_i(t|x^\ell|), \tag{3.1.6}$$

and note that

$$\overline{E}_i^\ell(\cdot) = \frac{1}{|x^\ell|}1_{[\overline{R}_i^\ell(0),\infty)}(\cdot) + \overline{N}_i^\ell((\cdot - \overline{R}_i^\ell(0))^+). \tag{3.1.7}$$

For future use, we split the scaled descriptors as follows. First, recall the definitions of $\mathcal{A}^{o,x}(t)$, $\mathcal{A}^{e,x}(t)$, $\mathcal{R}^{o,x}(t)$, $\mathcal{R}^{e,x}(t)$, $\mathcal{A}^{x}(t)$, $\mathcal{R}^{x}(t)$, $Q^{o,x}(t)$, $Q^{e,x}(t)$, $W^{o,x}(t)$ and $W^{e,x}(t)$ from Section 2.4. Then, for each integer $\ell \geq 1$, define the scaled random elements $\overline{\mathcal{A}}^{o,\ell}$, $\overline{\mathcal{A}}^{e,\ell}$, $\overline{\mathcal{R}}^{o,\ell}$ and $\overline{\mathcal{R}}^{e,\ell}$, taking values in $\mathbb{D}(\mathbf{M}^{\mathbb{I}})$ by letting, for each $t \in [0,\infty)$,

$$\overline{\mathcal{A}}^{o,\ell}(t) = \tfrac{1}{|x^\ell|}\mathcal{A}^{o,x^\ell}(t|x^\ell|), \tag{3.1.8}$$

$$\overline{\mathcal{A}}^{e,\ell}(t) = \tfrac{1}{|x^\ell|}\mathcal{A}^{e,x^\ell}(t|x^\ell|), \tag{3.1.9}$$

$$\overline{\mathcal{R}}^{o,\ell}(t) = \tfrac{1}{|x^\ell|}\mathcal{R}^{o,x^\ell}(t|x^\ell|), \tag{3.1.10}$$

$$\overline{\mathcal{R}}^{e,\ell}(t) = \tfrac{1}{|x^\ell|}\mathcal{R}^{e,x^\ell}(t|x^\ell|), \tag{3.1.11}$$

and note that

$$\overline{\mathcal{A}}^{\ell}(t) = \tfrac{1}{|x^\ell|}\mathcal{A}^{x^\ell}(|x^\ell|t) = \overline{\mathcal{A}}^{o,\ell}(t) + \overline{\mathcal{A}}^{e,\ell}(t), \tag{3.1.12}$$

$$\overline{\mathcal{R}}^{\ell}(t) = \tfrac{1}{|x^\ell|}\mathcal{R}^{x^\ell}(|x^\ell|t) = \overline{\mathcal{R}}^{o,\ell}(t) + \overline{\mathcal{R}}^{e,\ell}(t). \tag{3.1.13}$$

Next, for each integer $\ell \geq 1$, define the scaled random elements $\overline{Q}^{o,\ell}$, $\overline{Q}^{e,\ell}$, $\overline{W}^{o,\ell}$ and $\overline{W}^{e,\ell}$ taking values in $\mathbb{D}(\mathbb{R}_+^{\mathbb{I}})$ by letting, for each $t \in [0,\infty)$,

$$\overline{Q}^{o,\ell}(t) = \langle 1, \overline{\mathcal{A}}^{o,\ell}(t)\rangle, \tag{3.1.14}$$

$$\overline{Q}^{e,\ell}(t) = \langle 1, \overline{\mathcal{A}}^{e,\ell}(t)\rangle, \tag{3.1.15}$$

$$\overline{W}^{o,\ell}(t) = \langle \chi, \overline{\mathcal{A}}^{o,\ell}(t)\rangle, \tag{3.1.16}$$

$$\overline{W}^{e,\ell}(t) = \langle \chi, \overline{\mathcal{A}}^{e,\ell}(t)\rangle, \tag{3.1.17}$$

and note that

$$\overline{Q}^{\ell}(t) = \overline{Q}^{o,\ell}(t) + \overline{Q}^{e,\ell}(t), \tag{3.1.18}$$

$$\overline{Q}^{\ell}(t) = \overline{W}^{o,\ell}(t) + \overline{W}^{e,\ell}(t). \tag{3.1.19}$$

3.2 Weak limit points of scaled residual work descriptors

Here, we state a result that plays an important role in our proof of the main theorem. Recall that for each $x \in \mathbb{A}$ and $t \in [0,\infty)$, $\mathcal{R}^x(t)$ consists of the first $\mathbb{I}$ elements of $\mathbf{R}^x(t)$ and tracks residual work sizes for flows in the network at time t.

Theorem 3.2.1. *Let $t \in (0,\infty)$. Then, the sequence*

$$\{\overline{\mathcal{R}}^{\ell}(t+\cdot) : \ell = 1,2,3,\ldots\} \tag{3.2.1}$$

of random elements taking values in $\mathbb{D}(\mathbf{M}^{\mathbb{I}})$ is C-tight, and any weak limit point is almost surely a fluid model solution with finite initial workload.

Corollary 3.2.1. *Let $t \in (0,\infty)$, and let $\overline{\mathcal{R}}$ be a weak limit point of the sequence (3.2.1). Then, almost surely, for each $s \in [0,\infty)$*

$$\|\langle 1, \overline{\mathcal{R}}(s)\rangle\| \le \mathbb{I} + (t+s)\,\|\nu\|\,, \tag{3.2.2}$$

$$\|\langle \chi, \overline{\mathcal{R}}(s)\rangle\| \le \left(\max_{i=1}^{\mathbb{I}} \sup_{\alpha\in[0,\alpha_i)} \langle \chi, \vartheta_i^{\alpha}\rangle\right) + (t+s)\,\|\rho\|\,. \tag{3.2.3}$$

Our proof of Theorem 3.2.1 is broken into six steps, each of which corresponds to one of the next six subsections, and Corollary 3.2.1 will be proven after our proof of Theorem 3.2.1 is completed.

3.2.1 Convergence of scaled exogenous flow arrival descriptors

Recall that for each $x \in \mathbb{A}$ and $t \in [0,\infty)$, $R^x(t)$ consists of the last $\mathbb{I}$ elements of $\mathbf{R}^x(t)$ and for each route i, $R_i^x(t)$ describes the time (measured from time t) till the next exogenous flow arrival to route i for the network starting at x.

Lemma 3.2.1. *The sequence* $\{(\overline{E}^{\ell}, \overline{R}^{\ell})\}_{\ell=1}^{\infty}$ *of random elements taking values in* $\mathbb{D}(\mathbb{R}_+^{2\mathbb{I}})$ *converges in distribution to* $(\overline{E}, \overline{R})$*, where* $\overline{E}_i(t) = \nu_i t$ *and* $\overline{R}_i(t) = 0$ *for each route* i *and* $t \in [0, \infty)$*.*

Proof. First, note that $(\overline{E}, \overline{R}) \in \mathbb{C}(\mathbb{R}_+^{2\mathbb{I}})$ is deterministic and continuous. Hence, it is enough to show convergence in distribution for each component (cf. Theorem 11.4.5 in Whitt [28] and Proposition VI.2.2 in Jacod and Shiryaev [18]). In other words, it suffices to prove that for each route i, as $\ell \to \infty$, $\overline{E}_i^{\ell}$ converges in distribution to $\overline{E}_i$ and as $\ell \to \infty$, $\overline{R}_i^{\ell}$ converges in distribution to $\overline{R}_i$. For this, fix route i.

Step 1. For each integer $\ell \geq 1$, note that

$$\frac{1}{|x^{\ell}|}\mathbf{E}[R_i^{x^{\ell}}(0)] \leq \frac{1}{|x^{\ell}|} \sup_{\beta \leq |x^{\ell}|} \langle \chi, \varphi_i^{\beta} \rangle.$$

Then, by (2.2.4) in Assumption 2.2.2, we have

$$\lim_{\ell \to \infty} \mathbf{E}[\overline{R}_i^{\ell}(0)] \leq \limsup_{\ell \to \infty} \frac{1}{|x^{\ell}|} \sup_{\beta \leq |x^{\ell}|} \langle \chi, \varphi_i^{\beta} \rangle = 0,$$

and this shows that $\lim_{\ell \to \infty} \mathbf{E}[\overline{R}_i^{\ell}(0)] = 0$. So, by Markov's inequality, $\overline{R}_i^{\ell}(0)$ converges in distribution to 0 as $\ell \to \infty$.

Step 2. Recall our definition of $\{\overline{N}_i^{\ell}\}_{\ell=1}^{\infty}$ in (3.1.6). It is well known, by the functional law of large numbers for renewal processes, that the sequence $\{\overline{N}_i^{\ell}\}_{\ell=1}^{\infty}$ converges in distribution to $\overline{E}_i$ as $\ell \to \infty$. It follows using *Step 1* that the sequence $\{\overline{N}_i^{\ell}((\cdot - \overline{R}_i^{\ell}(0))^+)\}_{\ell=1}^{\infty}$ converges in distribution to $\overline{E}_i$. Therefore, by the identity in (3.1.7), as $\ell \to \infty$, the sequence $\{\overline{E}_i^{\ell}\}_{\ell=1}^{\infty}$ converges in distribution to $\overline{E}_i$.

Step 3. Note that for each integer $\ell \geq 1$ and $t \in [0, \infty)$,

$$t + \overline{R}_i^{\ell}(t) = \frac{1}{|x^{\ell}|}\tau_i^{x^{\ell}}(E_i^{x^{\ell}}(t|x^{\ell}|) + 1) = \overline{R}_i^{\ell}(0) + \frac{1}{|x^{\ell}|} \sum_{k=1}^{N_i(|x^{\ell}|(t - \overline{R}_i^{\ell}(0))^+)} u_{ik}. \tag{3.2.4}$$

The sequence $\{\overline{N}_i^{\ell}((\cdot - \overline{R}_i^{\ell}(0))^+)\}_{\ell=1}^{\infty}$ converges in distribution to $\overline{E}_i$ and the sequence $\{\overline{R}_i^{\ell}(0)\}_{\ell=1}^{\infty}$ converges in distribution to 0. Then, using Lemma 1 in Iglehart and

Whitt [16], we see that the sequence

$$\left\{ \frac{1}{|x^\ell|} \sum_{k=1}^{N_i(|x^\ell|(\cdot - \overline{R}_i^\ell(0))^+)} u_{ik} \right\}_{\ell=1}^{\infty}$$

converges in distribution to $(1/\nu_i)\overline{E}_i \in \mathbb{C}(\mathbb{R}_+)$, where $(1/\nu_i)\overline{E}_i(t) = t$ for all $t \in [0,\infty)$. Together with (3.2.4), we see that $\{\overline{R}_i^\ell\}_{\ell=1}^\infty$ converges in distribution to $\overline{R}_i \equiv 0$. □

3.2.2 Tightness of scaled state descriptors at fixed times

Recall that for each $x \in \mathbb{A}$ and $t \in [0,\infty)$, $\mathcal{A}^x(t)$ consists of the first $\mathbb{I}$ elements of $\mathbf{A}^x(t)$ and describes completed work for flows in the network at time t. Also, for each $x \in \mathbb{A}$ and $t \in [0,\infty)$, $Q^x(t) = \langle 1, \mathcal{R}^x(t)\rangle = \langle 1, \mathcal{A}^x(t)\rangle$ counts the number of flows in the network at time t, and $W^x(t) = \langle \chi, \mathcal{R}^x(t)\rangle$ describes the residual work for flows in the network at time t.

For our next lemma, we define

$$\mathbb{S} = \mathbf{M}^{\mathbb{I}} \times \mathbb{R}_+^{\mathbb{I}} \times \mathbb{R}_+^{\mathbb{I}} \times \mathbf{M}^{\mathbb{I}} \times \mathbb{R}_+^{\mathbb{I}} \times \mathbf{M}^{\mathbb{I}} \times \mathbf{M}^{\mathbb{I}} \times \mathbb{R}_+^{\mathbb{I}} \times \mathbb{R}_+^{\mathbb{I}},$$

and we equip the product space $\mathbb{S}$ with the product topology, where each $\mathbf{M}$ in the first $\mathbf{M}^{\mathbb{I}}$ is given the topology of vague convergence but any other $\mathbf{M}$ is equipped with the topology of weak convergence.

Lemma 3.2.2. *Let $t \in [0,\infty)$. The sequence*

$$\{(\overline{\mathcal{A}}^\ell(t), \overline{E}^\ell(t), \overline{Q}^\ell(t), \overline{\mathcal{R}}^\ell(t), \overline{W}^\ell(t), \overline{\mathcal{R}}^{o,\ell}(t), \overline{\mathcal{R}}^{e,\ell}(t), \overline{W}^{o,\ell}(t), \overline{W}^{e,\ell}(t))\}_{\ell=1}^\infty \qquad (3.2.5)$$

of random elements taking values in $\mathbb{S}$ is tight.

Proof. We divide our proof into two steps.

Step 1. Observe that for each integer $\ell \geq 1$ and route i, almost surely, $\overline{Q}_i^{o,\ell}(t) \leq \overline{Q}_i^{\ell}(0)$ and $\overline{W}_i^{o,\ell}(t) \leq \overline{W}_i^{o,\ell}(0)$, and then, note that

$$\overline{Q}_i^{\ell}(0) \leq 1, \tag{3.2.6}$$

$$\mathbf{E}[\overline{W}_i^{o,x^\ell}(0)] \leq \sup_{\alpha\in[0,\alpha_i)}\langle \chi, \vartheta_i^\alpha\rangle. \tag{3.2.7}$$

Therefore, for each integer $\ell \geq 1$,

$$\mathbf{E}[\|\overline{Q}^{o,\ell}(t)\|] = \textstyle\sum_{i=1}^{\mathbb{I}} \mathbf{E}[\overline{Q}_i^{o,\ell}(t)] \leq \mathbb{I}, \tag{3.2.8}$$

$$\mathbf{E}[\|\overline{W}^{o,\ell}(t)\|] = \textstyle\sum_{i=1}^{\mathbb{I}} \mathbf{E}[\overline{W}_i^{o,\ell}(t)] \leq \mathbb{I}\max_{i=1}^{\mathbb{I}} \sup_{\alpha\in[0,\alpha_i)}\langle \chi, \vartheta_i^\alpha\rangle. \tag{3.2.9}$$

Next, note that for each route i, almost surely,

$$\overline{Q}_i^{e,\ell}(t) \leq \overline{E}_i^{\ell}(t), \tag{3.2.10}$$

$$\overline{W}_i^{e,\ell}(t) \leq \tfrac{1}{|x^\ell|}\textstyle\sum_{k=1}^{E_i^{x^\ell}(|x^\ell|t)} v_{ik}, \tag{3.2.11}$$

and

$$\lim_{\ell\to\infty} \mathbf{E}[\overline{E}_i^{\ell}(t)] = t\nu_i, \tag{3.2.12}$$

$$\lim_{\ell\to\infty} \mathbf{E}[(1/|x^\ell|)\textstyle\sum_{k=1}^{E_i^{x^\ell}(|x^\ell|t)} v_{ik}] = t\rho_i. \tag{3.2.13}$$

Therefore, we have

$$\lim_{\ell\to\infty} \mathbf{E}[\|\overline{E}^{\ell}(t)\|] \leq \mathbb{I}t\max_{i=1}^{\mathbb{I}} \nu_i, \tag{3.2.14}$$

$$\limsup_{\ell\to\infty} \mathbf{E}[\|\overline{Q}^{e,\ell}(t)\| + \|\overline{W}^{e,\ell}(t)\|] \leq \mathbb{I}t\max_{i=1}^{\mathbb{I}}(\nu_i+\rho_i). \tag{3.2.15}$$

Step 2. Through Markov's inequality, one sees that for each integer $n \geq 1$, there exists $r_n \in (0,\infty)$ such that

$$\liminf_{\ell\to\infty} \mathbf{P}\left[(\overline{E}^{\ell}(t), \overline{Q}^{o,\ell}(t), \overline{W}^{o,\ell}(t), \overline{Q}^{e,\ell}(t), \overline{W}^{e,\ell}(t)) \in [0, r_n/2]^{5\mathbb{I}}\right] \geq 1 - \frac{1}{2n},$$

and since each $\overline{Q}^{\ell}(t)$, (respectively, $\overline{W}^{\ell}(t)$) is obtained by adding $\overline{Q}^{o,\ell}(t)$ to $\overline{Q}^{e,\ell}(t)$, (respectively $\overline{W}^{o,\ell}(t)$ to $\overline{W}^{e,\ell}(t)$), this implies that

$$\liminf_{\ell\to\infty} \mathbf{P}\left[\overline{E}^{\ell}(t) \in [0, r_n]^{\mathbb{I}}, \overline{Q}^{\ell}(t) \in [0, r_n]^{\mathbb{I}}, \overline{W}^{\ell}(t) \in [0, r_n]^{\mathbb{I}}\right] \geq 1 - \frac{1}{n}. \tag{3.2.16}$$

The inequality (3.2.16) implies that for each integer $n \geq 1$,

$$\liminf_{\ell\to\infty} \mathbf{P}\left[(\overline{\mathcal{A}}^\ell(t), \overline{\mathcal{R}}^\ell(t), \overline{\mathcal{R}}^{o,\ell}(t), \overline{\mathcal{R}}^{e,\ell}(t)) \in \widetilde{\mathbb{K}}_n \times \mathbb{K}_n^3\right] \geq 1 - \frac{1}{n}, \qquad (3.2.17)$$

where

$$\widetilde{\mathbb{K}}_n = \{\mu \in \mathbf{M}^{\mathbb{I}} : \textstyle\sum_{i=1}^{\mathbb{I}} \langle 1, \mu_i\rangle \leq r_n\},$$
$$\mathbb{K}_n = \{\mu \in \mathbf{M}^{\mathbb{I}} : \textstyle\sum_{i=1}^{\mathbb{I}} \langle 1, \mu_i\rangle \vee \langle \chi, \mu_i\rangle \leq r_n\}.$$

For each integer $n \geq 1$, by Alaoglu's theorem, the set $\widetilde{\mathbb{K}}_n$ is compact with respect to the topology of vague convergence and also, the set $\mathbb{K}_n$ is compact with respect to the topology of weak convergence (cf. Theorem 15.7.5 in Kallenberg [19]). So,

$$\widetilde{\mathbb{K}}_n \times [0, r_n]^{\mathbb{I}} \times [0, r_n]^{\mathbb{I}} \times \mathbb{K}_n \times [0, r_n]^{\mathbb{I}} \times \mathbb{K}_n \times \mathbb{K}_n \times [0, r_n]^{\mathbb{I}} \times [0, r_n]^{\mathbb{I}}$$

is a compact subset of $\mathbb{S}$. Then, the inequalities (3.2.16) and (3.2.17) imply tightness of the sequence (3.2.5). □

3.2.3 Limit points of scaled state descriptors at time zero

For our next lemma, recall the definition of $\Theta_1, \ldots, \Theta_{\mathbb{I}}$ in Assumption 2.2.1. The following lemma shows that the limit points of the scaled initial residual work descriptors can be written in terms of the limit points of the scaled initial completed work descriptors.

Lemma 3.2.3. *Let* $(\overline{\mathcal{A}}^0, \overline{E}^0, \overline{Q}^0, \overline{\mathcal{R}}^0, \overline{W}^0)$ *be a weak limit point of the sequence*

$$\{(\overline{\mathcal{A}}^\ell(0), \overline{E}^\ell(0), \overline{Q}^\ell(0), \overline{\mathcal{R}}^\ell(0), \overline{W}^\ell(0)) : \ell = 1, 2, \ldots\}.$$

Then, for each route i *and* $B \in \mathcal{B}(\mathbb{R}_+)$,

$$\overline{\mathcal{R}}_i^0(B) = \int_{[0,\alpha_i)} \vartheta_i^\alpha(B)\overline{\mathcal{A}}_i^0(d\alpha) + \left(\overline{Q}_i^0 - \langle 1_{[0,\alpha_i)}, \overline{\mathcal{A}}_i^0\rangle\right)\Theta_i(B). \qquad (3.2.18)$$

Proof. First, note that $\overline{\mathcal{A}}^0$ and $\overline{Q}^0$ are deterministic since $\{\overline{\mathcal{A}}^\ell(0)\}_{\ell=1}^\infty$ and $\{\overline{Q}^\ell(0)\}_{\ell=1}^\infty$ are deterministic. Next, it is well known that there exists a sequence $\{f_k\}_{k=1}^\infty$ in $\mathcal{C}_b^+(\mathbb{R}_+)$ such that for each $(\mu,\mu')\in\mathbf{M}^2$, $\mu=\mu'$ if and only if $\langle f_k,\mu\rangle=\langle f_k,\mu'\rangle$ for each integer $k\geq 1$. Hence, to prove (3.2.18), it suffices to show that for each $f\in\mathcal{C}_b^+(\mathbb{R}_+)$ and route i, almost surely,

$$\langle f,\overline{\mathcal{R}}_i^0\rangle=\int_{[0,\alpha_i)}\langle f,\vartheta_i^\alpha\rangle\,\overline{\mathcal{A}}_i^0(d\alpha)+\left(\overline{Q}_i^0-\langle 1_{[0,\alpha_i)},\overline{\mathcal{A}}_i^0\rangle\right)\langle f,\Theta_i\rangle. \tag{3.2.19}$$

For this, we fix $f\in\mathcal{C}_b^+(\mathbb{R}_+)$. Our proof consists of five steps.

Step 1. For each route i, define a function $H_i:[0,\alpha_i)\to\mathbb{R}_+$ by letting $H_i(\alpha)=\langle f,\vartheta_i^\alpha\rangle$ for each $\alpha\in[0,\alpha_i)$. Now, for each route i, we have that for each integer $\ell\geq 1$,

$$\langle H_i1_{[0,\alpha_i)},\overline{\mathcal{A}}_i^\ell(0)\rangle=\int_{[0,\alpha_i)}H_i(\alpha)\overline{\mathcal{A}}_i^\ell(0)(d\alpha)=\tfrac{1}{|x^\ell|}\sum_{k=1}^{Q_i^{x^\ell}(0)}\langle f,\vartheta_i^{c_{ik}^\ell}\rangle, \tag{3.2.20}$$

$$\mathbf{E}[\langle f,\overline{\mathcal{R}}_i^\ell(0)\rangle]=\mathbf{E}\left[\tfrac{1}{|x^\ell|}\sum_{k=1}^{Q_i^{x^\ell}(0)}f(\widetilde{v}_{ik}^{x^\ell})\right]=\tfrac{1}{|x^\ell|}\sum_{k=1}^{Q_i^{x^\ell}(0)}\langle f,\vartheta_i^{c_{ik}^\ell}\rangle. \tag{3.2.21}$$

Since $\langle f,\cdot\rangle\in\mathcal{C}^+(\mathbf{M})$, by continuous mapping, it follows that for each route i, the sequence $\{\langle f,\overline{\mathcal{R}}_i^\ell(0)\rangle\}_{\ell=1}^\infty$ converges in distribution to $\langle f,\overline{\mathcal{R}}_i^0\rangle$ as $\ell\to\infty$. Also, since each $Q_i^{x^\ell}(0)\leq|x^\ell|$, we have that

$$\sup_{\ell=1,2,3,\ldots}(\langle f,\overline{\mathcal{R}}_i^\ell(0)\rangle)^2\leq\|f\|_\infty^2<\infty.$$

For each route i, this implies that the sequence $\{\langle f,\overline{\mathcal{R}}_i^\ell(0)\rangle\}_{\ell=1}^\infty$ is uniformly integrable and then, we have that

$$\lim_{\ell\to\infty}\mathbf{E}[\langle f,\overline{\mathcal{R}}_i^\ell(0)\rangle]=\mathbf{E}[\langle f,\overline{\mathcal{R}}_i^0\rangle].$$

Hence, for each route i,

$$\lim_{\ell\to\infty}\langle H_i1_{[0,\alpha_i)},\overline{\mathcal{A}}_i^\ell(0)\rangle=\lim_{\ell\to\infty}\mathbf{E}[\langle f,\overline{\mathcal{R}}_i^\ell(0)\rangle]=\mathbf{E}[\langle f,\overline{\mathcal{R}}_i^0\rangle]. \tag{3.2.22}$$

Step 2. We claim that as $\ell \to \infty$,

$$\langle f, \overline{\mathcal{R}}_i^\ell(0)\rangle \Rightarrow \mathbf{E}[\langle f, \overline{\mathcal{R}}_i^0\rangle]. \tag{3.2.23}$$

In view of (3.2.20), (3.2.21) and (3.2.22), it suffices to show that for each route i and

$$\xi_i^\ell = \frac{1}{|x^\ell|}\sum_{k=1}^{Q_i^{x^\ell}(0)} \xi_{ik}^\ell,$$

where each $\xi_{ik}^\ell = f(\tilde{v}_{ik}^{x^\ell}) - \langle f, \vartheta_i^{c_{ik}^\ell}\rangle$, $k = 1, \ldots, Q_i^{x^\ell}(0)$, we have that

$$\lim_{\ell\to\infty} \mathbf{E}[(\xi_i^\ell)^2] = 0.$$

For each route i, by the independence and mean zero property of each sequence $\{\xi_{ik}^\ell\}_{k=1}^{Q_i^{x^\ell}(0)}$, we have that

$$\begin{aligned}\limsup_{\ell\to\infty} \mathbf{E}[(\xi_i^\ell)^2] &= \limsup_{\ell\to\infty} \frac{1}{|x^\ell|^2}\sum_{k=1}^{Q_i^{x^\ell}(0)} \mathbf{E}[(\xi_{ik}^\ell)^2] &(3.2.24)\\ &\le \limsup_{\ell\to\infty} \frac{1}{|x^\ell|} 4\|f\|_\infty^2 = 0, &(3.2.25)\end{aligned}$$

and this proves (3.2.23).

