Title
Higher Quasicoherent Sheaves

Author
Stefanich, German

Publication Date
2021

Peer reviewed|Thesis/dissertation

Higher Quasicoherent Sheaves

by

German Stefanich

A dissertation submitted in partial satisfaction of the

requirements for the degree of

Doctor of Philosophy

in

Mathematics

in the

Graduate Division

of the

University of California, Berkeley

Committee in charge:

Professor David Nadler, Chair
Professor Nicolai Reshetikhin
Professor Martin White

Spring 2021

Higher Quasicoherent Sheaves

Abstract

Higher Quasicoherent Sheaves

by

German Stefanich

Doctor of Philosophy in Mathematics

University of California, Berkeley

Professor David Nadler, Chair

This thesis consists of two parts. The first half concerns various foundational aspects of the theory of enriched ∞-categories. We develop the theory of adjunctions and weighted limits and colimits in enriched ∞-categories. We introduce theories of enriched ∞-props and operads, which provide a framework for the study of higher algebra in the enriched context. Finally, we study the theory of monads and monadic adjunctions in enriched $(\infty, 2)$-categories, and prove an enriched generalization of the Barr-Beck-Lurie monadicity theorem.

The second half of this thesis applies the results of the first half to the study of higher categorical sheaf theory in derived algebraic geometry. We introduce and study a theory of quasicoherent sheaves of presentable stable (∞, n)-categories on prestacks, generalizing the case $n = 1$ studied in [Gai15]. We prove a universal property for the $(\infty, n+1)$-category of correspondences, generalizing and providing a new approach to the case $n = 1$ from [GR17], and use it to show that our higher quasicoherent sheaves give rise to representations of the higher categories of correspondences of prestacks. We also introduce a notion of n-affineness for prestacks and provide a simple inductive criterion for checking n-affineness, which allows one to reduce affineness questions to the case $n = 1$ studied in [Gai15].

Contents

Acknowledgments

I would like to thank my advisor David Nadler for his support and encouragement these last few years. His suggestion to study categorified sheaf theory with the intent of applying it to geometric Langlands has turned out to be a fantastic match for my interests, and it has led to a research program which I am very excited about - I am grateful to him for pointing me in this direction.

I am thankful to Constantin Teleman for the inspiring seminars that he organized these last few years in Berkeley and for introducing me to the concept of anticategory, which I ended up using in this thesis under the name of categorical spectrum.

Many thanks to Dima Arinkin and Dennis Gaitsgory for their enthusiasm and for explaining their approach to the theory of ind-coherent sheaves of categories and their singular support, which informed some of my ideas on this subject.

I am grateful to Nick Rozenblyum for a stimulating conversation regarding higher presentable categories and higher categories of correspondences, and for sharing with me his thoughts about how some aspects of the theory could be approached.

I would like to thank Carlos di Fiore for many stimulating conversations about sheaves of categories, singular support, and related topics, which have greatly enriched my understanding of the subject.

Thanks to Christopher Kuo for sharing my enthusiasm about sheaf theory and derived phenomena - I learned a lot these last few years by discussing these topics with him.

Chapter 1

Introduction

This thesis consists of two parts. The first part deals with the foundations of the theory of enriched ∞-categories. The second part applies the material of part I to the study of higher categorical sheaf theory in derived algebraic geometry.

This work is part of a program aimed at constructing new examples of fully extended topological field theories. The following introduction explains how this thesis fits into this broader program. We also refer to the introduction of each part for a more in depth description of its contents.

1.1 Topological field theory and geometric Langlands

Let n be a nonnegative integer. Then, following [Lur09b] and [CS19], one may define a symmetric monoidal (∞, n)-category $n\mathrm{Cob}$ called the (∞, n)-category of (unoriented) cobordisms, with the following informal description:

- Objects of $n\mathrm{Cob}$ are finite collections of points, to be thought of as compact 0-dimensional manifolds.

- A morphism between two compact 0-dimensional manifolds S and T is a 1-dimensional compact manifold with boundary M equipped with a decomposition $\partial M = S \sqcup T$. In other words, this is a cobordism from S to T.

- In general, for every $1 \leq k \leq n$, a k-cell in $n\mathrm{Cob}$ is in particular a k-dimensional compact manifold with boundaries and arbitrary codimensional corners, which determines a cobordism between two pieces of its boundary.

- Composition in $n\mathrm{Cob}$ is given by gluing of cobordisms along shared boundary components.

- The symmetric monoidal structure on $n\mathrm{Cob}$ is given by taking disjoint unions of manifolds.

Let $\mathcal{D}$ be a symmetric monoidal $(\infty, n+1)$-category, whose objects we think about as being (∞, n)-categories of some sort. A (fully extended, unoriented) $(n+1)$-dimensional topological field theory[1] with target $\mathcal{D}$ is a symmetric monoidal functor $\chi : n\mathrm{Cob} \to \mathcal{D}$. This assigns in particular:

- To the empty zero dimensional manifold $\emptyset$, the unit $1_{\mathcal{D}}$ in $\mathcal{D}$.
- To the zero dimensional manifold consisting of one point, an object $\chi(\mathrm{pt})$ in $\mathcal{D}$.
- To the circle S^1, thought of as a cobordism from $\emptyset$ to itself, an object $\chi(S^1)$ in $\mathrm{End}_{\mathcal{D}}(1_{\mathcal{D}})$.
- To a closed 2-dimensional manifold M, thought of as a self-cobordism of the empty 1-dimensional cobordism, an object $\chi(M)$ in $\mathrm{End}_{\mathrm{End}_{\mathcal{D}}(1_{\mathcal{D}})}(\mathcal{D})$.
- In general, to a closed k-dimensional manifold, an object in the "k-fold looping" of $\mathcal{D}$.

The above data is subject to various constraints, which encode the locality of χ: the value of χ on a manifold can be recovered by expressing the manifold as a composition of cobordisms.

In recent years, topological field theories have been found to provide a powerful framework that organizes a number of structures arising in geometric representation theory. A central instance of these interactions is given by the geometric Langlands program. It was observed by Kapustin and Witten in [KW07] that the geometric Langlands correspondence can be understood as a consequence of an equivalence between two four dimensional topological field theories attached to a complex reductive group and its Langlands dual group. Although it is now understood that geometric Langlands in is usual formulation is more closely related to conformal field theory rather than topological field theory, Kapustin and Witten's work has led to creation of the so-called Betti version of geometric Langlands [BN18], which has a more topological flavour, and is conjectured to ultimately form part of an equivalence of four-dimensional topological field theories.

Although the topological field theories in Kapustin and Witten's work make sense physically, they do not yet admit a rigorous mathematical formulation. So far, only some traces of the full structure of a topological field theory have been put in firm mathematical footing. There is in fact a fairly limited supply of fully extended topological field theories that have been constructed beyond dimension 2. This raises the following natural question:

Question 1. How do we construct interesting examples of topological field theories, in particular those which are expected to underlie the Betti geometric Langlands program?

The cobordism hypothesis [Lur09b] provides one possible way to approach this question. It states roughly speaking that the data of χ can be recovered its value on a point: in fact, any object d of $\mathcal{D}$ gives rise to a unique topological field theory χ such that $\chi(\mathrm{pt}) = d$, as

[1] This is more typically called an n-dimensional topological field theory, as the dimension of the manifolds is bounded by n. We choose however to use here a terminology that more closely matches how these objects are named in the physics literature: the reader may think that we are discussing $(n+1)$-dimensional topological field theories whose values at $(n+1)$-dimensional manifolds diverge.

long as d satisfies appropriate finiteness conditions, and comes equipped with an $O(n)$-fixed point structure.

This essentially reduces the problem to constructing interesting examples of (∞, n)-categories (or, more generally, objects in a background $(\infty, n+1)$-category $\mathcal{D}$). In this thesis, we begin exploring the idea that higher categorical sheaf theory provides a rich source of (∞, n)-categories and topological field theories, and gives tools to begin answering question 1.

1.2 Two dimensional field theories via sheaf theory

We now explain the most basic instance of using sheaf theory to produce topological field theories, which serves as a guide for our categorified story.

Let X be a (quasicompact, quasiseparated) scheme over a field k, and let $\mathrm{QCoh}(X)$ be the dg-category of quasicoherent sheaves on X. We consider $\mathrm{QCoh}(X)$ as an object in the symmetric monoidal $(\infty, 2)$-category $2\mathscr{V}\mathrm{ect}$ of presentable dg-categories, described informally as follows:

- Objects in $2\mathscr{V}\mathrm{ect}$ are dg-categories admitting all colimits, satisfying a certain set theoretical tameness condition called presentability.

- Given two objects $\mathcal{C}, \mathcal{D}$ in $2\mathscr{V}\mathrm{ect}$, the ∞-category $\mathrm{Hom}_{2\mathscr{V}\mathrm{ect}}(\mathcal{C}, \mathcal{D})$ is the full subcategory of $\mathrm{Funct}(\mathcal{C}, \mathcal{D})$ on the colimit preserving functors.

- The unit of $2\mathscr{V}\mathrm{ect}$ is the dg-category Vect of (chain complexes of) k-vector spaces.

- Given two objects $\mathcal{C}, \mathcal{D}$, their tensor product $\mathcal{C} \otimes \mathcal{D}$ is the universal recipient of a functor $\mathcal{C} \times \mathcal{D} \to \mathcal{C} \otimes \mathcal{D}$ which preserves colimits in each coordinate and coequalizes the action of Vect on both factors.

It turns out that $\mathrm{QCoh}(X)$ is a dualizable object in $2\mathscr{V}\mathrm{ect}$, and can be equipped with a canonical structure of $O(1)$-fixed point. The cobordism hypothesis thus guarantees that $\mathrm{QCoh}(X)$ gives rise to a 2-dimensional topological field theory χ with target $2\mathscr{V}\mathrm{ect}$. In physics language, this is a version of the B-model of X.

By definition, the value of χ at the point recovers $\mathrm{QCoh}(X)$. Being a 2-dimensional topological field theory, it also makes sense to wonder about the value of χ at the circle. The most robust way to compute this, and in fact understand χ in its entirety, is provided by the following fundamental result, first proven in [GR17]:

Theorem 1.2.1. *The assignment $Y \mapsto \mathrm{QCoh}(Y)$ forms part of a symmetric monoidal functor*

$$\mathrm{QCoh} : 2\mathrm{Corr}(\mathrm{Sch}) \to 2\mathscr{V}\mathrm{ect}^{2\text{-op}}.$$

In the above theorem, $2\mathscr{V}\mathrm{ect}^{2\text{-op}}$ denotes the $(\infty, 2)$-category obtained from $2\mathscr{V}\mathrm{ect}$ by reversing the orientation of the 2-cells, and $2\mathrm{Corr}(\mathrm{Sch})$ denotes the $(\infty, 2)$-category of correspondences of schemes, described informally as follows:

- Objects of 2Corr(Sch) are (quasicompact, quasiseparated, derived) schemes over k.
- Given two schemes Y, Z, a morphism from Y to Z in 2Corr(Sch) is a third scheme S equipped with maps $Y \leftarrow S \rightarrow Z$.
- Given two composable morphisms $Y \leftarrow S \rightarrow Z \leftarrow S' \rightarrow W$, their composition is given by $Y \leftarrow S \times_Z S' \rightarrow W$.
- Given two parallel morphisms $Y \leftarrow S \rightarrow Z$ and $Y \leftarrow S' \rightarrow Z$, a 2-cell between them is a commutative diagram of schemes as follows:

$$\begin{tikzcd} & S \arrow[dl] \arrow[dr] \arrow[dd] & \\ Y & & Z \\ & S' \arrow[ul] \arrow[ur] & \end{tikzcd}$$

- Given two schemes Y, Z, the tensor product of Y and Z in 2Corr(Sch) is the object $Y \times Z$.

Concretely, theorem 1.2.1 associates to each morphism $Y \xleftarrow{p} S \xrightarrow{q} Z$ in 2Corr(Sch), the Fourier-Mukai type functor $q_* p^* : \mathrm{QCoh}(Y) \rightarrow \mathrm{QCoh}(Z)$. One can think about theorem 1.2.1 as providing an efficient way of encoding the functoriality of the theory of quasicoherent sheaves. For instance, compatibility with compositions encodes the base change property, and the functoriality under 2-cells encodes the fact that pushforwards are right adjoint to pullbacks.

Using theorem 1.2.1 one can give a very concise construction of the topological field theory χ associated to a fixed scheme X, as a composition of three different symmetric monoidal functors:

$$1\mathrm{Cob} \rightarrow 2\mathrm{Corr}(\mathrm{Spc}^{\mathrm{op}}) \rightarrow 2\mathrm{Corr}(\mathrm{Sch}) \xrightarrow{\mathrm{QCoh}} 2\mathscr{V}\mathrm{ect}^{2\text{-op}}.$$

Here the first arrow is the functor from cobordisms to cospans in spaces which arises from considering a cobordism as a cospan between its boundary components. The middle arrow is induced from the functor $X^{(-)} : \mathrm{Spc}^{\mathrm{op}} \rightarrow \mathrm{Sch}$ which maps each homotopy type M to the (derived) scheme X^M parametrizing maps from M into X.

The benefit of the above construction is that it provides a fairly direct way of computing the value of χ on the circle (we refer to [BN13] for more on this theme). Indeed, $\chi(S^1)$ is the endofunctor of Vect given by the composition

$$\mathrm{Vect} = \mathrm{QCoh}(\mathrm{Spec}(k)) \xrightarrow{\pi^*} \mathrm{QCoh}(X^{S^1}) \xrightarrow{\pi_*} \mathrm{QCoh}(\mathrm{Spec}(k)) = \mathrm{Vect}$$

where $\pi : X^{S^1} \rightarrow \mathrm{Spec}(k)$ denotes the projection. This recovers the endofunctor of Vect given by tensoring with $\mathcal{O}(X^{S^1})$. We may summarize this by saying that the value of χ on the circle recovers the space of functions on X^{S^1}.

The scheme X^{S^1} is called the (derived) loop space of X. Breaking the circle symmetry, one may present S^1 as the suspension of S^0, which leads to the description of X^{S^1} as the fiber product $X \times_{X \times X} X$. In the case when $X = \operatorname{Spec}(A)$ is affine, we may further rewrite this as $\operatorname{Spec}(A \otimes_{A \otimes A} A)$. In other words, the space of functions on X^{S^1} is the chain complex computing the Hochschild homology of A. This connection with Hochschild invariants makes the derived loop space an object of central importance in derived algebraic geometry (see [TV09], [BN12], [Pre15]).

1.3 Higher quasicoherent sheaves

One of the main goals of this thesis is to present a version of the above story which allows one to produce topological field theories of dimension greater than 2. In order to do so, one must leave the realm of sheaves with values in vector spaces, and work with sheaves with values in higher categories - indeed, in order to produce an $(n+1)$-dimensional topological field theory we need to replace $\operatorname{QCoh}(X)$ with some kind of (∞, n)-category.

The first problem to be solved is to construct categorifications of $2\mathscr{V}\text{ect}$. We accomplish this in chapter 12, where we introduce for each $n \geq 2$ a symmetric monoidal $(\infty, n+1)$-category $(n+1)\mathscr{V}\text{ect}$ of k-linear presentable stable (∞, n)-categories. To a first approximation, we may think about $(n+1)\mathscr{V}\text{ect}$ as follows:

- An object in $(n+1)\mathscr{V}\text{ect}$ is an (∞, n)-category which admits all colimits, such that all its Hom $(\infty, n-1)$-categories also admit all colimits, and all the Hom $(\infty, n-2)$-categories of those admit all colimits, and so on. At the very last level, we require the ∞-categories that arise to admit all colimits, and to come equipped with a k-linear structure on their Hom spaces.

- Given two objects $\mathcal{C}, \mathcal{D}$ in $(n+1)\mathscr{V}\text{ect}$, morphisms from $\mathcal{C}$ to $\mathcal{D}$ are k-linear functors which preserve colimits at all levels.

- The unit of $(n+1)\mathscr{V}\text{ect}$ is given by $n\mathscr{V}\text{ect}$.

- Given two objects $\mathcal{C}, \mathcal{D}$, their tensor product $\mathcal{C} \otimes \mathcal{D}$ is the universal recipient of a k-bilinear functor $\mathcal{C} \times \mathcal{D} \to \mathcal{C} \otimes \mathcal{D}$ which preserves colimits (at all levels) in each coordinate.

Although the above description provides useful intuition, our definition of $(n+1)\mathscr{V}\text{ect}$ proceeds along somewhat different lines. Instead of defining $(n+1)\mathscr{V}\text{ect}$ directly as a subcategory of the $(\infty, n+1)$-category of k-linear (∞, n)-categories, we first define its underlying symmetric monoidal ∞-category $(n+1)\text{Vect}$, and then produce an $(\infty, n+1)$-categorical enhancement of $(n+1)\text{Vect}$.

The definition of $(n+1)\text{Vect}$ is inductive. Given a definition of $n\text{Vect}$, we define $(n+1)\text{Vect}$ to be a certain tame full subcategory of the ∞-category of cocomplete ∞-categories equipped with an action of $n\text{Vect}$ (we refer to the introduction of part II and to chapter 12 for an explanation of this tameness condition).

To construct the $(\infty, n+1)$-categorical enhancement $(n+1)\mathscr{V}\mathrm{ect}$ we make use of the theory of enriched ∞-categories developed in [GH15] and [Hin20a], which we review in chapter 3. This theory allows us to pass from ∞-categories tensored over a symmetric monoidal ∞-category $\mathcal{M}$, to ∞-categories enriched in $\mathcal{M}$. For our purposes, we need in fact a functorial strengthening of the procedure of enrichment of tensored ∞-categories, which we develop in chapter 4.

The theory of higher presentable categories not only serves to define $(n+1)\mathscr{V}\mathrm{ect}$. In fact, for each commutative k-(dg)-algebra A, one may define a commutative algebra object $A\text{-}\mathcal{m}\mathrm{od}^n$ in $(n+1)\mathscr{V}\mathrm{ect}$, which we think about as the k-linear (∞, n)-category of A-linear presentable stable $(\infty, n-1)$-categories. As before, in order to define $A\text{-}\mathcal{m}\mathrm{od}^n$, one first defines its underlying module $A\text{-}\mathrm{mod}^n$ over $n\mathrm{Vect}$. This is done inductively by setting $A\text{-}\mathrm{mod}^0 = A$, and $A\text{-}\mathrm{mod}^n = (A\text{-}\mathrm{mod}^{n-1})\text{-}\mathrm{mod}(n\mathrm{Vect})$.

This leads us to the notion of quasicoherent sheaf of higher categories on an affine scheme:

Definition 1.3.1. *Let $X = \mathrm{Spec}(A)$ be an affine scheme. We let $n\mathcal{Q}\mathrm{Coh}(X) = A\text{-}\mathcal{m}\mathrm{od}^n$. We call this the presentable stable (∞, n)-category of quasicoherent sheaves of presentable stable $(\infty, n-1)$-categories on X.*

Unpacking the definition, we see that $n\mathcal{Q}\mathrm{Coh}(X)$ corresponds to the data of an $n\mathrm{Vect}$-module $n\mathrm{QCoh}(X)$, which is in turn defined inductively starting with $\mathrm{QCoh}(X)$ by setting $n\mathrm{QCoh}(X)$ to be $(n-1)\mathrm{QCoh}(X)\text{-}\mathrm{mod}(n\mathrm{Vect})$.

Having the notion of a higher quasicoherent sheaf on an affine scheme, one may formally extend it to any prestack X by defining $n\mathcal{Q}\mathrm{Coh}(X)$ to be the limit of $n\mathcal{Q}\mathrm{Coh}(S)$ over all affine schemes S equipped with a map to X.

In the case $n = 2$, our theory reduces to the theory of sheaves of categories studied in [Gai15]. A concept of central importance in that case is the notion of 1-affineness. Roughly speaking, a prestack X is 1-affine if $2\mathrm{QCoh}(X)$ is equivalent to $\mathrm{QCoh}(X)\text{-}\mathrm{mod}(2\mathrm{Vect})$. This is true by definition for affine schemes, but it also holds for a number of prestacks of interest, including schemes (see [Gai15]).

In chapter 14 we introduce a notion of $(n-1)$-affineness for arbitrary $n \geq 1$. Roughly speaking, a prestack is said to be $(n-1)$-affine if there is an equivalence

$$n\mathcal{Q}\mathrm{Coh}(X) = (n-1)\mathcal{Q}\mathrm{Coh}(X)\text{-}\mathrm{mod}(n\mathscr{V}\mathrm{ect})$$

as objects in $(n+1)\mathscr{V}\mathrm{ect}$. To make sense of the above, we use the theory of monads and monadic morphisms in enriched $(\infty, 2)$-categories which we develop in chapter 7, which builds upon a theory of enriched higher algebra that we introduce in chapter 6.

The following theorem is our main result on the theory of higher affineness. It allows one to reduce questions of higher affineness to the case $n = 2$, which was studied in [Gai15].

Theorem 1.3.2. *Let X be a prestack, and let $n \geq 3$. Assume that the diagonal map $X \to X \times X$ is $(n-2)$-affine. Then X is $(n-1)$-affine.*

In particular, it is a consequence of theorem 1.3.2 that prestacks with schematic diagonal are $(n-1)$-affine for all $n \geq 3$.

The proof of theorem 1.3.2 is given in chapter 14. One of its ingredients is the following enriched generalization of the Barr-Beck-Lurie monadicity theorem, which we prove in chapter 7:

Theorem 1.3.3. *Let $\mathcal{M}$ be a presentable symmetric monoidal ∞-category and let $G : \mathcal{C} \to \mathcal{D}$ be a functor of $\mathcal{M}$-enriched ∞-categories. Then the following are equivalent:*

(i) *There exists a monad M on $\mathcal{D}$ and a structure of M-module on G, such that G presents $\mathcal{C}$ as the Eilenberg-Moore object of M.*

(ii) *For every $\mathcal{M}$-enriched ∞-category $\mathcal{E}$, the functor of ∞-categories underlying the functor of $\mathcal{M}$-enriched ∞-categories*

$$G_* : \mathrm{Funct}(\mathcal{E}, \mathcal{C}) \to \mathrm{Funct}(\mathcal{E}, \mathcal{D})$$

is monadic.

(iii) *The functor G admits a left adjoint, is conservative, and creates conical geometric realizations of G-split simplicial objects.*

To formulate the third condition in theorem 1.3.3 we use the theory of conical colimits in enriched ∞-categories which we develop in chapter 5, as part of a general study of adjunctions and weighted limits in enriched ∞-categories.

1.4 Higher dimensional field theories via higher sheaf theory

Our second main result concerning the theory of higher quasicoherent sheaves is the following generalization of theorem 1.2.1, which we prove in chapter 14:

Theorem 1.4.1. *The assignment $Y \mapsto n\mathscr{Q}\mathrm{Coh}(Y)$ forms part of a symmetric monoidal functor*

$$n\mathscr{Q}\mathrm{Coh} : (n+1)\mathrm{Corr}(\mathrm{Sch}) \to (n+1)\mathscr{V}\mathrm{ect}^{(n+1)\text{-op}}.$$

In the above theorem, $(n+1)\mathscr{V}\mathrm{ect}^{(n+1)\text{-op}}$ denotes the $(\infty, n+1)$-category obtained from $(n+1)\mathscr{V}\mathrm{ect}$ by reversing the orientation of the $(n+1)$-dimensional cells, and $(n+1)\mathrm{Corr}(\mathrm{Sch})$ denotes the $(\infty, n+1)$-category of correspondences of schemes, defined informally by induction as follows:

- Objects of $(n+1)\mathrm{Corr}(\mathrm{Sch})$ are schemes over k.
- Given two schemes Y, Z, we have an equivalence of (∞, n)-categories

$$\mathrm{Hom}_{(n+1)\mathrm{Corr}(\mathrm{Sch})}(Y, Z) = n\mathrm{Corr}(\mathrm{Sch}_{/Y,Z}).$$

Concretely, theorem 1.4.1 associates to each morphism $Y \leftarrow S \rightarrow Z$ in $(n+1)\mathrm{Corr}(\mathrm{Sch})$ the functor of $n\mathrm{Vect}$-modules $n\mathrm{QCoh}(Y) \rightarrow n\mathrm{QCoh}(Z)$ given by tensoring with the $(n-1)\mathrm{QCoh}(Y) - n\mathrm{QCoh}(Z)$-bimodule $(n-1)\mathrm{QCoh}(S)$. Furthermore, it associates to each 2-cell

$$\begin{tikzcd} & S \arrow[dl] \arrow[dr] & \\ Y & T \arrow[u, "p"] \arrow[d, "q"] & Z \\ & S' \arrow[ul] \arrow[ur] & \end{tikzcd}$$

the natural transformation of functors resulting from the morphism of $(n-1)\mathrm{QCoh}(Y) - (n-1)\mathrm{QCoh}(Z)$-bimodules

$$q_*p^* : (n-1)\mathrm{QCoh}(S) \rightarrow (n-1)\mathrm{QCoh}(S').$$

We may think about theorem 1.4.1 as encoding the functoriality of the theory of higher quasicoherent sheaves. As n grows, we have more and more functoriality available at our disposal, having to do with the ability to form Fourier-Mukai transforms, and Fourier-Mukai transforms between Fourier-Mukai kernels, and so on.

Given a scheme X, we may use theorem 1.4.1 to produce an $(n+1)$-dimensional topological field theory χ with target $(n+1)\mathscr{V}\mathrm{ect}$, by composing three symmetric monoidal functors:

$$n\mathrm{Cob} \rightarrow (n+1)\mathrm{Corr}(\mathrm{Spc}^{\mathrm{op}}) \xrightarrow{X^{(-)}} (n+1)\mathrm{Corr}(\mathrm{Sch}) \xrightarrow{n\mathscr{Q}\mathrm{Coh}} (n+1)\mathscr{V}\mathrm{ect}^{(n+1)\mathrm{bp}}.$$

Assuming some expected facts[2] about the relation between $n\mathrm{Cob}$ and $(n+1)\mathrm{Corr}(\mathrm{Spc}^{\mathrm{op}})$, one may use the above description of χ to show that its value on a k-dimensional manifold M can be identified with $(n-k)\mathscr{Q}\mathrm{Coh}(X^M)$. We may think about this as a generalization of the geometric description of the Hochschild invariants of a commutative algebra.

For many purposes, it is useful to have a generalization of theorem 1.4.1 which applies to prestacks, and not only schemes. It turns out that when passing to prestacks one loses some functoriality, so the general statement involves the (∞, n)-category of correspondences rather than the $(\infty, n+1)$-categorical version:

Theorem 1.4.2. *The assignment $Y \mapsto n\mathrm{QCoh}(Y)$ forms part of a symmetric monoidal functor*

$$n\mathscr{Q}\mathrm{Coh} : n\mathrm{Corr}(\mathrm{PreStk}) \rightarrow (n+1)\mathscr{V}\mathrm{ect}.$$

In higher categorical sheaf theory, the relationship between different categorical levels is arguably just as important as the property of each categorical level on its own. In order to

[2]The first arrow in this composition is defined by the property that it sends the point to the point. In order to use this description of χ for computation, one needs to know that the first arrow can also be recovered by interpreting cobordisms as cospans of spaces.

organize the functoriality of the theories $n\mathcal{Q}\mathrm{Coh}$ for different values of n, we use in this thesis the theory of categorical spectra[3], which we develop in chapter 13. In the same way that in stable homotopy theory a spectrum is defined to be a sequence of pointed homotopy types compatible under looping, a categorical spectrum is a sequence of pointed (∞,∞)-categories compatible under passage to endomorphisms of the basepoint.

It turns out that one may assemble the sequence of (∞,n)-categories $n\mathrm{Corr}(\mathrm{PreStk})$ into a categorical spectrum $\underline{\mathrm{Corr}}(\mathrm{PreStk})$, called the categorical spectrum of correspondences of prestacks. Similarly, one may assemble the $(\infty,n+1)$-categories $(n+1)\mathscr{V}\mathrm{ect}$ into a categorical spectrum of higher vector spaces. The entire functoriality of the theory of higher quasicoherent sheaves may be summarized by saying that it gives rise to a morphism of categorical spectra from $\underline{\mathrm{Corr}}(\mathrm{PreStk})$ to the categorical spectrum of higher vector spaces.

Continuing along these lines, one could say that the subject of higher categorical sheaf theory consists more generally of the study of the representation theory of the categorical spectrum of correspondences - the theory of higher quasicoherent sheaves being the most basic such representation.

The proofs of theorems 1.4.1 and 1.4.2 depend on a universal property for the higher categories of correspondences, which we establish in chapter 11:

Theorem 1.4.3. *Let $\mathcal{C}$ be an ∞-category admitting finite limits, and let $\mathcal{D}$ be a symmetric monoidal $(\infty,n+1)$-category. Then restriction along the inclusion $\mathcal{C}\to(n+1)\mathrm{Corr}(\mathcal{C})$ induces an equivalence between the space of symmetric monoidal functors $(n+1)\mathrm{Corr}(\mathcal{C})\to\mathcal{D}$ and the space of symmetric monoidal functors $\mathcal{C}\to\mathcal{D}$ satisfying the left n-fold Beck-Chevalley condition.*

The left n-fold Beck-Chevalley condition is a minimalistic list of base change properties that we introduce in chapter 11. In the same way that verifying the usual left Beck-Chevalley condition for a functor $F:\mathcal{C}\to\mathcal{D}$ involves checking an adjointability statement for every cartesian square in $\mathcal{C}$, verifying the left n-fold Beck-Chevalley condition involves checking a series of n different adjointability statements for every such cartesian square.

Our proof of theorem 1.4.3 is inductive, and builds upon the case $n=2$. In the case $n=2$, theorem 1.4.3 reduces to the universal property of the $(\infty,2)$-category of correspondences established in [GR17]. We provide in chapter 10 of this thesis a new approach to the proof of this basic case, which builds upon the description of Hom functors for enriched ∞-categories from [Hin20a], and the theory of two-sided fibrations of ∞-categories which we study in chapter 9.

1.5 Organization

This thesis consists of two parts. Below we provide a brief description of the contents of each part. We refer also to the introduction of each part for an expanded description of its

[3]We first learned about the notion of categorical spectrum from Constantin Teleman, under the name of anticategory.

contents.

Part I deals with the foundations of the theory of enriched ∞-categories. We begin in chapter 3 by presenting a systematic treatment of the basics of the theory, and include background on the theory of (∞, ∞)-categories, which will be used throughout the thesis. Chapter 4 studies the theory of modules over algebroids, and provides a functorial enhancement of the procedure of enrichment of presentable modules, which forms the basis of our construction of the theory of higher presentable categories. Chapter 5 studies the theory of adjunctions between enriched ∞-categories, and introduces a theory of weighted colimits. Chapter 6 introduces a theory of enriched ∞-props, and a new approach to enriched ∞-operads. Finally, chapter 7 studies the theories of monads an monadic morphisms in enriched $(\infty, 2)$-categories, and provides a proof of theorem 1.3.3.

Part II is concerned with the foundations of the higher categorical sheaf theory. We begin in chapter 9 by discussing the theory of two-sided fibrations, and proving a universal property for the two-sided fibration of correspondences. In chapter 10 we collect a number of basic results concerning the $(\infty, 2)$-category of correspondences, and present our proof of the case $n = 2$ of theorem 1.4.3, building on the material from chapter 9. In chapter 11 we study the problem of constructing functors out of higher categories of correspondences. We include here a proof of theorem 1.4.3, as well as an extension theorem which forms the basis of the passage from affine schemes to prestacks involved in theorem 1.4.2. In chapter 12 we introduce the theory of higher presentable categories, and show that it satisfies the conditions of the extension theorem from chapter 11. Chapter 13 deals with the theory of categorical spectra, and discusses its relationship with the theory of symmetric monoidal higher categories. Finally, in chapter 14 we introduce the theory of higher quasicoherent sheaves, and supply proofs of theorems 1.3.2, 1.4.1 and 1.4.2.

Conventions and notation

We use the language of higher category theory and higher algebra as developed in [Lur09a] and [Lur17]. All of our notions will be assumed to be homotopical or ∞-categorical, and we suppress this from our notation - for instance, we use the word n-category to mean (∞, n)-category.

We work with a nested sequence of universes. Objects belonging to the first universe are called small, objects in the second universe are called large, and objects in the third universe are called very large.

We denote by Spc and Cat the categories of (small) spaces and categories. For each $n \geq 2$ we denote by $n\mathrm{Cat}$ the category of small n-categories. We denote by $\mathscr{C}at$ the 2-categorical enhancement of Cat, and in general by $n\mathscr{C}at$ the $(n+1)$-categorical enhancement of $n\mathrm{Cat}$. If X is one of those objects (or related), we denote by $\widehat{X}$ its large variant. For instance, $\widehat{\mathrm{Spc}}$ denotes the category of large spaces.

We denote by Pr^L the category of presentable categories and colimit preserving functors. We usually consider this as a symmetric monoidal category, with the tensor product constructed in [Lur17] section 4.8. By presentable (symmetric) monoidal category we mean a (commutative) algebra in Pr^L. In other words, this is a presentable category equipped with a (symmetric) monoidal structure compatible with colimits.

If $\mathcal{C}$ is an n-category and $k \geq 0$, we denote by $\mathcal{C}^{\leq k}$ the k-category obtained from $\mathcal{C}$ by discarding all cells of dimension greater than k, and by ${}^{\leq k}\mathcal{C}$ the k-category obtained from $\mathcal{C}$ by inverting all cells of dimension greater than k. In particular, for each category $\mathcal{C}$ we denote by $\mathcal{C}^{\leq 0}$ the space of objects of $\mathcal{C}$.

For each category $\mathcal{C}$ we denote by $\mathrm{Hom}_{\mathcal{C}}(-,-)$ the Hom-functor of $\mathcal{C}$. We denote by $\mathcal{P}(\mathcal{C})$ its presheaf category. We usually identify $\mathcal{C}$ with its image under the Yoneda embedding.

Given a pair of n-categories $\mathcal{C}, \mathcal{D}$ we denote by $\mathrm{Funct}(\mathcal{C}, \mathcal{D})$ the n-category of functors between $\mathcal{C}$ and $\mathcal{D}$. When we wish to only consider the space of functors between them we will use the notation $\mathrm{Hom}_{n\mathrm{Cat}}(\mathcal{C}, \mathcal{D})$ instead.

Given a right (resp. left) adjointable functor of categories $\beta : \mathcal{C} \to \mathcal{D}$, we will usually denote by β^R (resp. β^L) its right (resp. left) adjoint. More generally, we use a similar notation for adjoint morphisms in a 2-category. We say that a commutative square of categories

$$\begin{tikzcd} \mathcal{C}' \arrow[r, "\alpha'"] \arrow[d, "\beta'"] & \mathcal{C} \arrow[d, "\beta"] \\ \mathcal{D}' \arrow[r, "\alpha"] & \mathcal{D} \end{tikzcd}$$

is vertically right adjointable if β and β' admit right adjoints and the induced natural transformation $\alpha'\beta'^R \to \beta^R\alpha$ is an isomorphism. Similarly, we can talk about horizontal left adjointability, or vertical left / right adjointability. We will at various points use the connections between adjointability of squares and the theory of two-sided fibrations which we develop in chapter 9.

We make use at various points of the theory of operads. We use a language for speaking about these which is close in spirit to the classical language in terms of objects and operations which satisfy a composition rule. Namely, given an operad $\mathcal{O}$ with associated category of operators $p : \mathcal{O}^{\otimes} \to \mathrm{Fin}_*$, we call $p^{-1}(\langle 1\rangle)$ the category of objects of $\mathcal{O}$, and arrows in $\mathcal{O}^{\otimes}$ lying above an active arrow of the form $\langle n\rangle \to \langle 1\rangle$ are called operations of $\mathcal{O}$. We will for the most part work with $\mathcal{O}$ without making explicit reference to the fibration p, and make it clear when we need to refer to the category of operators instead.

We denote by Op the category of operads, and for each operad $\mathcal{O}$ we denote by $\mathrm{Op}_{\mathcal{O}}$ the category of $\mathcal{O}$-operads. We denote by $\mathrm{Assos}, \mathrm{LM}, \mathrm{BM}$ the operads for associative algebras, left modules, and bimodules, respectively.

Part I

Enriched ∞-category theory

Chapter 2

Introduction to part I

The theory of enriched ∞-categories, introduced in [GH15] and independently in [Hin20a], provides a unified approach to various important notions in higher category theory. As particular cases, it recovers the notions of (∞, n)-categories, spectral categories, and dg-categories. The first goal of part I is to present a roughly complete treatment of the basics of the subject, adapted to our needs. We include also an introduction to the theory of (∞, ∞)-categories from the point of view of iterated enrichment.

The second goal of part I is to make a number of contributions to the state of the art in enriched ∞-category theory, including:

- We provide an alternative approach to the definition of the operad which corepresents enrichment, which we show to be equivalent to previous approaches in the literature.
- We study the functoriality properties of the procedure of enrichment of presentable modules from [GH15] and [Hin20a].
- We study the theory of adjunctions and weighted limits and colimits in enriched ∞-categories, and show that an enriched ∞-category admits all weighted colimits if and only if it admits all conical colimits and copowers.
- We study higher algebra in the enriched context, and introduce a new approach to the theory of enriched ∞-operads, via enriched ∞-props.
- We study the theory of monads in the setting of enriched $(\infty, 2)$-categories, and prove an enriched generalization of the Barr-Beck-Lurie monadicity theorem, which provides a description of monadic functors of enriched ∞-categories.

The material in chapters 3, 4 and 5 is an expansion of the author's preprint [Ste20b] (except for its last section which is present in part II of this thesis as chapter 12).

Below we provide a more detailed description of the contents of part I. As usual in this thesis, we will use the convention where all objects are ∞-categorical by default, and suppress this from our notation from now on.

2.1 The notion of an enriched category

Let $\mathcal{M}$ be a monoidal category. Roughly speaking, an algebroid $\mathcal{A}$ in $\mathcal{M}$ with space of objects X consists of

- For every pair of objects x, y in X an object $\mathcal{A}(y, x)$ in $\mathcal{M}$.
- For every object x in X a morphism $1_{\mathcal{M}} \to \mathcal{A}(x, x)$.
- For every triple of objects x, y, z a morphism $\mathcal{A}(z, y) \otimes \mathcal{A}(y, x) \to \mathcal{A}(z, x)$.
- Associativity and unit isomorphisms, and an infinite list of higher coherence data.

Given an algebroid $\mathcal{A}$ with space of objects X, there is a Segal space underlying $\mathcal{A}$, with space of objects X and for each pair of objects y, x the space of morphism being given by $\mathrm{Hom}_{\mathcal{M}}(1_{\mathcal{M}}, \mathcal{A}(y, x))$. We say that $\mathcal{A}$ is an $\mathcal{M}$-enriched category if its underlying Segal space is complete.

The theory of enriched categories provides a unified approach to various different notions in category theory:

- In the case when $\mathcal{M} = \mathrm{Spc}$ is the cartesian symmetric monoidal category of spaces, an $\mathcal{M}$-enriched category is simply a category.
- In the case when $\mathcal{M} = n\mathrm{Cat}$ is the cartesian symmetric monoidal category of n-categories, an $\mathcal{M}$-enriched category is the same as an $(n+1)$-category.
- In the case when $\mathcal{M} = \mathrm{Sp}$ is the category of spectra with its smash symmetric monoidal structure, we obtain a notion of spectrally enriched category. Stable categories are examples of these.
- In the case when $\mathcal{M} = \mathrm{Vect}_k$ is the category of k-module spectra over a field k, we obtain a notion of k-linear category. This provides a robust approach to the theory of dg-categories, which is native to the ∞-categorical world (we refer to [Hau15] for the comparison with dg-categories).

It is worth noting that the theory of enriched categories not only unifies the above notions, but it provides a streamlined way of relating them: given a symmetric monoidal functor $F : \mathcal{M} \to \mathcal{M}'$ and an $\mathcal{M}$-enriched category $\mathcal{A}$, there is an $\mathcal{M}$-enriched category $F_!\mathcal{A}$ with the same space of objects, and such that $(F_!\mathcal{A})(y, x) = F(\mathcal{A}(y, x))$ for every pair of objects x, y. For instance, this allows one to obtain a spectrally enriched category from an (unenriched) category, by passing to free spectra Hom-wise.

The theory of $\mathcal{M}$-enriched algebroids and categories was introduced in [GH15], and an alternative approach was provided in [Hin20a]. The definition of $\mathcal{M}$-enriched algebroid is an instance of the general strategy of corepresenting higher structures. For each space X the assignment $\mathcal{M} \mapsto \mathrm{Algbrd}_X(\mathcal{M})$ that sends each monoidal category $\mathcal{M}$ to the category of algebroids with space of objects X turns out to be corepresented by a nonsymmetric operad Assos_X, with the following properties:

- The space of objects of Assos_X is $X \times X$.
- Given $n \geq 0$ and a sequence of objects $\{x_i\}_{0\leq i\leq n}$ of X, there is an operation with source $\{(x_i, x_{i+1})\}_{0\leq i\leq n-1}$ and target (x_0, x_n).

In the case when $X = [0]$, the operad Assos_X is equivalent to the associative operad - this reflects the fact that an algebroid with one object is the same as an associative algebra. In general, we think about Assos_X as a many object version of the associative operad.

The approaches to the definition of Assos_X from [GH15] and [Hin20a] are somewhat different, although they both produce equivalent operads, as explained in [Mac21]. In chapter 3 we provide yet another (equivalent) approach to the definition of Assos_X, inspired by that of [Hin20a], but somewhat different in its implementation. We show that the assignment $X \mapsto \mathrm{Assos}_X$ is corepresented by an associative cooperad $\mathfrak{C}$ internal to Cat, which is in turn determined by its underlying associative cooperad $\mathfrak{C}^{\mathrm{cl}}$ internal to 0-truncated categories. The cooperad $\mathfrak{C}^{\mathrm{cl}}$ is classical and it can therefore be defined by specifying a finite amount of data. It is in fact uniquely characterized by a surprisingly small amount of data: namely the 0-truncated categories of objects and operations, and the cosource and cotarget maps. This observation allows us to compare our definition to previous approaches in the literature. We also use similar arguments to provide concise definitions of the operads LM_X and BM_X that play a key role in [Hin20a].

The rest of chapter 3 is devoted to presenting the basics of the theory of enriched ∞-categories. For ease of reference, we include here some results which appear previously in [GH15] and [Hin20a].

We finish chapter 3 with an introduction to the theory of (∞, ∞)-categories, which we call ω-categories. Although for most of our purposes the ω-categories that we will encounter will be n-categories for some finite n, we consider the notion of ω-category to provide a convenient framework for doing higher category theory in cases where the exact bounds on the dimensions of the cells are irrelevant.

2.2 Enrichment of presentable modules

Let $\mathcal{M}$ be a presentable symmetric monoidal category and let $\mathcal{C}$ be a presentable module over $\mathcal{M}$. Given a pair of objects c, c' in $\mathcal{C}$, there is a Hom object $\mathscr{H}om_{\mathcal{C}}(c, c')$ in $\mathcal{M}$, defined by the property that it represents the presheaf $m \mapsto \mathrm{Hom}_{\mathcal{C}}(m \otimes c, c')$.

It was shown in [GH15] and [Hin20a] that one may in fact associate to $\mathcal{C}$ and $\mathcal{M}$-enriched category $\overline{\mathcal{C}}$ with the same space of objects as $\mathcal{C}$, and having the property that $\mathrm{Hom}_{\overline{\mathcal{C}}}(c, c') = \mathscr{H}om_{\mathcal{C}}(c, c')$ for every pair of objects c, c' in $\mathcal{C}$. This is a key construction in the theory of enriched categories. For instance, specializing to the case $\mathcal{C} = \mathcal{M}$, this allows one to obtain an $\mathcal{M}$-enriched category $\overline{\mathcal{M}}$ enhancing the category $\mathcal{M}$.

The main result of chapter 4 provides a functorial enrichment of the assignment $\mathcal{C} \mapsto \overline{\mathcal{C}}$.[1] We may informally summarize it as follows:

Theorem 2.2.1. *Let $\mathcal{M}$ be a presentable symmetric monoidal category, and denote by $\widehat{\mathrm{Cat}}^{\mathcal{M}}$ the category of large $\mathcal{M}$-enriched categories. Then there is a lax symmetric monoidal functor*

$$\theta_{\mathcal{M}} : \mathcal{M}\text{-mod}(\mathrm{Pr}^L) \to \widehat{\mathrm{Cat}}^{\mathcal{M}}$$

with the following properties:

(i) *The composition of $\theta_{\mathcal{M}}$ with the lax symmetric monoidal forgetful functor $\widehat{\mathrm{Cat}}^{\mathcal{M}} \to \widehat{\mathrm{Cat}}$ recovers the usual lax symmetric monoidal forgetful functor $\mathcal{M}\text{-mod}(\mathrm{Pr}^L) \to \widehat{\mathrm{Cat}}$*

(ii) *For every presentable module $\mathcal{C}$ over $\mathcal{M}$, we have an equivalence $\theta_{\mathcal{M}}(\mathcal{C}) = \overline{\mathcal{C}}$.*

This functoriality will be a key ingredient in chapter 12 when we construct the theories of presentable n-categories. As a more basic consequence, we mention the following:

Corollary 2.2.2. *Let $\mathcal{M}$ be a presentable symmetric monoidal category. Then there is a canonical symmetric monoidal structure on $\overline{\mathcal{M}}$ which recovers upon passage to underlying categories the original symmetric monoidal structure on $\mathcal{M}$.*

2.3 Adjunctions and weighted limits

In chapter 5 we generalize the theory of adjunctions and limits to the enriched context. The concept of adjunction between enriched categories behaves in a similar fashion as its unenriched counterpart: given a pair of functors of $\mathcal{M}$-enriched categories $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$, a natural transformation $\eta : 1_{\mathcal{C}} \to GF$ is said to present G as right adjoint to F if for each pair of objects c in $\mathcal{C}$ and d in $\mathcal{D}$, we have an induced equivalence

$$\mathrm{Hom}_{\mathcal{D}}(F(c), d) = \mathrm{Hom}_{\mathcal{C}}(c, G(d))$$

as objects in $\mathcal{M}$. We note that this condition is strictly stronger than the condition that η be the unit of an adjunction between the categories underlying $\mathcal{C}$ and $\mathcal{D}$.

The theory of limits has a somewhat different character in the enriched world. While in unenriched category theory limits for a diagram $F : \mathcal{I} \to \mathcal{C}$ are particular extensions of F to the category obtained from $\mathcal{I}$ by adjoining an initial object, in enriched category theory there is a greater variety of kinds of extensions that one may consider: for every copresheaf W on $\mathcal{I}$, there is an associated cone $\mathcal{I}_W^{\triangleleft}$, and the problem of extending F to $\mathcal{I}_W^{\triangleleft}$ leads to the notion of W-weighted limit.[2]

[1] While finishing this work we learned about recent work of H. Heine [Hei20] which yields a similar functorial strengthening of this procedure.

[2] We refer also to [Hin21] for a discussion of the related concept of weighted colimits for left modules over enriched categories. Conjecturally, these two notions should be related via the procedure of enrichment of presentable modules.

The simplest cases of weighted limits have their own name. In the case when $\mathcal{I}$ is induced from an unenriched category and the weight W is the constant functor with value $1_{\mathcal{M}}$, one speaks about conical limits. This is the kind of limit that one usually works with in unenriched category theory. The existence of conical limits in an enriched category is often used in combination with the following result, which is a basic consequence of the stability results for conical limits which we prove in chapter 5:

Theorem 2.3.1. *Let $\mathcal{M}$ be a presentable symmetric monoidal category. Let $\mathcal{I}$ be a category and denote by $\mathcal{I}_{\mathcal{M}}$ the induced $\mathcal{M}$-enriched category. Let $\mathcal{D}$ be an $\mathcal{M}$-enriched category admitting all conical limits and let $f : \mathcal{J} \to \mathcal{J}'$ be an epimorphism of $\mathcal{M}$-enriched categories. Let $X : \mathcal{I}_{\mathcal{M}} \to \mathrm{Funct}(\mathcal{J}, \mathcal{D})$ be a functor. Then:*

(i) *The diagram X admits a conical limit $X^{\lhd}$ which is preserved by the evaluation functors.*

(ii) *If X factors through $\mathrm{Funct}(\mathcal{J}', \mathcal{D})$ then $X^{\lhd}$ also factors through $\mathrm{Funct}(\mathcal{J}', \mathcal{D})$.*

A case of fundamental importance is when $\mathcal{M} = \mathrm{Cat}$ and $f : \mathcal{J} \to \mathcal{J}'$ is the inclusion of the universal right adjointable arrow in the universal adjunction Adj. In this case theorem 2.3.1 reduces to the assertion that a limit of adjointable arrows in a 2-category with conical limits is also adjointable, provided that certain base-change conditions are met.

Besides conical limits, another instance of weighted limits of particular importance occurs in the case when $\mathcal{I}$ is the unit $\mathcal{M}$-enriched category, but the copresheaf W is arbitrary. In this case, a weighted limit is called a power. Our main result concerning the theory of weighted limits guarantees that one may in fact reduce many questions about weighted limits to the case of conical limits and powers.

Theorem 2.3.2. *Let $\mathcal{M}$ be a presentable symmetric monoidal category, and let $\mathcal{C}$ be an $\mathcal{M}$-enriched category. Then $\mathcal{C}$ admits all weighted limits if and only if it admits all conical limits and powers. In this case, a functor of $\mathcal{M}$-enriched categories $G : \mathcal{C} \to \mathcal{D}$ preserves all weighted limits if and only if it preserves all conical limits and powers.*

As a consequence, we are able to conclude that the enriched categories underlying presentable modules admit all weighted limits and colimits:

Corollary 2.3.3. *Let $\mathcal{M}$ be a presentable symmetric monoidal category and let $\mathcal{C}$ be a presentable $\mathcal{M}$-module. Then the $\mathcal{M}$-enriched category $\theta_{\mathcal{M}}(\mathcal{C})$ admits all weighted limits and colimits.*

The theory of enriched adjunctions and weighted limits is used in the rest of the thesis in a fundamental way:

- Theorem 2.3.2 plays a role in our study of monads in chapter 7, where it is used show that the 2-category of $\mathcal{M}$-enriched categories admits all Eilenberg-Moore objects for monads. As we shall see, Eilenberg-Moore objects are special kinds of weighted limits, which are neither conical limits nor powers.

- The extension theorem for functors out of higher categories of correspondences from chapter 11 requires the existence of conical colimits in the target n-category - the role of this hypothesis is in fact mediated by theorem 2.3.1. This is then used in chapter 14 in our study of higher quasicoherent sheaves, where we use the fact, established in chapter 12, that higher presentable categories admit all conical colimits (and many conical limits).

2.4 Enriched higher algebra

Let $\mathcal{M}$ be a symmetric monoidal category. An $\mathcal{M}$-enriched pre-prop $\mathcal{P}$ consists of:

- A space of objects P.
- For every pair $\{x_s\}_{s\in S}$, $\{y_t\}_{t\in T}$ of finite families of elements of P, an object
$$\mathrm{Hom}_{\mathcal{P}}(\{x_s\}_{s\in S}, \{y_t\}_{t\in T})$$
in $\mathcal{M}$ of operations in $\mathcal{P}$ with source $\{x_s\}_{s\in S}$ and target $\{y_t\}_{t\in T}$.
- For every triple $\{x_s\}_{s\in S}$, $\{y_t\}_{t\in T}$, $\{z_u\}_{u\in U}$ of finite families of elements of P, a composition map
$$\mathrm{Hom}_{\mathcal{P}}(\{x_s\}_{s\in S}, \{y_t\}_{t\in T}) \otimes \mathrm{Hom}_{\mathcal{P}}(\{y_t\}_{t\in T}, \{z_u\}_{u\in U}) \to \mathrm{Hom}_{\mathcal{P}}(\{x_s\}_{s\in S}, \{z_u\}_{u\in U})$$
- For every object x in P, a unit map $1_{\mathcal{M}} \to \mathrm{Hom}_{\mathcal{P}}(x, x)$.
- For every quadruple $X = \{x_s\}_{s\in S}$, $Y = \{y_t\}_{t\in T}$, $Z = \{z_u\}_{u\in U}$, $W = \{w_v\}_{v\in V}$ of finite families of elements of P, a stacking map
$$\mathrm{Hom}_{\mathcal{P}}(X, Y) \otimes \mathrm{Hom}_{\mathcal{P}}(Z, W) \to \mathrm{Hom}_{\mathcal{P}}(X \cup Z, Y \cup W)$$
- Isomorphisms witnessing unitality and associativity of composition, compatibility with stacking, and an infinite family of higher coherence data.

An $\mathcal{M}$-enriched pre-prop $\mathcal{P}$ has an underlying $\mathcal{M}$-enriched algebroid, whose morphisms are operations in $\mathcal{P}$ with single source and target. We say that $\mathcal{P}$ is an $\mathcal{M}$-enriched prop if its underlying $\mathcal{M}$-enriched algebroid is an $\mathcal{M}$-enriched category. We say that $\mathcal{P}$ is an $\mathcal{M}$-enriched operad if it is an $\mathcal{M}$-enriched prop satisfying an extra condition, which roughly speaking states that arbitrary operations are determined by single target operations.

The theory of $\mathcal{M}$-enriched props and operads is fundamental in the study of $\mathcal{M}$-enriched structures admitting operations with multiple sources and targets. In chapter 6 we present one way of making the above definitions precise.

Our approach is similar in spirit to the approach to the theory of (unenriched) operads developed in [Lur17]. We may summarize the latter by saying that it studies operads $\mathcal{O}$ by means of their categories of operators. The key insight is that, in the classical context, an $(1,1)$-operad can be recovered from its category of operators, as a $(1,1)$-category over Fin_* satisfying a number of properties. One then defines an operad to be a category over Fin_* satisfying analogous conditions. This has the advantage of reducing the study of operads to questions in category theory, and furthermore allowing direct access to various constructions of interest.

In our case, when $\mathcal{M}$ is not cartesian there is not a good notion of categories of operations: classically, the category of operators of an $(1,1)$-operad is the free semicartesian monoidal category on it, and this is a notion that only makes sense in the cartesian context. Instead, we will access $\mathcal{M}$-enriched operads by means of their enveloping symmetric monoidal $\mathcal{M}$-enriched algebroids, which are simply defined as commutative algebras in the symmetric monoidal category $\mathrm{Algbrd}(\mathcal{M})_{\mathrm{Spc}}$. More precisely, we will define $\mathcal{M}$-enriched operads and props as commutative algebra objects in $\mathrm{Algbrd}(\mathcal{M})_{\mathrm{Spc}}$ equipped with a subspace of their space of objects, subject to a number of conditions.

Our approach differs from the previous approach to enriched operads from [CH20], where enriched operads are defined as objects of a localization of a certain category of presheaves. Our methods have the benefit of making the relations between enriched operads, props and symmetric monoidal categories explicit, allowing one to reduce questions about $\mathcal{M}$-enriched operads to questions about symmetric monoidal $\mathcal{M}$-enriched algebroids.

In our approach, we are able to show that the category of symmetric monoidal $\mathcal{M}$-enriched categories is equivalent to a subcategory of the category of $\mathcal{M}$-enriched operads. We can thus think about symmetric monoidal $\mathcal{M}$-enriched categories as being $\mathcal{M}$-enriched operads satisfying a certain representability condition. This gives access to a robust notion of lax symmetric monoidal functors in the enriched setting. Furthermore, the inclusion of symmetric monoidal $\mathcal{M}$-enriched categories into $\mathcal{M}$-enriched operads admits a left adjoint, which we can think about as sending each $\mathcal{M}$-enriched operad to its enveloping symmetric monoidal $\mathcal{M}$-enriched category.

2.5 Monadicity

The notions of monads and monadic functors are fundamental in category theory. Of central importance is the monadicity theorem:

Theorem 2.5.1 ([Lur17] theorem 4.7.3.5)**.** *Let $G : \mathcal{D} \to \mathcal{C}$ be a functor of categories. The following conditions are equivalent:*

(i) *The functor G is monadic: in other words, G admits a left adjoint F, and G is equivalent to the forgetful functor $\mathrm{LMod}_A(\mathcal{C}) \to \mathcal{C}$ for A the endomorphism monad of G.*

(ii) *There exists an algebra A in the monoidal category of endofunctors of $\mathcal{C}$ such that G is equivalent to the forgetful functor $\mathrm{LMod}_A(\mathcal{C}) \to \mathcal{C}$.*

(iii) *The functor G is conservative and creates geometric realizations of G-split simplicial objects.*

The notion of monadic functor only depends on the 2-categorical structure of $\mathscr{C}at$. We can therefore think about theorem 2.5.1 (in particular, the equivalence between the first two and the last item) as providing a characterization of monadic morphisms in $\mathscr{C}at$.

In chapter 7 we extend the theory of monads and monadic morphisms to (possibly enriched) 2-categories. The main result of this chapter is the following enriched generalization of theorem 2.5.1, which provides a characterization of monadic morphism in the 2-category of categories enriched over a presentable symmetric monoidal category.[3]

Theorem 2.5.2. *Let $\mathcal{M}$ be a presentable symmetric monoidal category and let $G : \mathcal{C} \to \mathcal{D}$ be a functor of $\mathcal{M}$-enriched categories. Then the following are equivalent:*

(i) *There exists a monad M on $\mathcal{D}$ and a structure of M-module on G, such that G presents $\mathcal{C}$ as the Eilenberg-Moore object of M.*

(ii) *For every $\mathcal{M}$-enriched category $\mathcal{E}$, the functor of categories underlying the functor of $\mathcal{M}$-enriched categories*
$$G_* : \mathrm{Funct}(\mathcal{E}, \mathcal{C}) \to \mathrm{Funct}(\mathcal{E}, \mathcal{D})$$
is monadic.

(iii) *The functor G admits a left adjoint, is conservative, and creates conical geometric realizations of G-split simplicial objects.*

2.6 Organization

We now describe the contents of part I in more detail. We refer the reader also to the introduction of each chapter for an expanded outline of its contents.

Chapter 3 is a general introduction to enriched category theory. We begin by describing our approach to the definition of the operad Assos_X, and the closely related operads LM_X and $\mathrm{BM}_{X,Y}$. We then review the notions of algebroids and enriched categories, and the general functoriality and multiplicativity properties of the theory. We finish this section with an introduction to the theory of n-categories and ω-categories via iterated enrichment.

Chapter 4 deals with the theory of left modules and bimodules over algebroids. The bulk of this section is devoted to the construction of a functorial enhancement of the procedure of enrichment of modules over presentable categories. We also outline here the construction of the Yoneda embedding via diagonal bimodules, following [Hin20a].

In chapter 5 we study the theory of adjunctions between enriched categories, and weighted limits and colimits in enriched categories. We introduce the notion of local right adjoint to a functor between enriched categories, and show that a right adjoint exists if and only if all local adjoints exist. We also establish stability results for adjunctions under limits and passage to functor categories. We then use the theory of adjunctions to study the theory of

[3] We refer to [Dub70] for a discussion of the monadicity theorem in the setting of classical enriched category theory.

weighted limits and colimits, and the important special case of conical limits and colimits. We finish this chapter by proving theorem 2.3.2, which we use to show that enriched categories that arise from presentable modules admit all weighted limits and colimits.

In chapter 6 we discuss a number of topics concerning higher algebra in the enriched setting. We discuss the theory of cartesian symmetric monoidal enriched categories, and show that it is equivalent to the theory of symmetric monoidal enriched categories with finite products. We introduce 2-categories of $\mathcal{O}$-monoidal enriched categories for any operad $\mathcal{O}$, and discuss their enrichment in the case when $\mathcal{M}$ is cartesian. We finish this chapter by introducing a theory of enriched props and operads.

In chapter 7 we study the theory of monads and monadic morphisms in enriched 2-categories. We provide here a proof of theorem 2.5.2. Specializing to the case $\mathcal{M} = \omega\mathrm{Cat}$, we recover notions of monads and monadic morphisms in arbitrary ω-categories.

Chapter 3

Enriched category theory

Let $\mathcal{M}$ be a monoidal category. An algebroid $\mathcal{A}$ in $\mathcal{M}$ with space of objects X consists of

- For every pair of objects x, y an object $\mathcal{A}(y, x)$ in $\mathcal{M}$.
- For every object x in X a morphism $1_{\mathcal{M}} \to \mathcal{A}(x, x)$.
- For every triple of objects x, y, z a morphism $\mathcal{A}(z, y) \otimes \mathcal{A}(y, x) \to \mathcal{A}(z, x)$.
- Associativity and unit isomorphisms, and an infinite list of higher coherence data.

Given an algebroid $\mathcal{A}$ with space of objects X, there is a Segal space underlying $\mathcal{A}$, with space of objects X and for each pair of objects y, x the space of morphism being given by $\mathrm{Hom}_{\mathcal{M}}(1_{\mathcal{M}}, \mathcal{A}(y, x))$. We say that $\mathcal{A}$ is an $\mathcal{M}$-enriched category if its underlying Segal space is complete. Our goal in this chapter is to review the theory of algebroids and enriched categories, and the approach to n-category theory via iterated enrichment.

For each space X the assignment $\mathcal{M} \mapsto \mathrm{Algbrd}_X(\mathcal{M})$ that sends each monoidal category $\mathcal{M}$ to the category of algebroids with space of objects X turns out to be corepresented by a nonsymmetric operad Assos_X, to be thought of as a many object version of the associative operad. The assignment $X \mapsto \mathrm{Assos}_X$ determines a functor from spaces to the category of associative operads, which is in turn corepresented by an associative cooperad $\mathfrak{C}$ internal to the category Cat.

We begin in 3.1 by reviewing the notion of internal operads and cooperads, and presenting the definition of the cooperad $\mathfrak{C}$. This is defined starting from a cooperad $\mathfrak{C}^{\mathrm{cl}}$ in the (classical) category of posets, which can be specified by a finite amount of data, namely the posets of objects and operations, with source, target, unit, and composition maps. We show that $\mathfrak{C}^{\mathrm{cl}}$ is in fact uniquely determined from its categories of objects and operations, together with source and target maps - this uniqueness criterion allows us later on to compare our approach to enrichment with other approaches in the literature.

In 3.2 we give the definition of the associative operad Assos_X for an arbitrary category X. Although for the purposes of enriched category theory the category X will always be a space, we will use this extra generality later on to give a direct description of the equivalence between Cat and the category of categories enriched in Spc. We also give definitions of the related operads LM_X and $\mathrm{BM}_{X,Y}$ which corepresent left modules and bimodules.

In 3.3 we review the definition and functoriality of the category of algebroids $\mathrm{Algbrd}(\mathcal{M})$ in an associative operad $\mathcal{M}$. We pay special attention to the case when $\mathcal{M}$ is a presentable monoidal category - in this case we have that $\mathrm{Algbrd}(\mathcal{M})$ is also presentable. We introduce two basic examples of algebroids: the trivial algebroid, and the cells - together these generate $\mathrm{Algbrd}(\mathcal{M})$.

In 3.4 we review the case of Spc-algebroids with a space of objects, and its equivalence with the category of Segal spaces. We then define the category of $\mathcal{M}$-enriched categories $\mathrm{Cat}^{\mathcal{M}}$ as the full subcategory of $\mathrm{Algbrd}(\mathcal{M})$ on those algebroids with a space of objects and whose underlying Segal space is complete and reprove the basic fact that if $\mathcal{M}$ is presentable monoidal then $\mathrm{Cat}^{\mathcal{M}}$ is an accessible localization of $\mathrm{Algbrd}(\mathcal{M})$.

In 3.5 we discuss the canonical symmetric monoidal structure in the category of algebroids over a symmetric monoidal category. In the presentable setting, this gives access in particular to a notion of functor algebroids and functor enriched categories. We prove here a basic result describing Hom objects in functor algebroids when the source algebroid is a cell, which will later on be used as a starting point for establishing various facts about general functor algebroids. We finish by studying the behavior of functor algebroids as we change the enriching category.

In 3.6 we review the approach to n-categories as categories enriched in $(n-1)$-categories. We discuss the various functors relating the categories $n\mathrm{Cat}$ for different values of n. In the limit as n tends to infinity we recover the category $\omega\mathrm{Cat}$ of ω-categories. Although for our purposes all of the ω-categories we will encounter will be n-categories for some finite n, the theory of ω-categories provides a convenient setting in which to work with n-categories in cases where the exact value of n is irrelevant or may vary. We show that the theory of ω-category is in fact a fixed point under enrichment: there is an equivalence between $\omega\mathrm{Cat}$ and the category of categories enriched in $\omega\mathrm{Cat}$.

Remark 3.0.1. The theory of algebroids and enriched categories was introduced in [GH15] and [Hin20a]. In this thesis we introduce a new approach to the subject based on the internal cooperad $\mathfrak{C}$, and show that this approach arrives at the same theory as that from [Hin20a].

Some of the basic facts about algebroids and enriched categories that we discuss in 3.3-3.5 (for instance, claims about presentability, existence of symmetric monoidal structures, functoriality of the theory) appear already in some way in the references. We chose to include statements and proofs of most of those facts for completeness and ease of reference, as our notation and conventions differ from other sources.

Another reason why we opted for a systematic treatment of the subject is that in many cases we in fact need tools that go beyond those which appear in the literature. For instance, in 3.5 we show that the category of algebroids over a symmetric operad admits the structure of a symmetric operad, and that this structure is functorial under morphisms of operads - this functoriality will be necessary in chapter 12 to construct the realization functor ψ_n. We also pay special attention throughout the chapter to enriched cells. We are able to obtain a good understanding of products and functor categories for cells, which is a basic building block for proving results for arbitrary functor categories later on. In particular, this will be crucial in chapter 5 when we discuss adjunctions between functor enriched categories.

3.1 The internal cooperad $\mathfrak{C}$

We begin with a general discussion of the procedure of internalization of objects of arbitrary presentable categories.

Definition 3.1.1. *Let $\mathcal{D}$ be a presentable category and let $\mathcal{C}$ be a complete category. A $\mathcal{D}$-object internal to $\mathcal{C}$ is a limit preserving functor $F : \mathcal{D}^{\mathrm{op}} \to \mathcal{C}$. We denote by $\mathcal{D}(\mathcal{C})$ the full subcategory of* $\mathrm{Funct}(\mathcal{D}^{\mathrm{op}}, \mathcal{C})$ *on the internal $\mathcal{D}$-objects.*

Example 3.1.2. Let $\mathcal{D}$ be a presentable category. Then it follows from [Lur09a] proposition 5.5.2.2 that $\mathcal{D}(\mathrm{Spc})$ is equivalent to $\mathcal{D}$.

Example 3.1.3. Let $\mathcal{C}$ be a complete category. Then $\mathrm{Spc}(\mathcal{C})$ is equivalent to $\mathcal{C}$. More generally, if $\mathcal{D}'$ is a small category then $(\mathcal{P}(\mathcal{D}'))(\mathcal{C})$ is equivalent to $\mathrm{Funct}(\mathcal{D}'^{\mathrm{op}}, \mathcal{C})$.

Remark 3.1.4. Let $\mathcal{D}$ be a presentable category and let $\mathcal{C}$ be a locally small complete category. It follows from [Lur09a] proposition 5.5.2.2 that a functor $F : \mathcal{D}^{\mathrm{op}} \to \mathcal{C}$ is limit preserving if and only if it has a left adjoint. In this context, the data of F is equivalent to the data of a functor $G : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ such that for every d in $\mathcal{D}$ the presheaf $\mathrm{Hom}_{\mathcal{D}}(d, G(-))$ on $\mathcal{C}$ is representable. If $\mathcal{C}$ is presentable then this condition is equivalent to G preserving limits, and so we conclude that $\mathcal{D}(\mathcal{C}) = \mathcal{C}(\mathcal{D})$. In other words, if $\mathcal{C}$ and $\mathcal{D}$ are presentable then $\mathcal{D}$-objects internal to $\mathcal{C}$ are the same as $\mathcal{C}$-objects internal to $\mathcal{D}$. Indeed, in this case the category $\mathcal{D}(\mathcal{C}) = \mathcal{C}(\mathcal{D})$ admits a symmetric presentation as $\mathcal{C} \otimes \mathcal{D}$ (see [Lur17] proposition 4.8.1.17).

Remark 3.1.5. Let $L : \mathcal{D}_1 \to \mathcal{D}_2$ be a localization functor between presentable categories and let $\mathcal{C}$ be a locally small complete category. Then the functor $\mathcal{D}_2(\mathcal{C}) \to \mathcal{D}_1(\mathcal{C})$ given by precomposition with L is a fully faithful embedding. A $\mathcal{D}_1$-object $F : \mathcal{D}_1^{\mathrm{op}} \to \mathcal{C}$ belongs to $\mathcal{D}_2(\mathcal{C})$ if and only if the associated functor $G : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}_1$ factors through $\mathcal{D}_2$.

Remark 3.1.6. Let $\mathcal{D}$ be a presentable category and let $\mathcal{C}$ be a classical locally small complete category. Let $\mathcal{D}_{\leq 0}$ be the full subcategory of $\mathcal{D}$ on the 0-truncated objects and denote by $\tau_{\leq 0} : \mathcal{D} \to \mathcal{D}_{\leq 0}$ the truncation functor. Then it follows from remark 3.1.5 that precomposition with $\tau_{\leq 0}$ induces an equivalence $\mathcal{D}_{\leq 0}(\mathcal{C}) = \mathcal{D}(\mathcal{C})$.

Example 3.1.7. Let $\mathcal{C}$ be a classical locally small complete category. Then $\mathrm{Set}(\mathcal{C})$ is equivalent to $\mathcal{C}$. If $\mathcal{C}$ is presentable then $\mathcal{C}(\mathrm{Set})$ is also equivalent to $\mathcal{C}$.

Remark 3.1.8. Let $\mathcal{D}$ be a presentable category. Let $\mathcal{D}'$ be a small category equipped with a localization functor $L : \mathcal{P}(\mathcal{D}') \to \mathcal{D}$, so that the right adjoint to L embeds $\mathcal{D}$ as a full subcategory of $\mathcal{P}(\mathcal{D}')$. Let $\mathcal{C}$ be a locally small complete category. Then it follows from a combination of example 3.1.3 and remark 3.1.5 that $\mathcal{D}(\mathcal{C})$ is equivalent to the full subcategory of $\mathrm{Funct}(\mathcal{D}'^{\mathrm{op}}, \mathcal{C})$ on those functors F such that the presheaf $\mathrm{Hom}_{\mathcal{C}}(c, F(-))$ belongs to $\mathcal{D}$ for every c in $\mathcal{C}$.

We now specialize the above discussion to the case of internal operads.

Notation 3.1.9. Denote by Op the category of operads. For each operad $\mathcal{O}$ we denote by $\mathrm{Op}_{\mathcal{O}}$ the category of operads over $\mathcal{O}$.

Definition 3.1.10. *Let $\mathcal{O}$ be an operad, and $\mathcal{C}$ be a complete category. An $\mathcal{O}$-operad internal to $\mathcal{C}$ is an $\mathrm{Op}_{\mathcal{O}}$-object internal to $\mathcal{C}$. If $\mathcal{C}'$ is a cocomplete category then an $\mathcal{O}$-cooperad internal is an $\mathcal{C}'$ is an $\mathcal{O}$-operad internal to $(\mathcal{C}')^{\mathrm{op}}$.*

Remark 3.1.11. Let $\mathcal{O}$ be an operad, and $\mathcal{C}$ be a presentable category. Then following remark 3.1.4, we see that an $\mathcal{O}$-cooperad internal to $\mathcal{C}$ is the same data as an accessible, limit preserving functor $G : \mathcal{C} \to \mathrm{Op}_{\mathcal{O}}$.

Remark 3.1.12. Let $\mathcal{O}$ be an operad and $\mathcal{C}$ be a classical locally small complete category. Then by virtue of remark 3.1.6 we have an equivalence $\mathrm{Op}_{\mathcal{O}}(\mathcal{C}) = (\mathrm{Op}_{\mathcal{O}})_{\leq 0}(\mathcal{C})$. In other words, $\mathcal{O}$-operads internal to $\mathcal{C}$ are the same as 0-truncated $\mathcal{O}$-operads internal to $\mathcal{C}$.

Assume now that $\mathcal{O}$ is a 0-truncated object of Op, so that it has a set V of objects, and a set M of operations. Consider the full subcategory $\mathcal{D}'$ of $(\mathrm{Op}_{\mathcal{O}})_{\leq 0}$ on the following objects:

- The trivial $\mathcal{O}$-operad $\mathfrak{v}_o$ for each o in V.
- For every operation m in M, the free $\mathcal{O}$-operad $\mathfrak{f}_m$ containing an m-operation.
- For every operation m in M with source objects $\{o_i\}_{i \in S}$, and every family of operations $\{m_i\}_{i\in S}$ where the target object of m_i is o_i, the $\mathcal{O}$-operad $\mathfrak{f}_{m_i,m}$ defined as the pushout

$$\left(\coprod_{i\in S} \mathfrak{f}_{m_i}\right) \bigcup_{\coprod_{i\in S} \mathfrak{v}_{o_i}} \mathfrak{f}_m.$$

Then $(\mathrm{Op}_{\mathcal{O}})_{\leq 0}$ is a localization of the category of set valued presheaves on $\mathcal{D}'$. Using remark 3.1.8 are able to obtain a concrete description of the category $\mathrm{Op}_{\mathcal{O}}(\mathcal{C})$. Namely, an $\mathcal{O}$-operad $\mathcal{O}'$ internal to $\mathcal{C}$ consist of the following data:

- For each o in V an object $\mathcal{O}'(\mathfrak{v}_o)$ in $\mathcal{C}$.
- For each m in M with source objects $\{o_i\}_{i\in S}$ and target object o, an object $\mathcal{O}'(\mathfrak{f}_m)$ in $\mathcal{C}$, equipped with source and target maps $\prod_{i\in S} \mathcal{O}'(\mathfrak{v}_{o_i}) \leftarrow \mathcal{O}'(\mathfrak{f}_m) \to \mathcal{O}'(\mathfrak{v}_o)$.
- For each o in V a unit map $\mathcal{O}'(\mathfrak{v}_o) \to \mathcal{O}'(\mathfrak{f}_{\mathrm{id}_o})$, where id_o denotes the identity 1-ary operation of o.

- For each m in M with source objects $\{o_i\}_{i\in S}$, and every family of operations $\{m_i\}_{i\in S}$ where the target object of m_i is o_i, a composition map

  $$\prod_{i\in S} \mathcal{O}'(\mathfrak{f}_{m_i}) \times_{\prod_{i\in S} \mathcal{O}'(\mathfrak{v}_{o_i})} \mathcal{O}'(\mathfrak{f}_m) \to \mathcal{O}(\mathfrak{f}_l).$$

  where l denotes the composite in $\mathcal{O}$ of family of operations $\{m_i\}_{i\in S}$ with m.

The above data is required to satisfy a finite list of standard compatibility conditions mimicking those of the category of 0-truncated $\mathcal{O}$-operads, built so that the data obtained from the above by applying a corepresentable functor $\mathcal{C} \to \mathrm{Set}$ defines an $\mathcal{O}$-operad in Set. In other words:

- Composition and unit maps are required to be compatible with sources and targets.
- Units are required to be compatible with compositions.
- Composition is required to be associative.
- For every o in V the unit map induces an isomorphism between $\mathcal{O}'(\mathfrak{v}_o)$ and the subobject of isomorphisms inside $\mathcal{O}'(\mathfrak{f}_{\mathrm{id}_o})$.

Example 3.1.13. Let $\mathcal{C}$ be a locally small complete category and let Assos be the operad governing associative algebras. Then Assos-operads internal to $\mathcal{C}$ will be called internal nonsymmetric operads. By virtue of remark 3.1.12, in the case when $\mathcal{C}$ is classical we can specify a nonsymmetric operad internal to $\mathcal{C}$ by giving a finite amount of information. Namely, a nonsymmetric operad $\mathcal{O}$ internal to $\mathcal{C}$ consists of the following data:

- An object V in $\mathcal{C}$ parametrizing objects of $\mathcal{O}$.
- For each $n \geq 0$ an object M_n in $\mathcal{C}$ parametrizing n-ary operations in $\mathcal{O}$, equipped with source and target maps $V^n \leftarrow M_n \to V$.
- A unit map $V \to M_1$.
- For each $n \geq 0$ and each sequence $\{n_i\}_{1\leq i\leq n}$ of nonnegative integers with sum N a composition map

  $$\prod_{1\leq i\leq n} M_{n_i} \times_{V^n} M_n \to M_N$$

  subject to the conditions described in remark 3.1.12.

We now present the construction of the internal nonsymmetric cooperad in strict categories $\mathfrak{C}^{\mathrm{cl}}$ which underlies the assignment $X \mapsto \mathrm{Assos}_X$. As discussed in example 3.1.13, we can do this by specifying a finite amount of information.

Construction 3.1.14. Let $\mathrm{Cat}_{\leq 0}$ be the category of 0-truncated categories. In other words, this is the category of strict categories with no nontrivial isomorphisms. We define a nonsymmetric cooperad $\mathfrak{C}^{\mathrm{cl}}$ internal to $\mathrm{Cat}_{\leq 0}$ as follows:

- The category V of objects of $\mathfrak{C}^{\mathrm{cl}}$ is the set with two elements $\{t, s\}$.
- For each $n \geq 0$ the category M_n of n-ary operations of $\mathfrak{C}^{\mathrm{cl}}$ is in fact a poset, and has objects t_i, s_i for $0 \leq i \leq n+1$, with $t_0 = s_0$ and $t_{n+1} = s_{n+1}$, and arrows $s_i \leftarrow t_{i+1}$ for $0 \leq i \leq n$. We depict this as follows:

$$t_0 = s_0 \longleftarrow t_1 \qquad s_1 \longleftarrow t_2 \qquad \dots \qquad s_n \longleftarrow t_{n+1} = s_{n+1}$$

- The cosource map $V^{\amalg n} \to M_n$ maps the i-th copy of t and s to t_i and s_i respectively. The cotarget map $M_n \leftarrow V$ maps t, s to t_0 and s_{n+1} respectively.
- The counit map $M_1 \to V$ maps t_0 and t_1 to t and s_1 and s_2 to s.
- Let $n \geq 0$ and let $\{n_j\}_{1\leq j\leq n}$ be a sequence of nonnegative integers with sum N. Denote the objects of $\coprod_j M_{n_j}$ by t^j_k, s^j_k, where $1 \leq j \leq n$ and $0 \leq k \leq n_j + 1$. The poset

$$\left(\coprod_{1\leq i\leq n} M_{n_i}\right) \bigcup_{V^{\amalg n}} M_n$$

has two extra objects which are in the image of the cotarget map $M_n \leftarrow V$. We denote these by t'_0 and s'_{n+1}. The cocomposition map

$$M_N \to \left(\coprod_i M_{n_i}\right) \bigcup_{V^{\amalg n}} M_n$$

sends t_0, s_{N+1} to t'_0 and s'_{n+1}, and for $i \neq 0, N$ sends t_i and s_i to t^j_k and s^j_k respectively, where (j, k) is the unique pair with $1 \leq k \leq n_j$ such that $(\sum_{1\leq l<j} n_l) + k = i$.

Remark 3.1.15. For each $n \geq 0$ the cosource and cotarget maps $V^n \to M_n \leftarrow V$ are jointly surjective. It follows from this, that $\mathfrak{C}^{\mathrm{cl}}$ is characterized uniquely by the first three items of construction 3.1.14. In other words, there is a unique way in which we could have defined the counit and cocomposition maps for $\mathfrak{C}^{\mathrm{cl}}$ once we are given the data of V, M_n and the cosource and cotarget maps.

Remark 3.1.16. The internal nonsymmetric cooperad $\mathfrak{C}^{\mathrm{cl}}$ defines a colimit preserving functor $(\mathrm{Op}_{\mathrm{Assos}})_{\leq 0} \to \mathrm{Cat}_{\leq 0}$ which we continue denoting by $\mathfrak{C}^{\mathrm{cl}}$. Composing this with the inclusion $\Delta \to (\mathrm{Cat})_{\leq 0} \to (\mathrm{Op}_{\mathrm{Assos}})_{\leq 0}$ we obtain a cosimplicial 0-truncated category $\Delta \to \mathrm{Cat}_{\leq 0}$. This satisfies the Segal conditions, and is in fact a cocategory object in $\mathrm{Cat}_{\leq 0}$. Inspecting construction 3.1.14 reveals that this is the functor that sends $[n]$ to the poset $[n]\coprod[n]^{\mathrm{op}}$.

Our next task is to extend $\mathfrak{C}^{\mathrm{cl}}$ to a nonsymmetric cooperad internal to Cat.

Proposition 3.1.17. *There is a unique internal cooperad* $\mathfrak{C} : \mathrm{Op}_{\mathrm{Assos}} \to \mathrm{Cat}$ *whose categories of objects and operations are* 0*-truncated, and making the following triangle commute*

$$\begin{tikzcd} \mathrm{Op}_{\mathrm{Assos}} \arrow[r, "\mathfrak{C}"] \arrow[dr, "\mathfrak{C}^{\mathrm{cl}}"'] & \mathrm{Cat} \arrow[d, "\tau_{\leq 0}"] \\ & \mathrm{Cat}_{\leq 0} \end{tikzcd}$$

where $\tau_{\leq 0}$ *is left adjoint to the inclusion.*

Proof. Recall the presentation of $\mathrm{Op}_{\mathrm{Assos}}$ in terms of complete Segal operads from [Bar18], as a localization of the presheaf category on the category $\Delta_{\mathbb{O}}$ of trees. This identifies $\mathrm{Op}_{\mathrm{Assos}}$ with the full subcategory of $\mathcal{P}(\Delta_{\mathbb{O}})$ on those presheaves satisfying suitable Segal and completeness conditions. Let $L : \mathcal{P}(\Delta_{\mathbb{O}}) \to \mathrm{Op}_{\mathrm{Assos}}$ be the localization functor, and $i : \Delta_{\mathbb{O}} \to \mathcal{P}(\Delta_{\mathbb{O}})$ be the inclusion. Let $\mathfrak{v}$ be the terminal associative operad, thought of as an object of $\Delta_{\mathbb{O}}$, and for each $n \geq 0$ let $\mathfrak{f}_n$ be the free associative operad on an operation of arity n, again thought of as an object of $\Delta_{\mathbb{O}}$.

Since L is a localization, and in particular an epimorphism, it suffices to show that there is a unique colimit preserving functor $\mathfrak{C}' : \mathcal{P}(\Delta_{\mathbb{O}}) \to \mathrm{Cat}$ which factors through $\mathrm{Op}_{\mathrm{Assos}}$, maps the objects $\mathfrak{v}$ and $\mathfrak{f}$ to 0-truncated categories, and makes the following triangle commute:

$$\begin{tikzcd} \mathcal{P}(\Delta_{\mathbb{O}}) \arrow[r, "\mathfrak{C}'"] \arrow[dr, "\mathfrak{C}^{\mathrm{cl}}L"'] & \mathrm{Cat} \arrow[d, "\tau_{\leq 0}"] \\ & \mathrm{Cat}_{\leq 0} \end{tikzcd}$$

For this it suffices to show that there is a unique functor $G : \Delta_{\mathbb{O}} \to \mathrm{Cat}$ which satisfies the dual Segal and completeness conditions, maps $\mathfrak{v}$ and $\mathfrak{f}$ to 0-truncated categories, and makes the following triangle commute:

$$\begin{tikzcd} \Delta_{\mathbb{O}} \arrow[r, "G"] \arrow[dr, "\mathfrak{C}^{\mathrm{cl}}Li"'] & \mathrm{Cat} \arrow[d, "\tau_{\leq 0}"] \\ & \mathrm{Cat}_{\leq 0} \end{tikzcd}$$

Let $j : \mathrm{Cat}_{\leq 0} \to \mathrm{Cat}$ be the inclusion. We claim that $G_0 = j\mathfrak{C}^{\mathrm{cl}}Li$ satisfies the dual Segal and completeness conditions. The fact that G_0 satisfies the dual completeness conditions follows from the description of the simplicial category underlying G_0 from 3.1.16. The fact that G_0 satisfies the dual Segal conditions follows from the fact that j preserves the pushouts involved in them.

Note that G_0 comes equipped with an identification $\epsilon : \tau_{\leq 0}G_0 = \mathfrak{C}^{\mathrm{cl}}Li$ given by the counit of the adjunction $\tau_{\leq 0} \dashv j$. It now suffices to show that pair (G, ρ) of a functor G

and an identification $\rho : \tau_{\leq 0}G = \mathfrak{C}^{\mathrm{cl}}Li$ as above is canonically equivalent to (G_0, ϵ). Let $\eta : G \to j\tau_{\leq 0}G$ be the unit map. The fact that G maps $\mathfrak{v}$ and $\mathfrak{f}_n$ to 0-truncated categories implies that η is an isomorphism on the full subcategory of $\Delta_{\mathbb{O}}$ on the objects $\mathfrak{v}$ and $\mathfrak{f}_n$. Since both G and $j\tau_{\leq 0}G = G_0$ satisfy the dual Segal conditions, we conclude that $(j\rho) \circ \eta$ gives us an isomorphism $G = G_0$. Our claim now follows from the fact that the diagram of functors and natural isomorphisms

$$\begin{tikzcd} \tau_{\leq 0}G \arrow[r, "\tau_{\leq 0}\eta"] \arrow[dr, "\rho"'] & \tau_{\leq 0}j\tau_{\leq 0}G \arrow[r, "\tau_{\leq 0}j\rho"] & \tau_{\leq 0}G_0 \arrow[dl, "\epsilon"] \\ & \mathfrak{C}^{\mathrm{cl}}Li & \end{tikzcd}$$

commutes in a natural way. □

Corollary 3.1.18. *Let $C : \mathrm{Op}_{\mathrm{Assos}} \to \mathrm{Cat}$ be an internal associative cooperad. Assume that the category of objects and category of operations of C are equivalent to those of $\mathfrak{C}$, with an equivalence that commutes with the cosource and cotarget maps. The C is equivalent to $\mathfrak{C}$.*

Proof. By remark 3.1.15 we have that $\tau_{\leq 0}C$ is equivalent to $\mathfrak{C}^{\mathrm{cl}}$. The claim now follows from proposition 3.1.17. □

Remark 3.1.19. Recall that Cat and $\mathrm{Op}_{\mathrm{Assos}}$ come equipped with involutions $(-)^{\mathrm{op}}$ and $(-)^{\mathrm{rev}}$, which correspond to actions of $\mathbb{Z}/2\mathbb{Z}$ on both categories. The internal cooperad $\mathfrak{C}^{\mathrm{cl}} : \mathrm{Op}_{\mathrm{Assos}} \to \mathrm{Cat}_{\leq 0}$ can be given a $\mathbb{Z}/2\mathbb{Z}$- equivariant structure, by switching the role of t and s in the category of objects, and of s_i, t_i with t_{n+1-i} and s_{n+1-i} in the categories of operations. It follows from proposition 3.1.17 that the internal cooperad $\mathfrak{C}$ inherits a $\mathbb{Z}/2\mathbb{Z}$ equivariant structure. In particular, we have a commutative square

$$\begin{tikzcd} \mathrm{Op}_{\mathrm{Assos}} \arrow[r, "\mathfrak{C}"] \arrow[d, "\mathrm{rev}"] & \mathrm{Cat} \arrow[d, "\mathrm{op}"] \\ \mathrm{Op}_{\mathrm{Assos}} \arrow[r, "\mathfrak{C}"] & \mathrm{Cat}\,. \end{tikzcd}$$

3.2 The operad Assos_X

We now introduce the operad Assos_X that corepresents the assignment $\mathcal{M} \mapsto \mathrm{Algbrd}_X(\mathcal{M})$.

Notation 3.2.1. Let $\mathrm{Assos}_{-} : \mathrm{Cat} \to \mathrm{Op}_{\mathrm{Assos}}$ be the right adjoint to $\mathfrak{C}$. This sends each category X to an associative operad Assos_X.

Remark 3.2.2. In the case when X is the terminal category, the operad Assos_X coincides with the associative operad Assos. In general, we think about Assos_X as a many object version of Assos. The category of objects of Assos_X is $X \times X^{\mathrm{op}}$. Given a nonempty sequence $\{(y_i, x_i)\}_{1\leq i\leq n}$ of source objects, and a target object (y, x), a multimorphism

$\{(y_i, x_i)\}_{1\leq i\leq n} \to (y, x)$ consists of a series of arrows $x_i \leftarrow y_{i+1}$ for $1 \leq i < n$, and arrows $x_n \leftarrow x$ and $y \leftarrow y_1$. A multimorphism from the empty sequence of objects to (y, x) consists of an arrow $y \leftarrow x$.

Remark 3.2.3. In [Hin20a], Hinich works in the language of categories of operators, and defines an assignment $\mathrm{Assos}^H_{-} : \mathrm{Cat} \to \mathrm{Op}_{\mathrm{Assos}}$ to be corepresented by a certain functor $\mathcal{F} : \Delta/\Delta^{\mathrm{op}} \to \mathrm{Cat}_{\leq 0}$. This functor is the categories of operators incarnation of the internal nonsymmetric cooperad $\mathfrak{C}$.

Indeed, note that the functor Assos^H_{-} is accessible and preserves limits, so by virtue of remark 3.1.11 it is corepresented by an internal nonsymmetric cooperad $\mathfrak{C}^H : \mathrm{Op}_{\mathrm{Assos}} \to \mathrm{Cat}$. Direct inspection of the definition of $\mathcal{F}$ reveals that the category of objects and operations of $\mathfrak{C}^H$ agree with those of $\mathfrak{C}^{\mathrm{cl}}$, in a way which is compatible with source and target maps. As observed in corollary 3.1.18 this implies that $\mathfrak{C}^H$ is equivalent to $\mathfrak{C}$. It follows that the functor Assos^H defined in [Hin20a] is equivalent to our functor Assos.

Example 3.2.4. Let $X = \{a, b\}$ be the set with two elements a, b. Then the associative operad Assos_X is classical, and can be computed explicitly from the definitions. We note that it has objects $(a, a), (b, b), (a, b), (b, a)$. The objects (a, a) and (b, b) are algebras in Assos_X. The object (a, b) is an $(a, a) - (b, b)$ bimodule and the object (b, a) is a $(b, b) - (a, a)$ bimodule.

As we shall see below, for each pair of categories X, Y, the category $X \times Y^{\mathrm{op}}$ has compatible (weak) actions of the associative operads Assos_X and Assos_Y on the left and on the right. This is the basis for the theory of bimodules over algebroids.

Notation 3.2.5. Denote by $\mathrm{BM}, \mathrm{LM}, \mathrm{RM}$ be the associative operads governing bimodules left modules, and right modules, respectively. Recall that we have canonical inclusions $\mathrm{LM} \to \mathrm{BM} \leftarrow \mathrm{RM}$. We denote by Assos^+ and Assos^- the copies of the associative operad in BM contained in LM and RM, respectively.

Construction 3.2.6. Let X and Y be categories. Consider the projection $X \sqcup Y \to \{a, b\}$ from the disjoint union of X and Y into the set with two elements a, b, that maps X to a and Y to b. Applying the functor Assos_{-} we obtain a map of associative operads $\mathrm{Assos}_{X\sqcup Y} \to \mathrm{Assos}_{\{a,b\}}$. We let $\mathrm{BM}_{X,Y}$ be the BM-operad obtained by pullback of $\mathrm{Assos}_{X\sqcup Y}$ along the map $\mathrm{BM} \to \mathrm{Assos}_{\{a,b\}}$ corresponding to the $(a, a) - (b, b)$ bimodule (a, b) (see example 3.2.4). The assignment $X \mapsto \mathrm{BM}_{X,Y}$ is functorial in X and Y. We denote by $\mathrm{BM}_{-,-} : \mathrm{Cat} \times \mathrm{Cat} \to \mathrm{Op}_{\mathrm{BM}}$ the corresponding functor.

Remark 3.2.7. Let X, Y be categories. The functor Assos_{-} preserves limits since it is a right adjoint. Applying it to the cartesian square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X \sqcup Y \arrow[d] \\ \{a\} \arrow[r] & \{a, b\} \end{tikzcd}$$

we conclude that the Assos^--component of $\mathrm{BM}_{X,Y}$ coincides with Assos_X. This equivalence is natural in X.

Similarly, from the cartesian square

$$\begin{array}{ccc} Y & \longrightarrow & X \sqcup Y \\ \downarrow & & \downarrow \\ \{b\} & \longrightarrow & \{a,b\} \end{array}$$

we see that the Assos^+-component of $\mathrm{BM}_{X,Y}$ coincides with Assos_Y.

Consider now the cartesian square

$$\begin{array}{ccc} (\mathrm{BM}_{X,Y})_m & \longrightarrow & \mathrm{Assos}_{X\sqcup Y} \\ \downarrow & & \downarrow \\ \{(a,b)\} & \longrightarrow & \mathrm{Assos}_{\{a,b\}} \end{array}$$

where the category $(\mathrm{BM}_{X,Y})_m$ is the fiber of $\mathrm{BM}_{X,Y}$ over the universal bimodule m in BM. Applying the (limit preserving) forgetful functor $\mathrm{Op}_{\mathrm{Assos}} \to \mathrm{Cat}$ we obtain a cartesian square

$$\begin{array}{ccc} (\mathrm{BM}_{X,Y})_m & \longrightarrow & (X\times X^{\mathrm{op}})\sqcup(Y\times Y^{\mathrm{op}})\sqcup(X\times Y^{\mathrm{op}})\sqcup(Y\times X^{\mathrm{op}}) \\ \downarrow & & \downarrow \\ \{(a,b)\} & \longrightarrow & \{(a,a),(b,b),(a,b),(b,a)\}. \end{array}$$

We conclude that the category $(\mathrm{BM}_{X,Y})_m$ is equivalent to $X \times Y^{\mathrm{op}}$. This equivalence is also natural in X, Y.

Example 3.2.8. Let X be a category. Then we have an equivalence

$$\mathrm{Assos}(X \sqcup X) = \mathrm{Assos}(X \times \{a,b\}) = \mathrm{Assos}(X) \times \mathrm{Assos}(\{a,b\})$$

which is natural in X. It follows that we have an equivalence $\mathrm{BM}_{X,X} = \mathrm{Assos}_X \times \mathrm{BM}$, which is natural in X.

Notation 3.2.9. Let X be a category. Denote by BM_X the associative operad $\mathrm{BM}_{X,[0]}$. Let LM_X be the LM-operad obtained by pullback of BM_X along the inclusion $\mathrm{LM} \to \mathrm{BM}$. We denote by $\mathrm{BM}_- : \mathrm{Cat} \to \mathrm{Op}_{\mathrm{BM}}$ the functor that assigns to each category X the BM-operad BM_X, and by LM_- the composition of BM_- with the functor of base change to LM.

Remark 3.2.10. In [Hin20a], Hinich defines an assignment $\mathrm{BM}^H_- : \mathrm{Cat} \to \mathrm{Op}_{\mathrm{BM}}$ in the language of categories of operators, by declaring it to be corepresented by a certain functor

$$\mathcal{F}_{\mathrm{BM}} : \Delta/(\Delta_{/[1]})^{\mathrm{op}} \to \mathrm{Cat}\,.$$

The functor BM^H_- is accessible and preserves limits, and therefore by remark 3.1.11 it is corepresented by an internal BM-cooperad $\mathfrak{C}^H_{\mathrm{BM}} : \mathrm{Op}_{\mathrm{BM}} \to \mathrm{Cat}$.

Likewise, our functor BM_- can be obtained as the composite functor

$$\mathrm{Cat} = \mathrm{Cat}_{/\{a\}} \xrightarrow{-\sqcup\{b\}} \mathrm{Cat}_{/\{a,b\}} \xrightarrow{\mathrm{Assos}_-} (\mathrm{Op}_{\mathrm{Assos}})_{/\,\mathrm{Assos}_{\{a,b\}}} \xrightarrow{(a,a)(a,b)(b,b)} (\mathrm{Op}_{\mathrm{Assos}})_{/\,\mathrm{BM}} = \mathrm{Op}_{\mathrm{BM}}$$

and each of the functors in the composition preserves limits and is accessible, so we have that BM_- is also corepresented by an internal BM-cooperad $\mathfrak{C}_{\mathrm{BM}} : \mathrm{Op}_{\mathrm{BM}} \to \mathrm{Cat}$. Direct inspection of the functor $\mathcal{F}_{\mathrm{BM}}$ reveals that $\mathfrak{C}_{\mathrm{BM}}$ and $\mathfrak{C}^H_{\mathrm{BM}}$ have equivalent categories of objects and operations, in a way compatible with sources and target. A variation of the arguments in proposition 3.1.17 and corollary 3.1.18 (where we work with $(\Delta_{\mathbb{O}})_{/\,\mathrm{BM}}$) shows that the cooperads $\mathfrak{C}_{\mathrm{BM}}$ and $\mathfrak{C}^H_{\mathrm{BM}}$ are equivalent, and thus our functor BM_- is equivalent to the functor BM^H_- from [Hin20a].

3.3 Algebroids

We now present the definition of an algebroid in an associative operad (also known as categorical algebras in [GH15] and enriched precategories in [Hin20a]) and review the basic funtoriality properties of the theory.

Definition 3.3.1. *Let $\mathcal{M}$ be an associative operad and X be a category. An algebroid on $\mathcal{M}$ with category of objects X is an* Assos_X*-algebra in $\mathcal{M}$.*

Remark 3.3.2. Let $\mathcal{M}$ be an associative operad. An algebroid with category of objects $[0]$ is an algebra in $\mathcal{M}$. In general, we think about an algebroid $\mathcal{A}$ in $\mathcal{M}$ as a many-object associative algebra. Indeed, an algebroid with category of objects X assigns to each pair of objects y, x in X an object $\mathcal{A}(y, x)$ in $\mathcal{M}$ and to every $n \geq 0$ and every sequence of arrows $y_0 = x_0 \leftarrow y_1, x_1 \leftarrow y_2, \dots, x_{n-1} \leftarrow y_n, x_n \leftarrow y_{n+1} = x_{n+1}$ it assigns a multimorphism

$$\{\mathcal{A}(y_1, x_1), \mathcal{A}(y_2, x_2), \dots, \mathcal{A}(y_n, x_n)\} \to \mathcal{A}(y_0, x_{n+1})$$

in $\mathcal{M}$. In the case when $\mathcal{M}$ is a monoidal category, this is the same as a morphism

$$\mathcal{A}(y_1, x_1) \otimes \mathcal{A}(y_2, x_2) \otimes \dots, \otimes \mathcal{A}(y_n, x_n) \to \mathcal{A}(y_0, x_{n+1}).$$

Specializing to the case $n = 0$ we obtain for every pair of objects x, y a map

$$\mathrm{Hom}_X(y, x) \to \mathrm{Hom}_{\mathcal{M}}(1_{\mathcal{M}}, \mathcal{A}(y, x)).$$

In particular, starting from the identity in $\mathrm{Hom}_X(x, x)$ we obtain a map $1_{\mathcal{M}} \to \mathcal{A}(x, x)$ (the unit at x).

In the case when $n = 2$ and all the arrows are identities we obtain a map

$$\mathcal{A}(z, y) \otimes \mathcal{A}(y, x) \to \mathcal{A}(z, x)$$

(the composition map) for every triple of objects x, y, z in $\mathcal{M}$. It follows from the definition of the composition rule in the cooperad $\mathfrak{C}$ that these maps satisfy the usual unitality and associativity rules up to homotopy.

Construction 3.3.3. Let X be a category and $\mathcal{M}$ be an associative operad. We denote by $\mathrm{Algbrd}_X(\mathcal{M})$ the category of Assos_X-algebras in $\mathcal{M}$. Denote by $\mathrm{Algbrd}_{\_}(-) : \mathrm{Cat}^{\mathrm{op}} \times \mathrm{Op}_{\mathrm{Assos}} \to \mathrm{Cat}$ the composite functor

$$\mathrm{Cat}^{\mathrm{op}} \times \mathrm{Op}_{\mathrm{Assos}} \xrightarrow{\mathrm{Assos}_{\_}^{\mathrm{op}} \times \mathrm{id}} \mathrm{Op}_{\mathrm{Assos}}^{\mathrm{op}} \times \mathrm{Op}_{\mathrm{Assos}} \xrightarrow{\mathrm{Alg}_{\_}(-)} \mathrm{Cat}\,.$$

For each $\mathcal{M}$ in $\mathrm{Op}_{\mathrm{Assos}}$ we denote by $\mathrm{Algbrd}(\mathcal{M})$ the total category of the cartesian fibration associated to the functor $\mathrm{Algbrd}_{\_}(\mathcal{M}) : \mathrm{Cat}^{\mathrm{op}} \to \mathrm{Cat}$. We call $\mathrm{Algbrd}(\mathcal{M})$ the category of algebroids in $\mathcal{M}$.

The assignment $\mathcal{M} \mapsto (\mathrm{Algbrd}(\mathcal{M}) \to \mathrm{Cat})$ yields a functor

$$\mathrm{Algbrd}(-) : \mathrm{Op}_{\mathrm{Assos}} \to \widehat{\mathrm{Cat}}$$

equipped with a natural transformation to the constant functor Cat. We denote by Algbrd the total category of the cocartesian fibration associated to $\mathrm{Algbrd}(-)$. This comes equipped with a projection $\mathrm{Algbrd} \to \mathrm{Cat} \times \mathrm{Op}_{\mathrm{Assos}}$ whose fiber over a pair $(X, \mathcal{M})$ is the category $\mathrm{Algbrd}_X(\mathcal{M})$. This is the two-sided fibration associated to the functor $\mathrm{Algbrd}(-)_{-}$.[1]

Notation 3.3.4. Let $i : X \to Y$ be a functor of categories, and let $j : \mathcal{M} \to \mathcal{N}$ be a map of associative operads. We denote by $i^! : \mathrm{Algbrd}_Y(\mathcal{M}) \to \mathrm{Algbrd}_X(\mathcal{M})$ the functor induced by i, and by $j_! : \mathrm{Algbrd}(\mathcal{M}) \to \mathrm{Algbrd}(\mathcal{N})$ the functor induced by j.

Example 3.3.5. Let $\mathcal{M}$ be a monoidal category. Then the unit $1_{\mathcal{M}}$ has an algebra structure, and therefore defines an algebroid with category of objects $[0]$. Since $1_{\mathcal{M}}$ is initial in $\mathrm{Algbrd}_{[0]}(\mathcal{M})$, the functor $\mathrm{Algbrd}(\mathcal{M}) \to \mathrm{Spc}$ corepresented by $1_{\mathcal{M}}$ is equivalent to the restriction along the projection $\mathrm{Algbrd}(\mathcal{M}) \to \mathrm{Cat}$ of the functor $\mathrm{Cat} \to \mathrm{Spc}$ corepresented by $[0]$. In particular, for every $\mathcal{M}$-algebroid $\mathcal{A}$ with category of objects X, the space $\mathrm{Hom}_{\mathrm{Algbrd}(\mathcal{M})}(1_{\mathcal{M}}, \mathcal{A})$ is equivalent to the space underlying X.