Step 3. To prepare for *Step 4*, for each route i, fix an increasing sequence $\{b_{ik}\}_{k=1}^\infty$ in $(0, \alpha_i)$ such that for each integer $k \ge 1$, $\overline{\mathcal{A}}_i^0(\{b_{ik}\}) = 0$ and

$$\langle 1_{(b_{ik},\alpha_i)}, \overline{\mathcal{A}}_i^0\rangle \|f\|_\infty + \int_{(b_{ik},\alpha_i)} H_i(\alpha)\overline{\mathcal{A}}_i^0(d\alpha) \le \frac{1}{2k}, \tag{3.2.26}$$

and $\lim_{k\to\infty} b_{ik} = \alpha_i$. Next, for each $\varepsilon \in (0,1)$ and route i, we fix $\alpha_{i,\varepsilon} \in [0, \alpha_i)$ such that $|\langle f, \vartheta_i^\alpha\rangle - \langle f, \Theta_i\rangle| < \varepsilon/4$ for each $\alpha \in [\alpha_{i,\varepsilon}, \alpha_i)$, and let $k_{i,\varepsilon}$ be an integer so that $b_{ik_{i,\varepsilon}} \ge \alpha_{i,\varepsilon}$ and $1/(2k_{i,\varepsilon}) \le \varepsilon/4$. Hence, for any integer $k \ge k_{i,\varepsilon}$, we have

$$\left(\frac{\langle 1_{(b_{ik},\alpha_i)}, \mathcal{A}_i^{x^\ell}(0)\rangle}{|x^\ell|}\left|\frac{\langle H_i 1_{(b_{ik},\alpha_i)}, \mathcal{A}_i^{x^\ell}(0)\rangle}{\langle 1_{(b_{ik},\alpha_i)}, \mathcal{A}_i^{x^\ell}(0)\rangle} - \langle f, \Theta_i\rangle\right|\right) \vee \left|\int_{(b_{ik},\alpha_i)} H_i(\alpha)\overline{\mathcal{A}}_i^0(d\alpha)\right| \le \frac{\varepsilon}{4},$$

where we take $0/0 = 0$. Recall that for each route i, the sequences $\{\overline{\mathcal{A}}_i^\ell(0)\}_{\ell=1}^\infty$ and $\{\overline{Q}_i^\ell(0)\}_{\ell=1}^\infty$ are deterministic, $\{\overline{\mathcal{A}}_i^\ell(0)\}_{\ell=1}^\infty$ converges to $\overline{\mathcal{A}}_i^0$ vaguely, and $\{\overline{Q}_i^\ell(0)\}_{\ell=1}^\infty$ converges to $\overline{Q}_i^0$. Next, note that for each route i, because ϑ_i has no atoms in $\mathbb{R}_+$, we have $H_i \in \mathcal{C}_b^+([0,\alpha_i))$. Since each $b_{ik} < \alpha_i$ is a continuity point of $\overline{\mathcal{A}}_i^0$,

$$\lim_{\ell\to\infty} \langle H_i 1_{[0,b_{ik}]}, \overline{\mathcal{A}}_i^\ell(0)\rangle = \int_{[0,b_{ik}]} H_i(\alpha)\overline{\mathcal{A}}_i^0(d\alpha). \tag{3.2.27}$$

Step 4. We claim that for each route i,

$$\lim_{\ell\to\infty} \langle H_i 1_{[0,\alpha_i)}, \overline{\mathcal{A}}_i^\ell(0)\rangle = \langle H_i 1_{[0,\alpha_i)}, \overline{\mathcal{A}}_i^0\rangle + (\overline{Q}_i^0 - \langle 1_{[0,\alpha_i)}, \overline{\mathcal{A}}_i^0\rangle)\langle f, \Theta_i\rangle. \tag{3.2.28}$$

First, note that all quantities in (3.2.28) are deterministic so that we can use almost sure limits (rather than distributional limits). To show (3.2.28), consider $\varepsilon \in (0,1)$ and route i. Then, for each integer $k \geq k_{i,\varepsilon}$,

$$\begin{aligned}
&\limsup_{\ell\to\infty} \left| \langle H_i 1_{[0,\alpha_i)}, \overline{\mathcal{A}}_i^\ell(0)\rangle - \int_{[0,\alpha_i)} H_i(\alpha)\overline{\mathcal{A}}_i^0(d\alpha) - (\overline{Q}_i^0 - \langle 1_{[0,\alpha_i)}, \overline{\mathcal{A}}_i^0\rangle)\langle f, \Theta_i\rangle \right| \\
\leq\ & \limsup_{\ell\to\infty} \left\{ \left| \left\langle H_i 1_{[0,b_{ik}]}, \overline{\mathcal{A}}_i^\ell(0)\right\rangle - \int_{[0,b_{ik}]} H_i(\alpha)\overline{\mathcal{A}}_i^0(d\alpha) \right| \right. \\
& + \left| \int_{(b_{ik},\alpha_i)} H_i(\alpha)\overline{\mathcal{A}}_i^0(d\alpha) \right| + \frac{\langle 1_{(b_{ik},\alpha_i)}, \mathcal{A}_i^{x^\ell}(0)\rangle}{|x^\ell|} \left| \frac{\left\langle H_i 1_{(b_{ik},\alpha_i)}, \mathcal{A}_i^{x^\ell}(0)\right\rangle}{\langle 1_{(b_{ik},\alpha_i)}, \mathcal{A}_i^{x^\ell}(0)\rangle} - \langle f, \Theta_i\rangle \right| \\
& \left. + \left| \langle 1_{(b_{ik},\alpha_i)}, \overline{\mathcal{A}}_i^\ell(0)\rangle - \left(\overline{Q}_i^0 - \left\langle 1_{[0,\alpha_i)}, \overline{\mathcal{A}}_i^0\right\rangle\right) \right| \langle f, \Theta_i\rangle \right\} \\
\leq\ & 0 + \varepsilon/4 + \varepsilon/4 \\
& + \limsup_{\ell\to\infty} \left(\left| \overline{Q}_i^\ell(0) - \overline{Q}_i^0 \right| + \left| \langle 1_{[0,b_{ik}]}, \overline{\mathcal{A}}_i^\ell(0)\rangle - \langle 1_{[0,\alpha_i)}, \overline{\mathcal{A}}_i^0\rangle \right| \right) \|f\|_\infty \\
\leq\ & \varepsilon/2 + 0 + \langle 1_{(b_{ik},\alpha_i)}, \overline{\mathcal{A}}_i^0\rangle \|f\|_\infty \leq \varepsilon/2 + \varepsilon/2 < \varepsilon.
\end{aligned}$$

Since $\varepsilon \in (0,1)$ and route i were arbitrarily chosen, (3.2.28) follows.

Step 5. Combining (3.2.22), (3.2.23) and (3.2.28) together, we obtain (3.2.19). $\square$

3.2.4 Limit points of scaled state descriptors for exogenous flows

In our proof of Theorem 3.2.1, for each $t \in (0,\infty)$, we will apply Theorem 4.1 of Gromoll and Williams [13] to the sequence $\{\overline{\mathcal{R}}^{\ell}(t+\cdot)\}_{\ell=1}^{\infty}$. To do this, among other things, we must verify that the weak limit points of the sequence $\{\overline{\mathcal{R}}^{\ell}(t)\}_{\ell=1}^{\infty}$ satisfy certain properties, and the next lemma is the first step in our verification of these properties.

Lemma 3.2.4. *The sequence $\{(\overline{\mathcal{R}}^{e,\ell}, \overline{Q}^{e,\ell}, \overline{W}^{e,\ell}) : \ell = 1, 2, \ldots\}$ of random elements taking values in $\mathbb{D}(\mathbf{M}^{\mathbb{I}} \times \mathbb{R}_+^{\mathbb{I}} \times \mathbb{R}_+^{\mathbb{I}})$ is C-tight, and if $(\overline{\mathcal{R}}^{e}, \overline{Q}^{e}, \overline{W}^{e})$ is a weak limit point, then almost surely, for each route i and $t \in (0,\infty)$,*

$$\langle 1_{\{x\}}, \overline{\mathcal{R}}_i^{e}(t)\rangle = 0, \text{ for each } x \in [0,\infty), \tag{3.2.29}$$

$$\langle \chi, \overline{\mathcal{R}}_i^{e}(t)\rangle = \overline{W}_i^{e}(t) < \infty. \tag{3.2.30}$$

Proof. The same proof technique as used to prove Theorem 4.1 in Gromoll and Williams [13] can be used to show our claim here. While there are some differences, they are all minor issues. In particular, the results in [13] are about measure-valued processes akin to $\overline{\mathcal{R}}^{\ell}$, but $\overline{\mathcal{R}}^{e,\ell}$ is only a portion of $\overline{\mathcal{R}}^{\ell}$. Also, dynamics are as if the system started empty, but bandwidth allocation is determined by $\overline{Q}^{\ell}$ rather than $\overline{Q}^{e,\ell}$. However, these differences cause no significant difficulty in adapting the proof of Theorem 4.1 in [13] to our case. In fact, since $\overline{\mathcal{R}}^{e,\ell}(0) = 0$ for each integer $\ell \geq 1$ the proof of Theorem 4.1 in [13] can be slightly simplified in our case. For example, Lemma 5.3 and Lemma 5.4 are important steps toward proving Theorem 4.1 in [13], but the first part of the proof of Lemma 5.3 and *Step 1* of the proof of Lemma 5.4 in [13] can be safely omitted in our case. Then, once a version of Lemma 5.4 is established for our case, an oscillation result similar to Lemma 5.6 of [13] can be obtained for our case, going through almost identical computations. This is because the dynamics behind both $\overline{\mathcal{R}}^{\ell}$ and $\overline{\mathcal{R}}^{e,\ell}$ have the same bandwidth sharing policy and for each route i, and so at each time t the point masses of $\overline{\mathcal{R}}_i^{e,\ell}(t)$ and $\overline{\mathcal{R}}_i^{\ell}(t)$ *move toward zero* at the same rate. For further details, we refer the reader to Gromoll and Williams [13]. □

3.2.5 Limit points of scaled state descriptors at fixed times for initial flows

By combining our main results in this subsection, i.e., Lemma 3.2.5 and Lemma 3.2.6, one can obtain results that apply to initial flows and that are akin to (3.2.29) and (3.2.30) in Lemma 3.2.4. Also, we point out here that while Lemma 3.2.5 holds for all $t \in [0, \infty)$, Lemma 3.2.6 is valid only for all $t \in (0, \infty)$. We will further explain this in Remark 3.2.1 which appears after our proof of Lemma 3.2.6.

Lemma 3.2.5. *Let $t \in [0, \infty)$, and let $(\overline{\mathcal{R}}^o(t), \overline{W}^o(t))$ be a weak limit point of the sequence $\{(\overline{\mathcal{R}}^{o,\ell}(t), \overline{W}^{o,\ell}(t))\}_{\ell=1}^{\infty}$ of random elements taking values in $\mathbf{M}^{\mathbb{I}} \times \mathbb{R}_+^{\mathbb{I}}$. Then, almost surely, $\overline{W}_i^o(t) < \infty$ for each route i and*

$$\langle \chi, \overline{\mathcal{R}}^o(t) \rangle = \overline{W}^o(t). \tag{3.2.31}$$

Proof. Our proof consists of four steps.

Step 1. For convenience of notation, we assume that we have already passed to a subsequence so that as $\ell \to \infty$,

$$(\overline{\mathcal{R}}^{o,\ell}(t), \overline{W}^{o,\ell}(t)) \Rightarrow (\overline{\mathcal{R}}^o(t), \overline{W}^o(t)). \tag{3.2.32}$$

Furthermore, by invoking the Skorohod representation theorem (cf. Whitt [28]), we may assume that the convergence in distribution is replaced by almost sure convergence. Next, to facilitate our computations, for each integer $k \geq 1$, we define $f_k : \mathbb{R}_+ \to [0, 1]$ by letting, for each $s \in \mathbb{R}_+$,

$$f_k(s) = 1 - 1 \wedge \inf\{|s - r| : r \in [0, k]\}.$$

Step 2. For each route i, $\mathbf{P}[\overline{W}_i^o(t) < \infty] = 1$ because

$$\begin{aligned}
\mathbf{E}[\lim_{\ell\to\infty} \overline{W}_i^{o,\ell}(t)] &\leq \liminf_{\ell\to\infty} \mathbf{E}[\overline{W}_i^{o,\ell}(t)] \\
&\leq \liminf_{\ell\to\infty} \mathbf{E}[\overline{W}_i^{o,\ell}(0)] \\
&\leq \liminf_{\ell\to\infty} \overline{Q}_i^{o,\ell}(0) \sup_{\alpha\in[0,\alpha_i)} \langle \chi, \vartheta_i^\alpha \rangle \\
&\leq \sup_{\alpha\in[0,\alpha_i)} \langle \chi, \vartheta_i^\alpha \rangle < \infty.
\end{aligned}$$

Step 3. Note that $f_k \in \mathcal{C}_b^+(\mathbb{R}_+)$ and $0 \leq f_k(s) \leq f_{k+1}(s) \leq 1$ for all $s \in \mathbb{R}_+$ and integers $k \geq 1$. Also, observe that $\lim_{k\to\infty} f_k(s) = 1$ for all $s \in \mathbb{R}_+$. Then, almost surely, for each integer $k \geq 1$ and route i,

$$\langle \chi f_k, \overline{\mathcal{R}}_i^o(t) \rangle \leq \langle \chi f_{k+1}, \overline{\mathcal{R}}_i^o(t) \rangle, \tag{3.2.33}$$

$$\overline{W}_i^o(t) - \langle \chi f_k, \overline{\mathcal{R}}_i^0(t) \rangle \geq \overline{W}_i^o(t) - \langle \chi f_{k+1}, \overline{\mathcal{R}}_i^o(t) \rangle, \tag{3.2.34}$$

and since $\chi f_k \in \mathcal{C}_b^+(\mathbb{R}_+)$, almost surely,

$$\overline{W}_i^o(t) = \lim_{\ell\to\infty} \langle \chi, \overline{\mathcal{R}}_i^{o,\ell}(t) \rangle \geq \lim_{\ell\to\infty} \langle \chi f_k, \overline{\mathcal{R}}_i^{o,\ell}(t) \rangle = \langle \chi f_k, \overline{\mathcal{R}}_i^o(t) \rangle, \tag{3.2.35}$$

so that

$$\begin{aligned}
\overline{W}_i^o(t) - \langle \chi f_k, \overline{\mathcal{R}}_i^o(t) \rangle &= \lim_{\ell\to\infty} \langle \chi, \overline{\mathcal{R}}_i^{o,\ell}(t) \rangle - \lim_{\ell\to\infty} \langle \chi f_k, \overline{\mathcal{R}}_i^{o,\ell}(t) \rangle & (3.2.36)\\
&= \lim_{\ell\to\infty} \langle \chi(1-f_k), \overline{\mathcal{R}}_i^{o,\ell}(t) \rangle. & (3.2.37)
\end{aligned}$$

Then, we have that almost surely, for each route i,

$$\begin{aligned}
0 &\leq \overline{W}_i^o(t) - \langle \chi, \overline{\mathcal{R}}_i^o(t) \rangle & (3.2.38)\\
&= \lim_{k\to\infty} \left(\overline{W}_i^o(t) - \langle \chi f_k, \overline{\mathcal{R}}_i^o(t) \rangle \right) & (3.2.39)\\
&= \lim_{k\to\infty} \left(\lim_{\ell\to\infty} \langle \chi(1-f_k), \overline{\mathcal{R}}_i^{o,\ell}(t) \rangle \right), & (3.2.40)
\end{aligned}$$

where existence of the monotone limit in (3.2.39) is by the inequality in (3.2.34) and the equality in (3.2.40) is by the equality in (3.2.37).

Step 4. Using Fatou's lemma,

$$\begin{aligned} 0 &\le \mathbf{E}\left[\lim_{k\to\infty}\left(\lim_{\ell\to\infty}\langle\chi(1-f_k),\overline{\mathcal{R}}_i^{o,\ell}(t)\rangle\right)\right] \\ &\le \liminf_{k\to\infty}\mathbf{E}\left[\lim_{\ell\to\infty}\langle\chi(1-f_k),\overline{\mathcal{R}}_i^{o,\ell}(t)\rangle\right] \\ &= \liminf_{k\to\infty}\liminf_{\ell\to\infty}\mathbf{E}\left[\langle\chi(1-f_k),\overline{\mathcal{R}}_i^{o,\ell}(t)\rangle\right] \\ &\le \liminf_{k\to\infty}\liminf_{\ell\to\infty}\mathbf{E}\left[\langle\chi 1_{[k,\infty)},\overline{\mathcal{R}}_i^{\ell}(0)\rangle\right] \\ &\le \liminf_{k\to\infty}\liminf_{\ell\to\infty}\sup_{\alpha\in[0,\alpha_i)}\langle\chi 1_{[k,\infty)},\vartheta_i^{\alpha}\rangle \\ &= \liminf_{k\to\infty}\sup_{\alpha\in[0,\alpha_i)}\langle\chi 1_{[k,\infty)},\vartheta_i^{\alpha}\rangle = 0, \end{aligned}$$

by Assumption 2.2.1. Therefore, together with (3.2.38)-(3.2.40), we see that for each route i, almost surely,

$$0 \le \overline{W}_i^o(t) - \langle\chi,\overline{\mathcal{R}}_i^o(t)\rangle = \lim_{k\to\infty}\lim_{\ell\to\infty}\langle\chi(1-f_k),\overline{\mathcal{R}}_i^{o,\ell}(t)\rangle = 0,$$

proving the identity in (3.2.31). □

Lemma 3.2.6. *Let $t\in(0,\infty)$, and let $\overline{\mathcal{R}}^o(t)$ be a weak limit point of the sequence $\{\overline{\mathcal{R}}^{o,\ell}(t)\}_{\ell=1}^{\infty}$ of random elements taking values in $\mathbf{M}^{\mathbb{I}}$. Then, almost surely, for each route i, $\overline{\mathcal{R}}_i^o(t)$ has no atoms in $\mathbb{R}_+$.*

Proof. Our proof consists of five steps.

Step 1. For each integer $\ell\ge 1$, define a random element $\overline{D}^{o,\ell}$ taking values in $\mathbb{D}^{\mathbb{I}}(\mathbb{R}_+)$ by letting, for each route i and $s\in[0,\infty)$,

$$\overline{D}_i^{o,\ell}(s) = \overline{Q}_i^{o,\ell}(0) - \overline{Q}_i^{o,\ell}(s) \le 1, \tag{3.2.41}$$

and note that

$$\overline{Q}_i^{\ell}(s) = \overline{Q}_i^{\ell}(0) - \overline{D}_i^{o,\ell}(s) + \overline{Q}_i^{e,\ell}(s). \tag{3.2.42}$$

By Lemma B.7.3, when the product space $\mathbb{D}^{\mathbb{I}}(\mathbb{R}_+) = \prod_{i=1}^{\mathbb{I}}\mathbb{D}(\mathbb{R}_+)$ is given the product topology, where each $\mathbb{D}(\mathbb{R}_+)$ is given the M_1' topology, the sequence $\{\overline{D}^{o,\ell}\}_{\ell=1}^{\infty}$ is tight.

Also, by Lemma 3.2.4, when $\mathbb{D}(\mathbb{R}_+^{\mathbb{I}})$ is given the J_1 topology, the sequence $\{\overline{Q}^{e,\ell}\}_{\ell=1}^{\infty}$ is C-tight. Then, since we have the tightness results of Lemma 3.2.2 (with $t=0$), the tightness of $\{\overline{D}^{o,\ell}\}_{\ell=1}^{\infty}$ and the C-tightness of $\{\overline{Q}^{e,\ell}\}_{\ell=1}^{\infty}$ established in Lemma 3.2.4, we may assume that we have already passed to a subsequence so that as $\ell \to \infty$, the sequence

$$\{(\overline{\mathcal{A}}^{\ell}(0), \overline{\mathcal{R}}^{\ell}(0), \overline{\mathcal{R}}^{o,\ell}(t), \overline{Q}^{\ell}(0), \overline{D}^{o,\ell}, \overline{Q}^{e,\ell})\}_{\ell=1}^{\infty} \tag{3.2.43}$$

of random elements taking values in

$$\mathbf{M}^{\mathbb{I}} \times \mathbf{M}^{\mathbb{I}} \times \mathbf{M}^{\mathbb{I}} \times [0,1]^{\mathbb{I}} \times \mathbb{D}^{\mathbb{I}}(\mathbb{R}_+) \times \mathbb{D}(\mathbb{R}_+^{\mathbb{I}}) \tag{3.2.44}$$

converges in distribution to the random element

$$(\overline{\mathcal{A}}^{0}, \overline{\mathcal{R}}^{0}, \overline{\mathcal{R}}^{o}(t), \overline{Q}^{0}, \overline{D}^{o}, \overline{Q}^{e}) \tag{3.2.45}$$

such that $\overline{Q}^{e}$ takes values almost surely in $\mathbb{C}(\mathbb{R}_+^{\mathbb{I}})$. Here, we cautiously remind the reader that the product space in (3.2.44) is given the product topology, where each $\mathbf{M}$ associated with $\{\overline{\mathcal{A}}^{\ell}(0)\}_{\ell=1}^{\infty}$ is given the topology of vague convergence whereas each $\mathbf{M}$ associated with $\{\overline{\mathcal{R}}^{\ell}(0)\}_{\ell=1}^{\infty}$ or $\{\overline{\mathcal{R}}^{o,\ell}(t)\}_{\ell=1}^{\infty}$ is given the topology of weak convergence. Also, while the set $\mathbb{D}(\mathbb{R}_+^{\mathbb{I}})$ is given the J_1 topology, each $\mathbb{D}(\mathbb{R}_+)$ of the product space $\mathbb{D}^{\mathbb{I}}(\mathbb{R}_+)$ is given the M_1' topology.

Then, by invoking the Skorohod representation theorem for random elements taking values in a separable metric space (cf. Whitt [28]), we may further assume that the convergence is almost sure rather than in distribution.

Step 2. Let $\overline{Q}^{o}(t) = \langle 1, \overline{\mathcal{R}}^{o}(t)\rangle$ and also define a random element $\overline{Q}$ by letting, for each route i and $s \in [0,\infty)$,

$$\overline{Q}_i(s) = \overline{Q}_i^{0} - \overline{D}_i^{o}(s) + \overline{Q}_i^{e}(s). \tag{3.2.46}$$

First, for each route i and integer $\ell \geq 1$, almost surely,

$$\langle 1, \overline{\mathcal{R}}_i^{o,\ell}(t)\rangle = \overline{Q}_i^{o,\ell}(t) = \overline{Q}_i^{\ell}(0) - \overline{D}_i^{o,\ell}(t),$$

whence

$$\overline{Q}_i^o(t) \equiv \langle 1, \overline{\mathcal{R}}_i^o(t)\rangle = \lim_{\ell\to\infty} \overline{Q}_i^{o,\ell}(t) = \lim_{\ell\to\infty} \left(\overline{Q}_i^\ell(0) - \overline{D}_i^{o,\ell}(t)\right). \tag{3.2.47}$$

Next, since $\overline{Q}^e$ takes values in $\mathbb{C}(\mathbb{R}_+^{\mathbb{I}})$ almost surely and $\{\overline{Q}^{e,\ell}\}_{\ell=1}^\infty$ converges almost surely to $\overline{Q}^e$ in the J_1 topology, we have that almost surely, $\{\overline{Q}^{e,\ell}\}_{\ell=1}^\infty$ converges to $\overline{Q}^e$ uniformly on compact intervals. Moreover, for each route i and almost every ω, there exists a sequence $\{a_{k,\omega}^i\}_{k=1}^\infty \subset [0,t]$ such that, for each $s \in [0,t] \setminus \{a_{k,\omega}^i\}_{k=1}^\infty$,

$$\lim_{\ell\to\infty} \overline{D}_i^{o,\ell}(s,\omega) = \overline{D}_i^o(s,\omega)$$

so that

$$\begin{aligned}
\lim_{\ell\to\infty} \overline{Q}_i^\ell(s,\omega) &= \lim_{\ell\to\infty} \left(\overline{Q}_i^\ell(0) - \overline{D}_i^{o,\ell}(s,\omega) + \overline{Q}_i^{e,\ell}(s,\omega)\right) && (3.2.48)\\
&= \overline{Q}_i^0 - \overline{D}_i^o(s,\omega) + \overline{Q}_i^e(s,\omega) && (3.2.49)\\
&= \overline{Q}_i(s,\omega). && (3.2.50)
\end{aligned}$$

Therefore, for each route i and almost every ω, if $s \in [0,t] \setminus \{a_{k,\omega}^i\}_{k=1}^\infty$, then by (3.2.47),

$$\overline{Q}_i(s,\omega) = \lim_{\ell\to\infty} \overline{Q}_i^\ell(s,\omega) \geq \lim_{\ell\to\infty} \left(\overline{Q}_i^\ell(0) - \overline{D}_i^{o,\ell}(t,\omega)\right) = \overline{Q}_i^o(t,\omega). \tag{3.2.51}$$

Step 3. For each integer $\ell \geq 1$, define a random element $\overline{S}^\ell(t)$ taking values in $\mathbb{R}_+^{\mathbb{I}}$ by letting, for each route i,

$$\overline{S}_i^\ell(t) = \int_0^t \frac{\Lambda_i(\overline{Q}^\ell(u))}{\overline{Q}_i^\ell(u)} du, \tag{3.2.52}$$

and define a random element $\overline{S}(t)$ by letting, for each route i,

$$\overline{S}_i(t) = \begin{cases} \int_0^t \frac{\Lambda_i(\overline{Q}(s))}{\overline{Q}_i(s)} ds, & \text{if } \overline{Q}_i^o(t) > 0, \\ 0, & \text{if } \overline{Q}_i^o(t) = 0. \end{cases} \tag{3.2.53}$$

We caution the reader that for each route i, $\overline{S}_i(t)$ is defined somewhat arbitrarily; in particular, when $\overline{Q}_i^o(t) = 0$, $\overline{S}_i(t)$ is not meant to be the limit of $\{\overline{S}_i^\ell(t)\}_{\ell=1}^\infty$.

Now, we claim that for each route i and almost every ω such that $\overline{Q}_i^o(t,\omega) > 0$,

$$\lim_{\ell\to\infty} \overline{S}_i^\ell(t,\omega) = \overline{S}_i(t,\omega) \in (0,\infty). \tag{3.2.54}$$

To see this, note that for each route i and almost every ω, if $\overline{Q}_i^o(t,\omega) > 0$, then for each $s \in [0,t] \setminus \{a_{k,\omega}^i\}_{k=1}^\infty$, by (3.2.51) and continuity of Λ_i at $n = (n_1, \dots, n_{\mathbb{I}})$ satisfying $n_i > 0$, we have that as $\ell \to \infty$,

$$\frac{\Lambda_i(\overline{Q}^\ell(s,\omega))}{\overline{Q}_i^\ell(s,\omega)} \to \frac{\Lambda_i(\overline{Q}(s,\omega))}{\overline{Q}_i(s,\omega)} > 0, \tag{3.2.55}$$

where the left member is uniformly bounded by $2\|C\|/\overline{Q}_i^o(t,\omega)$ for all ℓ sufficiently large (depending on ω). It then follows by bounded convergence that (3.2.54) holds.

Step 4. We claim that for each route i and $B \in \mathcal{B}(\mathbb{R}_+)$, almost surely,

$$\overline{\mathcal{R}}_i^o(t)(B) = \begin{cases} \int_{(\overline{S}_i^o(t),\infty)} 1_B(r - \overline{S}_i^o(t))\overline{\mathcal{R}}_i^0(dr), & \text{if } \overline{Q}_i^o(t) > 0, \\ 0, & \text{if } \overline{Q}_i^o(t) = 0. \end{cases} \tag{3.2.56}$$

Note that for each $f \in \mathcal{C}_b^+(\mathbb{R}_+)$, route i and integer $\ell \geq 1$,

$$\langle f, \overline{\mathcal{R}}_i^{o,\ell}(t)\rangle = \int_{(\overline{S}_i^\ell(t),\infty)} f(r - \overline{S}_i^\ell(t))\overline{\mathcal{R}}_i^\ell(0)(dr). \tag{3.2.57}$$

First, note that by Lemma 3.2.3, $\overline{\mathcal{R}}_i^0 \in \mathbf{M}$ is deterministic and since Θ_i and ϑ_i^α have no atoms in $(0,\infty)$, $\overline{\mathcal{R}}_i^o$ has no atoms in $(0,\infty)$. Next, we also have $\overline{S}_i(t,\omega) \in (0,\infty)$ for each route i and almost every ω such that $\overline{Q}_i^o(t,\omega) > 0$. Hence, for almost every ω such that $\overline{Q}_i^o(t,\omega) > 0$, we may apply Lemma B.6.2 to see that for each route i and $f \in \mathcal{C}_b^+(\mathbb{R}_+)$,

$$\langle f, \overline{\mathcal{R}}_i^o(t,\omega)\rangle = \int_{(\overline{S}_i^o(t,\omega),\infty)} f(r - \overline{S}_i^o(t,\omega))\overline{\mathcal{R}}_i^0(dr),$$

On the other hand, for almost every ω such that $\overline{Q}_i^o(t,\omega)=0$, we see that for each route i and $f \in \mathcal{C}_b^+(\mathbb{R}_+)$,

$$0 \le \langle f, \overline{\mathcal{R}}_i^o(t,\omega)\rangle \le \|f\|_\infty \overline{Q}_i^o(t,\omega) = 0. \tag{3.2.58}$$

Since any finite Borel measure on $\mathbb{R}_+$ is uniquely determined by its integrals against functions in $\mathcal{C}_b^+(\mathbb{R}_+)$, this implies the results in (3.2.56).

Step 5. We now prove $\overline{\mathcal{R}}_i^o(t)$ has no atoms almost surely for each route i. Recall that for each route i, ϑ_i has no atoms in $\mathbb{R}_+$ by Assumption 2.2.1, and Θ_i has no atoms in $(0,\infty)$ by Lemma B.2.1. Therefore, it follows that if $\overline{Q}_i^o(t,\omega)>0$, then by (3.2.56) and Lemma 3.2.3, for each $r \in \mathbb{R}_+$,

$$\begin{aligned}
\langle 1_{\{r\}}, \overline{\mathcal{R}}_i^o(t,\omega)\rangle &= \int_{(\overline{S}_i^o(t,\omega),\infty)} 1_{\{r\}}(w-\overline{S}_i^o(t,\omega))\overline{\mathcal{R}}_i^0(dw) && (3.2.59)\\
&\le \int_{[0,\alpha_i)} \vartheta_i^\alpha(\{r+\overline{S}_i^o(t,\omega)\})\overline{\mathcal{A}}_i^0(d\alpha) && (3.2.60)\\
&\qquad +(\overline{Q}_i^0 - \langle 1_{[0,\alpha_i)}, \overline{\mathcal{A}}_i^0\rangle)\Theta_i(\{r+\overline{S}_i^o(t,\omega)\}), && (3.2.61)\\
&= 0. && (3.2.62)
\end{aligned}$$

On the other hand, if ω is such that $\overline{Q}_i^o(t,\omega)=0$, then trivially, for each $r \in \mathbb{R}_+$, $\langle 1_{\{r\}}, \overline{\mathcal{R}}_i^o(t,\omega)\rangle = \langle 1_{\{r\}}, 0\rangle = 0$. This observation completes our proof. □

Remark 3.2.1. To see our reason for excluding the initial time in the statement of Lemma 3.2.6, imagine a case where for some route i, $\alpha_i < \infty$ and the sequence $\{\overline{\mathcal{A}}_i^\ell(0)\}_{\ell=1}^\infty$ converges to δ_{α_i}. In that case, using Lemma 3.2.3, one can show that through a subsequence, $\{\overline{\mathcal{R}}_i^\ell(0)\}_{\ell=1}^\infty$ converges in distribution to $q_i\delta_0$, where q_i is a limit point of the sequence $\{\overline{Q}_i^\ell(0)\}_{\ell=1}^\infty$. So, if $q_i>0$, then one sees that the conclusion of Lemma 3.2.6 need not hold for the initial time.

3.2.6 Proof of Theorem 3.2.1 and Corollary 3.2.1

Proof of Theorem 3.2.1. Fix $t_0 \in (0,\infty)$. The statement that the sequence

$$\{\overline{\mathcal{R}}^\ell(t_0+\cdot) : \ell = 1,2,3,\ldots\} \tag{3.2.63}$$

is C-tight and any weak limit point of (3.2.63) is almost surely a fluid model solution, is equivalent to the statement that each subsequence of the sequence (3.2.63) has a subsequence that converges in distribution and the weak limit point of this further subsequence is almost surely a (continuous) fluid model solution. We note that by Lemma 3.2.2, the sequence

$$\{(\overline{\mathcal{R}}^{o,\ell}(t_0), \overline{W}^{o,\ell}(t_0), \overline{\mathcal{R}}^{e,\ell}(t_0), \overline{W}^{e,\ell}(t_0), \overline{\mathcal{R}}^{\ell}(t_0), \overline{W}^{\ell}(t_0))\}_{\ell=1}^{\infty} \tag{3.2.64}$$

is tight. Thus, to prove Theorem 3.2.1, we consider a subsequence of (3.2.63), where by taking a further subsequence, we may assume that (3.2.64) converges in distribution to

$$(\overline{\mathcal{R}}^{o}(t_0), \overline{W}^{o}(t_0), \overline{\mathcal{R}}^{e}(t_0), \overline{W}^{e}(t_0), \overline{\mathcal{R}}(t_0), \overline{W}(t_0)), \tag{3.2.65}$$

along the subsequence. Then, it suffices to show that for this subsequence, the sequence

$$\{(\overline{\mathcal{R}}^{\ell}(t_0+\cdot), \overline{Q}^{\ell}(t_0+\cdot), \overline{T}^{\ell}(t_0+\cdot) - \overline{T}^{\ell}(t_0), \overline{U}^{\ell}(t_0+\cdot) - \overline{U}^{\ell}(t_0), \overline{W}^{\ell}(t_0+\cdot))\}_{\ell=1}^{\infty}$$

is C-tight, and any weak limit point $(\overline{\mathcal{R}}, \overline{Q}, \overline{T}, \overline{U}, \overline{W})$ is such that almost surely, $\overline{\mathcal{R}}$ is a fluid model solution with auxiliary functions $(\overline{Q}, \overline{W}, \overline{T}, \overline{U})$.

The key step in our proof is to verify the assumptions of Theorem 4.1 of [13] from which the result follows immediately. First, for each route i, and integers k and $\ell \geq 1$, let H_{ik}^{ℓ} be the original size of the document associated with the $(E_i^{x^\ell}(|x^\ell|t_0) + k)$-th exogenous flow for route i. Then, for each integer $\ell \geq 1$, the i.i.d. sequences

$$\{H_{1k}^{\ell}\}_{k=1}^{\infty}, \ldots, \{H_{\mathbb{I}k}^{\ell}\}_{k=1}^{\infty}$$

are mutually independent. Moreover, for each route i, the sequence $\{H_{ik}^{\ell}\}_{k=1}^{\infty}$ is equal in distribution to the sequence $\{v_{ik}\}_{k=1}^{\infty}$, and is independent from the random elements $\mathcal{R}^{x^\ell}(|x^\ell|t_0)$ and $E^{x^\ell}(|x^\ell|t_0+\cdot) - E^{x^\ell}(|x^\ell|t_0)$. From this observation, one sees that the assumptions (4.11)-(4.13) for Theorem 4.1 of Gromoll and Williams

[13] are trivially satisfied for our case, and also, since $\max_{i=1}^{\mathbb{I}} \langle 1_{\{0\}}, \vartheta_i \rangle = 0$ and $\max_{i=1}^{\mathbb{I}} \langle \chi, \vartheta_i \rangle < \infty$, the assumptions (4.8) and (4.9) for Theorem 4.1 of Gromoll and Williams [13] are also satisfied.

Next, since by Lemma 3.2.1, as $\ell \to \infty$,

$$\frac{E^{x^\ell}(|x^\ell|(t_0 + \cdot)) - E^{x^\ell}(|x^\ell| t_0)}{|x^\ell|} \Rightarrow \overline{E}(t_0 + \cdot) - \overline{E}(t_0) = \overline{E}(\cdot), \tag{3.2.66}$$

where $\overline{E} \in \mathbb{D}(\mathbb{R}_+^{\mathbb{I}})$ such that $\overline{E}_i(t) = \nu_i t$ for each route i, the assumption (4.10) for Theorem 4.1 of Gromoll and Williams [13] is satisfied.

Lastly, note that for each integer $\ell \geq 1$,

$$\frac{1}{|x^\ell|} \mathcal{R}^{x^\ell}(|x^\ell| t_0 + 0) = \overline{\mathcal{R}}^\ell(t_0) = \overline{\mathcal{R}}^{o,\ell}(t_0) + \overline{\mathcal{R}}^{e,\ell}(t_0),$$

and therefore, by the assumed convergence of (3.2.64) along the subsequence, we have

$$\overline{\mathcal{R}}(t_0) = \overline{\mathcal{R}}^o(t_0) + \overline{\mathcal{R}}^e(t_0), \tag{3.2.67}$$

$$\overline{W}(t_0) = \overline{W}^o(t_0) + \overline{W}^e(t_0). \tag{3.2.68}$$

Furthermore, by Lemma 3.2.4, Lemma 3.2.5 and Lemma 3.2.6, almost surely, for each route i, we have

$$\langle \chi, \overline{\mathcal{R}}_i(t_0) \rangle = \overline{W}_i(t_0) < \infty, \tag{3.2.69}$$

$$\langle 1_{\{x\}}, \overline{\mathcal{R}}_i(t_0) \rangle = 0, \text{ for each } x \in [0, \infty). \tag{3.2.70}$$

This verifies the assumptions (4.14)-(4.16) for Theorem 4.1 of Gromoll and Williams [13].

In summary, the hypotheses in Section 4.2 of Gromoll and Williams [13] are satisfied and so, the desired conclusion follows by Theorem 4.1 of Gromoll and Williams [13]. $\square$

Proof of Corollary 3.2.1. Fix $t_0 \in (0, \infty)$. By Lemma 3.2.1 and a standard functional law of large numbers,

$$(\overline{E}^{\ell}, \overline{Y}^{\ell}) \Rightarrow (\overline{E}, \overline{Y}), \tag{3.2.71}$$

where $\overline{Y}_i^{\ell}(s) = \sum_{i=1}^{\lfloor |x^\ell| s \rfloor} v_{ik}/|x^\ell|$ and $\overline{Y}_i(s) = \langle \chi, \vartheta_i \rangle s$ for each $s \in \mathbb{R}_+$ and route i, and so as $\ell \to \infty$,

$$\frac{1}{|x^\ell|} \sum_{k=1}^{E^x(|x^\ell| \cdot)} v_{ik} \Rightarrow \overline{E}_i(\cdot) \langle \chi, \vartheta_i \rangle = \rho_i(\cdot),$$

where $\rho_i(t) = \rho_i t$ for each $t \in \mathbb{R}_+$.

Now, let $\overline{\mathcal{R}}$ be a weak limit point of the sequence $\{\overline{\mathcal{R}}^{\ell}(t_0 + \cdot)\}_{\ell=1}^{\infty}$. Then, it is not hard to see that $(\overline{E}, \overline{\mathcal{R}})$ is a weak limit point of the sequence $\{(\overline{E}^{\ell}(\cdot), \overline{\mathcal{R}}^{\ell}(t_0 + \cdot))\}_{\ell=1}^{\infty}$. By taking a subsequence if necessary, assume that as $\ell \to \infty$, the sequence

$$\{(\overline{\mathcal{A}}^{\ell}(0), \overline{E}^{\ell}(0), \overline{Q}^{\ell}(0), \overline{\mathcal{R}}^{\ell}(0), \overline{W}^{\ell}(0)) : \ell = 1, 2, \ldots\}$$

converges in distribution to

$$(\overline{\mathcal{A}}^0, \overline{E}^0, \overline{Q}^0, \overline{\mathcal{R}}^0, \overline{W}^0).$$

For each route i, fix a sequence $\{f_{ik}\}_{k=1}^{\infty} \subset \mathcal{C}_c^+([0, \alpha_i))$ such that for each $x \in [0, \alpha_i)$, $0 \le f_{ik}(x) \le f_{i(k+1)}(x) \le 1$ for each integer $k \ge 1$ and $\lim_{k \to \infty} f_{ik}(x) = 1$, and then, note that

$$\begin{aligned}
\langle \chi, \Theta_i \rangle &= \lim_{k \to \infty} \langle \chi f_{ik}, \Theta_i \rangle \\
&= \lim_{k \to \infty} \lim_{\alpha \uparrow \alpha_i} \langle \chi f_{ik}, \vartheta_i^{\alpha} \rangle \\
&\le \sup_{\alpha \in [0, \alpha_i)} \langle \chi, \vartheta_i^{\alpha} \rangle.
\end{aligned}$$

Hence, by Lemma 3.2.3, we have

$$\begin{aligned}\langle \chi, \overline{\mathcal{R}}_i^0 \rangle &= \int_{[0,\alpha_i)} \langle \chi, \vartheta_i^\alpha \rangle \overline{\mathcal{A}}_i^0(d\alpha) + \left(\overline{Q}_i^0 - \langle 1_{[0,\alpha_i)}, \overline{\mathcal{A}}_i^0 \rangle \right) \langle \chi, \Theta_i \rangle \\ &\le \Big(\sup_{\alpha \in [0,\alpha_i)} \langle \chi, \vartheta_i^\alpha \rangle \Big) \langle 1_{[0,\alpha_i)}, \overline{\mathcal{A}}_i^0 \rangle + \left(\overline{Q}_i^0 - \langle 1_{[0,\alpha_i)}, \overline{\mathcal{A}}_i^0 \rangle \right) \langle \chi, \Theta_i \rangle \\ &\le \sup_{\alpha \in [0,\alpha_i)} \langle \chi, \vartheta_i^\alpha \rangle \left(\langle 1_{[0,\alpha_i)}, \overline{\mathcal{A}}_i^0 \rangle + \overline{Q}_i^0 - \langle 1_{[0,\alpha_i)}, \overline{\mathcal{A}}_i^0 \rangle \right) \\ &= \Big(\sup_{\alpha \in [0,\alpha_i)} \langle \chi, \vartheta_i^\alpha \rangle \Big) \overline{Q}_i^0.\end{aligned}$$

Now, note that for each $t \in [0, \infty)$, integer $\ell \ge 1$ and route i,

$$\begin{aligned}\frac{1}{|x^\ell|} \langle 1, \mathcal{R}_i^{x^\ell}(|x^\ell| t) \rangle &= \frac{1}{|x^\ell|} Q_i^{x^\ell}(|x^\ell| t) && (3.2.72)\\ &\le \frac{1}{|x^\ell|} Q_i^{o,x^\ell}(0) + \frac{1}{|x^\ell|} Q_i^{e,x^\ell}(|x^\ell| t), && (3.2.73)\\ &\le \frac{1}{|x^\ell|} Q_i^{o,x^\ell}(0) + \frac{1}{|x^\ell|} E_i^{x^\ell}(|x^\ell| t), && (3.2.74)\end{aligned}$$

and

$$\begin{aligned}\frac{1}{|x^\ell|} \langle \chi, \mathcal{R}_i^{x^\ell}(|x^\ell| t) \rangle &= \frac{1}{|x^\ell|} W_i^{x^\ell}(|x^\ell| t) && (3.2.75)\\ &\le \frac{1}{|x^\ell|} W_i^{o,x^\ell}(0) + \frac{1}{|x^\ell|} W_i^{e,x^\ell}(|x^\ell| t) && (3.2.76)\\ &\le \frac{1}{|x^\ell|} W_i^{o,x^\ell}(0) + \frac{1}{|x^\ell|} \sum_{k=1}^{E^x(|x^\ell| t)} v_{ik}. && (3.2.77)\end{aligned}$$

Therefore, it follows that almost surely, for each route i and $s \in [0, \infty)$,

$$\langle 1, \overline{\mathcal{R}}_i(s) \rangle \le 1 + \overline{E}_i(t_0 + s) = 1 + (t_0 + s)\nu_i,$$

and

$$\begin{aligned}\langle \chi, \overline{\mathcal{R}}_i(s) \rangle &\le \Big(\sup_{\alpha \in [0,\alpha_i)} \langle \chi, \vartheta_i^\alpha \rangle \Big) \overline{Q}_i^0 + \overline{E}_i(t_0 + s) \langle \chi, \vartheta_i \rangle \\ &\le \Big(\sup_{\alpha \in [0,\alpha_i)} \langle \chi, \vartheta_i^\alpha \rangle \Big) + (t_0 + s)\rho_i.\end{aligned}$$

□

Chapter 4

Proof of the main theorem

Throughout this chapter, we assume that the fluid model is stable. Then, we fix $t_0 \in (0, \infty)$ such that for each fluid model solution ζ satisfying the condition that

$$\|\langle 1, \zeta(0)\rangle\| \vee \|\langle \chi, \zeta(0)\rangle\| \leq (\mathbb{I} + \|\nu\|) \vee \left(\left(\max_{i=1}^{\mathbb{I}} \sup_{\alpha \in [0, \alpha_i)} \langle \chi, \vartheta_i^\alpha \rangle \right) + \|\rho\| \right), \quad (4.0.1)$$

we have

$$\zeta(t_0) = 0. \quad (4.0.2)$$

Let

$$t_1 = 1 + t_0. \quad (4.0.3)$$

Our proof of the main theorem is divided into three steps, each of which corresponds to one of the following three sections.

4.1 From fluid model stability to negative mean drift

Here, we show that (given that the fluid model is stable), a "negative mean drift" property is satisfied in the sense of the inequality (4.1.15) in Corollary 4.1.1, but first, we prove two key lemmas.

Lemma 4.1.1.

$$\lim_{r\to\infty} \sup_{|x|>r} \mathbf{E}\left[\frac{1}{|x|}\|Q^x(|x|t_1)\|\right] = 0. \tag{4.1.1}$$

Proof. For each $r \in [0,\infty)$ and route i, let

$$H_i(r) = \sup_{|x|>r} \mathbf{E}\left[\frac{1}{|x|}Q_i^x(|x|t_1)\right].$$

To prove (4.1.1), it suffices to show that for each route i,

$$\lim_{r\to\infty} H_i(r) = 0. \tag{4.1.2}$$

For this, fix route i. Now, if $r_1 \le r_2$, then $H_i(r_1) \ge H_i(r_2)$, whence $\lim_{r\to\infty} H_i(r)$ exists, although the limit could be ∞. Fix a sequence $\{x^\ell\}_{\ell=1}^\infty$ in $\mathbb{A}$ so that $|x^\ell| > \ell$ for each integer $\ell \ge 1$ and for each route i,

$$\lim_{\ell\to\infty} \mathbf{E}\left[\frac{1}{|x^\ell|}Q_i^{x^\ell}(|x^\ell|t_1)\right] = \lim_{r\to\infty} H_i(r). \tag{4.1.3}$$

Then, using Theorem 3.2.1, by taking a subsequence if necessary, we assume that the sequence

$$\left\{\frac{1}{|x^\ell|}\mathcal{R}^{x^\ell}(|x^\ell|(1+\cdot)) : \ell = 1,2,3,\ldots\right\} \tag{4.1.4}$$

of random elements taking values in $\mathbb{D}(\mathbf{M}^{\mathbb{I}})$ converges in distribution as $\ell \to \infty$ to a random element $\overline{\mathcal{R}}$ which is almost surely a fluid model solution, and by Corollary 3.2.1 with $t = 1$ and $s = 0$, we may also assume that almost surely, $\zeta(0) = \overline{\mathcal{R}}(0)$ satisfies the condition in (4.0.1). Then, by our choice of $t_0 \in (0,\infty)$ satisfying the property described in (4.0.2), we have that almost surely, $\overline{\mathcal{R}}(t_0) = 0$ so that $\langle 1, \overline{\mathcal{R}}(t_0)\rangle = 0$. Since almost surely, $\overline{\mathcal{R}}$ is continuous, the sequence

$$\left\{\frac{1}{|x^\ell|}\mathcal{R}^{x^\ell}(|x^\ell|(1+t_0)) : \ell = 1,2,3,\ldots\right\} \tag{4.1.5}$$

converges in distribution to $\overline{\mathcal{R}}(t_0) = 0$. Therefore, by the dominated convergence theorem combined with the domination established in (3.2.72)-(3.2.74), we have

$$\lim_{r\to\infty} H_i(r) = \lim_{\ell\to\infty} \mathbf{E}\left[\frac{1}{|x^\ell|}\langle 1, \mathcal{R}_i^{x^\ell}(|x^\ell|t_1)\rangle\right] = \mathbf{E}[\langle 1, \overline{\mathcal{R}}_i(t_0)\rangle] = 0. \tag{4.1.6}$$

□

Lemma 4.1.2.

$$\lim_{r\to\infty} \sup_{|x|>r} \mathbf{E}\left[\frac{1}{|x|}\|A^x(|x|t_1)\|\right] = 0. \tag{4.1.7}$$

Proof. To prove (4.1.7), it suffices to show that for each route i,

$$\lim_{r\to\infty} \sup_{|x|>r} \mathbf{E}\left[\frac{1}{|x|}A_i^x(|x|t_1)\right] = 0. \tag{4.1.8}$$

For this, we fix route i, and recall our definition of the renewal process N_i in (3.0.3). For each $r \in [0,\infty)$, let

$$H_i(r) = \sup_{|x|>r} \mathbf{E}\left[\frac{1}{|x|}A_i^x(|x|t_1)\right].$$

Note that if $r_1 \leq r_2$, then $H_i(r_1) \geq H_i(r_2)$, and hence, $\lim_{r\to\infty} H_i(r)$ exists while the limit could be ∞. We will show that $\lim_{r\to\infty} H_i(r) = 0$. For this, fix a sequence $\{x^\ell\}_{\ell=1}^\infty$ in $\mathbb{A}$ so that $|x^\ell| > \ell$ for each integer $\ell \geq 1$ and

$$\lim_{\ell\to\infty} \mathbf{E}\left[\frac{1}{|x^\ell|}A_i^{x^\ell}(|x^\ell|t_1)\right] = \lim_{r\to\infty} H_i(r). \tag{4.1.9}$$

Note that there is a functional central limit theorem for the sequence $\{u_{ik}\}_{k=1}^\infty$ that give the interevent times of the renewal process N_i, and recall that $R_i^{x^\ell}(0)/|x^\ell| \Rightarrow 0$ as $\ell \to \infty$. Then, using the proof of Lemma 6 in [16] (for the last equality below), we see that

$$\begin{aligned}
&\limsup_{\ell\to\infty} \frac{1}{|x^\ell|}\mathbf{E}\left[A_i^{x^\ell}(|x^\ell|t_1)1_{\{R_i^{x^\ell}(0)<|x^\ell|t_1\}}\right] \\
\leq\ &\limsup_{\ell\to\infty} \frac{1}{|x^\ell|}\mathbf{E}\left[u_{ik}1_{\{k=E_i^{x^\ell}(|x^\ell|t_1)\}}1_{\{R_i^{x^\ell}(0)<|x^\ell|t_1\}}\right] \\
\leq\ &\limsup_{\ell\to\infty} \frac{1}{|x^\ell|}\mathbf{E}\left[\max_{k=1,\dots,E_i^{x^\ell}(|x^\ell|t)} u_{ik}\right] = 0.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\limsup_{\ell\to\infty} \frac{1}{|x^\ell|}\mathbf{E}\left[A_i^{x^\ell}(|x^\ell|t_1)1_{\{R_i^{x^\ell}(0)\geq|x^\ell|t_1\}}\right]\\
\leq\;& \limsup_{\ell\to\infty} \frac{1}{|x^\ell|}\mathbf{E}\left[(A_i^{x^\ell}(0)+|x^\ell|t_1)1_{\{R_i^{x^\ell}(0)\geq|x^\ell|t_1\}}\right]\\
\leq\;& \limsup_{\ell\to\infty}\left((1+t_1)\mathbf{E}\left[1_{\{R_i^{x^\ell}(0)\geq|x^\ell|t_1\}}\right]\right)\\
=\;& (1+t_1)\limsup_{\ell\to\infty}\mathbf{P}\left[\overline{R}_i^\ell(0)\geq t_1\right]\\
=\;& 0,
\end{aligned}$$

where the last inequality is obtained by Markov's inequality and the last equality uses Lemma 3.2.1. In summary, we have the desired equality in (4.1.8). □

The following claim is the main result of this section.

Theorem 4.1.1.

$$\lim_{r\to\infty}\sup_{|x|>r}\mathbf{E}^x\left[\frac{1}{|x|}|\mathbf{A}^\dagger(|x|t_1)|\right]=0.$$

Proof. For each $x\in\mathbb{A}$ and $t\in[0,\infty)$, we have $|\mathbf{A}^x(t)|=\|Q^x(t)\|+\|A^x(t)\|$. Therefore, we see that

$$\begin{aligned}
&\lim_{r\to\infty}\sup_{|x|>r}\mathbf{E}^x\left[\frac{1}{|x|}|\mathbf{A}^\dagger(|x|t_1)|\right] && (4.1.10)\\
=\;& \lim_{r\to\infty}\sup_{|x|>r}\mathbf{E}\left[\frac{1}{|x|}|\mathbf{A}^x(|x|t_1)|\right] && (4.1.11)\\
=\;& \lim_{r\to\infty}\sup_{|x|>r}\mathbf{E}\left[\frac{1}{|x|}\|Q^x(|x|t_1)\|+\frac{1}{|x|}\|A^x(|x|t_1)\|\right] && (4.1.12)\\
\leq\;& \lim_{r\to\infty}\sup_{|x|>r}\mathbf{E}\left[\frac{1}{|x|}\|Q^x(|x|t_1)\|\right]+\lim_{r\to\infty}\sup_{|x|>r}\mathbf{E}\left[\frac{1}{|x|}\|A^x(|x|t_1)\|\right] && (4.1.13)\\
=\;& 0, && (4.1.14)
\end{aligned}$$

where we have used Lemma 4.1.1 and Lemma 4.1.2 to see that the terms in (4.1.13) are zero. □

Corollary 4.1.1. *For each $\varepsilon \in (0,1)$, there exists $r \in (0,\infty)$ such that for any $x \in \mathbb{A}$ with $|x| > r$,*

$$\mathbf{E}^x\left[|\mathbf{A}^\dagger(|x|t_1)|\right] - |x| \leq -\varepsilon|x|. \tag{4.1.15}$$

4.2 From negative mean drift to finite mean hitting time

Here, we show that (given that the fluid model is stable), a "finite mean hitting time" property is satisfied in the sense of Corollary 4.2.1. The main result of this subsection is Theorem 4.2.1 from which Corollary 4.2.1 follows, and our proof of this will take the "negative mean drift" property as the starting point. To prove this, first, we establish two elementary lemmas.