Example 3.3.6. Let $\mathcal{M}$ be monoidal category, and let m be an object in $\mathcal{M}$. Assume that $\mathcal{M}$ admits an initial object which is compatible with the monoidal structure. Let $X = \{a, b\}$ be the set with two elements. Then we may form the free Assos_X-algebra C_m in $\mathcal{M}$ equipped with a map $m \to C_m(a, b)$. The description of free algebras from [Lur17] definition 3.1.3.1 yields the following description of C_m:

- $C_m((a, a)) = C_m((b, b)) = 1_{\mathcal{M}}$.
- $C_m(a, b) = m$.
- $C_m(b, a)$ is the initial object of $\mathcal{M}$.

[1] We refer to chapter 9 for background on the theory of two-sided fibrations and bifibrations, and a general discussion of the Grothendieck construction which relates these to bifunctors into Cat.

We call the algebroids of the form C_m cells. These come equipped with two maps $1_{\mathcal{M}} \to C_m$, which pick out the objects (a, a) (the target) and (b, b) (the source). These maps can be combined into a single map out of the coproduct $1_{\mathcal{M}} \sqcup 1_{\mathcal{M}}$ in $\mathrm{Algbrd}(\mathcal{M})$ (note that this coproduct indeed exists and is given by the initial object in $\mathrm{Algbrd}_{\{a,b\}}(\mathcal{M})$, which agrees with the cell associated to the initial object in $\mathcal{M}$).

The formation of cells is functorial in m: it underlies a colimit preserving functor

$$C_{-} : \mathcal{M} \to \mathrm{Algbrd}_{\{a,b\}}(\mathcal{M})$$

given by operadic left Kan extension along the inclusion $\{(a, b)\} \to \mathrm{Assos}_{\{a,b\}}$. This assignment is furthermore functorial in $\mathcal{M}$. In other words, given another monoidal category $\mathcal{M}'$ with compatible initial object and a monoidal functor $F : \mathcal{M} \to \mathcal{M}'$ which preserves initial objects, then the commutative square of categories

$$\begin{tikzcd}
\mathrm{Algbrd}_{\{a,b\}}(\mathcal{M}) \arrow[r, "\mathrm{ev}_{(a,b)}"] \arrow[d, "F_!"] & \mathcal{M} \arrow[d, "F"] \\
\mathrm{Algbrd}_{\{a,b\}}(\mathcal{M}') \arrow[r, "\mathrm{ev}_{(a,b)}"] & \mathcal{M}'
\end{tikzcd}$$

is horizontally left adjointable.

Remark 3.3.7. Let $u : \mathrm{Op}_{\mathrm{Assos}} \to \mathrm{Cat}$ be the colocalization functor that attaches to each associative operad its category of objects. It follows from 3.1.16 that we have an equivalence $u\,\mathrm{Assos}_{-} = \mathrm{id} \times (\mathrm{id})^{\mathrm{op}}$ as endofunctors of Cat. In particular, for every category X and associative operad $\mathcal{M}$ we have a functor

$$\mathrm{Algbrd}_X(\mathcal{M}) \to \mathrm{Funct}(X \times X^{\mathrm{op}}, \mathcal{M})$$

which is natural in X and $\mathcal{M}$. We think about this as the functor which attaches to each algebroid $\mathcal{A}$ with category of objects X, the hom-functor of $\mathcal{A}$ restricted to X.

Remark 3.3.8. Recall from remark 3.1.19 that the cooperad $\mathfrak{C}$ intertwines the order reversing involution $(-)^{\mathrm{rev}}$ of $\mathrm{Op}_{\mathrm{Assos}}$ and the passing to opposites involution of Cat. It follows that the same is true for the functor $\mathrm{Assos}_{-} : \mathrm{Cat} \to \mathrm{Op}_{\mathrm{Assos}}$. We thus see that the functor $\mathrm{Algbrd}_{-}(-)$ admits the structure of a fixed point for the involution $(-)^{\mathrm{op}} \times (-)^{\mathrm{rev}}$ on $\mathrm{Cat}^{\mathrm{op}} \times \mathrm{Op}_{\mathrm{Assos}}$, which implies that there is an involution $(-)^{\mathrm{op}}$ on Algbrd and an enhancement of the projection $\mathrm{Algbrd} \to \mathrm{Cat} \times \mathrm{Op}_{\mathrm{Assos}}$ to a $\mathbb{Z}/2\mathbb{Z}$-equivariant map.

In particular, for every category X and associative operad $\mathcal{M}$, we have an equivalence $\mathrm{Algbrd}_X(\mathcal{M}) = \mathrm{Algbrd}_{X^{\mathrm{op}}}(\mathcal{M}^{\mathrm{rev}})$. In the case when X is a space and $\mathcal{M}$ is the associative operad underlying a symmetric operad, then the pair $(X, \mathcal{M})$ is has the structure of fixed point for the involution $(-)^{\mathrm{op}} \times (-)^{\mathrm{rev}}$. It follows that the involution $(-)^{\mathrm{op}} : \mathrm{Algbrd} \to \mathrm{Algbrd}$ restricts to an involution on $\mathrm{Algbrd}_X(\mathcal{M})$. In other words, if $\mathcal{M}$ underlies a symmetric operad, then any $\mathcal{M}$-algebroid $\mathcal{A}$ with a space of objects X has attached to it another $\mathcal{M}$-algebroid $\mathcal{A}^{\mathrm{op}}$ with space of objects X. Examining the fixed point structure on $\mathfrak{C}^{\mathrm{cl}}$ from remark 3.1.19 reveals that for every pair of objects y, x in X one has $\mathcal{A}^{\mathrm{op}}(y, x) = \mathcal{A}(x, y)$.

Example 3.3.9. Let $\mathcal{M}$ be a symmetric monoidal category compatible with initial object and let m be an object of $\mathcal{M}$. Then the cell C_m is equivalent to its opposite. This equivalence interchanges sources and targets - namely, there is a commutative diagram of $\mathcal{M}$-algebroids

$$\begin{tikzcd} 1_{\mathcal{M}} \bigsqcup 1_{\mathcal{M}} \arrow[r, "((a,a),(b,b))"] \arrow[d, "\mathrm{id}"] & C_m \arrow[d, "="] \\ 1_{\mathcal{M}} \bigsqcup 1_{\mathcal{M}} \arrow[r, "((b,b),(a,a))"] & (C_m)^{\mathrm{op}}. \end{tikzcd}$$

In general, for any $\mathcal{M}$-algebroid $\mathcal{A}$ we can think about $\mathcal{A}^{\mathrm{op}}$ as being obtained from $\mathcal{A}$ by reversing the direction of the cells.

Remark 3.3.10. Let $\mathcal{M}, \mathcal{M}'$ be associative operads. Then for every category X there is a functor

$$\mathrm{Alg}_{\mathcal{M}}(\mathcal{M}') \times \mathrm{Algbrd}_X(\mathcal{M}) \to \mathrm{Algbrd}_X(\mathcal{M}').$$

This is natural in X and therefore defines a functor

$$\mathrm{Alg}_{\mathcal{M}}(\mathcal{M}') \times \mathrm{Algbrd}(\mathcal{M}) \to \mathrm{Algbrd}(\mathcal{M}')$$

which enhances the functoriality of construction 3.3.3 to take into account natural transformations between morphisms of associative operads. This is compatible with composition: namely, given a third associative operad $\mathcal{M}''$, there is a commutative square

$$\begin{tikzcd} \mathrm{Alg}_{\mathcal{M}'}(\mathcal{M}'') \times \mathrm{Alg}_{\mathcal{M}}(\mathcal{M}') \times \mathrm{Algbrd}(\mathcal{M}) \arrow[r] \arrow[d] & \mathrm{Alg}_{\mathcal{M}'}(\mathcal{M}'') \times \mathrm{Algbrd}(\mathcal{M}') \arrow[d] \\ \mathrm{Alg}_{\mathcal{M}}(\mathcal{M}'') \times \mathrm{Algbrd}(\mathcal{M}) \arrow[r] & \mathrm{Algbrd}(\mathcal{M}''). \end{tikzcd}$$

This is part of the data that would arise from an enhancement of $\mathrm{Algbrd}(-)$ to a functor of 2-categories. We do not construct this enhancement here; however note that the above property is already enough to conclude that if $\mathcal{M}, \mathcal{M}'$ are monoidal categories and $F : \mathcal{M} \to \mathcal{M}'$ is a monoidal functor admitting a (lax monoidal) right adjoint F^R, then we have an induced adjunction

$$F_! : \mathrm{Algbrd}(\mathcal{M}) \rightleftarrows \mathrm{Algbrd}(\mathcal{M}') : (F^R)_!.$$

By working with monoidal envelopes and passing to presheaf categories, one can often reduce questions in enriched category theory to the case when the enriching category is a presentable monoidal category. We now study some of the features of this setting.

Remark 3.3.11. In construction 3.3.3 we implicitly assumed that all operads and categories were small. Passing to a larger universe, one can similarly discuss categories of algebroids in presentable monoidal categories. Given a presentable monoidal category $\mathcal{M}$, we will denote by $\mathrm{Algbrd}(\mathcal{M})$ the category of algebroids in $\mathcal{M}$ with a small category of objects. Its version where we allow large categories of objects will be denoted by $\widehat{\mathrm{Algbrd}}(\mathcal{M})$.

Proposition 3.3.12. *Let $\mathcal{M}$ be a presentable category equipped with a monoidal structure which is compatible with colimits. Then*

(i) *The category* $\mathrm{Algbrd}(\mathcal{M})$ *is presentable, and the projection* $\mathrm{Algbrd}(\mathcal{M}) \to \mathrm{Cat}$ *is a limit and colimit preserving cartesian and cocartesian fibration.*

(ii) *For every colimit preserving monoidal functor $F : \mathcal{M} \to \mathcal{M}'$ into another presentable monoidal category, the induced functor* $F_! : \mathrm{Algbrd}(\mathcal{M}) \to \mathrm{Algbrd}(\mathcal{M}')$ *preserves colimits.*

Proof. We note that item (ii) is a direct consequence of remark 3.3.10 together with the adjoint functor theorem. We now prove item (i). It follows from [Lur17] corollary 3.2.3.5 that for every category X the category $\mathrm{Algbrd}_X(\mathcal{M})$ is presentable. Moreover, using [Lur17] corollary 3.1.3.5 we see that for every functor $i : X \to Y$ the induced functor $i^! : \mathrm{Algbrd}_Y(\mathcal{M}) \to \mathrm{Algbrd}_X(\mathcal{M})$ admits a left adjoint, so that the projection $\mathrm{Algbrd}(\mathcal{M}) \to \mathrm{Cat}$ is both a cartesian and a cocartesian fibration. Since the functors Assos_- and $\mathrm{Alg}_-(\mathcal{M})$ are accessible, we conclude from [GHN17] theorem 10.3 that $\mathrm{Algbrd}(\mathcal{M})$ is a presentable category. The fact that the projection to Cat preserves limits and colimits is now a consequence of [Lur09a] corollary 4.3.1.11. □

Notation 3.3.13. For each associative operad $\mathcal{M}$ denote by $\mathrm{Algbrd}(\mathcal{M})_{\mathrm{Spc}}$ the full subcategory of $\mathrm{Algbrd}(\mathcal{M})$ on those algebroids which have a space of objects.

Remark 3.3.14. Let $\mathcal{M}$ be a presentable monoidal category. Let κ be a regular cardinal and let $\{m_i\}_{i\in\mathcal{I}}$ be a small family of κ-compact generators of $\mathcal{M}$. Then the cells C_{m_i} together with the unit algebroid $1_{\mathcal{M}}$ are a family of κ-compact generators of $\mathrm{Algbrd}(\mathcal{M})_{\mathrm{Spc}}$.

3.4 Enriched categories

Our next goal is to review the notion of enriched category. In order to do this, we will first need to study the category of algebroids in the case $\mathcal{M} = \mathrm{Spc}$ equipped with its cartesian monoidal structure.

Construction 3.4.1. Let $\mathcal{M} = \mathrm{Spc}$ be the category of spaces, equipped with its cartesian monoidal structure. Then for every category X the category $\mathrm{Algbrd}_X(\mathrm{Spc})$ is presentable ([Lur17] corollary 3.2.3.5), and in particular admits an initial object. Since the projection $\mathrm{Algbrd}(\mathrm{Spc}) \to \mathrm{Cat}$ is a cartesian fibration, there is a unique section $s : \mathrm{Cat} \to \mathrm{Algbrd}(\mathrm{Spc})$ such that for every category X we have that $s(X)$ is initial in $\mathrm{Algbrd}_X(\mathrm{Spc})$.

Consider the cartesian square

$$\begin{array}{ccc} \mathrm{Algbrd}(\mathrm{Spc})_{\mathrm{Spc}} & \xrightarrow{i'} & \mathrm{Algbrd}(\mathrm{Spc}) \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ \mathrm{Spc} & \xrightarrow{i} & \mathrm{Cat}\,. \end{array}$$

Since p is a cartesian fibration and i admits a right adjoint, we have that the above square is horizontally right adjointable. The right adjoint $i'^R : \mathrm{Algbrd}(\mathrm{Spc}) \to \mathrm{Algbrd}(\mathrm{Spc})_{\mathrm{Spc}}$ maps an algebroid $\mathcal{A} : \mathrm{Assos}_X \to \mathrm{Spc}$ to the algebroid defined by the composite map

$$\mathrm{Assos}_{X^{\leq 0}} \to \mathrm{Assos}_X \xrightarrow{\mathcal{A}} \mathrm{Spc}\,.$$

Denote by $\rho : \mathrm{Cat} \to \mathrm{Algbrd}(\mathrm{Spc})_{\mathrm{Spc}}$ the composite map $(i')^R s$.

Example 3.4.2. The algebroid $\rho([0])$ is the unit algebroid 1_{Spc}.

Example 3.4.3. Examining the description of free algebras from [Lur17] definition 3.1.3.1 yields the following description of $\rho([1])$:

- $\rho([1])$ has a set of objects with two elements $0, 1$.
- $\rho([1])(0,0) = \rho([1])(1,1) = \rho([1])(1,0)$ are the singleton set.
- $\rho([1])(0,1)$ is empty.

In other words, we have that $\rho([1])$ is equivalent to the cell $C_{[0]}$.

Lemma 3.4.4. *The section s from construction 3.4.1 admits a left adjoint.*

Proof. We continue with the notation from construction 3.4.1. It follows from remark 3.3.14 that $\mathrm{Algbrd}(\mathrm{Spc})_{\mathrm{Spc}}$ is generated under colimits by the cell $C_{[0]}$ and the trivial algebroid 1_{Spc}. To obtain a set of generators for $\mathrm{Algbrd}(\mathrm{Spc})$ it suffices to add the algebroid $s([1])$. Since Cat admits all colimits, in order to show that s has a left adjoint, it suffices to show that for each generator G, there is a category $\mathcal{C}$ and a morphism $\eta : G \to s(\mathcal{C})$ such that for every category $\mathcal{D}$ the composite map

$$\mathrm{Hom}_{\mathrm{Cat}}(\mathcal{C}, \mathcal{D}) \xrightarrow{s_*} \mathrm{Hom}_{\mathrm{Algbrd}(\mathrm{Spc})}(s(\mathcal{C}), s(\mathcal{D})) \xrightarrow{\eta^*} \mathrm{Hom}_{\mathrm{Algbrd}(\mathrm{Spc})}(G, s(\mathcal{D}))$$

is an equivalence. Note that the section s is fully faithful, so the first map in the above composition is always an isomorphism. Since 1_{Spc} and $s([1])$ belong to the image of s, the identity maps of 1_{Spc} and $s([1])$ satisfy the desired condition.

It remains to consider the case of the generator $C_{[0]}$. We take $\mathcal{C} = [1]$, and the map $\eta : C_{[0]} \to s(\mathcal{C})$ to be the morphism of algebroids associated to the image of the map $1_{\mathrm{Spc}} \to s([1])(1,0)$ induced by the unique arrow $1 \leftarrow 0$ in $[1]$. Let $\mathcal{D}$ be a category and consider the commutative triangle

$$\begin{tikzcd} \mathrm{Hom}_{\mathrm{Cat}}(\mathcal{C}, \mathcal{D}) \arrow[rr, "\eta^* s_*"] \arrow[dr] & & \mathrm{Hom}_{\mathrm{Algbrd}(\mathrm{Spc})}(C_{[0]}, s(\mathcal{D})) \arrow[dl] \\ & \mathcal{D}^{\leq 0} \times \mathcal{D}^{\leq 0} & \end{tikzcd}$$

where the diagonal maps are the source and target maps. In order to show that $\eta^* s_*$ is an equivalence, it suffices to show that it is an equivalence when restricted to the fiber over any point (x, y) in $\mathcal{D}^{\leq 0} \times \mathcal{D}^{\leq 0}$. This restriction recovers the map $\operatorname{Hom}_{\mathcal{D}}(x, y) \to s(\mathcal{D})(y, x)$ which assigns to each arrow $y \leftarrow x : \alpha$ in $\mathcal{D}$, the image of the induced map $1_{\mathrm{Spc}} \to s(\mathcal{D})(y, x)$. Our claim now follows from the fact that $s(\mathcal{D})$ is the free $\mathrm{Assos}_{\mathcal{D}}$-algebra in Spc on the unique algebra over the empty operad, together with the description of free algebras from [Lur17] definition 3.1.3.1. □

The following proposition is a slight rephrasing of [GH15] theorem 4.4.7 and the discussion in [Hin20a] section 5.

Proposition 3.4.5. *There is a unique equivalence between* $\mathrm{Algbrd}(\mathrm{Spc})_{\mathrm{Spc}}$ *and the category* $\mathcal{P}(\Delta)_{\mathrm{Seg}}$ *of Segal spaces which intertwines the map* ρ *and the canonical inclusion of* Cat *into* $\mathcal{P}(\Delta)_{\mathrm{Seg}}$ *as the subcategory of complete Segal spaces.*

Proof. First we note that this equivalence is unique, if it exists. Indeed, the same method of proof of [Lur09a] theorem 5.2.9.1 shows that the space of automorphisms of the category $\mathcal{P}(\Delta)_{\mathrm{Seg}}$ is a two element set, consisting of the identity and the orientation reversing automorphism. It follows that the space of automorphisms of $\mathcal{P}(\Delta)_{\mathrm{Seg}}$ that restrict to the identity on Cat is contractible.

The existence of an equivalence $F : \mathrm{Algbrd}(\mathrm{Spc})_{\mathrm{Spc}} \to \mathcal{P}(\Delta)_{\mathrm{Seg}}$ is the subject of [GH15] theorem 4.4.7. Denote by $i : \mathrm{Cat} \to \mathcal{P}(\Delta)_{\mathrm{Seg}}$ the inclusion. It remains to show that we have an equivalence $F\rho = i$. Note that by virtue of lemma 3.4.4, the map ρ admits a left adjoint. It therefore suffices to show that there is an equivalence $i^L = \rho^L F^{-1}$.

Both i^L and $\rho^L F^{-1}$ are colimit preserving functors $\mathcal{P}(\Delta)_{\mathrm{Seg}} \to \mathrm{Cat}$, and so they are determined by the data of a Segal cosimplicial category. In the case of i^L, this is the canonical inclusion $\Delta \to \mathrm{Cat}$. The proof of lemma 3.4.4 shows that ρ^L maps 1_{Spc} to $[0]$ and $C_{[0]}$ to $[1]$, in a way compatible with the source and target maps. Moreover, inspecting the construction of the equivalence F from [GH15] reveals that F^{-1} maps $[0]$ to 1_{Spc} and $[1]$ to $C_{[0]}$, in a way compatible with source and target maps. It follows that the Segal cosimplicial category induced by $\rho^L F^{-1}$ is the identity on the full subcategory of Δ on the objects $[0]$ and $[1]$. The Segal conditions imply that $\rho^L F^{-1}([n])$ is equivalent to $i^L([n])$ for all $[n]$, and it is in particular a 0-truncated category. Our claim now follows from the fact that the source and target maps $[0] \to [1] \leftarrow [0]$ are jointly surjective, using the same arguments as those that establish corollary 3.1.18 □

Remark 3.4.6. It follows from remark 3.3.8 that $\mathrm{Algbrd}(\mathrm{Spc})_{\mathrm{Spc}}$ comes equipped with an involution $(-)^{\mathrm{op}}$. The map ρ intertwines the involutions $(-)^{\mathrm{op}}$ on Cat and $\mathrm{Algbrd}(\mathrm{Spc})_{\mathrm{Spc}}$. It follows from the uniqueness statement in proposition 3.4.5 that the equivalence between $\mathcal{P}(\Delta)_{\mathrm{Seg}}$ and $\mathrm{Algbrd}(\mathrm{Spc})_{\mathrm{Spc}}$ admits a $\mathbb{Z}/2\mathbb{Z}$-equivariant structure.

We now review the definition of $\mathcal{M}$-enriched categories. These are $\mathcal{M}$-algebroids satisfying a suitable completeness condition.

Notation 3.4.7. Let $\mathcal{M}$ be an associative operad and equip Spc with its cartesian monoidal structure. We denote by $\tau_{\mathcal{M}} : \mathcal{M} \to \mathrm{Spc}$ the morphism of associative operads which maps each object m in $\mathcal{M}$ to the space of operations from the empty list into m.

Remark 3.4.8. Let $\mathcal{M}$ be monoidal category. If $\mathcal{M}$ is presentable monoidal then the lax symmetric monoidal functor $\tau_{\mathcal{M}} : \mathcal{M} \to \mathrm{Spc}$ is right adjoint to the unit map $\mathrm{Spc} \to \mathcal{M}$. In general, $\tau_{\mathcal{M}}$ can be obtained as the composition of the symmetric monoidal embedding $\mathcal{M} \to \mathcal{P}(\mathcal{M})$ together with the lax symmetric monoidal map $\tau_{\mathcal{P}(\mathcal{M})} : \mathcal{P}(\mathcal{M}) \to \mathrm{Spc}$.

Definition 3.4.9. *Let $\mathcal{M}$ be an associative operad. An object $\mathcal{A}$ in $\mathrm{Algbrd}(\mathcal{M})$ is said to be an enriched category if it has a space of objects, and the induced object $(\tau_{\mathcal{M}})_!\mathcal{A}$ in $\mathrm{Algbrd}(\mathrm{Spc})_{\mathrm{Spc}}$ belongs to the image of ρ. We denote by $\mathrm{Cat}^{\mathcal{M}}$ the full subcategory of $\mathrm{Algbrd}(\mathcal{M})$ on the enriched categories. Given an $\mathcal{M}$-enriched category $\mathcal{A}$ and a pair of objects x, y in $\mathcal{A}$, we will usually use the notation $\mathrm{Hom}_{\mathcal{A}}(x, y)$ instead of $\mathcal{A}(y, x)$.*

In other words, an $\mathcal{M}$-enriched category is a $\mathcal{M}$-algebroid whose underlying Segal space is a complete Segal space.

Example 3.4.10. Let $\mathcal{M}$ be a monoidal category such that the monoid $\mathrm{End}_{\mathcal{M}}(1_{\mathcal{M}})$ does not have nontrivial invertible elements (for instance, if $\mathcal{M}$ is a cartesian closed presentable category). Then the unit algebroid $1_{\mathcal{M}}$ is an $\mathcal{M}$-enriched category. If in addition $\mathcal{M}$ has an initial object which is compatible with the monoidal structure, and the space of maps from the unit to the initial object is empty, then for every m in $\mathcal{M}$ the cell C_m from example 3.3.6 is an $\mathcal{M}$-enriched category.

Remark 3.4.11. It follows from remark 3.4.6 that an algebroid $\mathcal{A}$ is an enriched category if and only if $\mathcal{A}^{\mathrm{op}}$ is an enriched category. In other words, the involution $(-)^{\mathrm{op}}$ restricts to an involution on the full subcategory of Algbrd on the enriched categories.

Proposition 3.4.12. *Let $\mathcal{M}$ be a presentable monoidal category. Then*

(i) *The inclusion $\mathrm{Cat}^{\mathcal{M}} \to \mathrm{Algbrd}(\mathcal{M})_{\mathrm{Spc}}$ exhibits $\mathrm{Cat}^{\mathcal{M}}$ as an accessible localization of $\mathrm{Algbrd}(\mathcal{M})_{\mathrm{Spc}}$. In particular, $\mathrm{Cat}^{\mathcal{M}}$ is presentable.*

(ii) *Let $F : \mathcal{M} \to \mathcal{M}'$ be a colimit preserving monoidal functor into another presentable monoidal category. Then the functor $F_! : \mathrm{Algbrd}(\mathcal{M})_{\mathrm{Spc}} \to \mathrm{Algbrd}(\mathcal{M}')_{\mathrm{Spc}}$ descends to a functor $\mathrm{Cat}^{\mathcal{M}} \to \mathrm{Cat}^{\mathcal{M}'}$.*

Proof. Recall that Cat embeds into the category of Segal spaces as the full subcategory of local objects for the projection α from the walking isomorphism to the terminal category. Since the lax monoidal functor $\tau_{\mathcal{M}}$ is right adjont to the unit map $1_{\mathcal{M}} : \mathrm{Spc} \to \mathcal{M}$, we obtain an adjunction

$$(1_{\mathcal{M}})_! : \mathrm{Algbrd}(\mathrm{Spc})_{\mathrm{Spc}} \rightleftarrows \mathrm{Algbrd}(\mathcal{M})_{\mathrm{Spc}} : (\tau_{\mathcal{M}})_! \ .$$

It follows that $\mathrm{Cat}^{\mathcal{M}}$ is the full subcategory of $\mathrm{Algbrd}(\mathcal{M})_{\mathrm{Spc}}$ of $i_!\alpha$ local objects, which proves item (i). Item (ii) now follows from the fact that $F_!(1_{\mathcal{M}})_!\alpha$ is equivalent to $(1_{\mathcal{M}'})_!\alpha$, which becomes an isomorphism upon projection to $\mathrm{Cat}^{\mathcal{M}'}$. □

Remark 3.4.13. Let $\mathcal{M}$ be a monoidal category. Then $\mathrm{Algbrd}(\mathcal{M})_{\mathrm{Spc}}$ is a full subcategory of $\mathrm{Algbrd}(\mathcal{P}(\mathcal{M}))_{\mathrm{Spc}}$, and moreover $\mathrm{Cat}^{\mathcal{M}}$ is the intersection of $\mathrm{Algbrd}(\mathcal{M})_{\mathrm{Spc}}$ with $\mathrm{Cat}^{\mathcal{P}(\mathcal{M})}$. Let $\mathcal{C}$ be an object in $\mathrm{Algbrd}(\mathcal{M})_{\mathrm{Spc}}$ and let $\mathcal{C}'$ be its image in $\mathrm{Cat}^{\mathcal{P}(\mathcal{M})}$. It follows from the description of local equivalences from [GH15] corollary 5.6.3 that $\mathcal{C}'$ belongs to $\mathrm{Cat}^{\mathcal{M}}$. It follows that $\mathrm{Cat}^{\mathcal{M}}$ is a localization of $\mathrm{Algbrd}(\mathcal{M})_{\mathrm{Spc}}$, and moreover a map $\mathcal{A} \to \mathcal{A}'$ in $\mathrm{Algbrd}(\mathcal{M})_{\mathrm{Spc}}$ is local if and only if it is fully faithful (i.e., cartesian for the projection $\mathrm{Algbrd}(\mathcal{M})_{\mathrm{Spc}} \to \mathrm{Cat}$) and surjective on objects.

Example 3.4.14. Let $\mathcal{M}$ be a presentable monoidal category. As a consequence of proposition 3.4.12 the unit map $1_{\mathcal{M}} : \mathrm{Spc} \to \mathcal{M}$ induces a functor $(1_{\mathcal{M}})_! : \mathrm{Cat} = \mathrm{Cat}^{\mathrm{Spc}} \to \mathrm{Cat}^{\mathcal{M}}$. In other words, any category defines an $\mathcal{M}$-enriched category.

Remark 3.4.15. Let $i : \mathcal{M} \to \mathcal{M}'$ be a colimit preserving monoidal functor between presentable monoidal categories. Assume that i is fully faithful, so that the functor $i_! : \mathrm{Algbrd}(\mathcal{M}) \to \mathrm{Algbrd}(\mathcal{M}')$ is fully faithful. Then for every $\mathcal{M}$-algebroid $\mathcal{A}$ with a space of objects, the Segal space underlying $i_!\mathcal{A}$ is equivalent to the Segal space underlying $\mathcal{A}$. In particular, $\mathcal{A}$ is an $\mathcal{M}$-enriched category if and only if $i_!\mathcal{A}$ is an $\mathcal{M}'$-enriched category. This implies that the commutative square

$$\begin{tikzcd}
\mathrm{Algbrd}(\mathcal{M})_{\mathrm{Spc}} \arrow[r] \arrow[d, "i_!"] & \mathrm{Cat}^{\mathcal{M}} \arrow[d, "i_!"] \\
\mathrm{Algbrd}(\mathcal{M}')_{\mathrm{Spc}} \arrow[r] & \mathrm{Cat}^{\mathcal{M}'}
\end{tikzcd}$$

arising from proposition 3.4.12 item (ii), is horizontally right adjointable.

Assume now that i admits a left adjoint, so that $\mathcal{M}$ is a localization of $\mathcal{M}'$. Then for every space X the functor $i_! : \mathrm{Algbrd}_X(\mathcal{M}) \to \mathrm{Algbrd}_X(\mathcal{M}')$ preserves limits and is accessible. It follows from [Lur09a] proposition 4.3.1.9 together with the fact that the projection $\mathrm{Algbrd}(\mathcal{M}')_{\mathrm{Spc}} \to \mathrm{Spc}$ is both a cartesian and a cocartesian fibration, that the functor $i_! : \mathrm{Algbrd}(\mathcal{M}) \to \mathrm{Algbrd}(\mathcal{M}')$ preserves limits and is accessible. It now follows from the adjoint functor theorem that the above square is in fact also vertically left adjointable. In particular, we have that $\mathrm{Cat}^{\mathcal{M}}$ is a localization of $\mathrm{Cat}^{\mathcal{M}'}$.

We can describe this in more concrete terms. Let $\mathcal{A}$ be an $\mathcal{M}'$-enriched category. Then $\mathcal{A}$ belongs to the image of $i_!$ if and only if for every pair of objects x, y in $\mathcal{A}$ we have that $\mathrm{Hom}_{\mathcal{A}}(x, y)$ belongs to $\mathcal{M}$. Equivalently, for every object m' in $\mathcal{M}'$, the map

$$\mathrm{Hom}_{\mathcal{M}'}(ii^L m', \mathrm{Hom}_{\mathcal{A}}(x, y)) \to \mathrm{Hom}_{\mathcal{M}'}(m', \mathrm{Hom}_{\mathcal{A}}(x, y))$$

given by precomposition with the unit $m' \to ii^L m'$, is an equivalence. It follows that $\mathcal{A}$ belongs to the image of $i_!$ if and only if it is local for the class of maps $C_{m'} \to C_{ii^L m'}$. Note that we can simplify this further: we may take m' to belong to a set of generators of $\mathcal{M}'$.

3.5 Multiplicativity

We now discuss the notion of tensor product of algebroids and enriched categories.

Proposition 3.5.1. *The category* Algbrd *has finite products, which are preserved by the projection to* $\mathrm{Cat} \times \mathrm{Op}_{\mathrm{Assos}}$.

Proof. Let X be a category. Then the functor $\mathrm{Algbrd}_X(-) : \mathrm{Op}_{\mathrm{Assos}} \to \mathrm{Cat}$ is limit preserving. It follows from [Lur09a] corollary 4.3.1.15, that the total category Algbrd_X of the associated cocartesian fibration has all finite products, which are preserved by the projection to $\mathrm{Op}_{\mathrm{Assos}}$. Furthermore, if $\{\mathcal{A}_i\}_{i\in\mathcal{I}}$ is a finite family of objects of Algbrd_X lying above a finite family of associative operads $\{\mathcal{M}_i\}_{i\in\mathcal{I}}$, then its product is the unique object $\mathcal{A}$ in $\mathrm{Algbrd}_X(\prod \mathcal{M}_i)$ equipped with cocartesian arrows to $\mathcal{A}_i$ lifting the projection $\prod \mathcal{M}_i \to \mathcal{M}_i$, for all i in $\mathcal{I}$.

Assume now given a functor of categories $f : Y \to X$. Then $f^!\mathcal{A}$ is an object in $\mathrm{Algbrd}_Y(\prod \mathcal{M}_i)$ which comes equipped with cocartesian arrows to $f^!\mathcal{A}_i$ lifting the projections $\prod \mathcal{M}_i \to \mathcal{M}_i$, for all i in $\mathcal{I}$. It follows that $f^!$ preserves finite products. By a combination of [Lur09a] propositions 4.3.1.9 and 4.3.1.10 we conclude that the projection $\mathrm{Algbrd} \to \mathrm{Cat}$ has all relative finite products, which are preserved by the map $\mathrm{Algbrd} \to \mathrm{Cat} \times \mathrm{Op}_{\mathrm{Assos}}$. Our result now follows from the fact that Cat has all finite products. $\square$

Remark 3.5.2. It follows from proposition 3.5.1 that the final object of Algbrd is the unique algebroid lying above the final object of $\mathrm{Cat} \times \mathrm{Op}_{\mathrm{Assos}}$. In other words, this is the unit algebroid of the final monoidal category.

Notation 3.5.3. Let $\mathcal{A}, \mathcal{B}$ be objects of Algbrd. We denote by $\mathcal{A} \boxtimes \mathcal{B}$ their product in Algbrd.

Remark 3.5.4. Let X, Y be categories and $\mathcal{M}, \mathcal{N}$ be associative operads. Let $\mathcal{A}, \mathcal{B}$ be objects in $\mathrm{Algbrd}_X(\mathcal{M})$ and $\mathrm{Algbrd}_Y(\mathcal{N})$, respectively. Denote by p_1, p_2 the projections from Algbrd to Cat and $\mathrm{Op}_{\mathrm{Assos}}$, respectively. It follows from the proof of proposition 3.5.1 that the span

$$\mathcal{A} \leftarrow \mathcal{A} \boxtimes \mathcal{B} \to \mathcal{B}$$

is the unique lift of the span

$$(X, \mathcal{M}) \leftarrow (X \times Y, \mathcal{M} \times \mathcal{N}) \to (Y, \mathcal{N})$$

such that its left and right legs can be written as the composition of a p_2-cocartesian followed by a p_1-cartesian morphism.

It follows that $\mathcal{A} \boxtimes \mathcal{B}$ is the algebroid defined by the map

$$\mathrm{Assos}_{X\times Y} = \mathrm{Assos}_X \times \mathrm{Assos}_Y \xrightarrow{\mathcal{A}\times\mathcal{B}} \mathcal{M} \times \mathcal{N}$$

and the projections to $\mathcal{A}$ and $\mathcal{B}$ are induced from the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Assos}_X \arrow[d, "\mathcal{A}"] & \mathrm{Assos}_X \times \mathrm{Assos}_Y \arrow[l] \arrow[r] \arrow[d, "\mathcal{A}\times\mathcal{B}"] & \mathrm{Assos}_Y \arrow[d, "\mathcal{B}"] \\
\mathcal{M} & \mathcal{M} \times \mathcal{N} \arrow[l] \arrow[r] & \mathcal{N}
\end{tikzcd}$$

In particular, $\mathcal{A} \boxtimes \mathcal{B}$ is an $(\mathcal{M} \times \mathcal{M}')$-algebroid with category of objects $X \times Y$, and for every pair of objects $(x', y'), (x, y)$ we have an equivalence

$$(\mathcal{A} \boxtimes \mathcal{B})((x', y'), (x, y)) = (\mathcal{A}(x', x), \mathcal{B}(y', y)).$$

The composition maps for $\mathcal{A} \boxtimes \mathcal{B}$ are obtained by taking the product of the composition maps of $\mathcal{A}$ and $\mathcal{B}$.

Proposition 3.5.5. *Let $f : \mathcal{A} \to \mathcal{A}'$ be a morphism in* Algbrd *and let $\mathcal{B}$ be another object of* Algbrd. *Denote by $p = (p_1, p_2)$ the projection* $\mathrm{Algbrd} \to \mathrm{Cat} \times \mathrm{Op}_{\mathrm{Assos}}$.

(i) *If f is p_1-cartesian then $f \boxtimes \mathrm{id}_{\mathcal{B}}$ is p_1-cartesian.*

(ii) *If f is p_2-cocartesian then $f \boxtimes \mathrm{id}_{\mathcal{B}}$ is p_2-cocartesian.*

Proof. Denote by X, X', Y the categories of objects of $\mathcal{A}, \mathcal{A}'$ and $\mathcal{B}$, respectively, and let $\mathcal{M}, \mathcal{M}', \mathcal{N}$ be their underlying associative operads. Consider the following commutative diagram in $\mathrm{Cat} \times \mathrm{Op}_{\mathrm{Assos}}$:

$$\begin{tikzcd}
(X \times Y, \mathcal{M} \times \mathcal{N}) \arrow[r, "(\mathrm{id}, p_{\mathcal{M}})"] \arrow[d, "(p_1 f \times \mathrm{id}, p_2 f \times \mathrm{id})"] & (X \times Y, \mathcal{M}) \arrow[r, "(p_X, \mathrm{id}_{\mathcal{M}})"] \arrow[d, "(p_1 f \times \mathrm{id}, p_2 f)"] & (X, \mathcal{M}) \arrow[d, "(p_1 f, p_2 f)"] \\
(X' \times Y, \mathcal{M}' \times \mathcal{N}) \arrow[r, "(\mathrm{id}, p_{\mathcal{M}'})"] & (X' \times Y, \mathcal{M}') \arrow[r, "(p_{X'}, \mathrm{id})"] & (X', \mathcal{M}')
\end{tikzcd}$$

This admits a lift to a commutative diagram

$$\begin{tikzcd}
\mathcal{A} \boxtimes \mathcal{B} \arrow[r] \arrow[d, "f \boxtimes \mathrm{id}_{\mathcal{B}}"] & \overline{\mathcal{A}} \arrow[r] \arrow[d, "\mu"] & \mathcal{A} \arrow[d, "f"] \\
\mathcal{A}' \boxtimes \mathcal{B} \arrow[r] & \overline{\mathcal{A}'} \arrow[r] & \mathcal{A}'
\end{tikzcd}$$

where the horizontal rows are the factorizations of the projections as p_2-cocartesian maps followed by p_1-cartesian maps.

Similarly, the commutative diagram

$$\begin{tikzcd}
(X \times Y, \mathcal{M} \times \mathcal{N}) \arrow[r, "(\mathrm{id}, p_{\mathcal{N}})"] \arrow[d, "(p_1 f \times \mathrm{id}, p_2 f \times \mathrm{id})"] & (X \times Y, \mathcal{N}) \arrow[r, "(p_Y, \mathrm{id})"] \arrow[d, "(p_1 f \times \mathrm{id}, \mathrm{id})"] & (Y, \mathcal{N}) \arrow[d, "(\mathrm{id}, \mathrm{id})"] \\
(X' \times Y, \mathcal{M}' \times \mathcal{N}) \arrow[r, "(\mathrm{id}, p_{\mathcal{N}})"] & (X' \times Y, \mathcal{N}) \arrow[r, "(p_Y, \mathrm{id})"] & (Y, \mathcal{N})
\end{tikzcd}$$

admits a lift to a commutative diagram

$$\begin{tikzcd}
\mathcal{A} \boxtimes \mathcal{B} \arrow[r] \arrow[d, "f \boxtimes \mathrm{id}_{\mathcal{B}}"] & \overline{\mathcal{B}} \arrow[r] \arrow[d, "\nu"] & \mathcal{B} \arrow[d, "\mathrm{id}"] \\
\mathcal{A}' \boxtimes \mathcal{B} \arrow[r] & \widetilde{\mathcal{B}} \arrow[r] & \mathcal{B}
\end{tikzcd}$$

where the horizontal rows consist of a p_2-cocartesian followed by a p_1-cartesian map.

Assume now that f is p_1-cartesian, so that $\mathcal{M} = \mathcal{M}'$. Then μ and ν are p_1-cartesian. Write $f \boxtimes \mathrm{id}_{\mathcal{B}} = \alpha\eta$ where α is p_1-cartesian and η is such that $(p_1, p_2)\eta$ is invertible. We have

$$\mu = (p_{\mathcal{M}})_!(f \boxtimes \mathrm{id}_{\mathcal{B}}) = (p_{\mathcal{M}})_!(\alpha)(p_{\mathcal{M}})_!(\eta).$$

Since $(p_{\mathcal{M}})_!$ is a morphism of cartesian fibrations, we have that $(p_{\mathcal{M}})_!(\alpha)$ is cartesian and therefore $(p_{\mathcal{M}})_!(\eta)$ is an isomorphism. Similarly, we can conclude that $(p_{\mathcal{N}})_!(\eta)$ is an isomorphism. Item (i) now follows from the fact that the projections

$$\mathrm{Algbrd}_{X\times Y}(\mathcal{N}) \xleftarrow{(p_{\mathcal{N}})_!} \mathrm{Algbrd}_{X\times Y}(\mathcal{M}\times\mathcal{N}) \xrightarrow{(p_{\mathcal{M}})_!} \mathrm{Algbrd}_{X\times Y}(\mathcal{M})$$

are jointly conservative.

We now prove item (ii). In this case, f is p_2-cocartesian, so that $X = X'$. We therefore have that ν is invertible. Furthermore, we have that $\mu = p^!_X f$ is p_2-cocartesian. As before, write $f \boxtimes \mathrm{id}_{\mathcal{B}} = \eta\alpha$ where α is p_2-cocartesian and η is such that $(p_1, p_2)\eta$ is invertible. The composition of η with the p_2-cocartesian map $p_{\overline{\mathcal{A}'}} : \mathcal{A}' \boxtimes \mathcal{B} \to \overline{\mathcal{A}'}$ is a lift of the projection $(\mathrm{id}_{X\times Y}, p_{\mathcal{M}'})$ whose composition with the p_2-cocartesian map α is p_2-cocartesian. It follows that $p_{\overline{\mathcal{A}'}}\eta$ is p_2-cocartesian, and therefore we have that $(p_{\mathcal{M}'})_!\eta$ is an isomorphism. A similar argument shows that $(p_{\mathcal{N}})_!\eta$ is an isomorphism. Our result now follows from the fact that the projections

$$\mathrm{Algbrd}_{X\times Y}(\mathcal{N}) \xleftarrow{(p_{\mathcal{N}})_!} \mathrm{Algbrd}_{X\times Y}(\mathcal{M}'\times\mathcal{N}) \xrightarrow{(p_{\mathcal{M}'})_!} \mathrm{Algbrd}_{X\times Y}(\mathcal{M}')$$

are jointly conservative. $\square$

Construction 3.5.6. We equip Algbrd and $\mathrm{Cat} \times \mathrm{Op}_{\mathrm{Assos}}$ with their cartesian symmetric monoidal structures, so that the projection $\mathrm{Algbrd} \to \mathrm{Cat} \times \mathrm{Op}_{\mathrm{Assos}}$ inherits a canonical symmetric monoidal structure by proposition 3.5.1. It follows from proposition 3.5.5 that the projection $\mathrm{Algbrd} \to \mathrm{Op}_{\mathrm{Assos}}$ is a cocartesian fibration of operads, which straightens to a lax symmetric monoidal structure on the functor $\mathrm{Algbrd}(-) : \mathrm{Op}_{\mathrm{Assos}} \to \widehat{\mathrm{Cat}}$. Given $\mathcal{M}$ and $\mathcal{N}$ two associative operads, this produces a functor

$$\mathrm{Algbrd}(\mathcal{M}) \times \mathrm{Algbrd}(\mathcal{N}) \to \mathrm{Algbrd}(\mathcal{M} \times \mathcal{N})$$

which sends a pair of algebroids $\mathcal{A}, \mathcal{B}$ to $\mathcal{A} \boxtimes \mathcal{B}$.

Let $\mathcal{M}$ be a symmetric monoidal category. We can think about $\mathcal{M}$ as a commutative algebra object in Alg(Cat), and hence as a commutative algebra object in $\mathrm{Op}_{\mathrm{Assos}}$. It follows that $\mathrm{Algbrd}(\mathcal{M})$ inherits a symmetric monoidal structure. We will usually denote by

$$\otimes : \mathrm{Algbrd}(\mathcal{M}) \times \mathrm{Algbrd}(\mathcal{M}) \to \mathrm{Algbrd}(\mathcal{M})$$

the resulting functor. We note that the assignment $\mathcal{M} \mapsto (\mathrm{Algbrd}(\mathcal{M}), \otimes)$ is part of a functor $\mathrm{CAlg}(\mathrm{Cat}) \to \mathrm{CAlg}(\widehat{\mathrm{Cat}})$.

Remark 3.5.7. Let $\mathcal{M}$ be a symmetric monoidal category. The unit of the symmetric monoidal structure on $\mathrm{Algbrd}(\mathcal{M})$ is the unit algebra in $\mathcal{M}$, thought of as an algebroid with category of objects $[0]$. The tensor product functor on $\mathrm{Algbrd}(\mathcal{M})$ can be computed as the composition

$$\mathrm{Algbrd}(\mathcal{M}) \times \mathrm{Algbrd}(\mathcal{M}) \xrightarrow{\boxtimes} \mathrm{Algbrd}(\mathcal{M} \times \mathcal{M}) \xrightarrow{m_!} \mathcal{M}$$

where $m : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ is the tensoring map. In particular, if $\mathcal{A}$ and $\mathcal{B}$ have category of objects X and Y respectively then $\mathcal{A} \otimes \mathcal{B}$ has category of objects $X \times Y$. Moreover, if x, x' are objects in X and y, y' are objects in Y, we have an equivalence

$$(\mathcal{A} \otimes \mathcal{B})((x', y'), (x, y)) = \mathcal{A}(x', x) \otimes \mathcal{A}(y', y).$$

Proposition 3.5.8. *Let $\mathcal{M}$ be a category admitting finite products, equipped with the cartesian symmetric monoidal structure. Then the symmetric monoidal structure on* $\mathrm{Algbrd}(\mathcal{M})$ *given by construction 3.5.6 is cartesian.*

Proof. As observed in remark 3.5.7, the unit $1_{\mathrm{Algbrd}(\mathcal{M})}$ of $\mathrm{Algbrd}(\mathcal{M})$ is the unit algebra in $\mathcal{M}$. To check that $1_{\mathrm{Algbrd}(\mathcal{M})}$ is final in $\mathrm{Algbrd}(\mathcal{M})$ we have to see that for every category X, the algebroid $\pi_X^! 1_{\mathrm{Algbrd}(\mathcal{M})}$ is final in $\mathrm{Algbrd}_X(\mathcal{M})$, where $\pi_X : X \to [0]$ denotes the projection. Indeed, for every pair of objects x, y in X we have

$$\pi_X^! 1_{\mathrm{Algbrd}(\mathcal{M})}(y, x) = 1_{\mathrm{Algbrd}(\mathcal{M})}(\pi_X y, \pi_X x) = 1_{\mathcal{M}}$$

which is final in $\mathcal{M}$. The fact that $\pi_X^! 1_{\mathrm{Algbrd}(\mathcal{M})}$ is final then follows from [Lur17] corollary 3.2.2.5.

It remains to check that for every pair of algebroids $\mathcal{A}, \mathcal{B}$ in $\mathcal{M}$, the projections

$$\mathcal{A} = \mathcal{A} \otimes 1_{\mathrm{Algbrd}(\mathcal{M})} \leftarrow \mathcal{A} \otimes \mathcal{B} \rightarrow 1_{\mathrm{Algbrd}(\mathcal{M})} \otimes \mathcal{B} = \mathcal{B}$$

exhibit $\mathcal{A} \otimes \mathcal{B}$ as the product of $\mathcal{A}$ and $\mathcal{B}$ in $\mathrm{Algbrd}(\mathcal{M})$. Let X, Y be the category of objects of $\mathcal{A}, \mathcal{B}$ respectively. We have to show that for every category Z equipped with functors $j : Z \to X$ and $j' : Z \to Y$, the projections

$$j^! \mathcal{A} \leftarrow (j \times j')^! (\mathcal{A} \otimes \mathcal{B}) \rightarrow j'^! \mathcal{B}$$

exhibit $(j \times j')^! (\mathcal{A} \otimes \mathcal{B})$ as the product of $j^! \mathcal{A}$ and $j'^! \mathcal{B}$ in the category $\mathrm{Algbrd}_Z(\mathcal{M})$. Let z, w be objects in Z. The induced diagram

$$j^! \mathcal{A}(z, w) \leftarrow (j \times j')^! (\mathcal{A} \otimes \mathcal{B})(z, w) \rightarrow j'^! \mathcal{B}(z, w)$$

is the equivalent to the diagram

$$\mathcal{A}(jz, jw) = \mathcal{A}(jz, jw) \otimes 1_{\mathcal{M}} \leftarrow \mathcal{A}(jz, jw) \otimes \mathcal{B}(j'z, j'w) \rightarrow 1_{\mathcal{M}} \otimes \mathcal{B}(j'z, j'w) = \mathcal{B}(j'z, j'w)$$

and therefore it exhibits $(j \times j')^! (\mathcal{A} \otimes \mathcal{B})(z, w)$ as the product of $j^! \mathcal{A}(z, w)$ and $j'^! \mathcal{B}(z, w)$. Our result now follows from another application of [Lur17] corollary 3.2.2.5. $\square$

The symmetric monoidal structure on algebroids from construction 3.5.6 restricts to algebroids with a space of objects. The next proposition shows that this induces a symmetric monoidal structure on enriched categories.

Proposition 3.5.9. *Let $\mathcal{M}$ be a symmetric monoidal category. Then the localization functor* $\mathrm{Algbrd}(\mathcal{M})_{\mathrm{Spc}} \to \mathrm{Cat}^{\mathcal{M}}$ *is compatible with the restriction of the symmetric monoidal structure of construction 3.5.6.*

Proof. Recall that a morphism $F : \mathcal{A} \to \mathcal{B}$ in $\mathrm{Algbrd}(\mathcal{M})_{\mathrm{Spc}}$ is local for the localization in the statement if and only if it is fully faithful and surjective on objects. Equivalently, this means that F is p_1-cartesian and surjective on objects.

Let $\mathcal{A}'$ be another object of $\mathrm{Algbrd}(\mathcal{M})_{\mathrm{Spc}}$. It follows from propositions 3.5.1 and 3.5.5 that

$$F \boxtimes \mathrm{id}_{\mathcal{A}'} : \mathcal{A} \boxtimes \mathcal{A}' \to \mathcal{B} \boxtimes \mathcal{A}'$$

is still fully faithful and surjective on objects. Therefore the map $F \otimes \mathrm{id}_{\mathcal{A}'} = m_!(F \boxtimes \mathrm{id}_{\mathcal{A}'})$ is also fully faithful and surjective on objects, so it is local for the localization in the statement, as desired. □

Corollary 3.5.10. *Let $\mathcal{M}$ be a symmetric monoidal category. Then* $\mathrm{Cat}^{\mathcal{M}}$ *inherits a symmetric monoidal structure from* $\mathrm{Algbrd}(\mathcal{M})_{\mathrm{Spc}}$*, and the localization* $\mathrm{Algbrd}(\mathcal{M})_{\mathrm{Spc}} \to \mathrm{Cat}^{\mathcal{M}}$ *has a canonical symmetric monoidal structure.*

Example 3.5.11. Let $\mathcal{M}$ be a category admitting finite products, equipped with its cartesian symmetric monoidal structure. Then it follows from proposition 3.5.8 that the induced symmetric monoidal structure on $\mathrm{Cat}^{\mathcal{M}}$ is cartesian.

For later purposes, we will need a generalization of the functoriality of construction 3.5.6 which deals with lax symmetric monoidal functors between symmetric monoidal categories. In fact, it turns out that for any symmetric operad $\mathcal{M}$ one can give $\mathrm{Algbrd}(\mathcal{M})$ and $\mathrm{Cat}^{\mathcal{M}}$ the structure of a symmetric operad, in a way that depends functorially on $\mathcal{M}$.

Construction 3.5.12. Denote by $\mathrm{Env} : \mathrm{Op} \to \mathrm{CAlg}(\mathrm{Cat})$ the functor that sends each symmetric operad to its enveloping symmetric monoidal category - in other words, this is left adjoint to the inclusion $\mathrm{CAlg}(\mathrm{Cat}) \to \mathrm{Op}$. Consider now the composite functor

$$\xi : \mathrm{Op} \xrightarrow{\mathrm{Env}} \mathrm{CAlg}(\mathrm{Cat}) \xrightarrow{\mathrm{Algbrd}(-)} \mathrm{CAlg}(\widehat{\mathrm{Cat}}).$$

Note that the composition of ξ with the forgetful functor $\mathrm{CAlg}(\widehat{\mathrm{Cat}}) \to \widehat{\mathrm{Cat}}$ receives a natural transformation η from the functor $\mathrm{Algbrd}(-)|_{\mathrm{Op}} : \mathrm{Op} \to \widehat{\mathrm{Cat}}$. For each symmetric operad $\mathcal{M}$, this induces a functor

$$\eta(\mathcal{M}) : \mathrm{Algbrd}(\mathcal{M}) \to \xi(\mathcal{M}) = \mathrm{Algbrd}(\mathrm{Env}(\mathcal{M})).$$

Since the unit map $\mathcal{M} \to \mathrm{Env}(\mathcal{M})$ is an inclusion of symmetric operads, we have that $\eta(\mathcal{M})$ is fully faithful. Its image consists of those $\mathrm{Env}(\mathcal{M})$-algebroids $\mathcal{A}$ such that $\mathcal{A}(y, x)$ belongs to $\mathcal{M}$ for each pair of objects y, x in $\mathcal{A}$. Since $\mathrm{Algbrd}(\mathrm{Env}(\mathcal{M}))$ has a symmetric monoidal structure, the full subcategory $\mathrm{Algbrd}(\mathcal{M})$ inherits the structure of a symmetric operad. This is compatible with morphisms of symmetric operads, so we obtain a lift of $\mathrm{Algbrd}(-)|_{\mathrm{Op}}$ to a functor

$$(\mathrm{Algbrd}(-)|_{\mathrm{Op}})^{\mathrm{enh}} : \mathrm{Op} \to \widehat{\mathrm{Op}}.$$

The following proposition shows that construction 3.5.12 extends the functoriality of the theory of algebroids on symmetric monoidal categories from construction 3.5.6.

Proposition 3.5.13. *The restriction of the functor* $(\mathrm{Algbrd}(-)|_{\mathrm{Op}})^{\mathrm{enh}}$ *to* $\mathrm{CAlg}(\mathrm{Cat})$ *factors through* $\mathrm{CAlg}(\widehat{\mathrm{Cat}})$*, and coincides with the functor arising from construction 3.5.6.*

Proof. Let $\mathcal{M}$ be a symmetric monoidal category. Then the inclusion $\mathcal{M} \to \mathrm{Env}(\mathcal{M})$ exhibits $\mathcal{M}$ as a symmetric monoidal localization of $\mathrm{Env}(\mathcal{M})$. It follows that the inclusion $\mathrm{Algbrd}(\mathcal{M}) \to \mathrm{Env}(\mathcal{M})$ exhibits $\mathrm{Algbrd}(\mathcal{M})$ (with its symmetric monoidal structure from construction 3.5.6) as a symmetric monoidal localization of $\mathrm{Algbrd}(\mathrm{Env}(\mathcal{M}))$. This shows that the operadic structure on $\mathrm{Algbrd}(\mathcal{M})$ from construction 3.5.12 coincides with the operadic structure underlying the symmetric monoidal structure given to in construction 3.5.6.

Assume now given a symmetric monoidal functor $F : \mathcal{M} \to \mathcal{M}'$ between symmetric monoidal categories. We have a commutative square of symmetric monoidal categories and symmetric monoidal functors

$$\begin{tikzcd} \mathrm{Algbrd}(\mathrm{Env}(\mathcal{M})) \arrow[r, "\mathrm{Env}(F)_!"] \arrow[d] & \mathrm{Algbrd}(\mathrm{Env}(\mathcal{M}')) \arrow[d] \\ \mathrm{Algbrd}(\mathcal{M}) \arrow[r, "F_!"] & \mathrm{Algbrd}(\mathcal{M}'). \end{tikzcd}$$

This is vertically right adjointable. Passing to right adjoints of the vertical arrows yields a commutative diagram of symmetric monoidal categories and lax symmetric monoidal functors

$$\begin{tikzcd} \mathrm{Algbrd}(\mathrm{Env}(\mathcal{M})) \arrow[r, "\mathrm{Env}(F)_!"] & \mathrm{Algbrd}(\mathrm{Env}(\mathcal{M}')) \\ \mathrm{Algbrd}(\mathcal{M}) \arrow[r, "F_!"] \arrow[u] & \mathrm{Algbrd}(\mathcal{M}'). \arrow[u] \end{tikzcd}$$

It follows that the structures of morphism of symmetric operads on $F_!$ arising from constructions 3.5.6 and 3.5.12 agree. In particular, we conclude that the restriction of $(\mathrm{Algbrd}(-)|_{\mathrm{Op}})^{\mathrm{enh}}$ to $\mathrm{CAlg}(\mathrm{Cat})$ factors through $\mathrm{CAlg}(\widehat{\mathrm{Cat}})$.

Consider now the lax commutative triangle

$$\begin{tikzcd} \mathrm{CAlg}(\mathrm{Cat}) \arrow[r, "\mathrm{Env}"] \arrow[dr, "\mathrm{Algbrd}(-)"'] & \mathrm{CAlg}(\mathrm{Cat}) \arrow[d, "\mathrm{Algbrd}(-)"] \arrow[dl, phantom, "\Leftarrow"] \\ & \mathrm{CAlg}(\widehat{\mathrm{Cat}}) \end{tikzcd}$$

obtained by applying $\mathrm{Algbrd}(-)$ to the counit of the defining adjunction for Env. Passing to right adjoints yields a commutative triangle

$$\begin{tikzcd}
\mathrm{CAlg}(\mathrm{Cat}) \arrow[r, "\mathrm{Env}"] \arrow[dr, "\mathrm{Algbrd}(-)"'] \arrow[dr, phantom, shift left=3, "\Nearrow"] & \mathrm{CAlg}(\mathrm{Cat}) \arrow[d, "\mathrm{Algbrd}(-)"] \\
 & \widehat{\mathrm{Op}}.
\end{tikzcd}$$

This identifies the diagonal arrow with the restriction of $(\mathrm{Algbrd}(-)|_{\mathrm{Op}})^{\mathrm{enh}}$ to $\mathrm{CAlg}(\mathrm{Cat})$. $\square$

Remark 3.5.14. Let $F : \mathcal{M} \to \mathcal{M}'$ be a symmetric monoidal functor between symmetric monoidal categories. Assume that F admits a right adjoint F^R, and equip F^R with its natural lax symmetric monoidal structure. Then F and F^R induce a symmetric monoidal adjunction

$$\mathrm{Env}(F) : \mathrm{Env}(\mathcal{M}) \rightleftarrows \mathrm{Env}(\mathcal{M}') : \mathrm{Env}(F^R).$$

This in turn induces a symmetric monoidal adjunction

$$\mathrm{Env}(F)_! : \mathrm{Algbrd}(\mathrm{Env}(\mathcal{M})) \rightleftarrows \mathrm{Algbrd}(\mathrm{Env}(\mathcal{M}')) : \mathrm{Env}(F^R)_!$$

which restricts to an adjunction with symmetric monoidal left adjoint

$$F_! : \mathrm{Algbrd}(\mathcal{M}) \rightleftarrows \mathrm{Algbrd}(\mathcal{M}') : F_!^R.$$

It follows from this that the lax symmetric monoidal structure on $F_!^R$ arising from construction 3.5.12 is equivalent to the one arising by passing to adjoints the symmetric monoidal structure on $F_!$.

Remark 3.5.15. Let $F : \mathcal{M} \to \mathcal{M}'$ be a lax symmetric monoidal functor between symmetric monoidal categories. Then the lax symmetric monoidal functor

$$F_! : \mathrm{Algbrd}(\mathcal{M}) \to \mathrm{Algbrd}(\mathcal{M}')$$

restricts to a lax symmetric monoidal functor

$$F_! : \mathrm{Algbrd}(\mathcal{M})_{\mathrm{Spc}} \to \mathrm{Algbrd}(\mathcal{M}')_{\mathrm{Spc}}$$

which in turn induces a lax symmetric monoidal functor $F_! : \mathrm{Cat}^{\mathcal{M}} \to \mathrm{Cat}^{\mathcal{M}'}$. This forms part of a functor

$$(\mathrm{Cat}^{(-)}|_{\mathrm{CAlg}(\mathrm{Cat})^{\mathrm{lax}}})^{\mathrm{enh}} : \mathrm{CAlg}(\mathrm{Cat})^{\mathrm{lax}} \to \mathrm{CAlg}(\widehat{\mathrm{Cat}})^{\mathrm{lax}}$$

where $\mathrm{CAlg}(\mathrm{Cat})^{\mathrm{lax}}$ denotes the category of symmetric monoidal categories and lax symmetric monoidal functors.

We now study the presentable symmetric monoidal case. The following result follows from a version of [GH15] corollary 4.3.16 - we refer the reader there for a proof.

Proposition 3.5.16. *Let $\mathcal{M}$ be a presentable symmetric monoidal category. Then the induced symmetric monoidal structures on* $\mathrm{Algbrd}(\mathcal{M})$ *and* $\mathrm{Cat}^{\mathcal{M}}$ *are compatible with colimits.*

Corollary 3.5.17. *Let $\mathcal{M}$ be a cartesian closed presentable category. Then* $\mathrm{Algbrd}(\mathcal{M})$ *and* $\mathrm{Cat}^{\mathcal{M}}$ *are cartesian closed.*

Proof. Combine proposition 3.5.16 with example 3.5.11. □

Notation 3.5.18. Let $\mathcal{M}$ be a presentable symmetric monoidal category. We denote by

$$\mathrm{Funct}(-,-) : \mathrm{Algbrd}(\mathcal{M})^{\mathrm{op}} \times \mathrm{Algbrd}(\mathcal{M}) \to \mathrm{Algbrd}(\mathcal{M})$$

the internal Hom for the closed symmetric monoidal category $\mathrm{Algbrd}(\mathcal{M})$.

Proposition 3.5.19. *Let $\mathcal{M}$ be a presentable symmetric monoidal category. Then the category* $\mathrm{Algbrd}(\mathcal{M})_{\mathrm{Spc}}$ *is both a symmetric monoidal localization and a symmetric monoidal colocalization of* $\mathrm{Algbrd}(\mathcal{M})$.

Proof. Consider the pullback square

$$\begin{tikzcd} \mathrm{Algbrd}(\mathcal{M})_{\mathrm{Spc}} \arrow[r, "i'"] \arrow[d, "p'"] & \mathrm{Algbrd}(\mathcal{M}) \arrow[d, "p"] \\ \mathrm{Spc} \arrow[r, "i"] & \mathrm{Cat}\,. \end{tikzcd}$$

Recall from proposition 3.3.12 that p is both a cartesian and cocartesian fibration. Since i has both left and right adjoints, we conclude that i' has both left and right adjoints as well.

Concretely, given an $\mathcal{M}$-algebroid $\mathcal{A}$ with category of objects X, the unit $\mathcal{A} \to i'i'^{L}\mathcal{A}$ is a p-cocartesian lift of the map $X \to {}^{\leq 0}X$ from X into its geometric realization, and the counit $i'i'^{R}\mathcal{A} \to \mathcal{A}$ is a p-cartesian lift of the map $X^{\leq 0} \to X$ which includes the space of objects of X inside X.

It remains to show that the adjoints to i' are compatible with the symmetric monoidal structure on $\mathrm{Algbrd}(\mathcal{M})$. Let $\mathcal{A}$ and $\mathcal{B}$ be a pair of $\mathcal{M}$-algebroids with category of objects X and Y, respectively. Denote by $\eta_{\mathcal{A}} : \mathcal{A} \to i'i'^{L}\mathcal{A}$ and $\epsilon_{\mathcal{A}} : i'i'^{R}\mathcal{A} \to \mathcal{A}$ the localization and colocalization of $\mathcal{A}$.

Applying propositions 3.5.1 and 3.5.5 together with remark 3.5.7 we see that the map

$$\mathrm{id}_{\mathcal{B}} \otimes \epsilon_{\mathcal{A}} : \mathcal{B} \otimes i'i'^{R}\mathcal{A} \to \mathcal{B} \otimes \mathcal{A}$$

is p-cartesian and lies above an i^{R}-colocal map. This implies that it is i'^{R}-colocal, and therefore we have that $\mathrm{Algbrd}(\mathcal{M})_{\mathrm{Spc}}$ is a symmetric monoidal colocalization of $\mathrm{Algbrd}(\mathcal{M})$.

Consider now the map

$$\mathrm{id}_{\mathcal{B}} \otimes \eta_{\mathcal{A}} : \mathcal{B} \otimes \mathcal{A} \to \mathcal{B} \otimes i' i'^L \mathcal{A}.$$

Its image under p is i^L-local by a combination of proposition 3.5.1 and remark 3.5.7, together with the fact that i^L preserves products. To prove that it $\mathrm{id}_{\mathcal{B}} \otimes \eta_{\mathcal{A}}$ is i'-local it now suffices to show that it is p-cocartesian. Using [GH15] lemma 3.6.15 we see that

$$\mathrm{id}_{\mathcal{B}} \boxtimes \eta_{\mathcal{A}} : \mathcal{B} \boxtimes \mathcal{A} \to \mathcal{B} \boxtimes i' i'^L \mathcal{A}$$

is cocartesian for the projection $\mathrm{Algbrd}(\mathcal{M} \times \mathcal{M}) \to \mathrm{Cat}$. In other words, $\mathrm{id}_{\mathcal{B}} \boxtimes \eta_{\mathcal{A}}$ exhibits $\mathcal{B} \boxtimes i' i'^L \mathcal{A}$ as the free $\mathrm{Assos}_{Y \times (\leq^0 X)}$-algebra on the $\mathrm{Assos}_{Y \times X}$-algebra $\mathcal{B} \boxtimes \mathcal{A}$. Our claim now follows from the fact that the multiplication map $m : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ preserves the operadic colimits involved in the description of this free algebra. □

Proposition 3.5.20. *Let $\mathcal{M}$ be a presentable symmetric monoidal category and let $\mathcal{A}, \mathcal{B}$ be two $\mathcal{M}$-algebroids.*

(i) *If $\mathcal{B}$ has a space of objects then* $\mathrm{Funct}(\mathcal{A}, \mathcal{B})$ *has a space of objects.*

(ii) *If $\mathcal{B}$ is an enriched category then* $\mathrm{Funct}(\mathcal{A}, \mathcal{B})$ *is an enriched category.*

Proof. Item (i) follows directly from the fact that $\mathrm{Algbrd}(\mathcal{M})_{\mathrm{Spc}}$ is a symmetric monoidal localization of $\mathrm{Algbrd}(\mathcal{M})$. Similarly, item (ii) follows from proposition 3.5.9. □

Corollary 3.5.21. *Let $\mathcal{M}$ be a presentable symmetric monoidal category. The functors* $\mathrm{Funct}(-,-)|_{\mathrm{Algbrd}(\mathcal{M})^{\mathrm{op}}_{\mathrm{Spc}} \times \mathrm{Algbrd}(\mathcal{M})_{\mathrm{Spc}}}$ *and* $\mathrm{Funct}(-,-)|_{(\mathrm{Cat}^{\mathcal{M}})^{\mathrm{op}} \times (\mathrm{Cat}^{\mathcal{M}})}$ *are equivalent to the internal Homs of* $\mathrm{Algbrd}(\mathcal{M})_{\mathrm{Spc}}$ *and* $\mathrm{Cat}^{\mathcal{M}}$*, respectively.*

Proof. This is a direct consequence of proposition 3.5.20. □

Remark 3.5.22. The involution $(-)^{\mathrm{op}} : \mathrm{Algbrd} \to \mathrm{Algbrd}$ is product preserving. It follows that if $\mathcal{M}$ is a symmetric monoidal category, then the involution $(-)^{\mathrm{op}}$ on $\mathrm{Algbrd}(\mathcal{M})$ respects the symmetric monoidal structure. In particular, given $\mathcal{M}$-algebroids $\mathcal{A}, \mathcal{B}$, there is an equivalence

$$\mathrm{Funct}(\mathcal{A}^{\mathrm{op}}, \mathcal{B}^{\mathrm{op}})^{\mathrm{op}} = \mathrm{Funct}(\mathcal{A}, \mathcal{B}).$$

Our next goal is to provide a concrete description of the product of cells, and use it to study functor algebroids in the case when the source is a cell.

Notation 3.5.23. Let $\mathcal{M}$ be a monoidal category with an initial object compatible with the monoidal structure and let m, m' be objects in $\mathcal{M}$. Let $X = \{a, b, c\}$ be the set with three elements. Let $C_{m,m'}$ be the free Assos_X-algebra equipped with maps $m \to C_{m,m'}(b, c)$ and $m' \to C_{m,m'}(a, b)$. This is characterized by the following properties:

- $C_{m,m'}(a, a) = C_{m,m'}(b, b) = C_{m,m'}(c, c) = 1_{\mathcal{M}}$.

- $C_{m,m'}(a,b) = m'$
- $C_{m,m'}(b,c) = m$.
- $C_{m,m'}(a,c) = m \otimes m'$
- $C_{m,m'}(b,a) = C_{m,m'}(c,a) = C_{m,m'}(c,b)$ is the initial object of $\mathcal{M}$.

We note that $C_{m,m'}$ fits into a pushout

$$\begin{tikzcd} 1_{\mathcal{M}} \arrow[r] \arrow[d] & C_m \arrow[d] \\ C_{m'} \arrow[r] & C_{m,m'} \end{tikzcd}$$

where the top horizontal arrow and left vertical arrows pick out the target and source objects, respectively.

Remark 3.5.24. Let $\mathcal{M}$ be a symmetric monoidal category with compatible initial object, and let m, m' be objects in $\mathcal{M}$. Then the algebroid $C_m \otimes C_{m'}$ has objects (i,j) for $0 \le i, j \le 1$. Its morphisms can be depicted schematically as follows:

$$\begin{tikzcd} (0,0) \arrow[r, "m"] \arrow[d, "m'"] \arrow[dr, "m\otimes m'"] & (1,0) \arrow[d, "m'"] \\ (0,1) \arrow[r, "m"] & (1,1) \end{tikzcd}$$

Every Hom-object which is not associated to an arrow in the above diagram is the initial object in $\mathcal{M}$. Note that $C_m \otimes C_{m'}$ fits into a commutative square

$$\begin{tikzcd} C_{m\otimes m'} \arrow[r] \arrow[d] & C_{m,m'} \arrow[d] \\ C_{m',m} \arrow[r] & C_m \otimes C_{m'} \end{tikzcd}$$

where:

- The right vertical arrow picks out the m-cell between $(0,0)$ and $(1,0)$ and the m'-cell between $(1,0)$ and $(1,1)$.
- The bottom horizontal arrow picks out the m'-cell between $(0,0)$ and $(0,1)$ and the m-cell between $(0,1)$ and $(1,1)$.
- The cell $C_{m,\otimes m'} \to C_m \otimes C_{m'}$ the $m \otimes m'$-cell between $(0,0)$ and $(1,1)$.

Proposition 3.5.25. *Let $\mathcal{M}$ be a symmetric monoidal category with compatible initial object, and let m, m' be objects in $\mathcal{M}$. Then the commutative square of $\mathcal{M}$-algebroids*

$$\begin{tikzcd}
C_{m\otimes m'} \arrow[r] \arrow[d] & C_{m,m'} \arrow[d] \\
C_{m',m} \arrow[r] & C_m \otimes C_{m'}
\end{tikzcd}$$

from remark 3.5.24, is a pushout square.

Proof. Observe first that the induced square at the level of objects is given by

$$\begin{tikzcd}
\{(0,0),(1,1)\} \arrow[r, "k"] \arrow[d, "l"] & \{(0,0),(1,0),(1,1)\} \arrow[d, "j"] \\
\{(0,0),(0,1),(1,1)\} \arrow[r, "i"] & \{0,1\} \times \{0,1\}
\end{tikzcd}$$

and is indeed a pushout square. Consider the following $\mathcal{M}$-algebroids:

$$\overline{C_{m\otimes m'}} = (jk)_! C_{m\otimes m'}$$
$$\overline{C_{m,m'}} = j_! C_{m,m'}$$
$$\overline{C_{m',m}} = i_! C_{m',m}$$

Here we denote with $(-)_!$ the functors induced from the fact that the projection map $\mathrm{Algbrd}(\mathcal{M}) \to \mathrm{Cat}$ is a cocartesian fibration. In other words, the above algebroids are obtained from the previous ones by adding extra objects so that they all have set of objects $\{0,1\}^2$, where the new Hom objects are declared to be the initial object of $\mathcal{M}$. Combining proposition 3.3.12 with [Lur09a] proposition 4.3.1.9, we reduce to showing that the induced square

$$\begin{tikzcd}
\overline{C_{m\otimes m'}} \arrow[r] \arrow[d] & \overline{C_{m,m'}} \arrow[d] \\
\overline{C_{m',m}} \arrow[r] & C_m \otimes C_{m'}
\end{tikzcd}$$

is a pushout square in $\mathrm{Algbrd}_{\{0,1\}^2}(\mathcal{M})$. Our claim now follows from the fact that the images of the above square under the evaluation functors $\mathrm{Algbrd}_{\{0,1\}^2}(\mathcal{M}) \to \mathcal{M}$ are pushout squares. □

Corollary 3.5.26. *Let $\mathcal{M}$ be a presentable symmetric monoidal category and let m be an object in $\mathcal{M}$. Let $\mathcal{A}$ be an $\mathcal{M}$-algebroid and let $\mu, \nu : C_m \to \mathcal{A}$ be two m-cells in $\mathcal{A}$. Denote by $\mathscr{H}om_{\mathcal{M}}$ the internal Hom functor for $\mathcal{M}$. Then there is a pullback square*

$$\begin{tikzcd}
\mathrm{Funct}(C_m, \mathcal{A})(\nu, \mu) \arrow[r] \arrow[d] & \mathcal{A}(\nu(0), \mu(0))) \arrow[d] \\
\mathcal{A}(\nu(1), \mu(1)) \arrow[r] & \mathscr{H}om_{\mathcal{M}}(m, \mathcal{A}(\nu(1), \mu(0)))
\end{tikzcd}$$

where the top horizontal and left vertical arrows are given by the source and target maps, and the right horizontal and bottom vertical arrows are induced by composition with the cells ν and μ, respectively.

Proof. Let m' be another object of $\mathcal{M}$. The pushout square of remark 3.5.24 can be enhanced to a colimit diagram

$$\begin{tikzcd}
1_{\mathcal{M}} \sqcup 1_{\mathcal{M}} \arrow[r] \arrow[d] \arrow[dr] & C_m \sqcup 1_{\mathcal{M}} \arrow[dr] & \\
1_{\mathcal{M}} \sqcup C_m \arrow[dr] & C_{m\otimes m'} \arrow[r] \arrow[d] & C_{m,m'} \arrow[d] \\
& C_{m',m} \arrow[r] & C_m \otimes C_{m'}
\end{tikzcd}$$

where map $1_{\mathcal{M}} \sqcup 1_{\mathcal{M}} \to C_{m\otimes m'}$ picks out the source and target objects, and the maps $1_{\mathcal{M}} \sqcup C_m \to C_{m',m}$ and $C_m \sqcup 1_{\mathcal{M}} \to C_{m,m'}$ pick out the m-cell and the object which is not contained in it.

Let X be the category of objects of $\mathcal{A}$. We have an induced limit diagram of spaces

$$\begin{tikzcd}
X^{\leq 0} \times X^{\leq 0} & \operatorname{Hom}(C_m, \mathcal{A}) \times X^{\leq 0} \arrow[l] & \\
X^{\leq 0} \times \operatorname{Hom}(C_m, \mathcal{A}) \arrow[u] & \operatorname{Hom}(C_{m\otimes m'}, \mathcal{A}) \arrow[ul] & \operatorname{Hom}(C_{m,m'}, \mathcal{A}) \arrow[l] \arrow[ul] \\
& \operatorname{Hom}(C_{m',m}, \mathcal{A}) \arrow[ul] \arrow[u] & \operatorname{Hom}(C_m \otimes C_{m'}, \mathcal{A}) \arrow[l] \arrow[u]
\end{tikzcd}$$

where all Homs are taken in $\operatorname{Algbrd}(\mathcal{M})$.

Consider now the commutative diagram of spaces

$$\begin{tikzcd}
{[0]} \arrow[r] \arrow[d, "{(\mu(0),\nu)}"] & {[0]} \arrow[d, "{(\mu(0),\nu(1))}"] & {[0]} \arrow[l] \arrow[d, "{(\mu,\nu(1))}"] \\
X^{\leq 0} \times \operatorname{Hom}_{\operatorname{Algbrd}(\mathcal{M})}(C_m, \mathcal{A}) \arrow[r] & X^{\leq 0} \times X^{\leq 0} & \operatorname{Hom}_{\operatorname{Algbrd}(\mathcal{M})}(C_m, \mathcal{A}) \times X^{\leq 0} \arrow[l]
\end{tikzcd}$$

where the bottom left horizontal arrow is the target map, and the bottom right horizontal arrow is the source map. Pulling back our previous diagram along this yields a cartesian square of spaces

$$\begin{tikzcd}
\operatorname{Hom}_{\mathcal{M}}(m', \mathscr{H}om_{\mathcal{M}}(m, \mathcal{A}(\nu(1), \mu(0)))) & \operatorname{Hom}_{\mathcal{M}}(m', \mathcal{A}(\nu(1), \mu(1))) \arrow[l] \\
\operatorname{Hom}_{\mathcal{M}}(m', \mathcal{A}(\nu(0), \mu(0))) \arrow[u] & \operatorname{Hom}_{\mathcal{M}}(m', \operatorname{Funct}(C_m, \mathcal{A})(\nu, \mu)). \arrow[l] \arrow[u]
\end{tikzcd}$$

Our claim now follows from the fact that the above square is natural in m'. $\square$

We finish by studying the behavior of functor algebroids under changes in the enriching categories.

Proposition 3.5.27. *Let $G : \mathcal{M} \to \mathcal{M}'$ be a colimit preserving symmetric monoidal functor between presentable symmetric monoidal categories. Let $\mathcal{C}$ be an $\mathcal{M}$-algebroid and let $\mathcal{D}$ be an $\mathcal{M}'$ algebroid. There is an equivalence of $\mathcal{M}$-algebroids*

$$\operatorname{Funct}(\mathcal{C}, (G^R)_!\mathcal{D}) = (G^R)_! \operatorname{Funct}(G_!\mathcal{C}, \mathcal{D})$$

which is natural in $\mathcal{C}$ and $\mathcal{D}$.

Proof. Let $\mathcal{E}$ be another $\mathcal{M}$-algebroid. Then we have a chain of equivalences

$$\begin{aligned}
\operatorname{Hom}_{\operatorname{Algbrd}(\mathcal{M})}(\mathcal{E}, \operatorname{Funct}(\mathcal{C}, (G^R)_!\mathcal{D})) &= \operatorname{Hom}_{\operatorname{Algbrd}(\mathcal{M})}(\mathcal{E} \otimes \mathcal{C}, (G^R)_!\mathcal{D}) \\
&= \operatorname{Hom}_{\operatorname{Algbrd}(\mathcal{M}')}(G_!(\mathcal{E} \otimes \mathcal{C}), \mathcal{D}) \\
&= \operatorname{Hom}_{\operatorname{Algbrd}(\mathcal{M}')}(G_!\mathcal{E} \otimes G_!\mathcal{C}, \mathcal{D}) \\
&= \operatorname{Hom}_{\operatorname{Algbrd}(\mathcal{M}')}(G_!\mathcal{E}, \operatorname{Funct}(G_!\mathcal{C}, \mathcal{D})) \\
&= \operatorname{Hom}_{\operatorname{Algbrd}(\mathcal{M})}(\mathcal{E}, (G^R)_! \operatorname{Funct}(G_!\mathcal{C}, \mathcal{D})).
\end{aligned}$$

Our claim follows from the fact that the above equivalences are natural in $\mathcal{E}, \mathcal{C}$ and $\mathcal{D}$. $\square$

Proposition 3.5.28. *Let $i : \mathcal{M} \to \mathcal{M}'$ be a lax symmetric monoidal functor between presentable symmetric monoidal categories. Assume that i is fully faithful and admits a left adjoint which is strictly symmetric monoidal. Let $\mathcal{C}$ be an $\mathcal{M}'$-algebroid, and $\mathcal{D}$ be an $\mathcal{M}$ algebroid. Then $\operatorname{Funct}(\mathcal{C}, i_!\mathcal{D})$ belongs to the image of $i_! : \operatorname{Algbrd}(\mathcal{M}) \to \operatorname{Algbrd}(\mathcal{M}')$.*

Proof. Our conditions guarantee that $i_! : \operatorname{Algbrd}(\mathcal{M}) \to \operatorname{Algbrd}(\mathcal{M}')$ is fully faithful, and admits a symmetric monoidal left adjoint. To show that $\operatorname{Funct}(\mathcal{C}, i_!\mathcal{D})$ belongs to the image of $i_!$, it suffices to show that it is local for the maps $q_\mathcal{E} : \mathcal{E} \to i_! i_!^L \mathcal{E}$, for each $\mathcal{M}'$-algebroid $\mathcal{E}$. Indeed, the map

$$\operatorname{Hom}_{\operatorname{Algbrd}(\mathcal{M}')}(i_! i_!^L \mathcal{E}, \operatorname{Funct}(\mathcal{C}, i_!\mathcal{D})) \to \operatorname{Hom}_{\operatorname{Algbrd}(\mathcal{M}')}(\mathcal{E}, \operatorname{Funct}(\mathcal{C}, i_!\mathcal{D}))$$

of precomposition with $q_\mathcal{E}$ is equivalent to the map

$$\operatorname{Hom}_{\operatorname{Algbrd}(\mathcal{M}')}(i_! i_!^L \mathcal{E} \otimes \mathcal{C}, i_!\mathcal{D}) \to \operatorname{Hom}_{\operatorname{Algbrd}(\mathcal{M}')}(\mathcal{E} \otimes \mathcal{C}, i_!\mathcal{D})$$

of precomposition with $q_\mathcal{E} \otimes \operatorname{id}_\mathcal{C}$. It therefore suffices to show that the induced map

$$i_!^L(\mathcal{E} \otimes \mathcal{C}) \to i_!^L(i_! i_!^L \mathcal{E} \otimes \mathcal{C})$$

is an equivalence. This follows from the fact that $i_!^L$ is a symmetric monoidal localization. $\square$

Corollary 3.5.29. *Let $i : \mathcal{M} \to \mathcal{M}'$ be a symmetric monoidal functor between presentable symmetric monoidal categories. Assume that i is fully faithful and has a left adjoint which is strictly symmetric monoidal. Let $\mathcal{C}, \mathcal{D}$ be $\mathcal{M}$-algebroids. Then there is an equivalence $i_! \operatorname{Funct}(\mathcal{C}, \mathcal{D}) = \operatorname{Funct}(i_!\mathcal{C}, i_!\mathcal{D})$, which is natural in $\mathcal{C}, \mathcal{D}$.*

Proof. This is a direct consequence of proposition 3.5.28, as $i_! \operatorname{Funct}(\mathcal{C}, \mathcal{D})$ and $\operatorname{Funct}(i_!\mathcal{C}, i_!\mathcal{D})$ both corepresent the same functor on $\operatorname{Algbrd}(\mathcal{M})$. $\square$

3.6 ω-categories

The theory of n-categories can be obtained as a special case of the general notion of enriched categories. We refer the reader to [GH15] and [Hin20a] for proofs that the following definition agrees with other models for the theory of n-categories.

Definition 3.6.1. *Let* $1\mathrm{Cat}$ *be the category of categories. We inductively define for each* $n \geq 2$ *the cartesian closed presentable category* $n\mathrm{Cat}$ *of* n*-categories to be the category* $\mathrm{Cat}^{(n-1)\mathrm{Cat}}$ *of categories enriched in the cartesian closed presentable category* $(n-1)\mathrm{Cat}$*.*

Construction 3.6.2. Let $i : \mathrm{Spc} \to \mathrm{Cat}$ be the inclusion. This admits a left adjoint, and in particular it has a canonical symmetric monoidal structure, where we equip Spc and Cat with their cartesian symmetric monoidal structures. Specializing the discussion of remark 3.4.15 we obtain a commutative square of presentable categories and colimit preserving morphisms

$$\begin{tikzcd} \mathrm{Algbrd}(\mathrm{Spc})_{\mathrm{Spc}} \arrow[r] \arrow[d, "i_!"] & \mathrm{Cat} \arrow[d, "i_!"] \\ \mathrm{Algbrd}(\mathrm{Cat})_{\mathrm{Spc}} \arrow[r] & 2\mathrm{Cat} \end{tikzcd}$$

which is both horizontally right adjointable and vertically left adjointable, and whose vertical arrows are fully faithful.

Denote by $i^{1,2}$ the functor $i_! : \mathrm{Cat} \to 2\mathrm{Cat}$. Arguing by induction, we obtain for all $n \geq 2$ a commutative square of presentable categories and colimit preserving functors

$$\begin{tikzcd} \mathrm{Algbrd}((n-1)\mathrm{Cat})_{\mathrm{Spc}} \arrow[r] \arrow[d, "(i^{n-1,n})_!"] & n\mathrm{Cat} \arrow[d, "i^{n,n+1}"] \\ \mathrm{Algbrd}(n\mathrm{Cat})_{\mathrm{Spc}} \arrow[r] & (n+1)\mathrm{Cat} \end{tikzcd}$$

which is both horizontally right adjointable and vertically left adjointable, and whose vertical arrows are fully faithful. In particular, we have a sequence of presentable categories and left adjointable colimit preserving fully faithful functors

$$0\mathrm{Cat} \xrightarrow{i^{0,1}} \mathrm{Cat} \xrightarrow{i^{1,2}} 2\mathrm{Cat} \xrightarrow{i^{2,3}} 3\mathrm{Cat} \ldots$$

where we set $0\mathrm{Cat} = \mathrm{Spc}$ and $i^{0,1} = i$. For each pair $m \geq n$ denote by $i^{n,m} : n\mathrm{Cat} \to m\mathrm{Cat}$ the corresponding inclusion.

Example 3.6.3. Let C_0 be the terminal object in Spc, and C_1 be the arrow category [1]. We inductively define for each $n \geq 2$ an n-category C_n as the cell associated to the $(n-1)$-category C_{n-1}. We call C_n the n-cell. It follows by induction that the category $n\mathrm{Cat}$ is compactly generated by the object C_n. Note that for each $0 \leq k < n$ there are source and target maps $i^{k,n}C_k \to C_n$.

In practice, we usually identify n-categories with their images under the functors $i^{n,m}$. In other words, many times we implicitly work in the direct limit of the categories $n\mathrm{Cat}$.

Notation 3.6.4. Let $\omega\mathrm{Cat}$ be the direct limit in Pr^L of the sequence of construction 3.6.2. We call it the category of ω-categories. For each $n \geq 0$ denote by $i^{n,\omega} : n\mathrm{Cat} \to \omega\mathrm{Cat}$ the induced map.

Remark 3.6.5. We can alternatively think about $\omega\mathrm{Cat}$ as the limit of the categories $n\mathrm{Cat}$ under the functors $(i^{n,n+1})^R : (n+1)\mathrm{Cat} \to n\mathrm{Cat}$. In other words, an ω-category is a compatible family of n-categories for all $n \geq 0$. The resulting projections $\omega\mathrm{Cat} \to n\mathrm{Cat}$ are right adjoint to the maps $i^{n,\omega}$.

Note that $i^{n,n+1}$ preserves compact objects for every $n \geq 0$. Therefore $\omega\mathrm{Cat}$ is compactly generated and the maps $i^{n,\omega}$ preserve compact objects. In particular, we have that the projections $(i^{n,\omega})^R$ preserve filtered colimits.

Remark 3.6.6. The sequence of categories from construction 3.6.2 yields a functor

$$(-)\mathrm{Cat} : \mathbb{N} \to \mathrm{Cat}$$

where $\mathbb{N}$ is the poset of natural numbers. Let $p : \mathcal{E} \to \mathbb{N}$ be the associated cocartesian fibration. Since the functors $i^{n,n+1}$ admit right adjoints, this is also a cartesian fibration. By virtue of remark 3.6.5, the category $\omega\mathrm{Cat}$ is the category of cartesian sections of p. Note that $\mathcal{E}$ is a full subcategory of the product $\mathbb{N} \times (\bigcup_{n\geq 0} n\mathrm{Cat})$. It follows that $\omega\mathrm{Cat}$ is the full subcategory of the functor category $\mathrm{Funct}(\mathbb{N}, \bigcup_{n\geq 0} n\mathrm{Cat})$ on those sequences of objects

$$\mathcal{C}_0 \xrightarrow{i_0} \mathcal{C}_1 \xrightarrow{i_1} \dots$$

such that $\mathcal{C}_n$ is an n-category for each $n \geq 0$, and the map $(i^{n,n+1})^R i_n$ is an isomorphism for each $n \geq 0$.

Proposition 3.6.7. *For each $n \geq 0$ the inclusion $i^{n,\omega} : n\mathrm{Cat} \to \omega\mathrm{Cat}$ is fully faithful and admits both a left and a right adjoint.*

Proof. The existence of a right adjoint was already observed in remark 3.6.5. By the same argument as in remark 3.6.6, we have an equivalence between $\omega\mathrm{Cat}$ and the full subcategory of the functor category $\mathrm{Funct}(\mathbb{N}_{\geq n}, \bigcup_{m\geq n} m\mathrm{Cat})$ on those sequences

$$\mathcal{C}_n \xrightarrow{i_n} \mathcal{C}_{n+1} \xrightarrow{i_{n+1}} \dots$$

such that $\mathcal{C}_m$ is an m-category for each $m \geq n$, and the map $(i^{m,m+1})^R i_m$ is an isomorphism for each $m \geq n$. In this language, the projection $(i^{n,\omega})^R$ is given by the corestriction to $n\mathrm{Cat}$ of the composite map

$$\omega\mathrm{Cat} \hookrightarrow \mathrm{Funct}(\mathbb{N}_{\geq n}, \bigcup_{m\geq n} m\mathrm{Cat}) \xrightarrow{\mathrm{ev}_n} \bigcup_{m\geq n} m\mathrm{Cat}\,.$$

The left adjoint to ev_n is given by left Kan extension along the inclusion $\{n\} \to \mathbb{N}_{\geq n}$, and is fully faithful. The fact that $i^{n,\omega}$ is fully faithful follows from the fact that the left adjoint to ev_n maps an n-category $\mathcal{C}$ to the constant diagram

$$\mathcal{C} \to \mathcal{C} \to \dots$$

and this belongs to $\omega\mathrm{Cat}$.

It remains to show that $i^{n,\omega}$ admits a left adjoint. We have that $i^{n,\omega}$ is given by the corestriction to $\omega\mathrm{Cat}$ of the composite map

$$n\mathrm{Cat} \hookrightarrow \bigcup_{m \geq n} m\mathrm{Cat} \to \mathrm{Funct}(\mathbb{N}_{\geq n}, \bigcup_{m \geq n} m\mathrm{Cat})$$

where the second arrow is the functor of precomposition with the projection $\mathbb{N}_{\geq n} \to \{n\}$. The composition of the two arrows above is limit and colimit preserving. Furthermore, the inclusion $\omega\mathrm{Cat} \to \mathrm{Funct}(\mathbb{N}_{\geq n}, \bigcup_{m \geq n} m\mathrm{Cat})$ is accessible and limit preserving. We conclude that $i^{n,\omega}$ is accessible and limit preserving, and the adjoint functor theorem guarantees that it admits a left adjoint, as desired. $\square$

Notation 3.6.8. Let $n \geq 0$. We denote by $(-)^{\leq n}$ and ${}^{\leq n}(-)$ the right and left adjoints to the inclusion $i^{n,\omega}$. We think about these as the functors that discard (resp. invert) cells of dimension greater than n.

Remark 3.6.9. We have a diagram

$$i^{0,\omega}(-)^{\leq 0} \to i^{1,\omega}(-)^{\leq 1} \to i^{2,\omega}(-)^{\leq 2} \to \dots \to \mathrm{id}_{\omega\mathrm{Cat}}$$

of endofunctors of $\omega\mathrm{Cat}$, where the transitions are induced by the counits of the adjunctions $i^{n,n+1} \dashv (i^{n,n+1})^R$. We note that for every $n \geq 0$ the composition of the above sequence with the functor $(-)^{\leq n}$ is eventually constant, and is therefore a filtered colimit diagram.

Recall from remark 3.6.5 that $\omega\mathrm{Cat}$ is the limit of the sequence of categories

$$0\mathrm{Cat} \xleftarrow{(i^{0,1})^R} 1\mathrm{Cat} \xleftarrow{(i^{1,2})^R} 2\mathrm{Cat} \xleftarrow{(i^{2,3})^R} \dots \leftarrow \omega\mathrm{Cat}\,.$$

Each of the transition functors above preserve filtered colimits. We conclude that $\mathrm{id}_{\omega\mathrm{Cat}}$ is the colimit of the endofunctors $i^{n,\omega}(-)^{\leq n}$. In other words, every ω-category is the colimit of its truncations.

Proposition 3.6.10. *The category* $\omega\mathrm{Cat}$ *is cartesian closed.*

Proof. Let $\mathcal{D}$ be an ω-category and let $X^{\triangleright} : \mathcal{I}^{\triangleright} \to \omega\mathrm{Cat}$ be a colimit diagram for $X = X^{\triangleright}|_{\mathcal{I}}$. We have to show that the induced diagram $X^{\triangleright} \times \mathcal{D}$ is also a colimit diagram. For each $n \geq 0$ let $X_n^{\triangleright}$ be an extension of $i^{n,\omega}(X^{\leq n})$ to a colimit diagram in $\omega\mathrm{Cat}$. Thanks to remark 3.6.9, we have $X^{\triangleright} = \mathrm{colim}_n X_n^{\triangleright}$. Since $\omega\mathrm{Cat}$ is compactly generated we have that products distribute over filtered colimits, and therefore we have

$$X^{\triangleright} \times \mathcal{D} = \mathrm{colim}_n (X_n^{\triangleright} \times \mathcal{D}).$$

It suffices to show that $X_n^{\triangleright} \times \mathcal{D}$ is a colimit diagram for each $n \geq 0$. Using remark 3.6.9 again, we have

$$X_n^{\triangleright} \times \mathcal{D} = \mathrm{colim}_m(X_n^{\triangleright} \times i^{m,\omega}\mathcal{D}^{\leq m}).$$

It therefore suffices to show that for each $n, m \geq 0$, the diagram $X_n^{\triangleright} \times i^{m,\omega}\mathcal{D}^{\leq m}$ is a colimit diagram in $\omega\mathrm{Cat}$. This follows from the fact that $(\max(n, m))\mathrm{Cat}$ is cartesian closed and closed under products and colimits inside $\omega\mathrm{Cat}$. $\square$

Notation 3.6.11. We denote by

$$\mathrm{Funct}(-,-) : \omega\mathrm{Cat}^{\mathrm{op}} \times \omega\mathrm{Cat} \to \omega\mathrm{Cat}$$

the internal Hom of $\omega\mathrm{Cat}$.

Remark 3.6.12. Equip the category $\omega\mathrm{Cat}$ with its cartesian symmetric monoidal structure. For each category X we have that $\mathrm{Algbrd}_X(\omega\mathrm{Cat})$ is equivalent to the limit of the categories $\mathrm{Algbrd}_X(n\mathrm{Cat})$ under the transition functors $(i^{n,n+1})_!^R$. Integrating over all such X we see that $\mathrm{Algbrd}(\omega\mathrm{Cat})$ is the limit of the categories $\mathrm{Algbrd}(n\mathrm{Cat})$ under the functors $(i^{n,n+1})_!^R$.

Using proposition 3.6.7 we see that the inclusion $i^{0,\omega}$ is the unit map for the presentable monoidal category $\omega\mathrm{Cat}$. It follows that that an object in $\mathrm{Algbrd}(\omega\mathrm{Cat})$ is an enriched category if and only if its image in $\mathrm{Algbrd}(n\mathrm{Cat})$ is an enriched category for all $n \geq 0$. Therefore the category $\mathrm{Cat}^{\omega\mathrm{Cat}}$ is the limit of the categories $\mathrm{Cat}^{n\mathrm{Cat}}$ under the transition functors $(i^{n,n+1})_!^R$. Passing to left adjoints we conclude that the colimit in Pr^L of the diagram

$$\mathrm{Cat}^{\mathrm{Spc}} \xrightarrow{i_!^{0,1}} \mathrm{Cat}^{\mathrm{Cat}} \xrightarrow{i_!^{1,2}} \mathrm{Cat}^{2\mathrm{Cat}} \xrightarrow{i_!^{2,3}} \dots$$

is $\mathrm{Cat}^{\omega\mathrm{Cat}}$. The above is however equivalent to the diagram in construction 3.6.2. It follows that there is an equivalence

$$\mathrm{Cat}^{\omega\mathrm{Cat}} = \omega\mathrm{Cat}$$

which makes the following diagram commute for all $n \geq 0$:

$$\begin{tikzcd}
\mathrm{Cat}^{n\mathrm{Cat}} \arrow[r, "i_!^{n,\omega}"] \arrow[d, "="] & \mathrm{Cat}^{\omega\mathrm{Cat}} \arrow[d] \\
(n+1)\mathrm{Cat} \arrow[r, "i^{n+1,\omega}"] & \omega\mathrm{Cat}
\end{tikzcd}$$

In other words, an ω-category can be thought of as a category enriched in ω-categories, in a way which is compatible with the definition of $(n+1)$-categories as categories enriched in n-categories. In particular, the internal Hom functor for $\omega\mathrm{Cat}$ fits into the framework of functor enriched categories from 3.5.

Proposition 3.6.13. *The functor* ${}^{\leq n}(-) : \omega\mathrm{Cat} \to n\mathrm{Cat}$ *preserves finite products for all* $n \geq 0$.

Proof. We first observe that the final object of ωCat is the image of the final object of nCat under the (limit preserving) inclusion $i^{n,\omega}$. Hence ${}^{\leq n}(-)$ preserves final objects.

We now show that ${}^{\leq n}(-)$ preserves binary products. Let $\mathcal{C}, \mathcal{D}$ be two ω-categories. Thanks to remark 3.6.9, we can write $\mathcal{C} = \operatorname{colim}_{m\geq 0} \mathcal{C}_m$ and $\mathcal{D} = \operatorname{colim}_{m\geq 0} \mathcal{D}_m$ where $\mathcal{C}_m$ and $\mathcal{D}_m$ are m-categories for each $m \geq 0$. Since products commute with filtered colimits in ωCat, we have

$$\mathcal{C} \times \mathcal{D} = \operatorname{colim}_{m\geq 0} \mathcal{C}_m \times \mathcal{D}_m.$$

Since ${}^{\leq n}(-)$ preserves colimits, we reduce to showing that for each $m \geq n$ the left adjoint to the inclusion $i^{n,m}$ preserves products. Arguing inductively, we may furthermore reduce to the case $n = 0$, $m = 1$, which follows from the fact that the geometric realization functor $\mathrm{Cat} \to \mathrm{Spc}$ preserves finite products. □

Corollary 3.6.14. *Let $\mathcal{C}$ and $\mathcal{D}$ be ω-categories, and assume that $\mathcal{D}$ is an n-category for some $n \geq 0$. Then* $\mathrm{Funct}(\mathcal{C}, \mathcal{D})$ *is an n-category. Moreover, if $\mathcal{C}$ is also an n-category then* $\mathrm{Funct}(\mathcal{C}, \mathcal{D})$ *can be identified with the internal Hom between $\mathcal{C}$ and $\mathcal{D}$ in* nCat.

Proof. Combine proposition 3.6.13, proposition 3.5.28 and corollary 3.5.29. □

Remark 3.6.15. Let $n \geq 0$. Passing to right adjoints in the commutative diagram of remark 3.6.12 yields a commutative square

$$\begin{tikzcd}
\mathrm{Cat}^{n\mathrm{Cat}} \arrow[d, "="] & \mathrm{Cat}^{\omega\mathrm{Cat}} \arrow[l, "(-)_!^{\leq n}"'] \arrow[d] \\
(n+1)\mathrm{Cat} & \omega\mathrm{Cat}\,. \arrow[l, "(-)^{\leq n+1}"']
\end{tikzcd}$$

It follows that for every ω-category $\mathcal{C}$, the category underlying $\mathcal{C}$ (thought of as an object of $\mathrm{Cat}^{\omega\mathrm{Cat}}$) is $\mathcal{C}^{\leq 1}$. In particular, its space of objects is $\mathcal{C}^{\leq 0}$. Furthermore, for each pair of objects x, y in $\mathcal{C}$, we have an equivalence

$$\mathrm{Hom}_{\mathcal{C}^{\leq n+1}}(x, y) = \mathrm{Hom}_{\mathcal{C}}(x, y)^{\leq n}.$$

Similarly, passing to left adjoints in the commutative diagram of remark 3.6.12 yields a commutative square

$$\begin{tikzcd}
\mathrm{Cat}^{n\mathrm{Cat}} \arrow[d, "="] & \mathrm{Cat}^{\omega\mathrm{Cat}} \arrow[l, "{}^{\leq n}(-)_!"'] \arrow[d] \\
(n+1)\mathrm{Cat} & \omega\mathrm{Cat}\,. \arrow[l, "{}^{\leq n+1}(-)"']
\end{tikzcd}$$

It follows that for every ω-category $\mathcal{C}$, the $(n+1)$-category ${}^{\leq n+1}\mathcal{C}$ is the image under the localization functor $\mathrm{Algbrd}(n\mathrm{Cat})_{\mathrm{Spc}} \to (n+1)\mathrm{Cat}$ of an algebroid ${}^{\leq n+1}\mathcal{C}^{\mathrm{pre}}$ with space of objects $\mathcal{C}^{\leq 0}$, and such that for every pair of objects x, y in $\mathcal{C}^{\leq 0}$ we have an equivalence

$$\mathrm{Hom}_{{}^{\leq n+1}\mathcal{C}^{\mathrm{pre}}}(x, y) = {}^{\leq n}\mathrm{Hom}_{\mathcal{C}}(x, y).$$

Example 3.6.16. Let $m > n \geq 1$. It follows by induction that ${}^{\leq n}C_m$ is the singleton set. On the other hand, $C_m^{\leq n}$ is the boundary of the $(n+1)$-cell ∂C_{n+1}, defined inductively by the fact that ∂C_0 is empty, and for $n \geq 1$ we have $\partial C_n = C_{\partial C_{n-1}}$.

Remark 3.6.17. Looking at the unit of the adjunction $i^{(n+1),\omega} \dashv (-)^{\leq n+1}$ through the equivalence given by the first commutative square in remark 3.6.15 shows that an ω-category $\mathcal{C}$ is an $(n+1)$-category for some $n \geq 0$ if and only for every pair of objects x, y in $\mathcal{C}$, the ω-category $\mathrm{Hom}_{\mathcal{C}}(x, y)$ is an n-category.

Remark 3.6.18. Recall from remarks 3.3.8 and 3.4.11 that we have an involution $(-)^{\mathrm{op}}$ on the full subcategory of Algbrd on the enriched categories, which restricts to an involution on the category of algebroids over any symmetric operad. In particular, for each $n \geq 1$ we have an induced involution on nCat by virtue of its description as $\mathrm{Cat}^{(n-1)\mathrm{Cat}}$.

It follows by induction that nCat comes equipped with n commuting involutions $(-)^{k\text{-op}}$ for $1 \leq k \leq n$. We think about $(-)^{k\text{-op}}$ as the involution that inverts the direction of all k-cells. These involutions are compatible with the inclusions $i^{n,n+1}$, and they therefore induce an infinite family of commuting involutions on the category ωCat.

Chapter 4

Modules over algebroids

Let $\mathcal{M}$ be a monoidal category and let $\mathcal{C}$ be a category left tensored over $\mathcal{M}$. A left module in $\mathcal{C}$ for an $\mathcal{M}$-algebroid $\mathcal{A}$ consists of:

- For each object x in $\mathcal{A}$ an object $\mathcal{P}(x)$ in $\mathcal{C}$.
- For every pair of objects x, y in $\mathcal{A}$ a morphism $\mathcal{A}(y, x) \otimes \mathcal{P}(x) \to \mathcal{P}(y)$.
- An infinite list of compatibility data between the above morphisms and the structure maps for $\mathcal{A}$.

More generally, assume given another monoidal category $\mathcal{M}'$, and an $\mathcal{M} - \mathcal{M}'$-bimodule category $\mathcal{C}$. If $\mathcal{A}$ and $\mathcal{B}$ are algebroids in $\mathcal{M}$ and $\mathcal{M}'$ respectively, an $\mathcal{A} - \mathcal{B}$-bimodule in $\mathcal{C}$ consists of:

- For each pair of objects x in $\mathcal{A}$ and y in $\mathcal{B}$, an object $\mathcal{P}(x, y)$ in $\mathcal{C}$.
- For every pair of objects x, x' in $\mathcal{A}$ and object y in $\mathcal{B}$, a morphism
$$\mathcal{A}(x', x) \otimes \mathcal{P}(x, y) \to \mathcal{P}(x', y).$$
- For every pair of objects y, y' in $\mathcal{B}$ and object x in $\mathcal{A}$, a morphism
$$\mathcal{P}(x, y) \otimes \mathcal{B}(y, y') \to \mathcal{P}(x, y')$$
- An infinite list of compatibility data between the above morphisms and the structure maps for $\mathcal{A}$ and $\mathcal{B}$.

Our goal in this chapter is to review the theory of left modules and bimodules, and provide a functorial enhancement of the procedure of enrichment of presentable modules over presentable monoidal categories.

We begin in 4.1 by using the operads LM_X from 3.2 to define the category of left modules over an algebroid. We show that there is a well behaved procedure of restriction of scalars along morphisms of algebroids. We record here two basic results regarding the multiplicativity properties of the theory of left modules, analogous the ones obtained in 3.5 for the theory of algebroids.

In 4.2 we construct, for each presentable symmetric monoidal category $\mathcal{M}$, a lax symmetric monoidal functor

$$\theta_{\mathcal{M}} : \mathcal{M}\text{-mod}(\mathrm{Pr}^L) \to \widehat{\mathrm{Cat}}^{\mathcal{M}}.$$

For each presentable $\mathcal{M}$-module $\mathcal{C}$, the enriched category $\theta_{\mathcal{M}}(\mathcal{C})$ has $\mathcal{C}$ as its underlying category, and for each pair of objects x, y in $\mathcal{C}$ one has an isomorphism between $\mathrm{Hom}_{\theta_{\mathcal{M}}(\mathcal{C})}(x, y)$ and the Hom object $\mathscr{H}om_{\mathcal{C}}(x, y)$ obtained from the action of $\mathcal{M}$ on $\mathcal{C}$. We show that the functor $\theta_{\mathcal{M}}$ is compatible with changes in the enriching category. In the particular case when $\mathcal{M}$ is the category of spaces with its cartesian symmetric monoidal structure, we prove that the functor $\theta_{\mathcal{M}}$ is equivalent, as a lax symmetric monoidal functor, to the usual forgetful functor from $\mathcal{M}\text{-mod}(\mathrm{Pr}^L)$ to $\widehat{\mathrm{Cat}}$. As a first consequence of the existence and properties of $\theta_{\mathcal{M}}$ we show that $\mathcal{M}$ admits a canonical enrichment over itself. This allows us in particular to construct an $(n+1)$-category of n-categories for each $n \geq 0$, and in the limit it provides a definition of the ω-category of ω-categories.

In 4.3 we review the notion of bimodule over an algebroid. We recall here the approach to the construction of the Yoneda embedding via the diagonal bimodule and the folding construction from [Hin20a], and record a basic result regarding the procedure of restriction of scalars in the context of bimodules over algebroids.

4.1 Left modules

We begin by reviewing the concept of left module over an algebroid.

Notation 4.1.1. For each LM-operad $\mathcal{M}$ we denote by $\mathcal{M}_l$ the its Assos-component, and by $\mathcal{M}_m$ the fiber of $\mathcal{M}$ over the module object in LM.

Definition 4.1.2. *Let $\mathcal{M}$ be a* LM-*operad. Let $\mathcal{A}$ be an algebroid in $\mathcal{M}_l$ with category of objects X. A left $\mathcal{A}$-module is a* LM_X*-algebra in $\mathcal{M}$, whose* Assos_X*-component is identified with $\mathcal{A}$.*

Remark 4.1.3. Let $\mathcal{M}$ be a LM-operad. Let $\mathcal{A}$ be an algebroid in $\mathcal{M}_l$ with category of objects X. A left $\mathcal{A}$-module $\mathcal{P}$ assigns to each object x in X an object $\mathcal{P}(x)$ in $\mathcal{M}_m$. For every $n \geq 0$ and every sequence of objects and arrows

$$y_0 = x_0 \leftarrow y_1, x_1 \leftarrow y_2, \ldots, x_{n-1} \leftarrow y_n, x_n \leftarrow y_{n+1}$$

in X, the left $\mathcal{A}$- module $\mathcal{P}$ induces a multimorphism

$$\{\mathcal{A}(y_1, x_1), \ldots, \mathcal{A}(y_n, x_n), \mathcal{P}(y_{n+1})\} \to \mathcal{P}(y_0)$$

in $\mathcal{M}$. In the case when $\mathcal{M}$ is a LM-monoidal category (in other words, $\mathcal{M}_l$ is a monoidal category and $\mathcal{M}_m$ is a left module for it), this induces a morphism

$$\mathcal{A}(y_1, x_1) \otimes \ldots \otimes \mathcal{A}(y_n, x_n) \otimes \mathcal{P}(y_{n+1}) \to \mathcal{P}(y_0).$$

In particular, in the case when $n = 1$ and the arrows are identities we obtain, for every pair of objects x_0, x_1 in X a map

$$\mathcal{A}(x_0, x_1) \otimes \mathcal{P}(x_1) \to \mathcal{P}(x_0).$$

This is compatible with the units and composition of $\mathcal{A}$, up to homotopy.

Example 4.1.4. Let X be a category and let x be an object of X. Then the functor $(\mathrm{id}, x) : X \sqcup [0] \to X$ induces a morphism of associative operads $\mathrm{LM}_X \to \mathrm{Assos}_X$. It follows that for every associative operad $\mathcal{M}$ and every $\mathcal{M}$-algebroid $\mathcal{A}$ with category of objects X, we have an induced left module $\mathcal{P}$ in $\mathcal{M}$. This has the following properties:

- For every object x' in X we have $\mathcal{P}(x') = \mathcal{A}(x', x)$.
- For every pair of objects x', x'' in X the action map

  $$\{\mathcal{A}(x'', x'), \mathcal{P}(x')\} \to \mathcal{P}(x'')$$

  is equivalent, under the identifications of the previous item, to the composition map

  $$\{\mathcal{A}(x'', x'), \mathcal{A}(x', x)\} \to \mathcal{A}(x'', x).$$

We call $\mathcal{P}$ the left module corepresented by x. We will usually use the notation $\mathcal{A}(-, x)$ for $\mathcal{P}$, and in the case when $\mathcal{A}$ is an $\mathcal{M}$-enriched category, we instead write $\mathrm{Hom}_{\mathcal{A}}(x, -)$.

Construction 4.1.5. Consider the functor $\mathrm{Alg}_{\mathrm{LM}_-}(-)$ defined by the composition

$$\mathrm{Cat}^{\mathrm{op}} \times \mathrm{Op}_{\mathrm{LM}} \xrightarrow{\mathrm{LM}_- \times \mathrm{id}_{\mathrm{Op}_{\mathrm{LM}}}} \mathrm{Op}_{\mathrm{LM}}^{\mathrm{op}} \times \mathrm{Op}_{\mathrm{LM}} \xrightarrow{\mathrm{Alg}_-(-)} \mathrm{Cat}\,.$$

For each object $\mathcal{M}$ in $\mathrm{Op}_{\mathrm{LM}}$ we denote by $\mathrm{LMod}(\mathcal{M})$ the total category of the cartesian fibration associated to the functor $\mathrm{Alg}_{\mathrm{LM}_-}(\mathcal{M}) : \mathrm{Cat}^{\mathrm{op}} \to \mathrm{Cat}$. This comes equipped with a forgetful functor $\mathrm{LMod}(\mathcal{M}) \to \mathrm{Algbrd}(\mathcal{M}_l)$. For each algebroid $\mathcal{A}$ in $\mathcal{M}_l$ we denote by $\mathrm{LMod}_{\mathcal{A}}(\mathcal{M})$ the fiber of $\mathrm{LMod}(\mathcal{M})$ over $\mathcal{A}$, and call it the category of left $\mathcal{A}$-modules.

The assignment $\mathcal{M} \mapsto \mathrm{LMod}(\mathcal{M})$ defines a functor $\mathrm{LMod}(-) : \mathrm{Op}_{\mathrm{BM}} \to \widehat{\mathrm{Cat}}$. Let LMod be the total category of the associated cocartesian fibration. In other words, LMod is the total category of the two-sided fibration associated to $\mathrm{Alg}_{\mathrm{LM}_-}(-)$.

Warning 4.1.6. Our usage of the terminology $\mathrm{LMod}(\mathcal{M})$ conflicts with that of [Lur17]. There only left modules over associative algebras are considered - this corresponds to the fiber of the projection $\mathrm{LMod}(\mathcal{M}) \to \mathrm{Cat}$ over $[0]$.

Remark 4.1.7. The category LMod fits into a commutative square

$$\begin{tikzcd}
\mathrm{Arr}_{\mathrm{oplax}}(\mathrm{Cat}) \arrow[d] & \mathrm{LMod} \arrow[l] \arrow[r] \arrow[d] & \mathrm{Algbrd} \arrow[d] \\
\mathrm{Cat} \times \mathrm{Cat} & \mathrm{Cat} \times \mathrm{Op}_{\mathrm{LM}} \arrow[l] \arrow[r] & \mathrm{Cat} \times \mathrm{Op}_{\mathrm{Assos}}\,.
\end{tikzcd}$$

Here the vertical arrows are the two-sided fibrations classified by the functors $\mathrm{Funct}(-,-)$, $\mathrm{Alg}_{\mathrm{LM}_-}(-)$ and $\mathrm{Algbrd}_-(-)$, and the horizontal arrows are the functors of "forgetting the algebra" and "forgetting the left module".

Proposition 4.1.8. *Let $\mathcal{M}$ be a* LM*-operad. Then the projection*

$$p : \mathrm{LMod}(\mathcal{M}) \to \mathrm{Algbrd}(\mathcal{M}_l)$$

is a cartesian fibration. Moreover, a morphism $F : (\mathcal{A}, \mathcal{P}) \to (\mathcal{B}, \mathcal{Q})$ in $\mathrm{LMod}(\mathcal{M})$ *is p-cartesian if and only if for every object x in $\mathcal{A}$ the induced map $\mathcal{P}(x) \to \mathcal{Q}(F(x))$ is an isomorphism.*

Proof. Let $\mathrm{Env}(\mathcal{M})$ be the LM-monoidal envelope of $\mathcal{M}$, and let $\mathcal{P}(\mathrm{Env}(\mathcal{M}))$ be the image of $\mathrm{Env}(\mathcal{M})$ under the symmetric monoidal functor $\mathcal{P} : \mathrm{Cat} \to \mathrm{Pr}^L$. We have a commutative square of categories

$$\begin{array}{ccc} \mathrm{LMod}(\mathcal{M}) & \longrightarrow & \mathrm{LMod}(\mathcal{P}(\mathrm{Env}(\mathcal{M}))) \\ \downarrow & & \downarrow \\ \mathrm{Algbrd}(\mathcal{M}_l) & \longrightarrow & \mathrm{Algbrd}(\mathcal{P}(\mathrm{Env}(\mathcal{M}))_l). \end{array}$$

Note that the horizontal arrows are fully faithful. Our result would follow if we are able to show that the right vertical arrow is a cartesian fibration, and that cartesian morphisms are given by the condition in the statement. In other words, it suffices to prove the result in the case when $\mathcal{M}$ is a presentable LM-monoidal category. We assume that this is the case from now on.

Let X be a category. Recall from [Hin20a] that LM_X is a flat LM-operad. It follows that there is a universal LM-operad $\mathcal{M}_X$ equipped with a morphism of LM-operads

$$\mathcal{M}_X \times_{\mathrm{LM}} \mathrm{LM}_X \to \mathcal{M}.$$

In particular, we have equivalences

$$\mathrm{Alg}_{\mathrm{LM}_X}(\mathcal{M}) = \mathrm{Alg}_{\mathrm{LM}}(\mathcal{M}_X)$$

and

$$\mathrm{Algbrd}_X(\mathcal{M}) = \mathrm{Alg}_{\mathrm{Assos}}((\mathcal{M}_X)_l).$$

The projection $p_X : \mathrm{Alg}_{\mathrm{LM}_X}(\mathcal{M}) \to \mathrm{Algbrd}_X(\mathcal{M}_l)$ becomes identified, under this dictionary, with the canonical projection

$$p'_X : \mathrm{Alg}_{\mathrm{LM}}(\mathcal{M}_X) \to \mathrm{Alg}_{\mathrm{Assos}}((\mathcal{M}_X)_l).$$

As discussed in [Hin20a] corollary 4.4.9, $\mathcal{M}_X$ is a presentable LM-monoidal category. It now follows from [Lur17] corollary 4.2.3.2 that p_X is a cartesian fibration, and moreover a morphism $F : (\mathcal{A}, \mathcal{P}) \to (\mathcal{B}, \mathcal{Q})$ in $\mathrm{Alg}_{\mathrm{LM}_X}$ is p_X-cartesian if and only if for every object x in X the induced map $\mathcal{P}(x) \to \mathcal{Q}(x)$ is an isomorphism.

Assume now that F is p_X-cartesian and let $g : Y \to X$ be a functor of categories. Consider the induced morphism

$$g^! F : (g^! \mathcal{A}, g^! \mathcal{P}) \to (g^! \mathcal{B}, g^! \mathcal{Q})$$

in $\mathrm{Alg}_{\mathrm{LM}_Y}(\mathcal{M})$. Let y be an object in Y. Then the induced map $g^! \mathcal{P}(y) \to g^! \mathcal{Q}(y)$ is equivalent to the map $\mathcal{P}(g(y)) \to \mathcal{Q}(g(y))$, and is therefore an isomorphism. It follows that $g^! F$ is also p_Y-cartesian, and hence $g^!$ is a morphism of cartesian fibrations. Combining [Lur09a] propositions 2.4.2.8 and 2.4.2.11 we conclude that p is a cartesian fibration.

Our characterization of p-cartesian morphisms follows from the above characterization of p_X-cartesian morphisms together with item (iii) in [Lur09a] proposition 2.4.2.11. □

Notation 4.1.9. Let $\mathcal{M}$ be an associative operad. We denote by $(\mathrm{Op}_{\mathrm{LM}})|_{\mathcal{M}}$ the fiber of the projection $\mathrm{Op}_{\mathrm{LM}} \to \mathrm{Op}_{\mathrm{Assos}}$ over $\mathcal{M}$, and by $\mathrm{LMod}\,|_{\mathcal{M}}$ the fiber over $\mathcal{M}$ of the projection $\mathrm{LMod} \to \mathrm{Op}_{\mathrm{Assos}}$.

Corollary 4.1.10. *Let $\mathcal{M}$ be an associative operad. Then the projection*

$$\mathrm{LMod}\,|_{\mathcal{M}} \to (\mathrm{Op}_{\mathrm{LM}})|_{\mathcal{M}} \times \mathrm{Algbrd}(\mathcal{M})$$

is a two-sided fibration from $(\mathrm{Op}_{\mathrm{LM}})|_{\mathcal{M}}$ *to* $\mathrm{Algbrd}(\mathcal{M})$.

Proof. By construction, the projection $\mathrm{LMod} \to \mathrm{Algbrd}$ is a morphism of cocartesian fibrations over the functor $\mathrm{Op}_{\mathrm{LM}} \to \mathrm{Op}_{\mathrm{Assos}}$. It follows that the projection in the statement is a morphism of cocartesian fibrations over $(\mathrm{Op}_{\mathrm{LM}})|_{\mathcal{M}}$. Its fiber over a given $\mathcal{M}$-module is a cartesian fibration, thanks to proposition 4.1.8. Our claim now follows from proposition 9.1.9. □

Proposition 4.1.11. *The categories* LMod *and* $\mathrm{Arr}_{\mathrm{oplax}}(\mathrm{Cat})$ *admit finite products. Moreover, all the maps in the diagram of remark 4.1.7 preserve finite products.*

Proof. The fact that LMod admits finite products which are preserved by the projection to $\mathrm{Cat} \times \mathrm{Op}_{\mathrm{LM}}$ follows by the same arguments as those from proposition 3.5.1. One similarly shows that $\mathrm{Arr}_{\mathrm{oplax}}(\mathrm{Cat})$ admits finite products which are preserved by the projection to $\mathrm{Cat} \times \mathrm{Cat}$. It remains to show that the projections from LMod to $\mathrm{Arr}_{\mathrm{oplax}}(\mathrm{Cat})$ and Algbrd preserve finite products. Both claims can be proven using similar arguments - below we present the case of Algbrd.

Observe first that the final object for LMod is the unique object lying above the final object in $\mathrm{Cat} \times \mathrm{Op}_{\mathrm{LM}}$. Its image in Algbrd is the unique algebroid lying above the final object in $\mathrm{Cat} \times \mathrm{Op}_{\mathrm{Assos}}$, which is indeed the final object of Algbrd.

It remains to show that the projection $\mathrm{LMod} \to \mathrm{Algbrd}$ preserves binary products. Let $\mathcal{M}, \mathcal{N}$ be two LM-operads, and let X, Y be two categories. Let $(\mathcal{A}, \mathcal{P})$ and $(\mathcal{B}, \mathcal{Q})$ be objects of LMod lying above $(X, \mathcal{M})$ and $(Y, \mathcal{N})$, respectively. A variant of the discussion from remark 3.5.4 shows that their product $(\mathcal{A}, \mathcal{P}) \boxtimes (\mathcal{B}, \mathcal{Q})$ fits into a diagram

$$(\mathcal{A}, \mathcal{P}) \xleftarrow{\beta_{(\mathcal{A},\mathcal{P})}} \overline{(\mathcal{A}, \mathcal{P})} \xleftarrow{\alpha_{(\mathcal{A},\mathcal{P})}} (\mathcal{A}, \mathcal{P}) \boxtimes (\mathcal{B}, \mathcal{Q}) \xrightarrow{\alpha_{(\mathcal{B},\mathcal{Q})}} \overline{(\mathcal{B}, \mathcal{Q})} \xrightarrow{\beta_{(\mathcal{B},\mathcal{Q})}} (\mathcal{B}, \mathcal{Q})$$

where $\alpha_{(\mathcal{A},\mathcal{P})}$ and $\alpha_{(\mathcal{B},\mathcal{Q})}$ are cocartesian for the projection $\mathrm{LMod} \to \mathrm{Op}_{\mathrm{LM}}$ and $\beta_{(\mathcal{A},\mathcal{P})}$ and $\beta_{(\mathcal{B},\mathcal{Q})}$ are cartesian for the projection $\mathrm{LMod} \to \mathrm{Cat}$. The image of the above diagram under the projection to Algbrd recovers a diagram

$$\mathcal{A} \xleftarrow{\beta_{\mathcal{A}}} \overline{\mathcal{A}} \xleftarrow{\alpha_{\mathcal{A}}} W \xrightarrow{\alpha_{\mathcal{B}}} \overline{\mathcal{B}} \xrightarrow{\beta_{\mathcal{B}}} \mathcal{B}$$

where $\alpha_{\mathcal{A}}$ and $\alpha_{\mathcal{B}}$ are cocartesian for the projection $\mathrm{Algbrd} \to \mathrm{Op}_{\mathrm{Assos}}$ and $\beta_{\mathcal{A}}$ and $\beta_{\mathcal{B}}$ are cartesian for the projection $\mathrm{Algbrd} \to \mathrm{Cat}$. Using remark 3.5.4 we conclude that the above diagram exhibits W as the product of $\mathcal{A}$ and $\mathcal{B}$ in Algbrd, as desired. □

Proposition 4.1.12. *Let $f : (\mathcal{A}, \mathcal{P}) \to (\mathcal{A}', \mathcal{P}')$ be a morphism in* LMod *and let $(\mathcal{B}, \mathcal{Q})$ be another object of* LMod. *Denote by $p = (p_1, p_2)$ the projection $\mathrm{LMod} \to \mathrm{Cat} \times \mathrm{Op}_{\mathrm{LM}}$.*

(i) *If f is p_1-cartesian then $f \boxtimes \mathrm{id}_{\mathcal{B}}$ is p_1-cartesian.*

(ii) *If f is p_2-cocartesian then $f \boxtimes \mathrm{id}_{\mathcal{B}}$ is p_2-cocartesian.*

Proof. Follows from the same arguments as those of proposition 3.5.5. □

4.2 Enrichment of presentable modules

Our next goal is to discuss the procedure of enrichment of presentable modules over presentable monoidal categories.

Notation 4.2.1. Recall the projection $\mathrm{LMod} \to \mathrm{Arr}_{\mathrm{oplax}}(\mathrm{Cat})$ from remark 4.1.7. Note that we have an inclusion $\mathrm{Funct}([1], \mathrm{Cat}) \to \mathrm{Arr}_{\mathrm{oplax}}(\mathrm{Cat})$ which is surjective on objects, which arises from straightening the natural transformation $\mathrm{Hom}_{\mathrm{Cat}}(-,-) \to \mathrm{Funct}(-,-)$ (see proposition 9.2.4). In other words, $\mathrm{Funct}([1], \mathrm{Cat})$ is the total category of the maximal bifibration contained inside the two-sided fibration $\mathrm{Arr}_{\mathrm{oplax}}(\mathrm{Cat}) \to \mathrm{Cat} \times \mathrm{Cat}$.

We denote by LMod' the fiber product $\mathrm{LMod} \times_{\mathrm{Arr}_{\mathrm{oplax}}} \mathrm{Funct}([1], \mathrm{Cat})$. For each $\mathcal{M}$ in $\mathrm{Op}_{\mathrm{LM}}$ we denote by $\mathrm{LMod}'(\mathcal{M})$ the fiber over $\mathcal{M}$ of the projection $\mathrm{LMod}' \to \mathrm{Op}_{\mathrm{LM}}$.

Remark 4.2.2. A variation of the argument in 4.1.11 shows that the inclusion of the arrow category $\mathrm{Funct}([1], \mathrm{Cat})$ inside $\mathrm{Arr}_{\mathrm{oplax}}(\mathrm{Cat})$ preserves finite products. It follows that LMod' admits finite products, which are preserved by its inclusion inside LMod.

Proposition 4.2.3. *Let $\mathcal{M}$ be a presentable* LM*-monoidal category (in other words, a pair of a monoidal category $\mathcal{M}_l$ and a presentable module $\mathcal{M}_m$). Then $\widehat{\mathrm{LMod}'}(\mathcal{M})$ has a final object. Moreover, a pair $(\mathcal{A}, \mathcal{P})$ of an $\mathcal{M}_l$-algebroid $\mathcal{A}$ with category of objects X and a left $\mathcal{A}$-module*

$\mathcal{P}$ in $\mathcal{M}_m$ is final if and only if the functor $X \to \mathcal{M}_m$ underlying $\mathcal{P}$ is an equivalence, and for every pair of objects x, y in X, the action map

$$\mathcal{A}(y,x) \otimes \mathcal{P}(x) \to \mathcal{P}(y)$$

exhibits $\mathcal{A}(y,x)$ as the Hom object between x and y.

Proof. Note that the composition of the projection $\widehat{\mathrm{LMod}}'(\mathcal{M}) \to \mathrm{Funct}([1], \widehat{\mathrm{Cat}})$ with the target map $\mathrm{Funct}([1], \widehat{\mathrm{Cat}}) \to \widehat{\mathrm{Cat}}$ is canonically equivalent to the constant functor $\mathcal{M}_m$. It follows that we have a commutative diagram

$$\begin{tikzcd}
\widehat{\mathrm{Cat}}_{/\mathcal{M}_m} \arrow[dr] & & \widehat{\mathrm{LMod}}'(\mathcal{M}) \arrow[ll] \arrow[dl] \\
& \widehat{\mathrm{Cat}} &
\end{tikzcd}$$

where the left vertical map is the forgetful functor, and the right vertical arrow picks out the category of objects of the underlying algebroid. Since the right vertical arrow is a cartesian fibration and the left vertical arrow is a right fibration, we have that the horizontal arrow is a cartesian fibration.

Let X be a category equipped with a map $f : X \to \mathcal{M}_m$ and recall the presentable LM-monoidal category $\mathcal{M}_X$ from the proof of proposition 4.1.8. The fiber of $\widehat{\mathrm{LMod}}'(\mathcal{M})$ over X is the category of LM_X-algebras in $\mathcal{M}_m$ whose underlying functor $X \to \mathcal{M}_m$ is f. This can equivalently be described as the category of associative algebras in $(\mathcal{M}_X)_l$ equipped with an action on f (thought of as an object of $(\mathcal{M}_X)_m$). By [Hin20a] corollary 6.3.4, we conclude that $(\widehat{\mathrm{LMod}}'(\mathcal{M}))_X$ admits a final object, and moreover a pair $(\mathcal{A}, \mathcal{P})$ of an $\mathcal{M}_l$-algebroid with category of objects X and a left $\mathcal{A}$-module $\mathcal{P}$ whose underlying functor $X \to \mathcal{M}_m$ is f is final if and only if for every pair of objects x, y in X the action map

$$\mathcal{A}(y,x) \otimes f(x) \to f(y)$$

exhibits $\mathcal{A}(y,x)$ as the Hom object between $f(x)$ and $f(y)$.

This description implies that if $(\mathcal{A}, \mathcal{P})$ is final in $(\widehat{\mathrm{LMod}}'(\mathcal{M}))_X$ and $g : Y \to X$ is a functor of categories, then $(g^!\mathcal{A}, g^!\mathcal{P})$ is final in $(\widehat{\mathrm{LMod}}'(\mathcal{M}))_Y$. The result now follows from an application of [Lur09a] proposition 4.3.1.10. □

Corollary 4.2.4. *Let $\mathcal{M}$ be a presentable* LM-*monoidal category. Then the projection* $\widehat{\mathrm{LMod}}'(\mathcal{M}) \to \widehat{\mathrm{Algbrd}}(\mathcal{M}_l)$ *is a representable right fibration.*

Proof. Combine propositions 4.1.8 and 4.2.3. □

Notation 4.2.5. Let $\mathcal{M}$ be a presentable monoidal category. We denote by

$$\widehat{\mathrm{LMod}}'|_{\mathcal{M}\text{-mod}(\mathrm{Pr}^L)}$$

the base change of the projection $\widehat{\mathrm{LMod}}' \to \widehat{\mathrm{Op}}_{\mathrm{LM}}$ along the inclusion $\mathcal{M}\text{-mod}(\mathrm{Pr}^L) \to \widehat{\mathrm{Op}}_{\mathrm{LM}}$.

For our next result, we need the notion of representable bifibration (see definition 9.2.7).

Corollary 4.2.6. *Let $\mathcal{M}$ be a presentable monoidal category. Then the projection*

$$\widehat{\mathrm{LMod}}'|_{\mathcal{M}\text{-mod}(\mathrm{Pr}^L)} \to \mathcal{M}\text{-mod}(\mathrm{Pr}^L) \times \widehat{\mathrm{Algbrd}}(\mathcal{M})$$

is a representable bifibration from $\mathcal{M}\text{-mod}(\mathrm{Pr}^L)$ to $\widehat{\mathrm{Algbrd}}(\mathcal{M})$.

Proof. It follows from corollary 4.1.10 together with the description of cartesian arrows from proposition 4.1.8 that the projection in the statement is the maximal bifibration contained inside the base change of the projection from corollary 4.1.10 along the inclusion

$$\mathcal{M}\text{-mod}(\mathrm{Pr}^L) \times \widehat{\mathrm{Algbrd}}(\mathcal{M}) \to (\widehat{\mathrm{Op}}_{\mathrm{LM}})|_{\mathcal{M}} \times \widehat{\mathrm{Algbrd}}(\mathcal{M}).$$

The fact that it is representable is the content of corollary 4.2.4. □

Our next goal is to study the dependence in $\mathcal{M}_l$ of the algebroid from proposition 4.2.3.

Construction 4.2.7. Consider the commutative diagram of categories

$$\begin{tikzcd}
\widehat{\mathrm{LMod}}'|_{\mathrm{Alg}_{\mathrm{LM}}(\mathrm{Pr}^L)} \arrow[r, "i'"] \arrow[d, "p'"] & \widehat{\mathrm{LMod}}' \arrow[r, "r"] \arrow[d, "p"] & \widehat{\mathrm{Algbrd}} \arrow[d, "q"] \\
\mathrm{Alg}_{\mathrm{LM}}(\mathrm{Pr}^L) \arrow[r, "i"] & \widehat{\mathrm{Op}}_{\mathrm{LM}} \arrow[r, "j"] & \widehat{\mathrm{Op}}_{\mathrm{Assos}}
\end{tikzcd}$$

where the left square is cartesian. We equip all four categories in the right square with their cartesian symmetric monoidal structure. By propositions 3.5.1 and 4.1.11 together with remark 4.2.2 we see that the right square has a canonical lift to a commutative square of cartesian symmetric monoidal categories. It follows from propositions 3.5.5 and 4.1.12 that p and q are in fact cocartesian fibrations of operads.

Equip $\mathrm{Alg}_{\mathrm{LM}}(\mathrm{Pr}^L)$ with its canonical symmetric monoidal structure, so that i inherits a lax symmetric monoidal structure. It follows from the above that p' has a canonical structure of cocartesian fibration of operads, and i' of lax symmetric monoidal morphism.

Using proposition 4.2.3 we see that that p' admits a fully faithful right adjoint p'^R, which comes equipped with a canonical lax symmetric monoidal structure. We therefore have a commutative diagram of symmetric monoidal categories and lax symmetric monoidal functors as follows:

$$\begin{tikzcd} & \widehat{\mathrm{Algbrd}} \arrow[d, "q"] \\ \mathrm{Alg}_{\mathrm{LM}}(\mathrm{Pr}^L) \arrow[ur, "ri'p'^R"] \arrow[r, "ji"] & \widehat{\mathrm{Op}_{\mathrm{Assos}}} \end{tikzcd}$$

Note that ji factors through the lax symmetric monoidal inclusion $\mathrm{Alg}_{\mathrm{Assos}}(\mathrm{Pr}^L) \to \widehat{\mathrm{Op}_{\mathrm{Assos}}}$, so we have an induced diagram of symmetric monoidal categories and lax symmetric monoidal functors

$$\begin{tikzcd} & \widehat{\mathrm{Algbrd}}|_{\mathrm{Alg}_{\mathrm{Assos}}(\mathrm{Pr}^L)} \arrow[d, "q'"] \\ \mathrm{Alg}_{\mathrm{LM}}(\mathrm{Pr}^L) \arrow[ur, "\theta'"] \arrow[r, "u"] & \mathrm{Alg}_{\mathrm{Assos}}(\mathrm{Pr}^L). \end{tikzcd}$$

Observe that the maps u and q' are cocartesian fibrations of operads. If $\mathcal{M}_l$ is a presentable symmetric monoidal category, thought of as a commutative algebra object in $\mathrm{Alg}_{\mathrm{Assos}}(\mathrm{Pr}^L)$, we obtain in particular an induced lax symmetric monoidal functor

$$\theta'_{\mathcal{M}} : \mathcal{M}\text{-mod}(\mathrm{Pr}^L) \to \widehat{\mathrm{Algbrd}}(\mathcal{M}).$$

Proposition 4.2.8. *The map θ' from construction 4.2.7 is a morphism of cartesian fibrations over* $\mathrm{Alg}_{\mathrm{Assos}}(\mathrm{Pr}^L)$*.*

Proof. We continue with the notation from construction 4.2.7. Observe that u and q' are indeed cartesian fibrations. For q' this follows from the adjoint functor theorem combined with remark 3.3.10, and for u this follows from [Lur17] corollary 4.2.3.2. The rest of the proof is devoted to showing that θ' maps u-cartesian arrows to q'-cartesian arrows.

Let $F : \mathcal{M} \to \mathcal{M}'$ be a u-cartesian arrow in $\mathrm{Alg}_{\mathrm{LM}}(\mathrm{Pr}^L)$, whose components consist of a morphism of presentable monoidal categories $F_l : \mathcal{M}_l \to \mathcal{M}'_l$ and an isomorphism of modules $F_m : \mathcal{M}_m \to \mathcal{M}'_m$. Let $(\mathcal{A}, \mathcal{P}) = p'^R(\mathcal{M})$ and $(\mathcal{A}', \mathcal{P}') = p'^R(\mathcal{M}')$. The morphism

$$p'^R F : (\mathcal{A}, \mathcal{P}) \to (\mathcal{A}', \mathcal{P}')$$

can be factored as $\eta\alpha$ where α is a p'-cocartesian lift of F, and $\eta : F_!(\mathcal{A}, \mathcal{P}) \to (\mathcal{A}', \mathcal{P}')$ is the unique map in $\widehat{\mathrm{LMod}}'(\mathcal{M}')$ from $F_!(\mathcal{A}, \mathcal{P})$ to the final object. Consider the morphism of algebroids $ri'\eta : (F_l)_!\mathcal{A} \to \mathcal{A}'$. We have to show that the induced map $\mu : \mathcal{A} \to (F_l^R)_!\mathcal{A}'$ is an isomorphism.

Let X and X' be the categories of objects of $\mathcal{A}$ and $\mathcal{A}'$, respectively. We have a commutative square

$$\begin{tikzcd} \mathcal{M}_m \arrow[r, "F_m"] & \mathcal{M}'_m \\ X \arrow[u, "\mathcal{P}"] \arrow[r, "g"] & X' \arrow[u, "\mathcal{P}'"'] \end{tikzcd}$$

where the bottom horizontal arrow is the projection of $p'^R F$ to Cat. It follows from proposition 4.2.3 that the vertical arrows are equivalences. Since F_m is also an isomorphism, we conclude that g is an equivalence. It follows that η and μ also induce equivalences at the level of categories of objects. To simplify notation, in the rest of the proof we identify X with $\mathcal{M}_m$ and X' with $\mathcal{M}'_m$ via the isomorphisms $\mathcal{P}$ and $\mathcal{P}'$.

Let x, y be two objects in $\mathcal{M}_m$. We have to show that the induced map

$$\mu_* : \mathcal{A}(y, x) \to F_l^R \mathcal{A}'(F_m y, F_m x)$$

is an isomorphism. The morphism η induces a commutative square in $\mathcal{M}'_m$ as follows:

$$\begin{tikzcd} F_l\mathcal{A}(y,x) \otimes F_m x \arrow[r] \arrow[d, "ri'\eta_*\otimes\mathrm{id}"] & F_m y \arrow[d, "\mathrm{id}"] \\ \mathcal{A}'(F_m y, F_m x) \otimes F_m x \arrow[r] & F_m y \end{tikzcd}$$

Applying the (lax symmetric monoidal) right adjoint to F yields a commutative square in $\mathcal{M}_m$ as follows:

$$\begin{tikzcd} F_l^R F_l\mathcal{A}(y,x) \otimes x \arrow[r] \arrow[d, "F_l^R ri'\eta_*\otimes\mathrm{id}"] & y \arrow[d, "\mathrm{id}"] \\ F_l^R\mathcal{A}'(F_m y, F_m x) \otimes x \arrow[r] & y \end{tikzcd}$$

Composing with the unit map $\mathcal{A}(y,x) \to F_l^R F_l \mathcal{A}(y,x)$ yields a commutative square

$$\begin{tikzcd} \mathcal{A}(y,x) \otimes x \arrow[r] \arrow[d, "\mu_*\otimes\mathrm{id}"] & y \arrow[d, "\mathrm{id}"] \\ F_l^R\mathcal{A}'(F_m y, F_m x) \otimes x \arrow[r] & y \end{tikzcd}$$

where the top horizontal arrow exhibits $\mathcal{A}(y,x)$ as the Hom object between x and y.

It now suffices to show that the bottom horizontal arrow exhibits $F_l^R\mathcal{A}'(F_m y, F_m x)$ as the Hom object between x and y. This follows from the right adjointability of the following commutative square of categories:

$$\begin{tikzcd} \mathcal{M}_l \arrow[r, "-\otimes x"] \arrow[d, "F_l"] & \mathcal{M}_m \arrow[d, "F_m"] \\ \mathcal{M}'_l \arrow[r, "-\otimes F_m x"] & \mathcal{M}'_m \end{tikzcd}$$

$\square$

Corollary 4.2.9. *Let $F : \mathcal{M} \to \mathcal{M}'$ be a colimit preserving symmetric monoidal functor between presentable symmetric monoidal categories. Then there is a commutative square of symmetric monoidal categories and lax symmetric monoidal functors*

$$\begin{tikzcd} \mathcal{M}'\text{-mod}(\mathrm{Pr}^L) \arrow[r, "\theta'_{\mathcal{M}'}"] \arrow[d, "F^*"] & \widehat{\mathrm{Algbrd}}(\mathcal{M}') \arrow[d, "F_!^R"] \\ \mathcal{M}\text{-mod}(\mathrm{Pr}^L) \arrow[r, "\theta'_{\mathcal{M}}"] & \widehat{\mathrm{Algbrd}}(\mathcal{M}) \end{tikzcd}$$

where F^* denotes the functor of restriction of scalars along F.

Proof. For each symmetric monoidal category X denote by $X^{\otimes}$ its category of operators. The map F corresponds to a morphism $F^{\otimes} : [1] \times \mathrm{Fin}_* \to \mathrm{Alg}_{\mathrm{Assos}}(\mathrm{Pr}^L)^{\otimes}$ over Fin_*. Base change of θ' along $F^{\otimes}$ yields a commutative diagram

$$\begin{tikzcd} & (\widehat{\mathrm{Algbrd}}|_{\mathrm{Alg}_{\mathrm{Assos}}(\mathrm{Pr}^L)})^{\otimes}_{F^{\otimes}} \arrow[d, "v=(v_1,v_2)"] \\ \mathrm{Alg}_{\mathrm{LM}}(\mathrm{Pr}^L)^{\otimes}_{F^{\otimes}} \arrow[ur, "\theta'_{F^{\otimes}}"] \arrow[r, "h=(h_1,h_2)"] & {[1] \times \mathrm{Fin}_*} \end{tikzcd}$$

where h and v are cocartesian fibrations. Observe that the maps h and v are also two-sided fibrations - in other words, the associated functors $[1] \times \mathrm{Fin}_* \to \mathrm{Cat}$ are right adjointable in the $[1]$ coordinate. In particular, we see that h_1 and v_1 are cartesian fibrations, and h and v are morphisms of cartesian fibrations over $[1]$. It follows from proposition 4.2.8 that $\theta'_{F^{\otimes}}$ is a morphism of cartesian fibrations over $[1]$. Straightening it yields a commutative square in $\widehat{\mathrm{Cat}}_{/\mathrm{Fin}_*}$. Tracing the construction of this square reveals that it is actually a commutative square of commutative operads, and satisfies the desired conditions. □

Proposition 4.2.10. *The lax symmetric monoidal functor*

$$\theta'_{\mathrm{Spc}} : \mathrm{Pr}^L = \mathrm{Spc}\text{-mod}(\mathrm{Pr}^L) \to \widehat{\mathrm{Algbrd}}(\mathrm{Spc})$$

factors through the image of the section s from construction 3.4.1. Furthermore, the composition of θ'_{Spc} with the symmetric monoidal projection $\widehat{\mathrm{Algbrd}}(\mathrm{Spc}) \to \widehat{\mathrm{Cat}}$ is equivalent to the forgetful functor $\mathrm{Pr}^L \to \widehat{\mathrm{Cat}}$ with its canonical lax symmetric monoidal structure.

Proof. Let $\mathcal{C}$ be a presentable category. Then $\theta'_{\mathrm{Spc}}(\mathcal{C})$ is a Spc-algebroid equipped with a left module in $\mathcal{C}$ which identifies its category of objects with $\mathcal{C}$, and such that for every pair of objects x, y the map

$$\theta'_{\mathrm{Spc}}(\mathcal{C})(y, x) \otimes x \to y$$

exhibits $\theta'_{\mathrm{Spc}}(\mathcal{C})(y,x)$ as the Hom object between x and y. Inspecting the unit morphism $\mathrm{Hom}_{\mathcal{C}}(x,y) \to \theta'_{\mathrm{Spc}}(\mathcal{C})(y,x)$ one obtains a commutative square

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(x,y) \otimes x \arrow[r, "\mathrm{ev}"] \arrow[d] & y \arrow[d, "\mathrm{id}"] \\
\theta'_{\mathrm{Spc}}(\mathcal{C})(y,x) \otimes x \arrow[r] & y.
\end{tikzcd}$$

The top horizontal arrow exhibits $\mathrm{Hom}_{\mathcal{C}}(x,y)$ as the Hom object between x and y, so we see that the unit map $\mathrm{Hom}_{\mathcal{C}}(x,y) \to \theta'_{\mathrm{Spc}}(\mathcal{C})(y,x)$ is an isomorphism. As discussed in the proof of lemma 3.4.4, this is also the case for the algebroid $s(\mathcal{C})$. It follows that the canonical map $s(\mathcal{C}) \to \theta'_{\mathrm{Spc}}(\mathcal{C})$ is an equivalence, and therefore θ'_{Spc} factors through the image of s.

Consider now the following diagram:

$$\begin{tikzcd}
\mathrm{Funct}([1],\mathrm{Cat}) \arrow[d, "(t_0,t_1)"] & \widehat{\mathrm{LMod}}' \arrow[l, "v"'] \arrow[r, "r"] \arrow[d, "(\alpha,p)"] & \widehat{\mathrm{Algbrd}} \arrow[d, "(\beta,q)"] \\
\widehat{\mathrm{Cat}} \times \widehat{\mathrm{Cat}} & \widehat{\mathrm{Cat}} \times \widehat{\mathrm{Op}}_{\mathrm{LM}} \arrow[l, "(\mathrm{id},w)"'] \arrow[r, "(\mathrm{id},j)"] & \widehat{\mathrm{Cat}} \times \widehat{\mathrm{Op}}_{\mathrm{Assos}}
\end{tikzcd}$$

We equip all categories above with their cartesian symmetric monoidal structure, so that all functors inherit a canonical symmetric monoidal structure.

The composition of θ'_{Spc} with the symmetric monoidal projection $\widehat{\mathrm{Algbrd}}(\mathrm{Spc}) \to \widehat{\mathrm{Cat}}$ is equivalent to the lax symmetric monoidal functor obtained by taking the fiber over Spc of the lax symmetric monoidal functor $(\beta,q)ri'p'^R$. This is equivalent to the composite lax symmetric monoidal functor

$$\mathrm{Pr}^L = \mathrm{Spc}\text{-}\mathrm{mod}(\mathrm{Pr}^L) \hookrightarrow \mathrm{Alg}_{\mathrm{LM}}(\mathrm{Pr}^L) \xrightarrow{p'^R} \widehat{\mathrm{LMod}}'_{\mathrm{Alg}(\mathrm{Pr}^L)} \xrightarrow{i'} \widehat{\mathrm{LMod}}' \xrightarrow{r} \widehat{\mathrm{Algbrd}} \xrightarrow{\beta} \widehat{\mathrm{Cat}}$$

which is in turn equivalent to the following composition:

$$\mathrm{Pr}^L = \mathrm{Spc}\text{-}\mathrm{mod}(\mathrm{Pr}^L) \hookrightarrow \mathrm{Alg}_{\mathrm{LM}}(\mathrm{Pr}^L) \xrightarrow{p'^R} \widehat{\mathrm{LMod}}'_{\mathrm{Alg}(\mathrm{Pr}^L)} \xrightarrow{i'} \widehat{\mathrm{LMod}}' \xrightarrow{v} \mathrm{Funct}([1],\mathrm{Cat}) \xrightarrow{t_0} \widehat{\mathrm{Cat}}$$

Meanwhile, the lax symmetric monoidal forgetful functor $\mathrm{Pr}^L \to \widehat{\mathrm{Cat}}$ can be obtained as the following composition:

$$\mathrm{Pr}^L = \mathrm{Spc}\text{-}\mathrm{mod}(\mathrm{Pr}^L) \hookrightarrow \mathrm{Alg}_{\mathrm{LM}}(\mathrm{Pr}^L) \xrightarrow{p'^R} \widehat{\mathrm{LMod}}'_{\mathrm{Alg}(\mathrm{Pr}^L)} \xrightarrow{i'} \widehat{\mathrm{LMod}}' \xrightarrow{v} \mathrm{Funct}([1],\mathrm{Cat}) \xrightarrow{t_1} \widehat{\mathrm{Cat}}.$$

We have to show that these agree. Note that they are both obtained by composing the lax symmetric monoidal functor

$$F : \mathrm{Pr}^L = \mathrm{Spc}\text{-}\mathrm{mod}(\mathrm{Pr}^L) \hookrightarrow \mathrm{Alg}_{\mathrm{LM}}(\mathrm{Pr}^L) \xrightarrow{p'^R} \widehat{\mathrm{LMod}}'_{\mathrm{Alg}(\mathrm{Pr}^L)} \xrightarrow{i'} \widehat{\mathrm{LMod}}' \xrightarrow{v} \mathrm{Funct}([1],\mathrm{Cat})$$

with either t_0 or t_1. However, thanks to the characterization of the image of p'^R from proposition 4.2.3, we have that the image of F belongs to the full subcategory $\mathrm{Funct}([1], \mathrm{Cat})_{\mathrm{iso}}$ of $\mathrm{Funct}([1], \mathrm{Cat})$ on the isomorphisms. Hence we can factor the lax symmetric monoidal functor F as follows:

$$\mathrm{Pr}^L \xrightarrow{F'} \mathrm{Funct}([1], \mathrm{Cat})_{\mathrm{iso}} \hookrightarrow \mathrm{Funct}([1], \mathrm{Cat}).$$

Our claim now follows from the fact that the restrictions of s_0 and s_1 to $\mathrm{Funct}([1], \mathrm{Cat})_{\mathrm{iso}}$ are equivalent. □

Corollary 4.2.11. *Let $\mathcal{M}$ be a presentable* LM*-monoidal category, and let $\gamma : \mathcal{M}_m^{\leq 0} \to \mathcal{M}_m$ be the inclusion. Then $\gamma^! \theta'(\mathcal{M})$ is an $\mathcal{M}_l$-enriched category.*

Proof. We have an equivalence $(\tau_{\mathcal{M}})_! \gamma^! \theta'(\mathcal{M}) = \gamma^! (\tau_{\mathcal{M}})_! \theta'(\mathcal{M})$. Thanks to corollary 4.2.9 we have that $(\tau_{\mathcal{M}})_!(\theta'(\mathcal{M}))$ is equivalent to $\theta'(\mathcal{N})$, where $\mathcal{N}$ is the presentable LM-monoidal category obtained by restricting the action of $\mathcal{M}_l$ on $\mathcal{M}_m$ along the unit map $\mathrm{Spc} \to \mathcal{M}_l$. Our claim now follows directly from proposition 4.2.10. □

We now construct a variant of the functor θ' which takes values in enriched categories.

Construction 4.2.12. We continue with the notation of construction 4.2.7. Consider the following commutative diagram:

$$\begin{tikzcd}
 & \widehat{\mathrm{Algbrd}} \arrow[d, "q"] & \widehat{\mathrm{Algbrd}}_{\mathrm{Spc}} \arrow[l, "h"'] \arrow[d, "q_{\mathrm{Spc}}"] \\
\mathrm{Alg}_{\mathrm{LM}}(\mathrm{Pr}^L) \arrow[ur, "ri'p'^R"] \arrow[r, "ji"] & \widehat{\mathrm{Op}}_{\mathrm{Assos}} & \widehat{\mathrm{Op}}_{\mathrm{Assos}} \arrow[l, "\mathrm{id}"']
\end{tikzcd}$$

Here $\widehat{\mathrm{Algbrd}}_{\mathrm{Spc}}$ is the full subcategory of $\widehat{\mathrm{Algbrd}}$ on those algebroids with a space of objects. We equip all categories on the right square with their cartesian symmetric monoidal structure. Note that q and q_{Spc} have canonical structures of cocartesian fibrations of operads, and h is a morphism of cocartesian fibrations of operads.

We observe that the category $\widehat{\mathrm{Algbrd}}_{\mathrm{Spc}}$ is obtained by base change of the cartesian fibration $\widehat{\mathrm{Algbrd}} \to \widehat{\mathrm{Cat}}$ along the inclusion $\widehat{\mathrm{Spc}} \to \widehat{\mathrm{Cat}}$. The latter admits a right adjoint, which implies that h admits a right adjoint h^R such that for every object $\mathcal{A}$ in $\widehat{\mathrm{Algbrd}}$ the canonical map $hh^R\mathcal{A} \to \mathcal{A}$ is cartesian. It follows that the right square in the above diagram is horizontally right adjointable, so we have a commutative diagram of symmetric monoidal categories and lax symmetric monoidal functors as follows:

$$\begin{tikzcd}
 & \widehat{\mathrm{Algbrd}} \arrow[d, "q"] \arrow[r, "h^R"] & \widehat{\mathrm{Algbrd}}_{\mathrm{Spc}} \arrow[d, "q_{\mathrm{Spc}}"] \\
\mathrm{Alg}_{\mathrm{LM}}(\mathrm{Pr}^L) \arrow[ur, "ri'p'^R"] \arrow[r, "ji"] & \widehat{\mathrm{Op}}_{\mathrm{Assos}} \arrow[r, "\mathrm{id}"] & \widehat{\mathrm{Op}}_{\mathrm{Assos}}
\end{tikzcd}$$

As before, we observe that ji factors through the lax symmetric monoidal inclusion of $\mathrm{Alg}_{\mathrm{Assos}}(\mathrm{Pr}^L)$ inside $\widehat{\mathrm{Op}}_{\mathrm{Assos}}$, so we have an induced diagram of symmetric monoidal categories and lax symmetric monoidal functors

$$\begin{tikzcd}
 & \widehat{\mathrm{Algbrd}}|_{\mathrm{Alg}_{\mathrm{Assos}}(\mathrm{Pr}^L)} \arrow[r, "h'^R"] \arrow[d, "q'"] & (\widehat{\mathrm{Algbrd}}_{\mathrm{Spc}})|_{\mathrm{Alg}_{\mathrm{Assos}}(\mathrm{Pr}^L)} \arrow[d, "q'_{\mathrm{Spc}}"] \\
\mathrm{Alg}_{\mathrm{LM}}(\mathrm{Pr}^L) \arrow[ur, "\theta'"] \arrow[r, "u"] & \mathrm{Alg}_{\mathrm{Assos}}(\mathrm{Pr}^L) \arrow[r, "\mathrm{id}"] & \mathrm{Alg}_{\mathrm{Assos}}(\mathrm{Pr}^L).
\end{tikzcd}$$

We denote by

$$\theta : \mathrm{Alg}_{\mathrm{LM}}(\mathrm{Pr}^L) \to \widehat{\mathrm{Algbrd}}_{\mathrm{Spc}}$$

the lax symmetric monoidal functor obtained by composing h'^R and θ'. It follows from corollary 4.2.11 that θ' factors through the full subcategory of $\widehat{\mathrm{Algbrd}}_{\mathrm{Spc}}$ on the enriched categories.

If $\mathcal{M}_l$ is a presentable symmetric monoidal category, thought of as a commutative algebra object in $\mathrm{Alg}_{\mathrm{Assos}}(\mathrm{Pr}^L)$, we obtain in particular a lax symmetric monoidal functor

$$\theta_{\mathcal{M}} : \mathcal{M}\text{-mod}(\mathrm{Pr}^L) \to \widehat{\mathrm{Cat}}^{\mathcal{M}}.$$

Proposition 4.2.13. *The map θ from construction 4.2.12 is a morphism of cartesian fibrations over* $\mathrm{Alg}_{\mathrm{Assos}}(\mathrm{Pr}^L)$.

Proof. Using proposition 4.2.8, we reduce to showing that the morphism

$$h'^R : \widehat{\mathrm{Algbrd}}_{\mathrm{Alg}_{\mathrm{Assos}}(\mathrm{Pr}^L)} \to (\widehat{\mathrm{Algbrd}}_{\mathrm{Spc}})|_{\mathrm{Alg}_{\mathrm{Assos}}(\mathrm{Pr}^L)}$$

is a morphism of cartesian fibrations over $\mathrm{Alg}_{\mathrm{Assos}}(\mathrm{Pr}^L)$. This is a direct consequence of the fact that it is right adjoint to a morphism of cocartesian fibrations. □

Corollary 4.2.14. *Let $F : \mathcal{M} \to \mathcal{M}'$ be a colimit preserving symmetric monoidal functor between presentable symmetric monoidal categories. Then there is a commutative square of symmetric monoidal categories and lax symmetric monoidal functors*

$$\begin{tikzcd}
\mathcal{M}'\text{-mod}(\mathrm{Pr}^L) \arrow[r, "\theta_{\mathcal{M}'}"] \arrow[d, "F^*"] & \widehat{\mathrm{Cat}}^{\mathcal{M}'} \arrow[d, "F_!^R"] \\
\mathcal{M}\text{-mod}(\mathrm{Pr}^L) \arrow[r, "\theta_{\mathcal{M}}"] & \widehat{\mathrm{Cat}}^{\mathcal{M}}
\end{tikzcd}$$

where F^* denotes the functor of restriction of scalars along F.

Proof. This is deduced from proposition 4.2.13 using similar arguments as those from the proof of corollary 4.2.9. □

Proposition 4.2.15. *The lax symmetric monoidal functor*

$$\theta_{\mathrm{Spc}} : \mathrm{Pr}^L = \mathrm{Spc}\text{-mod}(\mathrm{Pr}^L) \to \widehat{\mathrm{Cat}}$$

is equivalent to the (lax symmetric monoidal) forgetful functor $\mathrm{Pr}^L \to \widehat{\mathrm{Cat}}$.

Proof. This is a direct consequence of proposition 4.2.10. □

Notation 4.2.16. Let $\mathcal{M}$ be a presentable symmetric monoidal category. We denote by $\overline{\mathcal{M}}$ the image of the unit of $\mathcal{M}$-mod(Pr^L) under $\theta_{\mathcal{M}}$. This is a commutative algebra in $\widehat{\mathrm{Cat}}^{\mathcal{M}}$ whose underlying symmetric monoidal category is equivalent to $\mathcal{M}$.

In the special case $\mathcal{M} = \mathrm{Cat}$ we use the notation $\mathscr{C}at = \overline{\mathrm{Cat}}$. This is the symmetric monoidal 2-category of categories. More generally, for each $n \geq 1$ we set $n\mathscr{C}at = \overline{n\mathrm{Cat}}$. This is the symmetric monoidal $(n+1)$-category of n-categories. We also set $\omega\mathscr{C}at = \overline{\omega\mathrm{Cat}}$. This is the symmetric monoidal ω-category of ω-categories.

We finish by considering a variant of the functor $\theta_{\mathcal{M}}$ from construction 4.2.12 which admits a left adjoint.

Notation 4.2.17. Let κ be an uncountable regular cardinal. We denote by Pr^L_κ the subcategory of Pr^L on the κ-compactly generated categories and functors which preserve κ-compact objects. We equip Pr^L_κ with the restriction of the symmetric monoidal structure from Pr^L.

Let $\mathcal{M}$ be a commutative algebra in Pr^L_κ. We denote by $\widehat{\mathrm{LMod}'}|_{\mathcal{M}\text{-mod}(\mathrm{Pr}^L_\kappa)}$ the base change of the projection $\widehat{\mathrm{LMod}'} \to \widehat{\mathrm{Op}}_{\mathrm{LM}}$ along the inclusion $\mathcal{M}$-mod$(\mathrm{Pr}^L_\kappa) \to \widehat{\mathrm{Op}}_{\mathrm{LM}}$. We denote by $\mathrm{LMod}'^{\kappa}|_{\mathcal{M}}$ the full subcategory of $\widehat{\mathrm{LMod}'}|_{\mathcal{M}\text{-mod}(\mathrm{Pr}^L_\kappa)}$ consisting of those triples $(\mathcal{A}, \mathcal{P}, \mathcal{C})$ of an object $\mathcal{C}$ in $\mathcal{M}$-mod(Pr^L_κ), a small algebroid $\mathcal{A}$ in $\mathcal{M}$ with category of objects X, and a left module $\mathcal{P}$ such that for every x in X the object $\mathcal{P}(x)$ in $\mathcal{C}$ is κ-compact.

Proposition 4.2.18. *Let κ be an uncountable regular cardinal and let $\mathcal{M}$ be a commutative algebra in Pr^L_κ. Then the projection*

$$p^\kappa = (p_1^\kappa, p_2^\kappa) : \mathrm{LMod}'^{\kappa}|_{\mathcal{M}} \to \mathcal{M}\text{-mod}(\mathrm{Pr}^L_\kappa) \times \mathrm{Algbrd}(\mathcal{M})$$

is a representable bifibration.

Proof. Consider first the projection

$$\widehat{\mathrm{LMod}'}|_{\mathcal{M}\text{-mod}(\mathrm{Pr}^L_\kappa)} \to \mathcal{M}\text{-mod}(\mathrm{Pr}^L_\kappa) \times \widehat{\mathrm{Algbrd}}(\mathcal{M}).$$

This arises by base change from the projection of corollary 4.2.6 so we conclude that it is a representable bifibration. It follows directly from the definition that $\mathrm{LMod}'^{\kappa}|_{\mathcal{M}}$ is still a cocartesian fibration over $\mathcal{M}$-mod(Pr^L_κ), and for every $\mathcal{C}$ in $\mathcal{M}$-mod(Pr^L_κ) the projection $\mathrm{LMod}'^{\kappa}|_{\mathcal{M}}(\mathcal{C}) \to \mathrm{Algbrd}(\mathcal{M})$ is a right fibration. Note that this right fibration is represented by $j^! \theta'_{\mathcal{M}}(\mathcal{C})$, where j is the inclusion of the full subcategory of κ-compact objects inside $\mathcal{C}$. We conclude that p^κ is a representable bifibration, as desired. □

Notation 4.2.19. Let κ be an uncountable regular cardinal and let $\mathcal{M}$ be a commutative algebra in Pr^L_κ. We denote by

$$\theta'^{\kappa}_{\mathcal{M}} : \mathcal{M}\text{-mod}(\mathrm{Pr}^L_\kappa) \to \mathrm{Algbrd}(\mathcal{M}).$$

the functor classifying the projection p^κ from proposition 4.2.18.

Remark 4.2.20. Let κ be an uncountable regular cardinal and let $\mathcal{M}$ be a commutative algebra in Pr^L_κ. The functor $\theta'^{\kappa}_{\mathcal{M}}$ can be obtained as the composition

$$\mathcal{M}\text{-mod}(\mathrm{Pr}^L_\kappa) \xrightarrow{(p_1^\kappa)^R} \mathrm{LMod}'^{\kappa}|_{\mathcal{M}} \xrightarrow{p_2^\kappa} \mathrm{Algbrd}(\mathcal{M}).$$

Composing $(p_1^\kappa)^R$ with the inclusion of $\mathrm{LMod}'^{\kappa}|_{\mathcal{M}}$ inside $\widehat{\mathrm{LMod}'}|_{\mathcal{M}\text{-mod}(\mathrm{Pr}^L_\kappa)}$ yields a section of the cocartesian fibration

$$p_1'^{\kappa} : \widehat{\mathrm{LMod}'}|_{\mathcal{M}\text{-mod}(\mathrm{Pr}^L_\kappa)} \to \mathcal{M}\text{-mod}(\mathrm{Pr}^L_\kappa).$$

By corollary 4.2.6 the projection $p_1'^{\kappa}$ admits a right adjoint. It follows that there is a lax commutative triangle

$$\begin{tikzcd}
\mathcal{M}\text{-mod}(\mathrm{Pr}^L_\kappa) \arrow[r, "(p_1^\kappa)^R"] \arrow[dr, "(p_1'^{\kappa})^R"'] & \mathrm{LMod}'^{\kappa}|_{\mathcal{M}} \arrow[d] \arrow[dl, phantom, "\Leftarrow"] \\
 & \widehat{\mathrm{LMod}'}|_{\mathcal{M}\text{-mod}(\mathrm{Pr}^L_\kappa)}.
\end{tikzcd}$$

Composing with the projection $\mathrm{LMod}'|_{\mathcal{M}\text{-mod}(\mathrm{Pr}^L_\kappa)} \to \widehat{\mathrm{Algbrd}}(\mathcal{M})$ we obtain a natural transformation

$$\theta'^{\kappa}_{\mathcal{M}} \to \theta'_{\mathcal{M}}|_{\mathcal{M}\text{-mod}(\mathrm{Pr}^L_\kappa)}$$

of functors $\mathcal{M}\text{-mod}(\mathrm{Pr}^L_\kappa) \to \widehat{\mathrm{Algbrd}}(\mathcal{M})$. For each object $\mathcal{C}$ in $\mathcal{M}\text{-mod}(\mathrm{Pr}^L_\kappa)$, the morphism of algebroids

$$\theta'^{\kappa}_{\mathcal{M}}(\mathcal{C}) \to \theta'_{\mathcal{M}}|_{\mathcal{M}\text{-mod}(\mathrm{Pr}^L_\kappa)}(\mathcal{C})$$

is cartesian for the projection $\widehat{\mathrm{Algbrd}}(\mathcal{M}) \to \widehat{\mathrm{Cat}}$, and lies above the inclusion of the full subcategory of κ-compact objects inside $\mathcal{C}$.

Notation 4.2.21. Let κ be an uncountable regular cardinal and let $\mathcal{M}$ be a commutative algebra in Pr^L_κ. Consider the composite functor

$$\mathcal{M}\text{-mod}(\mathrm{Pr}^L_\kappa) \xrightarrow{\theta'^{\kappa}_{\mathcal{M}}} \mathrm{Algbrd}(\mathcal{M}) \to \mathrm{Algbrd}(\mathcal{M})_{\mathrm{Spc}}$$

where the second map is the colocalization functor. It follows from proposition 4.2.15 together with the description of $\theta'^{\kappa}_{\mathcal{M}}$ from remark 4.2.20 that the above composite map factors through $\mathrm{Cat}^{\mathcal{M}}$. We denote by

$$\theta^{\kappa}_{\mathcal{M}} : \mathcal{M}\text{-mod}(\mathrm{Pr}^L_\kappa) \to \mathrm{Cat}^{\mathcal{M}}$$

the resulting functor.

Remark 4.2.22. Let κ be an uncountable regular cardinal and let $\mathcal{M}$ be a commutative algebra in Pr^L_κ. It follows from remark 4.2.20 that there is a natural transformation

$$\theta^\kappa_{\mathcal{M}} \to \theta_{\mathcal{M}}$$

such that for every object $\mathcal{C}$ in $\mathcal{M}\text{-mod}(\mathrm{Pr}^L_\kappa)$, the functor of enriched categories $\theta^\kappa_{\mathcal{M}}(\mathcal{C}) \to \theta_{\mathcal{M}}(\mathcal{C})$ exhibits $\theta^\kappa_{\mathcal{M}}(\mathcal{C})$ as the full subcategory of $\theta_{\mathcal{M}}(\mathcal{C})$ on those objects which correspond to κ-compact objects in $\mathcal{C}$.

Lemma 4.2.23. *Let κ be an uncountable regular cardinal and let $\mathcal{M}$ be a commutative algebra in Pr^L_κ. Then the projection*

$$p_2^\kappa : \mathrm{LMod}'^\kappa|_{\mathcal{M}} \to \mathrm{Algbrd}(\mathcal{M})$$

preserves colimits.

Proof. It follows from a combination of [Lur09a] proposition 5.5.7.10 and [Lur17] lemma 4.8.4.2 and that Pr^L_κ is a presentable symmetric monoidal category. Using [Lur17] corollary 4.2.3.7 we see that $\mathcal{M}\text{-mod}(\mathrm{Pr}^L_\kappa)$ is also presentable.

Using proposition 4.2.18 we see that the fibers of the cocartesian fibration p_1^κ are presentable and for every map $F : \mathcal{C} \to \mathcal{D}$ in $\mathcal{M}\text{-mod}(\mathrm{Pr}^L_\kappa)$ the induced functor

$$F_! : \mathrm{LMod}'^\kappa_{\mathcal{M}}(\mathcal{C}) \to \mathrm{LMod}'^\kappa_{\mathcal{M}}(\mathcal{D})$$

preserves colimits. Hence p_1^κ admits all relative colimits. Note that the projection map $\mathrm{LMod}'^\kappa|_{\mathcal{M}} \to \mathrm{Algbrd}(\mathcal{M})$ sends p_1^κ-cocartesian arrows to invertible arrows, and for every $\mathcal{C}$ in $\mathcal{M}\text{-mod}(\mathrm{Pr}^L_\kappa)$ the functor $\mathrm{LMod}'^\kappa_{\mathcal{M}}(\mathcal{C}) \to \mathrm{Algbrd}(\mathcal{M})$ preserves colimits. Applying [Lur09a] proposition 4.3.1.9 and 4.3.1.10 we have that p_2^κ itself preserves colimits. $\square$

Proposition 4.2.24. *Let κ be an uncountable regular cardinal and let $\mathcal{M}$ be a commutative algebra in Pr^L_κ. Then the functor $\theta^\kappa_{\mathcal{M}}$ admits a left adjoint.*

Proof. Thanks to the adjoint functor theorem, it suffices now to show that $\theta^\kappa_{\mathcal{M}}$ is accessible and preserves limits. Since the inclusion $\mathrm{Cat}^{\mathcal{M}} \to \mathrm{Algbrd}(\mathcal{M})$ creates limits and sufficiently filtered colimits, it suffices to show that the functor $\theta'^\kappa_{\mathcal{M}}$ is accessible and limit preserving. The fact that $\theta'^\kappa_{\mathcal{M}}$ is accessible follows from the description of $\theta'^\kappa_{\mathcal{M}}$ from remark 4.2.20, together with lemma 4.2.23.

It remains to prove that $\theta'^\kappa_{\mathcal{M}}$ is limit preserving. Recall that the projection $\mathrm{Algbrd}(\mathcal{M}) \to \mathrm{Cat}$ admits all relative limits. We claim that the composite map

$$\mathcal{M}\text{-mod}(\mathrm{Pr}^L_\kappa) \xrightarrow{\theta'^\kappa_{\mathcal{M}}} \mathrm{Algbrd}(\mathcal{M}) \to \mathrm{Cat}$$

is limit preserving. Examining the commutative diagram from remark 4.1.7 shows that the composition of the above map with the inclusion of Cat into $\widehat{\mathrm{Cat}}$ admits a factorization as follows:

$$\mathcal{M}\text{-mod}(\mathrm{Pr}^L_\kappa) \xrightarrow{\mu} \mathrm{Funct}([1], \widehat{\mathrm{Cat}}) \xrightarrow{\mathrm{ev}_0} \widehat{\mathrm{Cat}}$$

The map μ is such that the composition

$$\mathcal{M}\text{-mod}(\mathrm{Pr}^L_\kappa) \xrightarrow{\mu} \mathrm{Funct}([1], \widehat{\mathrm{Cat}}) \xrightarrow{\mathrm{ev}_1} \widehat{\mathrm{Cat}}$$

recovers the canonical projection obtained by composing the following series of forgetful functors:

$$\mathcal{M}\text{-mod}(\mathrm{Pr}^L_\kappa) \to \mathrm{Pr}^L_\kappa \to \widehat{\mathrm{Cat}}$$

The description of $\theta'^{\kappa}_{\mathcal{M}}$ from remark 4.2.20 shows that $\mu\,\mathrm{ev}_0$ is the composite functor

$$\mathcal{M}\text{-mod}(\mathrm{Pr}^L_\kappa) \to \mathrm{Pr}^L_\kappa = \mathrm{Cat}^{\mathrm{rex}(\kappa)} \to \widehat{\mathrm{Cat}}$$

where the middle equivalence is given by passage to κ-compact objects ([Lur09a] proposition 5.5.7.8), and the last arrow is the usual forgetful functor. This composition is indeed limit preserving, so it follows that the composition of $\theta'^{\kappa}_{\mathcal{M}}$ with the forgetful functor to Cat is limit preserving, as we claimed.

Consider now a limit diagram $X^{\lhd} : \mathcal{I}^{\lhd} \to \mathcal{M}\text{-mod}(\mathrm{Pr}^L_\kappa)$. Denote by $*$ the initial object of $\mathcal{I}$. Let Y be the category of objects for the $\mathcal{M}$-algebroid $\theta'^{\kappa}_{\mathcal{M}}(X^{\lhd}(*))$. In other words, Y is the full subcategory of the category underlying the $\mathcal{M}$-module $X^{\lhd}(*)$ on the κ-compact objects.

Note that the composite map

$$\mathcal{I}^{\lhd} \xrightarrow{X^{\lhd}} \mathcal{M}\text{-mod}(\mathrm{Pr}^L_\kappa) \xrightarrow{\theta'^{\kappa}_{\mathcal{M}}} \mathrm{Algbrd}(\mathcal{M}) \to \mathrm{Cat}$$

factors through $\mathrm{Cat}_{Y/}$. Therefore we have that $\theta'^{\kappa}_{\mathcal{M}} X^{\lhd}$ factors through $\mathrm{Algbrd}(\mathcal{M}) \times_{\mathrm{Cat}} \mathrm{Cat}_{Y/}$. Since the projection $\mathrm{Algbrd}(\mathcal{M}) \times_{\mathrm{Cat}} \mathrm{Cat}_{Y/} \to \mathrm{Cat}_{Y/}$ is a cartesian fibration and Y is initial there, we have that the fiber of $\mathrm{Algbrd}(\mathcal{M})$ over Y is a colocalization of $\mathrm{Algbrd}(\mathcal{M}) \times_{\mathrm{Cat}} \mathrm{Cat}_{Y/}$. From this we may construct a diagram

$$Z^{\lhd} : \mathcal{I}^{\lhd} \to \mathrm{Algbrd}_Y(\mathcal{M})$$

equipped with a natural transformation

$$\begin{tikzcd} \mathcal{I}^{\lhd} \arrow[r, "Z^{\lhd}"] \arrow[dr, "\theta'^{\kappa}_{\mathcal{M}} X^{\lhd}"'] & \mathrm{Algbrd}_Y(\mathcal{M}) \arrow[d] \arrow[dl, phantom, "\Leftarrow"] \\ & \mathrm{Algbrd}(\mathcal{M}) \end{tikzcd}$$

such that the induced map $Z^{\lhd}(*) \to X^{\lhd}(*)$ is an isomorphism, and for every object i in $\mathcal{I}$ the morphism of algebroids $Z^{\lhd}(i) \to X^{\lhd}(i)$ is cartesian over Cat. Using [Lur09a] propositions 4.3.1.9 and 4.3.1.10 we see that in order to show that $\theta'^{\kappa}_{\mathcal{M}} X^{\lhd}$ is a limit diagram it suffices to show that that $Z^{\lhd}$ is a limit diagram.

Observe that $\mathrm{LMod}'^{\kappa}|_{\mathcal{M}}$ also has the structure of a cartesian fibration over Cat, and the projection $\mathrm{LMod}'^{\kappa}|_{\mathcal{M}} \to \mathrm{Algbrd}(\mathcal{M})$ is a morphism of cartesian fibrations over Cat. Repeating the above procedure, we may write $Z^{\lhd}$ as $p_2^{\kappa} W^{\lhd}$ where

$$W^{\lhd} : \mathcal{I}^{\lhd} \to (\mathrm{LMod}'^{\kappa}|_{\mathcal{M}})_Y$$

is a diagram which comes equipped with a natural transformation

$$\begin{tikzcd}
\mathcal{I}^{\lhd} \arrow[r, "W^{\lhd}"] \arrow[dr, "(p_1^{\kappa})^R X^{\lhd}"'] & (\mathrm{LMod}'^{\kappa}|_{\mathcal{M}})_Y \arrow[d] \arrow[dl, phantom, "\Leftarrow"] \\
& \mathrm{LMod}'^{\kappa}|_{\mathcal{M}}
\end{tikzcd}$$

such that the induced map $W^{\lhd}(*) \to (p_1^{\kappa})^R X^{\lhd}(*)$ is an isomorphism, and for every object i in $\mathcal{I}$ the morphism $W^{\lhd}(i) \to (p_1^{\kappa})^R X^{\lhd}(i)$ is cartesian over Cat.

We now observe that a diagram in $\mathrm{Algbrd}_Y(\mathcal{M})$ is a limit diagram if and only if its images in $\mathrm{Algbrd}_{\{a,b\}}(\mathcal{M})$ are limit diagrams for every map $\{a,b\} \to Y$, where $\{a,b\}$ denotes a two-element set. Fix one such map and let $U^{\lhd}$ be the composition of $W^{\lhd}$ with the induced morphism

$$(\mathrm{LMod}'^{\kappa}|_{\mathcal{M}})_Y \to (\mathrm{LMod}'^{\kappa}|_{\mathcal{M}})_{\{a,b\}}.$$

Our task is to show that the composite map

$$\mathcal{I}^{\lhd} \xrightarrow{U^{\lhd}} (\mathrm{LMod}'^{\kappa}|_{\mathcal{M}})_{\{a,b\}} \to \mathrm{Algbrd}_{\{a,b\}}(\mathcal{M})$$

is a limit diagram. Recall now from [Hin20a] that $\mathrm{Algbrd}_{\{a,b\}}(\mathcal{M})$ is the category of algebras in a certain presentable monoidal category $\mathcal{M}_{\{a,b\}}$. The category $(\mathrm{LMod}'^{\kappa}|_{\mathcal{M}})_{\{a,b\}}$ fits into a pullback square

$$\begin{tikzcd}
(\mathrm{LMod}'^{\kappa}|_{\mathcal{M}})_{\{a,b\}} \arrow[r] \arrow[d] & (\mathrm{LMod}'^{\kappa}|_{\mathcal{M}_{\{a,b\}}})_{[0]} \arrow[d] \\
\mathcal{M}\text{-mod}(\mathrm{Pr}^L_{\kappa}) \arrow[r] & \mathcal{M}_{\{a,b\}}\text{-mod}(\mathrm{Pr}^L_{\kappa})
\end{tikzcd}$$

where the bottom horizontal arrow maps a $\mathcal{M}$-module $\mathcal{C}$ to the $\mathcal{M}_{\{a,b\}}$-module $\mathrm{Funct}(\{a,b\},\mathcal{C})$. We note that the right vertical arrow admits a factorization through the category

$$\mathcal{M}_{\{a,b\}}\text{-mod}(\mathrm{Pr}^L_{\kappa})_{\mathcal{M}_{\{a,b\}}/}$$

of pointed $\mathcal{M}_{\{a,b\}}$-modules. The resulting projection has a fully faithful section

$$S : \mathcal{M}_{\{a,b\}}\text{-mod}(\mathrm{Pr}^L_{\kappa})_{\mathcal{M}_{\{a,b\}}/} \to (\mathrm{LMod}'^{\kappa}|_{\mathcal{M}_{\{a,b\}}})_{[0]}$$

whose image consists of those triples $(A, \mathcal{D}, M)$ of an $\mathcal{M}_{\{a,b\}}$-module $\mathcal{D}$, an algebra A in $\mathcal{M}_{\{a,b\}}$, and a κ-compact A-module M in $\mathcal{D}$ for which the structure map $A \otimes M \to M$

exhibits A as the endomorphism object of M. The description of endomorphism objects from [Hin20a] proposition 6.3.1 shows that in fact the composite map

$$\mathcal{I}^{\triangleleft} \xrightarrow{U^{\triangleleft}} (\mathrm{LMod}'^{\kappa}|_{\mathcal{M}})_{\{a,b\}} \to (\mathrm{LMod}'^{\kappa}|_{\mathcal{M}_{\{a,b\}}})_{[0]}$$

factors through the image of S. Furthermore, the resulting diagram $\mathcal{I}^{\triangleleft} \to \mathcal{M}_{\{a,b\}}\text{-mod}(\mathrm{Pr}^L_\kappa)$ is the image of $X^{\triangleleft}$ under the functor

$$\mathcal{M}\text{-mod}(\mathrm{Pr}^L_\kappa) \to \mathcal{M}_{\{a,b\}}\text{-mod}(\mathrm{Pr}^L_\kappa)$$

and is therefore a limit diagram. The fact that $\theta'^{\kappa}_{\mathcal{M}} X^{\triangleleft}$ is a limit diagram now follows from the fact that the composite map

$$\mathcal{M}_{\{a,b\}}\text{-mod}(\mathrm{Pr}^L_\kappa)_{\mathcal{M}_{\{a,b\}}/} \xrightarrow{S} (\mathrm{LMod}'^{\kappa}|_{\mathcal{M}_{\{a,b\}}})_{[0]} \to \mathrm{Alg}(\mathcal{M}_{\{a,b\}})$$

is the functor that sends a pointed $\mathcal{M}$-module to the endomorphism object of the basepoint, which admits a left adjoint. □

Remark 4.2.25. Let κ be an uncountable regular cardinal and let $\mathcal{M}$ be a commutative algebra in Pr^L_κ. Passing to κ-compact objects induces an equivalence between the symmetric monoidal category Pr^L_κ and the symmetric monoidal category $\mathrm{Cat}^{\mathrm{rex}(\kappa)}$ of small categories admitting κ-small colimits, and functors which preserve those colimits. In particular, we have an equivalence between $\mathcal{M}\text{-mod}(\mathrm{Pr}^L_\kappa)$ and $\mathcal{M}^{\kappa\text{-comp}}\text{-mod}(\mathrm{Cat}^{\mathrm{rex}(\kappa)})$, where $\mathcal{M}^{\kappa\text{-comp}}$ denotes the full subcategory of $\mathcal{M}$ on the κ-compact objects.

From this point of view, the left adjoint to $\theta^{\kappa}_{\mathcal{M}}$ maps small $\mathcal{M}$-enriched categories into κ-cocomplete categories tensored over $\mathcal{M}^{\kappa\text{-comp}}$. We think about this as a version of the functor of free κ-cocompletion in the context of enriched category theory. [1]

4.3 Bimodules

We now discuss the notion of bimodule between algebroids.

Notation 4.3.1. For each BM-operad $\mathcal{M}$ we denote by $\mathcal{M}_l$ and $\mathcal{M}_r$ the Assos^- and Assos^+ components of $\mathcal{M}$. We denote by $\mathcal{M}_m$ the fiber of $\mathcal{M}$ over the bimodule object in BM.

Definition 4.3.2. *Let $\mathcal{M}$ be a* BM*-operad. Let $\mathcal{A}, \mathcal{B}$ be algebroids in $\mathcal{M}_l$ and $\mathcal{M}_r$ respectively, with categories of objects X and Y respectively. An $\mathcal{A}-\mathcal{B}$-bimodule in $\mathcal{M}$ is a $\mathrm{BM}_{X,Y}$-algebra in $\mathcal{M}$, whose underlying* Assos_X *and* Assos_Y *algebras are identified with $\mathcal{A}$ and $\mathcal{B}$.*

Remark 4.3.3. Let $\mathcal{M}$ be a BM-operad. Let $\mathcal{A}, \mathcal{B}$ be algebroids in $\mathcal{M}_l$ and $\mathcal{M}_r$ respectively, with categories of objects X and Y respectively. A bimodule $\mathcal{P}$ between them assigns to

[1] We refer the reader to [Hin21] for a characterization of the free cocompletion as the Yoneda embedding.

each pair of objects (x, y) in $X \times Y^{\mathrm{op}}$ an object $\mathcal{P}(x, y)$ in $\mathcal{M}_m$. For every $n \geq 0, m \geq 0$, and every sequence of arrows

$$x'_0 = x_0 \leftarrow x'_1, x_1, \ldots, x_{n-1} \leftarrow x'_n, x_n \leftarrow x'_{n+1} = x_{n+1}$$

in X and

$$y'_0 = y_0 \leftarrow y'_1, y_2, \ldots, y_{m-1} \leftarrow y'_m, y_m \leftarrow y'_{m+1} = y_{m+1}$$

in Y, the bimodule $\mathcal{P}$ induces a multimorphism

$$\{\mathcal{A}(x'_1, x_1), \ldots, \mathcal{A}(x'_n, x_n), P(x'_{n+1}, y_0), \mathcal{B}(y'_1, y_1), \ldots, \mathcal{B}(y'_m, y_m)\} \to \mathcal{P}(x'_0, y_{m+1})$$

in $\mathcal{M}$. In the case when $\mathcal{M}$ is a BM-monoidal category (in other words, $\mathcal{M}_l$ and $\mathcal{M}_r$ are monoidal categories and $\mathcal{M}_m$ is a bimodule between them), this induces a morphism

$$\mathcal{A}(x'_1, x_1) \otimes \ldots \otimes \mathcal{A}(x'_n, x_n) \otimes P(x'_{n+1}, y_0) \otimes \mathcal{B}(y'_1, y_1) \otimes \ldots \otimes \mathcal{B}(y'_m, y_m) \to \mathcal{P}(x'_0, y_{m+1}).$$

In particular, in the cases when $n = 1, m = 0$ or $n = 0, m = 1$ and all arrows are identities we obtain, for each pair of objects x_1, x_2 in X and each pair of objects y_1, y_2 in Y, a map

$$\mathcal{A}(x_1, x_2) \otimes \mathcal{P}(x_2, y_1) \to \mathcal{P}(x_1, y_1)$$

(the left action of $\mathcal{A}$ on $\mathcal{P}$) and

$$\mathcal{P}(x_2, y_1) \otimes \mathcal{B}(y_1, y_2) \to \mathcal{P}(x_2, y_2).$$

(the right adjoint of $\mathcal{B}$ on $\mathcal{P}$). These actions commute with each other, and are compatible with the units and composition of $\mathcal{A}$ and $\mathcal{B}$, up to homotopy.

Example 4.3.4. Let $\mathcal{M}$ be an associative operad. Then every algebroid $\mathcal{A} : \mathrm{Assos}_X \to \mathcal{M}$ defines, by precomposition with the projection $\mathrm{BM}_{X,X} \to \mathrm{Assos}_X$ arising from the equivalence of example 3.2.8, an $\mathcal{A} - \mathcal{A}$-bimodule $\mathcal{P}$ in $\mathcal{M}$. This has the following properties:

- For every pair of objects x, x' in X we have $\mathcal{P}(x', x) = \mathcal{A}(x', x)$.
- For every triple of objects x, x', x'' in X, the action map

  $$\{\mathcal{A}(x'', x'), \mathcal{P}(x', x)\} \to \mathcal{P}(x'', x)$$

  is equivalent, under the identifications of the previous item, to the composition map

  $$\{\mathcal{A}(x'', x'), \mathcal{A}(x', x)\} \to \mathcal{A}(x'', x).$$

- For every triple of objects x, x', x'' in X, the action map

  $$\{\mathcal{P}(x', x), \mathcal{A}(x, x'')\} \to \mathcal{P}(x', x'')$$

  is equivalent, under the identifications of the first item, to the composition map

  $$\{\mathcal{A}(x', x), \mathcal{A}(x, x'')\} \to \mathcal{A}(x', x'').$$

We call $\mathcal{P}$ the diagonal bimodule of $\mathcal{A}$. We will usually use the notation $\mathcal{A}(-,-)$ for $\mathcal{P}$, and in the case when $\mathcal{A}$ is an $\mathcal{M}$-enriched category, we instead write $\mathrm{Hom}_{\mathcal{A}}(-,-)$.

Remark 4.3.5. Let $\mathcal{M}$ be an associative operad. As discussed in [Hin20a], an $\mathcal{M}$-algebroid $\mathcal{A}$ with category of objects X defines an associative algebra $\widetilde{\mathcal{A}}$ in a certain associative operad $\mathcal{M}_X$, and an $\mathcal{A}-\mathcal{A}$-bimodule is the same data as an $\widetilde{\mathcal{A}}-\widetilde{\mathcal{A}}$-bimodule in $\mathcal{M}_X$. Under this dictionary, the diagonal bimodule of $\mathcal{A}$ corresponds to the diagonal bimodule of $\widetilde{\mathcal{A}}$. Via the folding equivalence of [Hin20a] section 3.6, the diagonal bimodule of $\widetilde{\mathcal{A}}$ defines a left $\widetilde{\mathcal{A}}\boxtimes\widetilde{\mathcal{A}}^{\mathrm{op}}$-module in $\mathcal{M}_{X\times X^{\mathrm{op}}}$. If $\mathcal{M}$ is a presentable symmetric monoidal category, this defines a morphism

$$\mathcal{A}\otimes\mathcal{A}^{\mathrm{op}}\to\overline{\mathcal{M}}$$

with the property that each pair of objects (y,x) gets mapped to $\mathcal{A}(y,x)$. We call this the Hom functor of $\mathcal{A}$. This determines a morphism of algebroids

$$\mathcal{A}\to\mathrm{Funct}(\mathcal{A}^{\mathrm{op}},\overline{\mathcal{M}})$$

which is the Yoneda embedding. It was show in [Hin20a] corollary 6.2.7 that this map is fully faithful. Note that thanks to proposition 3.5.20 the algebroid $\mathrm{Funct}(\mathcal{A}^{\mathrm{op}},\overline{\mathcal{M}})$ is in fact an enriched category. We conclude that any $\mathcal{M}$-algebroid admits a fully faithful embedding into an $\mathcal{M}$-enriched category.

Construction 4.3.6. Consider the functor $\mathrm{Alg}_{\mathrm{BM}_{-,-}}(-)$ defined by the composition

$$\mathrm{Cat}^{\mathrm{op}}\times\mathrm{Cat}^{\mathrm{op}}\times\mathrm{Op}_{\mathrm{BM}}\xrightarrow{\mathrm{BM}_{-,-}\times\mathrm{id}_{\mathrm{Op}_{\mathrm{BM}}}}\mathrm{Op}_{\mathrm{BM}}^{\mathrm{op}}\times\mathrm{Op}_{\mathrm{BM}}\xrightarrow{\mathrm{Alg}_{-}(-)}\mathrm{Cat}\,.$$

For each BM-algebroid $\mathcal{M}$ we denote by $\mathrm{BMod}(\mathcal{M})$ the total category of the cartesian fibration associated to the functor $\mathrm{Alg}_{\mathrm{BM}_{-,-}}(\mathcal{M}):\mathrm{Cat}^{\mathrm{op}}\times\mathrm{Cat}^{\mathrm{op}}\to\mathrm{Cat}$. This comes equipped with a forgetful functor $\mathrm{BMod}(\mathcal{M})\to\mathrm{Algbrd}(\mathcal{M}_l)\times\mathrm{Algbrd}(\mathcal{M}_r)$. For each pair of algebroids $\mathcal{A}$ in $\mathcal{M}_l$ and $\mathcal{B}$ in $\mathcal{M}_r$ we denote by ${}_{\mathcal{A}}\mathrm{BMod}_{\mathcal{B}}(\mathcal{M})$ the fiber over $(\mathcal{A},\mathcal{B})$, and call it the category of $\mathcal{A}-\mathcal{B}$-bimodules.

The assignment $\mathcal{M}\mapsto\mathrm{BMod}(\mathcal{M})$ defines a functor $\mathrm{BMod}(-):\mathrm{Op}_{\mathrm{BM}}\to\widehat{\mathrm{Cat}}$. Let BMod be the total category of the associated cocartesian fibration. Note that this fits into a commutative square

$$\begin{tikzcd}
\mathrm{BMod} \arrow[r] \arrow[d] & \mathrm{Algbrd}\times\mathrm{Algbrd} \arrow[d] \\
\mathrm{Cat}\times\mathrm{Cat}\times\mathrm{Op}_{\mathrm{BM}} \arrow[r] & \mathrm{Cat}\times\mathrm{Cat}\times\mathrm{Op}_{\mathrm{Assos}}\times\mathrm{Op}_{\mathrm{Assos}}\,.
\end{tikzcd}$$

Here the left vertical arrow and right vertical arrow are the two-sided fibrations classified by the functors $\mathrm{Alg}_{\mathrm{BM}_{-,-}}(-)$ and $\mathrm{Algbrd}_{-}(-)$, and the horizontal arrows are the functors of “forgetting the bimodule”.

Proposition 4.3.7. *Let $\mathcal{M}$ be a presentable* BM*-monoidal category. Then the projection* $\mathrm{BMod}(\mathcal{M}) \to \mathrm{Algbrd}(\mathcal{M}_l) \times \mathrm{Algbrd}(\mathcal{M}_r)$ *is a cartesian fibration. Moreover, a morphism* $F : (\mathcal{A}, \mathcal{P}, \mathcal{B}) \to (\mathcal{A}', \mathcal{P}', \mathcal{B}')$ *is cartesian if and only if for every pair of objects x in $\mathcal{A}$ and y in $\mathcal{B}$, the induced map* $\mathcal{P}(x, y) \to \mathcal{P}'(F(x), F(y))$ *is an equivalence.*

Proof. This follows from a variation of the argument of proposition 4.1.8. □

Chapter 5

Enriched adjunctions and weighted limits

Let $\mathcal{M}$ be a presentable symmetric monoidal category. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of $\mathcal{M}$-enriched categories. Given a functor $G : \mathcal{D} \to \mathcal{C}$, a natural transformation $\epsilon : FG \to \mathrm{id}_{\mathcal{D}}$ is said to exhibit G as right adjoint to F if for every pair of objects c in $\mathcal{C}$ and d in $\mathcal{D}$ the induced map

$$\mathrm{Hom}_{\mathcal{C}}(c, G(d)) \xrightarrow{\epsilon(d)\circ F_*(-)} \mathrm{Hom}_{\mathcal{D}}(F(c), d)$$

is an isomorphism.

In 5.1 we study a local version of the notion of right adjoint to F, where the object $G(d)$ may only be well defined for specific values of d. We prove a criterion guaranteeing the existence of local adjoints to F in the case when F is obtained as the limit of a family of functors $F_i : D(i) \to D'(i)$. Given another $\mathcal{M}$-enriched category $\mathcal{J}$ and an object d in $\mathrm{Funct}(\mathcal{J}, \mathcal{D})$, we show that the right adjoint to $F_* : \mathrm{Funct}(\mathcal{J}, \mathcal{C}) \to \mathrm{Funct}(\mathcal{J}, \mathcal{D})$ at a functor d exists provided that the right adjoint to F at $\mathrm{ev}_j\, d$ exists for all j in $\mathcal{J}$.

In 5.2 we study the notion of adjoint functors between $\mathcal{M}$-enriched categories, as a special case of the notion of adjunction in a 2-category. We show that F admits a right adjoint if and only if it admits local right adjoints at every object in $\mathcal{D}$. We also discuss the notion of localization functors of $\mathcal{M}$-enriched categories.

In 5.3 we specialize to the case when F is the diagonal map $\mathcal{D} \to \mathrm{Funct}(\mathcal{I}_{\mathcal{M}}, \mathcal{D})$, where $\mathcal{I}_{\mathcal{M}}$ is the $\mathcal{M}$-enriched category obtained from a category $\mathcal{I}$ by pushforward along the unit map $\mathrm{Spc} \to \mathcal{M}$. A local right adjoint at an object X in $\mathrm{Funct}(\mathcal{I}_{\mathcal{M}}, \mathcal{D})$ is called a conical limit of X. We show that the data of a conical limit defines in particular an extension of X to a diagram $\mathcal{I}_{\mathcal{M}}^{\triangleleft} \to \mathcal{D}$. We study the interactions of the notion of conical limits with changes in the enriched category - in particular, we are able to conclude that a conical limit in $\mathcal{D}$ defines a limit diagram in the category underlying $\mathcal{D}$. Specializing our discussion of local adjoints we obtain basic results on the existence of conical limits on limits of enriched categories, and in enriched categories of functors.

In 5.4 we study how the notions of enriched adjunctions and conical limits interact with the procedure of enrichment of presentable modules over $\mathcal{M}$. We show that if $F : \mathcal{C} \to \mathcal{D}$ is a morphism of presentable modules then the induced functor of $\mathcal{M}$-enriched categories admits a right adjoint, and it also admits a left adjoint provided that F admits a left adjoint which strictly commutes with the action of $\mathcal{M}$. As a consequence, we are able to conclude that if $\mathcal{D}$ is a presentable module of $\mathcal{M}$, then the induced $\mathcal{M}$-enriched category admits all small conical limits and colimits. In particular, the canonical enrichment of $\mathcal{M}$ over itself is conically complete and cocomplete. We finish by applying this to show that conical limits are preserved by the Yoneda embedding. This provides in particular a characterization of the class of conical limits in an $\mathcal{M}$-enriched category $\mathcal{C}$ as those limits in the category underlying $\mathcal{C}$ which are preserved by all corepresentable enriched copresheaves.

The theory of conical limits is a particular case of the theory of weighted limits, which we explore in 5.5. We record here a proof of the fact that left adjoint functors preserve weighted colimits. Specializing to weighted limits and colimits over the unit $\mathcal{M}$-enriched category we recover the notions of powers and copowers. We show that in the case of $\mathcal{M}$-enriched categories arising from presentable modules, powers and copowers exist and are computed in the expected way.

In 5.6 we prove our main result on the theory of weighted colimits (theorem 5.6.1): an $\mathcal{M}$-enriched category $\mathcal{C}$ admits all weighted colimits if and only if it admits all conical colimits and copowers, and in this case a functor $F : \mathcal{C} \to \mathcal{D}$ into another $\mathcal{M}$-enriched category $\mathcal{D}$ preserves all weighted colimits if and only if it preserves all conical colimits and copowers. We use this to show that $\mathcal{M}$-enriched categories arising from presentable $\mathcal{M}$-modules admit all weighted limits and colimits, and that the Yoneda embedding detects weighted limits.

5.1 Local adjoints

We begin with a general discussion of the notion of locally defined adjoints for functors of enriched categories.

Definition 5.1.1. *Let $\mathcal{M}$ be a presentable monoidal category. Let $F : \mathcal{D} \to \mathcal{D}'$ be a functor of $\mathcal{M}$-categories, and let d' be an object of $\mathcal{D}'$. Let d be an object of $\mathcal{D}$ equipped with a morphism $\epsilon : F(d) \to d'$ in the category underlying $\mathcal{D}'$. We say that the pair (d, ϵ) is right adjoint to F at d' if for every object e in $\mathcal{D}$ the composite functor*

$$\mathrm{Hom}_{\mathcal{D}}(e, d) \xrightarrow{F_*} \mathrm{Hom}_{\mathcal{D}'}(F(e), F(d)) \xrightarrow{\epsilon} \mathrm{Hom}_{\mathcal{D}'}(F(e), d')$$

is an equivalence. Dually, we say that a pair (d, η) of an object d in $\mathcal{D}$ and a morphism $\eta : d' \to F(d)$ is left adjoint to F at d' if it is right adjoint to $F^{\mathrm{op}} : \mathcal{D}^{\mathrm{op}} \to \mathcal{D}'^{\mathrm{op}}$ at d'.

Remark 5.1.2. Let $G : \mathcal{M}' \to \mathcal{M}$ be a colimit preserving monoidal functor between presentable monoidal categories. Let $F : \mathcal{D} \to \mathcal{D}'$ be a functor of $\mathcal{M}$-categories, and let (d, ϵ) be right adjoint to F at an object d' in $\mathcal{D}'$. Then (d, ϵ) is also right adjoint to $(G^R)_! F : (G^R)_! \mathcal{D} \to (G^R)_! \mathcal{D}'$ at d'.

In the particular case when $\mathcal{M}' = \mathrm{Spc}$ and G is the unit map for $\mathcal{M}$ in Pr^L, the functor G^R is the functor $(\tau_{\mathcal{M}})_!$ that sends each $\mathcal{M}$-enriched category to its underlying category. We thus see that (d, ϵ) is right adjoint to $(\tau_{\mathcal{M}})_!F$ at d'.

Let $p : \mathcal{E} \to [1]$ be the cocartesian fibration associated to the functor $(\tau_{\mathcal{M}})_!F$. The pair (d, ϵ) induces a morphism α between $(0, d)$ and $(1, d')$ in $\mathcal{E}$. The condition that (d, ϵ) be right adjoint to $(\tau_{\mathcal{M}})_!F$ at d' is equivalent to the condition that α be a p-cartesian morphism. In particular, we see that the pair (d, ϵ) right adjoint to F at d' is unique if it exists.

Proposition 5.1.3. *Let $\mathcal{M}$ be a presentable monoidal category and let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ be functors of $\mathcal{M}$-enriched categories. Let e be an object of $\mathcal{E}$ and assume that G admits a left adjoint $\eta : e \to Gd$ at e. Let c be an object in $\mathcal{C}$ and $\eta' : d \to Fc$ be a morphism. Then η' presents c as left adjoint to F at d if and only if the composite map*

$$e \xrightarrow{\eta} Gd \xrightarrow{G_*\eta'} GFc$$

presents c as left adjoint to GF at c.

Proof. Let c' be an object in $\mathcal{C}$ and consider the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(c, c') \arrow[r, "F_*"] \arrow[dr, "(GF)_*"'] & \mathrm{Hom}_{\mathcal{D}}(Fc, Fc') \arrow[r, "\eta'^*"] \arrow[d, "G_*"] & \mathrm{Hom}_{\mathcal{D}}(d, Fc') \arrow[d, "G_*"] \\
& \mathrm{Hom}_{\mathcal{E}}(GFc, GFc') \arrow[r, "(G_*(\eta'))^*"] \arrow[dr, "(G_*(\eta')\eta)^*"'] & \mathrm{Hom}_{\mathcal{E}}(Gd, GFc') \arrow[d, "\eta^*"] \\
& & \mathrm{Hom}_{\mathcal{E}}(e, GFc').
\end{tikzcd}$$

Since (e, η) is left adjoint to G at e, we have that the composition of the right vertical arrows is an isomorphism. We therefore have that the composition of the top horizontal arrows is an isomorphism if and only if the composition of the diagonal arrows is an isomorphism. □

Definition 5.1.4. *Let $\mathcal{M}$ be a presentable monoidal category and consider a commutative square of $\mathcal{M}$-enriched categories*

$$\begin{tikzcd}
\mathcal{D} \arrow[r, "F"] \arrow[d, "T"] & \mathcal{D}' \arrow[d, "T'"] \\
\mathcal{E} \arrow[r, "G"] & \mathcal{E}'
\end{tikzcd} .$$

Let d' be an object of $\mathcal{D}'$. We say that the above square is horizontally right adjointable at d' if there is a pair (d, ϵ) right adjoint to F at d', and moreover the induced map $T'\epsilon : GTd \to T'd'$ is right adjoint to G at $T'd'$.

Remark 5.1.5. Let $\mathcal{M} = \mathrm{Spc}$ and consider a commutative diagram as in definition 5.1.4. This induces a morphism $\mathcal{T}$ of cocartesian fibrations over $[1]$ between the fibrations $\mathcal{E}_F$, $\mathcal{E}_G$ associated to F and G. The square is horizontally right adjointable at d' if there is a nontrivial cartesian arrow in $\mathcal{E}_F$ with target d', whose image under $\mathcal{T}$ is cartesian.

We now study the stability of local adjunctions under passage to limits and formation of functor enriched categories.

Proposition 5.1.6. *Let $\mathcal{M}$ be a presentable monoidal category and let $\mathcal{I}$ be a category. Let $D, D' : \mathcal{I} \to \mathrm{Cat}^{\mathcal{M}}$ be functors, and let $\eta : D \to D'$ be a natural transformation. Denote by $F : \mathcal{D} \to \mathcal{D}'$ the limit of η. For each i in $\mathcal{I}$ let $p_i : \mathcal{D} \to D(i)$ and $p'_i : \mathcal{D}' \to D'(i)$ be the projections. Assume that for each arrow $\alpha : i \to j$ in $\mathcal{I}$, the square*

$$\begin{tikzcd} D(i) \arrow[r, "\eta_i"] \arrow[d, "D(\alpha)"] & D'(i) \arrow[d, "D'(\alpha)"] \\ D(j) \arrow[r, "\eta_j"] & D'(j) \end{tikzcd}$$

is horizontally right adjointable at $p'_i(d')$. Then

(i) *There exists a right adjoint to F at d'.*

(ii) *A pair (d, ϵ) is right adjoint to F at d' if and only if $(p_i d, p'_i(\epsilon))$ is right adjoint to η_i at $p'_i(d')$ for every i in $\mathcal{I}$.*

Proof. Consider first the case $\mathcal{M} = \mathrm{Spc}$. Passing to cocartesian fibrations of the functors η_i and F, we obtain a diagram

$$\mathcal{E}_\eta : \mathcal{I} \to (\mathrm{Cat}^{\mathrm{cocart}}_{/[1]})_{\{1\}/}$$

with limit $\mathcal{E}_F$, where the right hand side denotes the undercategory of the category of cocartesian fibrations and morphisms of cocartesian fibrations over $[1]$, under the cocartesian fibration $\{1\} \to [1]$.

The adjointability of the square in the statement implies that the composition of $\mathcal{E}_\eta$ with the forgetful functor $(\mathrm{Cat}^{\mathrm{cocart}}_{/[1]})_{\{1\}/} \to (\mathrm{Cat}_{/[1]})_{\{1\}/}$ factors through the subcategory $(\mathrm{Cat}_{/[1]})_{[1]/^{\mathrm{cart}}}$ of categories over $[1]$ equipped with a cartesian section, and functors which preserve this section. The case $\mathcal{M} = \mathrm{Spc}$ of the proposition now follows from the fact that the projections

$$(\mathrm{Cat}^{\mathrm{cocart}}_{/[1]})_{\{1\}/} \to (\mathrm{Cat}_{/[1]})_{\{1\}/} \leftarrow (\mathrm{Cat}_{/[1]})_{[1]/^{\mathrm{cart}}} \to (\mathrm{Cat}_{/[1]})_{[1]/}$$

create limits.