For our next lemma, recall that the topology on $\mathbb{A} \subset \mathbf{M}^{\mathbb{I}} \times \mathbb{R}_+^{\mathbb{I}}$ is induced by the product topology on $\mathbf{M}^{\mathbb{I}} \times \mathbb{R}_+^{\mathbb{I}}$ where each $\mathbf{M}$ of $\mathbf{M}^{\mathbb{I}}$ is equipped with the topology of weak convergence.

Lemma 4.2.1. *For each $r \in [0,\infty)$, the set $\{x \in \mathbb{A} : |x| \leq r\}$ is a closed subset of $\mathbb{A}$.*

Proof. Fix $r \in [0,\infty)$, and consider a sequence $\{x_\ell\}_{\ell=1}^\infty$ in $\mathbb{A}$ such that $|x_\ell| \leq r$ for each integer $\ell \geq 1$. Suppose that there exists $x_0 \in \mathbb{A}$ such that $\lim_{\ell\to\infty} x_\ell = x_0$. Then, we have

$$\lim_{\ell\to\infty}(Q^{x_\ell}(0), A^{x_\ell}(0)) = (Q^{x_0}(0), A^{x_0}(0)).$$

so that

$$|x_0| = \sum_{i=1}^{\mathbb{I}}(Q_i^{x_0}(0) + A_i^{x_0}(0)) = \lim_{\ell\to\infty}\sum_{i=1}^{\mathbb{I}}(Q_i^{x_\ell}(0) + A_i^{x_\ell}(0)) = \lim_{\ell\to\infty}|x_\ell| \leq r.$$

□

Lemma 4.2.2. *For each* $r \in [0, \infty)$,

$$\sup_{|x|\le r} \mathbf{E}[|\mathbf{A}^x(t_1)|] < \infty. \tag{4.2.1}$$

Proof. Fix $r \in [0, \infty)$. Note that if $r = 0$, then (4.2.1) is clearly true since the set $\{x \in \mathbb{A} : |x| \le 0\}$ consists of a single element. So, assume that $r > 0$. Note that for each route i, if $\beta \le r$, then

$$\langle \chi, \varphi_i^\beta \rangle = \frac{1}{\varphi_i((\beta,\infty))} \int_{(\beta,\infty)} \chi(u-\beta)\varphi_i(du) \le \frac{1}{\varphi_i((r,\infty))} \langle \chi, \varphi_i \rangle.$$

Therefore, for each $x \in \mathbb{A}$ such that $|x| \le r$ and route i, we have

$$\mathbf{E}[A_i^x(t_1)1_{\{R_i^x(0)\ge t_1\}}] \le \mathbf{E}[A_i^x(0) + R_i^x(0)] \le r + \tfrac{1}{\varphi_i((r,\infty))}\langle \chi, \varphi_i \rangle, \tag{4.2.2}$$

$$\mathbf{E}[Q_i^x(t_1)1_{\{R_i^x(0)\ge t_1\}}] \le \mathbf{E}[Q_i^x(0)] \le r. \tag{4.2.3}$$

Recall our definition of N in (3.0.3). For each route i, we have that $\mathbf{E}[N_i(t_1)] < \infty$ (cf. Chung [7]) and that

$$A_i^x(t_1)1_{\{R_i^x(0)<t_1\}} \le t_1$$

$$Q_i^x(t_1)1_{\{R_i^x(0)<t_1\}} \le Q_i^x(0) + 1 + N_i((t_1 - R_i^x(0))^+) \le Q_i^x(0) + 1 + N_i(t_1).$$

Therefore, for each $x \in \mathbb{A}$ such that $|x| \le r$ and route i,

$$\mathbf{E}[A_i^x(t_1)1_{\{R_i^x(0)<t_1\}}] \le t_1, \tag{4.2.4}$$

$$\mathbf{E}[Q_i^x(t_1)1_{\{R_i^x(0)<t_1\}}] \le 1 + \mathbf{E}[Q_i^x(0) + N_i(t_1)] \le 1 + \mathbf{E}[r + N_i(t_1)]. \tag{4.2.5}$$

Then, (4.2.1) follows from (4.2.2)-(4.2.5). $\square$

Theorem 4.2.1. *There exist integers* $\widetilde{r} \ge 1, \widetilde{c} \ge 1$ *and* $\widetilde{d} \ge 1$ *such that for each* $x \in \mathbb{A}$,

$$\mathbf{E}^x[\widetilde{\tau}] \le \widetilde{c}|x| + \widetilde{d}, \tag{4.2.6}$$

where

$$\widetilde{\tau} = \inf\{t \ge t_1 : |\mathbf{A}^\dagger(t)| \le \widetilde{r}\}. \tag{4.2.7}$$

Proof. Using Theorem 4.1.1, we can fix an integer $\widetilde{r} \geq 1$ such that for each $x \in \mathbb{A}$ with $|x| > \widetilde{r}$,

$$\mathbf{E}^x \left[\left| \mathbf{A}^\dagger(|x| t_1) \right| \right] \leq |x|/2. \tag{4.2.8}$$

and let

$$\widetilde{\gamma} = \sup_{|x| \leq \widetilde{r}} \mathbf{E}^x [|\mathbf{A}^\dagger(t_1)|],$$

which is finite by Lemma 4.2.2. Next, for each $x \in \mathbb{A}$, define

$$H(x) = 2t_1|x|, \tag{4.2.9}$$

$$h(x) = \begin{cases} t_1, & \text{if } |x| \leq \widetilde{r}, \\ |x| t_1, & \text{if } |x| > \widetilde{r}, \end{cases} \tag{4.2.10}$$

and observe that

$$H(x) - h(x) + t_1(1 + 2\widetilde{\gamma}) 1_{[0,\widetilde{r}]}(|x|) = \begin{cases} 2t_1|x| + 2t_1\widetilde{\gamma}, & \text{if } |x| \leq \widetilde{r}, \\ |x| t_1, & \text{if } |x| > \widetilde{r}. \end{cases} \tag{4.2.11}$$

On the other hand,

$$\mathbf{E}^x \left[H(\mathbf{A}^\dagger(h(x)) \right] = \mathbf{E}^x \left[2t_1 |\mathbf{A}^\dagger(t_1)| \right] \leq 2t_1 \widetilde{\gamma}, \text{ if } |x| \leq \widetilde{r}, \tag{4.2.12}$$

$$\mathbf{E}^x \left[H(\mathbf{A}^\dagger(h(x)) \right] = \mathbf{E}^x \left[2t_1 |\mathbf{A}^\dagger(|x| t_1)| \right] \leq |x| t_1, \text{ if } |x| > \widetilde{r}. \tag{4.2.13}$$

Hence,

$$\mathbf{E}^x \left[H(\mathbf{A}^\dagger(h(x)) \right] \leq H(x) - h(x) + t_1 \left(1 + 2\widetilde{\gamma}\right) 1_{[0,\widetilde{r}]}(|x|). \tag{4.2.14}$$

Let $\eta^\dagger(0) = 0$ and $\Phi_0^\dagger = \mathbf{A}^\dagger(0)$. Then, for each integer $k \geq 0$, we inductively define $\eta^\dagger(k+1)$ and $\Phi^\dagger(k+1)$ as follows:

$$\eta^\dagger(k+1) = \eta^\dagger(k) + h(\Phi^\dagger(k)), \tag{4.2.15}$$

$$\Phi^\dagger(k+1) = \mathbf{A}^\dagger(\eta^\dagger(k+1)). \tag{4.2.16}$$

Note that for each integer $k \geq 0$, each $\eta^\dagger(k)$ is a stopping time for $\mathfrak{F}^\dagger$ and then, let

$$\mathcal{H}_k^\dagger = \mathcal{F}_{\eta^\dagger(k)}^\dagger.$$

Also, let

$$\xi^\dagger = \inf\left\{k \geq 1 : |\Phi^\dagger(k)| \leq \widetilde{r}\right\}, \tag{4.2.17}$$

$$\widetilde{\eta}^\dagger = \textstyle\sum_{k=0}^{(\xi^\dagger-1)^+} h(\Phi^\dagger(k)), \tag{4.2.18}$$

where we let $\inf \varnothing = \infty$.

Note that, on the event that $\{\xi^\dagger < \infty\}$,

$$\eta^\dagger(\xi^\dagger) = \textstyle\sum_{k=0}^{(\xi^\dagger-1)^+} h(\Phi^\dagger(k)), \tag{4.2.19}$$

$$|\mathbf{A}^\dagger(\eta^\dagger(\xi^\dagger))| = |\Phi^\dagger(\xi^\dagger)| \leq \widetilde{r}. \tag{4.2.20}$$

Since each sample path of $\mathbf{A}^\dagger$ is right continuous and the set $\{x \in \mathbb{A} : |x| \leq \widetilde{r}\}$ is a closed subset of $\mathbb{A}$, we also have $\{\widetilde{\tau} < \infty\} \subset \{|\mathbf{A}^\dagger(\widetilde{\tau})| \leq \widetilde{r}\}$. Therefore, we have

$$\widetilde{\tau} \leq \left(\sum_{k=0}^{(\xi^\dagger-1)^+} h(\Phi^\dagger(k))\right) = \widetilde{\eta}^\dagger. \tag{4.2.21}$$

Now, for each $x \in \mathbb{A}$, by the inequality (4.2.14) together with the strong Markov property of $\mathbf{A}^\dagger$ (see Lemma A.5.2),

$$\begin{aligned} & \mathbf{E}^x[H(\Phi^\dagger(k+1))|\mathcal{H}_k^\dagger] \\ = \; & \mathbf{E}^{\Phi^\dagger(k)}[H(\mathbf{A}^\dagger(h(\Phi^\dagger(k))))] \\ \leq \; & H(\Phi^\dagger(k)) - h(\Phi^\dagger(k)) + t_1(1+2\widetilde{\gamma})1_{[0,\widetilde{r}]}(|\Phi^\dagger(k)|), \end{aligned}$$

and then, by Proposition 11.3.2 in Meyn and Tweedie [23],

$$\mathbf{E}^x\left[\widetilde{\eta}^\dagger\right] = \mathbf{E}^x\left[\textstyle\sum_{k=0}^{(\xi^\dagger-1)^+} h(\Phi^\dagger(k))\right] \leq H(x) + t_1\left(1+2\widetilde{\gamma}\right). \tag{4.2.22}$$

Letting $\widetilde{c} = \lceil 2t_1 \rceil$ and $\widetilde{d} = \lceil t_1(1+2\widetilde{\gamma}) \rceil$, we have that for each $x \in \mathbb{A}$,

$$\mathbf{E}^x[\widetilde{\tau}] \leq \mathbf{E}^x[\widetilde{\eta}^\dagger] \leq \widetilde{c}|x| + \widetilde{d}, \tag{4.2.23}$$

where the first and the second inequality in (4.2.23) are by (4.2.21) and the third inequality in (4.2.23) is by (4.2.22). □

Corollary 4.2.1. *Fix an integer* $\widetilde{r} \geq 1$ *as in Theorem 4.2.1, and define* $\widetilde{\tau}$ *by (4.2.7). Then,*

$$\mathbf{P}^x[\widetilde{\tau} < \infty] = 1 \text{ for each } x \in \mathbb{A}, \tag{4.2.24}$$

$$\sup_{|x| \leq \widetilde{r}} \mathbf{E}^x[\widetilde{\tau}] < \infty. \tag{4.2.25}$$

4.3 Proof of positive Harris recurrence

Fix an integer $\widetilde{r} \geq 1$ as formulated in Theorem 4.2.1, and define $\widetilde{\tau}$ by (4.2.7). To prove positive Harris recurrence of the age process, we appeal to Theorem 1.1 and Theorem 1.2 of Meyn and Tweedie [22]. For this, by virtue of Lemma 4.2.1 and Corollary 4.2.1, the remaining fact to be verified is that the set $\{x \in \mathbb{A} : |x| \leq \widetilde{r}\}$ is *petite* (cf. [22]), i.e., there exist a non-trivial non-negative finite Borel measure μ_0 on $\mathbb{A}$ and a Borel probability measure λ_0 on $[0, \infty)$ such that

$$\inf_{\{x \in \mathbb{A} : |x| \leq \widetilde{r}\}} \int_{[0,\infty)} P_t(x, B) \lambda_0(dt) \geq \mu_0(B), \text{ for each } B \in \mathcal{B}(\mathbb{A}). \tag{4.3.1}$$

To show existence of the measures λ_0 and μ_0 satisfying (4.3.1), we will adapt the proof of Proposition 4.7 in Bramson [6] to our present situation.

Step 1. The condition (i) in Assumption 2.2.2 implies that for each route i, there exists $\iota(i) \in (\widetilde{r}, \infty)$ such that $\varphi_i((\widetilde{r}, \iota(i))) > 0$. Then, let

$$\iota_0 = \max_{i=1}^{\mathbb{I}} \iota(i), \tag{4.3.2}$$

and observe that for any $\beta \in [0, \widetilde{r}]$,

$$(\widetilde{r}, \iota_0) \subset (\beta, \beta + \iota_0). \tag{4.3.3}$$

Therefore, we have

$$\min_{i=1}^{\mathbb{I}} \inf_{\beta\in[0,\widetilde{r}]} \varphi_i^{\beta}((0,\iota_0)) = \min_{i=1}^{\mathbb{I}} \inf_{\beta\in[0,\widetilde{r}]} \frac{\varphi_i((\beta,\beta+\iota_0))}{\varphi_i((\beta,\infty))} \tag{4.3.4}$$

$$\geq \min_{i=1}^{\mathbb{I}} \inf_{\beta\in[0,\widetilde{r}]} \varphi_i((\beta,\beta+\iota_0)) \tag{4.3.5}$$

$$\geq \min_{i=1}^{\mathbb{I}} \varphi_i((\widetilde{r},\iota_0)) > 0. \tag{4.3.6}$$

Next, by Theorem VII.1.1 in Asmussen [1], the condition (ii) in Assumption 2.2.2 implies that there exists an integer $\kappa_0 \geq \max_{i=1}^{\mathbb{I}} \kappa_i$ along with $\varepsilon_0 \in (0,1)$ and $\ell_1, \ldots, \ell_{\mathbb{I}} \in (\iota_0, \infty)$ such that for each route i and interval

$$[s_1, s_2] \subset [\ell_i - \iota_0, \ell_i + 3],$$

we have

$$\varphi_i^{(*\kappa_0)}([s_1, s_2]) \geq (s_2 - s_1)\varepsilon_0. \tag{4.3.7}$$

Also, by the condition (iii) in Assumption 2.2.1, there exists an integer $n_0 \geq 1$ such that

$$\max_{i=1}^{\mathbb{I}} \sup_{\alpha\in[0,\alpha_i)} \left\langle \chi 1_{[n_0,\infty)}, \vartheta_i^{\alpha} \right\rangle \leq 1/2. \tag{4.3.8}$$

Now, let

$$\ell_0 = \max\nolimits_{i=1}^{\mathbb{I}} (\ell_i + 3), \tag{4.3.9}$$

$$\gamma_0 = (\widetilde{r} \vee (\kappa_0 + 1)) n_0 / b_0, \tag{4.3.10}$$

$$\varepsilon_1 = \min\nolimits_{i=1}^{\mathbb{I}} \varphi_i((\widetilde{r}, \iota_0)), \tag{4.3.11}$$

$$\varepsilon_2 = \min\nolimits_{i=1}^{\mathbb{I}} \varphi_i((\ell_0 + 2\gamma_0, \infty)), \tag{4.3.12}$$

where

$$b_0 = \min_{i=1}^{\mathbb{I}} \left(\inf \left\{ \Lambda_i(n) : n_i \geq 1, \|n\| \leq \widetilde{r} + \mathbb{I}(\kappa_0 + 1) \right\} \right) > 0; \tag{4.3.13}$$

the strict positivity holds since for each route i, the function Λ_i is continuous and strictly positive on the compact set $\{n \in \mathbb{R}_+^{\mathbb{I}} : n_i \geq 1, \|n\| \leq \widetilde{r} + \mathbb{I}(\kappa_0 + 1)\}$. Next, define $H_0 : \mathbb{R}_+^{\mathbb{I}} \to \mathbb{A}$ by letting, for each $a = (a_1, \dots, a_{\mathbb{I}}) \in \mathbb{R}_+^{\mathbb{I}}$,

$$H_0(a) = (0, \dots, 0, a_1, \dots, a_{\mathbb{I}}).$$

Finally, for each $s \in (\ell_0 + \gamma_0, \ell_0 + 2\gamma_0)$, let

$$\Gamma_s = [s - (\ell_1 + 3), s - \ell_1] \times \ldots \times [s - (\ell_{\mathbb{I}} + 3), s - \ell_{\mathbb{I}}],$$

and define a non-trivial non-negative subprobability measure μ_s on $(\mathbb{A}, \mathcal{B}(\mathbb{A}))$ by letting, for each $B \in \mathcal{B}(\mathbb{A})$,

$$\mu_s(B) = (\varepsilon_0\varepsilon_1\varepsilon_2)^{\mathbb{I}} \left(\frac{1}{2^{\mathbb{I}}}\right)^{\widetilde{r}+(\kappa_0+1)} \int_{\Gamma_s} 1_{H_0^{-1}(B)}(r_1, \dots, r_{\mathbb{I}}) dr_1 \cdots dr_{\mathbb{I}}.$$

Step 2. Fix $x \in \mathbb{A}$ such that $|x| \leq \widetilde{r}$ and fix $s \in (\ell_0 + \gamma_0, \ell_0 + 2\gamma_0)$. Let

$$\Omega_1^x = \textstyle\bigcap_{i=1}^{\mathbb{I}} \{\tau_i^x(\kappa_0 + 1) \in [\ell_i, \ell_i + 3]\}, \tag{4.3.14}$$

$$\Omega_2^x = \textstyle\bigcap_{i=1}^{\mathbb{I}} \{\tau_i^x(\kappa_0 + 2) \in (\ell_0 + 2\gamma_0, \infty)\}, \tag{4.3.15}$$

$$\Omega_3^x = \textstyle\bigcap_{i=1}^{\mathbb{I}} \left\{\max_{k=1}^{\kappa_0+1} v_{ik} \leq n_0\right\}, \tag{4.3.16}$$

$$\Omega_4^x = \textstyle\bigcap_{i=1}^{\mathbb{I}} \{W_i^x(0) \leq n_0 Q_i^x(0)\}. \tag{4.3.17}$$

First, we claim that that

$$\Omega_1^x \cap \Omega_2^x \cap \Omega_3^x \cap \Omega_4^x \subset \{\mathcal{A}^x(s) = 0\}. \tag{4.3.18}$$

This follows because on the event $\Omega_1^x \cap \Omega_2^x \cap \Omega_3^x \cap \Omega_4^x$, for each route i,

$$\begin{gathered} Q_i^{e,x}(s) \leq \kappa_0 + 1, \\ W_i^{e,x}(s) \leq \textstyle\sum_{k=1}^{\kappa_0+1} v_{ik}, \\ W_i^{e,x}(s) \leq \left(\textstyle\sum_{k=1}^{\kappa_0+1} v_{ik} - \int_{\ell_0}^{\ell_0+\gamma_0} \Lambda_i(Q^x(s))ds\right)^+ \leq ((\kappa_0+1)n_0 - \gamma_0 b_0)^+ = 0, \\ W_i^{o,x}(s) \leq (W_i^x(0) - \textstyle\int_0^s \Lambda_i(Q^x(u))du)^+ \leq (n_0 Q_i^x(0) - \gamma_0 b_0)^+ = 0, \\ \langle \chi, \mathcal{R}_i^x(s) \rangle = W_i^x(s) = W_i^{o,x}(s) + W_i^{x,e}(s) = 0, \end{gathered}$$

where we have used the fact that on the set in question, there are no new arrivals in $[\ell_0, \ell_0 + 2\gamma_0]$ and so if Q_i^x reaches zero in $[\ell_0, \ell_0 + 2\gamma_0]$, it stays there until after $\ell_0 + 2\gamma_0$.

Next, for each route i and interval $[s_1, s_2] \subset [\ell_i, \ell_i + 3]$,

$$\begin{aligned}
\mathbf{P}[\tau_i^x(\kappa_0+1) \in [s_1, s_2]] &= \mathbf{P}[\tau_i^x(\kappa_0+1) - \widetilde{u}_i^x + \widetilde{u}_i^x \in [s_1, s_2]] \\
&\geq \mathbf{E}[\varphi_i^{(*\kappa_0)}([s_1 - \widetilde{u}_i^x, s_2 - \widetilde{u}_i^x]); \widetilde{u}_i^x \leq \iota_0] \\
&\geq (s_2 - s_1)\varepsilon_0 \mathbf{P}[\widetilde{u}_i^x \leq \iota_0] \\
&\geq (s_2 - s_1)\varepsilon_0\varepsilon_1,
\end{aligned}$$

where the first equality is obtained by conditioning on $\widetilde{u}_i^x$ and then restricting to the event $\{\widetilde{u}_i^x \leq \iota_0\}$. So, more generally, for each $B \in \mathcal{B}(\mathbb{R}_+)$ and route i, we have

$$\mathbf{P}[\tau_i^x(\kappa_0+1) \in B] \geq \varepsilon_0\varepsilon_1 \int_{\ell_i}^{\ell_i+3} 1_B(u)du.$$

Now, consider $B \in \mathcal{B}(\mathbb{A})$ such that $H_0^{-1}(B) = B_1 \times \ldots \times B_{\mathbb{I}}$, where $B_1, \ldots, B_{\mathbb{I}} \in \mathcal{B}(\mathbb{R}_+)$. Then, we see that for each $s \in (\ell_0 + \gamma_0, \ell_0 + 2\gamma_0)$,

$$\begin{aligned}
\mathbf{P}\left[\mathbf{A}^x(s) \in B\right] &\geq \mathbf{P}\left[\mathcal{A}^x(s) = 0, A^x(s) \in H_0^{-1}(B)\right] && (4.3.19)\\
&\geq \mathbf{P}\left[\Omega_1^x \cap \Omega_2^x \cap \Omega_3^x \cap \Omega_4^x, A^x(s) \in B_1 \times \ldots \times B_{\mathbb{I}}\right] && (4.3.20)\\
&= \mathbf{P}[\Omega_3^x]\mathbf{P}[\Omega_4^x]\mathbf{P}[\Omega_1^x \cap \Omega_2^x, A^x(s) \in B_1 \times \ldots \times B_{\mathbb{I}}], && (4.3.21)
\end{aligned}$$

where the inequality in (4.3.20) is by (4.3.18) and the equality in (4.3.21) is by independence. Looking at the first two factors in (4.3.21), note that, by our choice of n_0, the inequality in (4.3.8) holds so that using Markov's inequality, we have

$$\begin{aligned}
\mathbf{P}\left[\Omega_3^x\right]\mathbf{P}\left[\Omega_4^x\right] &= \left(\prod_{i=1}^{\mathbb{I}} \mathbf{P}[\max_{k=1}^{\kappa_0+1} v_{ik} \le n_0]\right)\left(\prod_{i=1}^{\mathbb{I}} \mathbf{P}[W_i^x(0) \le n_0 Q_i^x(0)]\right) \\
&\ge \left(\prod_{i=1}^{\mathbb{I}}\prod_{k=1}^{\kappa_0+1} \mathbf{P}[v_{ik} \le n_0]\right)\left(\prod_{i=1}^{\mathbb{I}}\prod_{k=1}^{Q_i^x(0)} \mathbf{P}[\widetilde{v}_{ik}^x \le n_0]\right) \\
&\ge \left(\prod_{i=1}^{\mathbb{I}} \frac{1}{2^{\kappa_0+1}}\right)\left(\prod_{i=1}^{\mathbb{I}} \frac{1}{2^{Q_i^x(0)}}\right) \\
&\ge \left(\frac{1}{2^{\mathbb{I}}}\right)^{\widetilde{r}+(\kappa_0+1)}.
\end{aligned}$$

Next, looking at the third factor of (4.3.21) separately, we have

$$\begin{aligned}
&\mathbf{P}\left[\Omega_1^x \cap \Omega_2^x, A^x(s) \in B_1 \times \ldots \times B_{\mathbb{I}}\right] \\
= \;& \mathbf{P}\left[\bigcap_{i=1}^{\mathbb{I}}\{s - \tau_i^x(\kappa_0+1) \in B_i\}\,;\Omega_1^x \cap \Omega_2^x\right] \\
= \;& \prod_{i=1}^{\mathbb{I}} \mathbf{P}\left[s - \tau_i^x(\kappa_0+1) \in B_i, \tau_i^x(\kappa_0+1) \in [\ell_i, \ell_i+3], \tau_i^x(\kappa_0+2) \in (\ell_0+2\gamma_0, \infty)\right] \\
\ge \;& \varepsilon_2^{\mathbb{I}} \prod_{i=1}^{\mathbb{I}} \mathbf{P}\left[\tau_i^x(\kappa_0+1) \in s - B_i, \tau_i^x(\kappa_0+1) \in [\ell_i, \ell_i+3]\right] \\
\ge \;& (\varepsilon_0\varepsilon_1\varepsilon_2)^{\mathbb{I}} \prod_{i=1}^{\mathbb{I}} \int_{\ell_i}^{\ell_i+3} 1_{s-B_i}(u)du \\
= \;& (\varepsilon_0\varepsilon_1\varepsilon_2)^{\mathbb{I}} \prod_{i=1}^{\mathbb{I}} \int_{s-(\ell_i+3)}^{s-\ell_i} 1_{B_i}(u)du,
\end{aligned}$$

where $s - B_i$ denotes the set $\{s - r : r \in B_i\}$, and the second equality and the first inequality are obtained respectively by independence and by conditioning on the event

$$\bigcap_{i=1}^{\mathbb{I}}\{\tau_i^x(\kappa_0+2) - \tau_i^x(\kappa_0+1) > \ell_0 + 2\gamma_0\} \subset \bigcap_{i=1}^{\mathbb{I}}\{\tau_i^x(\kappa_0+2) \in (\ell_0+2\gamma_0, \infty)\}.$$

In summary, we have

$$\mathbf{P}\left[\mathbf{A}^x(s) \in B\right] \geq \left(\frac{1}{2^{\mathbb{I}}}\right)^{\widetilde{r}+(\kappa_0+1)} (\varepsilon_0\varepsilon_1\varepsilon_2)^{\mathbb{I}} \prod_{i=1}^{\mathbb{I}} \int_{s-(\ell_i+3)}^{s-\ell_i} 1_{B_i}(u)du \quad (4.3.22)$$

$$= \left(\frac{1}{2^{\mathbb{I}}}\right)^{\widetilde{r}+(\kappa_0+1)} (\varepsilon_0\varepsilon_1\varepsilon_2)^{\mathbb{I}} \int_{\Gamma_s} 1_{H_0^{-1}(B)}(u)du \quad (4.3.23)$$

$$= \mu_s(B). \quad (4.3.24)$$

Thus, we have

$$\mathbf{P}\left[\mathbf{A}^x(s) \in B\right] \geq \mu_s(B) \quad (4.3.25)$$

for each $B \in \mathcal{B}(\mathbb{A})$ such that $H_0^{-1}(B) = B_1 \times \ldots \times B_{\mathbb{I}} \in \mathcal{B}(\mathbb{R}_+^{\mathbb{I}})$, and hence, by Dynkin's π-λ theorem, we also have the inequality (4.3.25) for each $B \in \mathcal{B}(\mathbb{A})$.

Step 3. Choose any $s \in (\ell_0+\gamma_0, \ell_0+2\gamma_0)$, and then, define $\lambda_0 = \delta_s$, the unit point mass at s, and let $\mu_0 = \mu_s$. Now, by *Step 2*, the desired property stated in (4.3.1) is satisfied with these choices for λ_0 and μ_0. Thus, by Theorem 1.1 and Theorem 1.2 of Meyn and Tweedie [22], the age process is positive Harris recurrent, and so, Theorem 2.7.1 is proved. □

Appendix A

Borel right process

The main goal of Appendix A is to verify that the collection (2.5.2) is a Borel right process. When a queueing model's underlying dynamics can be reduced to a discrete event system, one can often rely on the theory of piecewise deterministic Markov processes (PDMPs) as described in Davis [9] to produce a Borel right process describing the queueing model. On the other hand, while it is shown in Davis [9] that each PDMP as defined in Davis [9] is a Borel right process, there are two reasons that stop us from simply applying relevant results in Davis [9] to our processes for the Internet congestion control model.

In our present situation, the first reason is that in order for the age process to be a PDMP as described in Davis [9], each of $\vartheta_1, \ldots, \vartheta_{\mathbb{I}}$ and $\varphi_1, \ldots, \varphi_{\mathbb{I}}$ should be absolutely continuous with respect to Lebesgue measure, but we do not necessarily want to assume this. The second reason is that in Davis [9], roughly speaking, when a PDMP *jumps* to a new state, the state to which the PDMP jumps depends on the state of the PDMP *just prior to* the jump time whereas for the age process, the relationship between times between *jumps* and states to which the age process *jumps* are slightly more intertwined.

Nonetheless, our proof here takes numerous hints from the discussion of PDMPs in Davis [9], and is broken into six stages in six sections below. In these six sections, the terminology developed up to Section 2.5 will be adopted.

A.1 Umbrella states

We introduce here the set $\mathbb{U}$ whose elements describe both residual quantities as well as age quantities, and hence, we use the symbol "$\mathbb{U}$" for "umbrella". For each route i, let $\mathbb{U}_{i,0} = \{0\}$ and for each integer $\ell \geq 1$, let

$$\mathbb{U}_{i,\ell} = \left\{ \sum_{k=1}^{\ell} \delta_{(c_{ik}, r_{ik})} : \max_{k=1}^{\ell}(c_{ik} + r_{ik}) < \alpha_i, 0 \leq c_{i1} < \ldots < c_{i\ell}, 0 < \min_{k=1}^{\ell} r_{ik} \right\},$$

and then, let

$$\mathbb{U} = \mathbb{U}_1 \times \ldots \times \mathbb{U}_{\mathbb{I}} \times (\mathbb{R}_+ \times (0, \infty))^{\mathbb{I}}, \tag{A.1.1}$$

where $\mathbb{U}_i = \cup_{\ell=0}^{\infty} \mathbb{U}_{i,\ell}$ for each route i.

We can use each element of $\mathbb{U}$ to describe the network at time zero. For instance, suppose that at time zero, for each route i, there are n_i flows on route i and the flow on route i with the k-th smallest completed work has completed work c_{ik}. Also, suppose that the last time that a flow has arrived to route i is a_i units of time ago for each route i. Then, when

$$x = \left(\sum_{k=1}^{n_1} \delta_{c_{1k}}, \ldots, \sum_{k=1}^{n_{\mathbb{I}}} \delta_{c_{\mathbb{I}k}}, a_1, \ldots, a_{\mathbb{I}}\right), \tag{A.1.2}$$

we can describe the network starting at x with the following random element of $\mathbb{U}$:

$$\left(\sum_{k=1}^{n_1} \delta_{(c_{1k}, \widetilde{v}_{1k}^x)}, \ldots, \sum_{k=1}^{n_{\mathbb{I}}} \delta_{(c_{\mathbb{I}k}, \widetilde{v}_{\mathbb{I}k}^x)}, (a_1, \widetilde{u}_1^x), \ldots, (a_{\mathbb{I}}, \widetilde{u}_{\mathbb{I}}^x)\right), \tag{A.1.3}$$

where any summation involving $\sum_{k=1}^{0}$ is taken to be the zero measure.

For each route i, the topology on $\mathbb{U}_i$ is the topology induced by the topology on $\mathbf{M}(\mathbb{R}_+ \times \mathbb{R}_+)$, i.e., the topology of weak convergence of non-negative finite Borel measures on $\mathbb{R}_+ \times \mathbb{R}_+$. Then, the set $\mathbb{U}$ is given the product topology. It can be shown that $\mathbb{U}$ is locally compact with countable base (cf. Appendix B.1), and in fact, the topology on $\mathbb{U}$ can be characterized by sequential convergence as described below.

Remark A.1.1. Consider a sequence $\{y_\ell\}_{\ell=1}^{\infty}$ in $\mathbb{U}$ and $y_0 \in \mathbb{U}$ such that

$$y_0 = \left(\sum_{k=1}^{n_1^0} \delta_{(c_{1k}^0, r_{1k}^0)}, \ldots, \sum_{k=1}^{n_{\mathbb{I}}^0} \delta_{(c_{\mathbb{I}k}^0, r_{\mathbb{I}k}^0)}, (a_1^0, r_1^0), \ldots, (a_{\mathbb{I}}^0, r_{\mathbb{I}}^0)\right), \tag{A.1.4}$$

and for each integer $\ell \geq 1$,

$$y_\ell = \left(\sum_{k=1}^{n_1^\ell} \delta_{(c_{1k}^\ell, r_{1k}^\ell)}, \ldots, \sum_{k=1}^{n_{\mathbb{I}}^\ell} \delta_{(c_{\mathbb{I}k}^\ell, r_{\mathbb{I}k}^\ell)}, (a_1^\ell, r_1^\ell), \ldots, (a_{\mathbb{I}}^\ell, r_{\mathbb{I}}^\ell)\right). \tag{A.1.5}$$

Here, we caution the reader that superscripts in (A.1.4) and (A.1.5) are only meant to be "labels" as opposed to being "exponents". Then, it can be shown that

$$y_0 = \lim_{\ell \to \infty} y_\ell$$

if and only if the following three conditions (i)-(iii) are satisfied:

(i) for each route i, there exists an integer $\ell_i \geq 1$ such that for each integer $\ell \geq \ell_i$,

$$n_i^0 = n_i^\ell, \tag{A.1.6}$$

(ii) for each route i such that $n_i^0 \geq 1$, if k is an integer such that $1 \leq k \leq n_i^0$, then

$$(c_{ik}^0, r_{ik}^0) = \lim_{\ell \to \infty} (c_{ik}^\ell, r_{ik}^\ell), \tag{A.1.7}$$

(iii) for each route i,

$$(a_i^0, r_i^0) = \lim_{\ell \to \infty} (a_i^\ell, r_i^\ell). \tag{A.1.8}$$

□

For each $y \in \mathbb{U}$, we let

$$\mathcal{I}_+(y) = \{i \in \{1, \dots, \mathbb{I}\} : n_i > 0\}, \tag{A.1.9}$$

where

$$y = \left(\sum_{k=1}^{n_1} \delta_{(c_{1k}, r_{1k})}, \dots, \sum_{k=1}^{n_{\mathbb{I}}} \delta_{(c_{\mathbb{I}k}, r_{\mathbb{I}k})}, (a_1, r_1), \dots, (a_{\mathbb{I}}, r_{\mathbb{I}})\right). \tag{A.1.10}$$

For future use in Appendix A.2, we define functions $\Delta : \mathbb{U} \to (0, \infty)$ and $\Psi : \mathbb{R}_+ \times \mathbb{U} \to \mathbb{U}$ by letting, for each $t \in \mathbb{R}_+$, $y \in \mathbb{U}$ and route i,

$$\Delta(y) = \begin{cases} (\min_{\iota=1}^{\mathbb{I}} r_\iota) \wedge \min_{\iota \in \mathcal{I}_+(y)} \left(\min_{k=1}^{n_\iota} \frac{r_{\iota k}}{\Lambda_\iota(n)/n_\iota}\right), & \text{if } \mathcal{I}_+(y) \neq \varnothing, \\ \min_{\iota=1}^{\mathbb{I}} r_\iota, & \text{if } \mathcal{I}_+(y) = \varnothing, \end{cases}$$

$$\Psi_i(t, y) = \begin{cases} \sum_{k=1}^{n_i} \delta_{(c_{ik} + \Lambda_i(n)t/n_i, r_{ik} - \Lambda_i(n)t/n_i)} 1_{(0, \alpha_i)}(r_{ik} - \Lambda_i(n)t/n_i), & \text{if } i \in \mathcal{I}_+(y), \\ 0, & \text{otherwise}, \end{cases}$$

$$\Psi_{\mathbb{I}+i}(t, y) = (a_i + t, r_i - t) 1_{(0, \infty)}(r_i - t).$$

where y is assumed to take the form in (A.1.10). Also, define a function $\psi : \mathbb{U} \to \mathbb{A}$ such that for each $y \in \mathbb{U}$,

$$\psi(y) = \left(\sum_{k=1}^{n_1} \delta_{c_{1k}}, \dots, \sum_{k=1}^{n_{\mathbb{I}}} \delta_{c_{\mathbb{I}k}}, a_1, \dots, a_{\mathbb{I}}\right), \tag{A.1.11}$$

where y takes the form in (A.1.10).

Lemma A.1.1. *The functions Δ, ψ and Ψ are Borel measurable.*

Proof. Our proof is broken down to four steps.

Step 1. For each integer $m \geq 1$, define a function $f_m : \mathbb{R} \to [0, 1]$ by letting, for each $s \in \mathbb{R}$,

$$f_m(s) = \Big(1 - \inf\{m|s - r| : r \in [1/m, \infty)\}\Big)^+,$$

and then, define a function $\Psi^m : \mathbb{R}_+ \times \mathbb{U} \to \left(\mathbf{M}(\mathbb{R}_+^2)\right)^{\mathbb{I}} \times \left(\mathbb{R}_+^2\right)^{\mathbb{I}}$ by letting, for each $(t, y) \in \mathbb{R}_+ \times \mathbb{U}$ and route i,

$$\Psi_i^m(t,y) = \begin{cases} \sum_{k=1}^{n_i} \delta_{(c_{ik}+\Lambda_i(n)t/n_i, r_{ik}-\Lambda_i(n)t/n_i)} f_m(r_{ik} - \Lambda_i(n)t/n_i), & \text{if } n_i > 0, \\ 0, & \text{otherwise}, \end{cases}$$

$$\Psi_{\mathbb{I}+i}^m(t,y) = (a_i + t, r_i - t) f_m(r_i - t),$$

where y is assumed to take the form in (A.1.10). Note that as $m \to \infty$, we have $\lim_{m\to\infty} f_m(s) = 1_{(0,\infty)}(s)$ for each $s \in \mathbb{R}$, and from this, it follows that for each $(t, y) \in \mathbb{R}_+ \times \mathbb{U}$, as $m \to \infty$, the sequence $\{\Psi^m(t, y)\}_{m=1}^\infty$ converges to $\Psi(t, y)$.

Step 2. Fix a sequence $\{y_\ell\}_{\ell=0}^\infty$ in $\mathbb{U}$ such that $\lim_{\ell\to\infty} y_\ell = y_0$. Assume that $y_0 \in \mathbb{U}$ takes the form in $(A.1.4)$ and also assume that for each integer $\ell \geq 1$, $y_\ell \in \mathbb{U}$ takes the form in $(A.1.5)$. Also, fix a sequence $\{t_\ell\}_{\ell=0}^\infty$ in $\mathbb{R}_+$ such that $t_0 = \lim_{\ell\to\infty} t_\ell$. Note that for all sufficiently large integers $\ell \geq 1$, we have $n^\ell = n^0$ so that by taking re-indexing if necessary, we assume that $n^\ell = n^0$ for all integer $\ell \geq 1$.

Step 3. For each integer $m \geq 1$, note that $f_m \in \mathcal{C}_b^+(\mathbb{R}_+)$. Then, for each route i, since $\lim_{\ell\to\infty}(a_i^\ell + t_\ell, r_i^\ell - t_\ell) = (a_i^0 + t_0, r_i^0 - t_0)$,

$$\lim_{\ell\to\infty} (a_i^\ell - t_\ell, r_i^\ell - t_\ell) f_m(r_i^\ell - t_\ell) = (a_i^0 - t_0, r_i^0 - t_0) f_m(r_i^0 - t_0),$$

and moreover, if $n_i^0 \geq 1$, then for each integer $m \geq 1$ and positive integer $k \leq n_i^0$,

$$\begin{aligned} & \lim_{\ell\to\infty} \delta_{\left(c_{ik}^\ell+\Lambda_i(n^\ell)t_\ell/n_i^\ell, r_{ik}^\ell-\Lambda_i(n^\ell)t_\ell/n_i^\ell\right)} f_m(r_{ik}^\ell - \Lambda_i(n^\ell)t_\ell/n_i^\ell) \\ = \; & \delta_{\left(c_{ik}^0+\Lambda_i(n^0)t_0/n_i^0, r_{ik}^0-\Lambda_i(n^0)t_0/n_i^0\right)} f_m(r_{ik}^0 - \Lambda_i(n^0)t_0/n_i^0). \end{aligned}$$

Therefore, it follows that for each integer $m \geq 1$,

$$\lim_{\ell\to\infty} \Psi^m(t_\ell, y_\ell) = \Psi(t_0, y_0). \tag{A.1.12}$$

Next, if $\mathcal{I}_+(y_0) = \varnothing$, then from (A.1.8), it follows that $\lim_{\ell\to\infty} \Delta(y_\ell) = \Delta(y_0)$. On the other hand, if $\mathcal{I}_+(y_0) \neq \varnothing$, then using (A.1.6)-(A.1.8), we also see that $\lim_{\ell\to\infty} \Delta(y_\ell) = \Delta(y_0)$. Similarly, we also see that $\lim_{\ell\to\infty} \psi(y_\ell) = \psi(y_0)$.

Step 4. Since the sequences $\{y_\ell\}_{\ell=1}^\infty$ and $\{t_\ell\}_{\ell=1}^\infty$ are chosen arbitrarily in *Step 2*, our computations in *Step 3* show that the functions Δ and ψ are continuous and for each integer $m \geq 1$, Ψ^m is a continuous function from $\mathbb{R}_+ \times \mathbb{U}$ to $\big(\mathbf{M}(\mathbb{R}_+^2)\big)^{\mathbb{I}} \times \big(\mathbb{R}_+^2\big)^{\mathbb{I}}$. Also, recall from *Step 1* that Ψ is the pointwise limit of the sequence $\{\Psi^m\}_{m=1}^\infty$. Therefore, it follows that Ψ is a Borel measurable function from $\mathbb{R}_+ \times \mathbb{U}$ to $\big(\mathbf{M}(\mathbb{R}_+^2)\big)^{\mathbb{I}} \times \big(\mathbb{R}_+^2\big)^{\mathbb{I}}$. Now, the function Ψ takes values only in the set $\mathbb{U}$, which is a Borel measurable subset of $\big(\mathbf{M}(\mathbb{R}_+^2)\big)^{\mathbb{I}} \times \big(\mathbb{R}_+^2\big)^{\mathbb{I}}$, and from this, it follows that Ψ is also a Borel measurable function from $\mathbb{R}_+ \times \mathbb{U}$ to $\mathbb{U}$. $\square$

A.2 Umbrella processes

We will define here a collection $\{\mathbf{U}^x(t) : x \in \mathbb{A}, t \in [0,\infty)\}$, where for each $t \in [0,\infty)$ and $x \in \mathbb{A}$, the random element $\mathbf{U}^x(t)$ describes the network at time t that starts at x. Our definition will be inductive. First, for each $x \in \mathbb{A}$, we let

$$\mathbf{U}^x(0) = \left(\textstyle\sum_{k=1}^{n_1} \delta_{(c_{1k}, \widetilde{v}_{1k}^x)}, \ldots, \sum_{k=1}^{n_{\mathbb{I}}} \delta_{(c_{\mathbb{I}k}, \widetilde{v}_{\mathbb{I}k}^x)}, (a_1, \widetilde{u}_1^x), \ldots, (a_{\mathbb{I}}, \widetilde{u}_{\mathbb{I}}^x)\right), \quad \text{(A.2.1)}$$

where x takes the form in (A.1.2), and set $\sigma_0^x = 0$. Now, let $x \in \mathbb{A}$, and to proceed inductively, suppose that for an integer $k \geq 0$, σ_k^x and $\mathbf{U}^x(\sigma_k^x)$ are well defined. Then, we let $\sigma_{k+1}^x = \sigma_k^x + \Delta(\mathbf{U}^x(\sigma_k^x))$ and then, for each $t \in (\sigma_k^x, \sigma_{k+1}^x]$, define

$$\mathbf{U}^x(t) = \Psi(t - \sigma_k^x, \mathbf{U}^x(\sigma_k^x)) + \mathbf{L}^x(t), \quad \text{(A.2.2)}$$

where each route i,

$$\mathbf{L}_i^x(t) = \textstyle\sum_{k=1}^\infty \delta_{(0,v_{ik})} 1_{\{t\}}(\tau_i^x(k)), \quad \text{(A.2.3)}$$

$$\mathbf{L}_{\mathbb{I}+i}^x(t) = \textstyle\sum_{k=1}^\infty (0, u_{ik}) 1_{\{t\}}(\tau_i^x(k)). \quad \text{(A.2.4)}$$

Next, define a function $\mathbf{L} : [0,\infty) \times \mathbb{A} \times \Omega \to \mathbb{U}$ by letting $\mathbf{L}(t,x,\omega) = \mathbf{L}^x(t,\omega)$ for each $(t,x,\omega) \in [0,\infty) \times \mathbb{A} \times \Omega$, where $\mathbf{L}^x(t,\omega)$ denotes the value of $\mathbf{L}^x(t)$ at ω. Similarly, define a function $\mathbf{U} : [0,\infty) \times \mathbb{A} \times \Omega \to \mathbb{U}$ by letting $\mathbf{U}(t,x,\omega) = \mathbf{U}^x(t,\omega)$

for each $(t, x, \omega) \in [0, \infty) \times \mathbb{A} \times \Omega$, where $\mathbf{U}^x(t, \omega)$ denotes the value of $\mathbf{U}^x(t)$ at ω. Lastly, define a function $\mathbf{A} : [0, \infty) \times \mathbb{A} \times \Omega \to \mathbb{A}$ by letting $\mathbf{A} = \psi \circ \mathbf{U}$.

The following observation is the main result of this section.

Lemma A.2.1. *The functions* $\mathbf{U} : [0, \infty) \times \mathbb{A} \times \Omega \to \mathbb{U}$ *and* $\mathbf{A} : [0, \infty) \times \mathbb{A} \times \Omega \to \mathbb{A}$ *are* $\mathcal{B}([0, \infty)) \times \mathcal{B}(\mathbb{A}) \times \mathcal{F}$*-measurable, and for each* $(t, x, \omega) \in [0, \infty) \times \mathbb{A} \times \Omega$, $\mathbf{A}^x(t, \omega) = \mathbf{A}(t, x, \omega)$.

Proof. Since the function $\mathbf{U}^x(\cdot, \omega)$ is right continuous for each $(x, \omega) \in \mathbb{A} \times \Omega$, for our proof for $\mathcal{B}([0, \infty)) \times \mathcal{B}(\mathbb{A}) \times \mathcal{F}$-measurability of $\mathbf{U}$, it suffices to show that for each $t \in [0, \infty)$, the function $\mathbf{U}(t, \cdot, \cdot)$ is $\mathcal{B}(\mathbb{A}) \times \mathcal{F}$-measurable, and this will follow essentially from the discrete event system dynamics of the network together with an observation that the random variables describing the initial flows depend "nicely" on initial states of the network.

For each route i, define $F_i : [0, \alpha_i) \times (0, 1) \to [0, \alpha_i)$ and $G_i : \mathbb{R}_+ \times (0, 1) \to \mathbb{R}_+$ by letting

$$F_i(\alpha, s) = \inf\{t \geq 0 : \vartheta_i^\alpha([0, t]) \geq s\}, \tag{A.2.5}$$

$$G_i(\beta, s) = \inf\{t \geq 0 : \varphi_i^\beta([0, t]) \geq s\}. \tag{A.2.6}$$

Next, for each route i and integer $k \geq 1$, define functions $H_{ik} : \mathbb{A} \times \Omega \to [0, \alpha_i) \times (0, 1)$ and $K_i : \mathbb{A} \times \Omega \to \mathbb{R}_+ \times (0, 1)$ by letting,

$$H_{ik}(x, \omega) = \begin{cases} (c_{ik}, \widetilde{V}_{ik}(\omega)) & \text{if } k \leq n_i, \\ (0, \widetilde{V}_{ik}(\omega)) & \text{otherwise,} \end{cases} \tag{A.2.7}$$

$$K_i(x, \omega) = (a_i, \widetilde{U}_i(\omega)), \tag{A.2.8}$$

where x is assumed to take the form in (A.1.11).

By Lemma B.3.1, for each route i, F_i is $\mathcal{B}([0, \alpha_i) \times (0, 1))$-measurable and G_i is $\mathcal{B}(\mathbb{R}_+ \times (0, 1))$-measurable; note that to use Lemma B.3.1, we are exploiting the fact that for each route i, by definition,

$$\vartheta_i^\alpha([0,t]) = \vartheta_i^\alpha((0,t]), \tag{A.2.9}$$

$$\varphi_i^\beta([0,t]) = \varphi_i^\beta((0,t]), \tag{A.2.10}$$

for each $\alpha \in [0, \alpha_i)$, $\beta \in \mathbb{R}_+$ and $t \in [0, \infty)$. Also, note that H_{ik} and K_i are $\mathcal{B}(\mathbb{A}) \times \mathcal{F}$-measurable. From this, we see that $\mathbf{U}(0,\cdot,\cdot) : \mathbb{A} \times \Omega \to \mathbb{U}$ is $\mathcal{B}(\mathbb{A}) \times \mathcal{F}$-measurable and $\mathbf{L} : [0,\infty) \times \mathbb{A} \times \Omega \to \mathbb{U}$ is $\mathcal{B}([0,\infty)) \times \mathcal{B}(\mathbb{A}) \times \mathcal{F}$-measurable by noting that for each $(x,\omega) \in \mathbb{A} \times \Omega$, route i and positive integer $k \le n_i$,

$$\widetilde{v}_{ik}^x(\omega) = F_i(H_{ik}(x,\omega)), \tag{A.2.11}$$

$$\widetilde{u}_i^x(\omega) = G_i(K_i(x,\omega)), \tag{A.2.12}$$

$$v_{ik}(\omega) = F_i(0, V_{ik}(\omega)), \tag{A.2.13}$$

$$u_i(\omega) = G_i(0, U_{ik}(\omega)). \tag{A.2.14}$$

Now, we fix $t \in (0, \infty)$. First, it is clear from our construction of $\{\mathbf{U}^x : x \in \mathbb{A}\}$ that for each $x \in \mathbb{A}$ and $\omega \in \Omega$,

$$\mathbf{U}^x(t,\omega) = \mathbf{L}^x(t,\omega) + \sum_{k=0}^{\infty} \Psi(t - \sigma_k^x(\omega), \mathbf{U}^x(\sigma_k^x(\omega),\omega)) 1_{(\sigma_k^x(\omega), \sigma_{k+1}^x(\omega)]}(t). \tag{A.2.15}$$

Hence, to prove that the function $\mathbf{U}(t,\cdot,\cdot)$ is $\mathcal{B}(\mathbb{A}) \times \mathcal{F}$-measurable, it is enough to show that each term of the infinite sum in (A.2.15) defines a measurable function from $\mathbb{A} \times \Omega$ to $\mathbb{U}$ when both x and ω are variables. In fact, it suffices to show that for each integer $k \ge 0$, when both x and ω are variables,

$$(\sigma_k^x(\omega), \mathbf{U}^x(\sigma_k^x(\omega),\omega)) \tag{A.2.16}$$

induces a measurable function from $\mathbb{A} \times \Omega$ to $[0,\infty) \times \mathbb{U}$, and we prove this by induction.

The initial step $k = 0$ for our induction is already done since $\sigma_0^x = 0$ for each $x \in \mathbb{A}$ and we have already proved $\mathcal{B}(\mathbb{A}) \times \mathcal{F}$-measurability of the function $\mathbf{U}(0,\cdot,\cdot) :$

$\mathbb{A} \times \Omega \to \mathbb{U}$. Next, for our induction step, suppose that our claim for (A.2.16) holds for an integer $k \geq 0$. Now, note that for each $(x, \omega) \in \mathbb{A} \times \Omega$,

$$\begin{aligned} \sigma_{k+1}^x(\omega) &= \sigma_k^x(\omega) + \Delta(\mathbf{U}^x(\sigma_k^x(\omega), \omega)) \\ \mathbf{U}^x(\sigma_{k+1}^x(\omega), \omega) &= \mathbf{L}^x(\sigma_{k+1}^x(\omega), \omega) + \Psi(\Delta(\mathbf{U}^x(\sigma_k^x(\omega), \omega)), \mathbf{U}^x(\sigma_k^x(\omega), \omega)). \end{aligned}$$

Since the functions Δ and Ψ are measurable by Lemma A.1.1, we then have that our claim in (A.2.16) holds true for $k+1$, and This completes our induction step.