We now consider the general case. By virtue of the above and the uniqueness claim from remark 5.1.2, it suffices to show that if (d, ϵ) is such that $(p_j d, p'_j \epsilon)$ is right adjoint to η_i at $p'_i d'$ for all i then it is right adjoint to F at d'. Let e be an object in $\mathcal{D}$. Note that there is a functor $R : \mathcal{I} \to \mathrm{Funct}([2], \mathcal{M})$ whose value on each index i is given by

$$\mathrm{Hom}_{D(i)}(p_i e, p_i d) \xrightarrow{(\eta_i)_*} \mathrm{Hom}_{D'(i)}(\eta_i(e), \eta_i(d)) \xrightarrow{p'_i \epsilon} \mathrm{Hom}_{D'(i)}(\eta_i(e), p'_i d')$$

and which has a limit given by

$$\mathrm{Hom}_{\mathcal{D}}(e, d) \xrightarrow{F_*} \mathrm{Hom}_{\mathcal{D}'}(F(e), F(d)) \xrightarrow{\epsilon} \mathrm{Hom}_{\mathcal{D}'}(F(e), d').$$

For each i in $\mathcal{I}$ the composition of the maps in $R(i)$ is an isomorphism, since $(p_i d, p'_i \epsilon)$ is right adjoint to η_i at $p'_i d'$. We conclude that the composition of the maps in $\lim_{\mathcal{I}} R$ is an isomorphism, which means that (d, ϵ) is right adjoint to F at d', as desired. $\square$

Proposition 5.1.7. *Let $\mathcal{M}$ be a presentable symmetric monoidal category. Let $\mathcal{J}$ be an $\mathcal{M}$-enriched category $F : \mathcal{D} \to \mathcal{D}'$ be a functor of $\mathcal{M}$-enriched categories. Let d' be an object in $\mathrm{Funct}(\mathcal{J}, \mathcal{D}')$ and assume that for all objects j in $\mathcal{J}$ there exists a right adjoint to F at $\mathrm{ev}_j\, d'$. Then*

(i) *There exists a right adjoint to $F_* : \mathrm{Funct}(\mathcal{J}, \mathcal{D}) \to \mathrm{Funct}(\mathcal{J}, \mathcal{D}')$ at d'.*

(ii) *A pair (d, ϵ) is right adjoint to F_* at d' if and only if for every j in $\mathcal{J}$ the pair $(\mathrm{ev}_j\, d, \mathrm{ev}_j\, \epsilon)$ is right adjoint to F at $\mathrm{ev}_j\, d'$.*

Proof. Let $\mathcal{S}$ be the full subcategory of $\mathrm{Cat}^{\mathcal{M}}$ on those $\mathcal{M}$-enriched categories $\mathcal{J}$ for which the proposition holds. We claim that $\mathcal{S}$ is closed under colimits in $\mathrm{Cat}^{\mathcal{M}}$. Let $J : \mathcal{I} \to \mathcal{S}$ be a diagram, and let $\mathcal{J}$ be its colimit in $\mathrm{Cat}^{\mathcal{M}}$. We then have that the functor

$$F_* : \mathrm{Funct}(\mathcal{J}, \mathcal{D}) \to \mathrm{Funct}(\mathcal{J}, \mathcal{D}')$$

is obtained by passage to the limit of the functors

$$\eta_i : \mathrm{Funct}(J(i), \mathcal{D}) \to \mathrm{Funct}(J(i), \mathcal{D}')$$

given by composition with F. Let d' be an object in $\mathrm{Funct}(\mathcal{J}, \mathcal{D}')$ and assume that there exists a right adjoint to F at $\mathrm{ev}_j\, d'$ for every j in $\mathcal{J}$. The fact that $J(i)$ belongs to $\mathcal{S}$ for all i implies that for every arrow $\alpha : i \to i'$ in $\mathcal{I}$ the square

$$\begin{tikzcd} \mathrm{Funct}(J(i'), \mathcal{D}) \arrow[r, "\eta_{i'}"] \arrow[d, "J(\alpha)^*"] & \mathrm{Funct}(J(i'), \mathcal{D}') \arrow[d, "J(\alpha)^*"] \\ \mathrm{Funct}(J(i), \mathcal{D}) \arrow[r, "\eta_i"] & \mathrm{Funct}(J(i'), \mathcal{D}') \end{tikzcd}$$

is horizontally right adjointable at $d'|_{J(i')}$. It then follows from proposition 5.1.6 that there is indeed a right adjoint to F_* at d', and moreover a pair (d, ϵ) is right adjoint to F_* at d' if and only if the associated pair $(d|_{J(i)}, \epsilon|_{J(i)})$ is right adjoint to η_i at $d'|_{J(i)}$ for all i in $\mathcal{I}$. Using again the fact that $J(i)$ belongs to $\mathcal{S}$ for every i, we see that this happens if and only if $(\mathrm{ev}_j\, d, \mathrm{ev}_j\, \epsilon)$ is right adjoint to F at $\mathrm{ev}_j\, d'$ for every j in $\mathcal{J}$ which is in the image of the map $J(i) \to \mathcal{J}$ for some i in $\mathcal{I}$. Since $\mathcal{J}$ is the colimit of the objects $J(i)$, we have that the maps $J(i) \to \mathcal{J}$ are jointly surjective, so we have that $\mathcal{J}$ belongs to $\mathcal{S}$.

Since $\mathrm{Cat}^{\mathcal{M}}$ is generated under colimits by cells, our result will follow if we show that for every m in $\mathcal{M}$ the enriched category underlying the cell C_m belongs to $\mathcal{S}$. Let $d' : C_m \to \mathcal{D}'$ be an m-cell in $\mathcal{D}'$, with source and target objects d'_0 and d'_1. Let (d_0, ϵ_0) and (d_1, ϵ_1) be right adjoint to F at d'_0 and d'_1 respectively.

We claim that there is a unique enhancement of this data to a pair (d, ϵ) of an m-cell $d : C_m \to \mathcal{D}$ and a morphism $\epsilon : F_*d \to d'$. Indeed, the data of a cell d between d_0 and d_1 corresponds to a map $m \to \mathrm{Hom}_{\mathcal{D}}(d_0, d_1)$. A map ϵ lifting ϵ_0 and ϵ_1 consists of an identification of m-cells $\epsilon_1 F_* d = d' \epsilon_0$. The fact that (d_1, ϵ_1) is right adjoint to F at d'_1 implies that there is a unique such pair (d, ϵ), as claimed.

It remains to show that (d, ϵ) is right adjoint to F_* at d'. Let $e : C_m \to \mathcal{D}$ be another m-cell with source and target objects e_0 and e_1. Recall from corollary 3.5.26 that we have a cartesian square

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{Funct}(C_m,\mathcal{D})}(e, d) \arrow[r, "\mathrm{ev}_0"] \arrow[d, "\mathrm{ev}_1"] & \mathrm{Hom}_{\mathcal{D}}(e_0, d_0) \arrow[d] \\
\mathrm{Hom}_{\mathcal{D}}(e_1, d_1) \arrow[r] & \mathscr{H}om_{\mathcal{M}}(m, \mathrm{Hom}_{\mathcal{D}}(e_0, d_1))
\end{tikzcd}$$

where the right and bottom arrows are given by composition with the cells d and e, respectively. The functor F induces a map from the above square to the cartesian square

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{Funct}(C_m,\mathcal{D}')}(F_*e, F_*d) \arrow[r, "\mathrm{ev}_0"] \arrow[d, "\mathrm{ev}_1"] & \mathrm{Hom}_{\mathcal{D}'}(Fe_0, Fd_0) \arrow[d] \\
\mathrm{Hom}_{\mathcal{D}'}(Fe_1, Fd_1) \arrow[r] & \mathscr{H}om_{\mathcal{M}}(m, \mathrm{Hom}_{\mathcal{D}'}(Fe_0, Fd_1))
\end{tikzcd}$$

where the right and bottom arrows are given by composition with F_*d and F_*e. Finally, composition with ϵ yields a map from the above to the cartesian square

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{Funct}(C_m,\mathcal{D}')}(F_*e, d') \arrow[r, "\mathrm{ev}_0"] \arrow[d, "\mathrm{ev}_1"] & \mathrm{Hom}_{\mathcal{D}'}(Fe_0, d'_0) \arrow[d] \\
\mathrm{Hom}_{\mathcal{D}'}(Fe_1, d'_1) \arrow[r] & \mathscr{H}om_{\mathcal{M}}(m, \mathrm{Hom}_{\mathcal{D}'}(Fe_0, d'_1))
\end{tikzcd}$$

where the right and bottom arrows are given by composition with d' and F_*e. We thus see that there is a commutative cube

$$\begin{tikzcd}
 & \mathrm{Hom}_{\mathrm{Funct}(C_m,\mathcal{D}')}(F_*e, d') \arrow[rr, "\mathrm{ev}_0"] \arrow[dd, "\mathrm{ev}_1"] & & \mathrm{Hom}_{\mathcal{D}'}(Fe_0, d'_0) \arrow[dd] \\
\mathrm{Hom}_{\mathrm{Funct}(C_m,\mathcal{D})}(e, d) \arrow[ur, "\epsilon F_*"] \arrow[rr, "\mathrm{ev}_0"] \arrow[dd, "\mathrm{ev}_1"] & & \mathrm{Hom}_{\mathcal{D}}(e_0, d_0) \arrow[ur, "\epsilon_0 F"] \arrow[dd] & \\
 & \mathrm{Hom}_{\mathcal{D}'}(Fe_1, d'_1) \arrow[rr] & & \mathscr{H}om_{\mathcal{M}}(m, \mathrm{Hom}_{\mathcal{D}'}(Fe_0, d'_1)) \\
\mathrm{Hom}_{\mathcal{D}}(e_1, d_1) \arrow[ur, "\epsilon_1 F"] \arrow[rr] & & \mathscr{H}om_{\mathcal{M}}(m, \mathrm{Hom}_{\mathcal{D}}(e_0, d_1)) \arrow[ur, "\epsilon_1 F"] &
\end{tikzcd}$$

with cartesian front and back faces. Since (d_0, ϵ_0) and (d_1, ϵ_1) are right adjoint to F at d'_0 and d'_1, the bottom left, bottom right, and top right diagonal arrows are isomorphisms. We conclude that the top left diagonal arrow is an isomorphism, which means that (d, ϵ) is right adjoint to F_* at d, as desired. $\square$

Corollary 5.1.8. *Let $\mathcal{M}$ be a presentable symmetric monoidal category. Let $\mathcal{J}$ and $\mathcal{D}$ be $\mathcal{M}$-enriched categories. Then a morphism $\epsilon : d \to d'$ in $\mathrm{Funct}(\mathcal{J}, \mathcal{D})$ is an isomorphism if and only if $\mathrm{ev}_j\, \epsilon$ is an isomorphism for every j in $\mathcal{J}$.*

Proof. Specialize proposition 5.1.7 to the case when F is the identity of $\mathcal{D}$. $\square$

Corollary 5.1.9. *Let $\mathcal{M}$ be a presentable symmetric monoidal category. Let $F : \mathcal{D} \to \mathcal{D}'$ be a functor of $\mathcal{M}$-enriched categories, and assume that F admits a right adjoint at d' for every d' in $\mathcal{D}'$. Then there is a unique functor $F^R : \mathcal{D}' \to \mathcal{D}$ equipped with a morphism $\epsilon : FF^R \to \mathrm{id}_{\mathcal{D}'}$ such that for every d' in $\mathcal{D}'$ the pair $(F^R(d'), \epsilon(d'))$ is right adjoint to F at $\mathcal{D}$.*

Proof. Specialize proposition 5.1.7 to the case when $\mathcal{J} = \mathcal{D}'$ and take (F^R, ϵ) to be right adjoint to F_* at $\mathrm{id}_{\mathcal{D}'}$. $\square$

5.2 Global adjoints

We now discuss the notion of adjunction between functors of enriched categories. We will obtain this as a particular case of the general notion of adjunction in a 2-category.

Definition 5.2.1. *Let $\mathcal{D}$ be a 2-category. An arrow $\alpha : d \to e$ in $\mathcal{D}$ is said to admit a right adjoint if there is an arrow $\alpha^R : e \to d$ and a pair of 2-cells $\eta : \mathrm{id}_d \to \alpha^R\alpha$ and $\epsilon : \alpha\alpha^R \to \mathrm{id}_e$ satisfying the following two conditions:*

- *The composite 2-cell*
$$\alpha = \alpha\,\mathrm{id}_d \xrightarrow{\mathrm{id}_\alpha \eta} \alpha\alpha^R\alpha \xrightarrow{\epsilon\, \mathrm{id}_\alpha} \mathrm{id}_e\,\alpha = \alpha$$
is equivalent to the identity.

- *The composite 2-cell*
$$\alpha^R = \mathrm{id}_d\,\alpha^R \xrightarrow{\eta\, \mathrm{id}_{\alpha^R}} \alpha^R\alpha\alpha^R \xrightarrow{\mathrm{id}_{\alpha^R}\epsilon} \alpha^R\,\mathrm{id}_e = \alpha^R$$
is equivalent to the identity.

In this situation, we say that α^R is right adjoint to α, and we call η and ϵ the unit and counit of the adjunction, respectively. We say that α admits a left adjoint if it admits a right adjoint as a morphism in $\mathcal{D}^{2\text{-op}}$.

Example 5.2.2. In the case $\mathcal{D} = \mathscr{C}at$, definition 5.2.1 recovers the usual notion of adjunction between functors of categories.

We refer to [RV16] for a proof of the following fundamental theorems.

Theorem 5.2.3. *There exists a 2-category* Adj *equipped with an epimorphism* $L : [1] \to \mathrm{Adj}$ *such that for every 2-category* $\mathcal{D}$ *composition with* L *induces an equivalence between the space of functors* $\mathrm{Adj} \to \mathcal{D}$ *and the space of functors* $[1] \to \mathcal{D}$ *which pick out a right adjointable arrow in* $\mathcal{D}$.

Theorem 5.2.4. *Let* $\mathcal{D}$ *be a 2-category. Then the following spaces are equivalent:*

(i) *The space of arrows in* $\mathcal{D}$ *which admit a right adjoint.*

(ii) *The space of triples* (α, α^R, η) *of an arrow* $\alpha : d \to e$ *in* $\mathcal{D}$*, an arrow* $\alpha^R : e \to d$ *in* $\mathcal{D}$*, and a 2-cell* $\eta : \mathrm{id}_d \to \alpha^R\alpha$ *which can be extended to an adjunction between* α *and* α^R.

The equivalence is given by mapping a triple (α, α^R, η) *to* α.

We now specialize to the case when $\mathcal{D}$ is the 2-category of categories enriched in a presentable symmetric monoidal category

Notation 5.2.5. Let $\mathcal{M}$ be a presentable symmetric monoidal category. We denote by $\mathscr{C}at^{\mathcal{M}}$ the image of $\mathrm{Cat}^{\mathcal{M}}$ under the composite functor

$$\mathrm{Cat}^{\mathcal{M}}\text{-mod}(\mathrm{Pr}^L) \xrightarrow{\theta_{\mathrm{Cat}^{\mathcal{M}}}} \widehat{\mathrm{Cat}}^{\mathrm{Cat}^{\mathcal{M}}} \xrightarrow{(\tau_{\mathcal{M}})_!} \widehat{\mathrm{Cat}}^{\mathrm{Cat}} \hookrightarrow \widehat{2\mathrm{Cat}}.$$

We call $\mathscr{C}at^{\mathcal{M}}$ the 2-category of $\mathcal{M}$-enriched categories. Note that the 1-category underlying $\mathscr{C}at^{\mathcal{M}}$ is equivalent to $\mathrm{Cat}^{\mathcal{M}}$.

Example 5.2.6. Let $n \geq 0$. Then $\mathscr{C}at^{n\mathrm{Cat}}$ is the 2-category underlying $(n+1)\mathscr{C}at$.

Remark 5.2.7. Let $\mathcal{M}$ be a presentable symmetric monoidal category and let $\mathcal{C}, \mathcal{D}$ be $\mathcal{M}$-enriched categories. Then the Hom category $\mathrm{Hom}_{\mathscr{C}at^{\mathcal{M}}}(\mathcal{C}, \mathcal{D})$ is the category underlying $\mathrm{Funct}(\mathcal{C}, \mathcal{D})$.

Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors, and $\eta : F \to G$ be a 2-cell. Let x, y be a pair of objects of $\mathcal{C}$. Then there is an induced morphism

$$C_{\mathrm{Hom}_{\mathcal{C}}(x,y)} \otimes C_{1_{\mathcal{M}}} \to \mathcal{D}.$$

Examining the description of the product of cells from remark 3.5.24 we obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(x,y) \arrow[r, "F_*"] \arrow[d, "G_*"] & \mathrm{Hom}_{\mathcal{D}}(Fx,Fy) \arrow[d, "\eta(y)\circ -"] \\
\mathrm{Hom}_{\mathcal{D}}(Gx,Gy) \arrow[r, "-\circ\eta(x)"] & \mathrm{Hom}_{\mathcal{D}}(Fx,Gy).
\end{tikzcd}$$

Definition 5.2.8. *Let* $\mathcal{M}$ *be a presentable symmetric monoidal category and let* $F : \mathcal{C} \to \mathcal{D}$ *be a functor of* $\mathcal{M}$*-enriched categories. We say that* F *admits a right adjoint if it admits a right adjoint when thought of as a morphism in the 2-category* $\mathscr{C}at^{\mathcal{M}}$.

Our next goal is to show that a functor of enriched categories admits a right adjoint if and only if it admits local right adjoints at every point.

Lemma 5.2.9. *Let $\mathcal{J}$ and $\mathcal{D}$ be 2-categories. Let $\eta : F \to G$ be a morphism in $\mathrm{Funct}(\mathcal{J}, \mathcal{D})$. Then η admits a right adjoint if and only if for every morphism $\alpha : j \to j'$ in $\mathcal{J}$, the commutative square*

$$\begin{tikzcd} F(j) \arrow[r, "\eta_j"] \arrow[d, "F\alpha"] & G(j) \arrow[d, "G\alpha"] \\ F(j') \arrow[r, "\eta_{j'}"] & G(j') \end{tikzcd}$$

is horizontally right adjointable.

Proof. Combine [Hau20] theorem 4.6 and corollary 3.15. □

Proposition 5.2.10. *Let $\mathcal{M}$ be a presentable symmetric monoidal category and let $F : \mathcal{C} \to \mathcal{D}$ be a functor of $\mathcal{M}$-enriched categories. Then F admits a right adjoint if and only if for every object d in $\mathcal{D}$ there exists a right adjoint for F at d.*

Proof. Assume first that F admits a right adjoint F^R. Denote by $\eta : \mathrm{id}_{\mathcal{C}} \to F^R F$ and $\epsilon : FF^R \to \mathrm{id}_{\mathcal{D}}$ the unit and counit of the adjunction, respectively. Let d be an object in $\mathcal{D}$. We claim that the morphism

$$\epsilon(d) : FF^R(d) \to \mathrm{id}_{\mathcal{D}}(d) = d$$

exhibits $F^R(d)$ as right adjoint to F at d. Let c be an object of $\mathcal{C}$. We have to show that the map V given by the composition

$$\mathrm{Hom}_{\mathcal{C}}(c, F^R(d)) \xrightarrow{F_*} \mathrm{Hom}_{\mathcal{D}}(F(c), FF^R(d)) \xrightarrow{\epsilon(d)\circ -} \mathrm{Hom}_{\mathcal{D}}(F(c), d)$$

is an isomorphism. Observe that there is a map W going in the opposite direction, given by the following composition:

$$\mathrm{Hom}_{\mathcal{D}}(F(c), d) \xrightarrow{(F^R)_*} \mathrm{Hom}_{\mathcal{C}}(F^R F(c), F^R(d)) \xrightarrow{-\circ\eta(c)} \mathrm{Hom}_{\mathcal{C}}(c, F^R(d))$$

We claim that V and W are inverse equivalences. Observe that the map WV is given by the composition

$$\mathrm{Hom}_{\mathcal{C}}(c, F^R(d)) \xrightarrow{(F^R F)_*} \mathrm{Hom}_{\mathcal{C}}(F^R F(c), F^R F F^R(d)) \xrightarrow{F^R\epsilon(d)\circ - \circ\eta(c)} \mathrm{Hom}_{\mathcal{C}}(c, F^R(d)).$$

Thanks to remark 5.2.7, we can rewrite the above as the composition

$$\mathrm{Hom}_{\mathcal{C}}(c, F^R(d)) \xrightarrow{\eta(F^R(d))\circ -} \mathrm{Hom}_{\mathcal{C}}(c, F^R F F^R(d)) \xrightarrow{F^R\epsilon(d)\circ -} \mathrm{Hom}_{\mathcal{C}}(c, F^R(d)).$$

This is equivalent to the identity thanks to the second condition in definition 5.2.1. The fact that VW is equivalent to the identity follows from similar arguments. We conclude that V is indeed an isomorphism, so $\epsilon(d)$ exhibits $F^R(d)$ as right adjoint to F at d.

Assume now that for every object d in $\mathcal{D}$ there exists a right adjoint for F at d. To show that F admits a right adjoint it suffices to show that its image under the Yoneda embedding

$$\mathscr{C}at^{\mathcal{M}} \to \operatorname{Funct}((\mathscr{C}at^{\mathcal{M}})^{\text{1-op}}, \mathscr{C}at)$$

admits a right adjoint. Applying lemma 5.2.9, we reduce to showing that for every functor $\alpha : \mathcal{J} \to \mathcal{J}'$ between $\mathcal{M}$-enriched categories, the commutative square of categories

$$\begin{tikzcd}
(\tau_{\mathcal{M}})_! \operatorname{Funct}(\mathcal{J}', \mathcal{C}) \arrow[r, "F_*"] \arrow[d, "\alpha^*"] & (\tau_{\mathcal{M}})_! \operatorname{Funct}(\mathcal{J}', \mathcal{D}) \arrow[d, "\alpha^*"] \\
(\tau_{\mathcal{M}})_! \operatorname{Funct}(\mathcal{J}, \mathcal{C}) \arrow[r, "F_*"] & (\tau_{\mathcal{M}})_! \operatorname{Funct}(\mathcal{J}, \mathcal{D})
\end{tikzcd}$$

is horizontally right adjointable. Using proposition 5.1.7 together with remark 5.1.2 we see that the above commutative square of categories is horizontally right adjointable at every object of $(\tau_{\mathcal{M}})_! \operatorname{Funct}(\mathcal{J}', \mathcal{D})$. Our claim now follows from the fact that local and global adjointability agree for functors between categories. $\square$

We finish with a discussion of localization functors of enriched categories.

Proposition 5.2.11. *Let $\mathcal{M}$ be a presentable symmetric monoidal category and let $F : \mathcal{C} \to \mathcal{D}$ be a functor of $\mathcal{M}$-enriched categories admitting a right adjoint F^R. Then the following conditions are equivalent:*

(i) *The functor F^R is fully faithful.*

(ii) *The counit $\epsilon : FF^R \to \mathrm{id}_{\mathcal{D}}$ is an isomorphism.*

(iii) *The functor F is surjective on objects, and for every object c in $\mathcal{C}$, the unit map $c \to F^R F c$ is inverted by F.*

Proof. The equivalence between items (ii) and (iii) follows directly from the triangle conditions. It remains to show the equivalence with item (i). Let d, d' be two objects in $\mathcal{D}$, and consider the composite map

$$\operatorname{Hom}_{\mathcal{D}}(d, d') \xrightarrow{F^R_*} \operatorname{Hom}_{\mathcal{C}}(F^R d, F^R d') \xrightarrow{F_*} \operatorname{Hom}_{\mathcal{D}}(FF^R d, FF^R d') \xrightarrow{\epsilon_*} \operatorname{Hom}_{\mathcal{D}}(FF^R d, d').$$

By naturality of ϵ, this is equivalent to the morphism

$$\operatorname{Hom}_{\mathcal{D}}(d, d') \xrightarrow{\epsilon^*} \operatorname{Hom}_{\mathcal{D}}(FF^R d, d').$$

It follows that $F^R_* : \operatorname{Hom}_{\mathcal{D}}(d, d') \to \operatorname{Hom}_{\mathcal{C}}(F^R d, F^R d')$ is an equivalence if and only if $\epsilon^* : \operatorname{Hom}_{\mathcal{D}}(d, d') \to \operatorname{Hom}_{\mathcal{D}}(FF^R d, d')$ is an equivalence. The result now follows from the fact that d, d' are arbitrary. $\square$

Definition 5.2.12. *Let $\mathcal{M}$ be a presentable symmetric monoidal category and let $F : \mathcal{C} \to \mathcal{D}$ be a functor of $\mathcal{M}$-enriched categories admitting a right adjoint F^R. We say that F is a localization functor if it satisfies the equivalent conditions of proposition 5.2.11.*

Remark 5.2.13. Let $\mathcal{M}$ be a presentable symmetric monoidal category and let $F : \mathcal{C} \to \mathcal{D}$ be a functor of $\mathcal{M}$-enriched categories admitting a right adjoint F^R. It follows from remark 5.1.2 together with the second characterization of localizations in proposition 5.2.11 that F is a localization functor if and only if the functor of categories $(\tau_{\mathcal{M}})_! F$ underlying F is a localization functor.

Remark 5.2.14. Let $\mathcal{M}$ be a presentable symmetric monoidal category. Let $F : \mathcal{C} \to \mathcal{D}$ be a localization functor of $\mathcal{M}$-enriched categories, and let $\mathcal{C}'$ be an $\mathcal{M}$-enriched category. Denote by $\mathrm{Funct}(\mathcal{C}, \mathcal{C}')_{\mathrm{loc}}$ the full subcategory of $\mathrm{Funct}(\mathcal{C}, \mathcal{C}')$ on those functors that invert the unit map $c \to F^R F c$ for every c in $\mathcal{C}$. Then we have functors

$$F^* : \mathrm{Funct}(\mathcal{D}, \mathcal{C}') \to \mathrm{Funct}(\mathcal{C}, \mathcal{C}')_{\mathrm{loc}}$$

and

$$(F^R)^* : \mathrm{Funct}(\mathcal{C}, \mathcal{C}')_{\mathrm{loc}} \to \mathrm{Funct}(\mathcal{D}, \mathcal{C}').$$

The unit and counit of the adjunction $F \dashv F^R$ induce equivalences $(F^R)^* F^* = \mathrm{id}_{\mathrm{Funct}(\mathcal{D}, \mathcal{C}')}$ and $F^*(F^R)^* = \mathrm{id}_{\mathrm{Funct}(\mathcal{C}, \mathcal{C}')_{\mathrm{loc}}}$. It follows that F^* and $(F^R)^*$ are inverse equivalences. In particular, we conclude that localization functors are epimorphisms in $\mathrm{Cat}^{\mathcal{M}}$.

5.3 Conical limits

We now specialize the notion of local adjoints to obtain a theory of conical limits and colimits.

Notation 5.3.1. Let $\mathcal{M}$ be a presentable monoidal category, and $\mathcal{I}$ be a category. We denote by $\mathcal{I}_{\mathcal{M}}$ the image of $\mathcal{I}$ under the functor $\mathrm{Cat} \to \mathrm{Cat}^{\mathcal{M}}$ induced by pushforward along the unit map $\mathrm{Spc} \to \mathcal{M}$. We note that for every $\mathcal{M}$-enriched category $\mathcal{D}$, there is a correspondence between functors $\mathcal{I}_{\mathcal{M}} \to \mathcal{D}$ and functors $\mathcal{I} \to (\tau_{\mathcal{M}})_! \mathcal{D}$.

Definition 5.3.2. *Let $\mathcal{M}$ be a presentable symmetric monoidal category. Let $\mathcal{I}$ be category and let $\mathcal{D}$ be an $\mathcal{M}$-enriched category. Let $X : \mathcal{I}_{\mathcal{M}} \to \mathcal{D}$ be a functor. We say that X admits a conical limit if the functor*

$$\Delta : \mathcal{D} = \mathrm{Funct}(1_{\mathcal{M}}, \mathcal{D}) \to \mathrm{Funct}(\mathcal{I}_{\mathcal{M}}, \mathcal{D})$$

of precomposition with the projection $\mathcal{I} \to [0]$ admits a right adjoint at X. In this case, we call its right adjoint at X the conical limit of X. We say that X admits a conical colimit if the induced diagram $X^{\mathrm{op}} : \mathcal{I}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$ admits a conical limit - in this case we define the conical colimit of X to be the conical limit of X^{op}.

Remark 5.3.3. Let $\mathcal{I}$ be a category and denote by $\mathcal{I}^{\triangleleft}$ the category obtained from $\mathcal{I}$ by adjoining a final object. We have a pushout diagram in Cat

$$\begin{tikzcd} \mathcal{I} \arrow[r, "\mathrm{id}_{\mathcal{I}} \times \{0\}"] \arrow[d] & \mathcal{I} \times [1] \arrow[d] \\ {[0]} \arrow[r] & \mathcal{I}^{\triangleleft}. \end{tikzcd}$$

Let $\mathcal{M}$ be a presentable symmetric monoidal category. It follows from the above that for every $\mathcal{M}$-enriched category $\mathcal{D}$ there is a pullback diagram of spaces

$$\begin{tikzcd} \mathrm{Hom}_{\mathrm{Cat}^{\mathcal{M}}}(\mathcal{I}^{\triangleleft}_{\mathcal{M}}, \mathcal{D}) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathrm{Cat}^{\mathcal{M}}}(C_{1_{\mathcal{M}}}, \mathrm{Funct}(\mathcal{I}_{\mathcal{M}}, \mathcal{D})) \arrow[d] \\ \mathrm{Hom}_{\mathrm{Cat}^{\mathcal{M}}}(1_{\mathcal{M}}, \mathcal{D}) \arrow[r] & \mathrm{Hom}_{\mathrm{Cat}^{\mathcal{M}}}(\mathcal{I}_{\mathcal{M}}, \mathcal{D}) \end{tikzcd}$$

where the right vertical arrow is given by evaluation at the source, and to bottom horizontal arrow is the diagonal map. Hence we see that a pair (d, ϵ) of an object d in $\mathcal{D}$ and a morphism $\epsilon : \Delta d \to X$ in $\mathrm{Funct}(\mathcal{I}, \mathcal{D})$ is the same data as a diagram $X^{\triangleleft} : \mathcal{I}^{\triangleleft}_{\mathcal{M}} \to \mathcal{D}$. In particular, we have that a conical limit for a diagram $X : \mathcal{I}_{\mathcal{M}} \to \mathcal{D}$ can be identified with a particular kind of extension of X to a diagram $X^{\triangleleft} : \mathcal{I}^{\triangleleft}_{\mathcal{M}} \to \mathcal{D}$.

Remark 5.3.4. Let $\mathcal{M}$ be a presentable symmetric monoidal category and let $F : \mathcal{D} \to \mathcal{D}'$ be a functor of $\mathcal{M}$-enriched categories. Let $\mathcal{I}$ be a category. Then we have a commutative square

$$\begin{tikzcd} \mathcal{D} \arrow[r, "\Delta"] \arrow[d, "F"] & \mathrm{Funct}(\mathcal{I}_{\mathcal{M}}, \mathcal{D}) \arrow[d, "F_*"] \\ \mathcal{D}' \arrow[r, "\Delta"] & \mathrm{Funct}(\mathcal{I}_{\mathcal{M}}, \mathcal{D}'). \end{tikzcd}$$

Let d be an object in $\mathcal{D}$ and $\epsilon : \Delta d \to X$ be a morphism in $\mathrm{Funct}(\mathcal{I}_{\mathcal{M}}, \mathcal{D})$, associated to a diagram $X^{\triangleleft} : \mathcal{I}^{\triangleleft}_{\mathcal{M}} \to \mathcal{D}$ under the equivalence of remark 5.3.3. Then the induced pair $(Fd, F_*\epsilon)$ is associated to $F_*X^{\triangleleft}$.

Remark 5.3.5. Let $G : \mathcal{M} \to \mathcal{M}'$ be a colimit preserving monoidal functor between presentable symmetric monoidal categories and let $\mathcal{D}$ be an $\mathcal{M}'$ enriched category. It follows from proposition 3.5.27 that we have an equivalence

$$\mathrm{Funct}(-_{\mathcal{M}}, (G^R)_!\mathcal{D}) = (G^R)_! \,\mathrm{Funct}(-_{\mathcal{M}'}, \mathcal{D})$$

of functors $\mathrm{Cat} \to \mathrm{Cat}^{\mathcal{M}}$. Let $\mathcal{I}$ be a category. Evaluating the above equivalence at the projection $\mathcal{I} \to [0]$ we obtain a commutative square

$$\begin{tikzcd} (G^R)_!\mathcal{D} \arrow[r, "(G^R)_!\Delta"] \arrow[d, "\mathrm{id}"] & (G^R)_! \,\mathrm{Funct}(\mathcal{I}_{\mathcal{M}'}, \mathcal{D}) \arrow[d, "="] \\ (G^R)_!\mathcal{D} \arrow[r, "\Delta"] & \mathrm{Funct}(\mathcal{I}_{\mathcal{M}}, (G^R)_!\mathcal{D}). \end{tikzcd}$$

Let d be an object in $\mathcal{D}$, and $\epsilon : \Delta d \to X$ be a morphism in $\mathrm{Funct}(\mathcal{I}_{\mathcal{M}}, \mathcal{D})$. Using the above square, we can also think about ϵ as a morphism in $\mathrm{Funct}(\mathcal{I}_{\mathcal{M}}, (G^R)_!\mathcal{D})$. If the pair (d, ϵ) corresponds to a diagram $X^{\triangleleft} : \mathcal{I}^{\triangleleft}_{\mathcal{M}'} \to \mathcal{D}$ under the equivalence of remark 5.3.3, then the diagram $\mathcal{I}^{\triangleleft}_{\mathcal{M}} \to (G^R)_!\mathcal{D}$ obtained from (d, ϵ) via the above commutative square is equivalent to the image of $X^{\triangleleft}$ under the canonical equivalence

$$\mathrm{Hom}_{\mathrm{Cat}^{\mathcal{M}'}}(\mathcal{I}^{\triangleleft}_{\mathcal{M}'}, \mathcal{D}) = \mathrm{Hom}_{\mathrm{Cat}^{\mathcal{M}}}(\mathcal{I}^{\triangleleft}_{\mathcal{M}}, (G^R)_!\mathcal{D}).$$

We now explore the behavior of conical limits under changes in the enriching category.

Proposition 5.3.6. *Let $G : \mathcal{M} \to \mathcal{M}'$ be a colimit preserving symmetric monoidal functor between presentable symmetric monoidal categories. Let $\mathcal{D}$ be an $\mathcal{M}$-enriched category and let $X'^{\triangleleft} : \mathcal{I}^{\triangleleft}_{\mathcal{M}'} \to \mathcal{D}$ be a conical limit diagram. Then the induced functor $X^{\triangleleft} : \mathcal{I}^{\triangleleft}_{\mathcal{M}} \to (G^R)_!\mathcal{D}$ is a conical limit diagram.*

Proof. This is a direct consequence of the discussion in remarks 5.1.2 and 5.3.5. □

Corollary 5.3.7. *Let $\mathcal{M}$ be a presentable symmetric monoidal category. Let $\mathcal{D}$ be an $\mathcal{M}$-enriched category and $X^{\triangleleft} : \mathcal{I}^{\triangleleft}_{\mathcal{M}} \to \mathcal{D}$ be a conical limit diagram in $\mathcal{D}$. Then the induced diagram $\mathcal{I}^{\triangleleft} \to (\tau_{\mathcal{M}})_!\mathcal{D}$ in the category underlying $\mathcal{D}$, is a limit diagram.*

Proof. Apply proposition 5.3.6 to the unit map $\mathrm{Spc} \to \mathcal{M}$. □

Proposition 5.3.8. *Let $i : \mathcal{M} \to \mathcal{M}'$ be a colimit preserving symmetric monoidal functor between presentable symmetric monoidal categories. Assume that i is fully faithful and admits a strictly symmetric monoidal left adjoint. Let $\mathcal{D}$ be an $\mathcal{M}$-enriched category and let $X^{\triangleleft} : \mathcal{I}^{\triangleleft}_{\mathcal{M}} \to \mathcal{D}$ be a conical limit diagram. Then $i_!X^{\triangleleft} : \mathcal{I}^{\triangleleft}_{\mathcal{M}'} \to i_!\mathcal{D}$ is a conical limit diagram*

Proof. It follows from corollary 3.5.29 that we have an equivalence

$$i_! \mathrm{Funct}(-_{\mathcal{M}}, \mathcal{D}) = \mathrm{Funct}(-_{\mathcal{M}'}, i_!\mathcal{D})$$

of functors $\mathrm{Cat}^{\mathrm{op}} \to \mathrm{Cat}^{\mathcal{M}'}$. Applying it to the projection $\mathcal{I} \to [0]$ we obtain a commutative square

$$\begin{array}{ccc} i_!\mathcal{D} & \xrightarrow{i_!\Delta} & i_! \mathrm{Funct}(\mathcal{I}_{\mathcal{M}}, \mathcal{D}) \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle =} \\ i_!\mathcal{D} & \xrightarrow{\Delta} & \mathrm{Funct}(\mathcal{I}_{\mathcal{M}'}, i_!\mathcal{D}). \end{array}$$

Let (d, ϵ) be the right adjoint to $\Delta : \mathcal{D} \to \mathrm{Funct}(\mathcal{I}_{\mathcal{M}}, \mathcal{D})$ at $X = X^{\triangleleft}|_{\mathcal{I}_{\mathcal{M}}}$. By virtue of remark 5.1.2, we have that (d, ϵ) is also right adjoint to $i_!\Delta : i_!\mathcal{D} \to i_! \mathrm{Funct}(\mathcal{I}_{\mathcal{M}}, \mathcal{D})$. Its image under the above equivalence is right adjoint to $\Delta : i_!\mathcal{D} \to \mathrm{Funct}(\mathcal{I}_{\mathcal{M}'}, \mathcal{D})$ at $i_!X$ - in other words, it is a conical limit for X. Our result now follows from the fact that the associated limit diagram $\mathcal{I}^{\triangleleft}_{\mathcal{M}'} \to i_!\mathcal{D}$ is given by $i_!X^{\triangleleft}$. □

Corollary 5.3.9. *Let $m \geq n \geq 1$ and let $\mathcal{D}$ be an n-category, thought of as a category enriched in $(n-1)$-categories. Then a diagram $X^{\triangleleft} : \mathcal{I}^{\triangleleft} \to \mathcal{D}$ is a conical limit diagram if and only if its image under the inclusion functor $i^{n,m} : n\mathrm{Cat} \to m\mathrm{Cat}$ is a conical limit diagram.*

Definition 5.3.10. *Let $\mathcal{M}$ be a presentable symmetric monoidal category and let $F : \mathcal{D} \to \mathcal{D}'$ be a functor of $\mathcal{M}$-enriched categories. We say that a conical limit diagram $X^{\triangleleft} : \mathcal{I}^{\triangleleft} \to \mathcal{D}$ is preserved by F if $FX^{\triangleleft}$ is a conical limit diagram in $\mathcal{D}'$. Similarly, a conical colimit diagram $Y^{\triangleright} : \mathcal{I}^{\triangleright} \to \mathcal{D}$ is said to be preserved by F if $FY^{\triangleright}$ is a conical colimit diagram in $\mathcal{D}'$.*

Remark 5.3.11. Let $\mathcal{M}$ be a presentable symmetric monoidal category and let $F : \mathcal{D} \to \mathcal{D}'$ be a functor of $\mathcal{M}$-enriched categories. Let $X^{\triangleleft} : \mathcal{I}^{\triangleleft}_{\mathcal{M}} \to \mathcal{D}$ be a conical limit diagram. Then it follows from remark 5.3.4 that $X^{\triangleleft}$ is preserved by F if and only if the commutative square

$$\begin{tikzcd}
\mathcal{D} \arrow[r, "\Delta"] \arrow[d, "F"] & \mathrm{Funct}(\mathcal{I}_{\mathcal{M}}, \mathcal{D}) \arrow[d, "F_*"] \\
\mathcal{D}' \arrow[r, "\Delta"] & \mathrm{Funct}(\mathcal{I}_{\mathcal{M}}, \mathcal{D}')
\end{tikzcd}$$

is horizontally right adjointable at X.

We now specialize propositions 5.1.6 and 5.1.7 to the case of conical limits.

Proposition 5.3.12. *Let $\mathcal{M}$ be a presentable symmetric monoidal category and let $\mathcal{K}$ be a category. Let $D : \mathcal{K} \to \mathrm{Cat}^{\mathcal{M}}$ be a functor, and denote by $\mathcal{D}$ its limit. For each j in $\mathcal{K}$ denote by $p_j : \mathcal{D} \to D(j)$ the projection. Let $X : \mathcal{I}_{\mathcal{M}} \to \mathcal{D}$ be a diagram in $\mathcal{D}$. Assume that for every j in $\mathcal{K}$ the diagram $p_j X : \mathcal{I} \to D(j)$ admits a conical limit, which is preserved by the functor $D(\alpha) : D(j) \to D(j')$ for every arrow $\alpha : j \to j'$ in $\mathcal{K}$. Then*

(i) *The diagram X admits a conical limit.*

(ii) *An extension $X^{\triangleleft} : \mathcal{I}^{\triangleleft}_{\mathcal{M}} \to \mathcal{D}$ is a conical limit diagram if and only if $p_j X^{\triangleleft}$ is a conical limit diagram in $D(j)$ for every j in $\mathcal{K}$.*

Proof. Combine proposition 5.1.6 together with remarks 5.3.4 and 5.3.11. □

Proposition 5.3.13. *Let $\mathcal{M}$ be a presentable symmetric monoidal category. Let $\mathcal{I}$ be a category, and let $\mathcal{J}$ and $\mathcal{D}$ be $\mathcal{M}$-enriched categories. Let $X : \mathcal{I}_{\mathcal{M}} \to \mathrm{Funct}(\mathcal{J}, \mathcal{D})$ be a diagram, and assume that for every object j in $\mathcal{J}$, the diagram $\mathrm{ev}_j X : \mathcal{I}_{\mathcal{M}} \to \mathcal{D}$ admits a conical limit. Then*

(i) *There exists a conical limit for X.*

(ii) *An extension $X^{\triangleleft} : \mathcal{I}^{\triangleleft}_{\mathcal{M}} \to \mathrm{Funct}(\mathcal{J}, \mathcal{D})$ is a conical limit for X if and only if for every object j in $\mathcal{J}$ the diagram $\mathrm{ev}_j X^{\triangleleft}$ is a conical limit.*

Proof. Apply proposition 5.1.7 to the diagonal map $\Delta : \mathcal{D} \to \mathrm{Funct}(\mathcal{I}, \mathcal{D})$. $\square$

Definition 5.3.14. *Let $\mathcal{M}$ be a presentable symmetric monoidal category and let $\mathcal{D}$ be an $\mathcal{M}$-enriched category. Let $\mathcal{I}$ be a category. We say that $\mathcal{D}$ admits all conical (co)limits of shape $\mathcal{I}$ if every diagram $X : \mathcal{I} \to \mathcal{D}$ admits a conical (co)limit. We say that $\mathcal{D}$ is conically (co)complete if it admits all conical (co)limits of shape $\mathcal{I}$ for every small category $\mathcal{I}$.*

Corollary 5.3.15. *Let $\mathcal{M}$ be a presentable symmetric monoidal category and let $\mathcal{D}$ be an $\mathcal{M}$-enriched category which admits all conical limits of shape $\mathcal{I}$. Then for every $\mathcal{M}$-enriched category $\mathcal{J}$, the $\mathcal{M}$-enriched category* $\mathrm{Funct}(\mathcal{J}, \mathcal{D})$ *admits all conical limits of shape $\mathcal{I}$.*

Proof. Follows directly from proposition 5.3.13. $\square$

Corollary 5.3.16. *Let $\mathcal{M}$ be a presentable symmetric monoidal category and let $\mathcal{D}$ be an $\mathcal{M}$-enriched category which admits all conical limits of shape $\mathcal{I}$. Then there is a functor* $\mathrm{Funct}(\mathcal{I}_{\mathcal{M}}, \mathcal{D}) \to \mathcal{D}$ *which maps each diagram $X : \mathcal{I} \to \mathcal{D}$ to the value of its conical limit at the cone point of $\mathcal{I}^{\lhd}$.*

Proof. Apply corollary 5.1.9 to the diagonal map $\Delta : \mathcal{D} \to \mathrm{Funct}(\mathcal{I}, \mathcal{D})$. $\square$

For later purposes we record the following basic consequence of proposition 5.3.13.

Proposition 5.3.17. *Let $\mathcal{M}$ be a presentable symmetric monoidal category. Let $\mathcal{D}$ be an $\mathcal{M}$-enriched category and let $f : \mathcal{J} \to \mathcal{J}'$ be an epimorphism of $\mathcal{M}$-enriched categories which is surjective on objects. Let $X : \mathcal{I}_{\mathcal{M}} \to$* $\mathrm{Funct}(\mathcal{J}, \mathcal{D})$ *be a diagram admitting a conical colimit $\mathscr{X}$ which is preserved by the evaluation functors* $\mathrm{ev}_j : \mathrm{Funct}(\mathcal{J}, \mathcal{D}) \to \mathcal{D}$ *for every object j in $\mathcal{J}$. Assume that for every arrow $\alpha : i \to i'$ in $\mathcal{I}$ the morphism $X(\alpha) : X(i) \to X(j)$ in* $\mathrm{Funct}(\mathcal{J}, \mathcal{D})$ *belongs to* $\mathrm{Funct}(\mathcal{J}', \mathcal{D})$. *Then*

(i) *The functor $\mathscr{X} : \mathcal{J} \to \mathcal{D}$ factors through $\mathcal{J}'$.*

(ii) *For every object i in $\mathcal{I}$ the morphism $X(i) \to \mathscr{X}$ belongs to* $\mathrm{Funct}(\mathcal{J}', \mathcal{D})$.

Proof. Note that since the map f is an epimorphism, the map

$$f^* : \mathrm{Funct}(\mathcal{J}', \mathcal{D}) \to \mathrm{Funct}(\mathcal{J}, \mathcal{D})$$

is indeed a monomorphism. Our assumptions imply that X factors through the image of f^*. Let $X' : \mathcal{I}_{\mathcal{M}} \to \mathrm{Funct}(\mathcal{J}', \mathcal{D})$ be the induced diagram. Since f is surjective on objects, we have that for every j in $\mathcal{J}'$ the diagram $\mathrm{ev}_j X'$ admits a conical colimit. It follows from (the dual version of) proposition 5.3.13 that X' can be extended to a conical colimit diagram $X'^{\rhd} : \mathcal{I}_{\mathcal{M}}^{\rhd} \to \mathrm{Funct}(\mathcal{J}', \mathcal{D})$. The diagram $f^* X'^{\rhd}$ is an extension of X whose image under all evaluation functors is a conical colimit diagram. Applying proposition 5.3.13 again we conclude that $f^* X'^{\rhd}$ is a conical limit diagram in $\mathcal{D}$. Item (i) now follows from the fact that $\mathscr{X}$ is equivalent to the value of $f^* X'^{\rhd}$ at the cone point $*$ in $\mathcal{I}^{\rhd}$. Item (ii) is a consequence of the fact that the morphism $X(i) \to \mathscr{X}$ is equivalent to the image under f^* of the morphism $X'^{\rhd}(i) \to X'^{\rhd}(*)$. $\square$

5.4 The case of presentable modules

We now study the interactions between adjunctions with the procedure of enrichment of modules over presentable symmetric monoidal categories.

Proposition 5.4.1. *Let $\mathcal{M}$ be a presentable symmetric monoidal category and let $F : \mathcal{C} \to \mathcal{D}$ be a morphism in $\mathcal{M}$-mod(Pr^L). Then*

(i) *The functor of enriched categories $\theta_{\mathcal{M}}(F) : \theta_{\mathcal{M}}(\mathcal{C}) \to \theta_{\mathcal{M}}(\mathcal{D})$ admits a right adjoint.*

(ii) *Assume that F admits a left adjoint F^L, and that the canonical structure of oplax morphism of $\mathcal{M}$-modules on F^L is strict. Then $\theta_{\mathcal{M}}(F)$ admits a left adjoint.*

Proof. We first prove item (i). Let $F^R : \mathcal{D} \to \mathcal{C}$ be the right adjoint to the functor underlying F. Let d be an object in $\mathcal{D}$ and let $\epsilon(d) : FF^R(d) \to d$ be the counit of the adjunction at d. We claim that $(F^R(d), \epsilon(d))$ is right adjoint to $\theta_{\mathcal{M}}(F)$ at d. To see this, we have to show that for every c in $\mathcal{C}$ the composite map

$$\mathrm{Hom}_{\theta_{\mathcal{M}}(\mathcal{C})}(c, F^R(d)) \xrightarrow{\theta_{\mathcal{M}}(F)_*} \mathrm{Hom}_{\theta_{\mathcal{M}}(\mathcal{D})}(F(c), FF^R(d)) \xrightarrow{\epsilon(d)} \mathrm{Hom}_{\theta_{\mathcal{M}}(\mathcal{D})}(F(c), d)$$

is an isomorphism. It suffices to show that for every m in $\mathcal{M}$ the image of the above composition under the functor $\mathrm{Hom}_{\mathcal{M}}(m, -)$ is an isomorphism. This is equivalent to

$$\mathrm{Hom}_{\mathcal{C}}(m \otimes c, F^R(d)) \xrightarrow{F_*} \mathrm{Hom}_{\mathcal{D}}(m \otimes F(c), FF^R(d)) \xrightarrow{\epsilon(d)} \mathrm{Hom}_{\mathcal{D}}(m \otimes F(c), d)$$

which is indeed an isomorphism, since $(F^R(d), \epsilon(d))$ is right adjoint to F at d. Item (i) now follows from proposition 5.2.10.

We now prove item (ii). Let d be an object in $\mathcal{D}$ and let $\eta(d) : d \to FF^L(d)$ be the counit of the adjunction at d. We claim that $(F^L(d), \eta(d))$ is left adjoint to $\theta_{\mathcal{M}}(F)$ at d. To see this, we have to show that for every c in $\mathcal{C}$ the composite map

$$\mathrm{Hom}_{\theta_{\mathcal{M}}(\mathcal{C})}(F^L(d), c) \xrightarrow{\theta_{\mathcal{M}}(F)_*} \mathrm{Hom}_{\theta_{\mathcal{M}}(\mathcal{D})}(FF^L(d), F(c)) \xrightarrow{\eta(d)} \mathrm{Hom}_{\theta_{\mathcal{M}}(\mathcal{D})}(d, F(c))$$

is an isomorphism. It suffices to show that for every m in $\mathcal{M}$ the image of the above composition under the functor $\mathrm{Hom}_{\mathcal{M}}(m, -)$ is an isomorphism. This is equivalent to the composite map

$$\mathrm{Hom}_{\mathcal{C}}(m \otimes F^L(d), c) \xrightarrow{F_*} \mathrm{Hom}_{\mathcal{D}}(m \otimes FF^L(d), F(c)) \xrightarrow{\mathrm{id}_m \otimes \eta(d)} \mathrm{Hom}_{\mathcal{D}}(m \otimes d, F(c)).$$

To show that the above is an equivalence, it suffices to show that $\mathrm{id}_m \otimes \eta(d)$ exhibits $m \otimes F^L(d)$ as left adjoint to F at $m \otimes d$. This is implied by the fact that F^L is a strict morphism of $\mathcal{M}$-modules. $\square$

Remark 5.4.2. Let $\mathcal{M}$ be a presentable symmetric monoidal category. Then the unit map $\mathrm{Spc} \to \mathcal{M}$ induces a symmetric monoidal colimit preserving functor of presentable symmetric monoidal categories $i : \mathrm{Cat} \to \mathrm{Cat}^{\mathcal{M}}$. We can think about this as a morphism in $\mathrm{Cat}\text{-mod}(\mathrm{Pr}^L)$. Applying the functor θ_{Cat} yields an enhancement of i to a symmetric monoidal functor of symmetric monoidal 2-categories $\bar{i} : \mathscr{C}at \to \mathscr{C}at^{\mathcal{M}}$. It follows from proposition 5.4.1 that $\bar{i}$ admits a right adjoint

$$\overline{(\tau_{\mathcal{M}})_!} : \mathscr{C}at^{\mathcal{M}} \to \mathscr{C}at.$$

Proposition 5.4.3. *Let $\mathcal{M}$ be a presentable symmetric monoidal category. Then the functor of categories $\mathrm{Cat}^{\mathcal{M}} \to \mathrm{Cat}$ underlying $\overline{(\tau_{\mathcal{M}})_!}$ is equivalent to $\tau_{\mathcal{M}}$.*

Proof. We continue with the notation from remark 5.4.2. Let

$$\epsilon : \bar{i}\overline{(\tau_{\mathcal{M}})_!} \to \mathrm{id}_{\mathscr{C}at^{\mathcal{M}}}$$

be the counit of the adjunction. We think about ϵ as a morphism in $\mathrm{Funct}(\mathscr{C}at^{\mathcal{M}}, \mathscr{C}at^{\mathcal{M}})$.

Consider the functor

$$((-)^{\leq 1})_* : \mathrm{Hom}_{2\mathrm{Cat}}(\mathscr{C}at^{\mathcal{M}}, \mathscr{C}at^{\mathcal{M}}) \to \mathrm{Hom}_{\mathrm{Cat}}(\mathrm{Cat}^{\mathcal{M}}, \mathrm{Cat}^{\mathcal{M}}).$$

This admits an enhancement to a functor

$$\varphi : \mathrm{Funct}(\mathscr{C}at^{\mathcal{M}}, \mathscr{C}at^{\mathcal{M}})^{\leq 1} \to \mathrm{Funct}(\mathrm{Cat}^{\mathcal{M}}, \mathrm{Cat}^{\mathcal{M}})$$

induced from the composite map

$$\mathrm{Cat}^{\mathcal{M}} \times \mathrm{Funct}(\mathscr{C}at^{\mathcal{M}}, \mathscr{C}at^{\mathcal{M}})^{\leq 1} = (\mathscr{C}at^{\mathcal{M}} \times \mathrm{Funct}(\mathscr{C}at^{\mathcal{M}}, \mathscr{C}at^{\mathcal{M}}))^{\leq 1} \xrightarrow{\mathrm{ev}^{\leq 1}} (\mathscr{C}at^{\mathcal{M}})^{\leq 1} = \mathrm{Cat}^{\mathcal{M}}.$$

The image of ϵ under φ is a map

$$\varphi(\epsilon) : (\bar{i}\overline{(\tau_{\mathcal{M}})_!})^{\leq 1} = i(\overline{(\tau_{\mathcal{M}})_!})^{\leq 1} \to \mathrm{id}_{\mathrm{Cat}^{\mathcal{M}}}.$$

For each object x in $\mathrm{Cat}^{\mathcal{M}}$, the morphism

$$\varphi(\epsilon)(x) : i(\overline{(\tau_{\mathcal{M}})_!})^{\leq 1}(x) \to x$$

can be identified with $\epsilon(x)$. Using corollary 5.1.9 we conclude that $\varphi(\epsilon)$ exhibits $(\overline{(\tau_{\mathcal{M}})_!})^{\leq 1}$ as right adjoint to i, as desired. □

Corollary 5.4.4. *Let $\mathcal{M}$ be a presentable symmetric monoidal category, and let $F : \mathcal{C} \to \mathcal{D}$ be a functor of $\mathcal{M}$-enriched categories. Assume that F admits a right adjoint F^R. Then $(\tau_{\mathcal{M}})_!(F^R)$ is right adjoint to $(\tau_{\mathcal{M}})_!(F)$.*

Proof. This follows directly from proposition 5.4.3, since functors of 2-categories preserve adjunctions. □

We now discuss the existence of conical limits and colimits in enriched categories arising from presentable modules.

Notation 5.4.5. Let $\mathcal{M}$ be a presentable monoidal category and let $\mathcal{I}$ be a category. We denote by $\mathcal{I}'_{\mathcal{M}}$ the image of $\mathcal{I}$ under the composite functor

$$\mathrm{Cat} \xrightarrow{s} \mathrm{Algbrd}(\mathrm{Spc}) \to \mathrm{Algbrd}(\mathcal{M})$$

where the first map is the functor s from construction 3.4.1, and the second map is given by pushforward along the unit map $\mathrm{Spc} \to \mathcal{M}$.

Lemma 5.4.6. *Let $\mathcal{M}$ be a presentable monoidal category. Let $\mathcal{D}$ be a presentable $\mathcal{M}$-module and let $\mathcal{I}$ be a category. Then the projection* $\mathrm{LMod} \to \mathrm{Arr}_{\mathrm{oplax}}(\mathrm{Cat})$ *induces an equivalence*

$$\mathrm{LMod}_{\mathcal{I}'_{\mathcal{M}}}(\mathcal{D}) = \mathrm{Funct}(\mathcal{I}, \mathcal{D}).$$

Proof. As in the proof of proposition 4.1.8, we let $\mathcal{M}_{\mathcal{I}}$ be the Assos-operad with the universal map $\mathcal{M}_{\mathcal{I}} \times_{\mathrm{Assos}} \mathrm{Assos}_{\mathcal{I}} \to \mathcal{M}$. Recall from [Hin20a] that this is a presentable monoidal category which acts on $\mathrm{Funct}(\mathcal{I}, \mathcal{D})$. The category $\mathrm{LMod}_{\mathcal{I}'_{\mathcal{M}}}(\mathcal{D})$ is then the category of modules for an algebra in $\mathcal{M}_{\mathcal{I}}$. As discussed in [Hin20a] 4.7, this algebra is in fact the unit in $\mathcal{M}_X$. We conclude that its category of modules is equivalent to $\mathrm{Funct}(\mathcal{I}, \mathcal{D})$, as desired. □

Lemma 5.4.7. *Let $\mathcal{M}$ be a presentable monoidal category. Let $\mathcal{D}$ be a presentable $\mathcal{M}$-module and let $\mathcal{I}$ be a category. Then restriction of scalars along the canonical map $\mathcal{I}_{\mathcal{M}} \to \mathcal{I}'_{\mathcal{M}}$ induces an equivalence*

$$\mathrm{LMod}_{\mathcal{I}'_{\mathcal{M}}}(\mathcal{D}) = \mathrm{LMod}_{\mathcal{I}_{\mathcal{M}}}(\mathcal{D}).$$

Proof. We continue with the notation from the proof of lemma 5.4.6. Recall from [Hin20a] 4.4.10 and 4.7.1 that the unit map $\mathrm{Spc} \to \mathcal{M}$ induces a symmetric monoidal functor $\mathrm{Spc}_{\mathcal{I}} \to \mathcal{M}_{\mathcal{I}}$, where $\mathrm{Spc}_{\mathcal{I}}$ is defined as $\mathcal{M}_{\mathcal{I}}$. The monoidal category $\mathrm{Spc}_{\mathcal{I}}$ thus acts on $\mathrm{Funct}(\mathcal{I}, \mathcal{D})$ by restriction of scalars, and the category $\mathrm{LMod}_{\mathcal{I}'_{\mathcal{M}}}(\mathcal{D})$ is the category of modules over the algebra in $\mathrm{Spc}_{\mathcal{I}}$ associated to $\mathcal{I}'_{\mathrm{Spc}}$. Similarly, we have that $\mathrm{LMod}_{\mathcal{I}_{\mathcal{M}}}(\mathcal{D})$ is equivalent to the category of modules in $\mathrm{Funct}(\mathcal{I}^{\leq 0}, \mathcal{D})$ for the algebra in $\mathrm{Spc}_{\mathcal{I}^{\leq 0}}$ associated to $\mathcal{I}_{\mathrm{Spc}}$.

It follows that to prove our lemma it suffices to assume that $\mathcal{M} = \mathrm{Spc}$. We claim that for every category $\mathcal{E}$ the induced functor

$$\mathrm{Hom}_{\mathrm{Cat}}(\mathcal{E}, \mathrm{LMod}_{\mathcal{I}'_{\mathcal{M}}}(\mathcal{D})) \to \mathrm{Hom}_{\mathrm{Cat}}(\mathcal{E}, \mathrm{LMod}_{\mathcal{I}_{\mathcal{M}}}(\mathcal{D}))$$

is an equivalence. Using proposition 4.2.15 together with [Hin20a] proposition 6.3.7 we see that the above is equivalent to the canonical map

$$\mathrm{Hom}_{\mathrm{Algbrd}(\mathrm{Spc})}(\mathcal{E} \times \mathcal{I}'_{\mathcal{M}}, \theta'_{\mathrm{Spc}}(\mathcal{D})) \to \mathrm{Hom}_{\mathrm{Algbrd}(\mathrm{Spc})}(\mathcal{E} \times \mathcal{I}_{\mathcal{M}}, \theta'_{\mathrm{Spc}}(\mathcal{D})).$$

This is an isomorphism thanks to propositions 4.2.10 and 3.4.5. □

Proposition 5.4.8. *Let $\mathcal{M}$ be a presentable symmetric monoidal category and let $\mathcal{C}$ be a presentable module over $\mathcal{M}$. Then $\theta_{\mathcal{M}}(\mathcal{C})$ is conically complete and cocomplete.*

Proof. Let $\mathcal{I}$ be a small category. We have to show that the diagonal map

$$\Delta : \theta_{\mathcal{M}}(\mathcal{C}) \to \mathrm{Funct}(\mathcal{I}_{\mathcal{M}}, \theta_{\mathcal{M}}(\mathcal{C}))$$

admits both right and left adjoints.

Let $h^R_{\mathcal{M}} : \mathrm{Algbrd}(\mathcal{M}) \to \mathrm{Algbrd}(\mathcal{M})_{\mathrm{Spc}}$ be the colocalization functor. We note that Δ is the image under $h^R_{\mathcal{M}}$ of the morphism of algebroids

$$\Delta' : \theta'_{\mathcal{M}}(\mathcal{C}) \to \mathrm{Funct}(\mathcal{I}_{\mathcal{M}}, \theta'_{\mathcal{M}}(\mathcal{C}))$$

of precomposition with the projection $\mathcal{I}_{\mathcal{M}} \to 1_{\mathcal{M}}$. Using the equivalence from [Hin20a] proposition 6.3.7, we see that the above map can be rewritten as

$$\theta'_{\mathcal{M}}(\Delta_{\mathrm{mod}}) : \theta'_{\mathcal{M}}(\mathcal{C}) \to \theta'_{\mathcal{M}}(\mathrm{LMod}_{\mathcal{I}_{\mathcal{M}}}(\mathcal{C}))$$

where:

- The category $\mathrm{LMod}_{\mathcal{I}_{\mathcal{M}}}(\mathcal{C})$ is equipped with the structure of presentable $\mathcal{M}$-module by virtue of its realization as a category of left modules for an algebra in the Assos^{-}-component of the BM-monoidal category $\mathrm{Funct}_{\mathrm{BM}}(\mathrm{BM}_{\mathcal{I}^{\leq 0},[0]}, \mathcal{C})$ (where we consider $\mathcal{C}$ as a $\mathcal{M} - \mathcal{M}$-bimodule in the canonical way).

- The functor Δ_{mod} denotes the functor of restriction of scalars

$$\mathcal{C} = \mathrm{LMod}_{1_{\mathcal{M}}}(\mathcal{C}) \to \mathrm{LMod}_{\mathcal{I}_{\mathcal{M}}}(\mathcal{C})$$

  along the projection $\mathcal{I}_{\mathcal{M}} \to 1_{\mathcal{M}}$, equipped with its canonical structure of morphism of $\mathcal{M}$-modules.

Using lemma 5.4.7 we may rewrite our map as

$$\theta'_{\mathcal{M}}(\Delta'_{\mathrm{mod}}) : \theta'_{\mathcal{M}}(\mathcal{C}) \to \theta'_{\mathcal{M}}(\mathrm{LMod}_{\mathcal{I}'_{\mathcal{M}}}(\mathcal{C}))$$

where Δ'_{mod} is defined as Δ_{mod}, except that using the projection $\mathcal{I}'_{\mathcal{M}} \to 1_{\mathcal{M}}$ instead.

Applying lemma 5.4.6 we see that the above is equivalent to

$$\theta'_{\mathcal{M}}(\Delta_{\mathrm{funct}}) : \theta'_{\mathcal{M}}(\mathcal{C}) \to \theta'_{\mathcal{M}}(\mathrm{Funct}(\mathcal{I}, \mathcal{C}))$$

where $\mathrm{Funct}(\mathcal{I}, \mathcal{C})$ is equipped with its canonical structure of module over $\mathcal{M}$, and Δ_{funct} denotes the diagonal functor $\mathcal{C} \to \mathrm{Funct}(\mathcal{I}, \mathcal{C})$.

We conclude that our original map Δ is equivalent to $\theta_{\mathcal{M}}(\Delta_{\mathrm{funct}})$. This admits both left and right adjoints thanks to proposition 5.4.1. □

We finish by giving an alternative characterization of the class of conical limits.

Proposition 5.4.9. *Let $\mathcal{M}$ be a presentable symmetric monoidal category and let $\mathcal{I}$ be a category. Let $\mathcal{D}$ be an $\mathcal{M}$-enriched category and denote by $i : \mathcal{D} \to \operatorname{Funct}(\mathcal{D}^{\mathrm{op}}, \overline{\mathcal{M}})$ the Yoneda embedding. Then a functor $X^{\lhd} : \mathcal{I}^{\lhd}_{\mathcal{M}} \to \mathcal{D}$ is a conical limit diagram if and only if $iX^{\lhd}$ is a conical limit diagram.*

Proof. If $iX^{\lhd}$ is a conical limit diagram, then $X^{\lhd}$ is a conical limit diagram, since i is fully faithful. It remains to show the converse. Denote by $*$ the cone point of $\mathcal{I}^{\lhd}$ and let $X = X^{\lhd}|_{\mathcal{I}_{\mathcal{M}}}$. Let $x = X^{\lhd}(*)$, and let $\epsilon : \Delta x \to X$ be the morphism presenting x as the conical limit of X.

It follows from [Hin20a] proposition 6.3.7 that $\operatorname{Funct}(\mathcal{D}^{\mathrm{op}}, \overline{\mathcal{M}})$ belongs to the image of $\theta_{\mathcal{M}}$, and therefore by proposition 5.4.8, we see that $\operatorname{Funct}(\mathcal{D}^{\mathrm{op}}, \overline{\mathcal{M}})$ admits all conical limits. Let $\epsilon' : y \to iX$ be the conical limit for iX, and let $\alpha : ix \to y$ be the unique morphism equipped with an identification $\epsilon'\Delta_*\alpha = i_*\epsilon$. We need to show that α is an isomorphism.

To see this, it suffices to show that for every object z in $\mathcal{D}$, the morphism

$$\alpha_* : \operatorname{Hom}_{\operatorname{Funct}(\mathcal{D}^{\mathrm{op}}, \overline{\mathcal{M}})}(iz, ix) \to \operatorname{Hom}_{\operatorname{Funct}(\mathcal{D}^{\mathrm{op}}, \overline{\mathcal{M}})}(iz, y)$$

is an isomorphism. This fits into a commutative diagram

$$\begin{tikzcd}
\operatorname{Hom}_{\operatorname{Funct}(\mathcal{D}^{\mathrm{op}}, \overline{\mathcal{M}})}(iz, ix) \arrow[r, "\alpha_*"] \arrow[d, "\Delta_*"] & \operatorname{Hom}_{\operatorname{Funct}(\mathcal{D}^{\mathrm{op}}, \overline{\mathcal{M}})}(iz, y) \arrow[d, "\Delta_*"] \\
\operatorname{Hom}_{\operatorname{Funct}(\mathcal{I}_{\mathcal{M}}, \operatorname{Funct}(\mathcal{D}^{\mathrm{op}}, \overline{\mathcal{M}}))}(\Delta iz, \Delta ix) \arrow[r, "(\Delta_*\alpha)_*"] \arrow[d, "(i_*\epsilon)_*"] & \operatorname{Hom}_{\operatorname{Funct}(\mathcal{I}_{\mathcal{M}}, \operatorname{Funct}(\mathcal{D}^{\mathrm{op}}, \overline{\mathcal{M}}))}(\Delta iz, \Delta y) \arrow[d, "\epsilon'_*"] \\
\operatorname{Hom}_{\operatorname{Funct}(\mathcal{I}_{\mathcal{M}}, \operatorname{Funct}(\mathcal{D}^{\mathrm{op}}, \overline{\mathcal{M}}))}(\Delta iz, iX) \arrow[r, "\mathrm{id}"] & \operatorname{Hom}_{\operatorname{Funct}(\mathcal{I}_{\mathcal{M}}, \operatorname{Funct}(\mathcal{D}^{\mathrm{op}}, \overline{\mathcal{M}}))}(\Delta iz, iX).
\end{tikzcd}$$

Since (y, ϵ') is a conical limit diagram, the composition of the right vertical arrows is an isomorphism. Since (x, ϵ) is a conical limit diagram and i is fully faithful, the composition of the left vertical arrows is an isomorphism. Since the bottom horizontal arrow is an isomorphism, we have that the top horizontal arrow is an isomorphism as well, as desired. □

Corollary 5.4.10. *Let $\mathcal{M}$ be a presentable symmetric monoidal category and let $\mathcal{I}$ be a category. Let $\mathcal{D}$ be an $\mathcal{M}$-enriched category and let $X^{\lhd} : \mathcal{I}^{\lhd}_{\mathcal{M}} \to \mathcal{D}$ be a diagram. Then $X^{\lhd}$ is a conical limit diagram in $\mathcal{D}$ if and only if for every d in $\mathcal{D}$ the composite map*

$$\mathcal{I}^{\lhd}_{\mathcal{M}} \xrightarrow{X^{\lhd}} \mathcal{D} \xrightarrow{\operatorname{Hom}_{\mathcal{D}}(d, -)} \overline{\mathcal{M}}$$

is a conical limit diagram.

Proof. This follows from proposition 5.4.9 together with proposition 5.3.13, by using the fact that $\overline{\mathcal{M}}$ admits all conical limit diagrams. □

Remark 5.4.11. Let $\mathcal{M}$ be a presentable symmetric monoidal category. Since $\overline{\mathcal{M}}$ admits all conical colimits, we have that a left cone in $\overline{\mathcal{M}}$ is a conical limit diagram if and only if the associated left cone in $\mathcal{M}$ is a limit diagram. We can thus informally summarize corollary 5.4.10 by saying that a conical limit in an $\mathcal{M}$-enriched category $\mathcal{D}$ is the same as a limit in the category underlying $\mathcal{D}$ which is preserved under all enriched corepresentable copresheaves.

5.5 Weighted limits and colimits

We begin by discussing the notion of join of enriched categories.

Notation 5.5.1. Let $\mathcal{M}$ be a presentable symmetric monoidal category and let $\mathcal{I}, \mathcal{I}'$ be $\mathcal{M}$-algebroids with categories of objects X and X', respectively. Let $W : \mathrm{BM}_{X,X'} \to \mathcal{M}$ be an $\mathcal{I} - \mathcal{I}'$-bimodule in $\mathcal{M}$. We denote by $\mathcal{I} \star_W^{\mathrm{Algbrd}} \mathcal{I}'$ the operadic left Kan extension of W along the inclusion $\mathrm{BM}_{X,X'} \to \mathrm{Assos}_{X\cup X'}$.

In the case when $\mathcal{I}$ and $\mathcal{I}'$ are $\mathcal{M}$-enriched categories, we will denote by $\mathcal{I} \star_W \mathcal{I}'$ the $\mathcal{M}$-enriched category underlying $\mathcal{I} \star_W^{\mathrm{Algbrd}} \mathcal{I}'$. We call this the join of $\mathcal{I}$ and $\mathcal{I}'$ weighted by W.

Remark 5.5.2. Let $\mathcal{M}$ be a presentable symmetric monoidal category and let $\mathcal{I}, \mathcal{I}'$ be $\mathcal{M}$-algebroids with categories of objects X and X', respectively. Let $W : \mathrm{BM}_{X,X'} \to \mathcal{M}$ be an $\mathcal{I} - \mathcal{I}'$-bimodule in $\mathcal{M}$. Then the $\mathcal{M}$-algebroid $\mathcal{I} \star_W^{\mathrm{Algbrd}} \mathcal{I}'$ has category of objects $X \cup X'$, and comes equipped with fully faithful morphisms of algebroids

$$i : \mathcal{I} \to \mathcal{I} \star_W^{\mathrm{Algbrd}} \mathcal{I}' \leftarrow \mathcal{I}' : i'$$

which are cartesian lifts of the inclusions $X \to X \cup X' \leftarrow X'$.

The unit map $W \to \mathcal{I} \star_W^{\mathrm{Algbrd}} \mathcal{I}'$ presents the bimodule W as the restriction of scalars of the diagonal bimodule of $\mathcal{I} \star_W^{\mathrm{Algbrd}} \mathcal{I}'$ along i and i'. Furthermore, for each pair of objects x in $\mathcal{I}$ and x' in $\mathcal{I}'$, we have that $\mathcal{I} \star_W^{\mathrm{Algbrd}} \mathcal{I}'(x', x)$ is the initial object of $\mathcal{M}$.

Remark 5.5.3. Let $T : \mathcal{M} \to \mathcal{M}'$ be a morphism of commutative algebras in Pr^L. Let X, X' be categories, and consider the commutative square of categories

$$\begin{tikzcd}
\mathrm{Alg}_{\mathrm{Assos}_{X\cup X'}}(\mathcal{M}) \arrow[r] \arrow[d, "T_!"] & \mathrm{Alg}_{\mathrm{BM}_{X,X'}}(\mathcal{M}) \arrow[d, "T_!"] \\
\mathrm{Alg}_{\mathrm{Assos}_{X\cup X'}}(\mathcal{M}') \arrow[r] & \mathrm{Alg}_{\mathrm{BM}_{X,X'}}(\mathcal{M})
\end{tikzcd}$$

where the horizontal arrows are the restriction maps. Since T preserves all operadic colimits involved in the construction of free algebras, we have that the above commutative square is horizontally left adjointable.

Let $\mathcal{I}, \mathcal{I}'$ be $\mathcal{M}$-algebroids with categories of objects X, X' respectively. Let W be an $\mathcal{I} - \mathcal{I}'$-bimodule in $\mathcal{M}$, and let $T_!W$ be the induced $T_!\mathcal{I} - T_!\mathcal{I}'$-bimodule in $\mathcal{M}'$. It follows from the above that we have an equivalence of $\mathcal{M}'$-algebroids

$$T_!(\mathcal{I} \star_W^{\mathrm{Algbrd}} \mathcal{I}') = (T_!\mathcal{I}) \star_{T_!W}^{\mathrm{Algbrd}} (T_!\mathcal{I}').$$

When $\mathcal{I}, \mathcal{I}'$ are $\mathcal{M}$-enriched categories, the above induces an equivalence of $\mathcal{M}'$-enriched categories

$$T_!(\mathcal{I} \star_W \mathcal{I}') = (T_!\mathcal{I}) \star_{T_!W} (T_!\mathcal{I}').$$

Of particular importance is the case when one of the two categories in the join is the unit $\mathcal{M}$-enriched category.

Notation 5.5.4. Let $\mathcal{M}$ be a presentable symmetric monoidal category. Let $\mathcal{I}$ be an $\mathcal{M}$-enriched category and let W be a presheaf on $\mathcal{I}$, which we think about as an $1_{\mathcal{M}} - \mathcal{I}$-bimodule. We denote by $\mathcal{I}^{\triangleright}_W$ the join of the unit $\mathcal{M}$-enriched category and $\mathcal{I}$, weighted by W. Similarly, if W' is a copresheaf on $\mathcal{I}$, we denote by $\mathcal{I}^{\triangleleft}_{W}$ the join of $\mathcal{I}$ and the unit $\mathcal{I}$-enriched category, weighted by W'. We call $\mathcal{I}^{\triangleright}_W$ (resp. $\mathcal{I}^{\triangleleft}_W$) the right (resp. left) cone of $\mathcal{I}$ weighted by W (resp. W').

Example 5.5.5. Let $\mathcal{M}$ be a presentable symmetric monoidal category and let $\mathcal{I}$ be the initial $\mathcal{M}$-enriched category (in other words, $\mathcal{I}$ is the unique $\mathcal{M}$-enriched algebroid with an empty space of objects). Let W be the unique presheaf on $\mathcal{I}$. Then $\mathcal{I}^{\triangleright}_W$ is the unit $\mathcal{M}$-enriched category.

Example 5.5.6. Let $\mathcal{M}$ be a presentable symmetric monoidal category and let m be an object of $\mathcal{M}$. Let $\mathcal{I}$ be the unit $\mathcal{M}$-enriched category and let $W : \mathcal{I}^{\mathrm{op}} \to \overline{\mathcal{M}}$ be the map that picks out the object m. Then $(\mathcal{I})^{\triangleright}_W$ is the enriched category underlying the m-cell C_m.

Example 5.5.7. Let $\mathcal{M} = \mathrm{Spc}$ equipped with its cartesian symmetric monoidal structure, and let $\mathcal{I}$ be a category. Let $W : \mathcal{I}^{\mathrm{op}} \to \mathrm{Spc}$ be the terminal presheaf. Then the cone point of $\mathcal{I}^{\triangleright}_W$ is a final object. Let $\mathcal{I}^{\triangleright}$ be the category obtained from $\mathcal{I}$ by freely adjoining a final object. Then the functor $\mathcal{I}^{\triangleright} \to \mathcal{I}^{\triangleright}_W$ induced from the inclusion $\mathcal{I} \to \mathcal{I}^{\triangleright}_W$ is an equivalence.

Remark 5.5.8. Let $T : \mathcal{M} \to \mathcal{M}'$ be a morphism of commutative algebras in Pr^L. Let $\mathcal{I}$ be an $\mathcal{M}$-enriched category and let W be a presheaf on $\mathcal{M}$. Denote by $T_!W$ the presheaf on $T_!\mathcal{I}$ defined by adjunction from the composite map

$$\mathcal{I}^{\mathrm{op}} \xrightarrow{W} \overline{\mathcal{M}} \to T_!^R\overline{\mathcal{M}'}.$$

Then it follows from remark 5.5.3 that there is an equivalence $(T_!\mathcal{I})^{\triangleright}_{T_!W} = T_!(\mathcal{I}^{\triangleright}_W)$.

Example 5.5.9. Let $\mathcal{M}$ be a presentable symmetric monoidal category. Let $\mathcal{I}$ be a category and let W be the presheaf on $\mathcal{I}_{\mathcal{M}}$ induced from the presheaf $\mathcal{I}^{\mathrm{op}} \to \mathcal{M}$ which is constant $1_{\mathcal{M}}$. Then it follows from a combination of remarks 5.5.3 and 5.5.8 together with example 5.5.7 that $\mathcal{I}^{\triangleright}_W$ is equivalent to $(\mathcal{I}^{\triangleright})_{\mathcal{M}}$.

We now discuss the notion of weighted limits and colimits.

Definition 5.5.10. *Let $\mathcal{M}$ be a presentable symmetric monoidal category and let $\mathcal{I}, \mathcal{C}$ be $\mathcal{M}$-enriched categories. Let $W : \mathcal{I}^{\mathrm{op}} \to \overline{\mathcal{M}}$ be a presheaf on $\mathcal{I}$, and let $X : \mathcal{I} \to \mathcal{C}$ be a functor. A right cone for X weighted by W (or W-weighted right cone, for short) is an extension $X^{\triangleright}_W : \mathcal{I}^{\triangleright}_W \to \mathcal{C}$ for X. Dually, a left cone for X weighted by a copresheaf W' is an extension $X^{\triangleleft}_{W'} : \mathcal{I}^{\triangleleft}_{W'} \to \mathcal{C}$ for X.*

Notation 5.5.11. Let $\mathcal{M}$ be a presentable symmetric monoidal category and let $X : \mathcal{I} \to \mathcal{C}$ be a functor of $\mathcal{M}$-enriched categories. We denote by $H_X : \mathcal{C} \to \operatorname{Funct}(\mathcal{I}^{\mathrm{op}}, \overline{\mathcal{M}})$ the functor induced from the composite functor

$$\mathcal{I}^{\mathrm{op}} \otimes \mathcal{C} \xrightarrow{X^{\mathrm{op}} \otimes \mathrm{id}_{\mathcal{C}}} \mathcal{C}^{\mathrm{op}} \otimes \mathcal{C} \xrightarrow{\operatorname{Hom}_{\mathcal{C}}(-,-)} \overline{\mathcal{M}}.$$

Remark 5.5.12. Let $\mathcal{M}$ be a presentable symmetric monoidal category and let $\mathcal{I}, \mathcal{C}$ be $\mathcal{M}$-enriched categories. Then the assignment $X \mapsto H_X$ forms part of a functor

$$H_{(-)} : \operatorname{Funct}(\mathcal{I}, \mathcal{C})^{\mathrm{op}} \to \operatorname{Funct}(\mathcal{C}, \operatorname{Funct}(\mathcal{I}^{\mathrm{op}}, \overline{\mathcal{M}})).$$

Assume now given another $\mathcal{M}$-enriched category $\mathcal{D}$ and a functor $G : \mathcal{C} \to \mathcal{D}$. Then the natural transformation $G_* : \operatorname{Hom}_{\mathcal{C}}(-,-) \to \operatorname{Hom}_{\mathcal{D}}(G-, G-)$ induces a natural transformation

$$G_* : H_{(-)} \to H_{G(-)} \circ G$$

of functors $\operatorname{Funct}(\mathcal{I}, \mathcal{C})^{\mathrm{op}} \to \operatorname{Funct}(\mathcal{C}, \operatorname{Funct}(\mathcal{I}^{\mathrm{op}}, \overline{\mathcal{M}}))$.

Remark 5.5.13. Let $\mathcal{M}$ be a presentable symmetric monoidal category and let $\mathcal{I}, \mathcal{C}$ be $\mathcal{M}$-enriched categories. Let $W : \mathcal{I}^{\mathrm{op}} \to \overline{\mathcal{M}}$ be a presheaf on $\mathcal{I}$, and let $X : \mathcal{I} \to \mathcal{C}$ be a functor. Then the space of right cones for X weighted by W is equivalent to the space of pairs of an object x in $\mathcal{C}$, and a morphism of presheaves

$$\eta : W(-) \to H_X(x).$$

Assume now given another $\mathcal{M}$-enriched category $\mathcal{D}$ and a functor $G : \mathcal{C} \to \mathcal{D}$. Let $X^{\triangleright}_W : \mathcal{I}^{\triangleright}_W \to \mathcal{C}$ be a right cone for X weighted by W, associated to a pair (x, η) as above. Then $GX^{\triangleright}_W$ is a right cone for GX weighted by W, which corresponds under the above identification to the pair (Gx, η'), where η' is given by the composite map

$$W(-) \xrightarrow{\eta} H_X(x) \xrightarrow{G_*} H_{GX}(Gx).$$

Definition 5.5.14. *Let $\mathcal{M}$ be a presentable symmetric monoidal category and let $\mathcal{I}, \mathcal{C}$ be $\mathcal{M}$-enriched categories. Let $W : \mathcal{I}^{\mathrm{op}} \to \overline{\mathcal{M}}$ be a presheaf on $\mathcal{I}$, and let $X : \mathcal{I} \to \mathcal{C}$ be a functor. Let $X^{\triangleright}_W$ be a right cone for X weighted by W, associated to an object x in $\mathcal{C}$ and a morphism of presheaves $\eta : W \to H_X(x)$. We say that $X^{\triangleright}_W$ is a colimit for X weighted by W (or W-weighted colimit, for short) if η presents x as left adjoint to H_X at W. Dually, given a copresheaf W' on $\mathcal{I}$, we say that a left cone $X^{\triangleleft}_{W'}$ for X weighted by W' is a limit for X weighted by W' (or W'-weighted limit, for short) if $(X^{\triangleleft}_{W'})^{\mathrm{op}}$ is a colimit for X^{op} weighted by W'.*

Example 5.5.15. Let $\mathcal{M}$ be a presentable symmetric monoidal category and let $\mathcal{I}$ be the initial $\mathcal{M}$-enriched category. Let W be the unique presheaf on $\mathcal{I}$, so that $\mathcal{I}^{\triangleright}_W$ is the unit $\mathcal{M}$-enriched category (see example 5.5.5).

Let $\mathcal{C}$ be an $\mathcal{M}$-enriched category and let $X : \mathcal{I} \to \mathcal{C}$ be the unique functor. A right cone for X weighted by W is the same data as an object x of $\mathcal{C}$. This defines a colimit of X weighted by W if and only if x is initial (in other words, $\operatorname{Hom}_{\mathcal{C}}(x, y)$ is a final object in $\mathcal{M}$ for every y in $\mathcal{C}$). Dually, we have that a limit for X weighted by W is the same a final object in $\mathcal{C}$.

Example 5.5.16. Let $\mathcal{M}$ be a presentable symmetric monoidal category and let $\mathcal{I}$ be the unit $\mathcal{M}$-enriched category. Let m be an object in $\mathcal{M}$ and let W be the associated presheaf on $\mathcal{I}$, so that $\mathcal{I}_W^{\triangleright}$ is the $\mathcal{M}$-enriched category underlying the m-cell (see example 5.5.6).

Let $\mathcal{C}$ be an $\mathcal{M}$-enriched category and let $X : \mathcal{I} \to \mathcal{C}$ be a functor that picks out an object x in $\mathcal{C}$. A right cone for X weighted by W consists of a pair of an object y in $\mathcal{C}$ and a morphism $\alpha : m \to \operatorname{Hom}_{\mathcal{C}}(x, y)$. This is a colimit for X weighted by W if it has the property that for every object z in $\mathcal{C}$ the composite map

$$m \otimes \operatorname{Hom}_{\mathcal{C}}(y, z) \xrightarrow{\alpha \otimes \mathrm{id}} \operatorname{Hom}_{\mathcal{C}}(x, y) \otimes \operatorname{Hom}_{\mathcal{C}}(y, z) \to \operatorname{Hom}_{\mathcal{C}}(x, z)$$

induces an isomorphism $\operatorname{Hom}_{\mathcal{C}}(y, z) = \mathscr{H}om_{\mathcal{M}}(m, \operatorname{Hom}_{\mathcal{C}}(x, z))$. In this case, we say that α presents y as the copower (or tensor) of x by m. Passing to opposites we obtain the dual notion of power (or cotensor) of an object in $\mathcal{C}$ by an object in $\mathcal{M}$.

Example 5.5.17. Let $\mathcal{M} = \mathrm{Spc}$. Let $\mathcal{I}$ be a category and let W be the terminal presheaf on $\mathcal{I}$. Then a diagram $\mathcal{I}_W^{\triangleright} \to \mathcal{C}$ is a W-weighted colimit if and only if it is a colimit diagram.

Example 5.5.18. Let $\mathcal{M}$ be a presentable symmetric monoidal category and let $\mathcal{I}$ be an $\mathcal{M}$-enriched category. Let j be an object in $\mathcal{I}$ and let $W = \operatorname{Hom}_{\mathcal{I}}(-, j)$ be the corresponding representable presheaf. The equivalence $W = \operatorname{Hom}_{\mathcal{I}}(-, j)$ induces a functor

$$r : \mathcal{I}_W^{\triangleright} \to \mathcal{I}$$

which is a retraction for the inclusion $i : \mathcal{I} \to \mathcal{I}_W^{\triangleright}$, and maps the cone point to j. Observe that the equivalence $ri = \mathrm{id}_{\mathcal{I}}$ presents i as left adjoint to r.

Let $\mathcal{C}$ be an $\mathcal{M}$-enriched category and let $X : \mathcal{I} \to \mathcal{C}$ be a functor. Then Xr is a right cone for X weighted by W. Unwinding the definitions, we see that this is the right cone which corresponds to the object $X(j)$ in $\mathcal{C}$ and the natural transformation

$$\eta : W(-) \to \operatorname{Hom}_{\mathcal{C}}(X(-), X(j))$$

is induced by the identity in $\operatorname{Hom}_{\mathcal{C}}(X(j), X(j))$. It now follows from the Yoneda lemma that Xr is a colimit for X weighted by W.

We conclude from the above discussion that $\mathcal{M}$-enriched categories admit all colimits weighted by a representable presheaf.

Remark 5.5.19. Let $T : \mathcal{M} \to \mathcal{M}'$ be a colimit preserving symmetric monoidal functor between presentable symmetric monoidal categories. Let $\mathcal{I}$ be an $\mathcal{M}$-enriched category and let W be a presheaf on $\mathcal{I}$. Let $\mathcal{C}$ be an $\mathcal{M}'$-enriched category and let

$$X^{\triangleright} : (T_!\mathcal{I})_{T_!W}^{\triangleright} \to \mathcal{C}$$

be a weighted colimit diagram. Then the induced diagram $X'^{\triangleright} : \mathcal{I}_W^{\triangleright} \to T_!^R\mathcal{C}$ is a weighted colimit diagram.

Recall that, in unenriched category theory, left adjoint functors preserve all colimits. We now discuss an enriched generalization of this fact.

Definition 5.5.20. *Let $\mathcal{M}$ be a presentable symmetric monoidal category and let $\mathcal{I}, \mathcal{C}$ be $\mathcal{M}$-enriched categories. Let $W : \mathcal{I}^{\mathrm{op}} \to \overline{\mathcal{M}}$ be a presheaf on $\mathcal{I}$, and let $X : \mathcal{I} \to \mathcal{C}$ be a functor. Let $G : \mathcal{C} \to \mathcal{C}'$ be a functor of $\mathcal{M}$-enriched categories. We say that a colimit $X_W^{\triangleright}$ for X weighted by W is preserved by G if $GX_W^{\triangleright}$ is a colimit for GX weighted by X. Passing to opposites, we similarly define the notion of a functor preserving a weighted limit.*

Example 5.5.21. Let $\mathcal{M}$ be a presentable symmetric monoidal category and let $\mathcal{I}$ be an $\mathcal{M}$-enriched category. Let j be an object in $\mathcal{I}$ and let $W = \mathrm{Hom}_{\mathcal{I}}(-, j)$ be the corresponding representable presheaf. It follows from the description of W-weighted colimits from example 5.5.18 that these are preserved under all functors of $\mathcal{M}$-enriched categories. We may summarize this by saying that colimits weighted by representable presheaves are examples of absolute colimits.

Notation 5.5.22. Let $\mathcal{M}$ be a presentable symmetric monoidal category and let $G : \mathcal{C} \to \mathcal{D}$ be a functor of $\mathcal{M}$-enriched categories. Let $\mathcal{D}'$ be the full subcategory of $\mathcal{D}$ on those objects d such that G admits a left adjoint at d. It follows from proposition 5.1.7 that the functor

$$G_* : \mathrm{Funct}(\mathcal{D}', \mathcal{C}) \to \mathrm{Funct}(\mathcal{D}', \mathcal{D})$$

admits a left adjoint at the inclusion $i : \mathcal{D}' \to \mathcal{D}$. We will usually denote the resulting functor $G^L : \mathcal{D}' \to \mathcal{C}$ and call it the (partially defined) left adjoint of G. We call $\mathcal{D}'$ the domain of definition of G^L.

Remark 5.5.23. Let $\mathcal{M}$ be a presentable symmetric monoidal category and let $G : \mathcal{C} \to \mathcal{D}$ be a functor of $\mathcal{M}$-enriched categories. Then G^L is characterized by the property that it comes equipped with a unit natural transformation $\eta : i \to GG^L$ such that $\eta(d)$ presents $G^L(d)$ as left adjoint to G at d for each d in $\mathcal{D}'$. In the case when G admits a left adjoint then $\mathcal{D}' = \mathcal{D}$ and η presents G^L as left adjoint to G.

Proposition 5.5.24. *Let $\mathcal{M}$ be a presentable symmetric monoidal category. Let $\mathcal{I}$ be an $\mathcal{M}$-enriched category and let W be a presheaf on $\mathcal{I}$. Let $G : \mathcal{C} \to \mathcal{D}$ be a functor of $\mathcal{M}$-enriched categories and let $X : \mathcal{I} \to \mathcal{D}$ be a functor admitting a W-weighted colimit $X^{\triangleright} : \mathcal{I}_W^{\triangleright} \to \mathcal{D}$. Assume that G admits a left adjoint at $X(i)$ for all i in $\mathcal{I}$. Then $G^L X$ admits a W-weighted colimit if and only if G admits a left adjoint at $X^{\triangleright}(*)$. Furthermore, in this case $G^L X^{\triangleright}$ is a W-weighted colimit for $G^L X$.*

Proof. Let $\mathcal{D}'$ be the domain of definition of G^L, and let η be the unit for the partial adjunction between G^L and G. Let $x = X^{\triangleright}(*)$ and let $\mu : W \to H_X(x)$ be the induced natural transformation which presents x as left adjoint to H_X at W.

Note that the composite natural transformation

$$H_{G^L X} \xrightarrow{G_*} H_{GG^L X}G \xrightarrow{\eta^*} H_X G$$

is an isomorphism. It follows from proposition 5.1.3 that $H_X G$ admits a left adjoint at W if and only if G admits a left adjoint at x. Since $H_X G$ is equivalent to $H_{G^L X}$, we conclude that $G^L X$ admits a colimit weighted by W if and only if G admits a left adjoint at x, as desired.

Assume now that G admits a left adjoint at x. We need to show that G^L preserves the colimit of X weighted by W. The weighted cone $G^L X^{\triangleright}$ corresponds to the pair of the object $G^L x$ and the composite map

$$W \xrightarrow{\mu} H_X(x) \xrightarrow{G^L_*} H_{G^L X}(G^L x).$$

Composing with the isomorphism $H_{G^L X} = H_X G$ described above we obtain the composite natural transformation

$$\mu' : W \xrightarrow{\mu} H_X(x) \xrightarrow{G^L_*} H_{G^L X}(G^L x) \xrightarrow{G_*} H_{GG^L X}(GG^L x) \xrightarrow{\eta^*} H_X(GG^L x).$$

To show that G^L preserves the W-weighted colimit of X, we have to show that μ' presents $G^L x$ as left adjoint to $H_X G$ at W.

The natural transformation μ' is obtained by composing μ with the natural transformation $H_X(x) \to H_X(GG^L x)$ induced from the composite natural transformation

$$\mathrm{Hom}_{\mathcal{D}'}(-,-) \xrightarrow{GG^L_*} \mathrm{Hom}_{\mathcal{D}}(GG^L-, GG^L-) \xrightarrow{\eta^*} \mathrm{Hom}_{\mathcal{D}}(-, GG^L-).$$

The naturality of η implies that the above is equivalent to

$$\eta_* : \mathrm{Hom}_{\mathcal{D}'}(-,-) \to \mathrm{Hom}_{\mathcal{D}}(-, GG^L-)$$

(see remark 5.2.7). Hence we see that μ' is equivalent to the composite map

$$W \xrightarrow{\mu} H_X(x) \xrightarrow{H_X(\eta)} H_X(GG^L(x)).$$

Using proposition 5.1.3 we see that the above presents $G^L x$ as left adjoint to $H_X G$ at W, as desired. $\square$

Corollary 5.5.25. *Let $\mathcal{M}$ be a presentable symmetric monoidal category. Let $G : \mathcal{C} \to \mathcal{D}$ be a functor of $\mathcal{M}$-enriched categories, and assume that $\mathcal{C}$ admits all weighted colimits. Then the full subcategory of $\mathcal{D}$ on those objects d such that G admits a left adjoint at d, is closed under weighted colimits.*

Corollary 5.5.26. *Let $\mathcal{M}$ be a presentable symmetric monoidal category. Let $G : \mathcal{C} \to \mathcal{D}$ be a functor of $\mathcal{M}$-enriched categories admitting a left adjoint G^L. Then G^L preserves all weighted colimits that exist in $\mathcal{D}$.*

Our next goal is to prove that enriched categories arising from presentable modules admit powers and copowers, which are computed in the expected way.

Proposition 5.5.27. *Let $\mathcal{M}$ be a presentable symmetric monoidal category and let $\mathcal{C}$ be a presentable $\mathcal{M}$-module. Let x be an object in $\mathcal{C}$ and let m be an object in $\mathcal{M}$. Then*

(i) *The morphism $m \to \operatorname{Hom}_{\theta_{\mathcal{M}}(\mathcal{C})}(x, m \otimes x)$ induced from the identity of $m \otimes x$ presents $m \otimes x$ as the copower of x by m in the $\mathcal{M}$-enriched category $\theta_{\mathcal{M}}(\mathcal{C})$.*

(ii) *Let x^m be the object representing the presheaf $\operatorname{Hom}_{\mathcal{C}}(m \otimes -, x)$. Then the morphism $m \to \operatorname{Hom}_{\theta_{\mathcal{M}}(\mathcal{C})}(x^m, x)$ induced from the canonical map $m \otimes x^m \to x$ presents x^m as the power of x by m in the $\mathcal{M}$-enriched category $\theta_{\mathcal{M}}(\mathcal{C})$.*

Proof. We first prove item (i). We need to show that for every object y in $\mathcal{C}$, the induced map

$$m \otimes \operatorname{Hom}_{\theta_{\mathcal{M}}\mathcal{C}}(m \otimes x, y) \to \operatorname{Hom}_{\theta_{\mathcal{M}}\mathcal{C}}(x, m \otimes x) \otimes \operatorname{Hom}_{\theta_{\mathcal{M}}\mathcal{C}}(m \otimes x, y) \to \operatorname{Hom}_{\theta_{\mathcal{M}}\mathcal{C}}(x, y)$$

presents $\operatorname{Hom}_{\theta_{\mathcal{M}}\mathcal{C}}(m \otimes x, y)$ as the Hom object $\mathscr{H}om_{\mathcal{M}}(m, \operatorname{Hom}_{\theta_{\mathcal{M}}\mathcal{C}}(x, y))$. Unwinding the definitions, we see that the above is equivalent to the map

$$\eta : m \otimes \mathscr{H}om_{\mathcal{C}}(m \otimes x, y) \to \mathscr{H}om_{\mathcal{C}}(x, y)$$

induced from the evaluation map $\operatorname{ev} : \mathscr{H}om_{\mathcal{C}}(m \otimes x, y) \otimes m \otimes x \to y$. Our task is to show that η presents $\mathscr{H}om_{\mathcal{C}}(m \otimes x, y)$ as the Hom object $\mathscr{H}om_{\mathcal{M}}(m, \mathscr{H}om_{\mathcal{C}}(x, y))$. This is a consequence of proposition 5.1.3 applied to the functors

$$\mathcal{M} \xrightarrow{-\otimes m} \mathcal{M} \xrightarrow{-\otimes x} \mathcal{C}.$$

We now prove item (ii). We need to show that for every object z in $\mathcal{C}$, the induced map

$$\operatorname{Hom}_{\theta_{\mathcal{M}}\mathcal{C}}(z, x^m) \otimes m \to \operatorname{Hom}_{\theta_{\mathcal{M}}\mathcal{C}}(z, x^m) \otimes \operatorname{Hom}_{\theta_{\mathcal{M}}\mathcal{C}}(x^m, x) \to \operatorname{Hom}_{\theta_{\mathcal{M}}\mathcal{C}}(z, x)$$

presents $\operatorname{Hom}_{\theta_{\mathcal{M}}\mathcal{C}}(z, x^m)$ as the Hom object $\mathscr{H}om_{\mathcal{M}}(m, \operatorname{Hom}_{\theta_{\mathcal{M}}\mathcal{C}}(z, x))$. We can identify the above map with the map

$$\eta' : m \otimes \mathscr{H}om_{\mathcal{C}}(z, x^m) \to \mathscr{H}om_{\mathcal{C}}(z, x)$$

induced by composing the evaluation maps

$$m \otimes z \otimes \mathscr{H}om_{\mathcal{C}}(z, x^m) \to m \otimes x^m \to x.$$

Our task is to show that η' induces an equivalence

$$\mathscr{H}om_{\mathcal{C}}(z, x^m) = \mathscr{H}om_{\mathcal{M}}(m, \mathscr{H}om_{\mathcal{C}}(z, x)).$$

Consider the following commutative square of categories:

$$\begin{tikzcd} \mathcal{M} \arrow[r, "-\otimes z"] \arrow[d, "m\otimes -"] & \mathcal{C} \arrow[d, "m\otimes -"] \\ \mathcal{M} \arrow[r, "-\otimes z"] & \mathcal{C} \end{tikzcd}$$

The evaluation maps $m \otimes x^m \to x$ and $\mathscr{H}om_{\mathcal{C}}(z, x^m) \otimes z \to x^m$ are right adjoint to the right vertical and top horizontal maps at x and x^m, respectively. Using proposition 5.1.3 (in its dual form) we see that the composite evaluation map $m \otimes z \otimes \mathscr{H}om_{\mathcal{M}}(z, x^m) \to x$ is right adjoint to the diagonal map $\mathcal{M} \xrightarrow{m\otimes - \otimes z} \mathcal{C}$ at x. The result now follows from another application of proposition 5.1.3, this time to the bottom horizontal and left vertical arrows in the above diagram. □

5.6 Weighted colimits via conical colimits and copowers

Our next goal is to give a proof of the following fundamental result, which allows one to reduce many questions in the theory of weighted colimits to questions about conical colimits and copowers.

Theorem 5.6.1. *Let $\mathcal{M}$ be a presentable symmetric monoidal category and let $\mathcal{C}$ be an $\mathcal{M}$-enriched category.*

(i) *Let $\mathcal{I}$ be a category and $X^{\triangleright} : \mathcal{I}^{\triangleright}_{\mathcal{M}} \to \mathcal{C}$ be a functor. Then $X^{\triangleright}$ is a conical colimit diagram if and only if it is a colimit diagram weighted by the presheaf W induced from the functor $\mathcal{I}^{\mathrm{op}} \to \mathcal{M}$ which is constant $1_{\mathcal{M}}$.*

(ii) *The $\mathcal{M}$-enriched category $\mathcal{C}$ admits all weighted colimits if and only if it admits all conical colimits and copowers.*

(iii) *Let $G : \mathcal{C} \to \mathcal{D}$ be a functor of $\mathcal{M}$-enriched categories, and assume that $\mathcal{C}$ admits all weighted colimits. Then G preserves all weighted colimits if and only if it preserves conical colimits and copowers.*

Corollary 5.6.2. *Let $\mathcal{M}$ be a presentable symmetric monoidal category and let $\mathcal{C}$ be a presentable module over $\mathcal{M}$. Then $\theta_{\mathcal{M}}(\mathcal{C})$ admits all weighted limits and colimits.*

Proof. This is a direct consequence of theorem 5.6.1, by virtue of propositions 5.4.8 and 5.5.27. □

Corollary 5.6.3. *Let $\mathcal{M}$ be a presentable symmetric monoidal category. Let $\mathcal{I}$ be an $\mathcal{M}$-enriched category and let W be a copresheaf on $\mathcal{I}$. Let $\mathcal{C}$ be an $\mathcal{M}$-enriched category and denote by $i : \mathcal{C} \to \mathrm{Funct}(\mathcal{C}^{\mathrm{op}}, \overline{\mathcal{M}})$ the Yoneda embedding. Then a weighted left cone $X^{\triangleleft} : \mathcal{I}^{\triangleleft}_{W} \to \mathcal{C}$ is a weighted limit diagram if and only if $iX^{\triangleleft}$ is a weighted limit diagram.*

Proof. Lemma 5.6.6 below shows that this corollary holds under the additional hypothesis that $\mathrm{Funct}(\mathcal{C}^{\mathrm{op}}, \overline{\mathcal{M}})$ admits all weighted limits. It follows from corollary 5.6.2 that this hypothesis always holds, since $\mathrm{Funct}(\mathcal{C}^{\mathrm{op}}, \overline{\mathcal{M}})$ belongs to the image of $\theta_{\mathcal{M}}$ (see [Hin20a] proposition 6.3.7). □

Our proof of theorem 5.6.1 will need a few preliminary lemmas.

Lemma 5.6.4. *Let $\mathcal{M}$ be a presentable symmetric monoidal category and let $G : \mathcal{C} \to \mathcal{D}$ be a functor of $\mathcal{M}$-enriched categories. Assume that $\mathcal{C}$ admits all conical colimits. Then the full subcategory of $\mathcal{D}$ on those objects on which the left adjoint to G is defined, is closed under conical colimits.*

Proof. Let $\mathcal{I}$ be a category and $X : \mathcal{I}_{\mathcal{M}} \to \mathcal{D}$ be a functor admitting a conical colimit with underlying object x. Assume that for every i in $\mathcal{I}$, there exists a left adjoint to G at $X(i)$. We have to show that there exists a left adjoint to G at x.

Consider the following commutative square of $\mathcal{M}$-enriched categories:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "G"] \arrow[d, "\Delta"] & \mathcal{D} \arrow[d, "\Delta"] \\
\mathrm{Funct}(\mathcal{I}_{\mathcal{M}}, \mathcal{C}) \arrow[r, "G_*"] & \mathrm{Funct}(\mathcal{I}_{\mathcal{M}}, \mathcal{D}).
\end{tikzcd}$$

Thanks to proposition 5.1.7, the bottom horizontal arrow in the above square admits a left adjoint at X. Since $\mathcal{C}$ admits all conical colimits, the left vertical arrow admits a left adjoint. Applying proposition 5.1.3 we conclude that the induced functor $\mathcal{C} \to \mathrm{Funct}(\mathcal{I}_{\mathcal{M}}, \mathcal{D})$ admits a left adjoint at X. Our claim now follows from another application of proposition 5.1.3. □

Lemma 5.6.5. *Let $\mathcal{M}$ be a presentable symmetric monoidal category and let $\mathcal{C}$ be a presentable $\mathcal{M}$-module. Let $\mathcal{I}$ be a small full subcategory of $\theta_{\mathcal{M}}(\mathcal{C})$ and assume that the family of copresheaves $\{\mathrm{Hom}_{\theta_{\mathcal{M}}(\mathcal{C})}(i, -)\}_{i \in \mathcal{I}}$ detects isomorphisms. Then the closure of $\mathcal{I}$ under conical colimits and copowers is the entire $\theta_{\mathcal{M}}(\mathcal{C})$.*

Proof. Denote by $\mathcal{D}$ the smallest subcategory of $\mathcal{C}$ closed under colimits, the action of $\mathcal{M}$, and containing the objects of $\mathcal{I}$. By a combination of proposition 5.5.27 and proposition 5.4.8, it suffices to show that $\mathcal{D}$ is the entire $\mathcal{C}$.

Let κ be a regular cardinal such that $\mathcal{M}$ is κ-compactly generated. Observe that $\mathcal{D}$ is generated under colimits by objects of the form $m \otimes i$ with i in $\mathcal{I}$ and m a κ-compact object of $\mathcal{M}$. This is a small collection of objects. Since $\mathcal{C}$ is presentable, we conclude that $\mathcal{D}$ is presentable as well. Hence the inclusion of $\mathcal{D}$ inside $\mathcal{C}$ admits a right adjoint which presents $\mathcal{D}$ as a colocalization of $\mathcal{C}$.

Let c be an object in $\mathcal{C}$ and let d be its image under the colocalization map. We claim that the counit map $\epsilon : d \to c$ is an isomorphism. To see this, it suffices to show that for every object i in $\mathcal{I}$, the induced morphism

$$\epsilon_* : \mathrm{Hom}_{\theta_{\mathcal{C}}}(i, d) \to \mathrm{Hom}_{\theta_{\mathcal{C}}}(i, c)$$

is an isomorphism. The above is equivalent to the map $\mathscr{H}om_{\mathcal{C}}(i,d) \to \mathscr{H}om_{\mathcal{C}}(i,c)$ induced from the composite map

$$\mathscr{H}om_{\mathcal{C}}(i,d) \otimes i \xrightarrow{\mathrm{ev}} d \xrightarrow{\epsilon} c.$$

It suffices then to show that for every object m in $\mathcal{M}$, the composite morphism of spaces

$$\mathrm{Hom}_{\mathcal{M}}(m, \mathscr{H}om_{\mathcal{C}}(i,d)) \xrightarrow{-\otimes i} \mathrm{Hom}_{\mathcal{C}}(m\otimes i, m \otimes \mathscr{H}om_{\mathcal{C}}(i,d)) \xrightarrow{(\epsilon\,\mathrm{ev})_*} \mathrm{Hom}_{\mathcal{C}}(m\otimes i, c)$$

is an isomorphism. This follows from the fact that $\epsilon_* : \mathrm{Hom}_{\mathcal{C}}(m\otimes i, d) \to \mathrm{Hom}_{\mathcal{C}}(m \otimes i, c)$ is an isomorphism, together with the fact that the composite map of spaces

$$\mathrm{Hom}_{\mathcal{M}}(m, \mathscr{H}om_{\mathcal{C}}(i,d)) \xrightarrow{-\otimes i} \mathrm{Hom}_{\mathcal{C}}(m\otimes i, m \otimes \mathscr{H}om_{\mathcal{C}}(i,d)) \xrightarrow{\mathrm{ev}_*} \mathrm{Hom}_{\mathcal{C}}(m\otimes i, d)$$

is an isomorphism. □

Lemma 5.6.6. *Let $\mathcal{M}$ be a presentable symmetric monoidal category. Let $\mathcal{I}$ be an $\mathcal{M}$-enriched category and let W be a copresheaf on $\mathcal{I}$. Let $\mathcal{C}$ be an $\mathcal{M}$-enriched category and denote by $i : \mathcal{C} \to \mathrm{Funct}(\mathcal{C}^{\mathrm{op}}, \overline{\mathcal{M}})$ the Yoneda embedding. Assume that $\mathrm{Funct}(\mathcal{C}^{\mathrm{op}}, \overline{\mathcal{M}})$ admits all weighted limits.*[1] *Then a weighted left cone $X^{\lhd} : \mathcal{I}^{\lhd}_W \to \mathcal{C}$ is a weighted limit diagram if and only if $iX^{\lhd}$ is a weighted limit diagram.*

Proof. Let $\widehat{\mathcal{I}} = \mathrm{Funct}(\mathcal{I}, \overline{\mathcal{M}})$. Given an $\mathcal{M}$-enriched category $\mathcal{D}$ and a functor $F : \mathcal{I} \to \mathcal{D}$, we denote by $H' : \mathcal{D}^{\mathrm{op}} \to \widehat{\mathcal{I}}$ the functor induced from the composite map

$$\mathcal{D}^{\mathrm{op}} \otimes \mathcal{I} \xrightarrow{\mathrm{id}\otimes F} \mathcal{D}^{\mathrm{op}} \otimes \mathcal{D} \xrightarrow{\mathrm{Hom}_{\mathcal{D}}(-,-)} \overline{\mathcal{M}}.$$

Let $x = X^{\lhd}(*)$ and let $\eta : W(-) \to H'_X(x)$ be the natural transformation induced by $X^{\lhd}$. It follows from a dual version of the discussion in remark 5.5.13 that the weighted cone $iX^{\lhd}$ is associated to the pair (ix, η'), where η' is given by the composition

$$W(-) \xrightarrow{\eta} H'_X(x) \xrightarrow{i_*} H'_{iX}(ix).$$

Note that since i is fully faithful, the second morphism above is an equivalence.

Let $X = X^{\lhd}|_{\mathcal{I}}$ and let

$$\mu : W(-) \to H'_{iX}(y)$$

be the limit of iX weighted by W. Let $\alpha : ix \to y$ be the unique morphism equipped with an identification $\eta' = \alpha^*\mu$.

Let z be an object of $\mathcal{C}$ and consider the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(z,x) \arrow[r, "(H'_X)_*"] \arrow[d, "i_*"] & \mathrm{Hom}_{\widehat{\mathcal{I}}}(H'_X(x), H'_X(z)) \arrow[r, "\eta^*"] \arrow[d, "i_*"] & \mathrm{Hom}_{\widehat{\mathcal{I}}}(W(-), H'_X(z)) \arrow[d, "i_*"] \\
\mathrm{Hom}_{\mathrm{Funct}(\mathcal{C}^{\mathrm{op}},\overline{\mathcal{M}})}(iz, ix) \arrow[r, "(H'_{iX})_*"] \arrow[d, "\alpha_*"] & \mathrm{Hom}_{\widehat{\mathcal{I}}}(H'_{iX}(ix), H'_{iX}(iz)) \arrow[r, "\eta'^*"] \arrow[d, "H_{iX}(\alpha)^*"] & \mathrm{Hom}_{\widehat{\mathcal{I}}}(W(-), H'_{iX}(iz)) \arrow[d, "\mathrm{id}"] \\
\mathrm{Hom}_{\mathrm{Funct}(\mathcal{C}^{\mathrm{op}},\overline{\mathcal{M}})}(iz, y) \arrow[r, "(H'_{iX})_*"] & \mathrm{Hom}_{\widehat{\mathcal{I}}}(H'_{iX}(y), H'_{iX}(iz)) \arrow[r, "\mu^*"] & \mathrm{Hom}_{\widehat{\mathcal{I}}}(W(-), H'_{iX}(iz))
\end{tikzcd}$$

[1] It is a consequence of theorem 5.6.1 that this condition is always verified, see corollary 5.6.3.

Here the top vertical arrows are the isomorphisms which arise from the fully faithfulness of i. The composition of the bottom horizontal arrows is an equivalence, since μ presents y as left adjoint to H'_{iX} at W. The composition of the right vertical arrows is also an equivalence, since both arrows are an equivalence.

The composition of the top horizontal arrows is an equivalence if and only if $X^{\triangleleft}$ is a weighted limit diagram. By the commutativity of the outer square, this happens if and only if α_* is an isomorphism. The Yoneda lemma implies that this happens if and only if α is an isomorphism. This is equivalent to (ix, η') being a weighted limit, as desired. $\square$

Lemma 5.6.7. *Let $\mathcal{M}$ be a presentable symmetric monoidal category. Let $\mathcal{I}$ be an $\mathcal{M}$-enriched category and let W be a presheaf on $\mathcal{I}$. Let $G, G' : \mathcal{C} \to \mathcal{D}$ be functors of $\mathcal{M}$-enriched categories, and let $\mu : G \to G'$ be a natural transformation. Assume that G and G' preserve W-weighted colimits. Then the space of objects x in $\mathcal{C}$ such that $\mu(x)$ is an isomorphism, is closed under W-weighted colimits.*

Proof. Let $\mathcal{C}'$ be the full subcategory of $\mathcal{C}$ on those objects x for which $\mu(x)$ is an isomorphism. Let $X : \mathcal{I} \to \mathcal{C}$ be a functor which factors through $\mathcal{C}'$, and assume that X admits a W-weighted colimit, corresponding to an object x in $\mathcal{C}$ and a natural transformation $\eta : W \to H_X(x)$. We need to show that x belongs to $\mathcal{C}'$.

Let y be an object in $\mathcal{D}$. Let $\mathcal{P}(\mathcal{I}) = \mathrm{Funct}(\mathcal{I}^{\mathrm{op}}, \overline{\mathcal{M}})$ and consider the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}(G'(x), y) \arrow[rr, "\mu(x)^*"] \arrow[d, "(H_{G'X})_*"] \arrow[dr, "(H_{GX})_*"] & & \mathrm{Hom}_{\mathcal{D}}(G(x), y) \arrow[dd, "(H_{GX})_*"] \\
& \mathrm{Hom}_{\mathcal{P}(\mathcal{I})}(H_{GX}(G'(x)), H_{GX}(y)) \arrow[dr, "H_{GX}(\mu(x))^*"] \arrow[dd, "(\mu^*)^*"] & \\
\mathrm{Hom}_{\mathcal{P}(\mathcal{I})}(H_{G'X}(G'(x)), H_{G'X}(y)) \arrow[dr, "(\mu^*)_*"] \arrow[dd, "(G'_*)^*"] & & \mathrm{Hom}_{\mathcal{P}(\mathcal{I})}(H_{GX}(G(x)), H_{GX}(y)) \arrow[dd, "(G_*)^*"] \\
& \mathrm{Hom}_{\mathcal{P}(\mathcal{I})}(H_{G'X}(G'(x)), H_{GX}(y)) \arrow[dr, "(G'_*)^*"] & \\
\mathrm{Hom}_{\mathcal{P}(\mathcal{I})}(H_X(x), H_{G'X}(y)) \arrow[rr, "(\mu^*)_*"] \arrow[d, "\eta^*"] & & \mathrm{Hom}_{\mathcal{P}(\mathcal{I})}(H_X(x), H_{GX}(y)) \arrow[d, "\eta^*"] \\
\mathrm{Hom}_{\mathcal{P}(\mathcal{I})}(W, H_{G'X}(y)) \arrow[rr, "(\mu^*)_*"] & & \mathrm{Hom}_{\mathcal{P}(\mathcal{I})}(W, H_{GX}(y))
\end{tikzcd}$$

Here the diagonal squares commute thanks to the naturality of μ. The composition of the left vertical arrows is an isomorphism since G' preserves W-weighted colimits. Similarly, the composition of the right vertical arrows is an isomorphism since G preserves W-weighted colimits. The bottom horizontal arrow is an isomorphism since X factors through $\mathcal{C}'$. We conclude that the top horizontal arrow is an isomorphism. Since this happens for all y in $\mathcal{C}$, we conclude that $\mu(x)$ is an isomorphism, as desired. □

Lemma 5.6.8. *Let $\mathcal{M}$ be a presentable symmetric monoidal category, and let $\mathcal{C}, \mathcal{D}$ be two $\mathcal{M}$-enriched categories. Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors, and let $\mu : \mathrm{id}_{\mathcal{C}} \to GF$ be a natural transformation. Let $\mathcal{I}$ be an $\mathcal{M}$-enriched category and W be a presheaf on $\mathcal{I}$. Assume that $\mathcal{C}$ admits all W-weighted colimits and that F preserves W-weighted colimits. Then the space of objects x in $\mathcal{C}$ such that $\mu(x)$ presents $F(x)$ as left adjoint to G at x, is closed under W-weighted colimits.*

Proof. Let $\mathcal{C}'$ be the full subcategory of $\mathcal{C}$ on those objects where the left adjoint to G is defined, and let $\mathcal{C}''$ be the full subcategory of $\mathcal{C}$ on those objects x such that $\eta(x)$ presents $F(x)$ as left adjoint to G at x. Note that $\mathcal{C}''$ is contained in $\mathcal{C}'$. Let $\alpha : G^L \to F|_{\mathcal{C}'}$ be the unique natural transformation equipped with an identification $\eta^* G_*(\alpha) = \mu$. Then for each x in $\mathcal{C}'$ we have that $\alpha(x)$ is an isomorphism if and only if x belongs to $\mathcal{C}''$. The lemma now follows from a combination of lemma 5.6.7 and corollary 5.5.25. □

Proof of theorem 5.6.1. We first show that the existence of conical colimits and copowers implies the existence of all weighted colimits. Assume that $\mathcal{C}$ admits conical colimits and copowers. Let $\mathcal{I}$ be an $\mathcal{M}$-enriched category and let $X : \mathcal{I} \to \mathcal{C}$ be a functor. We need to show that H_X admits a left adjoint. Thanks to example 5.5.18, we have that H_X admits a left adjoint at every representable presheaf. Using lemma 5.6.4 and proposition 5.5.24, we reduce to showing that the closure of the representable presheaves under conical colimits and copowers is the entire $\mathrm{Funct}(\mathcal{I}^{\mathrm{op}}, \overline{\mathcal{M}})$. This follows from a combination of the Yoneda lemma and lemma 5.6.5.

We now prove item (i). It follows from propositions 5.4.8 and 5.5.27 that $\mathrm{Funct}(\mathcal{C}, \overline{\mathcal{M}})^{\mathrm{op}}$ admits all conical colimits and copowers. By the above, we also know that it has all weighted colimits. Applying proposition 5.4.9 and lemma 5.6.6 we reduce to showing that W-weighted colimit diagrams and conical colimit diagrams agree when the target is $\mathrm{Funct}(\mathcal{C}, \overline{\mathcal{M}})^{\mathrm{op}}$. Using remark 5.5.19 and corollary 5.3.7, we reduce to the case when $\mathcal{M} = \mathrm{Spc}$. This is example 5.5.17.

Item (ii) now follows, since we have already showed that the existence of conical colimits and copowers implies the existence of weighted colimits.

It remains to prove item (iii). Let $\xi : \mathrm{Funct}(\mathcal{I}^{\mathrm{op}}, \overline{\mathcal{M}}) \to \mathcal{C}$ be the left adjoint to H_X, and let η be the unit of the adjunction. Consider the composite natural transformation

$$\mu : \mathrm{id}_{\mathrm{Funct}(\mathcal{C}^{\mathrm{op}}, \overline{\mathcal{M}})} \xrightarrow{\eta} H_X \xi \xrightarrow{G_* \circ \mathrm{id}_\xi} H_{GX} G \xi.$$

To show that G preserves weighted colimits we need to show that μ is the unit of an adjunction between $G\xi$ and H_{GX}.

Since ξ is a left adjoint, we have that it preserves all weighted colimits that exist in $\mathrm{Funct}(\mathcal{I}^{\mathrm{op}}, \overline{\mathcal{M}})$, thanks to corollary 5.5.26. This implies in particular that ξ preserves all conical colimits and copowers. Since G preserves all conical colimits and copowers, we have that $G\xi$ preserves all conical colimits and copowers.

It follows from the discussion in example 5.5.21 that G preserves colimits weighted by representable presheaves, and hence $\mu(W)$ presents $G\xi(W)$ as left adjoint to H_{GX} at W for every representable W. By lemma 5.6.8 we have that this is also the case for W in the closure of the representable presheaves under conical colimits and copowers. Our claim now follows from another application of lemma 5.6.5. $\square$

Chapter 6

Enriched higher algebra

Let $\mathcal{M}$ be a symmetric monoidal category. An $\mathcal{M}$-enriched pre-prop $\mathcal{P}$ consists of:

- A space of objects P.
- For every pair $\{x_s\}_{s\in S}$, $\{y_t\}_{t\in T}$ of finite families of elements of P, an object

$$\mathrm{Hom}_{\mathcal{P}}(\{x_s\}_{s\in S}, \{y_t\}_{t\in T})$$

  in $\mathcal{M}$ of operations in $\mathcal{P}$ with source $\{x_s\}_{s\in S}$ and target $\{y_t\}_{t\in T}$.
- For every triple $\{x_s\}_{s\in S}$, $\{y_t\}_{t\in T}$, $\{z_u\}_{u\in U}$ of finite families of elements of P, a composition map

$$\mathrm{Hom}_{\mathcal{P}}(\{x_s\}_{s\in S}, \{y_t\}_{t\in T}) \otimes \mathrm{Hom}_{\mathcal{P}}(\{y_t\}_{t\in T}, \{z_u\}_{u\in U}) \to \mathrm{Hom}_{\mathcal{P}}(\{x_s\}_{s\in S}, \{z_u\}_{u\in U})$$

- For every object x in P, a unit map $1_{\mathcal{M}} \to \mathrm{Hom}_{\mathcal{P}}(x, x)$.
- For every quadruple $X = \{x_s\}_{s\in S}$, $Y = \{y_t\}_{t\in T}$, $Z = \{z_u\}_{u\in U}$, $W = \{w_v\}_{v\in V}$ of finite families of elements of P, a stacking map

$$\mathrm{Hom}_{\mathcal{P}}(X, Y) \otimes \mathrm{Hom}_{\mathcal{P}}(Z, W) \to \mathrm{Hom}_{\mathcal{P}}(X \cup Z, Y \cup W)$$

- Isomorphisms witnessing unitality and associativity of composition, compatibility with stacking, and an infinite family of higher coherence data.

An $\mathcal{M}$-enriched pre-prop $\mathcal{P}$ has an underlying $\mathcal{M}$-enriched algebroid, whose morphisms are operations in $\mathcal{P}$ with single source and target. We say that $\mathcal{P}$ is an $\mathcal{M}$-enriched prop if its underlying $\mathcal{M}$-enriched algebroid is an $\mathcal{M}$-enriched category. An $\mathcal{M}$-enriched operad is an $\mathcal{M}$-enriched prop $\mathcal{P}$ satisfying an extra condition, which roughly speaking states that arbitrary operations are determined by single target operations.

The goal of this chapter is to discuss a number of topics in enriched higher algebra, and to provide a way of making the above notions precise.

We begin in 6.1 with a preliminary discussion on the theory of cartesian monoidal categories over an operad. This is a generalization of the theory of cartesian symmetric monoidal categories from [Lur17] section 2.4.1. Recall that each category with finite products carries a unique cartesian symmetric monoidal structure. We prove here a generalization of this fact: for any operad $\mathcal{O}$, there is an equivalence between the category of cartesian $\mathcal{O}$-symmetric monoidal categories and the category of cartesian $(\mathcal{O} \otimes \mathrm{Comm})$-symmetric monoidal categories.

In 6.2 we introduce, for a cartesian symmetric monoidal category $\mathcal{M}$, a notion of cartesian $\mathcal{O}$-monoidal $\mathcal{M}$-enriched category, and provide an enriched generalization of the results from 6.1. In particular, this provides an ample source of cartesian symmetric monoidal $\mathcal{M}$-enriched categories: any $\mathcal{M}$-enriched category with finite conical products carries a unique such structure.

In 6.3 we discuss the canonical enrichment on the category of $\mathcal{O}$-algebras on a cartesian symmetric monoidal category $\mathcal{C}$ enriched over a cartesian closed presentable category $\mathcal{M}$. We define this as an enriched analogue of the category of $\mathcal{O}$-monoids from [Lur17] section 2.4.2. We prove here two basic results that allow one to understand enriched categories of algebras over some simple operads. As a particular case of this theory, we are able to define an $\mathcal{M}$-enriched 2-category of $\mathcal{O}$-monoidal $\mathcal{M}$-enriched categories. We prove that when $\mathcal{M} = \mathrm{Spc}$ this agrees with the 2-category of $\mathcal{O}$-monoidal categories defined as a subcategory of the 2-category of categories over the category of operators of $\mathcal{O}$.

In 6.4 we discuss the notion of prop enriched over a presentable symmetric monoidal category $\mathcal{M}$. We show that the category of symmetric monoidal $\mathcal{M}$-enriched categories is a subcategory of the category of $\mathcal{M}$-enriched props, and that this inclusion admits a left adjoint, which we think about as sending each $\mathcal{M}$-enriched prop to its symmetric monoidal envelope.

In 6.5 we discuss the notion of operad enriched over a presentable symmetric monoidal category $\mathcal{M}$. We show that every enriched operad admits an universal enveloping enriched prop, and that every enriched prop has an underlying enriched operad. We show that the category of $\mathcal{M}$-enriched operads contains the category of $\mathcal{M}$-enriched symmetric monoidal categories as a subcategory. We finish by proving that our approach recovers in the case $\mathcal{M} = \mathrm{Spc}$ the usual notion of operad.

6.1 Cartesian $\mathcal{O}$-monoidal categories

We begin by introducing the notion of cartesian $\mathcal{O}$-monoidal category for an arbitrary operad $\mathcal{O}$.

Notation 6.1.1. Denote by $\mathrm{Cat}_{\text{fin prod}}$ the subcategory of Cat on the categories with finite products, and finite product preserving functors. We equip $\mathrm{Cat}_{\text{fin prod}}$ with its cartesian symmetric monoidal structure, and the inclusion $\mathrm{Cat}_{\text{fin prod}} \to \mathrm{Cat}$ with its unique symmetric monoidal structure.

Definition 6.1.2. *Let $\mathcal{O}$ be an operad and let $\mathcal{E}$ be an $\mathcal{O}$-monoidal category. We say that $\mathcal{E}$ is cartesian if the associated morphism of operads $\mathcal{O} \to \mathrm{Cat}$ factors through $\mathrm{Cat}_{\text{fin prod}}$. We call $\mathrm{Alg}_{\mathcal{O}}(\mathrm{Cat}_{\text{fin prod}})$ the category of cartesian $\mathcal{O}$-operads.*

Remark 6.1.3. Let $\mathcal{O}$ be an operad and let $\mathcal{E}$ be an $\mathcal{O}$-monoidal category. Unwinding the definitions, we see that $\mathcal{E}$ is cartesian if and only if the following two conditions hold:

- For every object x in $\mathcal{O}$ the category $\mathcal{E}(x)$ admits finite products.
- For every operation in $\mathcal{O}$ with source $\{x_s\}_{s\in S}$ and target x, the induced functor
$$\prod_{s\in S} \mathcal{E}(x_s) \to \mathcal{E}(x)$$
preserves finite products.

Assume now that $\mathcal{E}$ is cartesian, and let $F : \mathcal{E} \to \mathcal{E}'$ be a morphism of $\mathcal{O}$-monoidal categories, with $\mathcal{E}'$ also cartesian. Then F defines a morphism in $\mathrm{Alg}_{\mathcal{O}}(\mathrm{Cat}_{\text{fin prod}})$ if and only if for every object x in $\mathcal{O}$ the induced functor $\mathcal{E}(x) \to \mathcal{E}'(x)$ preserves finite products.

The following proposition singles out a minimalistic collection of products that need to be preserved for an $\mathcal{O}$-monoidal category to be cartesian.

Definition 6.1.4. *Let $\mathcal{O}$ be an operad. A collection of operations $\{\mu_j\}_{j\in J}$ of $\mathcal{O}$ is said to be dense if its closure under compositions (and identities) is the entire collection of operations of $\mathcal{O}$.*

Proposition 6.1.5. *Let $\mathcal{O}$ be an operad and let $A = \{\mu_j\}_{j\in J}$ be a dense collection of operations in $\mathcal{O}$. Let $\mathcal{E}$ be an $\mathcal{O}$-monoidal category. Then $\mathcal{E}$ is cartesian if and only if the following conditions hold:*

- *For every object x in $\mathcal{O}$ the category $\mathcal{E}(x)$ admits finite products.*
- *For every operation μ in A with source $\{x_s\}_{s\in S}$ and target x, the induced functor*
$$\mu_* : \prod_{s\in S} \mathcal{E}(x_s) \to \mathcal{E}(x)$$
preserves terminal objects.
- *Let μ be an operation in A with source $\{x_s\}_{s\in S}$ and target x. Let s_0 be an index in S and denote by by $i_{s_0} : \mathcal{E}(x_{s_0}) \to \prod_{s\in S} \mathcal{E}(x_s)$ the functor induced from the identity $\mathcal{E}(x_{s_0}) \to \mathcal{E}(x_{s_0})$ and the terminal maps $\mathcal{E}(x_{s_0}) \to \mathcal{E}(x_s)$ for $s \neq s_0$. Then the composite functor*
$$\mathcal{E}(x_{s_0}) \xrightarrow{i_{s_0}} \prod_{s\in S} \mathcal{E}(x_s) \xrightarrow{\mu_*} \mathcal{E}(x)$$
preserves binary products.

- *Let μ be an operation in A with source $\{x_s\}_{s\in S}$ and target x, and let $e = \{e_s\}_{s\in S}$ be a family of objects with e_s in $\mathcal{E}(x_s)$. For each i in S let $e^{(i)} = \{e_s^{(i)}\}_{s\in S}$ be the family of objects given by $e_i^{(i)} = e_i$ and $e_s^{(i)} = 1_{\mathcal{E}(x_s)}$ otherwise. Then the projection maps $e \to e^{(i)}$ induce an isomorphism*
$$\mu(e) = \prod_{i\in S} \mu(e^{(i)}).$$

Proof. It follows from remark 6.1.3 that if $\mathcal{E}$ is cartesian then the four conditions in the statement hold. Assume now that the four conditions above hold. By virtue of remark 6.1.3, we need to show that for every operation μ with source $\{x_s\}_{s\in S}$ and target x the induced functor
$$\mu_* : \prod_{s\in S} \mathcal{E}(x_s) \to \mathcal{E}(x)$$
preserves finite products. Since A is dense, it suffices to assume μ belongs to A. The second item in the statement guarantees that this functor preserves final objects, so it remains to show that it preserves binary products.

Let $e = \{e_s\}_{s\in S}$ and $e' = \{e'_s\}_{s\in S}$ be two objects in $\prod_{s\in S} \mathcal{E}(x_s)$, and let $e^{(i)}, e'^{(i)}$ be as in the statement. Then for each i in S we have a commutative diagram

$$\begin{tikzcd}
\mu(e) \arrow[d] & \mu(e\times e') \arrow[l] \arrow[r] \arrow[d] & \mu(e') \arrow[d] \\
\mu(e^{(i)}) & \mu(e^{(i)} \times e'^{(i)}) \arrow[l] \arrow[r] & \mu(e'^{(i)}).
\end{tikzcd}$$

Using the fourth condition in the statement we see that as we range over all i in S, the vertical arrows present the upper row as the product of the lower rows. Our claim now follows from the fact that the lower row presents $\mu(e^{(i)} \times e'^{(i)})$ as the product of $\mu(e^{(i)})$ and $\mu(e'^{(i)})$, which is itself a consequence of the third condition in the statement. $\square$

Example 6.1.6. The operad Comm admits a dense collection of operations consisting of the unique operations with arity 0 and 2. It follows that a symmetric monoidal category $\mathcal{E}$ is a cartesian Comm-monoidal category if and only if the following two conditions hold:

- The unit object of $\mathcal{E}$ is final.
- Let e, e' be two objects of $\mathcal{E}$. Then the maps
$$e = e \otimes 1_{\mathcal{E}} \xleftarrow{\mathrm{id}_e \otimes \pi_{e'}} e \otimes e' \xrightarrow{\pi_e \otimes \mathrm{id}_{e'}} 1_{\mathcal{E}} \otimes e' = e'$$
present $e \otimes e'$ as the product of e and e'.

In other words, $\mathcal{E}$ is cartesian in the sense of definition 6.1.2 if and only if its symmetric monoidal structure is cartesian.

Example 6.1.7. The operad CAlgMod governing pairs of a commutative algebra and a left module over it admits a dense collection of operations consisting of the two operations of arity 2, and the unique operation of arity 0.

Let $\mathcal{E}$ be a symmetric monoidal category and let $\mathcal{C}$ be a category tensored over $\mathcal{E}$, so that the pair $(\mathcal{E}, \mathcal{C})$ defines a monoidal category over the operad CAlgMod. Then it follows from proposition 6.1.5 that $(\mathcal{E}, \mathcal{C})$ is cartesian if and only if the following conditions hold:

- The symmetric monoidal structure on $\mathcal{E}$ is cartesian.
- The category $\mathcal{C}$ admits finite products.
- For every pair of objects e in $\mathcal{E}$ and c in $\mathcal{C}$ the maps
$$e \otimes 1_{\mathcal{C}} \xleftarrow{\mathrm{id}_e \otimes \pi_c} e \otimes c \xrightarrow{\pi_e \otimes \mathrm{id}_c} 1_{\mathcal{E}} \otimes c = c$$
present $e \otimes c$ as the product of $e \otimes 1_{\mathcal{C}}$ and c.
- The map $- \otimes 1_{\mathcal{C}} : \mathcal{E} \to \mathcal{C}$ preserves finite products.

It turns out in fact that the fourth condition is a consequence of the first three. Indeed, if e, e' are objects of $\mathcal{E}$, then the pair of morphisms
$$(e \times 1_{\mathcal{E}}) \otimes 1_{\mathcal{C}} \xleftarrow{(\mathrm{id}_e \times \pi_{e'}) \otimes \mathrm{id}_{1_{\mathcal{C}}}} (e \times e') \otimes 1_{\mathcal{C}} \xrightarrow{(\pi_e \times \mathrm{id}_{e'}) \otimes \mathrm{id}_{1_{\mathcal{C}}}} (1_{\mathcal{E}} \times e') \otimes 1_{\mathcal{C}}$$
is equivalent to
$$e \otimes 1_{\mathcal{C}} \xleftarrow{\mathrm{id}_e \otimes \pi_{e'} \otimes 1_{\mathcal{C}}} e \otimes (e' \otimes 1_{\mathcal{C}}) \xrightarrow{\pi_e \otimes \mathrm{id}_{e'} \otimes 1_{\mathcal{C}}} e' \otimes 1_{\mathcal{C}}$$
which is a product diagram thanks to the third condition.

Recall from [Lur17] corollary 2.4.1.9 that the category $\mathrm{CAlg}(\mathrm{Cat}_{\text{fin prod}})$ of cartesian symmetric monoidal categories is equivalent to $\mathrm{Cat}_{\text{fin prod}}$. The following proposition provides a generalization of this fact.

Proposition 6.1.8. *Let $\mathcal{O}$ be an operad. Then restriction along the morphism of operads $\mathcal{O} = \mathcal{O} \otimes [0] \to \mathcal{O} \otimes \mathrm{Comm}$ induces an equivalence of categories*
$$\mathrm{Alg}_{\mathcal{O}}(\mathrm{Cat}_{\text{fin prod}}) = \mathrm{Alg}_{\mathcal{O} \otimes \mathrm{Comm}}(\mathrm{Cat}_{\text{fin prod}}).$$

Proof. Recall that $\mathrm{Cat}_{\text{fin prod}}$ is equivalent to the full subcategory $\mathrm{CAlg}_{\mathrm{cart}}(\mathrm{Cat})$ of $\mathrm{CAlg}(\mathrm{Cat})$ on the cartesian symmetric monoidal categories. Since the cartesian symmetric monoidal structure on $\mathrm{CAlg}(\mathrm{Cat})$ is also cocartesian, we conclude that the same holds for $\mathrm{Cat}_{\text{fin prod}}$. Therefore precomposition with $[0] \to \mathrm{Comm}$ induces an equivalence of symmetric monoidal categories
$$\mathrm{Alg}_{\mathrm{Comm}}(\mathrm{Cat}_{\text{fin prod}}) = \mathrm{Alg}_{[0]}(\mathrm{Cat}_{\text{fin prod}}) = \mathrm{Cat}_{\text{fin prod}}\,.$$
The result now follows from the above by passing to categories of $\mathcal{O}$-algebras. $\square$

Corollary 6.1.9. *Let $\infty \geq n \geq 0$. Then precomposition with the unique map $[0] \to E_n$ induces an equivalence*
$$\mathrm{Alg}_{E_n}(\mathrm{Cat}_{\text{fin prod}}) = \mathrm{Cat}_{\text{fin prod}}\,.$$

Proof. Consider the commutative square of operads

$$\begin{tikzcd}
{[0]} \arrow[r] \arrow[d] & {[0] \otimes \mathrm{Comm}} \arrow[d] \\
E_n \arrow[r] & E_n \otimes \mathrm{Comm}
\end{tikzcd}$$

obtained by tensoring the unique map $[0] \to E_n$ with the unique map $[0] \to \mathrm{Comm}$. This induces a commutative square of categories

$$\begin{tikzcd}
\mathrm{Alg}_{E_n}(\mathrm{Cat}_{\text{fin prod}}) \arrow[r] \arrow[d] & \mathrm{Alg}_{E_n\otimes\mathrm{Comm}}(\mathrm{Cat}_{\text{fin prod}}) \arrow[d] \\
\mathrm{Alg}_{[0]}(\mathrm{Cat}_{\text{fin prod}}) \arrow[r] & \mathrm{Alg}_{[0]\otimes\mathrm{Comm}}(\mathrm{Cat}_{\text{fin prod}}).
\end{tikzcd}$$

It follows from proposition 6.1.8 that the horizontal arrows are equivalences. Since $E_n \otimes \mathrm{Comm} = [0] \otimes \mathrm{Comm} = \mathrm{Comm}$ we have that the right vertical arrow is an equivalence as well, and the result follows. □

Corollary 6.1.10. *Let $\infty \geq n \geq 1$ and let $\mathcal{O} = \mathrm{LMod} \cup_{\mathrm{Assos}} E_n$ be the operad governing pairs of an E_n-algebra and a left module over it. Then there is an equivalence*

$$\mathrm{Alg}_{\mathcal{O}}(\mathrm{Cat}_{\text{fin prod}}) = \mathrm{Funct}([1], \mathrm{Cat}_{\text{fin prod}}).$$

Proof. Note that we have equivalences of operads

$$\begin{aligned}
\mathcal{O} \otimes \mathrm{Comm} &= (\mathrm{LMod} \otimes \mathrm{Comm}) \cup_{\mathrm{Assos}\otimes\mathrm{Comm}} (E_n \otimes \mathrm{Comm}) \\
&= (\mathrm{LMod} \otimes \mathrm{Comm}) \cup_{\mathrm{Comm}} \mathrm{Comm} \\
&= \mathrm{LMod} \otimes \mathrm{Comm}\,.
\end{aligned}$$

The result now follows from proposition 6.1.8 using the fact that $\mathrm{LMod} \otimes \mathrm{Comm}$ and $[1] \otimes \mathrm{Comm}$ are equivalent. □

Remark 6.1.11. Let $\infty \geq n \geq 1$ and let $\mathcal{O} = \mathrm{LMod} \cup_{\mathrm{Assos}} E_n$ be the operad governing pairs of an E_n-algebra and a left module over it. Inspecting the proof of 6.1.10 reveals that the equivalence between cartesian $\mathcal{O}$-monoidal categories and morphisms in $\mathrm{Cat}_{\text{fin prod}}$ is such that:

- To each pair $(\mathcal{E}, \mathcal{C})$ of a cartesian E_n-monoidal category $\mathcal{E}$ and a cartesian module $\mathcal{C}$ over it, it assigns the finite product preserving functor

$$- \otimes 1_{\mathcal{C}} : \mathcal{E} \to \mathcal{C}.$$

- To each product preserving functor $F : \mathcal{E} \to \mathcal{C}$ between categories with finite products, it assigns the pair $(\mathcal{E}, \mathcal{C})$, where $\mathcal{C}$ is seen as an $\mathcal{E}$-module by restriction of scalars along F.

6.2 Cartesian $\mathcal{O}$-monoidal enriched categories

We now review the notion of $\mathcal{O}$-monoidal enriched category.

Definition 6.2.1. *Let $\mathcal{M}$ be a symmetric monoidal category and let $\mathcal{O}$ be an operad. An $\mathcal{O}$-monoidal $\mathcal{M}$-enriched category is an $\mathcal{O}$-algebra in the symmetric monoidal category $\mathrm{Cat}^{\mathcal{M}}$. We call $\mathrm{Alg}_{\mathcal{O}}(\mathrm{Cat}^{\mathcal{M}})$ the category of $\mathcal{O}$-monoidal $\mathcal{M}$-enriched categories.*

Given an $\mathcal{O}$-monoidal $\mathcal{M}$-enriched category $\mathcal{C}$, its image under the functor

$$(\tau_{\mathcal{M}})_! : \mathrm{Alg}_{\mathcal{O}}(\mathrm{Cat}^{\mathcal{M}}) \to \mathrm{Alg}_{\mathcal{O}}(\mathrm{Cat})$$

induced from $\tau_{\mathcal{M}}$ is called the $\mathcal{O}$-monoidal category underlying $\mathcal{C}$. Given an $\mathcal{O}$-operad $\mathcal{O}'$, we define the category of $\mathcal{O}'$-algebras in $\mathcal{C}$ to be the category $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}((\tau_{\mathcal{M}})_!)\mathcal{C})$ of $\mathcal{O}'$-algebras in the $\mathcal{O}$-monoidal category underlying $\mathcal{C}$.

Remark 6.2.2. Let $\mathcal{M}$ be a symmetric monoidal category. Specializing definition 6.2.1 to the case $\mathcal{O} = \mathrm{Comm}$ (resp. $\mathcal{O} = \mathrm{Assos}$) we obtain categories of symmetric monoidal (resp. monoidal) $\mathcal{M}$-enriched categories, and commutative algebras (resp. associative algebras) in those.

Example 6.2.3. Let $\mathcal{M}$ be a presentable symmetric monoidal category. Let $\mathcal{O}$ be an operad and let $\mathcal{C}$ be an $\mathcal{O}$-algebra in $\mathcal{M}\text{-mod}(\mathrm{Pr}^L)$. Composing with the lax symmetric monoidal functor

$$\theta_{\mathcal{M}} : \mathcal{M}\text{-mod}(\mathrm{Pr}^L) \to \widehat{\mathrm{Cat}}^{\mathcal{M}}$$

we obtain an $\mathcal{O}$-monoidal $\mathcal{M}$-enriched category $\theta_{\mathcal{M}}(\mathcal{C})$, whose underlying $\mathcal{O}$-monoidal category is equivalent to the image of $\mathcal{C}$ under the lax symmetric monoidal forgetful functor $\mathcal{M}\text{-mod}(\mathrm{Pr}^L) \to \widehat{\mathrm{Cat}}$.

In particular, taking $\mathcal{C}$ to be the unit commutative algebra in $\mathcal{M}\text{-mod}(\mathrm{Pr}^L)$ we obtain a symmetric monoidal $\mathcal{M}$-enriched category $\overline{\mathcal{M}}$ whose underlying symmetric monoidal category is $\mathcal{M}$.

We now specialize to the case when $\mathcal{M}$ is cartesian symmetric monoidal. In this case, $\mathcal{M}$-enriched categories with conical finite products provide examples of symmetric monoidal $\mathcal{M}$-enriched categories.

Definition 6.2.4. *Let $\mathcal{M}$ be cartesian symmetric monoidal category. Let $\mathcal{C}$ be a symmetric monoidal $\mathcal{M}$-enriched category. We say that $\mathcal{C}$ is cartesian if the symmetric monoidal category underlying $\mathcal{C}$ is cartesian, and $\mathcal{C}$ admits all conical finite products.*

Example 6.2.5. Let $\mathcal{M}$ be a cartesian closed presentable category, equipped with its cartesian symmetric monoidal structure. It follows from proposition 5.4.8 that the symmetric monoidal $\mathcal{M}$-enriched category $\overline{\mathcal{M}}$ is cartesian.

Remark 6.2.6. Let $\mathcal{M}$ be a cartesian symmetric monoidal category and let $\mathcal{C}$ be a symmetric monoidal $\mathcal{M}$-enriched category. We denote by $1_{\mathcal{C}}$ the unit object of $\mathcal{C}$, and by $-\otimes- : \mathcal{C}\otimes\mathcal{C} \to \mathcal{C}$ the tensoring map. Then $\mathcal{C}$ is cartesian if and only if the following two conditions hold:

- For every object c in $\mathcal{C}$ we have that $\operatorname{Hom}_{\mathcal{C}}(c, 1_{\mathcal{C}})$ is a final object of $\mathcal{M}$. In other words, $1_{\mathcal{C}}$ is a conical final object of $\mathcal{C}$.
- Let c, d in $\mathcal{C}$ be a pair of objects of $\mathcal{C}$, and denote by $\pi_c : c \to 1_{\mathcal{C}}$ and $\pi_d : d \to 1_{\mathcal{C}}$ the unique maps. Then the morphisms
$$c = c \otimes 1_{\mathcal{C}} \xleftarrow{\mathrm{id}_c \otimes \pi_d} c \otimes d \xrightarrow{\pi_c \otimes \mathrm{id}_d} 1_{\mathcal{C}} \otimes d = d$$
exhibit $c \otimes d$ as a conical product of c and d.

Notation 6.2.7. Let $\mathcal{M}$ be a cartesian symmetric monoidal category. We denote by $\operatorname{CAlg}_{\mathrm{cart}}(\mathrm{Cat}^{\mathcal{M}})$ the full subcategory of $\operatorname{CAlg}(\mathrm{Cat}^{\mathcal{M}})$ on the cartesian symmetric monoidal $\mathcal{M}$-enriched categories. We denote by $(\mathrm{Cat}^{\mathcal{M}})_{\text{fin prod}}$ the subcategory of $\mathrm{Cat}^{\mathcal{M}}$ on those $\mathcal{M}$-enriched categories admitting all conical finite products, and conical finite product preserving functors.

Theorem 6.2.8. *Let $\mathcal{M}$ be cartesian symmetric monoidal category. Then the forgetful functor* $\operatorname{CAlg}(\mathrm{Cat}^{\mathcal{M}}) \to \mathrm{Cat}^{\mathcal{M}}$ *restricts to an equivalence*
$$\operatorname{CAlg}_{\mathrm{cart}}(\mathrm{Cat}^{\mathcal{M}}) = (\mathrm{Cat}^{\mathcal{M}})_{\text{fin prod}}.$$

Our proof of theorem 6.2.8 will need some preliminaries.

Construction 6.2.9. Let $\mathcal{M}$ be a cartesian symmetric monoidal category. Consider the functor
$$H : \mathcal{M}^{\mathrm{op}} \to \operatorname{Funct}(\mathcal{M}, \mathrm{Spc})$$
induced from the Hom functor $\operatorname{Hom}_{\mathcal{M}}(-,-) : \mathcal{M}^{\mathrm{op}} \times \mathcal{M} \to \mathrm{Spc}$. We have that H factors through the full subcategory $\operatorname{Funct}^{\times}(\mathcal{M}, \mathrm{Spc})$ of $\operatorname{Funct}(\mathcal{M}, \mathrm{Spc})$ on the finite product preserving functors.

Let $\mathcal{O}$ be an operad, and consider the composite functor
$$\operatorname{Alg}_{\mathcal{O}}(\mathcal{M}) \times \mathcal{M}^{\mathrm{op}} \xrightarrow{\mathrm{id} \times H} \operatorname{Alg}_{\mathcal{O}}(\mathcal{M}) \times \operatorname{Funct}^{\times}(\mathcal{M}, \mathrm{Spc}) \to \operatorname{Alg}_{\mathcal{O}}(\mathrm{Spc})$$
where the second arrow is composition. We denote by
$$\iota_{\mathcal{O},\mathcal{M}} : \operatorname{Alg}_{\mathcal{O}}(\mathcal{M}) \to \operatorname{Funct}(\mathcal{M}^{\mathrm{op}}, \operatorname{Alg}_{\mathcal{O}}(\mathrm{Spc}))$$
the associated functor.

Lemma 6.2.10. *Let $\mathcal{M}$ be a cartesian symmetric monoidal category and let $\mathcal{O}$ be an operad. Then the functor $\iota_{\mathcal{O},\mathcal{M}}$ from construction 6.2.9 is fully faithful.*

Proof. Let $\mathcal{O}^{\otimes}$ be the category of operators of $\mathcal{O}$. Consider the composite functor

$$\mathrm{Funct}(\mathcal{O}^{\otimes}, \mathcal{M}) \times \mathcal{M}^{\mathrm{op}} \xrightarrow{\mathrm{id} \times H} \mathrm{Funct}(\mathcal{O}^{\otimes}, \mathcal{M}) \times \mathrm{Funct}(\mathcal{M}, \mathrm{Spc}) \to \mathrm{Funct}(\mathcal{O}^{\otimes}, \mathrm{Spc})$$

where the right arrow is composition. This induces a functor

$$\iota'_{\mathcal{O},\mathcal{M}} : \mathrm{Funct}(\mathcal{O}^{\otimes}, \mathcal{M}) \to \mathrm{Funct}(\mathcal{M}^{\mathrm{op}}, \mathrm{Funct}(\mathcal{O}^{\otimes}, \mathrm{Spc})).$$

Observe that the above is equivalent to the functor

$$\mathrm{Funct}(\mathcal{O}^{\otimes}, \mathcal{M}) \to \mathrm{Funct}(\mathcal{O}^{\otimes}, \mathrm{Funct}(\mathcal{M}^{\mathrm{op}}, \mathrm{Spc}))$$

of composition with the Yoneda embedding $\mathcal{M} \to \mathrm{Funct}(\mathcal{M}^{\mathrm{op}}, \mathrm{Spc})$. In particular, we have that $\iota'_{\mathcal{O},\mathcal{M}}$ is fully faithful.

We have a commutative square of categories

$$\begin{tikzcd}
\mathrm{Alg}_{\mathcal{O}}(\mathcal{M}) \arrow[r, "\iota_{\mathcal{O},\mathcal{M}}"] \arrow[d] & \mathrm{Funct}(\mathcal{M}^{\mathrm{op}}, \mathrm{Alg}_{\mathcal{O}}(\mathrm{Spc})) \arrow[d] \\
\mathrm{Funct}(\mathcal{O}^{\otimes}, \mathcal{M}) \arrow[r, "\iota'_{\mathcal{O},\mathcal{M}}"] & \mathrm{Funct}(\mathcal{M}^{\mathrm{op}}, \mathrm{Funct}(\mathcal{O}^{\otimes}, \mathrm{Spc}))
\end{tikzcd}$$

where the left vertical arrow is the inclusion, and the right vertical arrow is induced from the inclusion of $\mathrm{Alg}_{\mathcal{O}}(\mathrm{Spc})$ inside $\mathrm{Funct}(\mathcal{O}^{\otimes}, \mathrm{Spc})$. Our result now follows from the fact that both vertical arrows and the bottom horizontal arrow are fully faithful. □

Construction 6.2.11. Let $\mathcal{M}$ be a cartesian symmetric monoidal category. Consider the composite functor

$$\mathrm{Algbrd}(\mathcal{M}) \times \mathcal{M}^{\mathrm{op}} \xrightarrow{\mathrm{id} \times H} \mathrm{Algbrd}(\mathcal{M}) \times \mathrm{Funct}^{\times}(\mathcal{M}, \mathrm{Spc}) \to \mathrm{Algbrd}(\mathrm{Spc})$$

where the second arrow is induced by functoriality of algebroids under change of enrichment. We denote by

$$\iota_{\mathrm{Algbrd},\mathcal{M}} : \mathrm{Algbrd}(\mathcal{M}) \to \mathrm{Funct}(\mathcal{M}^{\mathrm{op}}, \mathrm{Algbrd}(\mathrm{Spc}))$$

the induced functor.

Lemma 6.2.12. *Let $\mathcal{M}$ be a cartesian symmetric monoidal category. Then the functor $\iota_{\mathrm{Algbrd},\mathcal{M}}$ from construction 6.2.11 is fully faithful and preserves finite products.*

Proof. Observe that the functor

$$\mathrm{Algbrd}(\mathcal{M}) \times \mathcal{M}^{\mathrm{op}} \to \mathrm{Algbrd}(\mathrm{Spc})$$

from construction 6.2.11 can be upgraded to a functor of cartesian fibrations over Cat. In particular, we have a commutative square of categories

$$\begin{tikzcd} \mathrm{Algbrd}(\mathcal{M}) \times \mathcal{M}^{\mathrm{op}} \arrow[r] \arrow[d] & \mathrm{Algbrd}(\mathrm{Spc}) \arrow[d] \\ \mathrm{Cat} \times \mathcal{M}^{\mathrm{op}} \arrow[r] & \mathrm{Cat} \end{tikzcd}$$

where the vertical arrows are the projections, and the bottom horizontal arrow is the projection to the first factor. The above induces a commutative square of categories

$$\begin{tikzcd} \mathrm{Algbrd}(\mathcal{M}) \arrow[r, "\iota_{\mathrm{Algbrd},\mathcal{M}}"] \arrow[d] & \mathrm{Funct}(\mathcal{M}^{\mathrm{op}}, \mathrm{Algbrd}(\mathrm{Spc})) \arrow[d] \\ \mathrm{Cat} \arrow[r, "\Delta"] & \mathrm{Funct}(\mathcal{M}^{\mathrm{op}}, \mathrm{Cat}) \end{tikzcd}$$

Since $\mathcal{M}$ admits a final object, we have that $\mathcal{M}^{\mathrm{op}}$ is contractible, and in particular the bottom horizontal arrow is fully faithful. To show that $\iota_{\mathrm{Algbrd},\mathcal{M}}$ is fully faithful it suffices therefore to show that the induced functor

$$\mathrm{Algbrd}(\mathcal{M}) \to \mathrm{Funct}(\mathcal{M}^{\mathrm{op}}, \mathrm{Algbrd}(\mathrm{Spc})) \times_{\mathrm{Funct}(\mathcal{M}^{\mathrm{op}},\mathrm{Cat})} \mathrm{Cat}$$

is fully faithful. The above has the structure of morphism of cartesian fibrations over Cat, so it suffices to show that for every category X the functor

$$\mathrm{Algbrd}_X(\mathcal{M}) \to \mathrm{Funct}(\mathcal{M}^{\mathrm{op}}, \mathrm{Algbrd}(\mathrm{Spc})) \times_{\mathrm{Cat}} \{X\}$$

is fully faithful. The above is equivalent to the functor

$$\iota_{\mathrm{Assos}_X,\mathcal{M}} : \mathrm{Algbrd}_X(\mathcal{M}) \to \mathrm{Funct}(\mathcal{M}^{\mathrm{op}}, \mathrm{Algbrd}_X(\mathrm{Spc}))$$

which is fully faithful thanks to lemma 6.2.10.

It remains to show that $\iota_{\mathrm{Algbrd},\mathcal{M}}$ preserves finite products. It suffices for this to show that for every m in $\mathcal{M}$ the composite functor

$$\mathrm{Algbrd}(\mathcal{M}) \xrightarrow{\iota_{\mathrm{Algbrd},\mathcal{M}}} \mathrm{Funct}(\mathcal{M}^{\mathrm{op}}, \mathrm{Algbrd}(\mathrm{Spc})) \xrightarrow{\mathrm{ev}_m} \mathrm{Algbrd}(\mathrm{Spc})$$

preserves finite products. The above is equivalent to the functor

$$H(m)_* : \mathrm{Algbrd}(\mathcal{M}) \to \mathrm{Algbrd}(\mathrm{Spc})$$

induced from the symmetric monoidal functor $H(m) : \mathcal{M} \to \mathrm{Spc}$. Our claim now follows from the functoriality of the symmetric monoidal structures on categories of algebroids from construction 3.5.6, together with proposition 3.5.8. $\square$

Notation 6.2.13. Let $\mathcal{M}$ be a cartesian symmetric monoidal category. We denote by

$$\iota_{\mathcal{M}} : \mathrm{Cat}^{\mathcal{M}} \to \mathrm{Funct}(\mathcal{M}^{\mathrm{op}}, \mathrm{Cat})$$

the composite functor

$$\mathrm{Cat}^{\mathcal{M}} \hookrightarrow \mathrm{Algbrd}(\mathcal{M})_{\mathrm{Spc}} \xrightarrow{\iota_{\mathrm{Algbrd},\mathcal{M}}} \mathrm{Funct}(\mathcal{M}^{\mathrm{op}}, \mathrm{Algbrd}(\mathrm{Spc})_{\mathrm{Spc}}) \xrightarrow{L_*} \mathrm{Funct}(\mathcal{M}^{\mathrm{op}}, \mathrm{Cat})$$

where the last arrow is induced by the localization functor $L : \mathrm{Algbrd}(\mathrm{Spc})_{\mathrm{Spc}} \to \mathrm{Cat}$.

Lemma 6.2.14. *Let $\mathcal{M}$ be a cartesian symmetric monoidal category. Then the functor $\iota_{\mathcal{M}}$ from notation 6.2.13 is fully faithful and preserves finite products.*

Proof. It follows from a combination of propositions 3.5.8 and 3.5.9 that the localization functor $L : \mathrm{Algbrd}(\mathrm{Spc})_{\mathrm{Spc}} \to \mathrm{Cat}$ preserves finite products.The fact that $\iota_{\mathcal{M}}$ preserves finite products is a consequence of the fact that $\iota_{\mathrm{Algbrd},\mathcal{M}}$ preserves finite products, together with the fact that $\mathrm{Cat}^{\mathcal{M}}$ is closed under finite products inside $\mathrm{Algbrd}(\mathcal{M})_{\mathrm{Spc}}$.

It remains to show that $\iota_{\mathcal{M}}$ is fully faithful. Let $\mathcal{C}, \mathcal{D}$ be a pair of $\mathcal{M}$-enriched categories. We need to show that the morphism

$$(\iota_{\mathcal{M}})_* : \mathrm{Hom}_{\mathrm{Cat}^{\mathcal{M}}}(\mathcal{C}, \mathcal{D}) \to \mathrm{Hom}_{\mathrm{Funct}(\mathcal{M}^{\mathrm{op}}, \mathrm{Cat})}(\iota_{\mathcal{M}}\mathcal{C}, \iota_{\mathcal{M}}\mathcal{D})$$

is an equivalence. Thanks to lemma 6.2.12, we reduce to showing that the map

$$L_* : \mathrm{Hom}_{\mathrm{Funct}(\mathcal{M}^{\mathrm{op}}, \mathrm{Algbrd}(\mathrm{Spc})_{\mathrm{Spc}})}(\iota_{\mathrm{Algbrd},\mathcal{M}}\mathcal{C}, \iota_{\mathrm{Algbrd},\mathcal{M}}\mathcal{D}) \to \mathrm{Hom}_{\mathrm{Funct}(\mathcal{M}^{\mathrm{op}}, \mathrm{Cat})}(\iota_{\mathcal{M}}\mathcal{C}, \iota_{\mathcal{M}}\mathcal{D})$$

is an equivalence. The above fits into a commutative triangle of spaces

$$\begin{tikzcd}
 & \mathrm{Hom}_{\mathrm{Funct}(\mathcal{M}^{\mathrm{op}}, \mathrm{Cat})}(\iota_{\mathcal{M}}\mathcal{C}, \iota_{\mathcal{M}}\mathcal{D}) \arrow[d] \\
\mathrm{Hom}_{\mathrm{Funct}(\mathcal{M}^{\mathrm{op}}, \mathrm{Algbrd}(\mathrm{Spc})_{\mathrm{Spc}})}(\iota_{\mathrm{Algbrd},\mathcal{M}}\mathcal{C}, \iota_{\mathrm{Algbrd},\mathcal{M}}\mathcal{D}) \arrow[ur, "L_*"] \arrow[r] & \mathrm{Hom}_{\mathrm{Funct}(\mathcal{M}^{\mathrm{op}}, \mathrm{Algbrd}(\mathrm{Spc})_{\mathrm{Spc}})}(\iota_{\mathrm{Algbrd},\mathcal{M}}\mathcal{C}, \iota_{\mathcal{M}}\mathcal{D})
\end{tikzcd}$$

where the vertical arrow is induced by precomposition with the unit $\eta_{\mathcal{C}} : \iota_{\mathrm{Algbrd},\mathcal{M}}\mathcal{C} \to \iota_{\mathcal{M}}\mathcal{C}$, and the lower diagonal arrow is induced by composition with the unit $\eta_{\mathcal{D}} : \iota_{\mathrm{Algbrd},\mathcal{M}}\mathcal{D} \to \iota_{\mathcal{M}}\mathcal{D}$. Note that the right vertical arrow is an equivalence. Hence it suffices to show that the lower diagonal arrow is an equivalence.

It follows from [Lur09a] proposition 3.1.2.1 that the projection

$$p : \mathrm{Funct}(\mathcal{M}^{\mathrm{op}}, \mathrm{Algbrd}(\mathrm{Spc})_{\mathrm{Spc}}) \to \mathrm{Funct}(\mathcal{M}^{\mathrm{op}}, \mathrm{Spc})$$

is a cartesian fibration, and $\eta_{\mathcal{D}}$ is a cartesian arrow. Hence we have a pullback square of spaces

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{Funct}(\mathcal{M}^{\mathrm{op}},\mathrm{Algbrd}(\mathrm{Spc})_{\mathrm{Spc}})}(\iota_{\mathrm{Algbrd},\mathcal{M}}\mathcal{C},\iota_{\mathrm{Algbrd},\mathcal{M}}\mathcal{D}) \arrow[r, "p_*"] \arrow[d, "(\eta_{\mathcal{D}})_*"] & \mathrm{Hom}_{\mathrm{Funct}(\mathcal{M}^{\mathrm{op}},\mathrm{Spc})}(p\iota_{\mathrm{Algbrd},\mathcal{M}}\mathcal{C},p\iota_{\mathrm{Algbrd},\mathcal{M}}\mathcal{D}) \arrow[d, "(p\eta_{\mathcal{D}})_*"] \\
\mathrm{Hom}_{\mathrm{Funct}(\mathcal{M}^{\mathrm{op}},\mathrm{Algbrd}(\mathrm{Spc})_{\mathrm{Spc}})}(\iota_{\mathrm{Algbrd},\mathcal{M}}\mathcal{C},\iota_{\mathcal{M}}\mathcal{D}) \arrow[r, "p_*"] & \mathrm{Hom}_{\mathrm{Funct}(\mathcal{M}^{\mathrm{op}},\mathrm{Spc})}(p\iota_{\mathrm{Algbrd},\mathcal{M}}\mathcal{C},p\iota_{\mathcal{M}}\mathcal{D})
\end{tikzcd}$$

To show that the left vertical arrow is an equivalence it suffices to show that the right vertical arrow is an equivalence.

Note that $p\iota_{\mathrm{Algbrd},\mathcal{M}}\mathcal{C}$ and $p\iota_{\mathrm{Algbrd},\mathcal{M}}\mathcal{D}$ belong to the image of the diagonal map $\Delta : \mathrm{Spc} \to \mathrm{Funct}(\mathcal{M}^{\mathrm{op}}, \mathrm{Spc})$. Note that since $\mathcal{M}^{\mathrm{op}}$ has an initial object it is contractible, and hence Δ is fully faithful and admits a right adjoint given by evaluation at the initial object $1_{\mathcal{M}}$. It therefore suffices to show that the composition of the map

$$\mathrm{Hom}_{\mathrm{Spc}}(p\iota_{\mathrm{Algbrd},\mathcal{M}}\mathcal{C}(1_{\mathcal{M}}), p\iota_{\mathrm{Algbrd},\mathcal{M}}\mathcal{D}(1_{\mathcal{M}})) \to \mathrm{Hom}_{\mathrm{Funct}(\mathcal{M}^{\mathrm{op}},\mathrm{Spc})}(p\iota_{\mathrm{Algbrd},\mathcal{M}}\mathcal{C}, p\iota_{\mathrm{Algbrd},\mathcal{M}}\mathcal{D})$$

induced by Δ with the map

$$\mathrm{Hom}_{\mathrm{Funct}(\mathcal{M}^{\mathrm{op}},\mathrm{Spc})}(p\iota_{\mathrm{Algbrd},\mathcal{M}}\mathcal{C}, p\iota_{\mathrm{Algbrd},\mathcal{M}}\mathcal{D}) \to \mathrm{Hom}_{\mathrm{Funct}(\mathcal{M}^{\mathrm{op}},\mathrm{Spc})}(p\iota_{\mathrm{Algbrd},\mathcal{M}}\mathcal{C}, p\iota_{\mathcal{M}}\mathcal{D})$$

induced by $p\eta_{\mathcal{D}}$, is an equivalence. In other words, we have reduced our task to showing that $p\eta_{\mathcal{D}}$ presents $p\iota_{\mathrm{Algbrd},\mathcal{M}}\mathcal{D}(1_{\mathcal{M}})$ as right adjoint to Δ at $p\iota_{\mathcal{M}}\mathcal{D}$. This is equivalent to showing that $p\eta_{\mathcal{D}}(1_{\mathcal{M}})$ is an isomorphism.

We claim that in fact $\eta_{\mathcal{D}}(1_{\mathcal{M}})$ is an isomorphism. This is a morphism

$$\eta_{\mathcal{D}}(1_{\mathcal{M}}) : \iota_{\mathrm{Algbrd},\mathcal{M}}\mathcal{D}(1_{\mathcal{M}}) \to \iota_{\mathcal{M}}\mathcal{D}(1_{\mathcal{M}})$$

that presents $\iota_{\mathcal{M}}\mathcal{D}(1_{\mathcal{M}})$ as the category underlying $\iota_{\mathrm{Algbrd},\mathcal{M}}\mathcal{D}(1_{\mathcal{M}})$. It follows from the definitions that $\iota_{\mathrm{Algbrd},\mathcal{M}}\mathcal{D}(1_{\mathcal{M}})$ is the Segal space underlying $\mathcal{D}$. Our claim now follows from the fact that $\mathcal{D}$ is an $\mathcal{M}$-enriched category. □

Proof of theorem 6.2.8. It follows from lemma 6.2.14 that we have a commutative square of categories

$$\begin{tikzcd}
\mathrm{CAlg}(\mathrm{Cat}^{\mathcal{M}}) \arrow[r] \arrow[d, "(\iota_{\mathcal{M}})_*"] & \mathrm{Cat}^{\mathcal{M}} \arrow[d, "\iota_{\mathcal{M}}"] \\
\mathrm{CAlg}(\mathrm{Funct}(\mathcal{M}^{\mathrm{op}}, \mathrm{Cat})) \arrow[r] & \mathrm{Funct}(\mathcal{M}^{\mathrm{op}}, \mathrm{Cat}).
\end{tikzcd}$$

where the horizontal arrows are the forgetful functors, and the vertical arrows are fully faithful.

The restriction of $\iota_{\mathcal{M}}$ to $(\mathrm{Cat}^{\mathcal{M}})_{\text{fin prod}}$ factors through $\mathrm{Funct}(\mathcal{M}^{\mathrm{op}}, \mathrm{Cat}_{\text{fin prod}})$. Furthermore, the restriction of $(\iota_{\mathcal{M}})_*$ to $\mathrm{CAlg}_{\mathrm{cart}}(\mathrm{Cat}^{\mathcal{M}})$ factors through the image of the subcategory $\mathrm{Funct}(\mathcal{M}^{\mathrm{op}}, \mathrm{CAlg}_{\mathrm{cart}}(\mathrm{Cat}))$ across the equivalence

$$\mathrm{Funct}(\mathcal{M}^{\mathrm{op}}, \mathrm{CAlg}(\mathrm{Cat})) = \mathrm{CAlg}(\mathrm{Funct}(\mathcal{M}^{\mathrm{op}}, \mathrm{Cat})).$$

It follows that we have a commutative square of categories

$$\begin{tikzcd}
\mathrm{CAlg}_{\mathrm{cart}}(\mathrm{Cat}^{\mathcal{M}}) \arrow[r] \arrow[d, "(\iota_{\mathcal{M}})_*"] & (\mathrm{Cat}^{\mathcal{M}})_{\text{fin prod}} \arrow[d, "\iota_{\mathcal{M}}"] \\
\mathrm{Funct}(\mathcal{M}^{\mathrm{op}}, \mathrm{CAlg}_{\mathrm{cart}}(\mathrm{Cat})) \arrow[r] & \mathrm{Funct}(\mathcal{M}^{\mathrm{op}}, \mathrm{Cat}_{\text{fin prod}})
\end{tikzcd}$$

where the horizontal arrows are the forgetful functors. Note that the vertical arrows are still fully faithful. The bottom horizontal arrow is an equivalence thanks to [Lur17] corollary 2.4.1.9. Therefore the top horizontal arrow is fully faithful as well.

It remains to show that the top horizontal arrow is surjective. Let $\mathcal{C}$ be an $\mathcal{M}$-enriched category admitting conical finite products. It follows from the above discussion that $\iota_{\mathcal{M}}(\mathcal{C})$ admits an enhancement to a commutative algebra $\iota_{\mathcal{M}}(\mathcal{C})^{\mathrm{enh}}$ in $\mathrm{Funct}(\mathcal{M}^{\mathrm{op}}, \mathrm{Cat})$ whose image under all evaluation functors is a cartesian symmetric monoidal category. Since $\iota_{\mathcal{M}}(\mathcal{C})$ lies in the image of $\iota_{\mathcal{M}}$, we have that $\iota_{\mathcal{M}}(\mathcal{C})^{\mathrm{enh}}$ may be written as $(\iota_{\mathcal{M}})_*\mathcal{C}^{\mathrm{enh}}$ for some symmetric monoidal $\mathcal{M}$-enriched category $\mathcal{C}^{\mathrm{enh}}$ with underlying $\mathcal{M}$-enriched category $\mathcal{C}$. Note that the symmetric monoidal category underlying $\mathcal{C}^{\mathrm{enh}}$ is equivalent to $\iota_{\mathcal{M}}(\mathcal{C})^{\mathrm{enh}}(1_{\mathcal{M}})$. This is cartesian symmetric monoidal, and hence $\mathcal{C}^{\mathrm{enh}}$ provides the desired cartesian symmetric monoidal structure on $\mathcal{C}$. $\square$

We now provide an enriched generalization of the theory of cartesian $\mathcal{O}$-monoidal categories from 6.1.

Definition 6.2.15. *Let $\mathcal{O}$ be an operad and let $\mathcal{M}$ be a cartesian symmetric monoidal category. An $\mathcal{O}$-monoidal $\mathcal{M}$-enriched category $\mathcal{C}$ is said to be cartesian if the underlying $\mathcal{O}$-monoidal category $(\tau_{\mathcal{M}})_!\mathcal{C}$ is cartesian, and for every object x in $\mathcal{O}$ the $\mathcal{M}$-enriched category $\mathcal{C}(x)$ admits all conical finite products.*

Remark 6.2.16. Let $\mathcal{O}$ be an operad and let $\mathcal{M}$ be a cartesian symmetric monoidal category. Unwinding the definition, we have that an $\mathcal{O}$-monoidal $\mathcal{M}$-enriched category $\mathcal{C}$ is cartesian if and only if the following conditions are satisfied:

- For every object x in $\mathcal{O}$ the $\mathcal{M}$-enriched category $\mathcal{C}(x)$ admits all conical finite products.
- For every operation in $\mathcal{O}$ with source $\{x_s\}_{s\in S}$ and target x, the induced functor

$$\prod_{s\in S} \mathcal{C}(x_s) \to \mathcal{C}(x)$$

  preserves conical finite products.

As before, the second condition may be reduced to a smaller list of assertions by applying proposition 6.1.5.

Remark 6.2.17. Let $\mathcal{M}$ be a cartesian symmetric monoidal category. Equip $\mathrm{Cat}^{\mathcal{M}}_{\text{fin prod}}$ with its cartesian symmetric monoidal structure, and the inclusion $\mathrm{Cat}^{\mathcal{M}}_{\text{fin prod}} \to \mathrm{Cat}^{\mathcal{M}}$ with its unique symmetric monoidal structure. Let $\mathcal{O}$ be an operad, and $\mathcal{C}$ be an $\mathcal{O}$-monoidal $\mathcal{M}$-enriched category. Then it follows from remark 6.2.16 that $\mathcal{C}$ is cartesian if and only if the associated morphism $\mathcal{O} \to \mathrm{Cat}^{\mathcal{M}}$ factors through $\mathrm{Cat}^{\mathcal{M}}_{\text{fin prod}}$.

Definition 6.2.18. *Let $\mathcal{O}$ be an operad and let $\mathcal{M}$ be a cartesian symmetric monoidal category. We call $\mathrm{Alg}_{\mathcal{O}}(\mathrm{Cat}^{\mathcal{M}}_{\text{fin prod}})$ the category of cartesian $\mathcal{O}$-monoidal $\mathcal{M}$-enriched categories.*

Remark 6.2.19. Let $\mathcal{O}$ be an operad and let $\mathcal{M}$ be a cartesian symmetric monoidal category. Then $\mathrm{Alg}_{\mathcal{O}}(\mathrm{Cat}^{\mathcal{M}}_{\text{fin prod}})$ is the subcategory of $\mathrm{Alg}_{\mathcal{O}}(\mathrm{Cat}^{\mathcal{M}})$ defined by the following conditions:

- An $\mathcal{O}$-monoidal $\mathcal{M}$-enriched category $\mathcal{C}$ belongs to $\mathrm{Alg}_{\mathcal{O}}(\mathrm{Cat}^{\mathcal{M}}_{\text{fin prod}})$ if and only if it is cartesian.
- An morphism $F : \mathcal{C} \to \mathcal{C}'$ between two cartesian $\mathcal{O}$-monoidal $\mathcal{M}$-enriched categories belongs to $\mathrm{Alg}_{\mathcal{O}}(\mathrm{Cat}^{\mathcal{M}}_{\text{fin prod}})$ if and only if for every object x in $\mathcal{O}$ the induced functor $\mathcal{C}(x) \to \mathcal{C}'(x)$ preserves conical finite products.

In the particular case when $\mathcal{O} = \mathrm{Comm}$, we have an equivalence

$$\mathrm{Alg}_{\mathrm{Comm}}(\mathrm{Cat}^{\mathcal{M}}) = \mathrm{CAlg}_{\mathrm{cart}}(\mathrm{Cat}^{\mathcal{M}}).$$

The following proposition is a joint generalization of proposition 6.1.8 and theorem 6.2.8.

Proposition 6.2.20. *Let $\mathcal{O}$ be an operad and let $\mathcal{M}$ be a cartesian symmetric monoidal category. Equip $\mathrm{Cat}^{\mathcal{M}}_{\text{fin prod}}$ with its cartesian symmetric monoidal structure. Then restriction along the morphism of operads $\mathcal{O} = \mathcal{O} \otimes [0] \to \mathcal{O} \otimes \mathrm{Comm}$ induces an equivalence of categories*

$$\mathrm{Alg}_{\mathcal{O}}(\mathrm{Cat}^{\mathcal{M}}_{\text{fin prod}}) = \mathrm{Alg}_{\mathcal{O}\otimes\mathrm{Comm}}(\mathrm{Cat}^{\mathcal{M}}_{\text{fin prod}}).$$

Proof. Since the cartesian symmetric monoidal structure on $\mathrm{CAlg}(\mathrm{Cat}^{\mathcal{M}})$ is also cocartesian, we have that the same holds for its full subcategory $\mathrm{CAlg}_{\mathrm{cart}}(\mathrm{Cat}^{\mathcal{M}})$. Thanks to theorem 6.2.8, this is also the case for the category $\mathrm{Cat}^{\mathcal{M}}_{\text{fin prod}}$. Therefore precomposition with the map $[0] \to \mathrm{Comm}$ induces an equivalence of symmetric monoidal categories

$$\mathrm{Alg}_{\mathrm{Comm}}(\mathrm{Cat}^{\mathcal{M}}_{\text{fin prod}}) = \mathrm{Alg}_{[0]}(\mathrm{Cat}^{\mathcal{M}}_{\text{fin prod}}) = \mathrm{Cat}^{\mathcal{M}}_{\text{fin prod}}.$$

The result now follows from the above by passing to categories of $\mathcal{O}$-algebras. □

Corollary 6.2.21. *Let $\mathcal{M}$ be a cartesian symmetric monoidal category and let $\infty \geq n \geq 0$. Then precomposition with the unique map $[0] \to E_n$ induces an equivalence*

$$\mathrm{Alg}_{E_n}(\mathrm{Cat}^{\mathcal{M}}_{\text{fin prod}}) = \mathrm{Cat}^{\mathcal{M}}_{\text{fin prod}}.$$

Proof. This follows from proposition 6.2.20 by the same arguments as in the proof of corollary 6.1.9. □

Corollary 6.2.22. *Let $\mathcal{M}$ be a cartesian symmetric monoidal category. Let $\infty \geq n \geq 1$ and let $\mathcal{O} = \mathrm{LMod} \cup_{\mathrm{Assos}} E_n$ be the operad governing pairs of an E_n-algebra and a left module over it. Then there is an equivalence*

$$\mathrm{Alg}_{\mathcal{O}}(\mathrm{Cat}^{\mathcal{M}}_{\text{fin prod}}) = \mathrm{Funct}([1], \mathrm{Cat}^{\mathcal{M}}_{\text{fin prod}}).$$

Proof. This follows from proposition 6.2.20 by the same arguments as in the proof of corollary 6.1.10. □

6.3 Enriched categories of $\mathcal{O}$-algebras

Given a cartesian symmetric monoidal enriched category $\mathcal{C}$, the category $\mathcal{O}$-algebras in $\mathcal{C}$ from definition 6.2.1 admits a canonical enrichment.

Notation 6.3.1. Let $\mathcal{O}$ be an operad with category of operators $\mathcal{O}^{\otimes}$. Let $\mathcal{M}$ be a cartesian closed presentable category. We equip $\mathcal{M}$ with its cartesian symmetric monoidal structure. Let $\mathcal{C}$ be a cartesian symmetric monoidal $\mathcal{M}$-enriched category. We denote by $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ the full subcategory of $\mathrm{Funct}((\mathcal{O}^{\otimes})_{\mathcal{M}}, \mathcal{C})$ on those functors whose associated functor $\mathcal{O}^{\otimes} \to (\tau_{\mathcal{M}})_!\mathcal{C}$ is an $\mathcal{O}$-algebra in $\mathcal{C}$.

Definition 6.3.2. *Let $\mathcal{O}$ be an operad. Let $\mathcal{M}$ be a cartesian closed presentable category and let $\mathcal{C}$ be a cartesian symmetric monoidal $\mathcal{M}$-enriched category. We call $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ the $\mathcal{M}$-enriched category of $\mathcal{O}$-algebras in $\mathcal{C}$.*

Remark 6.3.3. Let $\mathcal{O}$ be an operad. Let $\mathcal{M}$ be a cartesian closed presentable category and let $\mathcal{C}$ be a cartesian symmetric monoidal $\mathcal{M}$-enriched category. Then the category underlying $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ is the category of $\mathcal{O}$-algebras in $\mathcal{C}$ from definition 6.2.1.

Our next goal is to give a description of $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ for special values of $\mathcal{O}$.

Proposition 6.3.4. *Let $\mathcal{M}$ be a cartesian closed presentable category and let $\mathcal{C}$ be a cartesian symmetric monoidal $\mathcal{M}$-enriched category. Let $\mathcal{I}$ be a category and let $\mathcal{O}$ be the image of $\mathcal{I}$ under the embedding $\mathrm{Cat} \to \mathrm{Op}$. Let $\mathcal{O}^{\otimes}$ be the category of operators of $\mathcal{O}$ and $F : \mathcal{I} \to \mathcal{O}^{\otimes}$ be the inclusion of the fiber of $\mathcal{O}^{\otimes}$ over $\langle 1 \rangle$. Then restriction along F induces an equivalence*

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) = \mathrm{Funct}(\mathcal{I}_{\mathcal{M}}, \mathcal{C}).$$

Our proof of proposition 6.3.4 will need a preliminary lemma.

Lemma 6.3.5. *Let $\mathcal{E}$ be a presentable category. Equip $\mathcal{E}$ with its cartesian symmetric monoidal structure. Let $\mathcal{I}$ be a category and let $\mathcal{O}$ be the image of $\mathcal{I}$ under the embedding $\mathrm{Cat} \to \mathrm{Op}$. Let $\mathcal{O}^{\otimes}$ be the category of operators of $\mathcal{O}$ and $F : \mathcal{I} \to \mathcal{O}^{\otimes}$ be the inclusion of the fiber of $\mathcal{O}^{\otimes}$ over $\langle 1 \rangle$. Then:*

(i) *The inclusion* $\mathrm{Alg}_{\mathcal{O}}(\mathcal{E}) \to \mathrm{Funct}(\mathcal{O}^{\otimes}, \mathcal{E})$ *admits a left adjoint* L.

(ii) *The restriction map*
$$F^* : \mathrm{Funct}(\mathcal{O}^{\otimes}, \mathcal{E}) \to \mathrm{Funct}(\mathcal{I}, \mathcal{E})$$
factors through the localization functor L.

Proof. Item (i) is a direct consequence of the fact that $\mathrm{Alg}_{\mathcal{O}}(\mathcal{E})$ is closed under limits and filtered colimits in $\mathrm{Funct}(\mathcal{O}^{\otimes}, \mathcal{E})$, together with the adjoint functor theorem.

We now prove item (ii). Passing to right adjoints, we reduce to showing that the functor

$$\mathrm{Funct}(\mathcal{I}, \mathcal{E}) \to \mathrm{Funct}(\mathcal{O}^{\otimes}, \mathcal{E})$$

of right Kan extension along F, factors through the full subcategory on the $\mathcal{O}$-algebras. Let $X = \{X_s\}_{s \in S}$ be an object in $\mathcal{O}^{\otimes}$, corresponding to a family of objects in $\mathcal{I}$ indexed by a finite set S. Then the overcategory $\mathcal{I}_{X/}$ has no nontrivial morphisms, and its space of objects is the set S, where each index s in S corresponds to the inert arrow $X \to X_s$.

It follows that a functor $G : \mathcal{O}^{\otimes} \to \mathcal{E}$ belongs to the right Kan extension of F if and only if the induced maps $G(X) \to G(X_s)$ present G as the product of the objects $G(X_s)$. This agrees with the condition of being an $\mathcal{O}$-monoid in $\mathcal{E}$, as desired. □

Proof of proposition 6.3.4. Note that the functor of categories underlying $F^* : \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \to \mathrm{Funct}(\mathcal{I}_{\mathcal{M}}, \mathcal{C})$ is equivalent to the functor

$$\mathrm{Alg}_{\mathcal{O}}((\tau_{\mathcal{M}})_! \mathcal{C}) \to \mathrm{Funct}(\mathcal{I}, (\tau_{\mathcal{M}})_! \mathcal{C}).$$

of precomposition with F. This is an equivalence by [Lur17] example 2.1.3.5. It follows in particular that F^* is surjective. It remains to show that it is fully faithful.

Let $i : \mathcal{C} \to \mathrm{Funct}(\mathcal{C}^{\mathrm{op}}, \overline{\mathcal{M}})$ be the Yoneda embedding and consider the commutative square of $\mathcal{M}$-enriched categories

$$\begin{tikzcd}
\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \arrow[r, "F^*"] \arrow[d, "i_*"] & \mathrm{Funct}(\mathcal{I}_{\mathcal{M}}, \mathcal{C}) \arrow[d, "i_*"] \\
\mathrm{Alg}_{\mathcal{O}}(\mathrm{Funct}(\mathcal{C}^{\mathrm{op}}, \overline{\mathcal{M}})) \arrow[r, "F^*"] & \mathrm{Funct}(\mathcal{I}_{\mathcal{M}}, \mathrm{Funct}(\mathcal{C}^{\mathrm{op}}, \overline{\mathcal{M}})).
\end{tikzcd}$$

The vertical arrows are fully faithful since i is fully faithful. Hence it suffices to show that the bottom horizontal arrow in the above diagram is fully faithful. Thanks to [Hin20a] proposition 6.3.7, this is equivalent to the restriction to $\mathrm{Alg}_{\mathcal{O}}(\mathrm{Funct}(\mathcal{C}^{\mathrm{op}}, \overline{\mathcal{M}}))$ of the functor

$$F^* : \mathrm{Funct}(\mathcal{O}^{\otimes}_{\mathcal{M}}, \theta_{\mathcal{M}}(\mathrm{LMod}_{\mathcal{C}^{\mathrm{op}}}(\mathcal{M}))) \to \mathrm{Funct}(\mathcal{I}_{\mathcal{M}}, \theta_{\mathcal{M}}(\mathrm{LMod}_{\mathcal{C}^{\mathrm{op}}}(\mathcal{M}))).$$

As in the proof of proposition 5.4.8, we may identify the above with the functor

$$\theta_{\mathcal{M}}(F^*) : \theta_{\mathcal{M}}(\mathrm{Funct}(\mathcal{O}^{\otimes}, \mathrm{LMod}_{\mathcal{C}^{\mathrm{op}}}(\mathcal{M}))) \to \theta_{\mathcal{M}}(\mathrm{Funct}(\mathcal{I}, \mathrm{LMod}_{\mathcal{C}^{\mathrm{op}}}(\mathcal{M}))).$$

It therefore suffices to show that if A, B are two $\mathcal{O}$-algebras in the cartesian symmetric monoidal category $\mathrm{LMod}_{\mathcal{C}^{\mathrm{op}}}(\mathcal{M})$, then the induced map

$$\mathscr{H}om_{\mathrm{Funct}(\mathcal{O}^{\otimes}, \mathrm{LMod}_{\mathcal{C}^{\mathrm{op}}}(\mathcal{M}))}(A, B) \to \mathscr{H}om_{\mathrm{Funct}(\mathcal{I}, \mathrm{LMod}_{\mathcal{C}^{\mathrm{op}}}(\mathcal{M}))}(F^*A, F^*B)$$

is an equivalence, where the Hom objects are taken with respect to the action of $\mathcal{M}$. This is equivalent to the assertion that for every object m in $\mathcal{M}$ the morphism of spaces

$$\mathrm{Hom}_{\mathrm{Funct}(\mathcal{O}^{\otimes}, \mathrm{LMod}_{\mathcal{C}^{\mathrm{op}}}(\mathcal{M}))}(m \otimes A, B) \to \mathrm{Hom}_{\mathrm{Funct}(\mathcal{I}, \mathrm{LMod}_{\mathcal{C}^{\mathrm{op}}}(\mathcal{M}))}(m \otimes F^*A, F^*B)$$

is an equivalence. Let

$$L : \mathrm{Funct}(\mathcal{O}^{\otimes}, \mathrm{LMod}_{\mathcal{C}^{\mathrm{op}}}(\mathcal{M})) \to \mathrm{Alg}_{\mathcal{O}}(\mathrm{LMod}_{\mathcal{C}^{\mathrm{op}}}(\mathcal{M}))$$

be the localization functor. Since the functor

$$F^* : \mathrm{Funct}(\mathcal{O}^{\otimes}, \mathrm{LMod}_{\mathcal{C}^{\mathrm{op}}}(\mathcal{M})) \to \mathrm{Funct}(\mathcal{I}, \mathrm{LMod}_{\mathcal{C}^{\mathrm{op}}}(\mathcal{M}))$$

is fully faithful on the full subcategory of $\mathcal{O}$-algebras, we reduce to showing that the image under F^* of the unit $m \otimes A \to L(m \otimes A)$ is invertible. This is a consequence of lemma 6.3.5. □

Proposition 6.3.6. *Let $\mathcal{M}$ be a cartesian closed presentable category and let $\mathcal{C}$ be a cartesian symmetric monoidal $\mathcal{M}$-enriched category. Let $E_0^{\otimes}$ be the category of operators of the E_0-operad and let $F : [1] \to E_0^{\otimes}$ be the functor that picks out the active arrow $\langle 0 \rangle \to \langle 1 \rangle$. Then precomposition with F induces an equivalence between $\mathrm{Alg}_{E_0}(\mathcal{C})$ and the full subcategory of $\mathrm{Funct}([1]_{\mathcal{M}}, \mathcal{C})$ on those arrows in $\mathcal{C}$ with source $1_{\mathcal{C}}$.*

Our proof of proposition 6.3.6 will need a preliminary lemma.

Lemma 6.3.7. *Let $\mathcal{E}$ be a presentable category. Equip $\mathcal{E}$ with its cartesian symmetric monoidal structure. Let $F : [1] \to E_0^{\otimes}$ be as in the statement of proposition 6.3.6. Then right Kan extension along F sends arrows in $\mathcal{E}$ with source $1_{\mathcal{E}}$ to E_0-algebras in $\mathcal{E}$.*

Proof. Let $n \geq 0$. Then the overcategory $[1]_{\langle n \rangle /}$ has $n+2$ objects, namely:

- The inert map $a : \langle n \rangle \to \langle 0 \rangle$.
- For each $1 \leq i \leq n$ the inert map $b_i : \langle n \rangle \to \langle 1 \rangle$.
- The active map $c : \langle n \rangle \to \langle 1 \rangle$, obtained by composing a with the active map $\langle 0 \rangle \to \langle 1 \rangle$.

The only nontrivial map in $[1]_{\langle n \rangle /}$ is the map $a \to c$. Note that the full subcategory of $[1]_{\langle n \rangle /}$ on the objects a and b_i is final.

Assume now given a functor $A : [1] \to \mathcal{E}$ which picks out an arrow with source $1_{\mathcal{E}}$, and let $G : E_0^{\otimes} \to \mathcal{E}$ be the right Kan extension of A along F. It follows from the above description of $[1]_{\langle n \rangle /}$ that we have an isomorphism

$$G(\langle n \rangle) = A(\langle 0 \rangle) \times \prod_{1 \leq i \leq n} A(\langle 1 \rangle) = \prod_{1 \leq i \leq n} A(\langle 1 \rangle).$$

In particular, we have $G(\langle 1 \rangle) = A(\langle 1 \rangle)$. The above equivalence then becomes an isomorphism

$$G(\langle n \rangle) = \prod_{1 \leq i \leq n} G(\langle 1 \rangle).$$

Tracing the identifications, we see that the above is induced by the maps b_i. This means that G is an E_0-algebra in $\mathcal{E}$, as desired. $\square$

Proof of proposition 6.3.6. Note that the functor of categories underlying $F^* : \mathrm{Alg}_{E_0}(\mathcal{C}) \to \mathrm{Funct}([1]_{\mathcal{M}}, \mathcal{C})$ is equivalent to the functor

$$\mathrm{Alg}_{E_0}((\tau_{\mathcal{M}})_! \mathcal{C}) \to \mathrm{Funct}([1], (\tau_{\mathcal{M}})_! \mathcal{C})$$

of precomposition with F. It follows from [Lur17] proposition 2.1.3.9 that the above is an equivalence with the full subcategory of $\mathrm{Funct}([1], (\tau_{\mathcal{M}})_! \mathcal{C})$ on those arrows in $(\tau_{\mathcal{M}})_! \mathcal{C}$ with source $1_{\mathcal{C}}$. In particular, we have that the image of F^* consists of those arrows in $\mathcal{C}$ with source $1_{\mathcal{C}}$.

It remains to show that F^* is fully faithful. As in the proof of 6.3.4, we reduce to proving that if A, B are two E_0-algebras in $\mathrm{LMod}_{\mathcal{C}^{\mathrm{op}}}(\mathcal{M})$ and m is an object of $\mathcal{M}$, then the morphism of spaces

$$\mathrm{Hom}_{\mathrm{Funct}(E_0^{\otimes}, \mathrm{LMod}_{\mathcal{C}^{\mathrm{op}}}(\mathcal{M}))}(m \otimes A, B) \to \mathrm{Hom}_{\mathrm{Funct}([1], \mathrm{LMod}_{\mathcal{C}^{\mathrm{op}}}(\mathcal{M}))}(m \otimes F^* A, F^* B)$$

is an equivalence. Denote by

$$L : \mathrm{Funct}(E_0^{\otimes}, \mathrm{LMod}_{\mathcal{C}^{\mathrm{op}}}(\mathcal{M})) \to \mathrm{Alg}_{E_0}(\mathrm{LMod}_{\mathcal{C}^{\mathrm{op}}}(\mathcal{M}))$$

the localization functor. To complete the proof it suffices to show that the image under F^* of the unit map $m \otimes A \to L(m \otimes A)$ is left orthogonal to $F^* B$. This follows from lemma 6.3.7. $\square$

We now specialize the above theory to obtain an enrichment of the category of $\mathcal{O}$-monoidal enriched categories from definition 6.2.1.

Definition 6.3.8. *Let $\mathcal{M}$ be a cartesian closed presentable category. Equip the $\mathcal{M}$-enriched 2-category $\overline{\mathrm{Cat}^{\mathcal{M}}}$ with its cartesian symmetric monoidal structure. Let $\mathcal{O}$ be an operad. We call $\mathrm{Alg}_{\mathcal{O}}(\overline{\mathrm{Cat}^{\mathcal{M}}})$ the $\mathcal{M}$-enriched 2-category of $\mathcal{O}$-monoidal $\mathcal{M}$-enriched categories.*

Remark 6.3.9. Let $\mathcal{M}$ be a cartesian closed presentable category and let $\mathcal{O}$ be an operad. It follows from remark 6.3.3 that the category underlying $\mathrm{Alg}_{\mathcal{O}}(\overline{\mathrm{Cat}^{\mathcal{M}}})$ is equivalent to the category $\mathrm{Alg}_{\mathcal{O}}(\mathrm{Cat}^{\mathcal{M}})$ of $\mathcal{O}$-monoidal $\mathcal{M}$-enriched categories from definition 6.2.1.

Remark 6.3.10. Let $\mathcal{M}$ be a cartesian closed presentable category. Specializing definition 6.3.8 to the case $\mathcal{O} = \mathrm{Comm}$ (resp. $\mathcal{O} = \mathrm{Assos}$) we obtain an $\mathcal{M}$-enriched 2-category of symmetric monoidal (resp. monoidal) $\mathcal{M}$-enriched categories.

In the case $\mathcal{M} = \mathrm{Spc}$, definition 6.3.8 supplies a 2-category $\mathrm{Alg}_{\mathcal{O}}(\mathscr{C}at)$ of $\mathcal{O}$-monoidal categories for any operad $\mathcal{O}$. We finish by showing that this is equivalent to an alternative definition as a sub-2-category of the 2-category of categories over $\mathcal{O}^{\otimes}$.

Construction 6.3.11. Let $\mathcal{B}$ be a category. Consider the pullback functor

$$- \times \mathcal{B} : \mathrm{Cat} \to \mathrm{Cat}_{/\mathcal{B}}\,.$$

Equip Cat and $\mathrm{Cat}_{/\mathcal{B}}$ with their cartesian symmetric monoidal structures, and $- \times \mathcal{B}$ with its unique symmetric monoidal structure. Restriction of scalars along it endows $\mathrm{Cat}_{/\mathcal{B}}$ with the structure of category tensored over Cat.

For each category $\mathcal{I}$, we have an equivalence

$$(\mathcal{I} \times \mathcal{B}) \times_{\mathcal{B}} - = \mathcal{I} \times -$$

of functors $\mathrm{Cat}_{/\mathcal{B}} \to \mathrm{Cat}_{\mathcal{B}}$, and in particular we see that $(\mathcal{I} \times \mathcal{B}) \times_{\mathcal{B}} -$ is colimit preserving. It follows that $\mathrm{Cat}_{/\mathcal{B}}$ is a presentable module over Cat. We let

$$\mathscr{C}at_{/\mathcal{B}} = \theta_{\mathrm{Cat}}(\mathrm{Cat}_{/\mathcal{B}}).$$

We call this the 2-category of categories over $\mathcal{B}$.

Remark 6.3.12. Let $\mathcal{B}$ be a category and let $p : \mathcal{C} \to \mathcal{B}$ and $q : \mathcal{D} \to \mathcal{B}$ be two categories over $\mathcal{B}$. Then for each category $\mathcal{I}$ we have an equivalence

$$\mathrm{Hom}_{\mathrm{Cat}}(\mathcal{I}, \mathrm{Hom}_{\mathscr{C}at_{/\mathcal{B}}}(\mathcal{C}, \mathcal{D})) = \mathrm{Hom}_{\mathrm{Cat}_{/\mathcal{B}}}(\mathcal{I} \times \mathcal{C}, \mathcal{D}) = \mathrm{Hom}_{\mathrm{Cat}}(\mathcal{I} \times \mathcal{C}, \mathcal{D}) \times_{\mathrm{Hom}_{\mathrm{Cat}}(\mathcal{I}\times\mathcal{C}, \mathcal{B})} [0]$$

where the map $[0] \to \mathrm{Hom}_{\mathrm{Cat}}(\mathcal{I}\times\mathcal{C}, \mathcal{B})$ picks the composition of the projection map $\mathcal{I}\times\mathcal{C} \to \mathcal{C}$ and p. The above equivalence is natural in $\mathcal{I}$, and it therefore induces an equivalence

$$\mathrm{Hom}_{\mathscr{C}at_{/\mathcal{B}}}(\mathcal{C}, \mathcal{D}) = \mathrm{Funct}(\mathcal{C}, \mathcal{D}) \times_{\mathrm{Funct}(\mathcal{C}, \mathcal{B})} [0] = \mathrm{Funct}_{\mathcal{B}}(\mathcal{C}, \mathcal{D}).$$

Notation 6.3.13. Let $\mathcal{B}$ be a category. We denote by $\mathscr{C}at_{/\mathcal{B}}^{\mathrm{cocart}}$ the sub 2-category of $\mathscr{C}at_{/\mathcal{B}}$ on the cocartesian fibrations over $\mathcal{B}$, the morphisms of cocartesian fibrations, and all 2-cells. We call this the 2-category of cocartesian fibrations over $\mathcal{B}$.

Proposition 6.3.14. *Let $\mathcal{B}$ be a category. Then there is an equivalence*

$$\mathscr{C}at_{/\mathcal{B}}^{\mathrm{cocart}} = \mathrm{Funct}(\mathcal{B}, \mathscr{C}at)$$

which recovers the usual straightening equivalence $\mathrm{Cat}_{/\mathcal{B}}^{\mathrm{cocart}} = \mathrm{Funct}(\mathcal{B}, \mathrm{Cat})$ upon passage to underlying categories.

Proof. Recall from [Hin20a] proposition 6.3.7 that there is an equivalence of 2-categories

$$\mathrm{Funct}(\mathcal{B}, \mathscr{C}at) = \theta_{\mathrm{Cat}}(\mathrm{LMod}_{\mathcal{B}}(\mathrm{Cat})).$$

Combining lemmas 5.4.6 and 5.4.7, as in the proof of proposition 5.4.8, we conclude that there is an equivalence of 2-categories

$$\mathrm{Funct}(\mathcal{B}, \mathscr{C}at) = \theta_{\mathcal{C}}(\mathrm{Funct}(\mathcal{B}, \mathrm{Cat}))$$

where $\mathrm{Funct}(\mathcal{B}, \mathrm{Cat})$ is a $\mathcal{B}$-module by restriction of scalars along the diagonal map $\mathrm{Cat} \rightarrow \mathrm{Funct}(\mathcal{B}, \mathrm{Cat})$. Our claim now follows from the fact that the cartesian symmetric monoidal functor $\mathrm{Cat} \rightarrow \mathrm{Funct}(\mathcal{B}, \mathrm{Cat})$ is equivalent to the functor $- \times \mathcal{B} : \mathrm{Cat} \rightarrow \mathrm{Cat}_{/\mathcal{B}}$. □

Notation 6.3.15. Let $\mathcal{O}$ be an operad, with category of operators $\mathcal{O}^{\otimes}$. We denote by $\mathscr{O}p_{/\mathcal{O}}$ the 2-subcategory of $\mathscr{C}at_{/\mathcal{O}^{\otimes}}$ with objects the $\mathcal{O}$-operads, morphisms the morphisms of $\mathcal{O}$-operads, and all 2-cells. We call $\mathscr{O}p_{/\mathcal{O}}$ the 2-category of $\mathcal{O}$-operads. In the case when $\mathcal{O} = \mathrm{Comm}$ we will use the notation $\mathscr{O}p$ and call it the 2-category of operads.

Corollary 6.3.16. *Let $\mathcal{O}$ be an operad. Then the 2-category $\mathrm{Alg}_{\mathcal{O}}(\mathscr{C}at)$ is equivalent to the 2-subcategory of $\mathscr{O}p_{/\mathcal{O}}$ on the $\mathcal{O}$-monoidal categories and the strictly $\mathcal{O}$-monoidal functors.*

Proof. Let $\mathcal{O}^{\otimes}$ be the category of operators of $\mathcal{O}$. Using proposition 6.3.14 we obtain an equivalence $\mathscr{C}at_{/\mathcal{O}^{\otimes}} = \mathrm{Funct}(\mathcal{O}^{\otimes}, \mathscr{C}at)$. The result now follows by restricting this equivalence to the full subcategories on the $\mathcal{O}$-monoidal categories. □

6.4 Enriched props

The notions of enriched prop and operad will be particular cases of the notion of enriched envelope, which we now introduce.

Definition 6.4.1. *Let $\mathcal{M}$ be a presentable symmetric monoidal category. An $\mathcal{M}$-enriched pre-envelope is a pair $(\mathcal{P}, P)$ of a commutative algebra $\mathcal{P}$ in $\mathrm{Algbrd}(\mathcal{M})_{\mathrm{Spc}}$, together with a subspace P of the space of objects of $\mathcal{P}$. We say that an $\mathcal{M}$-enriched pre-envelope $(\mathcal{P}, P)$ is an $\mathcal{M}$-enriched envelope if the full subalgebroid of $\mathcal{P}$ on P is an $\mathcal{M}$-enriched category.*

Warning 6.4.2. Let $\mathcal{M}$ be a presentable symmetric monoidal category and let $(\mathcal{P}, P)$ be an $\mathcal{M}$-enriched envelope. Then $\mathcal{P}$ is in general only a (symmetric monoidal) $\mathcal{M}$-enriched algebroid - completeness is only required for its full subalgebroid on P.

Notation 6.4.3. Let $\mathcal{M}$ be a presentable symmetric monoidal category. For each object $\mathcal{A}$ in $\mathrm{Algbrd}(\mathcal{M})_{\mathrm{Spc}}$ we will denote by $\mathcal{A}^{\leq 0}$ the space of objects of $\mathcal{A}$. If $\mathcal{A}$ is the algebroid underlying a commutative algebra object in $\mathrm{Algbrd}(\mathcal{M})_{\mathrm{Spc}}$, we will equip $\mathcal{A}^{\leq 0}$ with its structure of commutative algebra in spaces arising from the symmetric monoidal structure of the projection $\mathrm{Algbrd}(\mathcal{M})_{\mathrm{Spc}} \rightarrow \mathrm{Spc}$.

Let $\mathrm{Spc}_{\mathrm{sub}}$ be the full subcategory of the arrow category $\mathrm{Funct}([1], \mathrm{Spc})$ on the monomorphisms. Let $\mathrm{preEnvlp}(\mathcal{M})$ be the category arising as the pullback

$$\begin{tikzcd} \mathrm{preEnvlp}(\mathcal{M}) \arrow[r] \arrow[d] & \mathrm{CAlg}(\mathrm{Algbrd}(\mathcal{M})_{\mathrm{Spc}}) \arrow[d, "(-)^{\leq 0}"] \\ \mathrm{Spc}_{\mathrm{sub}} \arrow[r, "\mathrm{ev}_1"] & \mathrm{Spc}\,. \end{tikzcd}$$

We call $\mathrm{preEnvlp}(\mathcal{M})$ the category of $\mathcal{M}$-enriched pre-envelopes. We will typically denote by

$$q : \mathrm{preEnvlp}(\mathcal{M}) \to \mathrm{CAlg}(\mathrm{Algbrd}(\mathcal{M})_{\mathrm{Spc}})$$

the projection. This admits a section

$$s : \mathrm{CAlg}(\mathrm{Algbrd}(\mathcal{M})_{\mathrm{Spc}}) \to \mathrm{preEnvlp}(\mathcal{M})$$

which is induced by pullback of the section of ev_1 obtained by corestriction of the diagonal map $\Delta : \mathrm{Spc} \to \mathrm{Funct}([1], \mathrm{Spc})$.

We denote by $\mathrm{Envlp}^{\mathcal{M}}$ the full subcategory of $\mathrm{preEnvlp}(\mathcal{M})$ on the $\mathcal{M}$-enriched envelopes. We call this the category of $\mathcal{M}$-enriched envelopes.

Remark 6.4.4. Let $\mathcal{M}$ be a presentable symmetric monoidal category and let $(\mathcal{P}, P)$ and $(\mathcal{P}', P')$ be a pair of $\mathcal{M}$-enriched pre-envelopes. Then a morphism from $(\mathcal{P}, P)$ to $(\mathcal{P}', P')$ in $\mathrm{preEnvlp}(\mathcal{M})$ is the same data as a symmetric monoidal functor $F : \mathcal{P} \to \mathcal{P}'$ with the property that $F(P)$ is contained in P'.

The projection q maps a pair $(\mathcal{P}, P)$ to $\mathcal{P}$. We call $\mathcal{P}$ the symmetric monoidal envelope of $(\mathcal{P}, P)$. The section s maps a symmetric monoidal $\mathcal{M}$-enriched algebroid $\mathcal{P}$ to the pair $(\mathcal{P}, (\mathcal{P})^{\leq 0})$. We call this the $\mathcal{M}$-enriched pre-envelope underlying $\mathcal{P}$. Observe that the identification $qs = \mathrm{id}_{\mathrm{CAlg}(\mathrm{Algbrd}(\mathcal{M})_{\mathrm{Spc}})}$ presents s as right adjoint to q, so that $\mathrm{CAlg}(\mathrm{Algbrd}(\mathcal{M})_{\mathrm{Spc}})$ is a localization of $\mathrm{preEnvlp}(\mathcal{M})$. It follows that we have a localization

$$q' : \mathrm{Envlp}^{\mathcal{M}} \rightleftarrows \mathrm{CAlg}(\mathrm{Cat}^{\mathcal{M}}) : s|_{\mathrm{CAlg}(\mathrm{Cat}^{\mathcal{M}})}$$

where the left adjoint q' is the composition of the localization map q with the localization functor $\mathrm{CAlg}(\mathrm{Algbrd}(\mathrm{Spc})_{\mathrm{Spc}}) \to \mathrm{CAlg}(\mathrm{Cat})$. For each $\mathcal{M}$-enriched envelope $(\mathcal{P}, P)$, we call $q'(\mathcal{P}, P)$ its symmetric monoidal envelope. Given a symmetric monoidal $\mathcal{M}$-enriched category $\mathcal{P}$, we call $s(\mathcal{P})$ the $\mathcal{M}$-enriched envelope underlying $\mathcal{P}$.