Next, measurability of the function $\mathbf{A}$ follows from measurability of $\mathbf{U}$ because the function ψ is measurable. Lastly, the fact that $\mathbf{A}^x(t, \omega) = \mathbf{A}(t, x, \omega)$ for each $(t, x, \omega) \in [0, \infty) \times \mathbb{A} \times \Omega$ follows from the fact that $\mathbf{U}^x(t, \omega) = \mathbf{U}(t, x, \omega)$ and our construction of $\mathbf{U}^x$. □

A.3 Probability measures

Let $\Omega' = \mathbb{D}(\mathbb{U})$. For each $t \in [0, \infty)$, let $\mathbf{U}'_t$ be the mapping from Ω' to $\mathbb{U}$ given by $\mathbf{U}'_t(\omega) = \omega(t)$, where for each $\omega \in \Omega'$, $\omega(t)$ is the value of the function ω at t. Next, let $\mathcal{F}' = \sigma(\mathbf{U}'_t : t \in [0, \infty))$ and define $\widetilde{\psi} : \Omega' \to \Omega^\dagger$ by letting, for each $\omega \in \Omega'$, $\widetilde{\psi}(\omega)$ to be the element $\widetilde{\omega}$ of $\Omega^\dagger$ such that for each $t \in [0, \infty)$, $\widetilde{\omega}(t) = \psi(\omega(t))$.

By Lemma A.2.1, the function $\mathbf{U}$ is a measurable function from $[0, \infty) \times \mathbb{A} \times \Omega$ into $\mathbb{U}$, and for each $t \in [0, \infty)$, $x \in \mathbb{A}$ and $\omega \in \Omega$, $\mathbf{U}^x(t, \omega) = \mathbf{U}(t, x, \omega)$. Then, it follows that the function $\mathfrak{U} : \mathbb{A} \times \Omega \to \mathbb{D}(\mathbb{U})$ obtained by letting $\mathfrak{U}(x, \omega) = \mathbf{U}(\cdot, x, \omega)$, for each $(x, \omega) \in \mathbb{A} \times \Omega$, is Borel measurable; the measurability of $\mathfrak{U}$ follows from the measurability of $\mathbf{U}$ since $\mathfrak{U}(x, \omega)(t) = \mathbf{U}(t, x, \omega)$ and $\mathcal{B}(\mathbb{D}(\mathbb{R}_+))$ is generated by the coordinate projection mappings. In particular, for each $x \in \mathbb{A}$, $\mathbf{U}^x = \mathfrak{U}(x, \cdot)$ is a random element taking values in $\mathbb{D}(\mathbb{U})$. Therefore, for each $x \in \mathbb{A}$, we may define a probability measure $\mathbb{P}^x$ on $(\Omega', \mathcal{F}')$ by letting, for each $B \in \mathcal{F}'$,

$$\mathbb{P}^x[B] = \mathbf{P}[\mathbf{U}^x \in B]. \tag{A.3.1}$$

Also, recall the definition of $\Omega^\dagger$ and $\mathcal{F}^\dagger$ from Section 2.5. Then, for each probability measure μ on $(\mathbb{A}, \mathcal{B}(\mathbb{A}))$, we may define a probability measure $\mathbf{P}^\mu$ on $(\Omega^\dagger, \mathcal{F}^\dagger)$ by letting, for each $B \in \mathcal{F}^\dagger$,

$$\mathbf{P}^\mu[B] = \int_{\mathbb{A}} \mu(dx)\mathbb{P}^x[\widetilde{\psi}^{-1}(B)], \tag{A.3.2}$$

where the measurability in x is by Fubini's theorem.

For each $x \in \mathbb{A}$, we denote by $\mathbb{E}^x$, the expectation operator for $\mathbb{P}^x$, and for each probability measure μ on $(\mathbb{A}, \mathcal{B}(\mathbb{A}))$, we will denote by $\mathbf{E}^\mu$, the expectation operator for $\mathbf{P}^\mu$. For simplicity, for each $x \in \mathbb{A}$, we denote by $\mathbf{P}^x$ the probability measure $\mathbf{P}^{\delta_x}$, and denote by $\mathbf{E}^x$ the expectation operator $\mathbf{E}^{\delta_x}$.

Lemma A.3.1. *For each $B \in \mathcal{B}(\mathbb{A})$ and probability Borel measure μ on $\mathbb{A}$,*

$$\mu(B) = \mathbf{P}^\mu[\mathbf{A}^\dagger(0) \in B]. \tag{A.3.3}$$

Proof. Note that for each $x \in \mathbb{A}$, we have $\mathbf{A}^x = \widetilde{\psi}(\mathbf{U}^x)$. Therefore, we have

$$\begin{aligned}
\mathbf{P}^\mu[\mathbf{A}^\dagger(0) \in B] &= \mathbf{P}^\mu[\{\omega^\dagger \in \Omega^\dagger : \omega^\dagger(0) \in B\}] \\
&= \int_{\mathbb{A}} \mu(dx)\mathbb{P}^x[\{\omega' \in \Omega' : \widetilde{\psi}(\omega')(0) \in B\}] \\
&= \int_{\mathbb{A}} \mu(dx)\mathbf{P}[\mathbf{U}^x \in \{\omega' \in \Omega' : \widetilde{\psi}(\omega')(0) \in B\}] \\
&= \int_{\mathbb{A}} \mu(dx)\mathbf{P}[\widetilde{\psi}(\mathbf{U}^x)(0) \in B] \\
&= \int_{\mathbb{A}} \mu(dx)\mathbf{P}[\mathbf{A}^x(0) \in B] \\
&= \mu(B).
\end{aligned}$$

□

A.4 Semigroup property

In this section, we will prove that the collection $\mathcal{P}^\dagger = \{P_t : t \in [0, \infty)\}$ is a Borel Markov semigroup. To facilitate our discussion, we define a function $\Psi^\dagger : \mathbb{R}_+ \times \mathbb{A} \to$

$\mathbf{M}^{\mathbb{I}} \times \mathbb{R}_+^{\mathbb{I}}$ by letting, for each $t \in [0, \infty)$ and $x \in \mathbb{A}$,

$$\Psi^\dagger(t, x) = \left(\sum_{k=1}^{n_1} \delta_{c_{1k}+\Lambda_1(n)t/n_1}, \dots, \sum_{k=1}^{n_{\mathbb{I}}} \delta_{c_{ik}+\Lambda_i(n)t/n_{\mathbb{I}}}, a_1 + t, \dots, a_{\mathbb{I}} + t \right), \quad \text{(A.4.1)}$$

when x takes the form in (A.1.11) and any summation involving $\sum_{k=1}^{0}$ is taken to be the zero measure. Also, for each $x \in \mathbb{A}$ and integer $m \geq 0$, let

$$\mathcal{H}_m^x = \sigma(\mathbf{A}^x(\sigma_0^x), \mathbf{A}^x(\sigma_1^x), \dots, \mathbf{A}^x(\sigma_m^x)).$$

Lemma A.4.1. *Let $x \in \mathbb{A}$ and $t \in [0, \infty)$. For each $B \in \mathcal{F}'$, $\mathbf{P}$-almost surely,*

$$\mathbf{P}[\mathbf{U}^x(t + \cdot) \in B | \sigma_1^x > t] 1_{\{\sigma_1^x > t\}} = \mathbb{P}^{\mathbf{A}^x(t)}[B] 1_{\{\sigma_1^x > t\}}. \quad \text{(A.4.2)}$$

Proof. Assume that x takes the form in (A.1.11). Let $X_1, \dots, X_{m_0}$ be an enumeration of the following random variables:

$$\{\widetilde{u}_1^x, \dots, \widetilde{u}_{\mathbb{I}}^x\} \cup \left\{ \frac{\widetilde{v}_{1k}}{\Lambda_1(n)/n_1} \right\}_{k=1}^{n_1} \cup \dots \cup \left\{ \frac{\widetilde{v}_{\mathbb{I}k}}{\Lambda_{\mathbb{I}}(n)/n_{\mathbb{I}}} \right\}_{k=1}^{n_{\mathbb{I}}}, \quad \text{(A.4.3)}$$

and let $\mathfrak{S}^e$ be the collection of random variables listed in (2.3.1)-(2.3.2). Note that

$$\{\sigma_1^x > t\} = \bigcap_{\ell=1}^{m_0} \{X_\ell > t\}.$$

On the event $\{\sigma_1^x > t\}$, the random element $\mathbf{U}^x(t)$ is determined by $\mathbf{A}^x(t) = \Psi^\dagger(t, x)$ and $\{X_1 - t, \dots, X_{m_0} - t\}$. Then, for each $B_1, \dots, B_{m_0} \in \mathcal{B}(\mathbb{R}_+)$ and $B^e \in \sigma(\mathfrak{S}^e)$,

$$\begin{aligned}
& \mathbf{E}[\sigma_1^x > t, X_1 - t \in B_1, \dots, X_{m_0} - t \in B_{m_0}; B^e] \\
= \; & \mathbf{P}[B^e] \prod_{\ell=1,\dots,m_0} \mathbf{P}[X_\ell - t \in B_\ell, X_\ell > t] \\
= \; & \mathbf{P}[B^e] \prod_{\ell=1,\dots,m_0} \mathbf{E}\left[\mathbf{P}[X_\ell - t \in B_\ell | X_\ell > t] 1_{\{X_\ell > t\}}\right] \\
= \; & \mathbf{E}\Big[1_{\{\sigma_1^x > t\}} \mathbf{P}[B^e] \prod_{\ell=1,\dots,m_0} \mathbf{P}[X_\ell - t \in B_\ell | X_\ell > t]\Big].
\end{aligned}$$

Now, when X_ℓ corresponds to $\widetilde{v}^x_{ik}/(\Lambda_i(n)/n_i)$ for some route i and a positive integer $k \le n_i$, note that for each $r \in [0,\infty)$,

$$\begin{aligned}
\mathbf{P}[X_\ell - t > r/(\Lambda_i(n)/n_i)|X_\ell > t] &= \frac{\mathbf{P}[\widetilde{v}^x_{ik} - t\Lambda_i(n)/n_i > r]}{\mathbf{P}[\widetilde{v}^x_{ik} - t\Lambda_i(n)/n_i > 0]} \\
&= \frac{\vartheta_i((c_{ik} + r + t\Lambda_i(n)/n_i, \infty))/\vartheta_i((c_{ik}, \infty))}{\vartheta_i((c_{ik} + t\Lambda_i(n)/n_i, \infty))/\vartheta_i((c_{ik}, \infty))} \\
&= \vartheta_i^{c_{ik} + t\Lambda_i(n)/n_i}((r, \infty)).
\end{aligned}$$

Similarly, when X_ℓ corresponds to $\widetilde{u}^x_i$ for some route i,

$$\begin{aligned}
\mathbf{P}[X_\ell - t > r|X_\ell > t] &= \frac{\mathbf{P}[\widetilde{u}^x_i - t > r]}{\mathbf{P}[\widetilde{u}^x_i - t > 0]} \\
&= \frac{\varphi_i((a_i + r + t, \infty))/\varphi_i((a_i, \infty))}{\vartheta_i((a_i + t, \infty))/\vartheta_i((a_i, \infty))} \\
&= \varphi_i^{a_i + t}((r, \infty)).
\end{aligned}$$

Therefore, $\mathbf{P}$-almost surely, for each $f \in \mathcal{C}^+_b(\mathbb{U})$, we have that

$$\mathbf{E}\left[f(\mathbf{U}^x(t)); B^e \,|\sigma^x_1 > t\right] 1_{\{\sigma^x_1 > t\}} = \mathbf{E}[f(\mathbf{U}^{\mathbf{A}^x(t)}(0)); B^e] 1_{\{\sigma^x_1 > t\}}.$$

Moreover, on the event $\{\sigma^x_1 > t\}$, the random element $\mathbf{U}^x(t+\cdot)$ taking values in $\mathbb{D}(\mathbb{U})$ is determined by $\mathbf{U}^x(t)$ and the random variables in $\mathfrak{G}^e$. So, we have the desired result. $\square$

The next lemma will be used to show that the age process has, roughly speaking, the strong Markov property at the first jump time σ^x_1.

Lemma A.4.2. *Let $x \in \mathbb{A}$ and $B \in \mathcal{F}'$. Then, $\mathbf{P}$-almost surely,*

$$\mathbf{P}[\mathbf{U}^x(\sigma^x_1 + \cdot) \in B|\mathbf{A}^x(\sigma^x_1)] = \mathbb{P}^{\mathbf{A}^x(\sigma^x_1)}[\mathbf{U}' \in B]. \tag{A.4.4}$$

Proof. Assume that x takes the form in (A.1.11).

Step 1. We partition the outcome space Ω into smaller pieces, each of which corresponds to a reason why there was a jump in the number of flows in the network at time σ^x_1.

Let Ω^e be the event that at time σ_1^x, there is no departure of initial flows and an exogenous flow has arrived to some route. Next, let Ω^o be the event that at time σ_1^x, there is no arrival of an exogenous flow and an initial flow has departed from some route. Then, $\Omega^e \cap \Omega^o = \varnothing$, and since ϑ_i has no atoms for each route i, $\mathbf{P}[\sigma_1^x < \infty] = 1$ and the random variables in (2.3.1)-(2.3.4) are mutually independent, we have $\mathbf{P}$-almost surely, $1 = 1_{\Omega^e} + 1_{\Omega^o}$.

Then, for each route i and $k \in \{1, \ldots, n_i\}$, let Ω_{ik}^o be the event that at time σ_1^x, there is no arrival of any exogenous flow and the k-th initial flow on route i is the only flow that has departed from the network. Then, the sets in $\cup_{i=1}^{\mathbb{I}}\{\Omega_{ik} : k = 1, \ldots, n_i\}$ are mutually disjoint, and again, since ϑ_i has no atoms for each route i, $\mathbf{P}[\sigma_1^x < \infty] = 1$ and the random variables in (2.3.1)-(2.3.4) are mutually independent, $\mathbf{P}$-almost surely, $1_{\Omega^o} = \sum_{i=1}^{\mathbb{I}} \sum_{k=1}^{n_i} 1_{\Omega_{ik}^o}$.

Next, we partition Ω^e into $(2^{\mathbb{I}} - 1)$ disjoint events according to patterns in which exogenous flows can arrive at time σ_1^x. In other words, for each $\mathbf{k} = (k_1, \ldots, k_{\mathbb{I}}) \in \{0,1\}^{\mathbb{I}}$ such that $\mathbf{k} \neq (0, \ldots, 0)$, let $\Omega_{\mathbf{k}}^e$ be the event that at time σ_1^x, there is no departure of initial flows and the routes with an exogenous arrival are exactly the routes whose corresponding coordinate of $\mathbf{k}$ is 1. Let $\{\mathbf{k}_\ell\}_{\ell=1}^{2^{\mathbb{I}}-1}$ be an enumeration of elements of $\{0,1\}^{\mathbb{I}} \setminus \{(0, \ldots, 0)\}$, and note that $\mathbf{P}$-almost surely, $1_{\Omega^e} = \sum_{\ell=1}^{2^{\mathbb{I}}-1} 1_{\Omega_{\mathbf{k}_\ell}^e}$.

Let $m^* = 2^{\mathbb{I}} - 1 + \sum_{i=1}^{\mathbb{I}} n_i$ and then, let $\Omega_1, \ldots, \Omega_{m^*}$ be an enumeration of the sets

$$\{\Omega_{\mathbf{k}_1}^e, \ldots, \Omega_{\mathbf{k}_{2^{\mathbb{I}}-1}}^e, \Omega_{11}^o, \ldots, \Omega_{1n_1}^o, \Omega_{1n_{\mathbb{I}}}^o, \ldots, \Omega_{\mathbb{I}n_{\mathbb{I}}}^o\}.$$

Step 2. We claim that $\Omega_\ell \in \sigma(\mathbf{A}^x(\sigma_0^x), \mathbf{A}^x(\sigma_1^x))$ for each $\ell \in \{1, \ldots, m^*\}$. For arrivals, note that the completed work of a flow is zero if and only if it just arrived; in other words, for route i, there is an exogenous flow arrival to route i at time σ_1^x if and only if $\mathcal{A}_i^x(\sigma_1^x)$ has a point mass at 0. Next, to see how a departure of a flow can be described by events in $\sigma(\mathbf{A}^x(\sigma_0^x), \mathbf{A}^x(\sigma_1^x))$, recall our discussion regarding the strict inequalities in "$0 \leq c_{i1} < c_{i2} < \ldots < \alpha_i$" for $x \in \mathbb{A}$ taking the form in (2.1.9) in Section 2.1. In particular, recall that for each route i, there can be either one or no exogenous flow arrival at σ_1^x, and if $n_i \geq 1$, then as time progresses, the point masses located initially at $\{c_{ik}\}_{k=1}^{n_i}$ move at the same rate $\Lambda_i(n)/n_i$ during the time interval $[0, \sigma_1^x)$. Then, for route i such that $n_i \geq 1$, there is no departure from route i if and only if $\mathcal{A}_i^x(\sigma_1^x)$ is one of the following:

$$\delta_0 + \sum_{k=1}^{n_i} \delta_{c_{ik}+\Lambda_i(n)/n_i} \text{ or } \sum_{k=1}^{n_i} \delta_{c_{ik}+\Lambda_i(n)/n_i}. \tag{A.4.5}$$

Step 3. Let $X_1, \ldots, X_{m_0}$ be an enumeration of the following random variables:

$$\mathfrak{S}^o = \{\widetilde{u}_1^x, \ldots, \widetilde{u}_{\mathbb{I}}^x\} \cup \left\{ \frac{\widetilde{v}_{1k}}{\Lambda_1(n)/n_1} \right\}_{k=1}^{n_1} \cup \ldots \cup \left\{ \frac{\widetilde{v}_{\mathbb{I}k}}{\Lambda_{\mathbb{I}}(n)/n_{\mathbb{I}}} \right\}_{k=1}^{n_{\mathbb{I}}}. \tag{A.4.6}$$

Let $\mathfrak{S}^e$ be the collection of random variables listed in (2.3.1) and (2.3.2).

First, consider the event $\widetilde{\Omega}$ that at time σ_1^x, route 1 is the only route with an arrival of an exogenous flow and this occurs prior to the departure of any initial flow. Then,

$$\widetilde{\Omega} = \left\{ \widetilde{u}_1^x < \min_{i=2}^{\mathbb{I}} \widetilde{u}_i^x \wedge \min_{i=1}^{\mathbb{I}} \min_{k=1}^{n_i} \frac{\widetilde{v}_{ik}^x}{\Lambda_i(n)/n_i} \right\} \subset \{\widetilde{u}_1^x = \sigma_1^x\},$$

where as a convention, if $n_i = 0$, then we take $\min_{k=1}^{n_i} \widetilde{v}_{ik}^x/\Lambda_i(n)/n_i = \infty$. More generally, for each $\ell \in \{1, \ldots, m^*\}$, there exist $X_\ell^1, \ldots, X_\ell^{k_\ell} \in \mathfrak{S}^o$ such that

$$\{X_\ell^1 = \ldots = X_\ell^{k_\ell} < T_\ell\} = \Omega_\ell \subset \{\eta_\ell = \sigma_1^x\}, \tag{A.4.7}$$

where $\eta_\ell = \max_{k=1}^{k_\ell} X_\ell^k$, and $T_\ell = \min\{X : X \in \mathfrak{S}^o \setminus \{X_\ell^1, \ldots, X_\ell^{k_\ell}\}\}$, and moreover, the random element $\mathbf{U}^x(\eta_\ell + \cdot)1_{\Omega_\ell}$ is determined by $\eta_\ell, 1_{\Omega_\ell}$, the collection of random elements $\mathfrak{S}^e$ and the collection of random elements $\mathfrak{S}^o \setminus \{X_\ell^1, \ldots, X_\ell^{k_\ell}\}$. For instance, on the event $\widetilde{\Omega}$, for each route i,

$$\mathcal{A}_i^x(\sigma_1^x) = \begin{cases} \delta_0 + \sum_{k=1}^{n_i} \delta_{c_{ik}+\sigma_1^x \Lambda_i(n)/n_i}, & \text{if } i = 1, \\ \sum_{k=1}^{n_i} \delta_{c_{ik}+\sigma_1^x \Lambda_i(n)/n_i}, & \text{if } i \neq 1, \end{cases}$$

$$A_i^x(\sigma_1^x) = \begin{cases} 0, & \text{if } i = 1 \\ a_i + \sigma_1^x, & \text{if } i \neq 1, \end{cases}$$

where any summation involving $\sum_{k=1}^{0}$ is the zero measure. Therefore, for each $B \in \mathcal{F}'$ and $f \in \mathcal{C}_b^+(\mathbb{A})$,

$$
\begin{aligned}
& \mathbf{E}\left[1_B(\mathbf{U}^x(\sigma_1^x + \cdot))f(\mathbf{A}^x(\sigma_1^x))\right] && \text{(A.4.8)} \\
= & \sum_{\ell=1}^{m^*} \mathbf{E}\left[1_B(\mathbf{U}^x(\sigma_1^x + \cdot))f(\mathbf{A}^x(\sigma_1^x))1_{\Omega_\ell}\right] && \text{(A.4.9)} \\
= & \sum_{\ell=1}^{m^*} \mathbf{E}\left[\mathbf{E}\left[1_B(\mathbf{U}^x(\eta_\ell + \cdot))\,|\eta_\ell, 1_{\Omega_\ell}\right] f(\mathbf{A}^x(\eta_\ell))1_{\Omega_\ell}\right] && \text{(A.4.10)} \\
= & \sum_{\ell=1}^{m^*} \mathbf{E}\left[\mathbb{P}^{\mathbf{A}^x(\eta_\ell)}\left[\mathbf{U}' \in B\right] f(\mathbf{A}^x(\eta_\ell))1_{\Omega_\ell}\right] && \text{(A.4.11)} \\
= & \sum_{\ell=1}^{m^*} \mathbf{E}\left[\mathbb{P}^{\mathbf{A}^x(\sigma_1^x)}\left[\mathbf{U}' \in B\right] f(\mathbf{A}^x(\sigma_1^x))1_{\Omega_\ell}\right] && \text{(A.4.12)} \\
= & \mathbf{E}\left[\mathbb{P}^{\mathbf{A}^x(\sigma_1^x)}\left[\mathbf{U}' \in B\right] f(\mathbf{A}^x(\sigma_1^x))\right], && \text{(A.4.13)}
\end{aligned}
$$

where the equality in (A.4.10) is by (A.4.7) and the equality in (A.4.11) can be seen arguing as in our proof of Lemma A.4.1 using Lemma B.4.1 and Lemma B.5.1. $\square$

Corollary A.4.1. *For each $x \in \mathbb{A}$, $B \in \mathcal{F}'$ and integer $m \geq 1$, $\mathbf{P}$-almost surely,*

$$\mathbf{P}[\mathbf{U}^x(\sigma_m^x + \cdot) \in B|\mathcal{H}_m^x] = \mathbb{P}^{\mathbf{A}^x(\sigma_m^x)}[\mathbf{U}' \in B]. \tag{A.4.14}$$

Proof. Our proof is by induction. The case $m = 1$ follows immediately from Lemma A.4.2. Next, for our induction step, suppose that for an integer $m \geq 1$, the equality (A.4.14) holds true for each $x \in \mathbb{A}$ and $B \in \mathcal{F}'$. Now, fix $x \in \mathbb{A}$ and $B \in \mathcal{F}'$. Consider $B_m \in \mathcal{H}_m^x$ and $B_m^+ \in \mathcal{B}(\mathbb{A})$. To facilitate our discussion, we let $\mathbf{A}'(t) = \psi(\mathbf{U}'(t))$ for each $t \in [0,\infty)$ and let $\sigma' = \inf\{t > 0 : \mathbf{A}'(t-) \neq \mathbf{A}'(t)\}$. We observe that

$$\begin{aligned}
& \mathbf{E}[1_B(\mathbf{U}^x(\sigma^x_{m+1} + \cdot))1_{B^+_m}(\mathbf{A}^x(\sigma^x_{m+1})); B_m] \\
= \; & \mathbf{E}[\mathbf{E}[1_B(\mathbf{U}^x(\sigma^x_m + (\sigma^x_{m+1} - \sigma^x_m) + \cdot))1_{B^+_m}(\mathbf{A}^x(\sigma^x_m + (\sigma^x_{m+1} - \sigma^x_m)))|\mathcal{H}^x_m]1_{B_m}] \\
= \; & \mathbf{E}[\mathbb{P}^{\mathbf{A}^x(\sigma^x_m)}[1_B(\mathbf{U}'(\sigma' + \cdot))1_{B^+_m}(\mathbf{A}'(\sigma'))]1_{B_m}] \\
= \; & \mathbf{E}[\mathbb{P}^{\mathbf{A}^x(\sigma^x_m)}[\mathbb{P}^{\mathbf{A}'(\sigma')}[\mathbf{U}' \in B]1_{B^+_m}(\mathbf{A}'(\sigma'))]1_{B_m}] \\
= \; & \mathbf{E}[\mathbf{E}[\mathbb{P}^{\mathbf{A}^x(\sigma^x_m + (\sigma^x_{m+1} - \sigma^x_m))}[\mathbf{U}' \in B]1_{B^+_m}(\mathbf{A}^x(\sigma^x_m + (\sigma^x_{m+1} - \sigma^x_m)))|\mathcal{H}^x_m]1_{B_m}] \\
= \; & \mathbf{E}[\mathbb{P}^{\mathbf{A}^x(\sigma^x_{m+1})}[\mathbf{U}' \in B]1_{B^+_m}(\mathbf{A}^x(\sigma^x_{m+1})); B_n],
\end{aligned}$$

where we have used the induction hypothesis for the second and fourth equality, and Lemma A.4.2 for the third equality.

Also, we see that $\mathcal{H}^x_{m+1} = \mathcal{H}^x_m \vee \sigma(\mathbf{A}^x(\sigma^x_{m+1}))$. Therefore, for each $B_{m+1} \in \mathcal{H}^x_{m+1}$, we have

$$\mathbf{E}[1_B(\mathbf{U}^x(\sigma^x_{m+1} + \cdot)); B_{m+1}] = \mathbf{E}[\mathbb{P}^{\mathbf{A}^x(\sigma^x_{m+1})}[\mathbf{U}' \in B]; B_{m+1}].$$

This completes our induction step. $\square$

The following is the main result of this subsection and it figures in our proof of the claim that $\mathcal{P}^\dagger$ satisfies the second requirement (1.3.3) for a Borel right semigroup in Definition 1.3.1.

Lemma A.4.3. *Let $x \in \mathbb{A}$, $B \in \mathcal{B}(\mathbb{A})$ and $s, t \in [0,\infty)$. Then, $\mathbf{P}^x$-almost surely,*

$$\mathbf{P}^x[\mathbf{A}^\dagger(t+s) \in B|\mathcal{F}^\dagger_t] = P_s(\mathbf{A}^\dagger(t), B). \tag{A.4.15}$$

Proof. Define $\sigma^\dagger = \inf\{t > 0 : \mathbf{A}^\dagger(t-) \neq \mathbf{A}^\dagger(t)\}$. Then, fix $x \in \mathbb{A}$, $t \in [0,\infty)$ and $\mathcal{F}^x_t$-measurable bounded non-negative random variable H, where $\mathcal{F}^x_t = \sigma(\mathbf{A}^x(s) : s \in [0,t])$. First, note that $\mathbf{P}$-almost surely,

$$1 = \sum_{m=0}^{\infty} 1_{[\sigma^x_m, \sigma^x_{m+1})}(t).$$

Next, note that for each integer $m \geq 0$, since $1_{[\sigma_m^x, \sigma_{m+1}^x)}(t)$ and H are $\mathcal{F}_t^x$-measurable, there exists an $\mathcal{H}_m^x$-measurable bounded non-negative random variable H_m such that

$$1_{[\sigma_m^x, \sigma_{m+1}^x)}(t) H_m = 1_{[\sigma_m^x, \sigma_{m+1}^x)}(t) H;$$

this is because each random element in $\{\mathbf{A}^x(s) : s \leq t\}$ is determined by the random elements $\mathbf{A}^x(\sigma_0^x) = x, \mathbf{A}^x(\sigma_1^x), \ldots, \mathbf{A}^x(\sigma_m^x)$ when $\sigma_m^x \leq t < \sigma_{m+1}^x$. Therefore, we see that for each $s \in [0, \infty)$ and $B \in \mathcal{B}(\mathbb{A})$,

$$\begin{aligned}
& \mathbf{E}[1_B(\mathbf{A}^x(s+t))H] \\
= & \sum_{m=0}^{\infty} \mathbf{E}[1_B(\mathbf{A}^x(s+t)) 1_{[\sigma_m^x, \sigma_{m+1}^x)}(t) H] \\
= & \sum_{m=0}^{\infty} \mathbf{E}[\mathbf{E}[1_B(\mathbf{A}^x(s+t)) 1_{[\sigma_m^x, \sigma_{m+1}^x)}(t) H_m \,|\mathcal{H}_m^x]] \\
= & \sum_{m=0}^{\infty} \mathbf{E}[\mathbf{E}[1_B(\mathbf{A}^x(s + (t - \sigma_m^x) + \sigma_m^x)) 1_{[0, \sigma_{m+1}^x - \sigma_m^x)}(t - \sigma_m^x) \,|\mathcal{H}_m^x] H_m] \\
= & \sum_{m=0}^{\infty} \mathbf{E}[\mathbf{E}^{\mathbf{A}^x(\sigma_m^x)}[1_B(\mathbf{A}^\dagger(s + (t - \sigma_m^x))) 1_{[0, \sigma^\dagger)}(t - \sigma_m^x)] H_m] \\
= & \sum_{m=0}^{\infty} \mathbf{E}[\mathbf{E}^{\mathbf{A}^x(\sigma_m^x)}[P_s(\mathbf{A}^\dagger(t - \sigma_m^x), B) 1_{[0, \sigma^\dagger)}(t - \sigma_m^x)] H_m] \\
= & \sum_{m=0}^{\infty} \mathbf{E}[\mathbf{E}[P_s(\mathbf{A}^x(t - \sigma_m^x + \sigma_m^x), B) 1_{[0, \sigma_{m+1}^x - \sigma_m^x)}(t - \sigma_m^x) \,|\mathcal{H}_m^x] H_m] \\
= & \sum_{m=0}^{\infty} \mathbf{E}[\mathbf{E}[P_s(\mathbf{A}^x(t), B) 1_{[\sigma_m^x, \sigma_{m+1}^x)}(t) H \,|\mathcal{H}_m^x]] \\
= & \mathbf{E}[P_s(\mathbf{A}^x(t), B) H],
\end{aligned}$$

where the fourth equality is by Corollary A.4.1 together with Lemma A.4.1. □

A.5 Strong Markov property

The main result in this subsection is Lemma A.5.2 and it contributes to showing that the third requirement (1.3.4) for a Borel right semigroup in Definition 1.3.1 is satisfied by $\mathcal{P}^\dagger$. Our computations here take hints from the discussion of *backward recurrence processes for renewal processes* in Asmussen [1], and the discussion of *piecewise deterministic Markov processes* in Davis [9].

For our next lemma, recall that by Lemma A.2.1, for each $B \in \mathcal{B}(\mathbb{A})$,

$$P_t(x, B) = \mathbf{P}[\mathbf{A}^x(t) \in B] = \int_{\mathbb{A}} 1_B(\mathbf{A}(t, x, \omega))\mathbf{P}[d\omega],$$

for each $t \in [0, \infty)$ and $x \in \mathbb{A}$, and since the function $\mathbf{A}$ is *jointly* measurable, by Fubini's theorem, the function

$$\int_{\mathbb{A}} 1_B(\mathbf{A}(\cdot, \cdot, \omega))\mathbf{P}[d\omega]$$

is a measurable function from $[0, \infty) \times \mathbb{A}$ into $[0, 1]$. For each $f \in \mathcal{C}_b^+(\mathbb{A})$ and $t \in [0, \infty)$, recall that we denote by $P_t f$ the function from $\mathbb{A}$ into $\mathbb{R}_+$ such that for each $x \in \mathbb{A}$, $(P_t f)(x) = \mathbf{E}[f(\mathbf{A}^x(t))]$.

Lemma A.5.1. *Let $x \in \mathbb{A}$, $f \in \mathcal{C}_b^+(\mathbb{A})$ and $\alpha \in (0, \infty)$. For each $t \in [0, \infty)$, let $X(t)$ be the random variable defined on $(\Omega^\dagger, \mathcal{F}^\dagger)$ such that*

$$X(t) = \int_0^\infty (P_s f)(\mathbf{A}^\dagger(t))e^{-\alpha s} ds. \tag{A.5.1}$$

Then, $\mathbf{P}^x$-almost surely, for each $t \in [0, \infty)$,

$$\lim_{r \downarrow 0} X(t + r) = X(t). \tag{A.5.2}$$

Proof. Let $\sigma^\dagger = \inf\{t > 0 : \mathbf{A}^\dagger(t-) \neq \mathbf{A}^\dagger(t)\}$. Fix $f \in \mathcal{C}_b^+(\mathbb{A})$. Then, for each $s \in [0, \infty)$, define a function $H_s : \mathbb{A} \to \mathbb{R}_+$ by letting $H_s(x) = \mathbf{E}^x[f(\mathbf{A}^\dagger(s))]$ for each $x \in \mathbb{A}$.

Step 1. We claim that for each $x \in \mathbb{A}$ and $s \in [0, \infty)$,

$$H_s(x) = \lim_{r \downarrow 0} H_s(\Psi^\dagger(r, x)).$$

To see this, let $x \in \mathbb{A}$ and $s \in [0,\infty)$. Note that $\mathbf{P}^x[\sigma^\dagger > 0] = 1$ and by right continuity of $\mathbf{A}^\dagger$, we have $H_s(x) = \lim_{r\downarrow 0} \mathbf{E}^x[f(\mathbf{A}^\dagger(s+r))]$. Then, for each $\varepsilon \in (0,1)$, there exists $r^x_\varepsilon \in (0,\infty)$ such that if $r \in [0, r^x_\varepsilon)$, then $\Psi^\dagger(r,x) \in \mathbb{A}$ and

$$|H_s(x) - \mathbf{E}^x[\mathbf{A}^\dagger(s+r)]| \vee \left(\|f\|_\infty \mathbf{P}^x\left[\sigma^\dagger \le r\right]\right) \le \varepsilon.$$

Now, for each $\varepsilon \in (0,1)$, if $r \in [0, r^x_\varepsilon)$, then

$$\begin{aligned}
& |H_s(x) - H_s(\Psi^\dagger(r,x))| \\
\le\ & |H_s(x) - \mathbf{E}^x[f(\mathbf{A}^\dagger(s+r))]| + |\mathbf{E}^x[f(\mathbf{A}^\dagger(s+r))] - \mathbf{E}^{\Psi^\dagger(r,x)}[f(\mathbf{A}^\dagger(s))]| \\
\le\ & \varepsilon + \mathbf{E}^x[\mathbf{E}^x[f(\mathbf{A}^\dagger(s+r))|\mathcal{F}^\dagger_r]; \sigma^\dagger \le r] \\
& \quad + \left|\mathbf{E}^x[\mathbf{E}^x[f(\mathbf{A}^\dagger(s+r))|\mathcal{F}^\dagger_r]; \sigma^\dagger > r] - \mathbf{E}^{\Psi^\dagger(r,x)}[f(\mathbf{A}^\dagger(s))]\right| \\
\le\ & \varepsilon + \|f\|_\infty \mathbf{P}^x[\sigma^\dagger \le r] \\
& \quad + \left|\mathbf{E}^x[\mathbf{E}^{\Psi^\dagger(r,x)}[f(\mathbf{A}^\dagger(s))]; \sigma^\dagger > r] - \mathbf{E}^x[\mathbf{E}^{\Psi^\dagger(r,x)}[f(\mathbf{A}^\dagger(s))]; \sigma^\dagger > r]\right| \\
& \qquad + \mathbf{E}^x[\mathbf{E}^{\Psi^\dagger(r,x)}[f(\mathbf{A}^\dagger(s))]; \sigma^\dagger \le r] \\
\le\ & 2\varepsilon + 0 + \|f\|_\infty \mathbf{P}^x[\sigma^\dagger \le r] \le 3\varepsilon,
\end{aligned}$$

where the third inequality uses Lemma A.4.3.

Step 2. Let $x \in \mathbb{A}$. Then, for each $t \in [0,\infty)$, let

$$\tau(t) = \inf\{s \in (0,\infty) : \mathbf{A}^\dagger(t+s) \ne \mathbf{A}^\dagger((s+t)-)\}.$$

Now, if $r \in [0, \tau(t))$, then $\mathbf{A}^\dagger(t+r) = \Psi^\dagger(r, \mathbf{A}^\dagger(t))$. Also, note that $\mathbf{P}^x$-almost surely, we have $\tau(t) > 0$ for each $t \in [0,\infty)$; this is because $\mathbf{P}$-almost surely, we have $1 = \sum_{k=0}^\infty 1_{[\sigma^x_k, \sigma^x_{k+1})}(t)$ and $\sigma^x_k < \sigma^x_{k+1}$ for each integer $k \ge 0$. Therefore, $\mathbf{P}^x$-almost surely,

$$\lim_{r\downarrow 0} H_s(\mathbf{A}^\dagger(t+r)) = \lim_{r\downarrow 0} H_s(\Psi^\dagger(r, \mathbf{A}^\dagger(t))) = H_s(\mathbf{A}^\dagger(t)).$$

Then, since $\|H_s\|_\infty \le \|f\|_\infty$ for each $s \in [0,\infty)$, by the dominated convergence theorem, we have that for each $x \in \mathbb{A}$, $\mathbf{P}^x$-almost surely, for each $t \in [0, \infty)$,

$$\begin{aligned} \lim_{r \downarrow 0} X(t+r) &= \lim_{r \downarrow 0} \int_0^\infty e^{-\alpha s} H_s(\mathbf{A}^\dagger(t+r)) ds \\ &= \int_0^\infty \lim_{r \downarrow 0} e^{-\alpha s} H_s(\mathbf{A}^\dagger(t+r)) ds \\ &= \int_0^\infty e^{-\alpha s} H_s(\mathbf{A}^\dagger(t)) ds = X(t). \end{aligned}$$

□

For our next result, recall that a non-negative random variable τ on $(\Omega^\dagger, \mathcal{F}^\dagger)$ is a *stopping time* for $\{\mathcal{F}^\dagger_{t+} : t \in [0, \infty)\}$ if $\{\tau \le t\} \in \mathcal{F}^\dagger_{t+}$ for each $t \in [0, \infty)$, and recall also that a stopping time may take on the value ∞.

Lemma A.5.2. *Let τ be a stopping time for $\{\mathcal{F}^\dagger_{t+} : t \in [0, \infty)\}$. Then, for each $(x, B) \in \mathbb{A} \times \mathcal{B}(\mathbb{A})$,*

$$\mathbf{P}^x[\mathbf{A}^\dagger(\tau + t) \in B, \tau < \infty | \mathcal{F}^\dagger_{\tau+}] = P_t(\mathbf{A}^\dagger(\tau), B) \mathbf{1}\{\tau < \infty\}, \tag{A.5.3}$$

where

$$\mathcal{F}^\dagger_{\tau+} = \{B \in \mathcal{F}^\dagger : \{\tau \le t\} \cap B \in \mathcal{F}_{t+}, \textit{ for each } t \in [0, \infty)\}.$$

Proof. For each $t \in [0, \infty)$, let $\theta^\dagger_t$ be the function from $\Omega^\dagger$ to $\Omega^\dagger$ such that for each $\omega \in \Omega^\dagger$, $\theta^\dagger_t(\omega)$ is the element $\widetilde{\omega}$ of $\Omega^\dagger$ such that $\widetilde{\omega}(s) = \omega(t+s)$ for each $s \in [0, \infty)$. Then, let $\theta^\dagger = \{\theta^\dagger_t : t \in [0, \infty)\}$. Also, let $\mathcal{M} = \{\mathbf{P}^x : x \in \mathbb{A}\}$. Then, in the terminology of Blumenthal and Getoor [3], *Axiom R* (Regularity Condition) is satisfied by Lemma A.2.1, *Axiom H* (Homogeneity) is satisfied by definition of $\mathbf{A}^\dagger$ and $\theta^\dagger$, and *Axiom M* (Markov Property) is satisfied by Lemma A.4.3, whence the collection $(\Omega^\dagger, \mathcal{F}^\dagger, \mathfrak{F}^\dagger, \mathbf{A}^\dagger, \theta^\dagger, \mathcal{M})$ is a right continuous Markov process with state space $(\mathbb{A}, \mathcal{B}(\mathbb{A}))$. Also, since $\mathbb{A}$ is locally compact with countable base, for each open subset B of $\mathbb{A}$, there exists a sequence $\{f_\ell\}_{\ell=1}^\infty$ in $\mathcal{C}^+_b(\mathbb{A})$ such that for each $x \in \mathbb{A}$, $f_\ell(x) \le f_{\ell+1}(x)$ for each integer $\ell \ge 1$ and $\lim_{\ell \to \infty} f_\ell(x) = 1_B(x)$.

Then, by Lemma A.5.1, (A.5.3) follows from Theorem I.8.11 of Blumenthal and Getoor [3]. □

A.6 Proof of Theorem 2.5.1

For each $t \in [0, \infty)$, by Lemma A.2.1 together with our definition of P_t, P_t is a Borel Markov kernel. Also, by Lemma A.4.3, $\mathcal{P}^\dagger$ is a Borel Markov semigroup. Now, to show that the Borel Markov semigroup $\mathcal{P}^\dagger$ is *right*, we fix a probability measure μ on $(\mathbb{A}, \mathcal{B}(\mathbb{A}))$. First, as we have observed in Lemma A.3.1, for each $B \in \mathcal{B}(\mathbb{A})$, we have $\mathbf{P}^\mu[\mathbf{A}^\dagger(0) \in B] = \mu(B)$. Next, for each $B \in \mathcal{B}(\mathbb{A})$, s and $t \in [0, \infty)$, we have

$$\mathbf{P}^\mu[\mathbf{A}^\dagger(t+s) \in B | \mathcal{F}_t^\dagger] = P_s(\mathbf{A}^\dagger(t), B)$$

since by Lemma A.4.3, for each $\mathcal{F}_t^\dagger$-measurable bounded random variable Z,

$$\begin{aligned}
\mathbf{E}^\mu[1_B(\mathbf{A}^\dagger(t+s))Z] &= \int_{\mathbb{A}} \mu(dx)\mathbf{E}^x[1_B(\mathbf{A}^\dagger(t+s))Z] \\
&= \int_{\mathbb{A}} \mu(dx)\mathbf{E}^x[P_s(\mathbf{A}^\dagger(t), B)Z] \\
&= \mathbf{E}^\mu[P_s(\mathbf{A}^\dagger(t), B)Z].
\end{aligned}$$

Finally, consider a stopping time τ for the filtration $\{\mathcal{F}_{t+}^\dagger : t \in [0, \infty)\}$. Then, for each $f \in \mathcal{C}_{b,u}^+(\mathbb{A})$, $s \in [0, \infty)$, we have

$$\mathbf{E}^\mu[f(\mathbf{A}^\dagger(\tau+s))1_{\{\tau<\infty\}} | \mathcal{F}_{\tau+}^\dagger] = P_s(\mathbf{A}^\dagger(\tau), f)1_{\{\tau<\infty\}},$$

since by Lemma A.5.2, for each $\mathcal{F}_{\tau+}^\dagger$-measurable bounded random variable Z,

$$\begin{aligned}
\mathbf{E}^\mu[f(\mathbf{A}^\dagger(\tau+s))1_{\{\tau<\infty\}}Z] &= \int_{\mathbb{A}} \mu(dx)\mathbf{E}^x[f(\mathbf{A}^\dagger(\tau+s))1_{\{\tau<\infty\}}Z] && \text{(A.6.1)} \\
&= \int_{\mathbb{A}} \mu(dx)\mathbf{E}^x[P_s(\mathbf{A}^\dagger(\tau), f)1_{\{\tau<\infty\}}Z] && \text{(A.6.2)} \\
&= \mathbf{E}^\mu[P_s(\mathbf{A}^\dagger(\tau), f)1_{\{\tau<\infty\}}Z]. && \text{(A.6.3)}
\end{aligned}$$

Thus, the properties (1.3.2)-(1.3.4) are satisfied. In summary, $\mathcal{P}^\dagger$ is a Borel right semigroup. $\square$

Appendix B

Elementary lemmas

B.1 Locally compact with countable base state space

The lemma proved in this section contributes to verifying that the age process is a Borel right process with a locally compact with countable base state space. We do this by showing that $\mathbb{A}$ is homeomorphic to a space that is known to be locally compact with countable base. It follows that $\mathbb{A}$ is a Polish space (cf. Theorem 7.6.1 of Bauer [2].)

Fix $\gamma_0 \in [0, \infty) \cup \{\infty\}$. For each integer $n \geq 1$, let

$$\mathbb{X}_n = \{(r_1, \ldots, r_n) \in [0, \gamma_0)^n : r_1 < r_2 < \ldots < r_n\}$$

and let $X_0 = \{\varnothing\}$. Then, let $\mathbb{X} = \bigcup_{n=0}^{\infty} \mathbb{X}_n$. Define a function $d : \mathbb{X} \to [0, \infty]$ as follows: for each $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ in $\mathbb{X} \setminus \{\varnothing\}$,

$$d(x, y) = \begin{cases} \infty, & \text{if } n \neq m; \\ \sum_{k=1}^{n} |x_k - y_k|, & \text{otherwise,} \end{cases}$$

and $d(x, \varnothing) = \infty$. Now, it is not hard to see that d is a metric on $\mathbb{X}$ and the topology on $\mathbb{X}$ induced by the metric d is locally compact with countable base; to see the local compact property, note that $\{\varnothing\}$ is a compact neighborhood of $\varnothing$ and for each $x \in \mathbb{X}$, where $x = (r_1, \ldots, r_n)$ for some integer $n \geq 1$ and $\varepsilon = \min_{k=1,\ldots n-1} r_{k+1} - r_k > 0$, a compact neighborhood of x is $\{y \in \mathbb{X} : y = (y_1, \ldots, y_n), d(x, y) \leq \varepsilon/4\}$.

Next, for each integer $n \geq 1$, let

$$E_n = \left\{ \sum_{k=1}^{n} \delta_{r_k} : (r_1, r_2, \ldots, r_n) \in \mathbb{X}_n \right\},$$

and let $E_0 = \{0\}$, where 0 denotes the zero measure on $[0, \gamma_0)$. Then, let $E = \bigcup_{n=0}^{\infty} E_n$, and we equip the space $E \subset \mathbf{M}([0, \gamma_0))$ with the topology of weak convergence of finite non-negative measures; a sequence $\{\mu_n\}_{n=1}^{\infty}$ in E converges to $\mu \in E$ if and only if for any bounded continuous function f on $[0, \gamma_0)$, $\lim_{n\to\infty} \langle \mu_n, f \rangle = \langle \mu, f \rangle$.

Lemma B.1.1. *The topological spaces E and $\mathbb{X}$ are homeomorphic. In particular, E is locally compact with countable base.*

Proof. Define a function $H : \mathbb{X} \to E$ by letting

$$H(x) = \begin{cases} \sum_{k=1}^{n} \delta_{r_k}, & \text{if } x = (r_1, \ldots, r_n) \in \mathbb{X} \setminus \{\varnothing\} \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that H is one-to-one and onto. Now, for each sequence $\{x_k\}_{k=1}^{\infty}$ in $\mathbb{X}$ and $x_0 \in \mathbb{X}$, it is straightforward to check that the following statements are equivalent:

(i) as $k \to \infty$, $d(x_k, x_0) \to 0$.

(ii) as $k \to \infty$, $H(x_k)$ converges weakly to $H(x_0)$.

In other words, H is a bijective continuous function from $\mathbb{X}$ to E. □

B.2 No atoms property

The result in this section shows that as a consequence of Assumption 2.2.1, the subprobability measure Θ_i has no atoms in $(0, \infty)$ for each route i, and this observation is used crucially in proving Theorem 2.7.1 through our proof of Lemma 3.2.6.

Let μ be a probability measure on $\mathbb{R}_+$ that has no atoms in $\mathbb{R}_+$, and let

$$b_\mu = \inf\{x \in \mathbb{R}_+ : \mu((x,\infty)) = 0\}.$$

Then, for each $x \in [0, b_\mu)$, let μ^x be the probability measure on $\mathbb{R}_+$ such that $\mu^x(\{0\}) = 0$ and for each $B \in \mathcal{B}((0,\infty))$,

$$\mu^x(B) = \frac{\mu(x+B)}{\mu((x,\infty))},$$

where $x + B = \{x + y : y \in B\}$. Note that $\mu^0 = \mu$.

Lemma B.2.1. *Suppose that there exists* $\overline{\mu} \in \mathbf{M}$ *such that for each* $f \in \mathcal{C}_c(\mathbb{R}_+)$, $\lim_{x\to b_\mu}\langle f, \mu^x\rangle = \langle f, \overline{\mu}\rangle$. *Then, the measure* $\overline{\mu}$ *has no atoms in* $(0,\infty)$, *i.e.,* $\overline{\mu}(\{x\}) = 0$ *for each* $x \in (0,\infty)$.

Proof. If $b_\mu < \infty$, then $\overline{\mu} = \delta_0$ so that in particular, the measure $\overline{\mu}$ has no atoms in $(0,\infty)$. Next, assume that $b_\mu = \infty$. Suppose that there exists $a \in (0,\infty)$ such that $\Theta_i(\{a\}) > 0$.

Aiming for a contradiction, we will show that $\overline{\mu}(\{a - r\}) > 0$ for each $r \in (0, a)$. For this, consider $r \in (0, a)$. First, for each $\varepsilon \in (0,1)$, fix $f_\varepsilon \in \mathcal{C}_c^+(\mathbb{R}_+)$ such that

(i) $f_\varepsilon(a) = 1$,

(ii) $f_\varepsilon(x) = 0$ for each $x \in \mathbb{R}_+$ such that $|x - a| > \varepsilon \wedge (r/2)$,

and then, define a function $f_{\varepsilon,r} : \mathbb{R}_+ \to \mathbb{R}_+$ by letting, for each $x \in \mathbb{R}_+$, $f_{\varepsilon,r}(x) = f_\varepsilon(x + r)$ so that

(iii) $f_{\varepsilon,r}(a - r) = f_\varepsilon(a) > 0$,

(iv) $f_{\varepsilon,r}(x) = 0$ for each $x \in \mathbb{R}_+$ such that $|x - (a - r)| > \varepsilon \wedge (r/2)$

We will further assume that the family $\{f_\varepsilon : \varepsilon \in (0,1)$ of functions is fixed so that for each ε and $\varepsilon' \in (0,1)$ such that $\varepsilon \le \varepsilon'$, $f_{\varepsilon'}(x) \le f_\varepsilon(x)$ for each $x \in \mathbb{R}_+$. In particular, this implies that for each $x \in \mathbb{R}_+$, $\lim_{\varepsilon\to 0} f_\varepsilon(x) = 1_{\{0\}}(x)$.

Then, note that for each $\varepsilon \in (0,1)$ and integer $k \geq 1$, we have

$$\begin{aligned}
\langle f_{\varepsilon,r}, \mu^{r+k}\rangle &= \frac{1}{\mu((r+k,\infty))}\int_0^\infty f_{\varepsilon,r}(x)\mu(r+k+dx) && \text{(B.2.1)}\\
&\geq \frac{1}{\mu((k,\infty))}\int_{a-3r/2}^{a-r/2} f_\varepsilon(x+r)\mu(r+k+dx) && \text{(B.2.2)}\\
&= \frac{1}{\mu((k,\infty))}\int_{a-r/2}^{a+r/2} f_\varepsilon(u)\mu(k+du) && \text{(B.2.3)}\\
&= \langle f_\varepsilon, \mu^k\rangle. && \text{(B.2.4)}
\end{aligned}$$

Now, using the fact that $\overline{\mu}$ is the *unique* vague limit point of $\{\mu^x : x \in [0, b_\mu)\}$, we see that

$$\begin{aligned}
\overline{\mu}(\{a-r\}) &= \lim_{\varepsilon\downarrow 0}\langle f_{\varepsilon,r}, \overline{\mu}\rangle && \text{(B.2.5)}\\
&= \lim_{\varepsilon\downarrow 0}\left(\lim_{k\to\infty}\langle f_{\varepsilon,r}, \mu^{r+k}\rangle\right) && \text{(B.2.6)}\\
&\geq \lim_{\varepsilon\downarrow 0}\left(\lim_{k\to\infty}\langle f_\varepsilon, \mu^{k}\rangle\right) && \text{(B.2.7)}\\
&= \lim_{\varepsilon\downarrow 0}\langle f_\varepsilon, \overline{\mu}\rangle && \text{(B.2.8)}\\
&= \overline{\mu}(\{a\}) > 0. && \text{(B.2.9)}
\end{aligned}$$

In summary, we have shown that if there exists $a \in (0,\infty)$ such that $\overline{\mu}(\{a\}) > 0$, then for each $r \in (0,a)$, $\overline{\mu}(\{a\}) > 0$ but this contradicts the fact that any non-negative Borel measure on $\mathbb{R}_+$ can have at most countably many atoms. $\square$

B.3 Joint measurability

The lemma proved in this section contributes to verifying that the age process is a Borel right process.

Lemma B.3.1. *Let F be a probability distribution function on $\mathbb{R}_+$. Then, the function $H : [0,b) \times (0,1) \to [0,b)$ given by letting*

$$H(c,u) = \inf\{x \in [0,b) : F_c(x) \geq u\}$$

is Borel measurable, where $b = \inf\{x \in [0,\infty) : 1 - F(x) = 0\}$ *and*

$$F_c(x) = \frac{F(x+c) - F(c)}{1 - F(c)}.$$

Proof. Let $\{b_k\}_{k=1}^{\infty}$ be a subset of $[0,b)$ such that $F(c) = F(c-)$ for each $c \in [0,b) \setminus \{b_k\}_{k=1}^{\infty}$, and then let $\Xi = [0,b) \setminus \{b_k\}_{k=1}^{\infty}$. Let $F_c^{-1}(\cdot) = H(c,\cdot)$.