Remark 6.4.5. The inclusion $\mathrm{Spc}_{\mathrm{sub}} \to \mathrm{Funct}([1], \mathrm{Spc})$ preserves filtered colimits, and has a left adjoint which sends a morphism of spaces $f : X \to Y$ to the pair $(Y, \mathrm{im}(f))$. In particular we see that $\mathrm{Spc}_{\mathrm{sub}}$ is an accessible localization of the presentable category $\mathrm{Funct}([1], \mathrm{Spc})$, and hence $\mathrm{Spc}_{\mathrm{sub}}$ is itself presentable.

Let $\mathcal{M}$ be a presentable symmetric monoidal category. Note that the projection $(-)^{\leq 0}$: $\mathrm{CAlg}(\mathrm{Algbrd}(\mathcal{M})_{\mathrm{Spc}}) \to \mathrm{Spc}$ admits a left adjoint, given by the composition

$$\mathrm{Spc} \to \mathrm{Algbrd}(\mathrm{Spc})_{\mathrm{Spc}} \to \mathrm{Algbrd}(\mathcal{M})_{\mathrm{Spc}} \to \mathrm{CAlg}(\mathrm{Algbrd}(\mathcal{M})_{\mathrm{Spc}})$$

where the first arrow is the canonical inclusion, the second arrow is induced by the unit map $\mathrm{Spc} \to \mathcal{M}$, and the third arrow is given by taking free commutative algebras. Furthermore, the projection $\mathrm{Spc}_{\mathrm{sub}} \xrightarrow{\mathrm{ev}_1} \mathrm{Spc}$ also admits a left adjoint, given by the diagonal map.

We therefore see that the commutative square from notation 6.4.3 takes place in Pr^R. In particular, $\mathrm{preEnvlp}(\mathcal{M})$ is a presentable category.

For later purposes, we record the following result which allows one to construct functors into the category of pre-envelopes.

Proposition 6.4.6. *Let $\mathcal{I}$ be a category and let $F : \mathcal{I} \to \mathrm{Spc}$ be a functor. Assume given for each object i in $\mathcal{I}$ a subspace $P(i)$ of $F(i)$, with the property that for every arrow $\alpha : i \to j$ in $\mathcal{I}$ the image of $P(i)$ under $F(\alpha)$ is contained in $F(j)$. Then there exists a unique lift of F along the projection $u : \mathrm{Spc}_{\mathrm{sub}} \to \mathrm{Spc}$ to a functor*

$$F^{\mathrm{enh}} : \mathcal{I} \to \mathrm{Spc}_{\mathrm{sub}}$$

such that $F^{\mathrm{enh}}(i) = (F(i), P(i))$ for all i.

Proof. It follows from our hypothesis that the composite functor

$$\Delta F : \mathcal{I} \to \mathrm{Spc}_{\mathrm{sub}}$$

admits a subfunctor which maps each object i in $\mathcal{I}$ to the pair $(F(i), P(i))$. This shows the existence of lift of F with the desired conditions.

Assume now given such a lift F^{enh}. Consider the natural transformation $F^{\mathrm{enh}} \to \Delta u F^{\mathrm{enh}} = \Delta F$ induced from the unit of the adjunction $u \dashv \Delta$. This presents F^{enh} as a subfunctor of ΔF which maps each object i in $\mathcal{I}$ to the pair $(F(i), P(i))$. Our claim now follows from the fact that there is a unique such subfunctor. ☐

Corollary 6.4.7. *Let $\mathcal{M}$ be a presentable symmetric monoidal category. Let $\mathcal{I}$ be a category and let $F : \mathcal{I} \to \mathrm{CAlg}(\mathrm{Algbrd}(\mathcal{M})_{\mathrm{Spc}})$ be a functor. Assume given for each object i in $\mathcal{I}$ a subspace $P(i)$ of the space of objects of $F(i)$, with the property that for every arrow $\alpha : i \to j$ in $\mathcal{I}$ the image of $P(i)$ under $F(\alpha)$ is contained in $F(j)$. Then there exists a unique lift of F along q to a functor*

$$F^{\mathrm{enh}} : \mathcal{I} \to \mathrm{preEnvlp}(\mathcal{M})$$

such that $F^{\mathrm{enh}}(i) = (F(i), P(i))$ for all i.

Example 6.4.8. Let $\mathcal{M}$ be a presentable symmetric monoidal category. Consider the commutative square of categories

$$\begin{tikzcd}
\mathrm{CAlg}(\mathrm{Algbrd}(\mathcal{M})_{\mathrm{Spc}}) \arrow[r] \arrow[d, "\mathrm{CAlg}((-)^{\leq 0})"] & \mathrm{Algbrd}(\mathcal{M})_{\mathrm{Spc}} \arrow[d, "(-)^{\leq 0}"] \\
\mathrm{CAlg}(\mathrm{Spc}) \arrow[r] & \mathrm{Spc}
\end{tikzcd}$$

where the horizontal arrows are the forgetful functors. It follows from proposition 3.3.12 that the above square is horizontally left adjointable. Let

$$\mathrm{Sym} : \mathrm{Algbrd}(\mathcal{M})_{\mathrm{Spc}} \to \mathrm{CAlg}(\mathrm{Algbrd}(\mathcal{M})_{\mathrm{Spc}})$$

be the left adjoint to the forgetful functor. Note that for each object $\mathcal{A}$ in $\mathrm{Algbrd}(\mathcal{M})_{\mathrm{Spc}}$ the space of objects of $\mathrm{Sym}(\mathcal{A})$ is the free commutative algebra in spaces on $\mathcal{A}^{\leq 0}$. An application of corollary 6.4.7 shows that Sym admits a unique lift to a functor

$$\mathrm{Sym}^{\mathrm{enh}} : \mathrm{Algbrd}(\mathcal{M})_{\mathrm{Spc}} \to \mathrm{preEnvlp}(\mathcal{M})$$

which maps each object $\mathcal{A}$ to $(\mathrm{Sym}(\mathcal{A}), \mathcal{A}^{\leq 0})$.

Observe that $\mathrm{Sym}^{\mathrm{enh}}$ is fully faithful. Let $(\mathcal{P}, P)$ be an $\mathcal{M}$-enriched pre-envelope, and denote by $\mathcal{A}$ the full subalgebroid of $\mathcal{P}$ on P. Then the morphism $\mathrm{Sym}^{\mathrm{enh}}(\mathcal{A}) \to (\mathcal{P}, P)$ induced from the inclusion $\mathcal{A} \to \mathcal{P}$ presents $\mathcal{A}$ as right adjoint to $\mathrm{Sym}^{\mathrm{enh}}$ at $(\mathcal{P}, P)$. We call $\mathcal{A}$ the $\mathcal{M}$-algebroid underlying $(\mathcal{P}, P)$.

We may summarize this by saying that the category of $\mathcal{M}$-algebroids with a space of objects is a colocalization of the category of $\mathcal{M}$-enriched pre-envelopes. Note that this restricts to a colocalization

$$\mathrm{Sym}^{\mathrm{enh}}|_{\mathrm{Cat}^{\mathcal{M}}} : \mathrm{Cat}^{\mathcal{M}} \underset{}{\overset{}{\longleftrightarrows}} \mathrm{Envlp}^{\mathcal{M}} .$$

Remark 6.4.9. Let $\mathcal{M}$ be a presentable symmetric monoidal category and let I be the walking isomorphism inside $\mathrm{Algbrd}(\mathrm{Spc})_{\mathrm{Spc}}$. Then the $\mathcal{M}$-enriched envelopes sit inside $\mathrm{preEnvlp}(\mathcal{M})$ as the objects which are local for the morphism $\mathrm{Sym}^{\mathrm{enh}}(I_{\mathcal{M}}) \to \mathrm{Sym}^{\mathrm{enh}}(1_{\mathcal{M}})$ induced from the projection $I \to [0]$, where $\mathrm{Sym}^{\mathrm{enh}}$ is as in example 6.4.8. It follows from this, together with remark 6.4.5 that $\mathrm{Envlp}^{\mathcal{M}}$ is an accessible localization of $\mathrm{preEnvlp}(\mathcal{M})$. In particular, $\mathrm{Envlp}^{\mathcal{M}}$ is presentable.

We now introduce the notion of $\mathcal{M}$-enriched prop.

Definition 6.4.10. *Let $\mathcal{M}$ be a presentable symmetric monoidal category. We say that an $\mathcal{M}$-enriched pre-envelope $(\mathcal{P}, P)$ is an $\mathcal{M}$-enriched pre-prop if the inclusion $P \to \mathcal{P}^{\leq 0}$ presents $\mathcal{P}^{\leq 0}$ as the free commutative algebra in spaces on the space P. We say that $(\mathcal{P}, P)$ is an $\mathcal{M}$-enriched prop if it is an $\mathcal{M}$-enriched pre-prop and an $\mathcal{M}$ enriched envelope.*

Notation 6.4.11. Let $\mathcal{M}$ be a presentable symmetric monoidal category. We denote by $\mathrm{preProp}(\mathcal{M})$ the full subcategory of $\mathrm{preEnvlp}(\mathcal{M})$ on the $\mathcal{M}$-enriched pre-props. We call this the category of $\mathcal{M}$-enriched pre-props. We let $\mathrm{Prop}^{\mathcal{M}} = \mathrm{preProp}(\mathcal{M}) \cap \mathrm{Envlp}^{\mathcal{M}}$, and call it the category of $\mathcal{M}$-enriched props.

Remark 6.4.12. Let $\mathcal{M}$ be a presentable symmetric monoidal category and let $(\mathcal{P}, P)$ be an $\mathcal{M}$-enriched pre-envelope. Then it follows from the characterization of free algebras from

[Lur17] definition 3.1.3.1 that the pair $(\mathcal{P}, P)$ is an $\mathcal{M}$-enriched pre-prop if and only if the maps $P^S \xrightarrow{\otimes} \mathcal{P}^{\leq 0}$ induce an equivalence

$$\mathcal{P}^{\leq 0} = \operatorname{colim}_{\mathrm{Fin}^{\leq 0}} P^S,$$

where $\mathrm{Fin}^{\leq 0}$ denotes the space of finite sets.

If $(\mathcal{P}, P)$ is an $\mathcal{M}$-enriched pre-prop we will usually identify objects of $\mathcal{P}$ with finite families of objects of P. We call P the space of objects of $(\mathcal{P}, P)$. Given a pair of finite sets S, T and families $\{x_s\}_{s\in S}$ and $\{y_t\}_{t\in T}$ of objects of P, we call

$$\mathrm{Hom}_{\mathcal{P}}(\{x_s\}_{s\in S}, \{y_t\}_{t\in T})$$

the object of operations in $(\mathcal{P}, P)$ with source $\{x_s\}_{s\in S}$ and target $\{y_s\}_{s\in S}$.

Definition 6.4.13. *Let $\mathcal{M}$ be a presentable symmetric monoidal category. We say that a morphism of $\mathcal{M}$-enriched pre-envelopes $F : (\mathcal{P}, P) \to (\mathcal{P}', P')$ is a P-equivalence if F induces an equivalence of spaces $P = P'$, and for every pair of finite families $\{x_s\}_{s\in S}$ and $\{y_t\}_{t\in T}$ of objects of $P = P'$, the induced morphism*

$$F_* : \mathrm{Hom}_{\mathcal{P}}\left(\bigotimes_{s\in S} x_s, \bigotimes_{t\in T} y_t\right) \to \mathrm{Hom}_{\mathcal{P}'}\left(\bigotimes_{s\in S} x_s, \bigotimes_{t\in T} y_t\right)$$

is an equivalence.

Remark 6.4.14. Let $\mathcal{M}$ be a presentable symmetric monoidal category and let $F : (\mathcal{P}, P) \to (\mathcal{P}', P')$ be a P-equivalence of $\mathcal{M}$-enriched pre-envelopes. If $(\mathcal{P}', P')$ is an $\mathcal{M}$-enriched envelope then $(\mathcal{P}, P)$ is also an $\mathcal{M}$-enriched envelope.

Proposition 6.4.15. *Let $\mathcal{M}$ be a presentable symmetric monoidal category. The inclusion $i : \mathrm{preProp}(\mathcal{M}) \to \mathrm{preEnvlp}(\mathcal{M})$ admits a right adjoint i^R. Furthermore, a morphism of $\mathcal{M}$-enriched pre-envelopes is inverted by i^R if and only if it is a P-equivalence.*

Proof. It follows from proposition 3.5.5 that the projection

$$(-)^{\leq 0} : \mathrm{Algbrd}(\mathcal{M})_{\mathrm{Spc}} \to \mathrm{Spc}$$

admits the structure of cartesian fibration of operads. Hence we have that the induced projection

$$\mathrm{CAlg}((-)^{\leq 0}) : \mathrm{CAlg}(\mathrm{Algbrd}(\mathcal{M})_{\mathrm{Spc}}) \to \mathrm{CAlg}(\mathrm{Spc})$$

is a cartesian fibration. Furthermore, a morphism of commutative algebras in $\mathrm{Algbrd}(\mathcal{M})_{\mathrm{Spc}}$ is cartesian for $\mathrm{CAlg}((-)^{\leq 0})$ if and only if the underlying morphism in $\mathrm{Algbrd}(\mathcal{M})_{\mathrm{Spc}}$ is cartesian for $(-)^{\leq 0}$ (in other words, fully faithful).

Let $\mathrm{CAlg}(\mathrm{Spc})_{\mathrm{sub}}$ be the category arising as the pullback

$$\begin{tikzcd} \mathrm{CAlg}(\mathrm{Spc})_{\mathrm{sub}} \arrow[r] \arrow[d] & \mathrm{CAlg}(\mathrm{Spc}) \arrow[d] \\ \mathrm{Spc}_{\mathrm{sub}} \arrow[r, "\mathrm{ev}_1"] & \mathrm{Spc}\,. \end{tikzcd}$$

where the right vertical arrow is the forgetful functor. We have a pullback square

$$\begin{tikzcd} \mathrm{preEnvlp}(\mathcal{M}) \arrow[r] \arrow[d, "p"] & \mathrm{CAlg}(\mathrm{Algbrd}(\mathcal{M})_{\mathrm{Spc}}) \arrow[d, "\mathrm{CAlg}((-)^{\leq 0})"] \\ \mathrm{CAlg}(\mathrm{Spc})_{\mathrm{sub}} \arrow[r] & \mathrm{CAlg}(\mathrm{Spc}). \end{tikzcd}$$

Since the right vertical arrow is a cartesian fibration, we have that the left vertical arrow p is a cartesian fibration as well. Furthermore, a morphism of $\mathcal{M}$-enriched pre-envelopes is cartesian for p if and only if the underlying functor of symmetric monoidal $\mathcal{M}$-enriched categories is fully faithful.

Let $\mathrm{Sym} : \mathrm{Spc} \to \mathrm{CAlg}(\mathrm{Spc})$ be the left adjoint to the forgetful functor. By proposition 6.4.6, this admits a unique lift $\mathrm{Sym}^{\mathrm{enh}} : \mathrm{Spc} \to \mathrm{CAlg}(\mathrm{Spc})_{\mathrm{sub}}$ along the projection $\mathrm{CAlg}(\mathrm{Spc})_{\mathrm{sub}} \to \mathrm{CAlg}(\mathrm{Spc})$, such that $\mathrm{Sym}^{\mathrm{enh}}(X)$ corresponds to $\mathrm{Sym}(X)$ together with the inclusion $X \to \mathrm{Sym}(X)$.

Observe that $\mathrm{Sym}^{\mathrm{enh}}$ is fully faithful. Furthermore, for every pair (X, Y) of a commutative algebra in spaces X and a subspace Y in X, the morphism of commutative algebras $\mathrm{Sym}(Y) \to X$ induced from the identity on Y presents Y as right adjoint to $\mathrm{Sym}^{\mathrm{enh}}$ at (X, Y). It follows in particular that $\mathrm{Sym}^{\mathrm{enh}}$ admits a right adjoint. Since p is a cartesian fibration, we conclude that the inclusion

$$j : \mathrm{Spc} \times_{\mathrm{CAlg}(\mathrm{Spc})_{\mathrm{sub}}} \mathrm{CAlg}(\mathrm{Algbrd}(\mathcal{M})_{\mathrm{Spc}})_{\mathrm{sub}} \to \mathrm{preEnvlp}(\mathcal{M})$$

admits a right adjoint as well, which maps an $\mathcal{M}$-enriched pre-envelope $(\mathcal{P}, P)$ to the source of the p-cartesian lift of the unit map $\mathrm{Sym}(P) \to (\mathcal{P})^{\leq 0}$.

Observe that j is in fact equivalent to the inclusion $i : \mathrm{preProp}(\mathcal{M}) \to \mathrm{preEnvlp}(\mathcal{M})$. Hence i admits a right adjoint. The characterization of morphisms which are inverted by i^R follows from the above description of j^R, together with the description of p-cartesian arrows. □

Corollary 6.4.16. *Let $\mathcal{M}$ be a presentable symmetric monoidal category. The inclusion $i' : \mathrm{Prop}^{\mathcal{M}} \to \mathrm{Envlp}^{\mathcal{M}}$ admits a right adjoint i'^R. Furthermore, a morphism of $\mathcal{M}$-enriched envelopes is inverted by i'^R if and only if it is a P-equivalence.*

Proof. It follows from remark 6.4.14 together with proposition 6.4.15 that the functor $i^R : \mathrm{preEnvlp}(\mathcal{M}) \to \mathrm{preProp}(\mathcal{M})$ maps $\mathcal{M}$-enriched envelopes to $\mathcal{M}$-enriched props. Hence i^R restricts to provide a right adjoint to i'. □

Corollary 6.4.17. *Let $\mathcal{M}$ be a presentable symmetric monoidal category. Then an $\mathcal{M}$-enriched (pre-) envelope $(\mathcal{P}, P)$ is an $\mathcal{M}$-enriched (pre-) prop if and only if for every P-equivalence of $\mathcal{M}$-enriched (pre-) envelopes $F : (\mathcal{P}', P') \to (\mathcal{P}'', P'')$, the induced morphism of spaces*

$$F_* : \mathrm{Hom}_{\mathrm{preEnvlp}(\mathcal{M})}((\mathcal{P}, P), (\mathcal{P}', P')) \to \mathrm{Hom}_{\mathrm{preEnvlp}(\mathcal{M})}((\mathcal{P}, P), (\mathcal{P}'', P''))$$

is an equivalence.

Example 6.4.18. Let $\mathcal{M}$ be a presentable symmetric monoidal category. It follows from corollary 6.4.17 that for any $\mathcal{A}$ in $\mathrm{Algbrd}(\mathcal{M})_{\mathrm{Spc}}$, the $\mathcal{M}$-enriched pre-envelope $\mathrm{Sym}^{\mathrm{enh}}(\mathcal{A})$ from example 6.4.8 is an $\mathcal{M}$-enriched pre-prop. We therefore see that $\mathrm{Algbrd}(\mathcal{M})_{\mathrm{Spc}}$ (resp. $\mathrm{Cat}^{\mathcal{M}}$) sits inside $\mathrm{preProp}(\mathcal{M})$ (resp. $\mathrm{Prop}^{\mathcal{M}}$) as a colocalization.

Example 6.4.19. Let $\mathcal{M}$ be a presentable symmetric monoidal category. Let S, T be two finite sets, and let m be an object in $\mathcal{M}$. Let $S_{\mathcal{M}} = (1_{\mathcal{M}})^{\sqcup S}$ and $T_{\mathcal{M}} = (1_{\mathcal{M}})^{\sqcup T}$, and denote by $\mathrm{Sym} : \mathrm{Algbrd}(\mathcal{M})_{\mathrm{Spc}} \to \mathrm{CAlg}(\mathrm{Algbrd}(\mathcal{M})_{\mathrm{Spc}})$ the left adjoint to the forgetful functor.

We let $C_{S,T,m}$ be the symmetric monoidal $\mathcal{M}$-algebroid obtained as the pushout

$$\begin{tikzcd}
\mathrm{Sym}(1_{\mathcal{M}} \sqcup 1_{\mathcal{M}}) \arrow[r] \arrow[d] & \mathrm{Sym}(S_{\mathcal{M}} \sqcup T_{\mathcal{M}}) \arrow[d] \\
\mathrm{Sym}(C_m) \arrow[r] & C_{S,T,m}
\end{tikzcd}$$

where the left vertical arrow is induced from the source-target map $1_{\mathcal{M}} \sqcup 1_{\mathcal{M}} \to C_m$, and the top horizontal arrow is induced from the map $1_{\mathcal{M}} \sqcup 1_{\mathcal{M}} \to \mathrm{Sym}(S_{\mathcal{M}} \sqcup T_{\mathcal{M}})$ which picks out the objects $\{s\}_{s \in S}$ and $\{t\}_{t \in T}$.

We equip $C_{S,T,m}$ with the subspace of its space of objects obtained as the image of the composite map

$$S \sqcup T \hookrightarrow (\mathrm{Sym}(S_{\mathcal{M}} \sqcup T_{\mathcal{M}}))^{\leq 0} \to (C_{S,T,m})^{\leq 0}.$$

It follows from corollary 6.4.17 that $C_{S,T,m}$ is an $\mathcal{M}$-enriched pre-prop. Observe that for every $\mathcal{M}$-enriched pre-prop $(\mathcal{P}, P)$, the data of a morphism $C_{S,T,m} \to (\mathcal{P}, P)$ is equivalent to the data of a pair of finite families $\{x_s\}_{s \in S}, \{y_t\}_{t \in T}$ of objects of P together with a map

$$m \to \mathrm{Hom}_{\mathcal{P}}(\{x_s\}_{s \in S}, \{y_t\}_{t \in T}).$$

We think about $C_{S,T,m}$ as the universal $\mathcal{M}$-enriched pre-prop with an m-operation of arity S, T.

Remark 6.4.20. Let $\mathcal{M}$ be a presentable symmetric monoidal category and let m be an object in $\mathcal{M}$. Then in the case when S, T are singleton sets, the $\mathcal{M}$-enriched pre-prop $C_{S,T,m}$ from example 6.4.19 recovers $\mathrm{Sym}^{\mathrm{enh}}(C_m)$.

Remark 6.4.21. Let $\mathcal{M}$ be a presentable symmetric monoidal category, and let κ be a regular cardinal such that $\mathcal{M}$ is κ-compactly generated. Then a morphism $F : (\mathcal{P}, P) \to (\mathcal{P}', P')$ of $\mathcal{M}$-enriched pre-envelopes is a P-equivalence if and only if the following two conditions are satisfied:

- Composition with F induces an equivalence

$$F_* : \mathrm{Hom}_{\mathrm{preEnvlp}(\mathcal{M})}(\mathrm{Sym}^{\mathrm{enh}}(1_{\mathcal{M}}), (\mathcal{P}, P)) \to \mathrm{Hom}_{\mathrm{preEnvlp}(\mathcal{M})}(\mathrm{Sym}^{\mathrm{enh}}(1_{\mathcal{M}}), (\mathcal{P}', P')).$$

- For every pair of finite sets S, T and every κ-compact object m in $\mathcal{M}$, composition with F induces an equivalence

$$F_* : \mathrm{Hom}_{\mathrm{preEnvlp}(\mathcal{M})}(C_{S,T,m}, (\mathcal{P}, P)) \to \mathrm{Hom}_{\mathrm{preEnvlp}(\mathcal{M})}(C_{S,T,m}, (\mathcal{P}', P')).$$

It follows from this together with remark 6.4.5 that $\mathrm{preProp}(\mathcal{M})$ is presentable, and generated under colimits by the objects $\mathrm{Sym}^{\mathrm{enh}}(1_{\mathcal{M}})$ and $C_{S,T,m}$.

As in remark 6.4.9, we have that $\mathrm{Prop}^{\mathcal{M}}$ sits inside $\mathrm{preProp}(\mathcal{M})$ as the objects which are local for the morphism $\mathrm{Sym}^{\mathrm{enh}}(I_{\mathcal{M}}) \to \mathrm{Sym}^{\mathrm{enh}}(1_{\mathcal{M}})$. We conclude that $\mathrm{Prop}^{\mathcal{M}}$ is an accessible localization of $\mathrm{preProp}^{\mathcal{M}}$, and in particular it is also presentable.

Our next goal is to show that the category of $\mathcal{M}$-enriched symmetric monoidal categories can be identified with a subcategory of the category of $\mathcal{M}$-enriched props.

Definition 6.4.22. *Let $\mathcal{M}$ be a presentable symmetric monoidal category and let $(\mathcal{P}, P)$ be an $\mathcal{M}$-enriched pre-envelope. We call its image under the colocalization map $i^R : \mathrm{preEnvlp}(\mathcal{M}) \to \mathrm{preProp}(\mathcal{M})$ the $\mathcal{M}$-enriched pre-prop underlying $(\mathcal{P}, P)$.*

Remark 6.4.23. Let $\mathcal{M}$ be a presentable symmetric monoidal category. It follows from a combination of remark 6.4.4 and proposition 6.4.15 that there is an adjunction

$$qi : \mathrm{preProp}(\mathcal{M}) \xrightleftharpoons{} \mathrm{CAlg}(\mathrm{Algbrd}(\mathcal{M})_{\mathrm{Spc}}) : i^R s.$$

Given an $\mathcal{M}$-enriched pre-prop $(\mathcal{P}, P)$, we call $qi(\mathcal{P}, P)$ its symmetric monoidal envelope. Given a symmetric monoidal $\mathcal{M}$-algebroid $\mathcal{P}$, we call $i^R s(\mathcal{P})$ the $\mathcal{M}$-enriched pre-prop underlying $\mathcal{C}$. It follows from corollary 6.4.16 that we also have an adjunction

$$q'i' : \mathrm{Prop}^{\mathcal{M}} \xrightleftharpoons{} \mathrm{CAlg}(\mathrm{Cat}^{\mathcal{M}}) : i'^R s|_{\mathrm{CAlg}(\mathrm{Cat}^{\mathcal{M}})}.$$

Proposition 6.4.24. *Let $\mathcal{M}$ be a presentable symmetric monoidal category and let $\mathcal{P}$ be an $\mathcal{M}$-enriched symmetric monoidal category. Then the counit map $q'i'i'^R s\mathcal{P} \to \mathcal{P}$ is a localization functor.*

Our proof of proposition 6.4.24 will use the following lemma:

Lemma 6.4.25. *Let $\mathcal{M}$ be a presentable symmetric monoidal category and let $F : \mathcal{C} \to \mathcal{D}$ be a functor of $\mathcal{M}$-enriched categories. Assume given a subspace D of the space of objects of $\mathcal{C}$ such that $F(D) = \mathcal{D}^{\leq 0}$ and for every object c in $\mathcal{C}$ and every object d in D the morphism*

$$F_* : \mathrm{Hom}_{\mathcal{C}}(c, d) \to \mathrm{Hom}_{\mathcal{D}}(Fc, Fd)$$

is an isomorphism. Then F is a localization functor.

Proof. Let $\mathcal{D}'$ be the full subcategory of $\mathcal{C}$ on D. The conditions guarantee that $F|_{\mathcal{D}'} : \mathcal{D}' \to \mathcal{D}$ is an equivalence. Let $G : \mathcal{D} \to \mathcal{C}$ be the composition of the inverse of $F|_{\mathcal{D}'}$ and the inclusion $\mathcal{D}' \to \mathcal{D}$. Then we have an equivalence $\epsilon : FG = \mathrm{id}_{\mathcal{D}}$. The conditions in the statement imply that ϵ presents G as right adjoint to F. Hence F admits a fully faithful right adjoint, as desired. □

Proof of proposition 6.4.24. The counit $q'i'i'^R s\mathcal{P} \to \mathcal{P}$ is the morphism of $\mathcal{M}$-enriched categories underlying the P-equivalence $\epsilon : i'i'^R s\mathcal{P} \to s\mathcal{P}$ obtained from the counit of the adjunction $i' \dashv i'^R$. The result follows from an application of lemma 6.4.25, where we equip $q'i'i'^R s\mathcal{P}$ with the subspace $\mathcal{P}^{\leq 0}$. □

Corollary 6.4.26. *Let $\mathcal{M}$ be a presentable symmetric monoidal category and let $\mathcal{P}$ be an $\mathcal{M}$-enriched symmetric monoidal category. Then the counit map $q'i'i'^R s\mathcal{P} \to \mathcal{P}$ is an epimorphism of $\mathcal{M}$-enriched symmetric monoidal categories.*

Proof. Combine remark 5.2.14, proposition 6.4.24, and lemma 11.2.7. □

Proposition 6.4.27. *Let $\mathcal{M}$ be a presentable symmetric monoidal category. Then the functor*

$$i'^R s|_{\mathrm{CAlg}(\mathrm{Cat}^{\mathcal{M}})} : \mathrm{CAlg}(\mathrm{Cat}^{\mathcal{M}}) \to \mathrm{Prop}^{\mathcal{M}}$$

from remark 6.4.23 is the inclusion of a subcategory.

Proof. Let $\mathcal{C}, \mathcal{D}$ be two $\mathcal{M}$-enriched symmetric monoidal categories. We have to show that the functor

$$(i'^R s)_* : \mathrm{Hom}_{\mathrm{CAlg}(\mathrm{Cat}^{\mathcal{M}})}(\mathcal{C}, \mathcal{D}) \to \mathrm{Hom}_{\mathrm{Prop}^{\mathcal{M}}}(i'^R s\mathcal{C}, i'^R s\mathcal{D})$$

is a monomorphism, and surjective on isomorphisms. Since $q'i'$ is left adjoint to $i'^R s$, we have that the composite map

$$\mathrm{Hom}_{\mathrm{Prop}^{\mathcal{M}}}(i'^R s\mathcal{C}, i'^R s\mathcal{D}) \xrightarrow{q'i'_*} \mathrm{Hom}_{\mathrm{Cat}^{\mathcal{M}}}(q'i'i'^R s\mathcal{C}, q'i'i'^R s\mathcal{D}) \xrightarrow{\epsilon_*} \mathrm{Hom}_{\mathrm{Cat}^{\mathcal{M}}}(q'i'i'^R s\mathcal{C}, \mathcal{D})$$

is an isomorphism, where the second arrow is induced by the counit of the adjunction. The induced map

$$\epsilon_*(q'i')_*(i'^R s)_* : \mathrm{Hom}_{\mathrm{CAlg}(\mathrm{Cat}^{\mathcal{M}})}(\mathcal{C}, \mathcal{D}) \to \mathrm{Hom}_{\mathrm{Cat}^{\mathcal{M}}}(q'i'i'^R s\mathcal{C}, \mathcal{D})$$

is equivalent to the map given by precomposition with the unit $q'i'i'^R s\mathcal{C} \to \mathcal{C}$. Applying corollary 6.4.26 we conclude that $(i'^R s)_*$ is a monomorphism.

It remains to show that $(i'^R s)_*$ is surjective on isomorphisms. In other words, we have to show that if we have an isomorphism of props $\alpha : i'^R s\mathcal{C} \to i'^R s\mathcal{D}$, the induced morphism of $\mathcal{M}$-enriched symmetric monoidal categories $\epsilon\alpha : q'i'i'^R s\mathcal{C} \to \mathcal{D}$ factors through $\mathcal{C}$. By lemma 11.2.7, it suffices to show that the underlying functor of $\mathcal{M}$-enriched categories factors through $\mathcal{C}$. Observe that we have a commutative square of $\mathcal{M}$-enriched categories

$$\begin{tikzcd}
\mathcal{C} \arrow[r] \arrow[d] & q'i'i'^R s\mathcal{C} \arrow[d, "\alpha"] \\
\mathcal{D} \arrow[r] & q'i'i'^R s\mathcal{D}
\end{tikzcd}$$

where the horizontal arrows are the inclusions of the categories underlying the $\mathcal{M}$-enriched props $i'^R s\mathcal{C}$ and $i'^R s\mathcal{D}$, and the vertical arrows are isomorphisms. Examining the proof of lemma 6.4.25, we see that the functor $\epsilon\alpha : q'i'i'^R s\mathcal{C} \to \mathcal{D}$ is left adjoint to the bottom horizontal arrow. Our claim now follows from the horizontal left adjointability of the above square. $\square$

6.5 Enriched operads

We now discuss the notion of enriched operads.

Definition 6.5.1. *Let $\mathcal{M}$ be a presentable symmetric monoidal category. We say that a morphism of $\mathcal{M}$-enriched pre-envelopes $F : (\mathcal{P}, P) \to (\mathcal{P}', P')$ is an O-equivalence if F induces an equivalence of spaces $P = P'$, and for every finite family $\{x_s\}_{s\in S}$ of objects of $P = P'$ and every object y in $P = P'$, the induced morphism*

$$F_* : \mathrm{Hom}_{\mathcal{P}}\left(\bigotimes_{s\in S} x_s, y\right) \to \mathrm{Hom}_{\mathcal{P}'}\left(\bigotimes_{s\in S} x_s, y\right)$$

is an equivalence. We say that an $\mathcal{M}$-enriched pre-envelope $(\mathcal{P}'', P'')$ is an $\mathcal{M}$-enriched pre-operad if for every O-equivalence $F : (\mathcal{P}, P) \to (\mathcal{P}', P')$ of $\mathcal{M}$-enriched pre-envelopes, the morphism

$$F_* : \mathrm{Hom}_{\mathrm{preEnvlp}(\mathcal{M})}((\mathcal{P}'', P''), (\mathcal{P}, P)) \to \mathrm{Hom}_{\mathrm{preEnvlp}(\mathcal{M})}((\mathcal{P}'', P''), (\mathcal{P}', P'))$$

induced by composition with F, is an isomorphism. We say that an $\mathcal{M}$-enriched pre-operad is an $\mathcal{M}$-enriched operad if it is an $\mathcal{M}$-enriched envelope.

Notation 6.5.2. Let $\mathcal{M}$ be a presentable symmetric monoidal category. We denote by $\mathrm{preOp}(\mathcal{M})$ the full subcategory of $\mathrm{preEnvlp}(\mathcal{M})$ on the $\mathcal{M}$-enriched pre-operads. We call this the category of $\mathcal{M}$-enriched pre-operads. We let $\mathrm{Op}^{\mathcal{M}} = \mathrm{preOp}(\mathcal{M}) \cap \mathrm{Envlp}^{\mathcal{M}}$ and call it the category of $\mathcal{M}$-enriched operads.

Remark 6.5.3. Let $\mathcal{M}$ be a presentable symmetric monoidal category. Then P-equivalences of $\mathcal{M}$-enriched pre-envelopes are also O-equivalences. It follows that $\mathcal{M}$-enriched (pre-) operads are also $\mathcal{M}$-enriched (pre-) props.

Remark 6.5.4. Let $\mathcal{M}$ be a presentable symmetric monoidal category and let $F : (\mathcal{P}, P) \to (\mathcal{P}', P')$ be an O-equivalence of $\mathcal{M}$-enriched pre-envelopes. If $(\mathcal{P}', P')$ is an $\mathcal{M}$-enriched envelope then $(\mathcal{P}, P)$ is also an $\mathcal{M}$-enriched envelope.

Example 6.5.5. Let $\mathcal{M}$ be a presentable symmetric monoidal category. Then for any $\mathcal{A}$ in $\mathrm{Algbrd}(\mathcal{M})_{\mathrm{Spc}}$, the $\mathcal{M}$-enriched pre-envelope $\mathrm{Sym}^{\mathrm{enh}}(\mathcal{A})$ from example 6.4.8 is an $\mathcal{M}$-enriched pre-operad. We therefore see that $\mathrm{Algbrd}(\mathcal{M})_{\mathrm{Spc}}$ (resp. $\mathrm{Cat}^{\mathcal{M}}$) sits inside $\mathrm{preOp}(\mathcal{M})$ (resp. $\mathrm{Op}^{\mathcal{M}}$) as a colocalization.

Example 6.5.6. Let $\mathcal{M}$ be a presentable symmetric monoidal category. Let S be a finite set and let m be an object in $\mathcal{M}$. Specializing example 6.4.19 to the case where T is the singleton set we obtain an $\mathcal{M}$-enriched pre-operad $C_{S,*,m}$. We think about $C_{S,*,m}$ as the universal $\mathcal{M}$-enriched pre-operad with an m-operation of arity S.

Remark 6.5.7. Let $\mathcal{M}$ be a presentable symmetric monoidal category, and let κ be a regular cardinal such that $\mathcal{M}$ is κ-compactly generated. Then a morphism $F : (\mathcal{P}, P) \to (\mathcal{P}', P')$ of $\mathcal{M}$-enriched pre-props is an O-equivalence if and only if the following two conditions are satisfied:

- Composition with F induces an equivalence

$$F_* : \mathrm{Hom}_{\mathrm{preEnvlp}(\mathcal{M})}(\mathrm{Sym}^{\mathrm{enh}}(1_{\mathcal{M}}), (\mathcal{P}, P)) \to \mathrm{Hom}_{\mathrm{preEnvlp}(\mathcal{M})}(\mathrm{Sym}^{\mathrm{enh}}(1_{\mathcal{M}}), (\mathcal{P}', P')).$$

- For every finite set S and every κ-compact object m in $\mathcal{M}$, composition with F induces an equivalence

$$F_* : \mathrm{Hom}_{\mathrm{preEnvlp}(\mathcal{M})}(C_{S,*,m}, (\mathcal{P}, P)) \to \mathrm{Hom}_{\mathrm{preEnvlp}(\mathcal{M})}(C_{S,*,m}, (\mathcal{P}', P')).$$

It follows from this together with remark 6.4.5 that $\mathrm{preOp}(\mathcal{M})$ is presentable, and generated under colimits by the objects $\mathrm{Sym}^{\mathrm{enh}}(1_{\mathcal{M}})$ and $C_{S,*,m}$. Moreover, the inclusion $j : \mathrm{preOp}(\mathcal{M}) \to \mathrm{preEnvlp}(\mathcal{M})$ admits a right adjoint j^R, and a morphism of $\mathcal{M}$-enriched pre-envelopes is inverted by j^R if and only if it is an O-equivalence.

As in remark 6.4.9, we have that $\mathrm{Op}^{\mathcal{M}}$ sits inside $\mathrm{preOp}(\mathcal{M})$ as the objects which are local for the morphism $\mathrm{Sym}^{\mathrm{enh}}(I_{\mathcal{M}}) \to \mathrm{Sym}^{\mathrm{enh}}(1_{\mathcal{M}})$. We conclude that $\mathrm{Op}^{\mathcal{M}}$ is an accessible localization of $\mathrm{Op}^{\mathcal{M}}$, and in particular it is also presentable. It follows from remark 6.5.4 that j^R maps $\mathcal{M}$-enriched envelopes to $\mathcal{M}$-enriched operads, and therefore it restricts to provide a right adjoint to the inclusion $j' : \mathrm{Op}^{\mathcal{M}} \to \mathrm{Envlp}^{\mathcal{M}}$.

Our next goal is to show that the category of $\mathcal{M}$-enriched symmetric monoidal categories can be identified with a subcategory of the category of $\mathcal{M}$-enriched operads.

Definition 6.5.8. *Let $\mathcal{M}$ be a presentable symmetric monoidal category and let $(\mathcal{P}, P)$ be an $\mathcal{M}$-enriched pre-envelope. We call its image under the colocalization map j^R : $\mathrm{preEnvlp}(\mathcal{M}) \to \mathrm{preOp}(\mathcal{M})$ the $\mathcal{M}$-enriched pre-operad underlying $(\mathcal{P}, P)$.*

Remark 6.5.9. Let $\mathcal{M}$ be a presentable symmetric monoidal category. It follows from remarks 6.4.4 and 6.5.7 that there is an adjunction

$$qj : \mathrm{preOp}(\mathcal{M}) \xrightleftharpoons{} \mathrm{CAlg}(\mathrm{Algbrd}(\mathcal{M})_{\mathrm{Spc}}) : j^R s.$$

Given an $\mathcal{M}$-enriched pre-operad $(\mathcal{P}, P)$, we call $qj(\mathcal{P}, P)$ its symmetric monoidal envelope. Given a symmetric monoidal $\mathcal{M}$-algebroid $\mathcal{P}$, we call $j^R s(\mathcal{P})$ the $\mathcal{M}$-enriched pre-operad underlying $\mathcal{C}$. Note that we also have an adjunction

$$q'j' : \mathrm{Op}^{\mathcal{M}} \xrightleftharpoons{} \mathrm{CAlg}(\mathrm{Cat}^{\mathcal{M}}) : j'^R s|_{\mathrm{CAlg}(\mathrm{Cat}^{\mathcal{M}})}.$$

Proposition 6.5.10. *Let $\mathcal{M}$ be a presentable symmetric monoidal category and let $\mathcal{P}$ be an $\mathcal{M}$-enriched symmetric monoidal category. Then the counit map $q'j'j'^R s\mathcal{P} \to \mathcal{P}$ is a localization functor.*

Proof. This follows from lemma 6.4.25, in the same way as proposition 6.4.24. □

Corollary 6.5.11. *Let $\mathcal{M}$ be a presentable symmetric monoidal category and let $\mathcal{P}$ be an $\mathcal{M}$-enriched symmetric monoidal category. Then the counit map $q'j'j'^R\mathcal{P} \to \mathcal{P}$ is an epimorphism of $\mathcal{M}$-enriched symmetric monoidal categories.*

Proof. Combine remark 5.2.14, proposition 6.5.10, and lemma 11.2.7. □

Corollary 6.5.12. *Let $\mathcal{M}$ be a presentable symmetric monoidal category. Then the functor*

$$j'^R s|_{\mathrm{CAlg}(\mathrm{Cat}^{\mathcal{M}})} : \mathrm{CAlg}(\mathrm{Cat}^{\mathcal{M}}) \to \mathrm{Op}^{\mathcal{M}}$$

from remark 6.5.9 is the inclusion of a subcategory.

Proof. This follows from corollary 6.5.11, using similar arguments to those in proposition 6.4.27. □

We now show that, in the case $\mathcal{M} = \mathrm{Spc}$, our notion of operad recovers the usual notion. The key ingredient is supplied by the theory of monoidal envelopes from [Lur17] section 2.2.4.

Notation 6.5.13. We denote by

$$\mathrm{Env} : \mathrm{Op} \to \mathrm{CAlg}(\mathrm{Cat})$$

the left adjoint to the forgetful functor from symmetric monoidal categories to operads. For each operad $\mathcal{O}$, we let $P(\mathcal{O})$ be the subspace of $\mathrm{Env}(\mathcal{O})$ obtained as the image of the unit map $\mathcal{O} \to \mathrm{Env}(\mathcal{O})$. Note that for every morphism of operads $\alpha : \mathcal{O} \to \mathcal{O}'$ we have that the image of $P(\mathcal{O})$ under $\mathrm{Env}(\alpha)$ is contained in $P(\mathcal{O}')$. It follows from corollary 6.4.7, that we have a unique lift

$$\mathrm{Env}^{\mathrm{enh}} : \mathrm{Op} \to \mathrm{preEnvlp}(\mathrm{Spc})$$

of Env along q such that $\mathrm{Env}^{\mathrm{enh}}(\mathcal{O}) = (\mathrm{Env}(\mathcal{O}), P(\mathcal{O}))$ for every operad $\mathcal{O}$.

Proposition 6.5.14. *The functor* $\mathrm{Env}^{\mathrm{enh}}$ *from notation 6.5.13 is fully faithful, and its image is* $\mathrm{Op}^{\mathrm{Spc}}$.

Our proof of proposition 6.5.14 will need some preliminary lemmas.

Lemma 6.5.15. *Let* $\mathcal{O}$ *be an operad. Then* $\mathrm{Env}^{\mathrm{enh}}(\mathcal{O})$ *is a* Spc*-enriched operad.*

Proof. We first show that $\mathrm{Env}^{\mathrm{enh}}(\mathcal{O})$ is a Spc-enriched prop. Since $\mathrm{Env}(\mathcal{O})$ is a symmetric monoidal category (as opposed to algebroid), we have that $\mathrm{Env}^{\mathrm{enh}}$ is a Spc-enriched envelope. The category underlying $\mathrm{Env}(\mathcal{O})$ is given by the wide subcategory $\mathcal{O}^{\otimes}_{\mathrm{act}}$ of the category of operators $\mathcal{O}^{\otimes}$ on the active morphisms. The embedding $\mathcal{O} \to \mathrm{Env}(\mathcal{O})$ is given, at the level of underlying categories, by the embedding of the fiber $p^{-1}(\langle 1 \rangle)$ inside $\mathcal{O}^{\otimes}_{\mathrm{act}}$. Given a finite set S and a collection of objects x_s of $\mathcal{O}$ indexed by S, the image of this collection under the tensoring map $\mathrm{Env}(\mathcal{O})^S \to \mathrm{Env}(\mathcal{O})$ is given by the object $\{x_s\}_{s\in S}$ in $\mathcal{O}^{\otimes}_{\mathrm{act}}$. It now follows from remark 6.4.12 that $\mathrm{Env}^{\mathrm{enh}}(\mathcal{O})$ is a Spc-enriched prop, as claimed.

We now show that $\mathrm{Env}^{\mathrm{enh}}(\mathcal{O})$ is a Spc-enriched operad. Let $F : (\mathcal{P}', P') \to (\mathcal{P}'', P'')$ be an O-equivalence of Spc-enriched envelopes. We have to show that the induced map of spaces

$$\mathrm{Hom}_{\mathrm{Envlp}^{\mathrm{Spc}}}(\mathrm{Env}^{\mathrm{enh}}(\mathcal{O}), (\mathcal{P}', P')) \xrightarrow{F_*} \mathrm{Hom}_{\mathrm{Envlp}^{\mathrm{Spc}}}(\mathrm{Env}^{\mathrm{enh}}(\mathcal{O}), (\mathcal{P}'', P''))$$

is an equivalence.

Let $\overline{\mathcal{P}'}$ and $\overline{\mathcal{P}''}$ be the symmetric monoidal categories underlying $\mathcal{P}'$ and $\mathcal{P}''$. The functor of Spc-algebroids $\mathcal{P}' \to \overline{\mathcal{P}'}$ is fully faithful, and moreover its restriction to the full subalgebroid of $\mathcal{P}'$ on P' is a fully faithful functor of categories, which means that it induces a monomorphism on spaces. Hence we can see P' as a subspace of $\overline{\mathcal{P}'}^{\leq 0}$. Similarly, we can see P'' as a subspace of $\overline{\mathcal{P}''}^{\leq 0}$. Denote by

$$\overline{F} : (\overline{\mathcal{P}'}, P') \to (\overline{\mathcal{P}''}, P'')$$

the induced morphism of Spc-enriched envelopes. Note that this fits into a commutative square of Spc-enriched envelopes

$$\begin{tikzcd}
(\mathcal{P}', P') \arrow[r, "F"] \arrow[d] & (\mathcal{P}'', P'') \arrow[d] \\
(\overline{\mathcal{P}'}, P') \arrow[r, "\overline{F}"] & (\overline{\mathcal{P}''}, P'').
\end{tikzcd}$$

Here the vertical arrows are P-equivalences and therefore the bottom horizontal arrow is an O-equivalence.

We now have a commutative square of spaces

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathrm{Envlp}^{\mathrm{Spc}}}(\mathrm{Env}^{\mathrm{enh}}(\mathcal{O}),(\mathcal{P}',P')) & \xrightarrow{F_*} & \mathrm{Hom}_{\mathrm{Envlp}^{\mathrm{Spc}}}(\mathrm{Env}^{\mathrm{enh}}(\mathcal{O}),(\mathcal{P}'',P'')) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{\mathrm{Envlp}^{\mathrm{Spc}}}(\mathrm{Env}^{\mathrm{enh}}(\mathcal{O}),(\overline{\mathcal{P}'},P')) & \xrightarrow{\overline{F}_*} & \mathrm{Hom}_{\mathrm{Envlp}^{\mathrm{Spc}}}(\mathrm{Env}^{\mathrm{enh}}(\mathcal{O}),(\overline{\mathcal{P}''},P'')).
\end{array}$$

Since $\mathrm{Env}^{\mathrm{enh}}(\mathcal{O})$ is a Spc-enriched prop, we have that the vertical arrows are equivalences. To show that the top horizontal arrow is an equivalence it suffices to show that the bottom horizontal arrow is an equivalence.

Let $\mathcal{O}'$ and $\mathcal{O}''$ be the full suboperads of $\overline{\mathcal{P}'}$ and $\overline{\mathcal{P}''}$ on P' and P'', respectively. The map $\overline{F}$ induces a morphism of operads $f : \mathcal{O}' \to \mathcal{O}''$. The bottom horizontal arrow in the above diagram is equivalent to the map

$$f_* : \mathrm{Hom}_{\mathrm{Op}}(\mathcal{O},\mathcal{O}') \to \mathrm{Hom}_{\mathrm{Op}}(\mathcal{O},\mathcal{O}'').$$

Since $\overline{F}$ is an O-equivalence, we have that f is an equivalence of operads. Hence f_* is an equivalence, and the lemma follows. □

Lemma 6.5.16. *Let $(\mathcal{P}, P)$ be a* Spc*-enriched operad. Then $(\mathcal{P}, P)$ belongs to the image of* $\mathrm{Env}^{\mathrm{enh}}$*.*

Proof. Let $\overline{\mathcal{P}}$ be the category underlying $\mathcal{P}$. As in the proof of lemma 6.5.15 we may see P as a subspace of $\overline{\mathcal{P}}$ and we have an induced P-equivalence of Spc-enriched envelopes $(\mathcal{P}, P) \to (\overline{\mathcal{P}}, P)$. Let $\mathcal{O}$ be the full suboperad of $\overline{\mathcal{P}}$ on P. Then the inclusion $\mathcal{O} \to \overline{\mathcal{P}}$ induces an O-equivalence of Spc-enriched envelopes $\overline{f} : \mathrm{Env}^{\mathrm{enh}}(\mathcal{O}) \to (\overline{\mathcal{P}}, P)$. Since $\mathrm{Env}^{\mathrm{enh}}(\mathcal{O})$ is a Spc-enriched prop, we may lift $\overline{f}$ to an O-equivalence of Spc-enriched envelopes $f : \mathrm{Env}^{\mathrm{enh}}(\mathcal{O}) \to (\mathcal{P}, P)$. The claim now follows from lemma 6.5.15 since O-equivalences between Spc-enriched operads are necessarily isomorphisms. □

Proof of proposition 6.5.14. Combining lemmas 6.5.15 and 6.5.16 we see that the image of $\mathrm{Env}^{\mathrm{enh}}$ consists of the Spc-enriched operads. It remains to show that $\mathrm{Env}^{\mathrm{enh}}$ is fully faithful. Let $\mathcal{O}, \mathcal{O}'$ be a pair of operads. We have maps of spaces

$$\mathrm{Hom}_{\mathrm{Op}}(\mathcal{O},\mathcal{O}') \xrightarrow{\mathrm{Env}^{\mathrm{enh}}_*} \mathrm{Hom}_{\mathrm{preEnvlp(Spc)}}(\mathrm{Env}^{\mathrm{enh}}(\mathcal{O}),\mathrm{Env}^{\mathrm{enh}}(\mathcal{O}'))$$

and

$$\mathrm{Hom}_{\mathrm{preEnvlp(Spc)}}(\mathrm{Env}^{\mathrm{enh}}(\mathcal{O}),\mathrm{Env}^{\mathrm{enh}}(\mathcal{O}')) \xrightarrow{q_*} \mathrm{Hom}_{\mathrm{CAlg(Cat)}}(\mathrm{Env}(\mathcal{O}),\mathrm{Env}(\mathcal{O}'))$$

whose composition recovers the map Env_* induced by Env. Observe that q_* presents $\mathrm{Hom}_{\mathrm{preEnvlp(Spc)}}(\mathrm{Env}^{\mathrm{enh}}(\mathcal{O}),\mathrm{Env}^{\mathrm{enh}}(\mathcal{O}'))$ as the subspace of $\mathrm{Hom}_{\mathrm{CAlg(Cat)}}(\mathrm{Env}(\mathcal{O}),\mathrm{Env}(\mathcal{O}'))$ on those morphisms of symmetric monoidal categories which map objects in $P(\mathcal{O})$ to objects in $P(\mathcal{O}')$. To show that $\mathrm{Env}^{\mathrm{enh}}$ is fully faithful, it suffices to show that Env_* presents

$\mathrm{Hom}_{\mathrm{Op}}(\mathcal{O}, \mathcal{O}')$ as the same subspace. This follows from the fact that Env_* is equivalent to the map

$$\mathrm{Hom}_{\mathrm{Op}}(\mathcal{O}, \mathcal{O}') \to \mathrm{Hom}_{\mathrm{Op}}(\mathcal{O}, \mathrm{Env}(\mathcal{O}'))$$

of composition with the unit $\mathcal{O}' \to \mathrm{Env}(\mathcal{O}')$, which is a fully faithful morphism of operads. $\square$

Chapter 7

Monadicity

The notions of monads and monadic functors are fundamental in category theory. Of central importance is the monadicity theorem:

Theorem 7.0.1 ([Lur17] theorem 4.7.3.5)**.** *Let $G : \mathcal{D} \to \mathcal{C}$ be a functor of categories. The following conditions are equivalent:*

(i) *The functor G is monadic: in other words, G admits a left adjoint F, and G is equivalent to the forgetful functor $\mathrm{LMod}_A(\mathcal{C}) \to \mathcal{C}$ for A the endomorphism monad of G.*

(ii) *There exists an algebra A in the monoidal category of endofunctors of $\mathcal{C}$ such that G is equivalent to the forgetful functor $\mathrm{LMod}_A(\mathcal{C}) \to \mathcal{C}$.*

(iii) *The functor G is conservative and creates geometric realizations of G-split simplicial objects.*

The notion of monadic functor only depends on the 2-categorical structure of $\mathscr{C}at$. We can therefore think about theorem 7.0.1 (in particular, the equivalence between the first two and the last item) as providing a characterization of monadic morphisms in $\mathscr{C}at$.

Our goal in this chapter is to extend the theory of monads and monadic morphisms to (possibly enriched) 2-categories, and prove a generalization of theorem 7.0.1 which characterizes monadic morphisms in the 2-category of $\mathcal{M}$-enriched categories for an arbitrary presentable symmetric monoidal category $\mathcal{M}$.

We begin in 7.1 with a general discussion of the theory of enriched categories of modules over an associative algebra in $\mathcal{M}$. We show here that if $\mathcal{D}$ is an $\mathcal{M}$-enriched category and y is an object of $\mathcal{D}$, then the representable functor $\mathrm{Hom}_{\mathcal{D}}(-, y)$ admits a canonical enhancement to a functor into the $\mathcal{M}$-enriched category of modules over $\mathrm{End}_{\mathcal{D}}(y)$.

In 7.2 we discuss the notions of monads and modules over a monad in an arbitrary 2-category $\mathcal{D}$. We discuss the notion of endomorphism monad of a morphism in $\mathcal{D}$, and show that morphisms admitting left adjoints admit an endomorphism monad. We study here the functoriality of the categories of modules over a monad.

In 7.3 we discuss the notion of Eilenberg-Moore object for a monad, as a classifying object for modules. We prove here a basic result on the equivalence of various notions of monadic morphism in $\mathcal{D}$, which generalizes the equivalence between items (i) and (ii) in theorem 7.0.1.

In 7.4 we specialize the notions of the previous section to the case when $\mathcal{D}$ is the 2-category of categories enriched in a presentable symmetric monoidal category $\mathcal{M}$. We prove here our main result (theorem 7.4.10), which generalizes the description of monadic morphisms in $\mathscr{C}at$ provided by item (iii) of theorem 7.0.1. Specializing to the case $\mathcal{M} = \omega\mathrm{Cat}$, we obtain a monadicity theorem for functors of ω-categories, which reduces to theorem 7.0.1 in the case when the ω-categories in question are 1-categories.

In 7.5 we generalize the notions of monads and monadic morphisms to an arbitrary enriched 2-category, and provide various equivalent conditions for a morphism to be monadic. We finish by specializing to the case of enrichment over $\omega\mathrm{Cat}$ to obtain a theory of monads and monadic morphisms in an arbitrary ω-category.

7.1 Enriched categories of modules

We now discuss the canonical enrichment of the category of modules over an associative algebra in a presentable symmetric monoidal category.

Notation 7.1.1. Let $\mathcal{M}$ be a presentable symmetric monoidal category and let A be an associative algebra object in $\mathcal{M}$. We denote by $B_{\mathcal{M}}A$ the image of A under the composite functor

$$\mathrm{Alg}(\mathcal{M}) = \mathrm{Algbrd}_{[0]}(\mathcal{M}) \to \mathrm{Algbrd}(\mathcal{M})_{\mathrm{Spc}} \to \mathrm{Cat}^{\mathcal{M}}$$

where the second arrow is the inclusion, and the last arrow is the localization map.

Let $\mathcal{C}$ be an $\mathcal{M}$-enriched category. We denote by $A\text{-}\mathrm{mod}^l(\mathcal{C})$ the $\mathcal{M}$-enriched category $\mathrm{Funct}(B_{\mathcal{M}}A, \mathcal{C})$. We call this the $\mathcal{M}$-enriched category of left A-modules in $\mathcal{C}$. We also set $A\text{-}\mathrm{mod}^r(\mathcal{C}) = A^{\mathrm{op}}\text{-}\mathrm{mod}^l(\mathcal{C})$ and call it the $\mathcal{M}$-enriched category of right A-modules in $\mathcal{C}$.

Remark 7.1.2. Let $\mathcal{M}$ be a presentable symmetric monoidal category and let $\mathcal{C}$ be a presentable module over $\mathcal{M}$. Let A be an algebra in $\mathcal{M}$. It follows from [Hin20a] proposition 6.3.7 that there is an equivalence

$$A\text{-}\mathrm{mod}^l(\theta_{\mathcal{M}}(\mathcal{C})) = \theta_{\mathcal{M}}(A\text{-}\mathrm{mod}^l(\mathcal{C}))$$

where on the right we equip $A\text{-}\mathrm{mod}^l(\mathcal{C})$ with its canonical structure of presentable $\mathcal{M}$-module. In particular, we see that the category underlying $A\text{-}\mathrm{mod}^l(\theta_{\mathcal{M}}(\mathcal{C}))$ is given by $A\text{-}\mathrm{mod}^l(\mathcal{C})$.

Assume now that $\mathcal{C} = \mathcal{M}$, so that $\theta_{\mathcal{M}}(\mathcal{C}) = \overline{\mathcal{M}}$ is the canonical $\mathcal{M}$-enrichment of $\mathcal{M}$. Then $A\text{-}\mathrm{mod}^l(\overline{\mathcal{M}})$ provides an $\mathcal{M}$-enrichment of the category of left A-modules in $\mathcal{M}$. In particular, when $\mathcal{M} = \omega\mathrm{Cat}$ we obtain ω-categories of modules over a monoidal ω-category.

Construction 7.1.3. Let $\mathcal{M}$ be a presentable symmetric monoidal category. Let $\mathcal{D}$ be an $\mathcal{M}$-enriched category, and let y be an object in $\mathcal{D}$. We equip the object $\mathrm{End}_{\mathcal{D}}(y)$ with the structure of associative algebra in $\mathcal{M}$ arising from its presentation as the object underlying the single object algebroid $i_y^! \mathcal{D}$, where i_y is the inclusion of y inside the space of objects of $\mathcal{D}$. Note that $B_{\mathcal{M}} \operatorname{End}_{\mathcal{D}}(y)$ is the full subcategory of $\mathcal{D}$ on those objects isomorphic to y.

Let $\operatorname{Hom}_{\mathcal{D}}(-, y)^{\mathrm{enh}}$ be the composite functor

$$\mathcal{D}^{\mathrm{op}} \to \operatorname{Funct}(\mathcal{D}, \overline{\mathcal{M}}) \to \operatorname{Funct}(B_{\mathcal{M}} \operatorname{End}_{\mathcal{D}}(y), \overline{\mathcal{M}}) = \operatorname{End}_{\mathcal{D}}(y)\text{-}\mathrm{mod}^{l}(\overline{\mathcal{M}})$$

where the first arrow is the Yoneda embedding, and the second arrow is given by restriction along the embedding $B \operatorname{End}_{\mathcal{D}}(y) \to \mathcal{D}$. Observe that the composition of $\operatorname{Hom}_{\mathcal{D}}(-, y)^{\mathrm{enh}}$ with the forgetful functor $\operatorname{End}_{\mathcal{D}}(y)\text{-}\mathrm{mod}^{l}(\overline{\mathcal{M}}) \to \overline{\mathcal{M}}$ recovers the functor $\mathcal{D}^{\mathrm{op}} \to \overline{\mathcal{M}}$ represented by y.

We conclude that for every object x in $\mathcal{D}$ there is a structure of left $\operatorname{End}_{\mathcal{D}}(y)$-module on $\operatorname{Hom}_{\mathcal{D}}(x, y)$, which is contravariantly functorial in x. Unpacking the definitions, we see that the action map

$$\operatorname{Hom}_{\mathcal{D}}(x, y) \otimes \operatorname{End}_{\mathcal{D}}(y) \to \operatorname{Hom}_{\mathcal{D}}(x, y)$$

is given by composition in $\mathcal{D}$.

Remark 7.1.4. Let $\mathcal{M}$ be a cartesian closed presentable category, and equip $\overline{\mathcal{M}}$ with its cartesian symmetric monoidal structure. Let A be an associative algebra in $\mathcal{M}$. ategory, and equip $\overline{\mathcal{M}}$ with its cartesian symmetric monoidal structure. Denote by $\operatorname{LMod}_A(\overline{\mathcal{M}})$ the fiber over A of the projection

$$\operatorname{Alg}_{\mathrm{LMod}}(\overline{\mathcal{M}}) \to \operatorname{Alg}_{\mathrm{Assos}}(\overline{\mathcal{M}})$$

induced by restriction along the inclusion Assos $\to$ LMod. Then there is an equivalence of $\mathcal{M}$-enriched categories

$$\operatorname{LMod}_A(\overline{\mathcal{M}}) = A\text{-}\mathrm{mod}^{l}(\overline{\mathcal{M}}).$$

which upon passage to underlying categories recovers the canonical equivalence

$$(\tau_{\mathcal{M}})_! \operatorname{LMod}_A(\overline{\mathcal{M}}) = \operatorname{LMod}_A(\mathcal{M}) = (\tau_{\mathcal{M}})_! A\text{-}\mathrm{mod}^{l}(\overline{\mathcal{M}}).$$

In particular, in the case $\mathcal{M} = \mathrm{Cat}$ we see that the 2-categories of modules arising from notation 7.1.1 agree with the usual 2-categories of categories tensored over a monoidal category.

7.2 Monads in a 2-category

We now discuss the notion of monad and module over a monad in an arbitrary 2-category.

Definition 7.2.1. *Let $\mathcal{D}$ be a 2-category and let y be an object in $\mathcal{D}$. The category of monads on y is the category $\operatorname{Alg}(\operatorname{End}_{\mathcal{D}}(y))$ of associative algebras in the monoidal category $\operatorname{End}_{\mathcal{D}}(y)$. Given a monad M on y, a module for M is a pair of an object x in $\mathcal{D}$ and an M-module in the $\operatorname{End}_{\mathcal{D}}(y)$-module category $\operatorname{Hom}_{\mathcal{D}}(x, y)$.*

Definition 7.2.2. *Let $\mathcal{D}$ be a 2-category. An endomorphism monad*[1] *for a morphism $g : x \to y$ in $\mathcal{D}$ is an endomorphism object for g, thought of as an object in the* $\mathrm{End}_{\mathcal{D}}(y)$*-module category* $\mathrm{Hom}_{\mathcal{D}}(x, y)$.

Remark 7.2.3. Let $\mathcal{D}$ be a 2-category. Let $g : x \to y$ be a morphism in $\mathcal{D}$, admitting an endomorphism monad $\mathrm{End}(g)$. Then g has the structure of $\mathrm{End}(g)$-module. Furthermore, for any monad M on y, restriction of scalars provides an equivalence between the space of morphisms of monads $M \to \mathrm{End}(g)$ and the space of M-module structures on g.

Adjunctions in $\mathcal{D}$ provide an abundant source of monads.

Proposition 7.2.4. *Let $\mathcal{D}$ be a 2-category and let $g : x \to y$ be a morphism in $\mathcal{D}$. Assume that g admits a left adjoint $g^L : y \to x$. Then g admits an endomorphism monad, whose underlying object of* $\mathrm{End}_{\mathcal{D}}(y)$ *is given by gg^L, and the action map $gg^Lg \to g$ is induced from the unit map $g^Lg \to \mathrm{id}_x$.*

Proof. Consider the adjunction of categories

$$g^* : \mathrm{End}_{\mathcal{D}}(y) \rightleftarrows \mathrm{Hom}_{\mathcal{D}}(x, y) : (g^L)^*$$

obtained from the adjunction $g^L \dashv g$ by applying the functor $\mathcal{D}^{1\text{-op}} \to \mathscr{C}at$ represented by y. For each object h in $\mathrm{End}_{\mathcal{D}}(y)$ we have an induced isomorphism of spaces

$$\mathrm{Hom}_{\mathrm{End}_{\mathcal{D}}(y)}(h, gg^L) = \mathrm{Hom}_{\mathrm{Hom}_{\mathcal{D}}(x,y)}(hg, g).$$

The above is obtained as the composite map

$$\mathrm{Hom}_{\mathrm{End}_{\mathcal{D}}(y)}(h, gg^L) \to \mathrm{Hom}_{\mathrm{Hom}_{\mathcal{D}}(x,y)}(hg, gg^Lg) \to \mathrm{Hom}_{\mathrm{Hom}_{\mathcal{D}}(x,y)}(hg, g)$$

where the first map is given by composition with g, and the second map is induced by the counit of the adjunction $g^L \dashv g$. We conclude that the morphism $gg^Lg \to g$ presents gg^L as the Hom object from g to g, and our result follows. $\square$

Remark 7.2.5. Let $\mathcal{D}$ be a 2-category. Let M be a monad on an object y of $\mathcal{D}$, and let $g : x \to y$ be a module for M. Assume that g admits a left adjoint g^L so that there is an endomorphism monad $\mathrm{End}(g)$ for g with underlying endomorphism gg^L. Then the morphism in $\mathrm{End}_{\mathcal{D}}(y)$ underlying the induced morphism of monads $M \to \mathrm{End}(g)$ can be written as the composition

$$M \to Mgg^L \to gg^L$$

where the first map is induced by the unit of the adjunction $g^L \dashv g$ and the second map is induced by the structure map $Mg \to g$.

We now study the functoriality properties of the categories of modules over a monad.

[1] An alternative name for this concept would have been codensity monad.

Construction 7.2.6. Let $\mathcal{D}$ be a 2-category and let y be an object in $\mathcal{D}$. Specializing construction 7.1.3 we obtain a functor of 2-categories

$$\mathrm{Hom}^{\mathrm{enh}}_{\mathcal{D}}(-,y) : \mathcal{D}^{1\text{-op}} \to \mathrm{End}_{\mathcal{D}}(y)\text{-mod}^{l}(\mathscr{C}at)$$

whose composition with the forgetful functor $\mathrm{End}_{\mathcal{D}}(y)\text{-mod}^{l}(\mathscr{C}at) \to \mathscr{C}at$ recovers the functor $\mathcal{D}^{1\text{-op}} \to \mathscr{C}at$ represented by y. We can think about $\mathrm{Hom}^{\mathrm{enh}}_{\mathcal{D}}(-,y)$ as a functor $\mathcal{D}^{1\text{-op}} \to \mathrm{Alg}_{\mathrm{LMod}}(\mathscr{C}at)$ whose composition with the forgetful functor $\mathrm{Alg}_{\mathrm{LMod}}(\mathscr{C}at) \to \mathrm{Alg}_{\mathrm{Assos}}(\mathscr{C}at)$ recovers the constant functor with value $\mathrm{End}_{\mathcal{D}}(y)$. Composing with the functor

$$\mathrm{Alg}_{\mathrm{LMod}}(\mathscr{C}at) \hookrightarrow \mathscr{O}p_{\mathrm{LMod}}(\mathscr{C}at) \xrightarrow{\mathrm{Alg}_{\mathrm{LMod}}(-)} \mathscr{C}at$$

induces a functor

$$F : \mathcal{D}^{1\text{-op}} \to \mathscr{C}at_{/\,\mathrm{Alg}_{\mathrm{Assos}}(\mathrm{End}_{\mathcal{D}}(y))}$$

which sends each object x in $\mathcal{D}$ to the category whose objects are pairs of a monad on y and a module for it in $\mathrm{Hom}^{\mathrm{enh}}_{\mathcal{D}}(x,y)$.

Observe that restriction along the inclusion of the module object in LMod induces a natural transformation

$$F \to \mathrm{Hom}_{\mathcal{D}}(-,y) \times \mathrm{Alg}_{\mathrm{Assos}}(\mathrm{End}_{\mathcal{D}}(y)).$$

Proposition 7.2.7. *Let $\mathcal{D}$ be a 2-category and let y be an object in $\mathcal{D}$. Then the functor*

$$F : \mathcal{D}^{1\text{-op}} \to \mathscr{C}at_{/\,\mathrm{Alg}_{\mathrm{Assos}}(\mathrm{End}_{\mathcal{D}}(y))}$$

from construction 7.2.6 factors through $\mathscr{C}at^{\mathrm{cart}}_{/\,\mathrm{Alg}_{\mathrm{Assos}}(\mathrm{End}_{\mathcal{D}}(y))}$.

Proof. For each object x in $\mathcal{D}$, we have that $F(x)$ is the category $\mathrm{Alg}_{\mathrm{LMod}}(\mathrm{Hom}^{\mathrm{enh}}_{\mathcal{D}}(x,y))$, equipped with its canonical forgetful functor to $\mathrm{Alg}_{\mathrm{Assos}}(\mathrm{End}_{\mathcal{D}}(y))$. This is a cartesian fibration thanks to [Lur17] corollary 4.2.3.2. Assume now given a morphism $\alpha : x \to x'$ in $\mathcal{D}$. Then we have a commutative square of categories

$$\begin{tikzcd}
\mathrm{Alg}_{\mathrm{LMod}}(\mathrm{Hom}^{\mathrm{enh}}_{\mathcal{D}}(x',y)) \arrow[r, "\alpha^*"] \arrow[d] & \mathrm{Alg}_{\mathrm{LMod}}(\mathrm{Hom}^{\mathrm{enh}}_{\mathcal{D}}(x,y)) \arrow[d] \\
\mathrm{Hom}_{\mathcal{D}}(x',y) \arrow[r, "\alpha^*"] & \mathrm{Hom}_{\mathcal{D}}(x,y).
\end{tikzcd}$$

where the vertical arrows are the forgetful functors. The top horizontal arrow is equivalent to the image of $F(\alpha)$ under the forgetful functor

$$\mathrm{Cat}_{/\,\mathrm{Alg}_{\mathrm{Assos}}(\mathrm{End}_{\mathcal{D}}(y))} \to \mathrm{Cat}\,.$$

Using [Lur17] corollary 4.2.3.2 we have that the vertical arrows create cartesian arrows for $F(x')$ and $F(x)$. We conclude that $F(\alpha)$ is a morphism of cartesian fibrations, and our result follows. $\square$

Construction 7.2.8. Let $\mathcal{D}$ be a 2-category. Composing the functor F from construction 7.2.6 with the straightening equivalence we obtain a functor

$$\mathcal{D}^{1\text{-op}} \rightarrow \mathrm{Funct}(\mathrm{Alg}_{\mathrm{Assos}}(\mathrm{End}_{\mathcal{D}}(y))^{\mathrm{op}}, \mathscr{C}at).$$

This induces a functor

$$(-)\text{-mod}^{l}(\mathrm{Hom}^{\mathrm{enh}}_{\mathcal{D}}(-,y)) : \mathcal{D}^{1\text{-op}} \times (\mathrm{Alg}_{\mathrm{Assos}}(\mathrm{End}_{\mathcal{D}}(y))^{\mathrm{op}} \rightarrow \mathscr{C}at.$$

Observe that the above comes equipped with a natural transformation

$$(-)\text{-mod}^{l}(\mathrm{Hom}^{\mathrm{enh}}_{\mathcal{D}}(-,y)) \rightarrow \mathrm{Hom}_{\mathcal{D}}(-,y).$$

In particular, for each monad M on y we have a functor

$$M\text{-mod}^{l}(\mathrm{Hom}^{\mathrm{enh}}_{\mathcal{D}}(-,y)) : \mathcal{D}^{1\text{-op}} \rightarrow \mathscr{C}at$$

equipped with a natural transformation into the representable presheaf $\mathrm{Hom}_{\mathcal{D}}(-,y)$.

Remark 7.2.9. Let $\mathcal{D}$ be a 2-category and let M be a monad on an object y of $\mathcal{D}$. Tracing the definitions reveals that the functor $M\text{-mod}^{l}(\mathrm{Hom}^{\mathrm{enh}}_{\mathcal{D}}(-,y))$ from construction 7.2.8 assigns to each object x in $\mathcal{D}$ the category of M-modules in the $\mathrm{End}_{\mathcal{D}}(y)$-module category $\mathrm{Hom}_{\mathcal{D}}(x,y)$. Furthermore, the natural transformation

$$M\text{-mod}^{l}(\mathrm{Hom}^{\mathrm{enh}}_{\mathcal{D}}(-,y)) \rightarrow \mathrm{Hom}_{\mathcal{D}}(-,y)$$

recovers the canonical forgetful functor into $\mathrm{Hom}_{\mathcal{D}}(x,y)$.

We can summarize the situation informally by saying that the assignment

$$x \mapsto M\text{-mod}^{l}(\mathrm{Hom}^{\mathrm{enh}}_{\mathcal{D}}(x,y))$$

is functorial on x, and this functoriality is compatible with the usual functoriality of the assignment $x \mapsto \mathrm{Hom}_{\mathcal{D}}(x,y)$.

Assume now given a morphism of monads $\rho : M \rightarrow M'$. Then construction 7.2.8 supplies a commutative square of presheaves on $\mathcal{D}$ as follows:

$$\begin{tikzcd} M'\text{-mod}^{l}(\mathrm{Hom}^{\mathrm{enh}}_{\mathcal{D}}(-,y)) \arrow[r, "\rho^*"] \arrow[d] & M\text{-mod}^{l}(\mathrm{Hom}^{\mathrm{enh}}_{\mathcal{D}}(-,y)) \arrow[d] \\ \mathrm{Hom}_{\mathcal{D}}(-,y) \arrow[r, "\mathrm{id}"] & \mathrm{Hom}_{\mathcal{D}}(-,y) \end{tikzcd}$$

Evaluating at an object x in $\mathcal{D}$ recovers a commutative square

$$\begin{tikzcd} M'\text{-mod}^{l}(\mathrm{Hom}^{\mathrm{enh}}_{\mathcal{D}}(x,y)) \arrow[r, "\rho^*"] \arrow[d] & M\text{-mod}^{l}(\mathrm{Hom}^{\mathrm{enh}}_{\mathcal{D}}(x,y)) \arrow[d] \\ \mathrm{Hom}_{\mathcal{D}}(x,y) \arrow[r, "\mathrm{id}"] & \mathrm{Hom}_{\mathcal{D}}(x,y) \end{tikzcd}$$

where the vertical arrows are the forgetful functors, and the top horizontal arrow is given by restriction of scalars along ρ.

7.3 Eilenberg-Moore objects and monadic morphisms

We now study the notion of Eilenberg-Moore object for a monad.

Definition 7.3.1. *Let $\mathcal{D}$ be a 2-category. Let M be a monad on an object y of $\mathcal{D}$ and let $g : x \to y$ be a module for M. We say that g presents x as the Eilenberg-Moore object of M if for every object z in $\mathcal{D}$, the functor*

$$g_*^{\mathrm{enh}} : \mathrm{Hom}_{\mathcal{D}}(z, x) \to M\text{-}\mathrm{mod}^l(\mathrm{Hom}_{\mathcal{D}}(z, y))$$

induced by composition with g using the functoriality of M-modules from construction 7.2.8, is an equivalence.

Remark 7.3.2. Let $\mathcal{D}$ be a 2-category. Let M be a monad on an object y of $\mathcal{D}$ and let x be another object of $\mathcal{D}$. Then the data of an M-module $g : x \to y$ is equivalent to the data of a morphism in $\mathrm{Funct}(\mathcal{D}^{1\text{-op}}, \mathscr{C}at)$ from the representable presheaf $\mathrm{Hom}_{\mathcal{D}}(-, x)$ to the presheaf $M\text{-}\mathrm{mod}^l(\mathrm{Hom}^{\mathrm{enh}}_{\mathcal{D}}(-, y))$ from construction 7.2.8. Such an M-module presents x as the Eilenberg-Moore object of M if and only if the induced morphism

$$\mathrm{Hom}_{\mathcal{D}}(-, x) \to M\text{-}\mathrm{mod}^l(\mathrm{Hom}^{\mathrm{enh}}_{\mathcal{D}}(-, y))$$

is an isomorphism of presheaves on $\mathcal{D}$. In other words, an Eilenberg-Moore object is a representing object for the presheaf from construction 7.2.8. In particular, we have that Eilenberg-Moore objects are unique if they exist.

Remark 7.3.3. Let $\mathcal{U}$ be the monoidal envelope of the associative operad and let $\mathcal{U}_m$ be the $\mathcal{U}$-module arising from the LMod-monoidal envelope of the LMod-operad. Let B_{Cat} be the 2-category with one object and monoidal category of endofunctors $\mathcal{U}$, and let W be the copresheaf on $B_{\mathrm{Cat}}\mathcal{U}$ associated to $\mathcal{U}_m$.

Let $\mathcal{D}$ be a 2-category, let M be a monad on an object y of $\mathcal{D}$, and let $g : x \to y$ be an M-module. The monad M defines a functor of 2-categories $F : B_{\mathrm{Cat}}\mathcal{U} \to \mathcal{D}$. Furthermore, g defines a morphism η from W to the copresheaf $\mathrm{Hom}_{\mathcal{D}}(x, F-)$. The M-module g presents x as the Eilenberg-Moore object of M if and only if η presents x as the limit of F weighted by W.

Definition 7.3.4. *Let $\mathcal{D}$ be a 2-category. We say that $\mathcal{D}$ admits Eilenberg-Moore objects if for all monads M in $\mathcal{D}$, there exists an Eilenberg-Moore object for M.*

Remark 7.3.5. Let $\mathcal{C}$ be a presentable Cat-module. It follows from corollary 5.6.2 together with remark 7.3.3 that the 2-category $\theta_{\mathrm{Cat}}(\mathcal{C})$ admits Eilenberg-Moore objects. In particular, we conclude that for every presentable symmetric monoidal category $\mathcal{M}$, the 2-category $\mathscr{C}at^{\mathcal{M}}$ of $\mathcal{M}$-enriched categories admits Eilenberg-Moore objects.

Remark 7.3.6. Let $\mathcal{C}$ be a category and let M be a monad on $\mathcal{C}$. Let $G : \mathcal{D} \to \mathcal{C}$ be an M-module which exhibits $\mathcal{D}$ as the Eilenberg-Moore object of M. Then we have an equivalence

$$\mathcal{D} = \mathrm{Hom}_{\mathscr{C}at}([0], \mathcal{D}) = M\text{-}\mathrm{mod}^l(\mathrm{Hom}^{\mathrm{enh}}_{\mathscr{C}at}([0], \mathcal{C})) = M\text{-}\mathrm{mod}^l(\mathcal{C}).$$

where on the right hand side we take modules over M with respect to the canonical $\mathrm{End}_{\mathscr{C}at}(\mathcal{C})$-module structure on $\mathcal{C}$. The above equivalence maps the functor $G : \mathcal{D} \to \mathcal{C}$ to the forgetful functor $M\text{-mod}^l(\mathcal{C}) \to \mathcal{C}$.

We may summarize this by saying that for every monad M on $\mathcal{C}$, the forgetful functor $M\text{-mod}^l(\mathcal{C}) \to \mathcal{C}$ admits a structure of M-module which presents it as the Eilenberg-Moore object for M.

Proposition 7.3.7. *Let $\mathcal{D}$ be a 2-category. Let M be a monad on an object y of $\mathcal{D}$ and let $g : x \to y$ be a module for M. Assume that g presents x as the Eilenberg-Moore object for M. Then g admits a left adjoint, and the induced morphism of monads $M \to \mathrm{End}(g)$ is an isomorphism.*

Proof. To show that g admits a left adjoint, it suffices to show that for every morphism $\alpha : z \to w$ in $\mathcal{D}$, the commutative square of categories

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{D}}(w, x) \arrow[r, "\alpha^*"] \arrow[d, "g_*"] & \mathrm{Hom}_{\mathcal{D}}(z, x) \arrow[d, "g_*"] \\ \mathrm{Hom}_{\mathcal{D}}(w, y) \arrow[r, "\alpha^*"] & \mathrm{Hom}_{\mathcal{D}}(z, y) \end{tikzcd}$$

is vertically left adjointable. The above square is equivalent to the outer square in the commutative diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{D}}(w, x) \arrow[r, "\alpha^*"] \arrow[d, "g_*^{\mathrm{enh}}"] & \mathrm{Hom}_{\mathcal{D}}(z, x) \arrow[d, "g_*^{\mathrm{enh}}"] \\ M\text{-mod}^l(\mathrm{Hom}_{\mathcal{D}}(w, y)) \arrow[r, "\alpha^*"] \arrow[d] & M\text{-mod}^l(\mathrm{Hom}_{\mathcal{D}}(z, y)) \arrow[d] \\ \mathrm{Hom}_{\mathcal{D}}(w, y) \arrow[r, "\alpha^*"] & \mathrm{Hom}_{\mathcal{D}}(z, y) \end{tikzcd}$$

which is induced from the pair of natural transformations

$$\mathrm{Hom}_{\mathcal{D}}(-, x) \to M\text{-mod}^l(\mathrm{Hom}^{\mathrm{enh}}_{\mathcal{D}}(-, y)) \to \mathrm{Hom}_{\mathcal{D}}(-, y)$$

described in remarks 7.2.9 and 7.3.2. Since g presents x as an Eilenberg-Moore object for M, we have in fact that the top vertical arrows in the above diagram are equivalences. We thus reduce to showing that the commutative square of categories

$$\begin{tikzcd} M\text{-mod}^l(\mathrm{Hom}_{\mathcal{D}}(w, y)) \arrow[r, "\alpha^*"] \arrow[d] & M\text{-mod}^l(\mathrm{Hom}_{\mathcal{D}}(z, y)) \arrow[d] \\ \mathrm{Hom}_{\mathcal{D}}(w, y) \arrow[r, "\alpha^*"] & \mathrm{Hom}_{\mathcal{D}}(z, y) \end{tikzcd}$$

is vertically left adjointable.

The existence of a left adjoint to the vertical arrows is guaranteed by the existence of free M-modules in $\mathrm{End}_{\mathcal{D}}(y)$-module categories (see [Lur17] proposition 4.2.4.2). Assume now given a morphism $h : w \to y$, and let $\eta : h \to h'$ be a morphism in $\mathrm{Hom}_{\mathcal{D}}(w, y)$ presenting an M-module $h' : w \to y$ as a free M-module on h. In other words, denoting by $\mu : M \otimes h' \to h'$ the structure map, we have that the composite map

$$M \otimes h \xrightarrow{\mathrm{id}_M \otimes \eta} M \otimes h' \xrightarrow{\mu} h'$$

is an equivalence in $\mathrm{Hom}_{\mathcal{D}}(w, y)$. Composing with α^* we see that the composite map

$$M \otimes \alpha^* h \xrightarrow{\mathrm{id}_M \otimes \alpha^* \eta} M \otimes \alpha^* h' \xrightarrow{\alpha^* \mu} \alpha^* h'$$

is an equivalence in $\mathrm{Hom}_{\mathcal{D}}(z, y)$. Observe that $\alpha^* h'$ has the structure of an M-module, and $\alpha^* \mu$ is its structure map. We conclude that $\alpha^* \eta : \alpha^* h \to \alpha^* h'$ presents h' as the free M-module on $\alpha^* h$. This shows that our commutative square is vertically left adjointable at h. Since h was arbitrary, we conclude that our commutative square is vertically left adjointable, as desired.

It remains to show that the induced morphism of monads $M \to \mathrm{End}(g)$ is an isomorphism. Denote by g^L the left adjoint to g. Then by virtue of remark 7.2.5, the underlying morphism in $\mathrm{End}_{\mathcal{D}}(y)$ to our map of monads is given by the composition

$$M \to M g g^L \to g g^L$$

where the first arrow is induced by the unit of the adjunction $g^L \dashv g$, and the second arrow is induced by composition with the structure map $Mg \to g$. To show that the above is an isomorphism, it suffices to check that the unit $\eta : \mathrm{id}_y \to g g^L$ presents $g g^L$ as the free M-module on id_y in the monoidal category $\mathrm{End}_{\mathcal{D}}(y)$. The map η is the same as the unit at id_y for the adjunction

$$g_* : \mathrm{Hom}_{\mathcal{D}}(y, x) \rightleftarrows \mathrm{Hom}_{\mathcal{D}}(y, y) : g_*^L.$$

Since g presents x as the Eilenberg-Moore object of M, the above adjunction is equivalent to the free-forgetful adjunction

$$M\text{-}\mathrm{mod}^l(\mathrm{Hom}_{\mathcal{D}}(y, y)) \rightleftarrows \mathrm{Hom}_{\mathcal{D}}(y, y).$$

We conclude that the map η presents $g g^L$ as the free M-module on id_y, as desired. $\square$

Proposition 7.3.8. *Let $\mathcal{D}$ be a 2-category and let $g : x \to y$ be a morphism in $\mathcal{D}$. The following are equivalent:*

(i) *There exists a monad M on y and an M-module structure on g, such that g presents x as the Eilenberg-Moore object of M.*

(ii) *The morphism g admits a left adjoint, and g presents x as the Eilenberg-Moore object of the monad* $\mathrm{End}(g)$.

(iii) *The morphism g admits a left adjoint, and for every object z in $\mathcal{D}$ the functor of categories*

$$g_* : \mathrm{Hom}_{\mathcal{D}}(z, x) \to \mathrm{Hom}_{\mathcal{D}}(z, y)$$

is monadic.

Proof. The equivalence between (i) and (ii) follows directly from proposition 7.3.7. Assume now that (i) holds. Then for every object z in $\mathcal{D}$ we can write the morphism

$$g_* : \mathrm{Hom}_{\mathcal{D}}(z, x) \to \mathrm{Hom}_{\mathcal{D}}(z, y)$$

as the composite map

$$\mathrm{Hom}_{\mathcal{D}}(z, x) \xrightarrow{g_*^{\mathrm{enh}}} M\text{-mod}^l(\mathrm{Hom}_{\mathcal{D}}(z, y)) \to \mathrm{Hom}_{\mathcal{D}}(z, y)$$

where the first arrow is given as in definition 7.3.1, and the second arrow is the forgetful functor. By assumption, the first arrow is an isomorphism. Furthermore, [Lur17] theorem 4.7.3.5 guarantees that the second arrow is monadic. Hence we conclude that (iii) holds.

It remains to show that (iii) implies (ii). Let z be an object of $\mathcal{D}$. We have to show that the induced map

$$g_*^{\mathrm{enh}} : \mathrm{Hom}_{\mathcal{D}}(z, x) \to \mathrm{End}(g)\text{-mod}^l(\mathrm{Hom}_{\mathcal{D}}(z, y))$$

is an equivalence. Note that the composition of the above with the (monadic) forgetful functor

$$u : \mathrm{End}(g)\text{-mod}^l(\mathrm{Hom}_{\mathcal{D}}(z, y)) \to \mathrm{Hom}_{\mathcal{D}}(z, y)$$

recovers the functor

$$g_* : \mathrm{Hom}_{\mathcal{D}}(z, x) \to \mathrm{Hom}_{\mathcal{D}}(z, y)$$

which is monadic by our assumption. Since g_* is conservative we have that g_*^{enh} is conservative as well.

Let $\mathcal{C}$ be the full subcategory of $\mathrm{End}(g)\text{-mod}^l(\mathrm{Hom}_{\mathcal{D}}(z, y))$ on those objects for which the left adjoint to g_*^{enh} is defined and the unit is an isomorphism. To show that g_*^{enh} is an equivalence, it suffices to show that that $\mathcal{C}$ is the entire $\mathrm{End}(g)\text{-mod}^l(\mathrm{Hom}_{\mathcal{D}}(z, y))$.

We first show that $\mathcal{C}$ contains the image of u^L. Let α be an object in $\mathrm{Hom}_{\mathcal{D}}(z, y)$, and let $\mu : u^L\alpha \to g_*^{\mathrm{enh}} g_*^L \alpha$ be the morphism induced by adjunction from the unit

$$\eta : \alpha \to u g_*^{\mathrm{enh}} g_*^L \alpha = g_* g_*^L \alpha.$$

It follows from proposition 5.1.3 that μ presents $g_*^L\alpha$ as left adjoint to g_*^{enh} at $u^L\alpha$.

We claim that η presents $g_* g_*^L \alpha$ as the free $\mathrm{End}(g)$-module on α. To see this, we have to verify that the composition

$$\mathrm{End}(g)\alpha \xrightarrow{\mathrm{id}_{\mathrm{End}(g)}\,\eta} \mathrm{End}(g) g_* g_*^L \alpha \to g_* g_*^L \alpha$$

is an equivalence, where the last morphism is induced by composition with the structure map $\mathrm{End}(g)g \to g$. This is a consequence of the description of $\mathrm{End}(g)$ from proposition 7.2.4, together with the triangle identities for the adjunction $g^L \dashv g$.

It follows from the above that η presents $g_*^{\mathrm{enh}} g_*^L \alpha$ as left adjoint to u at α. Hence the map μ is in fact an isomorphism. This shows that the image of u^L is indeed contained in $\mathcal{C}$.

By virtue of [Lur17] proposition 4.7.3.14, it suffices now to show that $\mathcal{C}$ is closed under u-split geometric realizations. Let $F^{\triangleright} : (\Delta^{\mathrm{op}})^{\triangleright} \to \mathrm{End}(g)\text{-}\mathrm{mod}^l(\mathrm{Hom}_{\mathcal{D}}(z,y))$ be an u-split geometric realization diagram, and assume that $F = F^{\triangleright}|_{\Delta^{\mathrm{op}}}$ factors through $\mathcal{C}$. Then $(g_*^{\mathrm{enh}})^L F$ is defined, and we have that

$$g_*(g_*^{\mathrm{enh}})^L F = u_* g_*^{\mathrm{enh}} (g_*^{\mathrm{enh}})^L F = u_* F$$

is a split simplicial object. We conclude that $(g_*^{\mathrm{enh}})^L F$ is a g_*-split simplicial object in $\mathrm{Hom}_{\mathcal{D}}(z,x)$. Since g_* was assumed to be monadic, $(g_*^{\mathrm{enh}})^L F$ admits a geometric realization. An application of proposition 5.5.24 shows that g_*^{enh} admits a left adjoint at $F^{\triangleright}(*)$. Furthermore, $(g_*^{\mathrm{enh}})^L F^{\triangleright}$ is a geometric realization diagram which is g_*-split, and therefore $g_*^{\mathrm{enh}}(g_*^{\mathrm{enh}})^L F^{\triangleright}$ is also a geometric realization diagram. Since F factors through $\mathcal{C}$ we have that the natural transformation $F \to g_*^{\mathrm{enh}}(g_*^{\mathrm{enh}})^L F$ induced from the unit of the (partially defined) adjunction $(g_*^{\mathrm{enh}})^L g_*^{\mathrm{enh}}$ is an isomorphism. This implies that the natural transformation $F^{\triangleright} \to g_*^{\mathrm{enh}}(g_*^{\mathrm{enh}})^L F^{\triangleright}$ induced from the unit is also an isomorphism, which means that $F^{\triangleright}$ factors through $\mathcal{C}$, as desired. □

Definition 7.3.9. *Let $\mathcal{D}$ be a 2-category. A morphism $g : x \to y$ in $\mathcal{D}$ is said to be monadic if it satisfies the equivalent conditions of proposition 7.3.8.*

For later purposes we record the following basic stability property of monadic morphisms.

Proposition 7.3.10. *Let $\mathcal{D}$ be a 2-category. Let $\mathcal{I}$ be a category and let $F : \mathcal{I} \to \mathrm{Funct}([1], \mathcal{D})$ be a functor. Assume that:*

- *For every object i in $\mathcal{I}$ the morphism $F(i)$ is monadic.*
- *For every arrow $\alpha : i \to j$ in $\mathcal{I}$, the induced commutative square*

$$\begin{array}{ccc} F(i,0) & \xrightarrow{F(\alpha,0)} & F(j,0) \\ \downarrow{\scriptstyle F(i)} & & \downarrow{\scriptstyle F(j)} \\ F(i,1) & \xrightarrow{F(\alpha,1)} & F(j,1) \end{array}$$

  is vertically left adjointable.
- *The functors $\mathrm{ev}_0 F, \mathrm{ev}_1 F : \mathcal{I} \to \mathcal{D}$ admit conical limits.*

Then the conical limit of F is a monadic morphism.

Proof. Denote by $g : x \to y$ the limit of F. For each i in $\mathcal{I}$ denote by $p_i : x \to F(i, 0)$ and $q_i : y \to F(i, 1)$ the projections.

The fact that g has a left adjoint follows from an application of proposition 5.3.17. It remains to show that for any object z in $\mathcal{D}$ the functor

$$g_* : \mathrm{Hom}_{\mathcal{D}}(z, x) \to \mathrm{Hom}_{\mathcal{D}}(z, y)$$

is monadic. Note that g_* is the limit over $\mathcal{I}$ of the monadic functors

$$F(i)_* : \mathrm{Hom}_{\mathcal{D}}(z, F(i, 0)) \to \mathrm{Hom}_{\mathcal{D}}(z, F(i, 1)).$$

We verify the hypothesis of the monadicity theorem for g_*. Note that g_* admits a left adjoint since g itself admits a left adjoint. The conservativity of g_* follows from the fact that it is the limit of a sequence of conservative functors.

Assume now given a g_*-split simplicial object $S : \Delta^{\mathrm{op}} \to \mathrm{Hom}_{\mathcal{D}}(z, x)$. For every i in $\mathcal{I}$ we have that

$$S(i)_*(p_i)_*S = (q_i)_*g_*S$$

is a split simplicial object in $\mathrm{Hom}_{\mathcal{D}}(z, F(i, 1))$. Hence $(p_i)_*S$ is $S(i)_*$-split, and it therefore admits a geometric realization

$$((p_i)_*S)^{\triangleright} : (\Delta^{\mathrm{op}})^{\triangleright} \to \mathrm{Hom}_{\mathcal{D}}(z, F(i, 0))$$

which is preserved by $S(i)_*$.

For every arrow $\alpha : i \to j$ in $\mathcal{I}$ we have

$$F(j)_*F(\alpha, 0)_*((p_i)_*S)^{\triangleright} = F(\alpha, 1)_*F(i)_*((p_i)_*S)^{\triangleright}.$$

The right hand side is the image under $F(\alpha, 1)_*$ of the geometric realization of a split simplicial object, hence it is a colimit diagram. It follows from this that $F(\alpha, 0)_*((p_i)_*S)^{\triangleright}$ is a colimit diagram in $\mathrm{Hom}_{\mathcal{D}}(z, F(i, 0))$. In other words, the geometric realization of $(p_i)_*S$ is preserved under $F(\alpha, 0)$.

The geometric realizations of $(p_i)_*S$ therefore pass to the limit to give a geometric realization $S^{\triangleright} : (\Delta^{\mathrm{op}})^{\triangleright} \to \mathrm{Hom}_{\mathcal{D}}(z, x)$ for S. It remains to show that the augmented simplicial object $g_*S^{\triangleright}$ is a colimit diagram. This follows from the fact that for every i in $\mathcal{I}$ the diagram

$$(q_i)_*g_*S^{\triangleright} = F(i)_*(p_i)_*S^{\triangleright}$$

is the geometric realization of a split simplicial object. □

7.4 Monadic functors of enriched categories

We now specialize the above theory to obtain a notion of monadic functor of enriched categories.

Notation 7.4.1. Let $\mathcal{M}$ be a presentable symmetric monoidal category. We denote by $\mathscr{C}at^{\mathcal{M}}$ the 2-category of $\mathcal{M}$-enriched categories (see notation 5.2.5 for a definition).

Definition 7.4.2. *Let $\mathcal{M}$ be a presentable symmetric monoidal category. A functor $G : \mathcal{D} \to \mathcal{D}'$ between $\mathcal{M}$-enriched categories is said to be monadic if the associated morphism in $\mathscr{C}at^{\mathcal{M}}$ is monadic.*

Remark 7.4.3. Let $\mathcal{M}$ be a presentable symmetric monoidal category. Unpacking definition 7.4.2 we see that a functor $G : \mathcal{D} \to \mathcal{D}'$ between $\mathcal{M}$-enriched categories is monadic if and only if it admits a left adjoint and for every $\mathcal{M}$-enriched category $\mathcal{I}$ the functor of categories

$$G_* : \mathrm{Funct}(\mathcal{I}, \mathcal{D})^{\leq 1} \to \mathrm{Funct}(\mathcal{I}, \mathcal{D}')^{\leq 1}$$

is monadic.

Remark 7.4.4. A straightforward application of the monadicity theorem [Lur17] theorem 4.7.3.5 shows that in the case when $\mathcal{M} = \mathrm{Spc}$ equipped with its cartesian symmetric monoidal structure, definition 7.4.2 recovers the usual notion of monadic functor of categories.

Remark 7.4.5. Let $\mathcal{M}$ be a presentable symmetric monoidal category and let $G : \mathcal{D} \to \mathcal{D}'$ be a functor of $\mathcal{M}$-enriched categories. Let $\mathcal{E}$ and $\mathcal{J}$ be $\mathcal{M}$-enriched categories. Then the map

$$\mathrm{Funct}(\mathcal{J}, \mathrm{Funct}(\mathcal{E}, \mathcal{D}))^{\leq 1} \to \mathrm{Funct}(\mathcal{J}, \mathrm{Funct}(\mathcal{E}, \mathcal{D}'))^{\leq 1}$$

induced by G is equivalent to the functor

$$\mathrm{Funct}(\mathcal{J} \times \mathcal{E}, \mathcal{D})^{\leq 1} \to \mathrm{Funct}(\mathcal{J} \times \mathcal{E}, \mathcal{D}')^{\leq 1}$$

induced by G. It follows that if G is monadic then the induced functor

$$G_* : \mathrm{Funct}(\mathcal{E}, \mathcal{D}) \to \mathrm{Funct}(\mathcal{E}, \mathcal{D}')$$

is also monadic.

Our next goal is to give a characterization of monadic functors between enriched categories, generalizing the usual description provided by the monadicity theorem.

Definition 7.4.6. *Let $\mathcal{M}$ be a presentable symmetric monoidal category. We say that a functor $G : \mathcal{D} \to \mathcal{D}'$ of $\mathcal{M}$-enriched categories is conservative if the functor of categories $(\tau_{\mathcal{M}})_! G : (\tau_{\mathcal{M}})_! \mathcal{D} \to (\tau_{\mathcal{M}})_! \mathcal{D}'$ is conservative.*

Notation 7.4.7. Let $\Delta_{-\infty}$ be the wide subcategory of Δ on the morphisms which preserve minimums. Let $i : \Delta \to \Delta_{-\infty}$ be the functor that maps each nonempty totally ordered finite set O to the totally ordered set $\{-\infty\} \cup O$ obtained from O by adjoining a minimum element.

Definition 7.4.8. *Let $\mathcal{M}$ be a presentable symmetric monoidal category and let $\mathcal{D}$ be an $\mathcal{M}$-enriched category. A simplicial object in $\mathcal{D}$ is a functor $S : \Delta^{\mathrm{op}}_{\mathcal{M}} \to \mathcal{D}$. We say that S admits a conical geometric realization if it admits a conical colimit. We say that S is split if it admits an extension along the inclusion $i : \Delta^{\mathrm{op}}_{\mathcal{M}} \to (\Delta^{\mathrm{op}}_{-\infty})_{\mathcal{M}}$. Given a functor $G : \mathcal{D} \to \mathcal{D}'$, we say that S is G-split if GS is split.*

Remark 7.4.9. Let $\mathcal{M}$ be a presentable symmetric monoidal category and let $\mathcal{D}$ be an $\mathcal{M}$-enriched category. Let $S : \Delta^{\mathrm{op}}_{\mathcal{M}} \to \mathcal{D}$ be a split simplicial object in $\mathcal{D}$, and let $T : (\Delta^{\mathrm{op}}_{-\infty})_{\mathcal{M}} \to \mathcal{D}$ be an extension of S. Consider the inclusion $j : \Delta^{\triangleleft} \to \Delta_{-\infty}$ which extends i and maps the cone point to the initial object of $\Delta_{-\infty}$. Then it follows from corollary 5.4.10 together with [Lur09a] lemma 6.1.3.16 that the augmented simplicial object obtained by restricting T along j is a conical colimit diagram for S. In particular, we see that split simplicial objects admit conical geometric realizations which are preserved by arbitrary functors.

Theorem 7.4.10. *Let $\mathcal{M}$ be a presentable symmetric monoidal category. Then a functor $G : \mathcal{D} \to \mathcal{D}'$ between $\mathcal{M}$-enriched categories is monadic if and only if it admits a left adjoint, it is conservative, and every G-split simplicial object in $\mathcal{D}$ admits a conical geometric realization which is preserved by G.*

Before giving a proof, we highlight a few important special cases of theorem 7.4.10.

Corollary 7.4.11. *Let $\mathcal{M}$ be a presentable symmetric monoidal category and let $G : \mathcal{D} \to \mathcal{D}'$ be a functor of $\mathcal{M}$-enriched categories. Assume that G is conservative, and that $\mathcal{D}$ admits and G preserves conical geometric realizations. Then G is monadic.*

Proof. This is a direct consequence of theorem 7.4.10. □

Corollary 7.4.12. *Let $\mathcal{M}$ be a presentable symmetric monoidal category. Let $G : \mathcal{D} \to \mathcal{D}'$ be a morphism in $\mathcal{M}$-mod(Pr^L). Assume that G is conservative and admits a left adjoint F, and that the canonical structure of oplax morphism of $\mathcal{M}$-modules on F is strict. Then the functor of $\mathcal{M}$-enriched categories $\theta_{\mathcal{M}}(G) : \theta_{\mathcal{M}}(\mathcal{D}) \to \theta_{\mathcal{M}}(\mathcal{D}')$ is monadic.*

Proof. This follows from a combination of theorem 7.4.10 and propositions 5.4.1 and 5.4.8. □

Our proof of theorem 7.4.10 will need some preliminaries.

Notation 7.4.13. Let $\mathcal{M}$ be a presentable symmetric monoidal category. Given a functor of $\mathcal{M}$-enriched categories $G : \mathcal{D} \to \mathcal{D}'$, we denote by $\mathrm{Funct}_{G\text{-split}}(\Delta^{\mathrm{op}}_{\mathcal{M}}, \mathcal{D})$ the $\mathcal{M}$-enriched category arising as the pullback

$$\begin{tikzcd}
\mathrm{Funct}_{G\text{-split}}(\Delta^{\mathrm{op}}_{\mathcal{M}}, \mathcal{D}) \arrow[r] \arrow[d] & \mathrm{Funct}((\Delta^{\mathrm{op}}_{-\infty})_{\mathcal{M}}, \mathcal{D}') \arrow[d, "i^*"] \\
\mathrm{Funct}(\Delta^{\mathrm{op}}_{\mathcal{M}}, \mathcal{D}) \arrow[r, "G_*"] & \mathrm{Funct}(\Delta^{\mathrm{op}}_{\mathcal{M}}, \mathcal{D}').
\end{tikzcd}$$

We denote by $\Delta_{G\text{-split}} : \mathcal{D} \to \mathrm{Funct}_{G\text{-split}}(\Delta^{\mathrm{op}}_{\mathcal{M}}, \mathcal{D})$ the functor arising from the following commutative square:

$$\begin{tikzcd}
\mathcal{D} \arrow[r, "G_*\Delta"] \arrow[d, "\Delta"] & \mathrm{Funct}((\Delta^{\mathrm{op}}_{-\infty})_{\mathcal{M}}, \mathcal{D}') \arrow[d, "i^*"] \\
\mathrm{Funct}(\Delta^{\mathrm{op}}_{\mathcal{M}}, \mathcal{D}) \arrow[r, "G_*"] & \mathrm{Funct}(\Delta^{\mathrm{op}}_{\mathcal{M}}, \mathcal{D}')
\end{tikzcd}$$

Note that the letter Δ is being used in two ways in the above square: to denote the simplex category, and to denote the diagonal functors.

Remark 7.4.14. Let $\mathcal{M}$ be a presentable symmetric monoidal category and let $G : \mathcal{D} \to \mathcal{D}'$ be a functor of $\mathcal{M}$-enriched categories. Then an object of $\mathrm{Funct}_{G\text{-split}}(\Delta^{\mathrm{op}}_{\mathcal{M}}, \mathcal{D})$ is the same as a triple (S, T, ν) of a simplicial object S in $\mathcal{D}$, a functor $T : (\Delta^{\mathrm{op}}_{-\infty})_{\mathcal{M}} \to \mathcal{D}'$, and an identification $\nu : G_*S = i^*T$. We think about T as a choice of splitting for G_*S.

The functor $\Delta_{G\text{-split}}$ assigns to each object d in $\mathcal{D}$ the constant simplicial object on d, where we equip its image under G_* with the constant splitting.

Lemma 7.4.15. *Let $\mathcal{M}$ be a presentable symmetric monoidal category and let $G : \mathcal{D} \to \mathcal{D}'$ be a functor of $\mathcal{M}$-enriched categories. Let d be an object in $\mathcal{D}$ and let $\mathcal{S} = (S, T, \nu)$ be an object of* $\mathrm{Funct}_{G\text{-split}}(\Delta^{\mathrm{op}}_{\mathcal{M}}, \mathcal{D})$*. Then the projection* $\mathrm{Funct}_{G\text{-split}}(\Delta^{\mathrm{op}}_{\mathcal{M}}, \mathcal{D}) \to \mathrm{Funct}(\Delta^{\mathrm{op}}_{\mathcal{M}}, \mathcal{D})$ *induces an isomorphism*

$$\mathrm{Hom}_{\mathrm{Funct}_{G\text{-split}}(\Delta^{\mathrm{op}}_{\mathcal{M}}, \mathcal{D})}(\mathcal{S}, \Delta_{G\text{-split}} d) = \mathrm{Hom}_{\mathrm{Funct}(\Delta^{\mathrm{op}}_{\mathcal{M}}, \mathcal{D})}(S, \Delta d).$$

Proof. We have a pullback square of objects of $\mathcal{M}$

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{Funct}_{G\text{-split}}(\Delta^{\mathrm{op}}_{\mathcal{M}}, \mathcal{D})}(\mathcal{S}, \Delta_{G\text{-split}} d) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathrm{Funct}((\Delta^{\mathrm{op}}_{-\infty})_{\mathcal{M}}, \mathcal{D}')}(T, \Delta^{(2)} G d) \arrow[d, "(i^*)_*"] \\
\mathrm{Hom}_{\mathrm{Funct}(\Delta^{\mathrm{op}}_{\mathcal{M}}, \mathcal{D})}(S, \Delta d) \arrow[r, "(G_*)_*"] & \mathrm{Hom}_{\mathrm{Funct}(\Delta^{\mathrm{op}}_{\mathcal{M}}, \mathcal{D}')}(i^*T, \Delta^{(1)} G d)
\end{tikzcd}$$

induced from the pullback square from notation 7.4.13. Here

$$\Delta^{(1)} : \mathcal{D}' \to \mathrm{Funct}(\Delta^{\mathrm{op}}_{\mathcal{M}}, \mathcal{D}')$$

and

$$\Delta^{(2)} : \mathcal{D}' \to \mathrm{Funct}((\Delta^{\mathrm{op}}_{-\infty})_{\mathcal{M}}, \mathcal{D}')$$

denote the corresponding diagonal maps.

Our task is to show that the left vertical arrow is an isomorphism. It suffices for this to show that the right vertical arrow is an isomorphism. This will follow if we can show that the canonical identification $i^*\Delta^{(2)}Gd = \Delta^{(1)}Gd$ presents $\Delta^{(2)}Gd$ as right adjoint to i^* at $\Delta^{(1)}Gd$.

Note that $i : \Delta \to \Delta_{-\infty}$ is left adjoint to the inclusion of $\Delta_{-\infty}$ inside Δ. It follows that

$$i^* : \mathrm{Funct}((\Delta^{\mathrm{op}}_{-\infty})_{\mathcal{M}}, \mathcal{D}') \to \mathrm{Funct}(\Delta^{\mathrm{op}}_{\mathcal{M}}, \mathcal{D}')$$

is left adjoint to the functor $\mathrm{Funct}(\Delta^{\mathrm{op}}_{\mathcal{M}}, \mathcal{D}') \to \mathrm{Funct}((\Delta^{\mathrm{op}}_{-\infty})_{\mathcal{M}}, \mathcal{D}')$ of restriction along the inclusion $\Delta_{-\infty} \to \Delta$. Given a simplicial object X in $\mathcal{D}'$, the counit of the adjunction is a morphism of simplicial objects

$$i^*(X|_{(\Delta^{\mathrm{op}}_{-\infty})_{\mathcal{M}}}) \to X$$

which upon evaluation at a simplex $[n]$ recovers the morphism

$$X([n+1]) \to X([n])$$

induced from the unique strictly increasing map $[n] \to [n+1]$ sending 0 to 1. In the particular case when $X = \Delta^{(1)}Gd$ is the constant simplicial object at Gd, we have that $X|_{(\Delta^{\mathrm{op}}_{-\infty})_{\mathcal{M}}} = \Delta^{(2)}Gd$ is the constant functor $(\Delta^{\mathrm{op}}_{-\infty})_{\mathcal{M}} \to \mathcal{D}'$ at Gd. Our claim now follows from the fact that there is a unique morphism $i^*\Delta^{(2)}Gd \to \Delta^{(1)}Gd$ which restricts to the identity at $[0]$. □

Lemma 7.4.16. *Let $\mathcal{M}$ be a presentable symmetric monoidal category and let $G : \mathcal{D} \to \mathcal{D}'$ be a functor of $\mathcal{M}$-enriched categories. Let $\mathcal{S} = (S, T, \nu)$ be an object of $\mathrm{Funct}_{G\text{-split}}(\Delta^{\mathrm{op}}_{\mathcal{M}}, \mathcal{D})$. Let d be an object in $\mathcal{D}$ and $\eta : \mathcal{S} \to \Delta_{G\text{-split}}d$ be a morphism. Let $\eta' : S \to \Delta d$ be the image of η under the projection map*

$$\mathrm{Funct}_{G\text{-split}}(\Delta^{\mathrm{op}}_{\mathcal{M}}, \mathcal{D}) \to \mathrm{Funct}(\Delta^{\mathrm{op}}_{\mathcal{M}}, \mathcal{D}).$$

Then η presents d as left adjoint to $\Delta_{G\text{-split}}$ at $\mathcal{S}$ if and only if η' presents d as left adjoint to Δ at S.

Proof. Consider for each object e in $\mathcal{D}$ the commutative diagram of objects of $\mathcal{M}$

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}(d, e) \arrow[r, "\mathrm{id}"] \arrow[d, "(\Delta_{G\text{-split}})_*"] & \mathrm{Hom}_{\mathcal{D}}(d, e) \arrow[d, "\Delta_*"] \\
\mathrm{Hom}_{\mathrm{Funct}_{G\text{-split}}(\Delta^{\mathrm{op}}_{\mathcal{M}}, \mathcal{D})}(\Delta_{G\text{-split}}d, \Delta_{G\text{-split}}e) \arrow[r] \arrow[d, "\eta^*"] & \mathrm{Hom}_{\mathrm{Funct}\,(\Delta^{\mathrm{op}}_{\mathcal{M}}, \mathcal{D})}(\Delta d, \Delta e) \arrow[d, "\eta'^*"] \\
\mathrm{Hom}_{\mathrm{Funct}_{G\text{-split}}(\Delta^{\mathrm{op}}_{\mathcal{M}}, \mathcal{D})}(\mathcal{S}, \Delta_{G\text{-split}}e) \arrow[r] & \mathrm{Hom}_{\mathrm{Funct}(\Delta^{\mathrm{op}}_{\mathcal{M}}, \mathcal{D})}(S, \Delta e)
\end{tikzcd}$$

where the middle and bottom horizontal arrows are induced by the projection map

$$\mathrm{Funct}_{G\text{-split}}(\Delta^{\mathrm{op}}_{\mathcal{M}}, \mathcal{D}) \to \mathrm{Funct}(\Delta^{\mathrm{op}}_{\mathcal{M}}, \mathcal{D}).$$

It follows from lemma 7.4.15 that the horizontal arrows in the above diagram are all isomorphisms. We conclude that the composition of the two left vertical arrows is an isomorphism if and only if the composition of the two right vertical arrows is an isomorphism. The lemma now follows directly from the definition of local adjoints to functors. □

Lemma 7.4.17. *Let $G : \mathcal{D} \to \mathcal{D}'$ be a monadic functor of categories. Then the functor $\Delta_{G\text{-split}}$ admits a left adjoint.*

Proof. Let $\mathcal{S} = (S, T, \nu)$ be an object of $\mathrm{Funct}_{G\text{-split}}(\Delta^{\mathrm{op}}, \mathcal{D})$. We need to show that $\Delta_{G\text{-split}}$ admits a left adjoint at $\mathcal{S}$. Since G is monadic and S is G-split, we have that S admits a geometric realization. Let $\eta' : S \to \Delta d$ be a morphism in $\mathrm{Funct}(\Delta^{\mathrm{op}}, \mathcal{D})$ that presents an object d as the geometric realization of S. It follows from lemma 7.4.15 that η' admits a lift to a morphism $\eta : \mathcal{S} \to \Delta_{G\text{-split}} d$. Applying lemma 7.4.16 we conclude that η presents d as left adjoint to $\Delta_{G\text{-split}}$ at $\mathcal{S}$. □

Lemma 7.4.18. *Let $\mathcal{M}$ be a presentable symmetric monoidal category and let $G : \mathcal{D} \to \mathcal{D}'$ be a monadic functor of $\mathcal{M}$-enriched categories. Then the functor $\Delta_{G\text{-split}}$ admits a left adjoint.*

Proof. Applying lemma 5.2.9 to the image of the Yoneda embedding for $\mathscr{C}at^{\mathcal{M}}$ as in the proof of proposition 5.2.10, we reduce to showing that for every functor $F : \mathcal{J} \to \mathcal{J}'$ of $\mathcal{M}$-enriched categories, the commutative square of categories

$$\begin{tikzcd}
(\tau_{\mathcal{M}})_! \mathrm{Funct}(\mathcal{J}', \mathcal{D}) \arrow[r, "F^*"] \arrow[d, "(\Delta_{G\text{-split}})_*"] & (\tau_{\mathcal{M}})_! \mathrm{Funct}(\mathcal{J}, \mathcal{D}) \arrow[d, "(\Delta_{G\text{-split}})_*"] \\
(\tau_{\mathcal{M}})_! \mathrm{Funct}(\mathcal{J}', \mathrm{Funct}_{G\text{-split}}(\Delta^{\mathrm{op}}_{\mathcal{M}}, \mathcal{D})) \arrow[r, "F^*"] & (\tau_{\mathcal{M}})_! \mathrm{Funct}(\mathcal{J}, \mathrm{Funct}_{G\text{-split}}(\Delta^{\mathrm{op}}_{\mathcal{M}}, \mathcal{D}))
\end{tikzcd}$$

is vertically left adjointable. The above square is equivalent to the commutative square of categories

$$\begin{tikzcd}
(\tau_{\mathcal{M}})_! \mathrm{Funct}(\mathcal{J}', \mathcal{D}) \arrow[r, "F^*"] \arrow[d, "(\Delta_{G_{\mathcal{J}'}\text{-split}})_*"] & (\tau_{\mathcal{M}})_! \mathrm{Funct}(\mathcal{J}, \mathcal{D}) \arrow[d, "(\Delta_{G_{\mathcal{J}}\text{-split}})_*"] \\
\mathrm{Funct}_{G_{\mathcal{J}'}\text{-split}}(\Delta^{\mathrm{op}}, (\tau_{\mathcal{M}})_! \mathrm{Funct}(\mathcal{J}', \mathcal{D})) \arrow[r, "F^*"] & \mathrm{Funct}_{G_{\mathcal{J}}\text{-split}}(\Delta^{\mathrm{op}}, (\tau_{\mathcal{M}})_! \mathrm{Funct}(\mathcal{J}, \mathcal{D}))
\end{tikzcd}$$

where

$$G_{\mathcal{J}} : (\tau_{\mathcal{M}})_! \mathrm{Funct}(\mathcal{J}, \mathcal{D}) \to (\tau_{\mathcal{M}})_! \mathrm{Funct}(\mathcal{J}, \mathcal{D}')$$

and

$$G_{\mathcal{J}'} : (\tau_{\mathcal{M}})_! \mathrm{Funct}(\mathcal{J}', \mathcal{D}) \to (\tau_{\mathcal{M}})_! \mathrm{Funct}(\mathcal{J}', \mathcal{D}')$$

are the functors induced by G. Since G is monadic, we have that $G_{\mathcal{J}}$ and $G_{\mathcal{J}'}$ are monadic. Applying lemma 7.4.17 we conclude that the vertical arrows in the above square are left adjointable.

We now enlarge the above commutative square as follows:

$$\begin{tikzcd}
(\tau_{\mathcal{M}})_! \operatorname{Funct}(\mathcal{J}', \mathcal{D}) \arrow[r, "F^*"] \arrow[d, "(\Delta_{G_{\mathcal{J}'}\text{-split}})_*"] & (\tau_{\mathcal{M}})_! \operatorname{Funct}(\mathcal{J}, \mathcal{D}) \arrow[d, "(\Delta_{G_{\mathcal{J}}\text{-split}})_*"] \\
\operatorname{Funct}_{G_{\mathcal{J}'}\text{-split}}(\Delta^{\mathrm{op}}, (\tau_{\mathcal{M}})_! \operatorname{Funct}(\mathcal{J}', \mathcal{D})) \arrow[r, "F^*"] \arrow[d] & \operatorname{Funct}_{G_{\mathcal{J}}\text{-split}}(\Delta^{\mathrm{op}}, (\tau_{\mathcal{M}})_! \operatorname{Funct}(\mathcal{J}, \mathcal{D})) \arrow[d] \\
\operatorname{Funct}(\Delta^{\mathrm{op}}, (\tau_{\mathcal{M}})_! \operatorname{Funct}(\mathcal{J}', \mathcal{D})) \arrow[r, "F^*"] & \operatorname{Funct}(\Delta^{\mathrm{op}}, (\tau_{\mathcal{M}})_! \operatorname{Funct}(\mathcal{J}, \mathcal{D}))
\end{tikzcd}$$

where the bottom vertical arrows are the canonical projections. Using lemma 7.4.16 we reduce to showing that the outer commutative square is vertically left adjointable at every $G_{\mathcal{J}'}$-split simplicial object. In other words, we need to show that the top horizontal arrow preserves geometric realizations of $G_{\mathcal{J}'}$-split simplicial objects. This is a direct consequence of the commutativity of the square

$$\begin{tikzcd}
(\tau_{\mathcal{M}})_! \operatorname{Funct}(\mathcal{J}', \mathcal{D}) \arrow[r, "F^*"] \arrow[d, "G_{\mathcal{J}'}"] & (\tau_{\mathcal{M}})_! \operatorname{Funct}(\mathcal{J}, \mathcal{D}) \arrow[d, "G_{\mathcal{J}}"] \\
(\tau_{\mathcal{M}})_! \operatorname{Funct}(\mathcal{J}', \mathcal{D}') \arrow[r, "F^*"] & (\tau_{\mathcal{M}})_! \operatorname{Funct}(\mathcal{J}, \mathcal{D}')
\end{tikzcd}$$

together with the fact that both vertical arrows are monadic. □

Proof of theorem 7.4.10. Assume first that G admits a left adjoint, is conservative, and every G-split simplicial object in $\mathcal{D}$ admits a conical geometric realization which is preserved by G. Let $\mathcal{J}$ be an $\mathcal{M}$-enriched category. We need to show that the functor

$$G_* : (\tau_{\mathcal{M}})_! \operatorname{Funct}(\mathcal{J}, \mathcal{D}) \to (\tau_{\mathcal{M}})_! \operatorname{Funct}(\mathcal{J}, \mathcal{D}')$$

is monadic.

We verify the conditions of the monadicity theorem for G_*. Conservativity of G_* follows from the fact that G is conservative, together with corollary 5.1.8. Assume now given a G_*-split simplicial object S. Then for every object i in $\mathcal{I}$ we have that $\operatorname{ev}_i S$ is a G-split simplicial object in $\mathcal{D}$, and it therefore admits a conical geometric realization which is preserved by G. Applying proposition 5.3.13 we conclude that S admits a conical geometric realization which is preserved by G_*.

Assume now that G is monadic. In particular, G admits a left adjoint. Furthermore, the functor of categories $(\tau_{\mathcal{M}})_! G : (\tau_{\mathcal{M}})_! \mathcal{D} \to (\tau_{\mathcal{M}})_! \mathcal{D}'$ is monadic. Hence we have that G is conservative. Moreover, $(\tau_{\mathcal{M}})_! \mathcal{D}$ admits geometric realizations to all $(\tau_{\mathcal{M}})_! G$-split simplicial objects, and these are preserved by $(\tau_{\mathcal{M}})_! G$. It therefore suffices to show that all G-split simplicial objects in $\mathcal{D}$ admit a conical geometric realization.

Let S be a G-split simplicial object of $\mathcal{D}$. Choose a functor $T : (\Delta^{\mathrm{op}}_{-\infty})_{\mathcal{M}} \to \mathcal{D}'$ and an identification $\nu : G_* S = i^* T$, so that the triple $\mathcal{S} = (S, T, \nu)$ defines an enhancement of S

to an object in $\mathrm{Funct}_{G\text{-split}}(\Delta^{\mathrm{op}}_{\mathcal{M}}, \mathcal{D})$. Thanks to lemma 7.4.18 we know that there exists a morphism $\eta : \mathcal{S} \to \Delta_{G\text{-split}} d$ which presents an object d in $\mathcal{D}$ as left adjoint to $\Delta_{G\text{-split}}$ at $\mathcal{S}$. It now follows from lemma 7.4.16 that d is left adjoint to Δ at S, and therefore S admits a conical geometric realization, as desired. □

7.5 Monads and monadic morphisms in an enriched 2-category

We now provide an enriched generalization of the material from 7.2 and 7.3.

Definition 7.5.1. *Let $\mathcal{M}$ be a presentable symmetric monoidal category and let $\mathcal{D}$ be an $\mathcal{M}$-enriched 2-category. A monad on an object y in $\mathcal{D}$ is a monad on y in the 2-category underlying $\mathcal{D}$. A module for a monad in $\mathcal{D}$ is a module for the underlying monad in the 2-category underlying $\mathcal{D}$.*

Definition 7.5.2. *Let $\mathcal{M}$ be a presentable symmetric monoidal category and let $\mathcal{D}$ be an $\mathcal{M}$-enriched 2-category. An endomorphism monad for a morphism $g : x \to y$ in $\mathcal{D}$ is a monad $\mathrm{End}(g)$ on y equipped with an action on g, such that for every endomorphism h of y, the induced map*

$$\mathrm{Hom}_{\mathrm{End}_{\mathcal{D}}(y)}(h, \mathrm{End}(g)) \to \mathrm{Hom}_{\mathrm{Hom}_{\mathcal{D}}(x,y)}(hg, \mathrm{End}(g)g) \to \mathrm{Hom}_{\mathrm{Hom}_{\mathcal{D}}(x,y)}(hg, g)$$

is an isomorphism, where the first arrow is given by precomposition with g, and the second arrow is induced by the structure map $\mathrm{End}(g)g \to g$.

Remark 7.5.3. Let $\mathcal{M}$ be a presentable symmetric monoidal category and let $\mathcal{D}$ be an $\mathcal{M}$-enriched 2-category. Let g be a morphism in $\mathcal{D}$ admitting an endomorphism monad $\mathrm{End}(g)$. Then $\mathrm{End}(g)$ is also an endomorphism monad for g in the 2-category underlying $\mathcal{D}$.

Definition 7.5.4. *Let $\mathcal{M}$ be a presentable symmetric monoidal category and let $\mathcal{D}$ be an $\mathcal{M}$-enriched 2-category. We say that a morphism g in $\mathcal{D}$ admits a left adjoint if it admits a left adjoint in the 2-category underlying $\mathcal{D}$.*

Proposition 7.5.5. *Let $\mathcal{M}$ be a presentable symmetric monoidal category and let $\mathcal{D}$ be an $\mathcal{M}$-enriched 2-category. Let $g : x \to y$ be a morphism in $\mathcal{D}$ admitting a left adjoint g^L. Then g admits an endomorphism monad.*

Proof. By proposition 7.2.4, the morphism g admits an endomorphism monad $\mathrm{End}(g)$ in the 2-category underlying $\mathcal{D}$, with underlying endomorphism gg^L, and action map $gg^Lg \to g$ induced by the counit of the adjunction. It remains to show that $\mathrm{End}(g)$ is in fact an endomorphism monad for g in $\mathcal{D}$.

Consider the adjunction of $\mathcal{M}$-enriched categories

$$g^* : \mathrm{End}_{\mathcal{D}}(y) \underset{\longrightarrow}{\longleftarrow} \mathrm{Hom}_{\mathcal{D}}(x, y) : (g^L)^*$$

obtained from the adjunction $g^L \dashv g$ by applying the functor of 2-categories $(\tau_{\mathcal{M}})_! \mathcal{D}^{1\text{-op}} \to \mathscr{C}at^{\mathcal{M}}$ underlying the functor of $\mathcal{M}$-enriched 2-categories $\mathcal{D}^{1\text{-op}} \to \overline{\mathrm{Cat}^{\mathcal{M}}}$ represented by y. For each object h in $\mathrm{End}_{\mathcal{D}}(y)$ we have an induced isomorphism

$$\mathrm{Hom}_{\mathrm{End}_{\mathcal{D}}(y)}(h, gg^L) = \mathrm{Hom}_{\mathrm{Hom}_{\mathcal{D}}(x,y)}(hg, g).$$

The above is obtained as the composite map

$$\mathrm{Hom}_{\mathrm{End}_{\mathcal{D}}(y)}(h, gg^L) \to \mathrm{Hom}_{\mathrm{Hom}_{\mathcal{D}}(x,y)}(hg, gg^L g) \to \mathrm{Hom}_{\mathrm{Hom}_{\mathcal{D}}(x,y)}(hg, g)$$

where the first map is given by composition with g, and the second map is induced by the counit of the adjunction $g^L \dashv g$. The fact that this composite map is an isomorphism shows that $\mathrm{End}(g)$ is an endomorphism monad for g, as desired. □

Notation 7.5.6. Let $\mathcal{M}$ be a presentable symmetric monoidal category. Let $\mathcal{C}$ be an $\mathcal{M}$-enriched category and let M be a monad on $\mathcal{C}$. We denote by $M\text{-mod}^l(\mathcal{D})$ the Eilenberg-Moore object for M in $\mathscr{C}at^{\mathcal{M}}$.

Definition 7.5.7. *Let $\mathcal{M}$ be a presentable symmetric monoidal category and let $\mathcal{D}$ be an $\mathcal{M}$-enriched 2-category. Let M be a monad on an object y of $\mathcal{D}$ and let $g : x \to y$ be a module for M. We say that g presents x as the Eilenberg-Moore object of M if for every object z in $\mathcal{D}$, the functor*

$$g_*^{\mathrm{enh}} : \mathrm{Hom}_{\mathcal{D}}(z, x) \to M\text{-mod}^l(\mathrm{Hom}_{\mathcal{D}}(z, y))$$

induced by composition with g is an equivalence.

Remark 7.5.8. Let $\mathcal{M}$ be a presentable symmetric monoidal category and let $\mathcal{D}$ be an $\mathcal{M}$-enriched 2-category. Let M be a monad on an object y of $\mathcal{D}$ and let $g : x \to y$ be an Eilenberg-Moore object for M. Then g is an Eilenberg-Moore object for M in the 2-category underlying $\mathcal{D}$. In particular, Eilenberg-Moore objects for M are unique if they exist.

Proposition 7.5.9. *Let $\mathcal{M}$ be a presentable symmetric monoidal category and let $\mathcal{D}$ be an $\mathcal{M}$-enriched 2-category. Let $g : x \to y$ be a morphism in $\mathcal{D}$. Then the following are equivalent:*

(i) *There exists a monad M on y and an M-module structure on g, such that g presents x as the Eilenberg-Moore object of M.*

(ii) *The morphism g admits a left adjoint, and g presents x as the Eilenberg-Moore object of the monad* $\mathrm{End}(g)$.

(iii) *The morphism g admits a left adjoint, and for every object z in $\mathcal{D}$ the functor of $\mathcal{M}$-enriched categories*

$$g_* : \mathrm{Hom}_{\mathcal{D}}(z, x) \to \mathrm{Hom}_{\mathcal{D}}(z, y)$$

is monadic.

Proof. This follows along the same lines as propositions 7.3.7 and 7.3.8. □

Definition 7.5.10. *Let $\mathcal{M}$ be a presentable symmetric monoidal category and let $\mathcal{D}$ be an $\mathcal{M}$-enriched 2-category. Then a morphism $g : x \to y$ in $\mathcal{D}$ is said to be monadic if it satisfies the equivalent conditions of proposition 7.5.9.*

Remark 7.5.11. Let $\mathcal{M}$ be a presentable symmetric monoidal category and let $\mathcal{D}$ be an $\mathcal{M}$-enriched 2-category. Let $g : x \to y$ be a monadic morphism in $\mathcal{D}$. Then g is also a monadic morphism in the 2-category underlying $\mathcal{D}$.

Remark 7.5.12. Let $\mathcal{M}$ be a presentable symmetric monoidal category and let $F : \mathcal{C} \to \mathcal{D}$ be a functor of $\mathcal{M}$-enriched categories. Then it follows from theorem 7.4.10 that F is monadic if and only if it is a monadic morphism in the $\mathcal{M}$-enriched 2-category $\overline{\mathscr{C}at^{\mathcal{M}}}$.

We finish by specializing to the case of enrichment over $\omega\mathrm{Cat}$.

Definition 7.5.13. *Let $G : \mathcal{D} \to \mathcal{D}'$ be a functor of ω-categories. We say that G is monadic if it corresponds to a monadic functor of $\omega\mathrm{Cat}$-enriched categories under the equivalence $\mathrm{Cat}^{\omega\mathrm{Cat}} = \omega\mathrm{Cat}$ from remark 3.6.12.*

Remark 7.5.14. Let $G : \mathcal{D} \to \mathcal{D}'$ be a functor of ω-categories. Then G is monadic if and only if it admits a left adjoint and for every ω-category $\mathcal{I}$ the functor of categories

$$G_* : \mathrm{Funct}(\mathcal{I}, \mathcal{D})^{\leq 1} \to \mathrm{Funct}(\mathcal{I}, \mathcal{D}')^{\leq 1}$$

is monadic.

Remark 7.5.15. Let $n \geq 1$ and let $G : \mathcal{D} \to \mathcal{D}'$ be a functor of n-categories. Then G is monadic according to definition 7.5.13 if and only if it admits a left adjoint, and for any n-category $\mathcal{J}$ the functor of categories

$$G_* : \mathrm{Funct}(\mathcal{J}, \mathcal{D})^{\leq 1} \to \mathrm{Funct}(\mathcal{J}, \mathcal{D}')^{\leq 1}$$

is monadic. In other words, G is monadic as a functor of ω-categories if and only if it is monadic as a functor of categories enriched in $(n-1)\mathrm{Cat}$. In particular, setting $n = 1$ we conclude that in the case of functor between 1-categories definition 7.5.13 specializes to the usual notion of monadic functor of categories.

Proposition 7.5.16. *Let $G : \mathcal{D} \to \mathcal{D}'$ be a functor of ω-categories. Then G is monadic if and only if for every $n \geq 1$ the functor $G^{\leq n} : \mathcal{D}^{\leq n} \to \mathcal{D}'^{\leq n}$ is monadic.*

Proof. Assume first that G is monadic. Let $n \geq 1$. It follows from a combination of proposition 5.4.1 and corollary 5.4.4 that the truncation functor $(-)^{\leq n} : \omega\mathrm{Cat} \to n\mathrm{Cat}$ admits an enhancement to a functor of 2-categories. It follows from this that $G^{\leq n}$ admits a left adjoint. Furthermore, if $\mathcal{J}$ is an n-category then the functor

$$G^{\leq n}_* : \mathrm{Funct}(\mathcal{J}, \mathcal{D}^{\leq n})^{\leq 1} \to \mathrm{Funct}(\mathcal{J}, \mathcal{D}'^{\leq n})^{\leq 1}$$

is equivalent to the functor

$$G_* : \mathrm{Funct}(\mathcal{J}, \mathcal{D})^{\leq 1} \to \mathrm{Funct}(\mathcal{J}, \mathcal{D})^{\leq 1}$$

which is monadic. Hence $G^{\leq n}$ is monadic.

Assume now that $G^{\leq n}$ is monadic for all $n \geq 1$. Let $\mathcal{J}$ be an ω-category. Recall from remark 3.6.9 that $\mathcal{J}$ is the colimit of its truncations $\mathcal{J}^{\leq n}$. Hence the functor

$$G_* : \mathrm{Funct}(\mathcal{J}, \mathcal{D})^{\leq 1} \to \mathrm{Funct}(\mathcal{J}, \mathcal{D}')^{\leq 1}$$

is the limit of the functors

$$G_* : \mathrm{Funct}(\mathcal{J}^{\leq n}, \mathcal{D})^{\leq 1} \to \mathrm{Funct}(\mathcal{J}^{\leq n}, \mathcal{D}')^{\leq 1}$$

for $n \geq 1$. Applying proposition 7.3.10 we reduce to showing that the latter is monadic for every $n \geq 1$. Indeed, this is equivalent to the functor

$$G_*^{\leq n} : \mathrm{Funct}(\mathcal{J}^{\leq n}, \mathcal{D}^{\leq n})^{\leq 1} \to \mathrm{Funct}(\mathcal{J}^{\leq n}, \mathcal{D}'^{\leq n})^{\leq 1}$$

which is monadic, as desired. □

Definition 7.5.17. *Let $\mathcal{D}$ be an ω-category. We define the notions of monads, modules over a monad, Eilenberg-Moore object, and monadic morphisms in $\mathcal{D}$ by interpreting $\mathcal{D}$ as a 2-category enriched over $\omega\mathrm{Cat}$, using the equivalence $2\mathrm{Cat}^{\omega\mathrm{Cat}} = \omega\mathrm{Cat}$ that results from iterating the equivalence $\mathrm{Cat}^{\omega\mathrm{Cat}} = \omega\mathrm{Cat}$ from remark 3.6.12.*

Remark 7.5.18. Specializing remark 7.5.12 we see that a functor of ω-categories $G : \mathcal{D} \to \mathcal{D}'$ is monadic if and only if it defines a monadic morphism in the ω-category $\omega\mathscr{C}at$.

Remark 7.5.19. Let $\mathcal{D}$ be an ω-category. Then a morphism $g : x \to y$ in $\mathcal{D}$ is monadic if and only if for every object z in $\mathcal{D}$, the functor of ω-categories

$$g_* : \mathrm{Hom}_{\mathcal{D}}(z, x) \to \mathrm{Hom}_{\mathcal{D}}(z, y)$$

is monadic. It follows from remark 7.5.15 that if $\mathcal{D}$ is an n-category for some $n \geq 2$ then g is monadic if and only if it is monadic when interpreting $\mathcal{D}$ as a 2-category enriched in $(n-2)$-categories. We conclude in particular that in the case when $\mathcal{D}$ is a 2-category, the notions of monadic morphism from definitions 7.3.9 and 7.5.17 coincide.

Remark 7.5.20. Let $\mathcal{D}$ be an ω-category. It follows from proposition 7.5.16 that a morphism g in $\mathcal{D}$ is monadic if and only if it is monadic as a morphism in $\mathcal{D}^{\leq n}$ for all $n \geq 2$.

Remark 7.5.21. Let $\mathcal{D}$ be an ω-category. Let M be a monad on an object y of $\mathcal{D}$, and let $g : x \to y$ be a module over M. Then g presents x as the Eilenberg-Moore object of M if and only if g is monadic, and the induced morphism of monads $M \to \mathrm{End}(g)$ is an equivalence. It follows from remark 7.5.20 that g presents x as the Eilenberg-Moore object of M if and only if the same assertion holds in $\mathcal{D}^{\leq n}$ for every $n \geq 2$. In particular, when $\mathcal{D}$ is a 2-category, the notions of Eilenberg-Moore object from definitions 7.3.1 and 7.5.17 agree.

Part II

Higher quasicoherent sheaves

Chapter 8

Introduction to part II

Part II of this thesis forms part of a program aimed at setting up the foundations of the theory of sheaves of (∞, n)-categories in derived algebraic geometry. The goal of the program is to construct and study various theories of sheaves of (∞, n)-categories on prestacks, and use them to produce interesting examples of fully extended topological field theories.

We begin part II by studying a general framework for higher categorical sheaf theories in the language of categories of correspondences. As emphasized in [GR17], much of the functorial behaviour of a sheaf theory is encoded in its realization as a functor out of an $(\infty, 2)$-category of correspondences. The existence of this realization has direct consequences in terms of dualizability properties of the sheaf theory, and is a helpful tool in the computation of various field-theoretic invariants [BN13].

We therefore take the point of view that a higher sheaf theory on a category of geometric objects $\mathcal{C}$ is a (possibly symmetric monoidal) functor out of a higher category of correspondences on $\mathcal{C}$, into a target $(\infty, n+1)$-category $\mathcal{D}$ whose objects are to be thought of as being (∞, n)-categories of some sort. The first goal of part II is to provide tools for the construction of such functors.

In the case of non-categorical sheaf theories, the target $\mathcal{D}$ is usually taken to be the $(\infty, 2)$-category of presentable stable ∞-categories. The second goal of part II is to introduce (∞, n)-categorifications of the theory of presentable ∞-categories, which will be the target for our higher categorical sheaf theories.

Our starting point is the observation that the ∞-category Pr^L of presentable ∞-categories is not itself a presentable ∞-category. We argue that Pr^L should instead be considered as a presentable $(\infty, 2)$-category - we in fact define this notion so that Pr^L is the unit in the symmetric monoidal $(\infty, 2)$-category of presentable $(\infty, 2)$-categories. In general, we find that for each $n \geq 1$ the collection of all presentable (∞, n)-categories can be organized, not into a presentable (∞, n)-category, but rather into a presentable $(\infty, n+1)$-category.

The third goal of part II is to construct and study the most basic higher categorical sheaf theory: the theory of quasicoherent sheaves of presentable stable (∞, n)-categories. This turns out to satisfy very strong functoriality and dualizability properties, which we encode in its formulation as a symmetric monoidal functor out of a higher category of correspondences of prestacks.

In the case $n = 1$, our definition specializes to the notion of sheaves of categories studied in [Gai15]. A fundamental concept in that setting is that of 1-affineness. Roughly speaking, a prestack X is 1-affine if the ∞-category of sheaves of categories on X can be recovered by taking modules over the symmetric monoidal ∞-category $\mathrm{QCoh}(X)$. In part II we introduce generalizations of this notion for all values of n, and provide a simple inductive criterion that allows one to reduce higher affineness questions to the case $n = 1$.

The material in chapters 9, 10 and 11 is a slight revision of the author's preprint [Ste20a]. The material in chapter 12 is a slight revision of the last section in the author's preprint [Ste20b].

Below we provide a more detailed description of the contents of part II. As usual in this thesis, we will use the convention where all objects are ∞-categorical by default, and suppress this from our notation from now on.

8.1 Sheaf theories and the 2-category of correspondences

We begin by reviewing the case of ordinary, 1-categorical sheaf theory. Let $\mathcal{C}$ be a category admitting pullbacks, whose objects we think about as geometric spaces of some sort (for instance, $\mathcal{C}$ can be the category of affine schemes, stacks, etc). Then one can define a 2-category $2\mathrm{Corr}(\mathcal{C})$ with the following properties:

- The objects of $2\mathrm{Corr}(\mathcal{C})$ are the objects of $\mathcal{C}$.
- For each pair of objects c, c' in $\mathcal{C}$, the category $\mathrm{Hom}_{2\mathrm{Corr}(\mathcal{C})}(c, c')$ is the category $\mathcal{C}_{/c,c'}$ whose objects are spans $c \leftarrow s \rightarrow c'$ in $\mathcal{C}$.
- The composition of two spans $c \leftarrow s \rightarrow c'$ and $c' \leftarrow t \rightarrow c''$ is given by $c \leftarrow s \times_{c'} t \rightarrow c''$.

In its most basic form [1], a sheaf theory on $\mathcal{C}$ consists of a functor $F : 2\mathrm{Corr}(\mathcal{C}) \rightarrow \mathcal{D}$ into a 2-category $\mathcal{D}$ whose objects are to be thought of as categories of some sort. This assigns to each object c in $\mathcal{C}$ an object $F(c)$ in $\mathcal{D}$, subject to the following functoriality:

- For every morphism $\alpha : c \rightarrow c'$ in $\mathcal{C}$ a morphism $\alpha_! : F(c) \rightarrow F(c')$ associated to the following span:

$$\begin{tikzcd} & c \arrow[dl, "\mathrm{id}"'] \arrow[dr, "\alpha"] & \\ c & & c' \end{tikzcd}$$

[1]Many sheaf theories of interest can be formulated in this form if we allow ourselves to work with variants of $2\mathrm{Corr}(\mathcal{C})$ where the legs of the spans are required to belong to certain predetermined classes, and change the notion of 2-cell. For our purposes, this basic version will suffice.

- For every morphism $\alpha : c \to c'$ in $\mathcal{C}$ a morphism $\alpha^! : F(c') \to F(c)$ associated to the following span:

$$\begin{tikzcd} & c \arrow[dl, "\alpha"'] \arrow[dr, "\mathrm{id}"] & \\ c' & & c \end{tikzcd}$$

- For every morphism $\alpha : c \to c'$, a 2-cell $\alpha_! \alpha^! \to \mathrm{id}_{c'}$, associated to the following morphism of spans:

$$\begin{tikzcd} & c \arrow[ddl, "\alpha"'] \arrow[d, "\alpha"] \arrow[ddr, "\alpha"] & \\ & c' \arrow[dl, "\mathrm{id}"] \arrow[dr, "\mathrm{id}"'] & \\ c' & & c' \end{tikzcd}$$

It can be shown that the 2-cell from the third item is the counit of an adjunction between $\alpha_!$ and $\alpha^!$. Furthermore, the composition rule of $2\mathrm{Corr}(\mathcal{C})$ implies that for every pair of maps $\alpha : c \to c' \leftarrow c'' : \beta$ the square

$$\begin{tikzcd} F(c \times_{c'} c'') \arrow[d, "\beta'_!"] & F(c'') \arrow[l, "\alpha'^!"'] \arrow[d, "\beta_!"] \\ F(c) & F(c') \arrow[l, "\alpha^!"'] \end{tikzcd}$$

which is in principle only commutative up to a natural transformation, is in fact strictly commutative.

There is a natural inclusion $\iota_{\mathcal{C}} : \mathcal{C} \to 2\mathrm{Corr}(\mathcal{C})$, which is the identity on objects, and sends each arrow α to the span with right leg α and identity left leg. It was proven in [GR17] that the functor F can be recovered just from the knowledge of its restriction to $\mathcal{C}$. Moreover, given a functor $f : \mathcal{C} \to \mathcal{D}$ one can find an extension of F to $2\mathrm{Corr}(\mathcal{C})$ if and only if $f(\alpha)$ admits a right adjoint for every arrow α in $\mathcal{C}$, and for every pair of maps $\alpha : c \to c' \leftarrow c'' : \beta$ in $\mathcal{C}$ the associated lax commutative square is strictly commutative:

Theorem 8.1.1 ([GR17])**.** *Let $\mathcal{C}$ be a category admitting pullbacks and let $\mathcal{D}$ be a 2-category. Then precomposition with the inclusion $\iota_{\mathcal{C}}$ induces an equivalence between the space of functors $F : 2\mathrm{Corr}(\mathcal{C}) \to \mathcal{D}$ and the space of functors $f : \mathcal{C} \to \mathcal{D}$ satisfying the left Beck-Chevalley condition.*

In chapter 10 we give an alternative approach to this result.[2] Our proof relies on two main ideas:

[2] We refer also to [Mac20] for a related approach to theorem 8.1.1, which proves it conditional on the existence of a 2-categorical Grothendieck construction.

- Let $2\mathrm{Corr}^{\mathrm{univ}}(\mathcal{C})$ be the 2-category satisfying the universal property of theorem 8.1.1. Then the restriction of the hom functor

$$\mathrm{Hom}_{2\mathrm{Corr}^{\mathrm{univ}}(\mathcal{C})} : 2\mathrm{Corr}^{\mathrm{univ}}(\mathcal{C})^{1\text{-op}} \times 2\mathrm{Corr}^{\mathrm{univ}}(\mathcal{C}) \to \mathscr{C}at$$

  to $2\mathrm{Corr}^{\mathrm{univ}}(\mathcal{C})^{1\text{-op}} \times \mathcal{C}$ is shown to satisfy a universal property - roughly speaking, we show that it can be obtained by left Kan extension of $\mathrm{Hom}_{\mathcal{C}} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Spc}$ along the inclusion of $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ inside $2\mathrm{Corr}^{\mathrm{univ}}(\mathcal{C})^{1\text{-op}} \times \mathcal{C}$. Our proof of this relies on the description of Hom functors for enriched categories in terms of diagonal bimodules from [Hin20a].

- We prove a version of the Grothendieck construction which relates functors $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Cat}$ and the so-called two-sided fibrations over $\mathcal{C} \times \mathcal{C}$, and identify the image of $\mathrm{Funct}(2\mathrm{Corr}^{\mathrm{univ}}(\mathcal{C})^{1\text{-op}} \times \mathcal{C}, \mathscr{C}at)$ under this equivalence. The problem then gets reduced to establishing a universal property for the two-sided fibration whose objects are spans in $\mathcal{C}$. The proof of this fact, which is carried out in chapter 9, ultimately relies on the description of free fibrations from [GHN17].

8.2 Higher sheaf theories and the n-category of correspondences

For each $n > 2$ one can define an n-category $n\mathrm{Corr}(\mathcal{C})$ with the following properties:

- The objects of $n\mathrm{Corr}(\mathcal{C})$ are the objects of $\mathcal{C}$.
- For each pair of objects c, c' we have $\mathrm{Hom}_{n\mathrm{Corr}(\mathcal{C})}(c, c') = (n-1)\mathrm{Corr}(\mathrm{Hom}_{2\mathrm{Corr}(\mathcal{C})}(c, c'))$.

In its most basic form, a higher sheaf theory on $\mathcal{C}$ is a functor $F : n\mathrm{Corr}(\mathcal{C}) \to \mathcal{D}$ into an n-category $\mathcal{D}$ whose objects we think about as being $(n-1)$-categories of some sort. Such a functor satisfies a large list of adjointability and base change properties. As before, its restriction to $\mathcal{C}$ satisfies the left Beck-Chevalley condition. However, as soon as $n > 2$ it turns out that it also satisfies the right Beck-Chevalley condition: in fact, pushforwards and pullbacks form part of an ambidextrous adjunction. Furthermore, when $n > 2$ we have, for each pair of objects c, c' in $\mathcal{C}$, a functor

$$F_* : \mathcal{C}_{/c,c'} \to \mathrm{Hom}_{\mathcal{D}}(F(c), F(c'))$$

which also satisfies the left Beck-Chevalley condition. When $n > 3$, these functors also satisfy the right Beck-Chevalley condition, and for every pair of spans $S = (c \leftarrow s \to c')$ and $T = (c \leftarrow t \to c')$ there is an induced functor

$$(F_*)_* : (\mathcal{C}_{/c,c'})_{/s,t} \to \mathrm{Hom}_{\mathrm{Hom}_{\mathcal{D}}(F(c),F(c'))}(F_*(S), F_*(T))$$

which also satisfies the left Beck-Chevalley condition.

Our first main result in chapter 11 is a generalization of theorem 8.1.1 which singles out a minimalistic list of base change properties to verify for a functor $\mathcal{C} \to \mathcal{D}$ to admit an extension to $n\mathrm{Corr}(\mathcal{C})$:

Theorem 8.2.1. *Let $n \geq 1$. Let $\mathcal{C}$ be a category admitting pullbacks, and let $\mathcal{D}$ be an n-category. Then precomposition with the inclusion $\mathcal{C} \to n\mathrm{Corr}(\mathcal{C})$ induces an equivalence between the space of functors $n\mathrm{Corr}(\mathcal{C}) \to \mathcal{D}$ and the space of functors $\mathcal{C} \to \mathcal{D}$ satisfying the left $(n-1)$-fold Beck-Chevalley condition.*

In the same way that verifying the left Beck-Chevalley condition for a functor $f : \mathcal{C} \to \mathcal{D}$ involves checking an adjointability statement for every cartesian square in $\mathcal{C}$, verifying the left $(n-1)$-fold Beck-Chevalley condition involves checking a series of $n-1$ different adjointability statements for every such cartesian square. We refer the reader to section 11.2 for a precise definition of the left $(n-1)$-fold Beck-Chevalley condition.

In the case when $\mathcal{C}$ admits finite limits, the n-category $n\mathrm{Corr}(\mathcal{C})$ comes equipped with a symmetric monoidal structure, and the inclusion $\mathcal{C} \to n\mathrm{Corr}(\mathcal{C})$ can be enhanced to a symmetric monoidal functor, where $\mathcal{C}$ is given the cartesian symmetric monoidal structure. In chapter 11 we also prove a version of theorem 8.2.1 that takes into account this structure:

Corollary 8.2.2. *Let $n \geq 1$. Let $\mathcal{C}$ be a category admitting finite limits, and let $\mathcal{D}$ be a symmetric monoidal n-category. Then restriction along the inclusion $\mathcal{C} \to n\mathrm{Corr}(\mathcal{C})$ induces an equivalence between the space of symmetric monoidal functors $n\mathrm{Corr}(\mathcal{C}) \to \mathcal{D}$ and the space of symmetric monoidal functors $\mathcal{C} \to \mathcal{D}$ which satisfy the left $(n-1)$-fold Beck-Chevalley condition.*

It was shown in [Hau18] that every object of $n\mathrm{Corr}(\mathcal{C})$ is fully dualizable in the $(n-1)$-category underlying $n\mathrm{Corr}(\mathcal{C})$. Combining this fact with corollary 8.2.2 we are able to conclude that if $F : \mathcal{C} \to \mathcal{D}$ is a symmetric monoidal functor satisfying the $(n-1)$-fold left Beck-Chevalley condition and c is an object of $\mathcal{C}$, then $F(c)$ is a fully dualizable object in the $(n-1)$-category underlying $\mathcal{D}$. In other words, $F(c)$ can be considered as the object of boundary conditions for an $(n-1)$-dimensional topological field theory.

8.3 Extension along the Yoneda embedding

The second main result of chapter 11 concerns the interaction of the higher Beck-Chevalley condition and the procedure of Kan extension along the Yoneda embedding $\mathcal{C} \to \mathcal{P}(\mathcal{C})$. It gives conditions under which one can deduce that a colimit preserving functor $\mathcal{P}(\mathcal{C}) \to \mathcal{D}$ satisfies a higher Beck-Chevalley condition, from knowing that its restriction to $\mathcal{C}$ does.

This is instrumental in the study of higher sheaf theories in algebraic geometry, as one usually first gives a definition for affine schemes, and then one Kan-extends the theory to more general stacks. It can often be checked without too much difficulty that a higher Beck-Chevalley condition holds on affine schemes, but this is not so easy to do by hand for arbitrary stacks.

There are two conditions that are required on $\mathcal{D}$ to be able to do this. The first one is that $\mathcal{D}$ be conically cocomplete (see chapter 5). In other words, we want that colimits in the 1-category underlying $\mathcal{D}$ be compatible with the n-categorical structure. We use this to reduce base change properties for cartesian squares in $\mathcal{P}(\mathcal{C})$ to those in which the final vertex belongs to $\mathcal{C}$. We require that this holds not only for $\mathcal{D}$, but also for all Hom $(n-1)$-categories of $\mathcal{D}$, all Hom $(n-2)$-categories of those, and so on.

The second condition is called the passage to adjoints property, and is modeled on the well-known fact that colimits of right adjointable arrows in Pr^L can be computed as limits after passage to right adjoints:

Definition 8.3.1. *Let $\mathcal{D}$ be an n-category, and denote by $(\mathcal{D}^{\leq 1})^{\mathrm{radj}}$ (resp. $(\mathcal{D}^{\leq 1})^{\mathrm{ladj}}$) the subcategory of the category underlying $\mathcal{D}$ containing all objects, and only those morphisms which are right (resp. left) adjointable in $\mathcal{D}$. We say that $\mathcal{D}$ satisfies the passage to adjoints property if the following conditions are satisfied:*

- *The category $(\mathcal{D}^{\leq 1})^{\mathrm{radj}}$ has all colimits, and the inclusion $(\mathcal{D}^{\leq 1})^{\mathrm{radj}} \to \mathcal{D}$ preserves conical colimits.*
- *The category $(\mathcal{D}^{\leq 1})^{\mathrm{ladj}}$ has all limits, and the inclusion $(\mathcal{D}^{\leq 1})^{\mathrm{ladj}} \to \mathcal{D}$ preserves conical limits.*

This condition is used to ensure that the image of any arrow in $\mathcal{P}(\mathcal{C})$ will admit a right adjoint, from knowing that this is true for $\mathcal{C}$. Again the passage to adjoints property is required inductively on all Hom-categories as well.

The following is a simplified version of the second main result of chapter 11 - we refer the reader to section 11.3 for a slightly stronger statement:

Theorem 8.3.2. *Let $n \geq 1$. Let $\mathcal{C}$ be a category admitting pullbacks and let $\mathcal{D}$ be an n-category. Assume that $\mathcal{D}$ is $(n-1)$-fold conically cocomplete, and satisfies the $(n-1)$-fold passage to adjoints property. Let $F : \mathcal{P}(\mathcal{C}) \to \mathcal{D}$ be a conical colimit preserving functor and assume that $F|_{\mathcal{C}}$ and $F|_{\mathcal{C}}^{n\text{-op}}$ satisfy the left $(n-1)$-fold Beck-Chevalley condition. Then F satisfies the left $(n-1)$-fold Beck-Chevalley condition.*

We remark that if $\mathcal{D}$ is the n-category underlying an $(n+1)$-category $\mathcal{D}^+$ and $F|_{\mathcal{C}}$ satisfies the left n-fold Beck-Chevalley condition (as a functor into $\mathcal{D}^+$), then $F|_{\mathcal{C}}$ and $F|_{\mathcal{C}}^{n\text{-op}}$ satisfy the left $(n-1)$-fold Beck-Chevalley condition. In other words, under the stated assumptions on $\mathcal{D}$, any functor $(n+1)\mathrm{Corr}(\mathcal{C}) \to \mathcal{D}^+$ induces a functor $n\mathrm{Corr}(\mathcal{P}(\mathcal{C})) \to \mathcal{D}$.

8.4 Presentable n-categories

The theory of presentable categories is one of the cornerstones of category theory. Many categories arising in nature are presentable, and many categorical constructions preserve presentability. The category Pr^L of presentable categories and colimit preserving functors can be given a symmetric monoidal structure, and its full subcategory of stable presentable categories is the target of various sheaf theories of interest.

In chapter 12 we introduce higher categorical versions of the theory of presentable categories. This takes the form of a symmetric monoidal category $(n+1)$-category $n\mathscr{P}r^L$

of presentable n-categories for each $n \geq 1$, which recovers in the case $n = 1$ a 2-categorical enhancement of Pr^L.

In order to define $n\mathscr{P}r^L$, one first defines its underlying symmetric monoidal category $n\mathrm{Pr}^L$. Recall the following features of Pr^L:

- The category Pr^L admits all small colimits.
- For each pair of objects $\mathcal{C}, \mathcal{D}$ in Pr^L, there is a presentable category $\mathrm{Funct}(\mathcal{C}, \mathcal{D})$ parametrizing functors between $\mathcal{C}$ and $\mathcal{D}$. This endows Pr^L with the structure of a category enriched over itself.

Any presentable 2-category will share the same features. If Pr^L were itself presentable, then any module over Pr^L inside Pr^L would have these properties. In that case, we could simply define $2\mathrm{Pr}^L$ to be the category of modules for Pr^L inside itself.

Since Pr^L is not presentable, it does not make sense to consider its category of modules in Pr^L. However, Pr^L is still a cocomplete category, and its symmetric monoidal structure is compatible with colimits. It therefore makes sense to consider the category of modules for Pr^L inside the symmetric monoidal category $\widehat{\mathrm{Cat}}_{\mathrm{cocompl}}$ of large cocomplete categories admitting small colimits and colimit preserving functors.

The category $\mathrm{Pr}^L\text{-mod}(\widehat{\mathrm{Cat}}_{\mathrm{cocompl}})$ is however still not a good candidate for $2\mathrm{Pr}^L$. A first issue is that its objects turn out to be enriched in $\widehat{\mathrm{Cat}}_{\mathrm{cocompl}}$, rather than Pr^L. A related problem is that $\mathrm{Pr}^L\text{-mod}(\widehat{\mathrm{Cat}}_{\mathrm{cocompl}})$ is too big. Recall that presentable categories are controlled by a small amount of data, which in particular implies that Pr^L is a large category (as opposed to very large). However $\mathrm{Pr}^L\text{-mod}(\widehat{\mathrm{Cat}}_{\mathrm{cocompl}})$ is a very large category, and in particular it does not belong to $\widehat{\mathrm{Cat}}_{\mathrm{cocompl}}$, but to the category $\mathrm{CAT}_{\mathrm{cocompl}}$ of very large categories admitting large colimits. A theory of presentable n-categories which simply iteratively takes categories of modules would need a long sequence of nested universes, and would require one to carefully keep track of the relative sizes of different objects.

The main observation that leads to our definition of higher presentable categories is that, while $\mathrm{Pr}^L\text{-mod}(\widehat{\mathrm{Cat}}_{\mathrm{cocompl}})$ is a very large category, it is controlled by a large subcategory of objects we call presentable. More precisely, $\mathrm{Pr}^L\text{-mod}(\widehat{\mathrm{Cat}}_{\mathrm{cocompl}})$ is presentable as a very large category, and it is in fact κ_0-compactly generated, for κ_0 the smallest large cardinal. The category $2\mathrm{Pr}^L$ is then defined to be the full subcategory of $\mathrm{Pr}^L\text{-mod}(\widehat{\mathrm{Cat}}_{\mathrm{cocompl}})$ on the κ_0-compact objects.

Iterating the above discussion, one may obtain a symmetric monoidal category $n\mathrm{Pr}^L$ for each $n \geq 1$:

Definition 8.4.1. *We inductively define, for each $n \geq 2$, the symmetric monoidal category $n\mathrm{Pr}^L$ to be the full subcategory of $(n-1)\mathrm{Pr}^L\text{-mod}(\widehat{\mathrm{Cat}}_{\mathrm{cocompl}})$ on the κ_0-compact objects.*

The first main result of chapter 12 is the construction of an $(n+1)$-categorical enhancement of $n\mathrm{Pr}^L$, which we denote by $n\mathscr{P}r^L$. This is obtained as a consequence of the existence of a lax symmetric monoidal realization functor which maps objects in $(n-1)\mathrm{Pr}^L\text{-mod}(\widehat{\mathrm{Cat}}_{\mathrm{cocompl}})$

to n-categories, which ultimately depends on the procedure of enrichment of presentable modules from chapter 4. In other words, although our approach to presentable n-categories starts out being 1-categorical, we are in fact able to consider these as higher categories. To formulate the statement, it is convenient to set $0\mathrm{Pr}^L$ to be the category of spaces.

Theorem 8.4.2. *Let $n \geq 1$. There exists a lax symmetric monoidal functor*

$$\psi_n : (n-1)\mathrm{Pr}^L\text{-mod}(\widehat{\mathrm{Cat}}_{\mathrm{cocompl}}) \to \widehat{n\mathrm{Cat}}$$

with the following properties:

(i) *For each $(n-1)\mathrm{Pr}^L$-module $\mathcal{C}$, the 1-category underlying $\psi_n(\mathcal{C})$ is equivalent to the category underlying $\mathcal{C}$.*

(ii) *Assume $n > 1$ and let c, d be objects in $\mathcal{C}$. Then*

$$\mathrm{Hom}_{\psi_n(\mathcal{C})}(c, d) = \psi_{n-1}(\mathscr{H}om_{\mathcal{C}}(c, d))$$

where $\mathscr{H}om_{\mathcal{C}}(c, d)$ denotes the Hom object between c and d obtained from the action of $(n-1)\mathrm{Pr}^L$ on $\mathcal{C}$.

Remark 8.4.3. Although our main interest is in objects of $n\mathrm{Pr}^L$, we need to have access to the realization functor on the bigger category $(n-1)\mathrm{Pr}^L\text{-mod}(\widehat{\mathrm{Cat}}_{\mathrm{cocompl}})$. The main reason is that if $n > 2$ we do not know whether the Hom objects in part (ii) belong to $(n-1)\mathrm{Pr}^L$, even if $\mathcal{C}$ itself belongs to $n\mathrm{Pr}^L$.

Corollary 8.4.4. *Let $n \geq 1$. There exists a symmetric monoidal $(n+1)$-category $n\mathscr{P}\!r^L = \psi_{n+1}(n\mathrm{Pr}^L)$ whose underlying category is $n\mathrm{Pr}^L$, and such that for every pair of objects $\mathcal{C}, \mathcal{D}$, we have*

$$\mathrm{Hom}_{n\mathscr{P}\!r^L}(\mathcal{C}, \mathcal{D}) = \psi_n(\mathscr{H}om_{n\mathrm{Pr}^L}(\mathcal{C}, \mathcal{D})).$$

In particular, there is an equivalence

$$\mathrm{End}_{n\mathscr{P}\!r^L}((n-1)\mathrm{Pr}^L) = (n-1)\mathscr{P}\!r^L.$$

8.5 Colimits and the passage to adjoints property

A direct consequence of item (i) in theorem 8.4.2 is that for every object $\mathcal{C}$ in the category $(n-1)\mathrm{Pr}^L\text{-mod}(\widehat{\mathrm{Cat}}_{\mathrm{cocompl}})$, the category underlying $\psi_n(\mathcal{C})$ admits all small colimits. The second main result of chapter 12 provides a strengthening of this fact:

Theorem 8.5.1. *Let $n \geq 1$ and let $\mathcal{C}$ be an object in $(n-1)\mathrm{Pr}^L\text{-mod}(\widehat{\mathrm{Cat}}_{\mathrm{cocompl}})$. Then $\psi_n(\mathcal{C})$ admits all conical colimits.*

The following result is a typical application of theorem 8.5.1 (see also theorem 2.3.1).

Corollary 8.5.2. *Let $\mathcal{C}$ be an object in $(n-1)\mathrm{Pr}^L\text{-mod}(\widehat{\mathrm{Cat}}_{\mathrm{cocompl}})$, and let $\mathcal{I}$ be a small category. Let $F, G : \mathcal{I} \to \mathcal{C}$ be functors, and $\eta : F \to G$ be a natural transformation. Assume that for every arrow $\alpha : i \to j$ in $\mathcal{I}$ the commutative square*

$$\begin{tikzcd} F(i) \arrow[r, "\eta_i"] \arrow[d, "F(\alpha)"] & G(i) \arrow[d, "G(\alpha)"] \\ F(j) \arrow[r, "\eta_j"] & G(j) \end{tikzcd}$$

is horizontally right adjointable in $\psi_n(\mathcal{C})$ (in other words, the horizontal arrows admit right adjoints η_i^R and η_j^R in $\psi_n(\mathcal{C})$, and the induced 2-cell $F(\alpha)\eta_i^R \to \eta_j^R G(\alpha)$ is an isomorphism). Then the morphism

$$\mathrm{colim}_{\mathcal{I}}\, \eta : \mathrm{colim}_{\mathcal{I}}\, F \to \mathrm{colim}_{\mathcal{I}}\, G$$

admits a right adjoint in $\psi_n(\mathcal{C})$.

In other words, a colimit of right adjointable arrows in $\psi_n(\mathcal{C})$ is right adjointable, as long as various base change properties hold.

A fundamental feature of the theory of presentable categories is that passage to adjoints interchanges colimits and limits. Concretely, if $F : \mathcal{I} \to \mathrm{Pr}^L$ is a diagram, then the colimit of F is equivalent to the limit of the diagram $\mathcal{I}^{\mathrm{op}} \to \widehat{\mathrm{Cat}}$ obtained from F by passing to right adjoints of every arrow.

In the case when the right adjoint to $F(\alpha)$ belongs to Pr^L for every arrow α in $\mathcal{I}$, this is also the same as the limit of the resulting functor $F^R : \mathcal{I}^{\mathrm{op}} \to \mathrm{Pr}^L$. The properties of $2\mathscr{P}r^L$ which make this fact hold are in fact encoded in the passage to adjoints property (definition 8.3.1).

The third main result of chapter 12 states that the n-categories obtained from the realization functors of theorem 8.4.2 satisfy the passage to adjoints property.

Theorem 8.5.3. *Let $n \geq 2$ and let $\mathcal{C}$ be an object in $(n-1)\mathrm{Pr}^L\text{-mod}(\widehat{\mathrm{Cat}}_{\mathrm{cocompl}})$. Then $\psi_n(\mathcal{C})$ satisfies the passage to adjoints property.*

In particular, although we do not know whether $n\mathrm{Pr}^L$ admits all small limits when $n > 1$, we are able to conclude that it has limits of left adjointable diagrams.

Combining theorems 8.2.1 and 8.3.2 with theorems 8.5.1 and 8.5.3 yields the following result which forms the base of our approach to constructing representations of higher categories of correspondences of prestacks.

Corollary 8.5.4. *Let $\mathcal{C}$ be a category admitting pullbacks and let $\mathcal{D}$ be the $(n+1)$-category underlying a presentable $(n+1)$-category. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor such that $F|_{\mathcal{C}}$ and $F|_{\mathcal{C}}^{n\text{-op}}$ satisfy the left $(n-1)$-fold Beck-Chevalley condition. Then there exists a unique extension of F to a functor $n\mathrm{Corr}(\mathcal{P}(\mathcal{C})) \to \mathcal{D}$ whose restriction to $\mathcal{P}(\mathcal{C})$ preserves conical colimits.*

8.6 Categorical spectra

Let $\mathcal{C}$ be a category admitting finite products. Then for each $n \geq 1$ the n-category $n\mathrm{Corr}(\mathcal{C})$ comes equipped with a distinguished object, corresponding to the terminal object $1_{\mathcal{C}}$ of $\mathcal{C}$. Furthermore, there are equivalences

$$\mathrm{End}_{1_{\mathcal{C}}}((n+1)\mathrm{Corr}(\mathcal{C})) = n\mathrm{Corr}(\mathcal{C})$$

which map the unit endomorphism to $1_{\mathcal{C}}$.

A similar situation occurs in the theory of higher presentable categories. For each $n \geq 1$ the $(n+1)$-category $n\mathscr{P}r^L$ comes equipped with a distinguished object, given by $(n-1)\mathrm{Pr}^L$. Furthermore, there are equivalences

$$\mathrm{End}_{(n+1)\mathscr{P}r^L}(n\mathrm{Pr}^L) = n\mathscr{P}r^L$$

which map the unit endomorphism to $(n-1)\mathrm{Pr}^L$.

It is often the case that higher sheaf theories produce not only a functor out of a single higher category of correspondences, but a sequence of functors which are compatible with the above isomorphisms. Furthermore, the multiplicative structure of a higher sheaf theory is usually encoded in this sequence of functors.

To abstract and provide context for this situation, we introduce in chapter 13 the notion of a categorical spectrum.[3] The situation is reminiscent of stable homotopy theory: in the same way that a spectrum is a sequence of pointed homotopy types compatible under looping, a categorical spectrum consists of a sequence of ω-categories $\mathcal{C}_n$ equipped with basepoints x_n, and pointed equivalences

$$(\mathrm{End}_{\mathcal{C}_{n+1}}(x_{n+1}), \mathrm{id}_{x_{n+1}}) = (\mathcal{C}_n, x_n)$$

for all $n \geq 1$. In the same way that the passage from spaces to spectra amounts to allowing negative homotopy groups, we may think about the passage from ω-categories to categorical spectra as allowing cells of negative dimensions.

The theory of categorical spectra has intimate connections to the theory of symmetric monoidal ω-categories. The fundamental observation is that sometimes a symmetric monoidal structure on an ω-category is more naturally studied by studying a sequence of deloopings of it.[4] In fact, the theory of categorical spectra may be considered as a joint generalization of the theory of symmetric monoidal ω-categories and the theory of spectra. This is justified by the fact, explained in chapter 13, that the category CatSp of categorical spectra sits in a commutative square of categories and fully faithful functors

$$\begin{tikzcd}
\mathrm{CAlg}_{\mathrm{grpllike}}(\mathrm{Spc}) \arrow[r] \arrow[d] & \mathrm{Sp} \arrow[d] \\
\mathrm{CAlg}(\omega\mathrm{Cat}) \arrow[r] & \mathrm{CatSp}\,.
\end{tikzcd}$$

[3] We first heard about a version of the notion of categorical spectrum from Constantin Teleman.

[4] We refer to [Sch14] section 1.6 for a discussion of this theme in the setting of n-fold Segal spaces.

We can summarize this by saying that the theory of symmetric monoidal ω-categories is the same as the theory of connective categorical spectra. In the context of higher sheaf theory, one should arguably think about the symmetric monoidal ω-categories $n\mathrm{Corr}(\mathcal{C})$ and $n\mathscr{P}r^L$ as being connective covers of more fundamental, nonconnective categorical spectra.

8.7 Higher quasicoherent sheaves

Let $n \geq 1$. Then starting from a commutative ring spectrum A, the theory from chapter 12 produces a presentable stable n-category $A\text{-mod}^n$ of A-linear presentable stable $(n-1)$-categories. In chapter 14 we interpret this object geometrically:

Definition 8.7.1. *Let $X = \mathrm{Spec}(A)$ be an affine scheme. We let $n\mathrm{QCoh}(X) = A\text{-mod}^n$. We call this the presentable stable n-category of quasicoherent sheaves of $(n-1)$-categories on X.*

The assignment $A \mapsto A\text{-mod}^n$ is functorial in A, and therefore the assignment $X \mapsto n\mathrm{QCoh}(X)$ is also functorial in X. A right Kan extension procedure allows us to extend definition 8.7.1 to a functor $n\mathrm{QCoh}_{\mathrm{PreStk}}$ on the category of prestacks. The first main result of chapter 14 is an application of theorems 8.2.1, 8.3.2, 8.5.1 and 8.5.3, and states that the functor $n\mathrm{QCoh}_{\mathrm{PreStk}}$ admits an extension to a functor out of a higher category of correspondences of prestacks:

Theorem 8.7.2. *Let $n \geq 2$. Then there is a unique extension of $n\mathrm{QCoh}_{\mathrm{PreStk}}$ along the inclusion $\mathrm{PreStk}^{\mathrm{op}} \to n\mathrm{Corr}(\mathrm{PreStk})$ to a functor*

$$n\mathrm{QCoh}_{n\mathrm{Corr}(\mathrm{PreStk})} : n\mathrm{Corr}(\mathrm{PreStk}) \to n\mathscr{P}r^L_{\mathrm{St}}.$$

Furthermore, if $\mathrm{PreStk}_{\mathrm{rep}}$ denotes the wide subcategory of PreStk on the affine-schematic morphisms, we have that the restriction of $n\mathrm{QCoh}_{n\mathrm{Corr}(\mathrm{PreStk})}$ to $n\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{rep}})$ admits a unique extension to a functor

$$n\mathrm{QCoh}_{(n+1)\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{rep}})} : (n+1)\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{rep}}) \to (n\mathscr{P}r^L_{\mathrm{St}})^{(n+1)\text{-op}}.$$

As we shall see, the above functors are compatible as we change the number n. We may summarize the situation by saying that the theory of higher quasicoherent sheaves gives rise to a representation of the categorical spectrum of correspondences of prestacks.

The second main result of chapter 14 has to do with the notion of affineness. As observed in [Gai15], many stacks of interest are 1-affine: roughly speaking, they behave as if they were affine schemes, for the purposes of categorified sheaf theory. Building upon the theory of monads and monadic morphisms from chapter 7, we generalize this notion as follows:

Definition 8.7.3. *Let $n \geq 2$ and let X be a prestack. We say that X is n-affine if the global sections morphism*

$$\Gamma(X, -) : (n+1)\mathrm{QCoh}(X) \to n\mathrm{Pr}^L_{\mathrm{St}}$$

is a monadic morphism in $(n+1)\mathscr{P}r^L_{\mathrm{St}}$.

The second main result of chapter 14 is the following inductive criterion that allows one to reduce questions in higher affineness to the case $n = 1$ studied in [Gai15].

Theorem 8.7.4. *Let $n \geq 2$ and let X be a prestack with $(n-1)$-affine diagonal. Then X is n-affine.*

8.8 Organization

We now describe the contents of part II in more detail. We refer the reader also to the introduction of each chapter for an expanded outline of its contents.

In chapter 9 we study the theory of two-sided fibrations. We introduce a two-sided variant of the Grothendieck construction, and study the classes of two-sided fibrations which are classified by functors with various adjointability properties. We discuss two main examples: the arrow fibration and the span fibration, and prove universal properties for them.

Chapter 10 deals with the 2-category of correspondences. We recall its definition, and collect a series of basic functoriality, adjointness and duality results, for later reference. We finish this section by applying the theory of two-sided fibrations to the proof of theorem 8.1.1

In chapter 11 we study the n-category of correspondences. We begin by giving a definition adapted to our purposes using the language of enriched category theory, and review its main adjointness and duality properties. We then introduce the higher Beck-Chevalley conditions, and prove theorem 8.2.1. We finish this section with a proof of theorem 8.3.2.

In chapter 12 we construct the symmetric monoidal categories $n\mathrm{Pr}^L$ and the realization functor from theorem 8.4.2. We then give a proof of theorem 8.5.1. Finally, we study the procedure of passage to adjoints and prove theorem 8.5.3.

Chapter 13 deals with the theory of categorical spectra. We discuss a number of basic concepts such as shifts, cells, opposites, and connective objects. We present a number of examples of categorical spectra of interest for the rest of the thesis. We finish this chapter by studying the relation to the theory of symmetric monoidal ω-categories.

Finally, in chapter 14 we introduce the theory of higher quasicoherent sheaves, and study the functoriality, descent, and affineness properties of this theory. We supply here proof of theorems 8.7.2 and 8.7.4.

Chapter 9

Two-sided fibrations

Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A two-sided fibration from $\mathcal{C}$ to $\mathcal{D}$ is a functor $p : \mathcal{E} \to \mathcal{C} \times \mathcal{D}$ which behaves as a cocartesian fibration along the $\mathcal{C}$-directions, and as a cartesian fibration along the $\mathcal{D}$-directions. In 9.1 we review the notion of two-sided fibration,[1] and we present a two-sided analog of the straightening-unstraightening equivalence, which identifies the category of two-sided fibrations over $\mathcal{C} \times \mathcal{D}$ with the category of functors $\mathcal{C} \times \mathcal{D}^{\mathrm{op}} \to \mathrm{Cat}$.

In 9.2 we study a fundamental example of a two-sided fibration: the target-source projection $\mathrm{Funct}([1], \mathcal{C}) \to \mathcal{C} \times \mathcal{C}$. We show that this is classified by the Hom functor $\mathrm{Hom}_{\mathcal{C}} : \mathcal{C} \times \mathcal{C}^{\mathrm{op}} \to \mathrm{Spc}$. This is a two-sided counterpart to the fact that the twisted arrow category of $\mathcal{C}$ is the pairing of categories classified by the Hom functor of $\mathcal{C}$. As an application, we provide a concrete description of the so-called representable bifibrations, which are classified by functors $\mathcal{C} \times \mathcal{D}^{\mathrm{op}} \to \mathrm{Spc}$ of the form $\mathrm{Hom}_{\mathcal{D}}(F(-), -)$ for some functor $F : \mathcal{C} \to \mathcal{D}$. We also establish a universal property for the arrow bifibration - in its most basic form it states that the arrow category is the free bifibration on the diagonal functor $\Delta : \mathcal{C} \to \mathcal{C} \times \mathcal{C}$.

In 9.3 we introduce the class of bivariant fibrations. These are functors $p : \mathcal{E} \to \mathcal{C} \times \mathcal{D}$ which are simultaneously cocartesian fibrations, cartesian fibrations, and two-sided fibrations over both $\mathcal{C} \times \mathcal{D}$ and $\mathcal{D} \times \mathcal{C}$. In the same way that a two-sided fibration over $\mathcal{C} \times \mathcal{D}$ is classified by a functor $\mathcal{C} \times \mathcal{D}^{\mathrm{op}} \to \mathrm{Cat}$, we show that a bivariant fibration is equivalent to the data of four functors

$$\begin{array}{cc} \mathcal{C} \times \mathcal{D} \to \mathrm{Cat} & \mathcal{C} \times \mathcal{D}^{\mathrm{op}} \to \mathrm{Cat} \\ \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathrm{Cat} & \mathcal{C}^{\mathrm{op}} \times \mathcal{D}^{\mathrm{op}} \to \mathrm{Cat} \end{array}$$

each of which determines the rest by passage to right or left adjoints in one or both coordinates.

In 9.4 we specialize to the case when $\mathcal{C}$ and $\mathcal{D}$ admit pullbacks, and study the class of bivariant fibrations satisfying the so-called Beck-Chevalley condition. We show that these are classified by functors which satisfy familiar base change properties in each variable. We finish with a fundamental example of a bivariant fibration satisfying the Beck-Chevalley condition:

[1] The notion of (lax) two-sided fibration which we discuss here agrees with the notion of (lax) bifibration discussed previously in [Hin20b] section 3.1.2 and [Hin20a] section 2.2.6.

the source-target projection $\mathrm{Funct}(\Lambda^2_0, \mathcal{C}) \to \mathcal{C} \times \mathcal{C}$, where $\mathrm{Funct}(\Lambda^2_0, \mathcal{C})$ denotes the category whose objects are spans in $\mathcal{C}$. We show that this enjoys a universal property: it is the free cocartesian and two-sided fibration which satisfies the Beck-Chevalley condition in the first coordinate, on the arrow two-sided fibration.

9.1 The two-sided Grothendieck construction

We begin by reviewing the notion of two-sided fibration.

Definition 9.1.1. *Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be categories, and let $p = (p_1, p_2) : \mathcal{E} \to \mathcal{C} \times \mathcal{D}$ be a functor. We say that p is a lax two-sided fibration from $\mathcal{C}$ to $\mathcal{D}$ if the following conditions hold:*

- *Let e be an object of $\mathcal{E}$ and write $p(e) = (c, d)$. Then for every arrow $\alpha : c \to c'$ in $\mathcal{C}$ there is a p-cocartesian lift of (α, id_d) with source e.*

- *Let e be an object of $\mathcal{E}$ and write $p(e) = (c, d)$. Then for every arrow $\beta : d \to d'$ in $\mathcal{C}$ there is a p-cartesian lift of (id_c, β) with target e.*

Remark 9.1.2. The condition that the functor p be a lax two-sided fibration depends on the decomposition of $\mathcal{C} \times \mathcal{D}$ as an ordered product. It will usually be clear from context what this decomposition is. Unless otherwise stated, we will work in this chapter with the convention where a (lax) two-sided fibration has cocartesian lifts of arrows in the first factor, and cartesian lifts of arrows in the second factor. In other words, unless otherwise stated, if we say that a functor $p : \mathcal{E} \to \mathcal{C} \times \mathcal{D}$ is a (lax) two-sided fibration we mean that it is a (lax) two-sided fibration from $\mathcal{C}$ to $\mathcal{D}$.

Remark 9.1.3. Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be categories, and let $p = (p_1, p_2) : \mathcal{E} \to \mathcal{C} \times \mathcal{D}$ be a functor. Let $\alpha : c \to c'$ be an arrow in $\mathcal{C}$ and d be an object in $\mathcal{D}$. It follows from [Lur09a] proposition 4.3.1.5 item (2) that a lift of (α, id_d) to $\mathcal{E}$ is p-cocartesian if and only if it is p_1-cocartesian. Similarly, if $\beta : d \to d'$ is a morphism in $\mathcal{D}$ and c is an object in $\mathcal{C}$, a lift of (id_c, β) to $\mathcal{E}$ is p-cartesian if and only if it is p_2-cartesian.

In particular, we see that p is a lax two-sided fibration if and only if the following two conditions hold:

- p_1 is a cocartesian fibration and p is a morphism of cocartesian fibrations over $\mathcal{C}$.

- p_2 is a cartesian fibration and p is a morphism of cartesian fibrations over $\mathcal{D}$.

Definition 9.1.4. *Let $\mathcal{C}, \mathcal{D}$ be categories, and let $p = (p_1, p_2) : \mathcal{E} \to \mathcal{C} \times \mathcal{D}$ be a lax two-sided fibration from $\mathcal{C}$ to $\mathcal{D}$. Let $f : e \to e'$ be an arrow in $\mathcal{E}$, lying above an arrow $(\alpha, \beta) : (c, d) \to (c', d')$. Let $\overline{(\alpha, \mathrm{id}_d)} : e \to m$ be a p-cocartesian lift of (α, id_d), and $\overline{(\mathrm{id}'_c, \beta)} : m' \to e$ be a p-cartesian lift of (id_c, β). We say that f is p-bicartesian if the induced map $m \to m'$ is an isomorphism. We say that p is a two-sided fibration from $\mathcal{C}$ to $\mathcal{D}$ if composition of p-bicartesian arrows is p-bicartesian.*

Remark 9.1.5. Let $\mathcal{C}, \mathcal{D}$ be categories, and let $p = (p_1, p_2) : \mathcal{E} \to \mathcal{C} \times \mathcal{D}$ be a lax two-sided fibration. An arrow $f : e \to e'$ in $\mathcal{E}$ is p-bicartesian if and only if it can be written as a composition of a p_1-cocartesian arrow followed by a p_2-cartesian arrow. The projection p is a two-sided fibration if and only if any arrow of the form $f_1 \circ f_2$ where f_2 is p_2-cartesian and f_1 is p_1-cocartesian, is bicartesian.

Definition 9.1.6. *Let $\mathcal{C}, \mathcal{D}$ be categories, and let $p : \mathcal{E} \to \mathcal{C} \times \mathcal{D}$ and $q : \mathcal{E}' \to \mathcal{C} \times \mathcal{D}$ be two-sided fibrations. A functor $F : \mathcal{E} \to \mathcal{E}'$ equipped with an identification $qF = p$ is said to be a morphism of two-sided fibrations if it maps bicartesian arrows to bicartesian arrows.*

Remark 9.1.7. Let $\mathcal{C}, \mathcal{D}$ be categories, and let $p : \mathcal{E} \to \mathcal{C} \times \mathcal{D}$ and $q : \mathcal{E}' \to \mathcal{C} \times \mathcal{D}$ be two-sided fibrations. Let $F : \mathcal{E} \to \mathcal{E}'$ be a functor equipped with an identification $qF = p$. It follows from remark 9.1.5 that F is a morphism of two-sided fibrations if and only if it is a morphism of cocartesian fibrations over $\mathcal{C}$ and a morphism of cartesian fibrations over $\mathcal{D}$.

Notation 9.1.8. Let $\mathcal{C}$ be a category. We denote by $\mathrm{Cat}_{/\mathcal{C}}^{\mathrm{cart}}$ (resp. $\mathrm{Cat}_{/\mathcal{C}}^{\mathrm{cocart}}$) the subcategory of the overcategory $\mathrm{Cat}_{/\mathcal{C}}$ on the (co)cartesian fibrations and morphisms of (co)cartesian fibrations. Given another category $\mathcal{D}$, we denote by $\mathrm{Cat}_{/\mathcal{C},\mathcal{D}}^{\text{two-sided}}$ the subcategory of $\mathrm{Cat}_{/\mathcal{C},\mathcal{D}}$ on the two-sided fibrations and morphisms of two-sided fibrations.

If we allow ourselves to break symmetry, we can give an alternative characterization of two-sided fibrations and morphisms of two-sided fibrations.

Proposition 9.1.9. *Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be categories, and let $p = (p_1, p_2) : \mathcal{E} \to \mathcal{C} \times \mathcal{D}$ be a functor. Then*

(i) *The map p is a lax two-sided fibration if and only if p_1 is a cocartesian fibration, the functor p is a morphism of cocartesian fibrations over $\mathcal{C}$, and for every object c in $\mathcal{C}$ the projection $p_1^{-1}(c) \to \mathcal{D}$ is a cartesian fibration.*

(ii) *The map p is a two-sided fibration if and only if it is a lax two-sided fibration and the induced functor $\mathcal{C} \to \mathrm{Cat}_{/\mathcal{D}}$ factors through $\mathrm{Cat}_{/\mathcal{D}}^{\mathrm{cart}}$.*

(iii) *Assume that p is a two-sided fibration and let $q : \mathcal{E}' \to \mathcal{C} \times \mathcal{D}$ be another two-sided fibration. Then a functor $F : \mathcal{E} \to \mathcal{E}'$ equipped with an identification $qF = p$ is a morphism of two-sided fibrations if and only if it is a morphism of cocartesian fibrations over $\mathcal{C}$, and for every object c in $\mathcal{C}$ the induced functor $p^{-1}(c) \to q^{-1}(c)$ is a morphism of cartesian fibrations over $\mathcal{D}$.*

Proof. This follows from a combination of remarks 9.1.3 and 9.1.5, together with (the dual of) [Lur09a] corollary 4.3.1.15. □

Our next goal is to present a two-sided analog of the Grothendieck construction. We first recall the notion of weighted colimits of categories.[2]

[2]This should be a special case of the notion of weighted colimit from chapter 5. The fact that these two definitions are compatible will not be needed in this thesis.

Definition 9.1.10. *Let $\mathcal{B}$ be a category and denote by* $\mathrm{Tw}(\mathcal{B})$ *the twisted arrow category of* $\mathcal{B}$*. Let* $H : \mathcal{B} \to \mathrm{Cat}$ *and* $W : \mathcal{B}^{\mathrm{op}} \to \mathrm{Cat}$ *be functors. The colimit of* H *weighted by* W *is defined to be the colimit of the composite functor*

$$\mathrm{Tw}(\mathcal{B}) \to \mathcal{B}^{\mathrm{op}} \times \mathcal{B} \xrightarrow{W(-)\times H(-)} \mathrm{Cat}\,.$$

The lax colimit of H *is the colimit of* H *weighted by the functor* $\mathcal{B}_{-/} : \mathcal{B}^{\mathrm{op}} \to \mathrm{Cat}$ *obtained by straightening of the source fibration* $\mathrm{Funct}([1], \mathcal{B}) \to \mathcal{B}$*. The oplax colimit of* H *is the colimit of* H *weighted by the functor* $(\mathcal{B}_{-/})^{\mathrm{op}} = (\mathcal{B}^{\mathrm{op}})_{/-}$*.*

It is proven in [GHN17] that the lax colimit of a functor $H : \mathcal{B} \to \mathrm{Cat}$ recovers the total category of the cocartesian fibration associated to H. Similarly, the oplax colimit of H recovers the total category of the cartesian fibration associated to H. We now define our two-sided Grothendieck construction to be a mixed lax-oplax colimit.

Definition 9.1.11. *Let* $\mathcal{C}, \mathcal{D}$ *be categories and* $H : \mathcal{C} \times \mathcal{D}^{\mathrm{op}} \to \mathrm{Cat}$ *be a functor. We define the bilax colimit of* H *to be the colimit of* H *weighted by the functor* $\mathcal{C}_{-/} \times \mathcal{D}_{/-} : \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathrm{Cat}$*. We denote it by* $\int_{\mathcal{C}\times\mathcal{D}^{\mathrm{op}}} H$*.*

Example 9.1.12. Let $\mathcal{C}$ be a category and $H : \mathcal{C} = \mathcal{C} \times [0]^{\mathrm{op}} \to \mathrm{Cat}$ be a functor. Then we have

$$\int_{\mathcal{C}\times[0]^{\mathrm{op}}} \mathcal{C} = \mathrm{colim}_{\mathrm{Tw}(\mathcal{C}\times[0]^{\mathrm{op}})}\, \mathcal{C}_{-/} \times [0]_{/-} \times H = \mathrm{colim}_{\mathrm{Tw}(\mathcal{C})}\, \mathcal{C}_{-/} \times H$$

Using [GHN17] theorem 7.4 we see that $\int_{\mathcal{C}\times[0]^{\mathrm{op}}} H$ is the total category of the cocartesian fibration associated to H.

Example 9.1.13. Let $\mathcal{D}$ be a category and $H : \mathcal{D}^{\mathrm{op}} = [0] \times \mathcal{D}^{\mathrm{op}} \to \mathrm{Cat}$ be a functor. Then by [GHN17] corollary 7.6 we have that $\int_{[0]\times\mathcal{D}^{\mathrm{op}}} H$ is the total category of the cartesian fibration associated to H.

Example 9.1.14. Let $\mathcal{C}, \mathcal{D}$ be categories and $H : \mathcal{C} \times \mathcal{D}^{\mathrm{op}} \to \mathrm{Cat}$ be the functor which is constant $[0]$. We have

$$\int_{\mathcal{C}\times\mathcal{D}^{\mathrm{op}}} H = \mathrm{colim}_{\mathrm{Tw}(\mathcal{C}\times\mathcal{D}^{\mathrm{op}})}\, \mathcal{C}_{-/} \times \mathcal{D}_{/-} = \mathrm{colim}_{\mathrm{Tw}(\mathcal{C})}\, \mathcal{C}_{-/} \times \mathrm{colim}_{\mathrm{Tw}(\mathcal{D}^{\mathrm{op}})}\, \mathcal{D}_{/-}$$

Using [GHN17] corollary 7.5 we conclude that $\int_{\mathcal{C}\times\mathcal{D}^{\mathrm{op}}} H = \mathcal{C} \times \mathcal{D}^{\mathrm{op}}$.

Remark 9.1.15. Let $\mathcal{C}, \mathcal{D}$ be categories. The assignment $H \mapsto \int_{\mathcal{C}\times\mathcal{D}^{\mathrm{op}}} H$ can be enhanced to a functor $\mathrm{Funct}(\mathcal{C} \times \mathcal{D}^{\mathrm{op}}, \mathrm{Cat}) \to \mathrm{Cat}$. The functor constant $[0]$ is the terminal object in $\mathrm{Funct}(\mathcal{C} \times \mathcal{D}^{\mathrm{op}}, \mathrm{Cat})$. It follows from example 9.1.14 that the bilax colimit functor can be enhanced to a functor

$$\int_{\mathcal{C}\times\mathcal{D}^{\mathrm{op}}} : \mathrm{Funct}(\mathcal{C} \times \mathcal{D}^{\mathrm{op}}, \mathrm{Cat}) \to \mathrm{Cat}_{/\mathcal{C}\times\mathcal{D}^{\mathrm{op}}}\,.$$

Proposition 9.1.16. *Let $\mathcal{C}, \mathcal{D}$ be categories. The functor $\int_{\mathcal{C}\times\mathcal{D}^{\mathrm{op}}}$ factors through $\mathrm{Cat}^{\text{two-sided}}_{/\mathcal{C},\mathcal{D}}$, and induces an equivalence*

$$\mathrm{Funct}(\mathcal{C} \times \mathcal{D}^{\mathrm{op}}, \mathrm{Cat}) = \mathrm{Cat}^{\text{two-sided}}_{/\mathcal{C},\mathcal{D}}.$$

Proof. Let $H : \mathcal{C} \times \mathcal{D}^{\mathrm{op}} \to \mathrm{Cat}$ be a functor. We have

$$\begin{aligned}\int_{\mathcal{C}\times\mathcal{D}^{\mathrm{op}}} H &= \mathrm{colim}_{\mathrm{Tw}(\mathcal{C}\times\mathcal{D}^{\mathrm{op}})} \mathcal{C}_{-/} \times \mathcal{D}_{/-} \times H \\ &= \mathrm{colim}_{\mathrm{Tw}(\mathcal{C})}(\mathrm{colim}_{\mathrm{Tw}(\mathcal{D})} \mathcal{D}_{/-} \times H) \times \mathcal{C}_{-/} \\ &= \int_{\mathcal{C}\times[0]^{\mathrm{op}}} \left(\int_{[0]\times\mathcal{D}^{\mathrm{op}}} H\right).\end{aligned}$$

It follows that the functor $\int_{\mathcal{C}\times\mathcal{D}^{\mathrm{op}}} : \mathrm{Funct}(\mathcal{C} \times \mathcal{D}^{\mathrm{op}}, \mathrm{Cat}) \to \mathrm{Cat}_{/\mathcal{C}\times\mathcal{D}^{\mathrm{op}}}$ is equivalent to the composite functor

$$\mathrm{Funct}(\mathcal{C} \times \mathcal{D}^{\mathrm{op}}, \mathrm{Cat}) = \mathrm{Funct}(\mathcal{C}, \mathrm{Funct}(\mathcal{D}^{\mathrm{op}}, \mathrm{Cat})) \to \mathrm{Funct}(\mathcal{C}, \mathrm{Cat}_{/\mathcal{D}}) \to \mathrm{Cat}_{/\mathcal{C}\times\mathcal{D}}$$

where the middle arrow is the cartesian Grothendieck construction, and the last arrow is the cocartesian Grothendieck construction. This is an embedding, and it follows from proposition 9.1.9 that its image coincides with $\mathrm{Cat}^{\text{two-sided}}_{/\mathcal{C}\times\mathcal{D}}$. ◻

Remark 9.1.17. The proof of proposition 9.1.16 shows that the two-sided Grothendieck construction can be computed in two steps, by first doing cartesian unstraightening along $\mathcal{D}$, and then cocartesian unstraightening along $\mathcal{C}$. Alternatively, we can also compute it by first doing cocartesian unstraightening along $\mathcal{C}$ and then cartesian straightening along $\mathcal{D}$.

9.2 The arrow bifibration

The notion of two-sided fibration specializes in the case when the fibers are groupoids to the notion of bifibration introduced in [Lur09a].

Definition 9.2.1. *Let $\mathcal{C}, \mathcal{D}$ be categories. A two-sided fibration $p = (p_1, p_2) : \mathcal{E} \to \mathcal{C} \times \mathcal{D}$ is said to be a bifibration if for every pair (c, d) in $\mathcal{C} \times \mathcal{D}$ the fiber $p^{-1}((c, d))$ is a groupoid.*

Remark 9.2.2. If $p : \mathcal{E} \to \mathcal{C} \times \mathcal{D}$ is a functor whose fibers are groupoids, then p is a two-sided fibration if and only if it is a lax two-sided fibration. Moreover, if p is a bifibration and $q : \mathcal{E}' \to \mathcal{C} \times \mathcal{D}$ is another bifibration, then any functor $F : \mathcal{E} \to \mathcal{E}'$ equipped with an identification $qF = p$ is automatically a morphism of two-sided fibrations.

Remark 9.2.3. Let $\mathcal{C}, \mathcal{D}$ be categories. Under the equivalence of proposition 9.1.16, the full subcategory of $\mathrm{Cat}^{\text{two-sided}}_{/\mathcal{C}\times\mathcal{D}}$ on the bifibrations gets identified with $\mathrm{Funct}(\mathcal{C} \times \mathcal{D}^{\mathrm{op}}, \mathrm{Spc})$.

For each category $\mathcal{C}$, the arrow category of $\mathcal{C}$ equipped with its target-source projection turns out to be a bifibration. As the following proposition shows, it in fact plays a similar role in the theory of bifibrations as the twisted arrow category does in the theory of pairings of categories (see [Lur17] section 5.2.1).

Proposition 9.2.4. *Let $\mathcal{C}$ be a category. Then the bifibration associated to the functor $\mathrm{Hom}_{\mathcal{C}} : \mathcal{C} \times \mathcal{C}^{\mathrm{op}} \to \mathrm{Spc}$ is equivalent to the projection $p = (\mathrm{ev}_1, \mathrm{ev}_0) : \mathrm{Funct}([1], \mathcal{C}) \to \mathcal{C}$.*

Proof. The fact that p is a bifibration follows directly from the criteria of remark 9.1.3 together with remark 9.2.2. Let $\mathrm{ev}_1^\vee : \mathrm{Funct}([1], \mathcal{C})^\vee \to \mathcal{C}^{\mathrm{op}}$ be the cartesian fibration classified by the same functor as the cocartesian fibration ev_1. This comes equipped with a projection $p^\vee = (\mathrm{ev}_1^\vee, \mathrm{ev}_0^\vee) : \mathrm{Funct}([1], \mathcal{C})^\vee \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$, which is a right fibration classified by the same functor that classifies the bifibration p. According to [Lur17] proposition 5.2.1.11, the right fibration associated to $\mathrm{Hom}_{\mathcal{C}} : \mathcal{C} \times \mathcal{C}^{\mathrm{op}} \to \mathrm{Spc}$ is equivalent to the canonical projection $\lambda = (t, s) : \mathrm{Tw}(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$. Hence it suffices to show that $p^\vee$ is equivalent to λ.

We will use the description of dual fibrations from [BGN18] (translated to the language of simplicial spaces rather than simplicial sets). The space of n-simplices of the category $\mathrm{Funct}([1], \mathcal{C})^\vee$ is the space of functors $\sigma : \mathrm{Tw}([n])^{\mathrm{op}} \to \mathrm{Funct}([1], \mathcal{C})$ such that for each $0 \leq i < j \leq n$ the cospan

$$
\begin{tikzcd}
 & \sigma(i \to j) & \\
\sigma(i \to j-1) \arrow[ur] & & \sigma(i+1 \to j) \arrow[ul]
\end{tikzcd}
$$

has a cocartesian right leg, and the image of its left leg under ev_1 is an isomorphism. Under the isomorphism $\mathrm{Hom}_{\mathrm{Cat}}(\mathrm{Tw}([n])^{\mathrm{op}}, \mathrm{Funct}([1], \mathcal{C})) = \mathrm{Hom}_{\mathrm{Cat}}(\mathrm{Tw}([n])^{\mathrm{op}} \times [1], \mathcal{C})$ we have that the space of n-simplices in $\mathrm{Funct}([1], \mathcal{C})^\vee$ is the space of maps $\tau : \mathrm{Tw}([n])^{\mathrm{op}} \times [1] \to \mathcal{C}$ such that for every $0 \leq i < j \leq n$ the morphisms $\tau(i \to j-1, 1) \to \tau(i \to j, 1)$ and $\tau(i+1 \to j, 0) \to \tau(i \to j, 0)$ are isomorphisms. In other words, this is the space of maps $\tau : \mathrm{Tw}([n])^{\mathrm{op}} \times [1] \to \mathcal{C}$ which factor through the localization of $\mathrm{Tw}([n])^{\mathrm{op}} \times [1]$ at the collection S_n of arrows of the form $(i \to j-1, 1) \to (i \to j, 1)$ and $(i+1 \to j, 0) \to (i \to j, 0)$.

Consider for each $n \geq 0$ the functor $\mathrm{Tw}([n])^{\mathrm{op}} \times [1] \to [2n+1]$ that maps $(i \to j, 0)$ to j and $(i \to j, 1)$ to $2n+1-i$. This induces an isomorphism $S_n^{-1}(\mathrm{Tw}([n])^{\mathrm{op}} \times [1]) = [2n+1]$. This isomorphism is natural in n - namely, we have an isomorphism of simplicial categories $[n] \mapsto S_n^{-1}(\mathrm{Tw}([n]^{\mathrm{op}} \times [1])$ and $n \mapsto [n] \star [n]^{\mathrm{op}} = [2n+1]$. The latter corepresents the twisted arrow category construction, hence we see that $\mathrm{Funct}([1], \mathcal{C})^\vee$ is equivalent to $\mathrm{Tw}(\mathcal{C})$.

The projection $\mathrm{ev}_1^\vee : \mathrm{Funct}([1], \mathcal{C})^\vee \to \mathcal{C}^{\mathrm{op}}$ sends an n-simplex $\sigma : \mathrm{Tw}([n])^{\mathrm{op}} \to \mathrm{Funct}([1], \mathcal{C})$ to the composition of $\mathrm{ev}_1 \sigma$ with the functor $[n]^{\mathrm{op}} \to \mathrm{Tw}([n])^{\mathrm{op}}$ given by the formula $i \mapsto (i \to n)$. The projection $\mathrm{ev}_0^\vee : \mathrm{Funct}([1], \mathcal{C})^\vee \to \mathcal{C}$ sends an n-simplex $\sigma : \mathrm{Tw}([n])^{\mathrm{op}} \to \mathrm{Funct}([1], \mathcal{C})$ to the composition of $\mathrm{ev}_0 \sigma$ with the functor $[n] \to \mathrm{Tw}([n])^{\mathrm{op}}$ given by the formula $j \mapsto (0 \to j)$. Under the isomorphism $\mathrm{Funct}([1], \mathcal{C})^\vee([n]) = \mathcal{C}([2n+1]) = \mathcal{C}([n] \star [n]^{\mathrm{op}})$ these assignments amount to precomposing with the natural inclusions $[n]^{\mathrm{op}} \to [n] \star [n]^{\mathrm{op}}$ and $[n] \to [n] \star [n]^{\mathrm{op}}$. We conclude that under the isomorphism $\mathrm{Funct}([1], \mathcal{C})^{\vee} = \mathrm{Tw}(\mathcal{C})$, the projection $p^{\vee} = (\mathrm{ev}_1^{\vee}, \mathrm{ev}_0^{\vee})$ agrees with the projection λ, as desired. □

The following proposition shows that the arrow category enjoys a universal property in the category of two-sided fibrations. Specializing it to the category of bifibrations, we are able to conclude that the arrow bifibration is the free bifibration on the diagonal functor $\Delta : \mathcal{C} \to \mathcal{C} \times \mathcal{C}$.

Proposition 9.2.5. *Let $\mathcal{C}$ be a category and let $p = (\mathrm{ev}_1, \mathrm{ev}_0) : \mathrm{Funct}([1], \mathcal{C}) \to \mathcal{C} \times \mathcal{C}$. Denote by $\psi : \mathcal{C} = \mathrm{Funct}([0], \mathcal{C}) \to \mathrm{Funct}([1], \mathcal{C})$ the functor given by precomposition with the projection $[1] \to [0]$ and by $\Delta = p\psi : \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ the diagonal map. Let $r = (r_1, r_2) : \mathcal{E} \to \mathcal{C} \times \mathcal{C}$ be a two-sided fibration. Then precomposition with ψ induces an embedding*

$$\mathrm{Hom}_{\mathrm{Cat}^{\text{two-sided}}_{/\mathcal{C}\times\mathcal{C}}}(p, r) \to \mathrm{Hom}_{\mathrm{Cat}_{/\mathcal{C}\times\mathcal{C}}}(\Delta, r)$$

which identifies $\mathrm{Hom}_{\mathrm{Cat}^{\text{two-sided}}_{/\mathcal{C}\times\mathcal{C}}}(p, r)$ with the space of maps $\Delta \to r$ which send arrows in $\mathcal{C}$ to bicartesian arrows in $\mathcal{E}$.

Proof. Recall from [GHN17] section 4 that the functor ψ presents ev_1 as the free cocartesian fibration on $\mathrm{id}_{\mathcal{C}}$. It follows that precomposition with ψ induces an equivalence

$$\mathrm{Hom}_{(\mathrm{Cat}^{\mathrm{cocart}}_{/\mathcal{C}})_{/\mathcal{C}\times\mathcal{C}}}(p, r) = \mathrm{Hom}_{\mathrm{Cat}_{/\mathcal{C}\times\mathcal{C}}}(\Delta, r).$$

The space $\mathrm{Hom}_{\mathrm{Cat}^{\text{two-sided}}_{/\mathcal{C}\times\mathcal{C}}}(p, r)$ is the subspace of $\mathrm{Hom}_{(\mathrm{Cat}^{\mathrm{cocart}}_{/\mathcal{C}})_{/\mathcal{C}\times\mathcal{C}}}(p, r)$ containing those maps $F : p \to r$ which map ev_0-cartesian arrows to r_2-cartesian arrows.

Let F be an object in $\mathrm{Hom}_{(\mathrm{Cat}^{\mathrm{cocart}}_{/\mathcal{C}})_{/\mathcal{C}\times\mathcal{C}}}(p, r)$. We have to show that F maps ev_0-cartesian arrows to r_2-cartesian arrows if and only if $F\psi$ maps arrows in $\mathcal{C}$ to bicartesian arrows in $\mathcal{E}$.

Assume first that $F\psi$ maps arrows in $\mathcal{C}$ to bicartesian arrows in r. Let $f : \sigma \to \sigma'$ be an ev_0-cartesian arrow in $\mathrm{Funct}([1], \mathcal{C})$. In other words, we have $\mathrm{ev}_1 f$ invertible. We have a commutative diagram

$$\begin{tikzcd} \psi\sigma(0) \arrow[r] \arrow[d] & \sigma \arrow[d] \\ \psi\sigma'(0) \arrow[r] & \sigma' \end{tikzcd}$$

where the horizontal arrows are ev_1-cocartesian. Applying the functor F we obtain a commutative diagram

$$\begin{tikzcd} F\psi\sigma(0) \arrow[r] \arrow[d] & F\sigma \arrow[d] \\ F\psi\sigma'(0) \arrow[r] & F\sigma'. \end{tikzcd}$$

The horizontal arrows are r_1-cocartesian, and the left vertical arrow is bicartesian. Since the image of the right vertical arrow under r_1 is an isomorphism and r is a two-sided fibration we conclude that the right vertical arrow is r_2-cartesian, as desired.