Step 1. We claim that for each $c \in \Xi$ and $u \in (0,1)$, the function $H(\cdot,u)$ is lower semi continuous at c. For this, we fix $u_0 \in (0,1)$ and a sequence $\{c_n\}_{n=1}^{\infty}$ in $[0,b)$ such that $\lim_{n\to\infty} c_n = c_0 \in \Xi$. Since F is right continuous on $\mathbb{R}_+$, by the definition of $F_{c_n}^{-1}(u_0)$, we have that $F_{c_n}(F_{c_n}^{-1}(u_0)) \geq u_0$. Let

$$x_0 = \liminf_{\ell\to\infty} F_{c_\ell}^{-1}(u_0).$$

Then, since $c_0 + x_0 = \liminf_{\ell\to\infty}\left(c_\ell + F_{c_\ell}^{-1}(u_0)\right)$, we have

$$\begin{aligned} F(c_0 + x_0) &\geq \liminf_{\ell\to\infty} F(c_\ell + F_{c_\ell}^{-1}(u_0)) && \text{(B.3.1)}\\ &\geq \liminf_{\ell\to\infty} \left(F(c_\ell) + (1 - F(c_\ell))u_0\right) && \text{(B.3.2)}\\ &= F(c_0) + (1 - F(c_0))u_0, && \text{(B.3.3)} \end{aligned}$$

where the last equality follows because $c_0 \in \Xi$, the second inequality is by the definition of $F_{c_\ell}^{-1}(u_0)$, and the first inequality uses the fact that the function F is non-decreasing, right continuous on $\mathbb{R}_+$ and has left limits on $(0,\infty)$. So, by the definition of $F_{c_0}^{-1}(u_0)$, we have $x_0 \geq F_{c_0}^{-1}(u_0)$; in other words,

$$\liminf_{\ell\to\infty} F_{c_\ell}^{-1}(u_0) \geq F_{c_0}^{-1}(u_0).$$

Step 2. We claim that for each $u \in (0,1)$, $H(\cdot,u)$ is measurable. For this, fix $u_0 \in (0,1)$, and for each $c \in [0,b)$, define $K_0 : [0,b) \to [0,b)$ and $G_0 : [0,b) \to [0,b)$ by letting, for each $c \in [0,b)$,

$$G_0(c) = F_c^{-1}(u_0) - \textstyle\sum_{k=1}^{\infty} F_{b_k}^{-1}(u_0) 1_{\{b_k\}}(c),$$
$$K_0(c) = \textstyle\sum_{k=1}^{\infty} F_{b_k}^{-1}(u_0) 1_{\{b_k\}}(c).$$

Since the function K_0 is identically zero on Ξ and the set $[0,b)\setminus\Xi$ is countable, it follows that K_0 is Borel measurable. On the other hand, since K_0 is identically zero on Ξ, it follows from *Step 1* that G_0 is lower semi continuous at each $c\in\Xi$. Also, since the function G_0 is identically zero on $[0,b)\setminus\Xi$, G_0 is lower semi continuous at each $c\in[0,b)\setminus\Xi$. In summary, G_0 is Borel measurable. Note that $H(\cdot,u_0)=K_0(\cdot)+G_0(\cdot)$, and hence, $H(\cdot,u_0)$ is also Borel measurable.

Step 3. For each $c\in[0,b)$, it is well known (cf. Section 2.5.2 in Resnick [26]) that the function $H(c,\cdot)=F_c^{-1}(\cdot)$ is left continuous on $(0,1)$. Also, from *Step 2*, the function $H(\cdot,u)$ is measurable for each $u\in(0,1)$. From these observations, it follows that the function H is *jointly* measurable. $\square$

B.4 Conditional distribution

The lemma proved in this section contributes to verifying that the age process is a Borel right process.

Let X and Y be $\mathcal{F}$-measurable random variables, and let $\mathcal{H}\subset\mathcal{F}$ be a σ-algebra on Ω. Denote by F_X, the distribution function of X and let

$$\mathcal{H}'=\sigma\left(1_{\{X>Y\}}\right)\vee\mathcal{H}.$$

Lemma B.4.1. *Suppose that X is independent of $\mathcal{H}$ and that Y is measurable with respect to $\mathcal{H}$. For any $r\in[0,\infty)$,*

$$\mathbf{E}\left[1_{\{X>Y+r\}}\,|\mathcal{H}'\right]=\frac{1-F_X(Y+r)}{1-F_X(Y)}1_{\{X>Y\}}.$$

Proof. First, since X is independent of the σ-algebra $\mathcal{H}$, for any $\mathcal{H}$-measurable non-negative bounded random variable Z, we have $\mathbf{P}\left[X>Z\,|\mathcal{H}\right]=1-F_X(Z)$.

Next, we claim that for any $\mathcal{H}$-measurable non-negative bounded random variable Z and $r\in[0,\infty)$,

$$\mathbf{E}\left[\frac{1-F_X(Y+r)}{1-F_X(Y)}1_{\{X>Y\}}Z\right]=\mathbf{E}\left[1_{\{X>Y+r\}}1_{\{X>Y\}}Z\right].$$

For this, fix an $\mathcal{H}$-measurable non-negative bounded random variable Z and note that since X and Y are independent, we have

$$\frac{1-F_X(Y+r)}{1-F_X(Y)}1_{(0,\infty)}(X-Y) = \frac{\mathbf{P}\left[X > Y+r\,|Y\right]}{\mathbf{P}\left[X > Y\,|Y\right]}1_{(0,\infty)}(X-Y),$$

where we take $0/0 = 0$ as a matter of notational convention. Now, we see that

$$\begin{aligned}
\mathbf{E}\left[\frac{1-F_X(Y+r)}{1-F_X(Y)}1_{\{X>Y\}}Z\right] &= \mathbf{E}\left[\mathbf{E}\left[\frac{1-F_X(Y+r)}{1-F_X(Y)}1_{\{X>Y\}}Z\,|\mathcal{H}\right]\right] \\
&= \mathbf{E}\left[\mathbf{E}\left[1_{\{X>Y\}}\,|\mathcal{H}\right]\frac{1-F_X(Y+r)}{1-F_X(Y)}Z\right] \\
&= \mathbf{E}\left[(1-F_X(Y))\frac{1-F_X(Y+r)}{1-F_X(Y)}Z\right] \\
&= \mathbf{E}\left[(1-F_X(Y+r))Z\right] \\
&= \mathbf{E}\left[\mathbf{P}\left[X > Y+r\,|\mathcal{H}\right]Z\right] \\
&= \mathbf{E}\left[1_{\{X>Y+r\}}Z\right],
\end{aligned}$$

and

$$\mathbf{E}\left[1_{\{X>Y+r\}}1_{\{X\le Y\}}Z\right] = \mathbf{E}\left[\frac{1-F_X(Y+r)}{1-F_X(Y)}1_{\{X>Y\}}1_{\{X\le Y\}}Z\right].,$$

where the equality follows since both sides are trivially zero.

Next, let

$$\begin{aligned}
\mathfrak{S}' &= \bigcup_{B\in\mathcal{H}}\{B, \{X>Y\}\cap B, \{X\le Y\}\cap B\}, \\
\mathfrak{S} &= \left\{B\in\mathcal{H}' : \mathbf{E}\left[1_{\{X>Y+r\}}1_B\right] = \mathbf{E}\left[\frac{1-F_X(Y+r)}{1-F_X(Y)}1_{\{X>Y\}}1_B\right] \text{ for all } r\ge 0\right\}
\end{aligned}$$

Then, from our definition of $\mathcal{H}'$, it is immediate that $\mathfrak{S}'\subset\mathcal{H}'$, and moreover, $\mathfrak{S}'$ is a π-system such that $\sigma(\mathfrak{S}') = \mathcal{H}'$. Also, our computation shows that $\mathfrak{S}$ is a λ-system such that $\mathfrak{S}'\subset\mathfrak{S}$, whence by Dynkin's π-λ theorem, we have $\sigma(\mathfrak{S}') = \sigma(\mathfrak{S}) = \mathcal{H}'$. $\square$

B.5 Conditional independence

The lemma proved in this section contributes to verifying that the age process is a Borel right process.

Fix integers ℓ and $\ell_* \geq 1$. Suppose that $X_1, \ldots, X_\ell$ and $Y_1, \ldots, Y_{\ell_*}$ are $\mathcal{F}$-measurable mutually independent random variables, each of which is strictly positive $\mathbf{P}$-almost surely. For each $m \in \{1, \ldots, \ell\}$, denote by F_m the distribution function of X_m. Then, let

$$\eta_* = \max_{k=1}^{\ell_*} Y_k, \tag{B.5.1}$$

$$\Omega_* = \{Y_1 = \ldots = Y_{\ell_*} < \min_{k=1,\ldots,\ell} X_k\}, \tag{B.5.2}$$

$$\mathcal{H}_* = \sigma(\eta_*, 1_{\Omega_*}). \tag{B.5.3}$$

Lemma B.5.1. *For each $r_1, \ldots, r_\ell \in \mathbb{R}_+$,*

$$\mathbf{E}\left[\prod_{m=1}^{\ell} 1_{\{X_m - \eta_* \in (0, r_m]\}} \middle| \mathcal{H}_*\right] = \prod_{m=1}^{\ell} \frac{F_m(\eta_* + r_m) - F_m(\eta_*)}{1 - F_m(\eta_*)} 1_{\{X_m > \eta_*\}}. \tag{B.5.4}$$

Proof. For each $m \in \{1, \ldots, \ell\}$, let

$$\mathcal{H}_m = \mathcal{H}_* \vee \sigma(X_k : k \in \{1, \ldots, \ell\}, k \neq m),$$
$$\xi_m = \mathbf{E}\left[1_{\{X_m - \eta_* \in (0, r_m]\}} \,|\mathcal{H}_m\right].$$

First, note that

$$\begin{aligned}
\mathbf{E}\left[\prod_{m=1}^{\ell} 1_{\{X_m - \eta_* \in (0, r_m]\}} \middle| \mathcal{H}_*\right] &= \mathbf{E}\left[\mathbf{E}\left[\prod_{m=1}^{\ell} 1_{\{X_m - \eta_* \in (0, r_m]\}} \middle| \mathcal{H}_1\right] \middle| \mathcal{H}_*\right] \\
&= \mathbf{E}\left[\xi_1 \prod_{m=2}^{\ell} 1_{\{X_m - \eta_* \in (0, r_m]\}} \middle| \mathcal{H}_*\right] \\
&= \xi_1 \mathbf{E}\left[\prod_{m=2}^{\ell} 1_{\{X_m - \eta_* \in (0, r_m]\}} \middle| \mathcal{H}_*\right],
\end{aligned}$$

where the second equality follows since for each $m \in \{2, \ldots, \ell\}$, $X_m \in \mathcal{H}_1$. Proceeding inductively, we then have

$$\mathbf{E}\left[\prod_{m=1}^{\ell} 1_{\{X_m - \eta_* \in (0, r_m]\}} \middle| \mathcal{H}_*\right] = \prod_{m=1}^{\ell} \xi_m. \tag{B.5.5}$$

Next, by Lemma B.4.1, for each $m \in \{1, \ldots, \ell\}$, we have

$$\begin{aligned} \xi_m &= \mathbf{E}\left[1_{\{X_m - \eta_* \in (0, r_m]\}} \,|\mathcal{H}_m\right] \\ &= \mathbf{E}\left[1_{\{X_m - \eta_* > 0\}} \,|\mathcal{H}_m\right] - \mathbf{E}\left[1_{\{X_m - \eta_* > r_m\}} \,|\mathcal{H}_m\right] \\ &= \frac{F_m(\eta_* + r_m) - F_m(\eta_*)}{1 - F_m(\eta_*)} 1_{\{X_m > \eta_m\}} \end{aligned}$$

so that together with (B.5.5), we have

$$\mathbf{E}\left[\prod_{m=1}^{\ell} 1_{\{X_m - \eta_* \in (0, r_m]\}} \middle| \mathcal{H}_*\right] = \prod_{m=1}^{\ell} \xi_m = \prod_{m=1}^{\ell} \left(1_{\{X_m > \eta_*\}} \frac{F_m(\eta_* + r_m) - F_m(\eta_*)}{1 - F_m(\eta_*)} \right).$$

□

B.6 On weak convergence

The lemmas proved in this section justify our application of Theorem 4.1 in Gromoll and Williams [13] in our proof of Lemma 3.2.6.

Define a function $H : \mathbb{R}_+ \times \mathbf{M} \to \mathbf{M}$ by letting, for each $t \in \mathbb{R}_+$ and $\mu \in \mathbf{M}$,

$$H(t, \mu)(B) = \int_{(t,\infty)} 1_B(s - t)\mu(ds),$$

where $\mathbf{M}$ is given the topology of weak convergence.

Lemma B.6.1. *The function H is $\mathcal{B}(\mathbb{R}_+) \times \mathcal{B}(\mathbf{M})$-measurable.*

Proof. For each integer $k \geq 1$, define a function $g_k : \mathbb{R}_+ \to [0, 1]$ by letting, for each $s \in \mathbb{R}_+$,

$$g_k(s) = \max\{0, 1 - k \inf\{|r - s| : r \in [1/k, \infty)\}\}.$$

Note that $g_k \in \mathcal{C}_b^+(\mathbb{R}_+)$ and for each $s \in \mathbb{R}_+$,

$$\lim_{k\to\infty} g_k(s) = 1_{(0,\infty)}(s).$$

Now, for each integer $k \geq 1$, define a function $H_k : \mathbb{R}_+ \times \mathbf{M} \to \mathbf{M}$ by letting, for each $t \in \mathbb{R}_+$ and $\mu \in \mathbf{M}$,

$$H_k(t,\mu)(B) = \int_{\mathbb{R}_+} g_k((s-t)^+)1_B(s-t)\mu(ds),$$

First, we claim that for each integer $k \in \mathbb{R}_+$, H_k is Borel measurable. For this, fix integer $k \geq 1$, and consider a sequence $\{\mu_n\}_{n=1}^\infty$ in $\mathbf{M}$ that converges to $\mu_0 \in \mathbf{M}$ and a sequence $\{t_n\}_{n=1}^\infty$ in $\mathbb{R}_+$ that converges to 0. For each $t_0 \in \mathbb{R}_+$ and $f \in \mathcal{C}_b^+(\mathbb{R}_+)$, since $g_k((\cdot - t)^+)f(\cdot - t) \in \mathcal{C}_b^+(\mathbb{R}_+)$,

$$\begin{aligned}
\lim_{n\to\infty} \langle f, H_k(t_0,\mu_n)\rangle &= \lim_{n\to\infty} \int_{\mathbb{R}_+} g_k((s-t_0)^+)f(s-t_0)\mu_n(ds) \\
&= \int_{\mathbb{R}_+} g_k((s-t_0)^+)f(s-t_0)\mu_0(ds) \\
&= \langle f, H_k(t_0,\mu_0)\rangle,
\end{aligned}$$

and then, by the bounded convergence theorem,

$$\begin{aligned}
\lim_{n\to\infty} \langle f, H_k(t_0+t_n,\mu_0)\rangle &= \lim_{n\to\infty} \int_{\mathbb{R}_+} g_k((s-t_0-t_n)^+)f(s-t_0-t_n)\mu_0(ds) \\
&= \int_{\mathbb{R}_+} g_k((s-t_0)^+)f(s-t_0)\mu_0(ds) \\
&= \langle f, H_k(t_0,\mu_0)\rangle,
\end{aligned}$$

In other words, for each $t \in \mathbb{R}_+$, $H_k(t,\cdot)$ is continuous and then, for each $\mu \in \mathbf{M}$, $H_k(\cdot,\mu)$ is right continuous. Therefore, H_k is Borel measurable.

Now, note that for each $t \in \mathbb{R}_+$ and $\mu \in \mathbf{M}$, by the bounded convergence theorem, for each $f \in \mathcal{C}_b^+(\mathbb{R}_+)$,

$$\begin{aligned}
\lim_{k\to\infty} \langle f, H_k(t,\mu)\rangle &= \lim_{k\to\infty} \int_{\mathbb{R}_+} g_k((s-t)^+)f(s-t)\mu(ds) \\
&= \int_{\mathbb{R}_+} 1_{(0,\infty)}((s-t)^+)f(s-t)\mu(ds) \\
&= \langle f, H(t,\mu)\rangle.
\end{aligned}$$

This shows that H is the limit of Borel measurable functions $\{H_k\}_{k=1}^{\infty}$, and therefore, this shows that H is also Borel measurable. $\square$

The following lemma shows that H is jointly continuous at certain points having suitable properties.

Lemma B.6.2. *Let $\{\mu_\ell\}_{\ell=1}^{\infty}$ be a sequence in $\mathbf{M}$ and $\{s_\ell\}_{\ell=1}^{\infty}$ be a sequence in $\mathbb{R}_+$. Suppose that as $\ell \to \infty$, $\{\mu_\ell\}_{\ell=1}^{\infty}$ converges weakly to $\mu_0 \in \mathbf{M}$ and $\{s_\ell\}_{\ell=1}^{\infty}$ converges to s_0, where $s_0 \in (0, \infty)$ and $\mu_0(\{s_0\}) = 0$. Then, for each $f \in \mathcal{C}_b^+(\mathbb{R}_+)$,*

$$\lim_{\ell\to\infty} \langle f, H(s_\ell, \mu_\ell)\rangle = \langle f, H(s_0, \mu_0)\rangle. \tag{B.6.1}$$

Proof. Our claim is trivial if $\|f\|_\infty = 0$, and so, we assume that $\|f\|_\infty > 0$. Since $s_0 > 0$, by re-indexing if necessary, we assume that $s_\ell > 0$ for each integer $\ell \geq 1$. Let $\varepsilon \in (0, 1)$. First, fix $K \in [s_0 + 1, \infty)$ and an integer $k_\varepsilon \geq 1$ such that

$$s_0 - 1/k_\varepsilon > 0, \tag{B.6.2}$$

$$\mu_0([s_0 - 1/k_\varepsilon, s_0 + 1/k_\varepsilon]) < \varepsilon/\|f\|_\infty, \tag{B.6.3}$$

$$\mu_0([K, \infty)) < \varepsilon/(2\|f\|_\infty), \tag{B.6.4}$$

and an integers $\ell_\varepsilon \geq 1$ such that for each integer $\ell \geq \ell_\varepsilon$,

$$\sup_{w\in[s_0-1/k_\varepsilon, K]} |f(w - s_\ell) - f(w - s_0)| \leq \varepsilon/(\mu_0((s_0, \infty)) + 1), \tag{B.6.5}$$

$$|\mu_\ell([s_0, \infty)) - \mu_0([s_0, \infty))| \leq \varepsilon/\|f\|_\infty, \tag{B.6.6}$$

$$\mu_\ell([s_0 - 1/k_\varepsilon, s_0 + 1/k_\varepsilon]) \leq \mu_0([s_0 - 1/k_\varepsilon, s_0 + 1/k_\varepsilon]) + \varepsilon/\|f\|_\infty, \tag{B.6.7}$$

$$\mu_\ell((s_0, \infty)) < \mu_0((s_0, \infty)) + 1, , \tag{B.6.8}$$

$$\mu_\ell([K, \infty)) < \varepsilon/(2\|f\|_\infty), \tag{B.6.9}$$

$$\left| \int_{(s_0,\infty)} f(w - s_0)\mu_\ell(dw) - \int_{(s_0,\infty)} f(w - s_0)\mu_0(dw) \right| \leq \varepsilon, \tag{B.6.10}$$

where existence of ℓ_ε and k_ε satisfying respectively (B.6.6) and (B.6.3) follows from the fact that $\mu_0(\{s_0\}) = 0$, existence of k_ε satisfying (B.6.5) follows from uniform continuity of f on compact sets, and (B.6.9) follows by weak convergence and the fact that $\mu_0(\{s_0\}) = 0$. Now, for each integer $\ell \geq \ell_\varepsilon$ such that $s_\ell \in (s_0 - 1/k_\varepsilon, s_0 + 1/k_\varepsilon)$, we have

$$\begin{aligned}
& \left| \int_{(s_\ell,\infty)} f(w - s_\ell)\mu_\ell(dw) - \int_{(s_0,\infty)} f(w - s_0)\mu_0(dw) \right| \\
\leq \; & \int_{(s_0-1/k_\varepsilon, s_0+1/k_\varepsilon)} \|f\|_\infty \mu_\ell(dw) \\
& \quad + \left| \int_{(s_0,\infty)} f(w - s_\ell)\mu_\ell(dw) - \int_{(s_0,\infty)} f(w - s_0)\mu_0(dw) \right| \\
\leq \; & 2\varepsilon + \int_{(s_0,\infty)} |f(w - s_\ell) - f(w - s_0)|\mu_\ell(dw) \\
& \quad + \left| \int_{(s_0,\infty)} f(w - s_0)\mu_\ell(dw) - \int_{(s_0,\infty)} f(w - s_0)\mu_0(dw) \right| \\
\leq \; & 2\varepsilon + \int_{(s_0,K)} |f(w - s_\ell) - f(w - s_0)|\mu_\ell(dw) \\
& \quad + \int_{[K,\infty)} |f(w - s_\ell) - f(w - s_0)|\mu_\ell(dw) + \varepsilon \\
\leq \; & 2\varepsilon + \frac{\varepsilon}{\mu_0((s_0,\infty)) + 1}\mu_0((s_0,\infty)) + 2\|f\|_\infty \frac{\varepsilon/2}{\|f\|_\infty} + \varepsilon \\
\leq \; & 5\varepsilon.
\end{aligned}$$

□

B.7 On the M_1'-topology

While we refer the reader to Section 2 of [25] and Section 13.6.2 of [28] for the definition of the M_1' topology on $\mathbb{D}(\mathbb{R}_+)$, we will review a few relevant facts here. The space $\mathbb{D}(\mathbb{R}_+)$ with the M_1' topology is a separable metric space [25]. Also, as mentioned in [25], the M_1' topology is weaker than both the J_1 topology and the M_1 topology so that in particular, any sequence that converges either in the J_1 topology or in the M_1 topology also converges in the M_1' topology. The key difference between the M_1 topology and the M_1' topology is that with the M_1' topology, convergence of a sequence $\{x_\ell\}_{\ell=1}^\infty$ in $\mathbb{D}(\mathbb{R}_+)$ to $x \in \mathbb{D}(\mathbb{R}_+)$ does not require that $\lim_{\ell\to\infty} x_\ell(0) = x(0)$ but with the M_1 topology, it does. On the other hand, the Borel σ-algebra generated by the M_1' topology coincides with the Borel σ-algebra generated by the M_1 topology and the J_1 topology, and in fact, it is the Kolmogorov σ-algebra generated by the coordinate projections (cf. Lemma 2.7 of Whitt [27]).

For our next lemma, let $\mathcal{K}_+$ be the σ-algebra on $\mathbb{D}(\mathbb{R}_+)$ generated by the coordinate projection maps, i.e., the Kolmogorov σ-algebra. Also, define a function $H : \mathbb{R}_+ \times \mathbb{D}(\mathbb{R}_+) \to \mathbb{R}_+$ by letting, for each $t \in \mathbb{R}_+$ and $g \in \mathbb{D}(\mathbb{R}_+)$,

$$H(t, g) = \int_0^t g(s)ds.$$

Lemma B.7.1. *The function $H : \mathbb{R}_+ \times \mathbb{D}(\mathbb{R}_+) \to \mathbb{R}_+$ is $\mathcal{B}(\mathbb{R}_+) \times \mathcal{K}_+$-measurable.*

Proof. Fix $t \in \mathbb{R}_+$ and equip $\mathbb{D}(\mathbb{R}_+)$ with the J_1 topology. It is easy to see that $H(t, \cdot)$ is a continuous function from $\mathbb{D}(\mathbb{R}_+)$ to $\mathbb{R}_+$ (cf. Theorem 11.5.1 in Whitt [28]). On the other hand, by Theorem 11.5.2 in Whitt [28], the Borel σ-algebra generated by the J_1 topology coincides with the σ-algebra $\mathcal{K}_+$. Hence, $H(t, \cdot)$ is $\mathcal{K}_+$-measurable. On the other hand, for each $g \in \mathbb{D}(\mathbb{R}_+)$, since $\|g\|_{[0,t]} < \infty$ for each $t \in \mathbb{R}_+$, it is clear that $H(\cdot, g)$ is continuous. Therefore, H is $\mathcal{B}(\mathbb{R}_+) \times \mathcal{K}_+$-measurable. $\square$

For our next two lemmas, let $\mathcal{H} \subset \mathbb{D}(\mathbb{R}_+)$ be the set of all r.c.l.l. functions that are non-negative and non-decreasing.

Lemma B.7.2. *Let $\{g_\ell\}_{\ell=1}^\infty$ be a sequence in $\mathcal{H}$ and let $g_0 \in \mathcal{H}$. Then, the sequence $\{g_\ell\}_{\ell=1}^\infty$ converges to g_0 in the M_1'-topology if there exists a dense subset G of $(0, \infty)$ such that for each $s \in G$, $\lim_{\ell\to\infty} g_\ell(s) = g_0(s)$.*

Proof. See the proof of Theorem 13.6.3 in [28]. $\square$

Lemma B.7.3. *The set $\mathcal{H}_1 = \{h \in \mathcal{H} : \|h\|_\infty \le 1\}$ is a compact subset of $\mathbb{D}(\mathbb{R}_+)$, when $\mathbb{D}(\mathbb{R}_+)$ is equipped with the M_1' topology.*

Proof. Since with the M_1'-topology, the space $\mathbb{D}(\mathbb{R}_+)$ is a separable metric space, it is sufficient to show that $\mathcal{H}_1$ is sequentially compact (cf. page 130 in Folland [10]). For this, fix a sequence $\{h_k\}_{k=1}^{\infty}$ in $\mathcal{H}_1$. Then, by Helly's selection principle (cf. Theorem 4.3.3 in Chung [7]), there exist $g_0 \in \mathcal{H}$ and a subsequence $\{g_\ell\}_{\ell=1}^{\infty}$ of $\{h_k\}_{k=1}^{\infty}$ such that $\|g_0\|_\infty \leq 1$ and $g_0(t) = \lim_{\ell \to \infty} g_\ell(t)$ for each continuity point t of g_0. Note here that $t = 0$ need not be a continuity point of g_0. Also, since $g_0 \in \mathbb{D}(\mathbb{R}_+)$, the set of points at which g_0 is discontinuous is at most countable. Therefore, by Lemma B.7.2, the sequence $\{g_\ell\}_{\ell=1}^{\infty}$ converges to g_0 in the M_1'-topology as $n \to \infty$. This shows that $\mathcal{H}_1$ is compact. $\square$

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