Assume now that F maps ev_0-cartesian arrows to r_2-cartesian arrows. Since p is a bifibration and F also maps ev_1-cocartesian arrows to r_1-cocartesian arrows, we conclude that F maps all arrows in $\mathrm{Funct}([1], \mathcal{C})$ to bicartesian arrows. It follows that the same is true for $F\psi$, as desired. □

Corollary 9.2.6. *Let $\mathcal{C}$ be a category, and let p, ψ be as in the statement of proposition 9.2.5. Let $r : \mathcal{E} \to \mathcal{C} \times \mathcal{C}$ be a bifibration. Then precomposition with ψ induces an equivalence*

$$\mathrm{Hom}_{\mathrm{Cat}^{\text{two-sided}}_{/\mathcal{C}\times\mathcal{C}}}(p, r) = \mathrm{Hom}_{\mathrm{Cat}_{/\mathcal{C}\times\mathcal{C}}}(\Delta, r).$$

We now study the notion of representable bifibration, which is a two-sided analog of the notion of representable pairing of categories from [Lur17].

Definition 9.2.7. *Let $\mathcal{C}, \mathcal{D}$ be categories. We say that a bifibration $p = (p_1, p_2) : \mathcal{E} \to \mathcal{C} \times \mathcal{D}$ is representable if for every object c in $\mathcal{C}$ the fiber $p_1^{-1}(c)$ has a final object.*

Remark 9.2.8. Let $\mathcal{C}, \mathcal{D}$ be categories. Recall that the full subcategory of $\mathrm{Cat}^{\text{two-sided}}_{/\mathcal{C}\times\mathcal{D}}$ on the bifibrations corresponds under the equivalence of proposition 9.1.16 to the category $\mathrm{Funct}(\mathcal{C}, \mathcal{P}(\mathcal{D})) = \mathrm{Funct}(\mathcal{C} \times \mathcal{D}^{\mathrm{op}}, \mathrm{Spc}) \subseteq \mathrm{Funct}(\mathcal{C} \times \mathcal{D}^{\mathrm{op}}, \mathrm{Cat})$. The full subcategory of $\mathrm{Cat}^{\text{two-sided}}_{/\mathcal{C}\times\mathcal{D}}$ on the representable bifibrations agrees under this correspondence with the subcategory $\mathrm{Funct}(\mathcal{C}, \mathcal{D}) \subseteq \mathrm{Funct}(\mathcal{C}, \mathcal{P}(\mathcal{D}))$.

Notation 9.2.9. Let $\mathcal{C}, \mathcal{D}$ be categories and $F : \mathcal{C} \to \mathcal{D}$ be a functor. We define the mapping cocylinder of F as the following pullback:

$$\begin{tikzcd}
\mathrm{Cocyl}(F) \arrow[r, "\overline{F}"] \arrow[d, "\overline{\mathrm{ev}_1}"] & \mathrm{Funct}([1], \mathcal{D}) \arrow[d, "\mathrm{ev}_1"] \\
\mathcal{C} \arrow[r, "F"] & \mathcal{D}.
\end{tikzcd}$$

Note that $\mathrm{Cocyl}(F)$ comes equipped with a projection $p^F = (p_1^F, p_2^F) : \mathrm{Cocyl}(F) \to \mathcal{C} \times \mathcal{D}$ given by $p_1^F = \overline{\mathrm{ev}_1}$ and $p_2^F = \mathrm{ev}_0 \overline{F}$.

Proposition 9.2.10. *Let $\mathcal{C}, \mathcal{D}$ be categories and $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then the projection $p^F : \mathrm{Cocyl}(F) \to \mathcal{C} \times \mathcal{D}$ is a representable bifibration, and the induced functor $\mathcal{C} \to \mathcal{D}$ is equivalent to F.*

Proof. The fact that p^F is a bifibration follows directly from the fact that it is the base change of the bifibration $p : \mathrm{Funct}([1], \mathcal{D}) \to \mathcal{D} \times \mathcal{D}$ along the functor $(F, \mathrm{id}_{\mathcal{D}}) : \mathcal{C} \times \mathcal{D} \to \mathcal{D} \times \mathcal{D}$. The functor $\mathcal{C} \to \mathrm{Cat}^{\mathrm{cart}}_{/\mathcal{D}}$ classifying p_1^F is the composition

$$\mathcal{C} \xrightarrow{F} \mathcal{D} \to \mathrm{Cat}^{\mathrm{cart}}_{/\mathcal{D}}$$

where the second arrow is the functor classifying the projection $\mathrm{ev}_1 : \mathrm{Funct}([1], \mathcal{D}) \to \mathcal{D}$. By proposition 9.2.4, this corresponds under the equivalence $\mathrm{Cat}^{\mathrm{cart}}_{\mathcal{D}} = \mathrm{Funct}(\mathcal{D}^{\mathrm{op}}, \mathrm{Cat})$ to the composition of the Yoneda embedding $\mathcal{D} \to \mathcal{P}(\mathcal{D})$ and the inclusion $\mathcal{P}(\mathcal{D}) \to \mathrm{Funct}(\mathcal{D}, \mathrm{Cat})$. Hence we see that the functor $\mathcal{C} \to \mathcal{P}(\mathcal{D})$ associated to p^F is the composition of F with the Yoneda embedding of $\mathcal{D}$. This means that p^F is representable and the associated functor $\mathcal{C} \to \mathcal{D}$ is equivalent to F, as desired. □

9.3 Bivariant fibrations

We now introduce the class of bivariant fibrations.

Definition 9.3.1. *Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be categories, and $p : \mathcal{E} \to \mathcal{C} \times \mathcal{D}$ be a functor. We say that p is a lax bivariant fibration if it is both a cocartesian and a cartesian fibration.*

Remark 9.3.2. Let $\mathcal{C}, \mathcal{D}$ be categories, and $p = (p_1, p_2) : \mathcal{E} \to \mathcal{C} \times \mathcal{D}$ be a lax bivariant fibration. Then p is a lax two-sided fibration from $\mathcal{C}$ to $\mathcal{D}$, and a lax two-sided fibration from $\mathcal{D}$ to $\mathcal{C}$.

Definition 9.3.3. *Let $\mathcal{C}, \mathcal{D}$ be categories, and $p = (p_1, p_2) : \mathcal{E} \to \mathcal{C} \times \mathcal{D}$ be a lax bivariant fibration. We say that p is a bivariant fibration if p is a two-sided fibration from $\mathcal{C}$ to $\mathcal{D}$ and also from $\mathcal{D}$ to $\mathcal{C}$.*

Definition 9.3.4. *Let $\mathcal{C}, \mathcal{D}$ be categories and $p : \mathcal{E} \to \mathcal{C} \times \mathcal{D}$ and $p' : \mathcal{E}' \to \mathcal{C} \times \mathcal{D}$ be bivariant fibrations. A functor $F : \mathcal{E} \to \mathcal{E}'$ equipped with an identification $p'F = p$ is said to be a morphism of bivariant fibrations if it is a morphism of cartesian and cocartesian fibrations over $\mathcal{C} \times \mathcal{D}$. We denote by $\mathrm{Cat}^{\mathrm{bivar}}_{/\mathcal{C},\mathcal{D}}$ the subcategory of $\mathrm{Cat}_{/\mathcal{C}\times\mathcal{D}}$ on the bivariant fibrations and morphisms of bivariant fibrations.*

Our next goal is to identify the image of $\mathrm{Cat}^{\mathrm{bivar}}_{/\mathcal{C},\mathcal{D}}$ across the different versions of the straightening equivalence. We first review the notion of adjointable diagram of categories.

Definition 9.3.5. *We say that a commutative diagram of categories*

$$\begin{array}{ccc} \mathcal{E}_{00} & \xrightarrow{\alpha'} & \mathcal{E}_{10} \\ \Big\downarrow{\scriptstyle \beta'} & & \Big\downarrow{\scriptstyle \beta} \\ \mathcal{E}_{01} & \xrightarrow{\alpha} & \mathcal{E}_{11} \end{array}$$

is vertically right adjointable if the following conditions hold:

- *The functors β and β' have right adjoints β^R and β'^R.*
- *The natural transformation*

$$\alpha'(\beta')^R \to \beta^R\beta\alpha'(\beta')^R = \beta^R\alpha\beta'(\beta')^R \to \beta^R\alpha$$

built from the unit $\mathrm{id}_{\mathcal{E}_{10}} \to \beta^R\beta$ and the counit $\beta'(\beta')^R \to \mathrm{id}_{\mathcal{E}_{01}}$, is a natural isomorphism.

We say that the above diagram is horizontally right adjointable if its transpose is vertically right adjointable. We say that it is right adjointable if it is both horizontally and vertically right adjointable. We say that it is (vertically / horizontally) left adjointable if the diagram obtained by taking opposites of all the categories involved is (vertically / horizontally) right adjointable.

Remark 9.3.6. Let

$$\begin{tikzcd} \mathcal{E}_{00} \arrow[r, "\alpha'"] \arrow[d, "\beta'"] & \mathcal{E}_{10} \arrow[d, "\beta"] \\ \mathcal{E}_{01} \arrow[r, "\alpha"] & \mathcal{E}_{11} \end{tikzcd}$$

be a vertically right adjointable commutative diagram of categories. Then the natural isomorphism $\alpha'(\beta')^R = \beta^R\alpha$ exhibits the following diagram as commutative:

$$\begin{tikzcd} \mathcal{E}_{00} \arrow[r, "\alpha'"] & \mathcal{E}_{10} \\ \mathcal{E}_{01} \arrow[r, "\alpha"] \arrow[u, "(\beta')^R"] & \mathcal{E}_{11}. \arrow[u, "\beta^R"'] \end{tikzcd}$$

We say that this diagram arises from the first one by passage to right adjoints of the vertical arrows. We can similarly talk about diagrams obtained by passage to right adjoints of horizontal arrows, or left adjoints of vertical / horizontal arrows, by requiring that the original diagram satisfy the appropriate adjointability condition.

The following proposition provides a link between the notion of adjointability of commutative squares of categories and the theory of two-sided fibrations.

Proposition 9.3.7. *Let $\mathcal{C}, \mathcal{D}$ be categories and $H : \mathcal{C} \times \mathcal{D} \to \mathrm{Cat}$ be a functor. The following conditions are equivalent:*

(i) *For every pair of arrows $\alpha : c \to c'$ in $\mathcal{C}$ and $\beta : d \to d'$ in $\mathcal{D}$ the commutative diagram of categories*

$$\begin{tikzcd} H(c,d) \arrow[r, "H(\alpha,\mathrm{id}_d)"] \arrow[d, "H(\mathrm{id}_c,\beta)"] & H(c',d) \arrow[d, "H(\mathrm{id}_{c'},\beta)"] \\ H(c,d') \arrow[r, "H(\alpha,\mathrm{id}_{d'})"] & H(c',d') \end{tikzcd}$$

is vertically right adjointable.

(ii) *The cocartesian fibration $p^{\mathrm{cocart}} : \mathcal{E}^{\mathrm{cocart}} \to \mathcal{C} \times \mathcal{D}$ classified by H is a two-sided fibration.*

(iii) *The two-sided fibration $p^{\text{two-sided}} : \mathcal{E}^{\text{two-sided}} \to \mathcal{D} \times \mathcal{C}^{\mathrm{op}}$ classified by H is a cartesian fibration.*

(iv) *Let* $G : \mathcal{C} \to \mathrm{Cat}_{/\mathcal{D}}$ *be the image of* H *under the composite map*

$$\mathrm{Funct}(\mathcal{C} \times \mathcal{D}, \mathrm{Cat}) = \mathrm{Funct}(\mathcal{C}, \mathrm{Cat}_{/\mathcal{D}}^{\mathrm{cocart}}) \to \mathrm{Funct}(\mathcal{C}, \mathrm{Cat}_{/\mathcal{D}}).$$

Then G *factors through the subcategory* $\mathrm{Cat}_{/\mathcal{D}}^{\mathrm{cart}}$.

Proof. The equivalence between conditions (ii) and (iv) is a direct consequence of proposition 9.1.9. To show that conditions (iii) and (iv) are equivalent, note that the two-sided fibration $p^{\text{two-sided}}$ is obtained by applying the cartesian unstraightening construction to the functor G. The equivalence then follows from a combination of [Lur09a] propositions 2.4.2.8 and 2.4.2.11.

We now establish the equivalence between conditions (i) and (ii). Let c be an object in $\mathcal{C}$, and $\beta : d \to d'$ be an arrow in $\mathcal{D}$. The functor $H(\mathrm{id}_c, \beta)$ admits a right adjoint if and only if the arrow (id_c, β) in $\mathcal{C} \times \mathcal{D}$ admits a locally p^{cocart}-cartesian lift. By (the dual of) [Lur09a] corollary 4.3.1.15, this happens if and only if (id_c, β) admits a p^{cocart}-cartesian lift. This holds for all pairs c, β if and only if p^{cocart} is a lax two-sided fibration.

Assume now that p^{cocart} is indeed a lax two-sided fibration. Let $\alpha : c \to c'$ and $\beta : d \to d'$ be a pair of arrows in $\mathcal{C}$ and $\mathcal{D}$ and let e be an object in $(p^{\mathrm{cocart}})^{-1}(c, d') = H(c, d')$. Consider the commutative diagram

$$\begin{tikzcd}
e' \arrow[r, "\overline{(\alpha,\mathrm{id}_d)}"] \arrow[d, "\widehat{(\mathrm{id}_c,\beta)}"] & f \arrow[r, "\zeta"] & f' \arrow[d, "\widehat{(\mathrm{id}_{c'},\beta)}"] \\
e \arrow[rr, "\overline{(\alpha,\mathrm{id}_{d'})}"] & & e''
\end{tikzcd}$$

where $\widehat{A}$ denotes a p^{cocart}-cocartesian lift of the arrow A, and $\overline{A}$ denotes a p^{cocart}-cartesian lift of A . We have $f = H(\alpha, \mathrm{id}_d)H(\mathrm{id}_c, \beta)^R e$ and $f' = H(\mathrm{id}_{c'}, \beta)^R H(\alpha, \mathrm{id}_{d'})e$. The projection p^{cocart} is a two-sided fibration if and only if for every choice of α, β and e, the resulting map ζ is an isomorphism.

We now enlarge the above diagram as follows:

$$\begin{tikzcd}
e' \arrow[rr, "\overline{(\alpha,\mathrm{id}_d)}"] \arrow[dr, "\overline{(\mathrm{id}_c,\beta)}"] \arrow[dd, "\widehat{(\mathrm{id}_c,\beta)}"'] & & f \arrow[r, "\zeta_1"] \arrow[dr, "\overline{(\mathrm{id}_{c'},\beta)}"'] & g \arrow[r, "\zeta_2"] \arrow[d, "\widehat{(\mathrm{id}_{c'},\beta)}"] & f' \arrow[dd, "\widehat{(\mathrm{id}_{c'},\beta)}"] \\
& x \arrow[rr, "\overline{(\alpha,\mathrm{id}_{d'})}"'] \arrow[dl, "\eta"] & & y \arrow[dr, "\eta'"] & \\
e \arrow[rrrr, "\overline{(\alpha,\mathrm{id}_{d'})}"] & & & & e''.
\end{tikzcd}$$

The map ζ from the previous diagram is now decomposed as the composite map $f \xrightarrow{\zeta_1} g \xrightarrow{\zeta_2} f'$. We have equivalences

$$H(\mathrm{id}_{c'}, \beta)^R H(\mathrm{id}_{c'}, \beta) H(\alpha, \mathrm{id}_d) H^R(\mathrm{id}_c, \beta)e = g = H(\mathrm{id}_{c'}, \beta)^R H(\alpha, \mathrm{id}_{d'}) H(\mathrm{id}_c, \beta) H^R(\mathrm{id}_c, \beta)e.$$

Under these equivalences, the map ζ_1 is the unit of the adjunction $H(\mathrm{id}_{c'}, \beta) \dashv H(\mathrm{id}_{c'}, \beta)^R$ evaluated at f, and the map ζ_2 is the image under $H(\mathrm{id}_{c'}, \beta)^R H(\alpha, \mathrm{id}_{d'})$ of the counit of the adjunction $H(\mathrm{id}_c, \beta) \dashv H(\mathrm{id}_c, \beta)^R$ applied to e. We conclude that the map ζ is a component of the natural transformation witnessing the lax commutativity of the diagram obtained by passing to right adjoints of the vertical arrows of the square in the statement. It follows that the square in the statement is vertically right adjointable if and only if for every choice of α, β and e the resulting map ζ is an isomorphism, which we already observed is equivalent to p^{cocart} being a two-sided fibration. □

Corollary 9.3.8. *Let $\mathcal{C}, \mathcal{D}$ be categories and $H : \mathcal{C} \times \mathcal{D} \to \mathrm{Cat}$ be a functor. The following conditions are equivalent:*

(i) *For every pair of arrows $\alpha : c \to c'$ in $\mathcal{C}$ and $\beta : d \to d'$ in $\mathcal{D}$ the commutative diagram of categories*

$$\begin{tikzcd} H(c,d) \arrow[r, "H(\alpha,\mathrm{id}_d)"] \arrow[d, "H(\mathrm{id}_c,\beta)"] & H(c',d) \arrow[d, "H(\mathrm{id}_{c'},\beta)"] \\ H(c,d') \arrow[r, "H(\alpha,\mathrm{id}_{d'})"] & H(c',d') \end{tikzcd}$$

is vertically left adjointable.

(ii) *The cartesian fibration $p^{\mathrm{cart}} : \mathcal{E}^{\mathrm{cart}} \to \mathcal{D}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}}$ classified by H is a two-sided fibration.*

(iii) *The two-sided fibration $p^{\text{two-sided}} : \mathcal{E}^{\text{two-sided}} \to \mathcal{C} \times \mathcal{D}^{\mathrm{op}}$ classified by H is a cocartesian fibration.*

(iv) *Let $G : \mathcal{C} \to \mathrm{Cat}_{/\mathcal{D}^{\mathrm{op}}}$ be the image of H under the composite map*

$$\mathrm{Funct}(\mathcal{C} \times \mathcal{D}, \mathrm{Cat}) = \mathrm{Funct}(\mathcal{C}, \mathrm{Cat}^{\mathrm{cart}}_{/\mathcal{D}^{\mathrm{op}}}) \to \mathrm{Funct}(\mathcal{C}, \mathrm{Cat}_{/\mathcal{D}^{\mathrm{op}}}).$$

Then G factors through the subcategory $\mathrm{Cat}^{\mathrm{cocart}}_{/\mathcal{D}^{\mathrm{op}}}$.

Proof. This follows from proposition 9.3.7 applied to the functor $H^{\mathrm{op}} : \mathcal{C} \times \mathcal{D} \to \mathrm{Cat}$. □

Definition 9.3.9. *Let $\mathcal{C}, \mathcal{D}$ be categories and $H : \mathcal{C} \times \mathcal{D} \to \mathrm{Cat}$ be a functor. We say that H is right adjointable in the $\mathcal{D}$ coordinate if the equivalent conditions of proposition 9.3.7 are satisfied. We say that H is left adjointable in the $\mathcal{D}$ coordinate if the equivalent conditions of corollary 9.3.8 are satisfied. By switching the role of $\mathcal{C}$ and $\mathcal{D}$ we can similarly talk about right/left adjointability in the $\mathcal{C}$ coordinate.*

Notation 9.3.10. Let $\mathcal{C}, \mathcal{D}$ be categories and $H : \mathcal{C} \times \mathcal{D} \to \mathrm{Cat}$ be a functor. If H is right adjointable in the $\mathcal{D}$ coordinate then condition (iv) in proposition 9.3.7 provides a functor $\mathcal{C} \to \mathrm{Cat}^{\mathrm{cart}}_{/\mathcal{D}} = \mathrm{Funct}(\mathcal{D}^{\mathrm{op}}, \mathrm{Cat})$. We denote by $H^{R_{\mathcal{D}}} : \mathcal{C} \times \mathcal{D}^{\mathrm{op}} \to \mathrm{Cat}$ the induced functor. We say that this is obtained from H by passage to right adjoints in the $\mathcal{D}$ coordinate. Similarly, if H is left adjointable then out of equivalence (iv) in corollary 9.3.8 we obtain a functor $H^{L_{\mathcal{D}}} : \mathcal{C} \times \mathcal{D}^{\mathrm{op}} \to \mathrm{Cat}$ which is said to arise from H by passage to left adjoints in the $\mathcal{D}$ coordinate.

Notation 9.3.11. Let $\mathcal{C}$ be a category and $H : \mathcal{C} \to \mathrm{Cat}$ be a functor. Identifying $\mathcal{C}$ with $[0] \times \mathcal{C}$, it makes sense to talk about right and left adjointability of H in the coordinate $\mathcal{C}$. In this situation, we simply say that H is right (resp. left) adjointable if it is right (resp. left) adjointable in the coordinate $\mathcal{C}$. We will use the notation H^R (resp. H^L) instead of $H^{R_\mathcal{C}}$ (resp. $H^{L_\mathcal{C}}$).

Remark 9.3.12. Let $\mathcal{C}, \mathcal{D}$ be categories. Then passage to right and left adjoints in the $\mathcal{D}$ coordinate define inverse equivalences between the space of functors $\mathcal{C} \times \mathcal{D} \to \mathrm{Cat}$ which are right adjointable in the $\mathcal{D}$ coordinate, and the space of functors $\mathcal{C} \times \mathcal{D}^{\mathrm{op}} \to \mathrm{Cat}$ which are left adjointable in the $\mathcal{D}$ coordinate.

Remark 9.3.13. Let $\mathcal{C}, \mathcal{D}$ be categories and $H : \mathcal{C} \times \mathcal{D} \to \mathrm{Cat}$ be a functor. Let $p^{\mathrm{cocart}} : \mathcal{E}^{\mathrm{cocart}} \to \mathcal{C} \times \mathcal{D}$ be the cocartesian fibration classified by H and $p^{\text{two-sided}} : \mathcal{E}^{\text{two-sided}} \to \mathcal{D} \times \mathcal{C}^{\mathrm{op}}$ be the two-sided fibration classified by H. Assume that H is right adjointable in the $\mathcal{D}$ coordinate. Then the two-sided fibration p^{cocart} and the cartesian fibration $p^{\text{two-sided}}$ are both classified by the functor $H^{R_\mathcal{D}}$. Moreover it follows from the proof of proposition 9.3.7 that for every pair of arrows $\alpha : c \to c'$ and $\beta : d \to d'$ in $\mathcal{C}$ and $\mathcal{D}$ respectively, the commutative diagram

$$\begin{tikzcd} H^{R_\mathcal{D}}(c,d) \arrow[r, "H^{R_\mathcal{D}}(\alpha,\mathrm{id}_d)"] & H^{R_\mathcal{D}}(c',d) \\ H^{R_\mathcal{D}}(c,d') \arrow[u, "H^{R_\mathcal{D}}(\mathrm{id}_c,\beta)"] \arrow[r, "H^{R_\mathcal{D}}(\alpha,\mathrm{id}_{d'})"] & H^{R_\mathcal{D}}(c',d') \arrow[u, "H^{R_\mathcal{D}}(\mathrm{id}_{c'},\beta)"'] \end{tikzcd}$$

is equivalent to the diagram obtained from

$$\begin{tikzcd} H(c,d) \arrow[r, "H(\alpha,\mathrm{id}_d)"] \arrow[d, "H(\mathrm{id}_c,\beta)"] & H(c',d) \arrow[d, "H(\mathrm{id}_{c'},\beta)"] \\ H(c,d') \arrow[r, "H(\alpha,\mathrm{id}_{d'})"] & H(c',d') \end{tikzcd}$$

by passage to right adjoints of vertical arrows (see remark 9.3.6).

Remark 9.3.14. Let $\mathcal{C}, \mathcal{D}$ be categories and $H : \mathcal{C} \times \mathcal{D} \to \mathrm{Cat}$ be a functor. Let $p^{\mathrm{cart}} : \mathcal{E}^{\mathrm{cart}} \to \mathcal{D}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}}$ be the cartesian fibration classified by H and $p^{\text{two-sided}} : \mathcal{E}^{\text{two-sided}} \to \mathcal{C} \times \mathcal{D}^{\mathrm{op}}$ be the two-sided fibration classified by H. Assume that H is left adjointable in the $\mathcal{D}$ coordinate. Then the two-sided fibration p^{cart} and the cocartesian fibration $p^{\text{two-sided}}$ are both classified by the functor $H^{L_\mathcal{D}}$. Moreover, for every pair of arrows $\alpha : c \to c'$ and $\beta : d \to d'$ in $\mathcal{C}$ and $\mathcal{D}$ respectively, the commutative diagram

$$\begin{tikzcd} H^{L_\mathcal{D}}(c,d) \arrow[r, "H^{L_\mathcal{D}}(\alpha,\mathrm{id}_d)"] & H^{L_\mathcal{D}}(c',d) \\ H^{L_\mathcal{D}}(c,d') \arrow[u, "H^{L_\mathcal{D}}(\mathrm{id}_c,\beta)"] \arrow[r, "H^{L_\mathcal{D}}(\alpha,\mathrm{id}_{d'})"] & H^{L_\mathcal{D}}(c',d') \arrow[u, "H^{L_\mathcal{D}}(\mathrm{id}_{c'},\beta)"'] \end{tikzcd}$$

is equivalent to the diagram obtained from

$$\begin{tikzcd} H(c,d) \arrow[r, "H(\alpha,\mathrm{id}_d)"] \arrow[d, "H(\mathrm{id}_c,\beta)"] & H(c',d) \arrow[d, "H(\mathrm{id}_{c'},\beta)"] \\ H(c,d') \arrow[r, "H(\alpha,\mathrm{id}_{d'})"] & H(c',d') \end{tikzcd}$$

by passage to left adjoints of vertical arrows.

The notion of bivariant fibration arises naturally when studying functors of two variables which have adjointability properties with respect to both of them. We consider first the case of functors which are either right or left adjointable with respect to both variables.

Proposition 9.3.15. *Let $\mathcal{C}, \mathcal{D}$ be categories and $p : \mathcal{E} \to \mathcal{C} \times \mathcal{D}$ be a cocartesian fibration classified by a functor $H : \mathcal{C} \times \mathcal{D} \to \mathrm{Cat}$. Then p is a bivariant fibration if and only if H is right adjointable in both the $\mathcal{C}$ coordinate and the $\mathcal{D}$ coordinate.*

Proof. According to proposition 9.3.7, the functor H is right adjointable in both the $\mathcal{C}$ and $\mathcal{D}$ coordinates if and only if p is a two-sided fibration from $\mathcal{C}$ to $\mathcal{D}$ and from $\mathcal{D}$ to $\mathcal{C}$. In this case, for every arrow $(\alpha, \beta) : (c, d) \to (c', d')$ in $\mathcal{C} \times \mathcal{D}$, the functor $H(\alpha, \beta)$ has a right adjoint, since it can be written as the composition of the right adjointable functors $H(\alpha, \mathrm{id}_d)$ and $H(\mathrm{id}_{c'}, \beta)$. We conclude that in this case p is also a cartesian fibration, and therefore a bivariant fibration, as desired. □

Corollary 9.3.16. *Let $\mathcal{C}, \mathcal{D}$ be categories and $p : \mathcal{E} \to \mathcal{C} \times \mathcal{D}$ be a cartesian fibration classified by a functor $H : \mathcal{C}^{\mathrm{op}} \times \mathcal{D}^{\mathrm{op}} \to \mathrm{Cat}$. Then p is a bivariant fibration if and only if H is left adjointable in both the $\mathcal{C}$ coordinate and the $\mathcal{D}$ coordinate.*

Proof. Apply proposition 9.3.15 to the functor $H^{\mathrm{op}} : \mathcal{C} \times \mathcal{D} \to \mathrm{Cat}$. □

We now deal with the case of functors which enjoy mixed adjointability properties.

Proposition 9.3.17. *Let $\mathcal{C}, \mathcal{D}$ be categories and $p : \mathcal{E} \to \mathcal{C} \times \mathcal{D}$ be a two-sided fibration classified by a functor $H : \mathcal{C} \times \mathcal{D}^{\mathrm{op}} \to \mathrm{Cat}$. Then the following are equivalent*

(i) *The map p is a bivariant fibration.*

(ii) *The functor H is left adjointable in the $\mathcal{D}^{\mathrm{op}}$ coordinate and the functor $H^{L_{\mathcal{D}}}$ is right adjointable in the $\mathcal{C}$ coordinate.*

(iii) *The functor H is right adjointable in the $\mathcal{C}$ coordinate and the functor $H^{R_{\mathcal{C}}}$ is left adjointable in the $\mathcal{D}^{\mathrm{op}}$ coordinate.*

Proof. By corollary 9.3.8, H is left adjointable in the $\mathcal{D}^{\mathrm{op}}$ coordinate if and only if p is a cocartesian fibration. In this case, p is the cocartesian fibration classified by $H^{L_{\mathcal{D}}}$. The equivalence between conditions (i) and (ii) is now a direct consequence of proposition 9.3.15.

Similarly, proposition 9.3.7 shows that H is right adjointable in the $\mathcal{C}$ coordinate if and only if p is a cartesian fibration, and moreover in this case p is the cartesian fibration classified by $H^{R_{\mathcal{C}}}$. The equivalence between conditions (i) and (iii) now follows from corollary 9.3.16. $\square$

Remark 9.3.18. Let $\mathcal{C}, \mathcal{D}$ be categories and $p : \mathcal{E} \to \mathcal{C} \times \mathcal{D}$ be a bivariant fibration. Then the different versions of the straightening equivalence yield four functors

$$\begin{aligned} H_{\mathrm{cocart}} &: \mathcal{C} \times \mathcal{D} \to \mathrm{Cat} & H_{\mathrm{cocart,cart}} &: \mathcal{C} \times \mathcal{D}^{\mathrm{op}} \to \mathrm{Cat} \\ H_{\mathrm{cart,cocart}} &: \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathrm{Cat} & H_{\mathrm{cart}} &: \mathcal{C}^{\mathrm{op}} \times \mathcal{D}^{\mathrm{op}} \to \mathrm{Cat}\,. \end{aligned}$$

It follows from remarks 9.3.13 and 9.3.14 that each of these four functors determines the rest, by passage to right or left adjoints in the appropriate coordinates. In particular, the operations of passing to adjoints in different coordinates commute.

We now examine the image of the class of morphisms of bivariant fibrations under the different versions of the straightening equivalence.

Remark 9.3.19. Let $\mathcal{I}, \mathcal{C}, \mathcal{D}$ be categories, and $H : \mathcal{I} \times \mathcal{C} \times \mathcal{D} \to \mathrm{Cat}$ be a functor. Identifying $\mathcal{I} \times \mathcal{C} \times \mathcal{D}$ with $(\mathcal{I} \times \mathcal{C}) \times \mathcal{D}$, we may talk about right adjointability of H with respect to the $\mathcal{D}$ coordinate. Since any map in $\mathcal{I} \times \mathcal{C}$ is a composition of maps which are constant in one coordinate, the functor H is right adjointable in the $\mathcal{D}$ coordinate if and only if the following two conditions hold:

- For every object i in $\mathcal{I}$ the induced functor $H(i, -, -) : \mathcal{C} \times \mathcal{D} \to \mathrm{Cat}$ is right adjointable in the $\mathcal{D}$ coordinate.
- For every object c in $\mathcal{C}$ the induced functor $H(-, c, -) : \mathcal{I} \times \mathcal{D} \to \mathrm{Cat}$ is right adjointable in the $\mathcal{D}$ coordinate.

The above can be adapted in a straightforward way to yield characterizations for left adjointability in the $\mathcal{D}$ coordinate, or right/left adjointability in the $\mathcal{C}$ coordinate.

Proposition 9.3.20. *Let $\mathcal{C}, \mathcal{D}$ be categories. Let $p : \mathcal{E} \to \mathcal{C} \times \mathcal{D}$ and $p' : \mathcal{E} \to \mathcal{C} \times \mathcal{D}$ be bivariant fibrations, and let $F : p \to p'$ be a morphism in $\mathrm{Cat}_{/\mathcal{C}\times\mathcal{D}}$. The following are equivalent:*

(i) *The map F is a morphism of bivariant fibrations.*

(ii) *The map F is a morphism of cocartesian fibrations, and the induced functor $[1] \times \mathcal{C} \times \mathcal{D} \to \mathrm{Cat}$ is right adjointable in the $\mathcal{C}$ and $\mathcal{D}$ coordinates.*

(iii) *The map F is a morphism of two-sided fibrations from $\mathcal{C}$ to $\mathcal{D}$, and the induced functor $[1] \times \mathcal{C} \times \mathcal{D}^{\mathrm{op}} \to \mathrm{Cat}$ is right adjointable in the $\mathcal{C}$ coordinate, and left adjointable in the $\mathcal{D}^{\mathrm{op}}$ coordinate.*

(iv) *The map F is a morphism of cartesian fibrations, and the induced functor $[1] \times \mathcal{C}^{\mathrm{op}} \times \mathcal{D}^{\mathrm{op}} \to \mathrm{Cat}$ is left adjointable in the $\mathcal{C}$ and $\mathcal{D}$ coordinates.*

(v) *The map F is a morphism of two-sided fibrations from $\mathcal{D}$ to $\mathcal{C}$, and the induced functor* $[1] \times \mathcal{D} \times \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}$ *is right adjointable in the $\mathcal{D}$ coordinate, and left adjointable in the $\mathcal{C}^{\mathrm{op}}$ coordinate.*

Proof. We show that conditions (i) and (ii) are equivalent - the equivalence between (i) and each of the items (iii)-(v) follows along similar lines. Assume that F is a morphism of cocartesian fibrations and denote by $G : [1] \times \mathcal{C} \times \mathcal{D} \to \mathrm{Cat}$ the induced functor. Since p is a bivariant fibration, the functors $\mathcal{C} \times \mathcal{D} \to \mathrm{Cat}$ classifying p and p' are right adjointable in the $\mathcal{C}$ and $\mathcal{D}$ coordinates. By virtue of remark 9.3.19, the functor G is right adjointable in the $\mathcal{C}$ and $\mathcal{D}$ coordinates if and only if the following conditions are satisfied:

- For every object c in $\mathcal{C}$ the induced functor $G(-, c, -) : [1] \times \mathcal{D} \to \mathrm{Cat}$ is right adjointable in the $\mathcal{D}$ coordinate.
- For every object d in $\mathcal{D}$ the induced functor $G(-, -, d) : [1] \times \mathcal{C} \to \mathrm{Cat}$ is right adjointable in the $\mathcal{C}$ coordinate.

The result now follows from the characterization of adjointability given by item (iv) in proposition 9.3.7. □

9.4 The Beck-Chevalley condition

We now specialize to the class of bivariant fibrations satisfying an extra base change property.

Definition 9.4.1. *Let $\mathcal{B}$ be a category admitting pullbacks, and $p : \mathcal{E} \to \mathcal{B}$ be a functor which is both a cocartesian and a cartesian fibration. We say that p satisfies the Beck-Chevalley condition if for every cartesian square $C : [1] \times [1] \to \mathcal{B}$ the base change of p along C is a bivariant fibration.*

Remark 9.4.2. Let $\mathcal{C}, \mathcal{D}$ be categories admitting pullbacks. Let $\alpha : c \to c'$ and $\beta : d \to d'$ be arrows in $\mathcal{C}$ and $\mathcal{D}$ respectively. Then the commutative diagram

$$\begin{tikzcd}
(c, d) \arrow[r, "(\alpha, \mathrm{id}_d)"] \arrow[d, "(\mathrm{id}_c, \beta)"] & (c', d) \arrow[d, "(\mathrm{id}_{c'}, \beta)"] \\
(c, d') \arrow[r, "(\alpha, \mathrm{id}_{d'})"] & (c', d')
\end{tikzcd}$$

is a cartesian square in $\mathcal{C} \times \mathcal{D}$. It follows that if $p : \mathcal{E} \to \mathcal{C} \times \mathcal{D}$ is a lax bivariant fibration satisfying the Beck-Chevalley condition then p is a bivariant fibration.

Proposition 9.4.3. *Let $\mathcal{C}, \mathcal{D}$ be categories admitting pullbacks, and $p : \mathcal{E} \to \mathcal{C} \times \mathcal{D}$ be a bivariant fibration. Then p satisfies the Beck-Chevalley condition if and only if the following conditions are satisfied:*

- *For every cartesian square $C : [1] \times [1] \to \mathcal{C}$ and every object d in $\mathcal{D}$ the base change of p along $C \times d$ is a bivariant fibration.*
- *For every cartesian square $C : [1] \times [1] \to \mathcal{D}$ and every object c in $\mathcal{C}$ the base change of p along $c \times C$ is a bivariant fibration.*

Proof. The pair of conditions given in the statement are evidently implied by the Beck-Chevalley condition. Assume now that p satisfies the two conditions in the statement, and consider a general cartesian square $C : [1] \times [1] \to \mathcal{C} \times \mathcal{D}$ depicted as follows:

$$\begin{tikzcd}
(c' \times_c c'', d' \times_d d'') \arrow[r] \arrow[d] & (c'', d'') \arrow[d] \\
(c', d') \arrow[r] & (c, d).
\end{tikzcd}$$

We can see C as the outer boundary of a commutative diagram $G : [2] \times [2] \to \mathcal{C} \times \mathcal{D}$, as follows:

$$\begin{tikzcd}
(c' \times_c c'', d' \times_d d'') \arrow[r] \arrow[d] & (c'', d' \times_d d'') \arrow[r] \arrow[d] & (c'', d'') \arrow[d] \\
(c', d' \times_d d'') \arrow[r] \arrow[d] & (c, d' \times_d d'') \arrow[r] \arrow[d] & (c, d'') \arrow[d] \\
(c', d') \arrow[r] & (c, d') \arrow[r] & (c, d).
\end{tikzcd}$$

To show that the base change C^*p is a bivariant fibration it suffices to show that G^*p is a bivariant fibration. To see this we must show that the base change of p along each of the four small commutative diagrams inside G is a bivariant fibration. The fact that this holds for the lower left and the upper right squares is a direct consequence of the fact that p itself is a bivariant fibration. The fact that this holds for the upper left and lower right squares follows from our assumptions on p. $\square$

Corollary 9.4.4. *Let $\mathcal{C}, \mathcal{D}$ be categories admitting pullbacks, and $p : \mathcal{E} \to \mathcal{C} \times \mathcal{D}$ be a bivariant fibration classified by a functor $H : \mathcal{C} \times \mathcal{D}^{\mathrm{op}} \to \mathrm{Cat}$. Then p satisfies the Beck-Chevalley condition if and only if the following conditions are satisfied:*

- *For every pair of maps $c' \to c \leftarrow c''$ in $\mathcal{C}$ and every object d in $\mathcal{D}$ the commutative diagram of categories*

$$\begin{tikzcd}
H(c' \times_c c'', d) \arrow[r] \arrow[d] & H(c'', d) \arrow[d] \\
H(c', d) \arrow[r] & H(c, d)
\end{tikzcd}$$

  is right adjointable.

- *For every object c in $\mathcal{C}$ and every pair of arrows $d' \to d \leftarrow d''$ in $\mathcal{D}$ the commutative diagram of categories*

$$\begin{tikzcd} H(c,d) \arrow[r] \arrow[d] & H(c,d') \arrow[d] \\ H(c,d'') \arrow[r] & H(c, d' \times_d d'') \end{tikzcd}$$

is left adjointable.

Proof. Combine propositions 9.4.3 and 9.3.7. □

We now study a fundamental example of a bivariant fibration satisfying the Beck-Chevalley condition, which plays a role analogous to that of the arrow category in the theory of two-sided fibrations.

Notation 9.4.5. Denote by Λ^2_0 the category with objects $0, 1, 2$ and nontrivial arrows $0 \to 1$ and $0 \to 2$. For each category $\mathcal{C}$ and i in Λ^2_0 we let $\mathrm{ev}_i : \mathrm{Funct}(\Lambda^2_0, \mathcal{C}) \to \mathcal{C}$ be the functor of evaluation at i.

Proposition 9.4.6. *Let $\mathcal{C}$ be a category admitting pullbacks. Then*

$$q = (\mathrm{ev}_1, \mathrm{ev}_2) : \mathrm{Funct}(\Lambda^2_0, \mathcal{C}) \to \mathcal{C} \times \mathcal{C}$$

is a bivariant fibration which satisfies the Beck-Chevalley condition.

Proof. Let $\rho : \mu \to \mu'$ be an arrow in $\mathrm{Funct}(\Lambda^2_0, \mathcal{C})$. Then

- ρ is q-cocartesian if and only if the induced map $\mu(0) \to \mu'(0)$ is an isomorphism.
- ρ is q-cartesian if and only if the diagram

$$\begin{tikzcd} & \mu(0) \arrow[dl] \arrow[d] \arrow[dr] & \\ \mu(1) \arrow[d] & \mu'(0) \arrow[dl] \arrow[dr] & \mu(2) \arrow[d] \\ \mu'(1) & & \mu'(2) \end{tikzcd}$$

is a limit diagram.

We thus see that q has all cocartesian lifts, and that the existence of cartesian lifts is guaranteed by the fact that $\mathcal{C}$ has pullbacks. Therefore q is a lax bivariant fibration.

It follows from the above characterization of cartesian and cocartesian arrows that a morphism $\mu \to \mu'$ in $\mathrm{Funct}(\Lambda^2_0, \mathcal{C})$ is bicartesian (from $\mathcal{C}$ to $\mathcal{D}$) if and only if the square

$$\begin{tikzcd} \mu(0) \arrow[r] \arrow[d] & \mu'(0) \arrow[d] \\ \mu(2) \arrow[r] & \mu'(2) \end{tikzcd}$$

is cartesian. We thus see that the class of bicartesian arrows is stable under composition, and therefore q is a two-sided fibration from $\mathcal{C}$ to $\mathcal{D}$. Switching the roles of $\mathcal{C}$ and $\mathcal{D}$ we conclude that q is also a two-sided fibration from $\mathcal{D}$ to $\mathcal{C}$, and therefore q is a bivariant fibration.

It remains to show that q satisfies the Beck-Chevalley condition. Consider a commutative diagram $C : [1] \times [1] \to \mathrm{Funct}(\Lambda^2_0, \mathcal{C})$ depicted as follows

$$\begin{tikzcd} \nu \arrow[r] \arrow[d] & \mu'' \arrow[d] \\ \mu' \arrow[r] & \mu \end{tikzcd}$$

and such that qC is cartesian. Assume that the horizontal arrows are q-cocartesian and that the left vertical arrow is q-cartesian. We have to show that the right vertical arrow is q-cartesian. By virtue of proposition 9.4.3, and since the bivariant fibrations q and $(\mathrm{ev}_2, \mathrm{ev}_1)$ are equivalent, it suffices to consider the case when the image of the above square under ev_2 is constant.

Consider the commutative diagram

$$\begin{tikzcd} \nu(0) \arrow[r] \arrow[d] & \nu(1) \arrow[r] \arrow[d] & \mu''(1) \arrow[d] \\ \mu'(0) \arrow[r] & \mu'(1) \arrow[r] & \mu(1). \end{tikzcd}$$

The right inner square is $\mathrm{ev}_1(C)$ which is cartesian since qC is cartesian. The left inner square is cartesian since the morphism $\nu \to \mu'$ is ev_1-cartesian. Hence the outmost square is cartesian. Since the maps $\nu \to \mu''$ and $\mu' \to \mu$ are cocartesian, the outermost square is equivalent to

$$\begin{tikzcd} \mu''(0) \arrow[r] \arrow[d] & \mu''(1) \arrow[d] \\ \mu(0) \arrow[r] & \mu(1). \end{tikzcd}$$

Since the map $\mu'' \to \mu$ is constant under ev_2 this shows that it is in fact ev_1-cartesian, as desired. $\square$

We finish by proving a universal property for $\mathrm{Funct}(\Lambda^2_0, \mathcal{C})$. Although it is possible to formulate a universal property for it in the category of bivariant fibrations, for our purposes we will need a version which treats it as a cocartesian and two-sided fibration.

Notation 9.4.7. Let $\mathcal{C}, \mathcal{D}$ be categories. We denote by $\mathrm{Cat}^{\text{cocart,two-sided}}_{/\mathcal{C}\times\mathcal{D}}$ the intersection of $\mathrm{Cat}^{\text{cocart}}_{/\mathcal{C}\times\mathcal{D}}$ and $\mathrm{Cat}^{\text{two-sided}}_{/\mathcal{C}\times\mathcal{D}}$ inside $\mathrm{Cat}_{/\mathcal{C}\times\mathcal{D}}$.

Proposition 9.4.8. *Let $\mathcal{C}$ be a category admitting pullbacks. Let*

$$p = (\mathrm{ev}_1, \mathrm{ev}_0) : \mathrm{Funct}([1], \mathcal{C}) \to \mathcal{C} \times \mathcal{C}$$

and

$$q = (\mathrm{ev}_1, \mathrm{ev}_2) : \mathrm{Funct}(\Lambda^2_0, \mathcal{C}) \to \mathcal{C} \times \mathcal{C}.$$

Let $\phi : p \to q$ be the map given by precomposition with the functor $\Lambda^2_0 \to [1]$ sending 0 to 0, 1 to 1, and 2 to 0. Then

(i) *The map ϕ is a morphism of two-sided fibrations.*

(ii) *Let $r = (r_1, r_2) : \mathcal{E} \to \mathcal{C} \times \mathcal{C}$ be a cocartesian and two-sided fibration such that for every c in $\mathcal{C}$ the base change of $\mathcal{E}$ along $c \times \mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ satisfies the Beck-Chevalley condition. Then precomposition with ϕ induces an equivalence*

$$\mathrm{Hom}_{\mathrm{Cat}^{\text{cocart,two-sided}}_{/\mathcal{C}\times\mathcal{C}}}(q, r) = \mathrm{Hom}_{\mathrm{Cat}^{\text{two-sided}}_{/\mathcal{C}\times\mathcal{C}}}(p, r).$$

Proof. Let $g : \sigma \to \sigma'$ be an arrow in $\mathrm{Funct}([1], \mathcal{C})$. Then

- The arrow g is ev_1-cocartesian if and only if the induced map $\sigma(0) \to \sigma'(0)$ is an isomorphism.
- The arrow g is ev_0-cartesian if and only if the induced map $\sigma(1) \to \sigma'(1)$ is an isomorphism.

It follows from this together with the description of q-cocartesian and q-cartesian arrows from the proof of proposition 9.4.6 that the map ϕ is a morphism of two-sided fibrations.

It remains to check the universality of ϕ. Let $\Delta : \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ be the diagonal map. Let $\psi : \Delta \to p$ be the map induced by precomposition with the projection $[1] \to [0]$. It follows from [GHN17] section 4 that the map ψ presents ev_1 as the free cocartesian fibration on $\mathrm{id}_{\mathcal{C}}$, and that $\phi\psi$ presents q as the free cocartesian fibration on Δ. We therefore have equivalences

$$\mathrm{Hom}_{\mathrm{Cat}^{\text{cocart}}_{/\mathcal{C}\times\mathcal{C}}}(q, r) = \mathrm{Hom}_{\mathrm{Cat}_{/\mathcal{C}\times\mathcal{C}}}(\Delta, r) = \mathrm{Hom}_{(\mathrm{Cat}^{\text{cocart}}_{/\mathcal{C}})_{/\mathcal{C}\times\mathcal{C}}}(p, r).$$

Under the above equivalence, the space $\mathrm{Hom}_{\mathrm{Cat}^{\text{two-sided}}_{/\mathcal{C}\times\mathcal{C}}}(p, r)$ becomes identified with the space of morphisms of cocartesian fibrations $F : q \to r$ whose composition with ϕ is a morphism of two-sided fibrations. Since ϕ is a morphism of two-sided fibrations, we see that precomposition with ϕ induces an inclusion

$$\mathrm{Hom}_{\mathrm{Cat}^{\mathrm{cocart,two\text{-}sided}}_{/\mathcal{C}\times\mathcal{C}}}(q,r) \subseteq \mathrm{Hom}_{\mathrm{Cat}^{\mathrm{two\text{-}sided}}_{/\mathcal{C}\times\mathcal{C}}}(p,r).$$

It remains to show that the above inclusion is an equivalence. Let $F : q \to r$ be a morphism of cocartesian fibrations and assume that $F\phi$ is a morphism of two-sided fibrations. We have to show that F is a morphism of two-sided fibrations. Let $\rho : \mu \to \mu'$ be an ev_2-cartesian arrow in $\mathrm{Funct}(\Lambda^2_0, \mathcal{C})$. Consider the commutative diagram C in $\mathcal{E}$ given as follows:

$$\begin{tikzcd} F\phi(\mu(1) \leftarrow \mu(0)) \arrow[r] \arrow[d] & F\mu \arrow[d] \\ F\phi(\mu'(1) \leftarrow \mu'(0)) \arrow[r] & F\mu'. \end{tikzcd}$$

Since the maps $\phi(\mu(1) \leftarrow \mu(0)) \to \mu$ and $\phi(\mu'(1) \leftarrow \mu'(0)) \to \mu'$ are q-cocartesian, we see that the horizontal arrows in C are r-cocartesian. Moreover, since ρ is ev_2-cartesian the map $(\mu(1) \leftarrow \mu(0)) \to (\mu'(1) \leftarrow \mu'(0))$ is ev_0-cartesian. The fact that $F\phi$ is a morphism of two-sided fibrations then implies that the left vertical arrow in C is r_2-cartesian. The image of C under r is the commutative square

$$\begin{tikzcd} (\mu(1), \mu(0)) \arrow[r] \arrow[d] & (\mu(1), \mu(2)) \arrow[d] \\ (\mu'(1), \mu'(0)) \arrow[r] & (\mu'(1), \mu'(2)) \end{tikzcd}$$

which is cartesian and has constant first coordinate since ρ was taken to be ev_2-cartesian. Since r satisfies the Beck-Chevalley condition in the second coordinate we conclude that the right vertical arrow in C is r-cartesian, as desired. $\square$

Chapter 10

The 2-category of correspondences

Let $\mathcal{C}$ be a category admitting pullbacks. We can attach to $\mathcal{C}$ a 2-category $2\mathrm{Corr}(\mathcal{C})$ called the 2-category of correspondences of $\mathcal{C}$. Its space of objects coincides with the space of objects of $\mathcal{C}$, and for each pair of objects c, c' in $\mathcal{C}$, the hom category $\mathrm{Hom}_{2\mathrm{Corr}(\mathcal{C})}(c, c')$ is the category of diagrams in $\mathcal{C}$ of the form

$$\begin{tikzcd} & s \arrow[dl] \arrow[dr] & \\ c & & c'. \end{tikzcd}$$

Our goal in this chapter is to review the definition and main properties of the 2-category of correspondences[1], and to provide a new proof of its universal property.

We begin in 10.1 by recalling the definition and basic properties of $2\mathrm{Corr}(\mathcal{C})$. We define $2\mathrm{Corr}(\mathcal{C})$ first as a simplicial category, and show that it is in fact a complete Segal object in Cat, so it defines a 2-category. We provide here a description of the degeneracies and composition maps for $2\mathrm{Corr}(\mathcal{C})$.

In 10.2 we study the functoriality of the assignment $\mathcal{C} \mapsto 2\mathrm{Corr}(\mathcal{C})$. We construct 2Corr as a functor on the category of categories with pullbacks and pullback preserving morphisms. We show that this is in fact a limit preserving functor - in particular, if $\mathcal{C}$ comes equipped with a symmetric monoidal structure which is compatible with pullbacks, we obtain an induced symmetric monoidal structure on $2\mathrm{Corr}(\mathcal{C})$.

The 2-category $2\mathrm{Corr}(\mathcal{C})$ comes equipped with inclusions $\iota_{\mathcal{C}} : \mathcal{C} \to 2\mathrm{Corr}(\mathcal{C})$ and $\iota_{\mathcal{C}}^R : \mathcal{C}^{\mathrm{op}} \to 2\mathrm{Corr}(\mathcal{C})$. We show that these also depend functorially on $\mathcal{C}$. In particular, in the presence of a symmetric monoidal structure on $\mathcal{C}$ compatible with pullbacks, the inclusions $\iota_{\mathcal{C}}$ and $\iota_{\mathcal{C}}^R$ inherit canonical symmetric monoidal structures.

[1] Many of the basic properties of $2\mathrm{Corr}(\mathcal{C})$ that we discuss in 10.1 - 10.3 (namely, its construction, symmetric monoidal structure, adjunctions, duals) can be found in some way in [Hau18] or [GR17]. We chose to include statements and proofs of these facts for completenes and for ease of reference, as our notation differs from that of previous sources. Some of our proofs of these facts contain some level of novelty - for instance we prove the adjointness and duality properties of $2\mathrm{Corr}(\mathcal{C})$ by appeal to the functoriality of 2Corr, therefore reducing to checking that they hold in the universal examples, which is often manageable.

In 10.3 we record two basic dualizability and adjointness properties of the 2-category of correspondences. We show that if α is an arrow in $\mathcal{C}$ then the maps $\iota_{\mathcal{C}}(\alpha)$ and $\iota^R_{\mathcal{C}}(\alpha)$ are adjoint to each other. In the case when $\mathcal{C}$ admits finite limits, we show that every object of $2\mathrm{Corr}(\mathcal{C})$ is dualizable under the symmetric monoidal structure inherited from the cartesian symmetric monoidal structure on $\mathcal{C}$. The proof of these results appeals in a fundamental way to the functoriality properties of 2Corr: rather than showing that a candidate (co)unit indeed defines a duality or adjunction on $2\mathrm{Corr}(\mathcal{C})$ for an arbitrary $\mathcal{C}$, one proves it in the universal example, in which case the verification of the triangle identities becomes simple.

In 10.4 we review the so-called Beck-Chevalley condition, and provide a new proof of the fact that the inclusion $\iota_{\mathcal{C}} : \mathcal{C} \to 2\mathrm{Corr}(\mathcal{C})$ is the universal embedding of $\mathcal{C}$ into a 2-category satisfying the left Beck-Chevalley condition. This provides a concrete way of constructing functors out of $2\mathrm{Corr}(\mathcal{C})$: given a 2-category $\mathcal{D}$, a functor $2\mathrm{Corr}(\mathcal{C}) \to \mathcal{D}$ is the same data as a functor $\mathcal{C} \to \mathcal{D}$ satisfying familiar base change properties.

10.1 Construction and basic properties

We begin by reviewing the construction of the 2-category of correspondences.

Notation 10.1.1. Let $n \geq 0$ and let $\mathrm{Tw}([n])$ be the twisted arrow category of $[n]$. We identify the objects in $\mathrm{Tw}([n])$ with pairs (i, j) in $[n] \times [n]$ such that $i \leq j$, so that there is a unique arrow $(i, j) \to (i', j')$ whenever $i \leq i' \leq j' \leq j$. We denote by $\mathrm{Tw}([n])_{\mathrm{el}}$ the full subcategory of $\mathrm{Tw}([n])$ on the objects of the form $(i, i + 1)$ for $0 \leq i < n$.

Definition 10.1.2. *Let $\mathcal{C}$ be a category admitting pullbacks and let $n \geq 0$. We say that a functor $\mathrm{Tw}([n]) \to \mathcal{C}$ is cartesian if it is the right Kan extension of its restriction to $\mathrm{Tw}([n])_{\mathrm{el}}$.*

Proposition 10.1.3. *Let $\mathcal{C}$ be a category admitting pullbacks and let $n \geq 0$. Let $S : \mathrm{Tw}([n]) \to \mathcal{C}$ be a functor. Then the following conditions are equivalent:*

(i) *The functor S is cartesian.*

(ii) *For every object (i, j) in $\mathrm{Tw}([n])$ such that $j \geq i + 2$ the commutative square*

$$\begin{tikzcd} S(i,j) \arrow[r] \arrow[d] & S(i,j-1) \arrow[d] \\ S(i+1,j) \arrow[r] & S(i+1,j-1) \end{tikzcd}$$

is cartesian.

Proof. For each integer $1 \leq k \leq n$ let $\mathrm{Tw}([n])_{\leq k}$ be the full subcategory of $\mathrm{Tw}([n])$ on the objects of the form (i, j) with $j - i \leq k$. Note that if $2 \leq k \leq n$ and (i, j) is such that $j - i = k$ then the undercategory $((\mathrm{Tw}([n])_{\leq k-1})_{(i,j)/}$ contains the diagram

$$(i + 1, j) \to (i + 1, j - 1) \leftarrow (i, j - 1).$$

This diagram is in fact final in $((\mathrm{Tw}([n])_{\leq k-1})_{(i,j)/}$, and therefore we have that the restriction of S to $(\mathrm{Tw}([n]))_{\leq k}$ is the right Kan extension of its restriction to $(\mathrm{Tw}([n]))_{\leq k-1}$ if and only if the diagram in the statement is cartesian for every (i,j) such that $j-i=k$. The result now follows by induction on k. □

Notation 10.1.4. Let $\mathcal{C}$ be a category admitting pullbacks. Let $\overline{\mathrm{2Corr}}(\mathcal{C}) : \Delta^{\mathrm{op}} \to \mathrm{Cat}$ be the simplicial category given by the formula $\overline{\mathrm{2Corr}}(\mathcal{C})([n]) = \mathrm{Funct}(\mathrm{Tw}([n]), \mathcal{C})$. By virtue of proposition 10.1.3, for each map $[n] \to [n']$ in Δ, the induced map $\overline{\mathrm{2Corr}}(\mathcal{C})([n']) \to \overline{\mathrm{2Corr}}(\mathcal{C})([n]))$ sends cartesian objects to cartesian objects. We denote by $\mathrm{2Corr}'(\mathcal{C})$ the sub-simplicial category of $\overline{\mathrm{2Corr}}(\mathcal{C})$ such that for every $[n]$ in Δ the category $\mathrm{2Corr}'(\mathcal{C})([n])$ is the full subcategory of $\overline{\mathrm{2Corr}}(\mathcal{C})([n])$ on the cartesian objects.

Proposition 10.1.5. *Let $\mathcal{C}$ be a category admitting pullbacks. Then $\mathrm{2Corr}'(\mathcal{C})$ is a Segal category.*

Proof. Let $n \geq 0$ and denote by $\mathrm{sp}([n])$ the spine of $[n]$, that is, the union inside $\mathcal{P}(\Delta)$ of all the edges of $[n]$ of the form $i \to i+1$. Any simplicial category $\mathcal{S}$ determines by right Kan extension a functor $\mathcal{P}(\Delta)^{\mathrm{op}} \to \mathrm{Cat}$, which we will also denote by $\mathcal{S}$.

We have a commutative diagram

$$\begin{tikzcd}
\overline{\mathrm{2Corr}}(\mathcal{C})([n]) \arrow[r] & \overline{\mathrm{2Corr}}(\mathcal{C})(\mathrm{sp}[n]) \\
\mathrm{2Corr}'(\mathcal{C})([n]) \arrow[r] \arrow[u] & \mathrm{2Corr}'(\mathcal{C})(\mathrm{sp}[n]). \arrow[u]
\end{tikzcd}$$

Since $\mathrm{2Corr}'(\mathcal{C})$ and $\overline{\mathrm{2Corr}}(\mathcal{C})$ agree on simplices of dimension at most 1, the right vertical arrow is an isomorphism. Observe that the left Kan extension along $\Delta \to \mathcal{P}(\Delta)$ of the cosimplicial category $\mathrm{Tw}|_{\Delta}$ maps the inclusion $\mathrm{sp}([n]) \to [n]$ to the inclusion $\mathrm{Tw}([n])_{\mathrm{el}} \to \mathrm{Tw}([n])$. It follows that the top horizontal arrow in the above diagram is equivalent to the restriction map

$$\mathrm{Funct}(\mathrm{Tw}([n]), \mathcal{C}) \to \mathrm{Funct}(\mathrm{Tw}([n])_{\mathrm{el}}, \mathcal{C}).$$

It follows from the definition of $\mathrm{2Corr}'(\mathcal{C})([n])$ that restriction along the inclusion $\mathrm{Tw}([n])_{\mathrm{el}} \to \mathrm{Tw}([n])$ provides an equivalence $\mathrm{2Corr}'(\mathcal{C})([n]) = \mathrm{Funct}(\mathrm{Tw}([n])_{\mathrm{el}}, \mathcal{C})$. Therefore the bottom horizontal arrow in the diagram is an equivalence, which means that $\mathrm{2Corr}'(\mathcal{C})$ satisfies the Segal conditions, as desired. □

Note that the category $\mathrm{2Corr}'(\mathcal{C})([0])$ is equivalent to $\mathcal{C}$. We now consider the Segal category obtained from $\mathrm{2Corr}'(\mathcal{C})$ by discarding noninvertible arrows in $\mathrm{2Corr}'(\mathcal{C})([0])$.

Notation 10.1.6. Denote by $\mathrm{codisc} : \mathrm{Cat} = \mathrm{Funct}([0], \mathrm{Cat}) \to \mathrm{Funct}(\Delta^{\mathrm{op}}, \mathrm{Cat})$ the functor of right Kan extension along the inclusion $\{[0]\} \to \Delta^{\mathrm{op}}$. We have a diagram of endofunctors of $\mathrm{Funct}(\Delta^{\mathrm{op}}, \mathrm{Cat})$ as follows

$$\begin{tikzcd} & \mathrm{id}_{\mathrm{Funct}(\Delta^{\mathrm{op}},\mathrm{Cat})} \arrow[d] \\ \mathrm{codisc}\,\mathrm{ev}_{[0]}^{\leq 0} \arrow[r] & \mathrm{codisc}\,\mathrm{ev}_{[0]} \end{tikzcd}$$

where the vertical arrow is the unit of the adjunction $\mathrm{ev}_{[0]} \dashv \mathrm{codisc}$ and the horizontal arrow is the canonical inclusion. We let $(-)_{\mathrm{red}} : \mathrm{Funct}(\Delta^{\mathrm{op}}, \mathrm{Cat}) \to \mathrm{Funct}(\Delta^{\mathrm{op}}, \mathrm{Cat})$ be the fiber product of the above diagram.

Remark 10.1.7. The functor codisc can alternatively described as the functor induced from the composite map

$$\Delta^{\mathrm{op}} \times \mathrm{Cat} \xrightarrow{(-)^{\leq 0} \times \mathrm{id}_{\mathrm{Cat}}} \mathrm{Cat}^{\mathrm{op}} \times \mathrm{Cat} \xrightarrow{\mathrm{Funct}(-,-)} \mathrm{Cat}\,.$$

In other words, for each category $\mathcal{C}$ the simplicial category $\mathrm{codisc}(\mathcal{C})$ has $\mathcal{C}^n$ as its category of $[n]$-simplices. Note that this is in fact a Segal category.

Remark 10.1.8. The functor $(-)_{\mathrm{red}}$ sends a simplicial category $\mathcal{S}$ to a simplicial subcategory $\mathcal{S}_{\mathrm{red}}$ of $\mathcal{S}$ such that for each $n \geq 0$ we have that $\mathcal{S}_{\mathrm{red}}([n])$ is the subcategory of $\mathcal{S}([n])$ containing all objects, and only those arrows whose images under the $n+1$ functors $\mathcal{S}([n]) \to \mathcal{S}([0])$ are invertible. It follows from its presentation as the fiber product $\mathrm{codisc}(\mathcal{S}([0])^{\leq 0}) \times_{\mathrm{codisc}(\mathcal{S}([0]))} \mathcal{S}$ that the map $\mathcal{S}_{\mathrm{red}} \to \mathcal{S}$ is universal among the maps of simplicial categories $\mathcal{S}' \to \mathcal{S}$ such that $\mathcal{S}'([0])$ is a space. Moreover, since codiscrete simplicial categories are Segal, we see that if $\mathcal{S}$ is a Segal category then $\mathcal{S}_{\mathrm{red}}$ is also a Segal category.

Notation 10.1.9. Let $\mathcal{C}$ be a category admitting pullbacks. We denote by $2\mathrm{Corr}(\mathcal{C})$ the Segal category $2\mathrm{Corr}'(\mathcal{C})_{\mathrm{red}}$.

Remark 10.1.10. Let $\mathcal{C}$ be a category admitting pullbacks. The space of objects of $2\mathrm{Corr}(\mathcal{C})$ is the space of maps $\mathrm{Tw}([0]) = [0] \to \mathcal{C}$, and it therefore agrees with the space of objects of $\mathcal{C}$. Given two objects, c, c' in $\mathcal{C}$, the category of morphisms $\mathrm{Hom}_{2\mathrm{Corr}(\mathcal{C})}(c, c')$ is the overcategory

$$\mathcal{C}_{/c,c'} = \mathrm{Funct}(\mathrm{Tw}([1]), \mathcal{C}) \times_{\mathcal{C}\times\mathcal{C}} \{(c, c')\}$$

where the projection $\mathrm{Funct}(\mathrm{Tw}([1]), \mathcal{C}) \to \mathcal{C} \times \mathcal{C}$ is given by evaluation at $(0,0)$ and $(1,1)$.

The objects of $\mathrm{Hom}_{2\mathrm{Corr}(\mathcal{C})}(c, c')$ are therefore spans $c \leftarrow s \to c'$, and a morphism $(c \leftarrow s \to c') \to (c \leftarrow t \to c')$ is a commutative diagram

$$\begin{tikzcd} & s \arrow[d] \arrow[ddl] \arrow[ddr] & \\ & t \arrow[dl] \arrow[dr] & \\ c & & c'. \end{tikzcd}$$

The degeneracy map of $2\mathrm{Corr}(\mathcal{C})$ is given by precomposition with the projection $\mathrm{Tw}([1]) \to \mathrm{Tw}([0]) = [0]$. We thus see that for every object c in $\mathcal{C}$ the identity in $\mathrm{Hom}_{\mathcal{C}}(c, c)$ is given by the span $c \xleftarrow{\mathrm{id}_c} c \xrightarrow{\mathrm{id}_c} c$.

Consider the diagram of simplicial spaces $\mathrm{sp}([2]) \to [2] \leftarrow [1]$, where the right arrow is the unique active map from $[1]$ to $[2]$, and the left arrow is the inclusion of the spine $\mathrm{sp}([2]) = [1] \cup_{[0]} [1]$ inside $[2]$. As in the proof of proposition 10.1.5, via right Kan extension along the inclusion $\Delta^{\mathrm{op}} \to \mathcal{P}(\Delta)^{\mathrm{op}}$, we allow ourselves to evaluate Segal categories on such a diagram. Applying this to the maps $2\mathrm{Corr}'(\mathcal{C}) \to \overline{2\mathrm{Corr}}(\mathcal{C}) \to \mathrm{codisc}(\mathcal{C})$ we obtain a commutative diagram of categories

$$\begin{tikzcd}
2\mathrm{Corr}'(\mathrm{sp}([2])) \arrow[d] & 2\mathrm{Corr}'(\mathcal{C})([2]) \arrow[l] \arrow[r] \arrow[d] & 2\mathrm{Corr}'(\mathcal{C})([1]) \arrow[d] \\
\overline{2\mathrm{Corr}}(\mathrm{sp}([2])) \arrow[d] & \overline{2\mathrm{Corr}}(\mathcal{C})([2]) \arrow[l] \arrow[r] \arrow[d] & \overline{2\mathrm{Corr}}(\mathcal{C})([1]) \arrow[d] \\
\mathrm{codisc}(\mathcal{C})(\mathrm{sp}([2])) & \mathrm{codisc}(\mathcal{C})([2]) \arrow[l] \arrow[r] & \mathrm{codisc}(\mathcal{C})([1]).
\end{tikzcd}$$

Here the top left and bottom left horizontal arrows are isomorphisms, the top left and top right vertical arrows are isomorphisms, and the top middle vertical arrow is a monomorphism. The top and bottom left squares are horizontally right adjointable. Using the canonical identification $\mathrm{Tw}([2])_{\mathrm{el}} = \mathrm{Tw}([1]) \cup_{[0]} \mathrm{Tw}([1])$ we see that the bottom two rows of the diagram can be rewritten as follows:

$$\begin{tikzcd}
2\mathrm{Corr}'(\mathcal{C})(\mathrm{sp}[2]) \arrow[d] & 2\mathrm{Corr}'(\mathcal{C})([2]) \arrow[l] \arrow[r] \arrow[d] & 2\mathrm{Corr}'(\mathcal{C})([1]) \arrow[d] \\
\mathrm{Funct}(\mathrm{Tw}([2])_{\mathrm{el}}, \mathcal{C}) \arrow[d] & \mathrm{Funct}(\mathrm{Tw}([2]), \mathcal{C}) \arrow[l] \arrow[r] \arrow[d] & \mathrm{Funct}(\mathrm{Tw}([1]), \mathcal{C}) \arrow[d] \\
\mathcal{C} \times \mathcal{C} \times \mathcal{C} & \mathcal{C} \times \mathcal{C} \times \mathcal{C} \arrow[l] \arrow[r] & \mathcal{C} \times \mathcal{C}.
\end{tikzcd}$$

Here the bottom vertical arrows are given by evaluation at the objects of the form (i, i), the bottom left vertical arrow is the identity, the bottom right arrow is the projection onto the first and third coordinate, the middle left horizontal arrow is restriction along the inclusion $\mathrm{Tw}([2])_{\mathrm{el}} \to \mathrm{Tw}([2])$, and the middle right horizontal arrow is induced from the active map $[1] \to [2]$.

The top and bottom left commutative squares in the above diagram are horizontally right adjointable. Passing to horizontal right adjoints of these yields a commutative diagram

$$\begin{tikzcd}
2\mathrm{Corr}'(\mathcal{C})(\mathrm{sp}([2])) \arrow[r] \arrow[d] & 2\mathrm{Corr}'(\mathcal{C})([2]) \arrow[r] \arrow[d] & 2\mathrm{Corr}'(\mathcal{C})([1]) \arrow[d] \\
\mathrm{Funct}(\mathrm{Tw}([2])_{\mathrm{el}}, \mathcal{C}) \arrow[r] \arrow[d] & \mathrm{Funct}(\mathrm{Tw}([2]), \mathcal{C}) \arrow[r] \arrow[d] & \mathrm{Funct}(\mathrm{Tw}([1]), \mathcal{C}) \arrow[d] \\
\mathcal{C} \times \mathcal{C} \times \mathcal{C} \arrow[r] & \mathcal{C} \times \mathcal{C} \times \mathcal{C} \arrow[r] & \mathcal{C} \times \mathcal{C}.
\end{tikzcd}$$

Here the middle left horizontal arrow is given by right Kan extension along the inclusion $\mathrm{Tw}([2])_{\mathrm{el}} \to \mathrm{Tw}([2])$. The top row recovers the composition map for the Segal category $2\mathrm{Corr}'(\mathcal{C})$. The composition map for $2\mathrm{Corr}(\mathcal{C})$ can be obtained from the above by base change along the inclusion of

$$\mathcal{C}^{\leq 0} \times \mathcal{C}^{\leq 0} \times \mathcal{C}^{\leq 0} \longrightarrow \mathcal{C}^{\leq 0} \times \mathcal{C}^{\leq 0} \times \mathcal{C}^{\leq 0} \longrightarrow \mathcal{C}^{\leq 0} \times \mathcal{C}^{\leq 0}$$

into the bottom row.

Let c, c', c'' be a triple of objects in $\mathcal{C}$. The bottom row receives a map from the final row $[0] \to [0] \to [0]$ that picks out the triple (c, c', c'') inside $\mathcal{C} \times \mathcal{C} \times \mathcal{C}$. Base change along this map yields a commutative diagram

$$\begin{tikzcd}
\mathrm{Funct}(\mathrm{Tw}([2])_{\mathrm{el}}, \mathcal{C})_{(c,c',c'')} \arrow[r] \arrow[d] & 2\mathrm{Corr}(\mathcal{C})([2])_{(c,c',c'')} \arrow[r] \arrow[d] & \mathrm{Funct}(\mathrm{Tw}([1]), \mathcal{C})_{(c,c'')} \arrow[d] \\
\mathrm{Funct}(\mathrm{Tw}([2])_{\mathrm{el}}, \mathcal{C})_{(c,c',c'')} \arrow[r] & \mathrm{Funct}(\mathrm{Tw}([2]), \mathcal{C})_{(c,c',c'')} \arrow[r] & \mathrm{Funct}(\mathrm{Tw}([1]), \mathcal{C})_{(c,c'')}
\end{tikzcd}.$$

The composition of the two maps in the top row recovers the composition map

$$\mathrm{Hom}_{2\mathrm{Corr}(\mathcal{C})}(c, c') \times \mathrm{Hom}_{2\mathrm{Corr}(\mathcal{C})}(c', c'') \to \mathrm{Hom}_{2\mathrm{Corr}(\mathcal{C})}(c, c'').$$

In particular, we see that the composition map for $2\mathrm{Corr}(\mathcal{C})$ sends a pair of spans $c \leftarrow s \to c'$ and $c' \leftarrow s' \to c''$ to the span $c \leftarrow s \times_{c'} s' \to c''$. Given a pair of morphisms of spans

$$\begin{tikzcd}
 & s \arrow[d] \arrow[ddl] \arrow[ddr] & \\
 & t \arrow[dl] \arrow[dr] & \\
c & & c'
\end{tikzcd}
\qquad
\begin{tikzcd}
 & s' \arrow[d] \arrow[ddl] \arrow[ddr] & \\
 & t' \arrow[dl] \arrow[dr] & \\
c' & & c''
\end{tikzcd}$$

their composition is the morphism $(c \leftarrow s \times_{c'} s' \to c'') \to (c \leftarrow t \times_{c'} t' \to c'')$ that arises from the unique commutative diagram of the form

$$\begin{tikzcd}
 & & s \times_{c'} s' \arrow[dl] \arrow[d] \arrow[dr] & & \\
 & s \arrow[d] \arrow[ddl] \arrow[ddr] & t \times_{c'} t' \arrow[dl] \arrow[dr] & s' \arrow[d] \arrow[ddl] \arrow[ddr] & \\
 & t \arrow[dl] \arrow[dr] & & t' \arrow[dl] \arrow[dr] & \\
c & & c' & & c''
\end{tikzcd}$$

that extends the given morphisms of spans.

Proposition 10.1.11. *Let $\mathcal{C}$ be a category admitting pullbacks. Then a span $c \leftarrow s \to c'$ is invertible in* $2\mathrm{Corr}(\mathcal{C})$ *if and only if both legs are isomorphisms.*

Proof. If both legs of a span $c \leftarrow s \to c'$ are invertible, then it is equivalent to an identity span, which is invertible. Conversely, assume that the span $c \leftarrow s \to c'$ is invertible and let $c' \leftarrow t \to c$ be the inverse. Consider the composite span

$$\begin{tikzcd}
 & & u \arrow[dl] \arrow[dr] & & \\
 & s \arrow[dl] \arrow[dr] & & t \arrow[dl] \arrow[dr] & \\
c & & c' & & c
\end{tikzcd}$$

where $u = s \times_{c'} t$. The maps $c \leftarrow u$ and $u \to c''$ are invertible, and therefore we see that the projections $c \leftarrow s$ and $t \to c$ admit sections. Considering the composition in the opposite order reveals that the projections $s \to c'$ and $c' \leftarrow t$ also admit sections. In particular we conclude that the projection $s \leftarrow u$ admits a section, and therefore the map $c \leftarrow s$ is an isomorphism. Similarly, the projection $u \to t$ admits a section, and thus the map $t \to c$ is also an isomorphism. $\square$

Corollary 10.1.12. *Let $\mathcal{C}$ be a category admitting pullbacks. Then the Segal space underlying* $2\mathrm{Corr}(\mathcal{C})$ *is complete.*

Proof. Denote by $2\mathrm{Corr}(\mathcal{C})([1])^{\mathrm{iso}}$ the space of invertible 1-morphisms in $2\mathrm{Corr}(\mathcal{C})$ and by $\mathcal{C}([1])^{\mathrm{iso}}$ the space of invertible arrows in $\mathcal{C}$. Proposition 10.1.11 implies that $2\mathrm{Corr}(\mathcal{C})([1])^{\mathrm{iso}}$ is equivalent to the fiber product $\mathcal{C}([1])^{\mathrm{iso}} \times_{\mathcal{C}^{\leq 0}} \mathcal{C}([1])^{\mathrm{iso}}$, where the fiber product is taken with respect to the source projection in both coordinates. Our claim now follows from the fact that the degeneracy $\mathcal{C}^{\leq 0} \to \mathcal{C}([1])^{\mathrm{iso}}$ is an equivalence. $\square$

In other words, corollary 10.1.12 states that $2\mathrm{Corr}(\mathcal{C})$ belongs to the image of the inclusion $2\mathrm{Cat} \to \mathrm{Funct}(\Delta^{\mathrm{op}}, \mathrm{Cat})$.

Definition 10.1.13. *Let $\mathcal{C}$ be a category admitting pullbacks. We call $2\mathrm{Corr}(\mathcal{C})$ the 2-category of correspondences of $\mathcal{C}$.*

Remark 10.1.14. Let $\mathcal{C}$ be a category admitting pullbacks. For each $n \geq 0$ we have source and target projections $[n] \leftarrow \mathrm{Tw}([n]) \to [n]^{\mathrm{op}}$ that map the pair (i, j) to i and j respectively. These projections are natural in $[n]$, and precomposing with them yields functors

$$\mathcal{C} \xrightarrow{\iota_{\mathcal{C}}} 2\mathrm{Corr}(\mathcal{C}) \xleftarrow{\iota_{\mathcal{C}}^R} \mathcal{C}^{\mathrm{op}}.$$

The above functors are the identity on objects. Moreover, for every arrow $\alpha : c \to c'$ in $\mathcal{C}$, we have $\iota_{\mathcal{C}}(\alpha) = (c \xleftarrow{\mathrm{id}_c} c \xrightarrow{\alpha} c')$ and $\iota_{\mathcal{C}}^R(\alpha) = (c' \xleftarrow{\alpha} c \xrightarrow{\mathrm{id}_c} c)$.

10.2 Functoriality of 2Corr

We now examine the functoriality of the assignment $\mathcal{C} \mapsto 2\mathrm{Corr}(\mathcal{C})$.

Construction 10.2.1. We denote by $\mathrm{Cat}_{\mathrm{pb}}$ the subcategory of Cat on the categories admitting pullbacks, and functors which preserve pullbacks. Let $\overline{2\mathrm{Corr}} : \mathrm{Cat}_{\mathrm{pb}} \to \mathrm{Funct}(\Delta^{\mathrm{op}}, \mathrm{Cat})$ be the functor induced from the composite map

$$\Delta^{\mathrm{op}} \times \mathrm{Cat}_{\mathrm{pb}} \xrightarrow{\mathrm{Tw}(-)\times \mathrm{id}_{\mathrm{Cat}_{\mathrm{pb}}}} \mathrm{Cat}^{\mathrm{op}} \times \mathrm{Cat}_{\mathrm{pb}} \subset \mathrm{Cat}^{\mathrm{op}} \times \mathrm{Cat} \xrightarrow{\mathrm{Funct}(-,-)} \mathrm{Cat}\,.$$

It follows from proposition 10.1.3 that for every morphism $\mathcal{C} \to \mathcal{D}$ in $\mathrm{Cat}_{\mathrm{pb}}$ the induced morphism of simplicial categories $\overline{2\mathrm{Corr}}(\mathcal{C}) \to \overline{2\mathrm{Corr}}(\mathcal{D})$ restricts to a morphism $2\mathrm{Corr}'(\mathcal{C}) \to 2\mathrm{Corr}'(\mathcal{D})$. This in turn restricts to a functor $2\mathrm{Corr}(\mathcal{C}) \to 2\mathrm{Corr}(\mathcal{D})$. We therefore have that the assignment $\mathcal{C} \mapsto 2\mathrm{Corr}(\mathcal{C})$ extends to a functor $2\mathrm{Corr} : \mathrm{Cat}_{\mathrm{pb}} \to 2\mathrm{Cat}$ whose composition with the embedding $2\mathrm{Cat} \to \mathrm{Funct}(\Delta^{\mathrm{op}}, \mathrm{Cat})$ is a subfunctor of $\overline{2\mathrm{Corr}}$.

Proposition 10.2.2. *The functor $2\mathrm{Corr} : \mathrm{Cat}_{\mathrm{pb}} \to 2\mathrm{Cat}$ preserves limits.*

Proof. Denote by $i : 2\mathrm{Cat} \to \mathrm{Funct}(\Delta^{\mathrm{op}}, \mathrm{Cat})$ the inclusion. To show that 2Corr preserves limits, it suffices to show that the functor $\mathrm{ev}_{[n]}\, i 2\mathrm{Corr} : \mathrm{Cat}_{\mathrm{pb}} \to \mathrm{Cat}$ preserves limits for every $n \geq 0$. We now fix a category $\mathcal{J}$ and a limit diagram $F : \mathcal{J}^{\triangleleft} \to \mathrm{Cat}_{\mathrm{pb}}$. Denote by $*$ the initial object of $\mathcal{J}^{\triangleleft}$.

By construction, the functor $\mathrm{ev}_{[n]}\, \overline{2\mathrm{Corr}}$ is given by the composition of the forgetful functor $\mathrm{Cat}_{\mathrm{pb}} \to \mathrm{Cat}$ and the functor $\mathrm{Funct}(\mathrm{Tw}([n]), -) : \mathrm{Cat} \to \mathrm{Cat}$. Both of these preserve limits, so we see that $\mathrm{ev}_{[n]}\, \overline{2\mathrm{Corr}} F$ is a limit diagram. The natural functor

$$\lim(\mathrm{ev}_{[n]}\, i 2\mathrm{Corr} F|_{\mathcal{J}}) \to \mathrm{ev}_{[n]}\, \overline{2\mathrm{Corr}} F(*)$$

is a limit of monomorphisms, and is therefore a monomorphism.

An object S in $\mathrm{ev}_{[n]}\,\overline{2\mathrm{Corr}}F(*)$ belongs to $\lim(\mathrm{ev}_{[n]}\, i2\mathrm{Corr}F|_{\mathcal{J}})$ if and only if its projection to $\mathrm{ev}_{[n]}\,\overline{2\mathrm{Corr}}F(j)$ belongs to $\mathrm{ev}_{[n]}\, i2\mathrm{Corr}F(j)$ for every j in $\mathcal{J}$. It follows from proposition 10.1.3 and the fact that the transition maps in F preserve pullbacks that this happens if and only if S belongs to $\mathrm{ev}_{[n]}\, i2\mathrm{Corr}F(*)$.

Similarly, a morphism $g : S \to S'$ in $\mathrm{ev}_{[n]}\,\overline{2\mathrm{Corr}}F(*)$ belongs to $\lim(\mathrm{ev}_{[n]}\, i2\mathrm{Corr}F|_{\mathcal{J}})$ if and only if its projection to $\mathrm{ev}_{[n]}\,\overline{2\mathrm{Corr}}F(j)$ belongs to $\mathrm{ev}_{[n]}\, i2\mathrm{Corr}F(j)$ for every j in $\mathcal{J}$. This again happens if and only if g belongs to $\mathrm{ev}_{[n]}\, i2\mathrm{Corr}F(*)$.

We have thus seen that the functor $\lim(\mathrm{ev}_{[n]}\, i2\mathrm{Corr}F|_{\mathcal{J}}) \to \mathrm{ev}_{[n]}\,\overline{2\mathrm{Corr}}F(*)$ is a monomorphism, and its image coincides with $\mathrm{ev}_{[n]}\, i2\mathrm{Corr}F(*)$. This implies that $\mathrm{ev}_{[n]}\, i2\mathrm{Corr}F$ is indeed a limit diagram. □

Remark 10.2.3. Equip $\mathrm{Cat}_{\mathrm{pb}}$ and 2Cat with their cartesian symmetric monoidal structures. It follows from proposition 10.2.2 that 2Corr has a canonical symmetric monoidal structure. As a consequence, if $\mathcal{C}$ is a symmetric monoidal category admitting pullbacks and such that the tensor product functor $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves pullbacks, there is an induced symmetric monoidal structure on $2\mathrm{Corr}(\mathcal{C})$. In particular, given any finitely complete category $\mathcal{C}$ there is a symmetric monoidal structure on $2\mathrm{Corr}(\mathcal{C})$ inherited from the cartesian symmetric monoidal structure on $\mathcal{C}$. This assignment is functorial in $\mathcal{C}$ - namely, it can be enhanced to yield functors

$$\mathrm{Cat}_{\mathrm{lex}} \to \mathrm{CAlg}(\mathrm{Cat}_{\mathrm{pb}}) \to \mathrm{CAlg}(2\mathrm{Cat})$$

where $\mathrm{Cat}_{\mathrm{lex}}$ denotes the category if categories with finite limits and left exact functors.

The inclusions $\iota_{\mathcal{C}}$ and $\iota_{\mathcal{C}}^R$ from remark 10.1.14 turn out to be compatible with the symmetric monoidal structure of remark 10.2.3. To show this, we will need to make the transformations $\iota_{\mathcal{C}}$ and $\iota_{\mathcal{C}}^R$ functorial in $\mathcal{C}$.

Construction 10.2.4. Consider the commutative diagram of cosimplicial categories

$$\begin{tikzcd} \mathrm{Tw}|_{\Delta} \arrow[r] \arrow[d] & i_{\Delta}^{\mathrm{op}} \arrow[d] \\ i_{\Delta} \arrow[r] & {[0]} \end{tikzcd}$$

where $i_{\Delta} : \Delta \to \mathrm{Cat}$ is the canonical inclusion, and the left vertical and top horizontal maps are given by the source and target projections (see remark 10.1.14). This induces a commutative square of functors $\mathrm{Cat}_{\mathrm{pb}} \to \mathrm{Funct}(\Delta^{\mathrm{op}}, \mathrm{Cat})$ as follows

$$\begin{tikzcd} \mathrm{disc} \arrow[r] \arrow[d] & j(-)^{\mathrm{op}} \arrow[d] \\ j \arrow[r] & \overline{2\mathrm{Corr}} \end{tikzcd}$$

where disc denotes the functor that maps each category $\mathcal{C}$ to the constant simplicial category on $\mathcal{C}$, the functor j is given by the formula $j\mathcal{C}([n]) = \mathrm{Funct}([n], \mathcal{C})$, and $j(-)^{\mathrm{op}}$ denotes the composition of j with the functor $(-)^{\mathrm{op}} : \mathrm{Cat} \to \mathrm{Cat}$. The bottom horizontal and right vertical arrows factor through $2\mathrm{Corr}'$, so we obtain a commutative square of functors $\mathrm{Cat}_{\mathrm{pb}} \to \mathrm{Funct}(\Delta^{\mathrm{op}}, \mathrm{Cat})$ as follows:

$$\begin{tikzcd} \mathrm{disc} \arrow[r] \arrow[d] & j(-)^{\mathrm{op}} \arrow[d] \\ j \arrow[r] & 2\mathrm{Corr}' \end{tikzcd}$$

Composing with the functor $(-)_{\mathrm{red}}$ from notation 10.1.6 we obtain a commutative square of functors $\mathrm{Cat}_{\mathrm{pb}} \to 2\mathrm{Cat}$

$$\begin{tikzcd} i_{\mathrm{Spc}}(-)^{\leq 0} \arrow[r] \arrow[d] & i^{\mathrm{op}}_{\mathrm{Cat}_{\mathrm{pb}}} \arrow[d] \\ i_{\mathrm{Cat}_{\mathrm{pb}}} \arrow[r] & 2\mathrm{Corr} \end{tikzcd}$$

where $i_{\mathrm{Cat}_{\mathrm{pb}}}$ denotes the canonical inclusion $\mathrm{Cat}_{\mathrm{pb}} \to 2\mathrm{Cat}$, the functor $i^{\mathrm{op}}_{\mathrm{Cat}_{\mathrm{pb}}}$ is the composition of $i_{\mathrm{Cat}_{\mathrm{pb}}}$ with the functor $(-)^{\mathrm{op}} : \mathrm{Cat}_{\mathrm{pb}} \to \mathrm{Cat}_{\mathrm{pb}}$, and the top left corner is the composition of the truncation functor $(-)^{\leq 0} : \mathrm{Cat} \to \mathrm{Spc}$ and the canonical inclusion $i_{\mathrm{Spc}} : \mathrm{Spc} \to 2\mathrm{Cat}$.

We denote by ι the bottom horizontal arrow of the above diagram, and ι^R the right vertical arrow. When evaluated at a category $\mathcal{C}$ in $\mathrm{Cat}_{\mathrm{pb}}$, the above diagram recovers the commutative diagram

$$\begin{tikzcd} \mathcal{C}^{\leq 0} \arrow[r] \arrow[d] & \mathcal{C}^{\mathrm{op}} \arrow[d, "\iota^R_{\mathcal{C}}"] \\ \mathcal{C} \arrow[r, "\iota_{\mathcal{C}}"] & 2\mathrm{Corr}(\mathcal{C}) \end{tikzcd}$$

from remark 10.1.14.

Remark 10.2.5. We can think about the commutative diagram

$$\begin{tikzcd} i_{\mathrm{Spc}}(-)^{\leq 0} \arrow[r] \arrow[d] & i^{\mathrm{op}}_{\mathrm{Cat}_{\mathrm{pb}}} \arrow[d, "\iota^R"] \\ i_{\mathrm{Cat}_{\mathrm{pb}}} \arrow[r, "\iota"] & 2\mathrm{Corr} \end{tikzcd}$$

as a functor $\mathrm{Cat}_{\mathrm{pb}} \to \mathrm{Funct}([1] \times [1], 2\mathrm{Cat})$. Thanks to proposition 10.2.2, this functor is limit preserving, so it can be given a canonical symmetric monoidal structure, where we equip $\mathrm{Cat}_{\mathrm{pb}}$ and $2\mathrm{Cat}$ with their cartesian symmetric monoidal structures. It follows that if $\mathcal{C}$ is a symmetric monoidal category admitting pullbacks and such that the tensor product functor $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves pullbacks, the inclusions $\iota_{\mathcal{C}}$ and $\iota^R_{\mathcal{C}}$ can be given symmetric monoidal structures, and we have a commutative diagram of symmetric monoidal categories and symmetric monoidal functors

$$\begin{tikzcd}
\mathcal{C}^{\leq 0} \arrow[r] \arrow[d] & \mathcal{C}^{\mathrm{op}} \arrow[d, "\iota^R_{\mathcal{C}}"] \\
\mathcal{C} \arrow[r, "\iota_{\mathcal{C}}"] & 2\mathrm{Corr}(\mathcal{C}).
\end{tikzcd}$$

Remark 10.2.6. The span of cosimplicial categories

$$\begin{tikzcd}
\mathrm{Tw}\,|_{\Delta} \arrow[r] \arrow[d] & i^{\mathrm{op}}_{\Delta} \arrow[d] \\
i_{\Delta} \arrow[r] & [0]
\end{tikzcd}$$

from construction 10.2.4 is equivalent to the transpose of the diagram obtained by composing the above with the functor $\mathrm{Funct}(\Delta, \mathrm{Cat}) \to \mathrm{Funct}(\Delta, \mathrm{Cat})$ induced from the order reversing automorphism of Δ. It follows that the commutative square of functors $\mathrm{Cat}_{\mathrm{pb}} \to 2\mathrm{Cat}$

$$\begin{tikzcd}
i_{\mathrm{Spc}}(-)^{\leq 0} \arrow[r] \arrow[d] & i^{\mathrm{op}}_{\mathrm{Cat}_{\mathrm{pb}}} \arrow[d, "\iota^R"] \\
i_{\mathrm{Cat}_{\mathrm{pb}}} \arrow[r, "\iota"] & 2\mathrm{Corr}
\end{tikzcd}$$

is equivalent to the transpose of the square

$$\begin{tikzcd}
i_{\mathrm{Spc}}(-)^{\leq 0} \arrow[r] \arrow[d] & i_{\mathrm{Cat}_{\mathrm{pb}}} \arrow[d, "\iota^{R,1\text{-op}}"] \\
i^{\mathrm{op}}_{\mathrm{Cat}_{\mathrm{pb}}} \arrow[r, "\iota^{1\text{-op}}"] & 2\mathrm{Corr}^{1\text{-op}}
\end{tikzcd}$$

where $(-)^{1\text{-op}} : 2\mathrm{Cat} \to 2\mathrm{Cat}$ denotes the functor that reverses the directions of 1-arrows. In other words, for every object $\mathcal{C}$ in $\mathrm{Cat}_{\mathrm{pb}}$ we have an equivalence $2\mathrm{Corr}(\mathcal{C}) = 2\mathrm{Corr}(\mathcal{C})^{1\text{-op}}$ which is the identity on objects, and exchanges $i_{\mathcal{C}}(\alpha)$ and $i^R_{\mathcal{C}}(\alpha)$ for each arrow α in $\mathcal{C}$.

10.3 Adjointness and duality in $2\mathrm{Corr}(\mathcal{C})$

We now review some basic adjointness and duality properties of morphisms and objects in the 2-category of correspondences.

Proposition 10.3.1. *Let $\mathcal{C}$ be a category admitting pullbacks and let $\alpha : c \to c'$ be an arrow in $\mathcal{C}$. Then the morphism $\eta_\alpha : \iota_{\mathcal{C}}(\alpha)\iota^R_{\mathcal{C}}(\alpha) \to \mathrm{id}_{\iota_{\mathcal{C}}(c')}$ given by the following diagram*

$$\begin{tikzcd}
& c \arrow[d, "\alpha"] \arrow[ddl, "\alpha"'] \arrow[ddr, "\alpha"] & \\
& c' \arrow[dl, "\mathrm{id}_{c'}"] \arrow[dr, "\mathrm{id}_{c'}"'] & \\
c' & & c'
\end{tikzcd}$$

is the counit of an adjunction between $\iota_{\mathcal{C}}(\alpha)$ and $\iota^R_{\mathcal{C}}(\alpha)$.

Proof. Recall from (the dual version of) [Lur09a] proposition 5.3.6.2 that the forgetful functor $\mathrm{Cat}_{\mathrm{pb}} \to \mathrm{Cat}$ has a left adjoint $\mathcal{F}$, which maps a category $\mathcal{I}$ to the smallest full subcategory of the free completion of $\mathcal{I}$ (namely, $\mathcal{P}(\mathcal{I}^{\mathrm{op}})^{\mathrm{op}}$) containing $\mathcal{I}$ and closed under pullbacks. Consider the pullback preserving functor $\mathcal{F}([1]) \to \mathcal{C}$ induced from the map $[1] \to \mathcal{C}$ that sends the unique arrow $a : 0 \to 1$ in $[1]$ to α. The arrows $\iota_{\mathcal{C}}(\alpha)$ and $\iota^R_{\mathcal{C}}(\alpha)$ are the images of $\iota_{\mathcal{F}([1])}(a)$ and $\iota^R_{\mathcal{F}([1])}(a)$ under the induced functor $2\mathrm{Corr}(\mathcal{F}([1])) \to 2\mathrm{Corr}(\mathcal{C})$. Moreover, the morphism of spans in the statement is the image of the morphism of spans

$$\begin{tikzcd}
& 0 \arrow[d] \arrow[ddl] \arrow[ddr] & \\
& 1 \arrow[dl] \arrow[dr] & \\
1 & & 1.
\end{tikzcd}$$

We have thus reduced to proving the result in the case when $\mathcal{C} = \mathcal{F}([1])$ and $\alpha = a : 0 \to 1$. Note that in $2\mathrm{Corr}(\mathcal{F}([1]))$ there is a unique map $\iota_{\mathcal{F}([1])}(a) \to \iota_{\mathcal{F}([1])}(a)$ and a unique map $\iota^R_{\mathcal{F}([1])}(a) \to \iota^R_{\mathcal{F}([1])}(a)$. Therefore any morphism

$$\epsilon_a : \mathrm{id}_{\iota_{\mathcal{F}([1])}(0)} \to \iota^R_{\mathcal{F}([1])}(a)\iota_{\mathcal{F}([1])}(a)$$

will satisfy the triangle identities with η_a. Such a morphism is unique, and given by the following diagram:

$$\begin{tikzcd}
& 0 \arrow[d] \arrow[ddl] \arrow[ddr] & \\
& 0 \times_1 0 \arrow[dl] \arrow[dr] & \\
0 & & 0
\end{tikzcd}$$

□

Remark 10.3.2. Let $\mathcal{C}$ be a category admitting pullbacks and let $\alpha : c \to c'$ be an arrow in $\mathcal{C}$. The composition $\iota^R_{\mathcal{C}}(\alpha)\iota_{\mathcal{C}}(\alpha)$ is given by the span $c \leftarrow c \times_d c \to c$. It follows from the proof of proposition 10.3.1 that the counit of the adjunction $\iota_{\mathcal{C}}(\alpha) \dashv \iota^R_{\mathcal{C}}(\alpha)$ is given by the diagram

$$\begin{tikzcd} & c \arrow[d] \arrow[ddl, "\mathrm{id}_c"'] \arrow[ddr, "id_c"] & \\ & c \times_d c \arrow[dl] \arrow[dr] & \\ c & & c \end{tikzcd}$$

where the map $c \to c \times_d c$ is the diagonal map.

Proposition 10.3.3. *Let $\mathcal{C}$ be a category admitting finite limits and let c be an object in $\mathcal{C}$. Denote by $\Delta : c \to c \times c$ the diagonal map, and by $\pi : c \to 1_{\mathcal{C}}$ the map into the final object of $\mathcal{C}$. Equip $2\mathrm{Corr}(\mathcal{C})$ and the inclusion $\iota_{\mathcal{C}}$ with the symmetric monoidal structures from remark 10.2.5. Then the morphism*

$$\eta_c : 1_{2\mathrm{Corr}(\mathcal{C})} = \iota_{\mathcal{C}}(1_{\mathcal{C}}) \to \iota_{\mathcal{C}}(c \times c) = \iota_{\mathcal{C}}(c) \otimes \iota_{\mathcal{C}}(c)$$

given by the span

$$\begin{tikzcd} & c \arrow[dl, "\pi"'] \arrow[dr, "\Delta"] & \\ 1_{\mathcal{C}} & & c \times c \end{tikzcd}$$

is the unit of a self duality for $\iota_{\mathcal{C}}(c)$.

Proof. Let $\mathrm{Spc}_{\mathrm{fin}}$ be the category of finite spaces. This is obtained from $[0]$ by adjoining finite colimits, and therefore there is a unique left exact functor $F : \mathrm{Spc}^{\mathrm{op}}_{\mathrm{fin}} \to \mathcal{C}$ that maps the point $*$ to c. As observed in remark 10.2.5, the functor

$$2\mathrm{Corr}(F) : 2\mathrm{Corr}(\mathrm{Spc}^{\mathrm{op}}_{\mathrm{fin}}) \to 2\mathrm{Corr}(\mathcal{C})$$

inherits a symmetric monoidal structure, which is compatible the transformations ι and ι^R. The map η_c is the image under $2\mathrm{Corr}(F)$ of the functor

$$\eta_* : 1_{2\mathrm{Corr}(\mathrm{Spc}^{\mathrm{op}}_{\mathrm{fin}})} = \iota_{\mathrm{Spc}^{\mathrm{op}}_{\mathrm{fin}}}(\emptyset) \to \iota_{\mathrm{Spc}^{\mathrm{op}}_{\mathrm{fin}}}(* \amalg *) = \iota_{\mathrm{Spc}^{\mathrm{op}}_{\mathrm{fin}}}(*) \otimes \iota_{\mathrm{Spc}^{\mathrm{op}}_{\mathrm{fin}}}(*)$$

defined by the span

$$\begin{tikzcd} & * \arrow[dl] \arrow[dr] & \\ \emptyset & & * \amalg *. \end{tikzcd}$$

It therefore suffices to prove the proposition in the case $\mathcal{C} = \mathrm{Spc}^{\mathrm{op}}_{\mathrm{fin}}$ and $c = *$. Since there is a unique map $* \to *$ in $\mathrm{Spc}^{\mathrm{op}}_{\mathrm{fin}}$, any map $\epsilon_* : \iota_{\mathrm{Spc}^{\mathrm{op}}_{\mathrm{fin}}}(*) \otimes \iota_{\mathrm{Spc}^{\mathrm{op}}_{\mathrm{fin}}}(*) \to 1_{2\mathrm{Corr}(\mathrm{Spc}^{\mathrm{op}}_{\mathrm{fin}})}$ satisfies the triangle identities with η_*. Such a map exists and is unique, and is given by the following span

$$\begin{tikzcd} & * \arrow[dl] \arrow[dr] & \\ * \amalg * & & \emptyset. \end{tikzcd}$$

□

Remark 10.3.4. Let $\mathcal{C}$ be a category admitting finite limits and let c be an object in $\mathcal{C}$. Then the proof of proposition 10.3.3 shows that the morphism

$$\epsilon_c : \iota_{\mathcal{C}}(c) \otimes \iota_{\mathcal{C}}(c) = \iota_{\mathcal{C}}(c \times c) \to \iota_{\mathcal{C}}(1_{\mathcal{C}}) = 1_{2\mathrm{Corr}(\mathcal{C})}$$

given by the span

$$\begin{tikzcd} & c \arrow[dl, "\Delta"'] \arrow[dr, "\pi"] & \\ c \times c & & 1_{\mathcal{C}} \end{tikzcd}$$

is the counit of the self duality of proposition 10.3.3.

Proposition 10.3.5. *Let $\mathcal{C}$ be a category admitting finite limits. Let c, c' be objects of $\mathcal{C}$, and let $\sigma : \iota_{\mathcal{C}}(c) \to \iota_{\mathcal{C}}(c')$ be a morphism between them, represented by a span*

$$\begin{tikzcd} & s \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ c & & c'. \end{tikzcd}$$

Then the morphism $\sigma^\vee : \iota_{\mathcal{C}}(c') \to \iota_{\mathcal{C}}(c)$ dual to σ under the self duality of proposition 10.3.3 is given by the span

$$\begin{tikzcd} & s \arrow[dl, "\beta"'] \arrow[dr, "\alpha"] & \\ c' & & c. \end{tikzcd}$$

Proof. Note that the morphism σ is equivalent to $\iota_{\mathcal{C}}(\beta)\iota^R_{\mathcal{C}}(\alpha)$, and furthermore we have an equivalence $\sigma^\vee = (\iota^R_{\mathcal{C}}(\alpha))^\vee \iota_{\mathcal{C}}(\beta)^\vee$. It suffices therefore to show that there are equivalences $\iota_{\mathcal{C}}(\beta)^\vee = \iota^R_{\mathcal{C}}(\beta)$ and $\iota^R_{\mathcal{C}}(\alpha)^\vee = \iota_{\mathcal{C}}(\alpha)$. We may furthermore restrict to showing the first identity only - the second one follows from the first one by replacing β with α and passing to adjoints.

Let $\mathcal{F}([1])$ be the free finitely complete category with limits on the arrow category. The morphism β is the image of the walking arrow $0 \to 1$ under a finite limit preserving functor $\mathcal{F}([1]) \to \mathcal{C}$. It therefore suffices to prove our proposition in the case when $\mathcal{C} = \mathcal{F}([1])$ and β

is the walking arrow β_{univ}. In this case, the dual morphism to $\iota_{\mathcal{F}([1])}(\beta_{\mathrm{univ}})$ can be computed as the following composition:

$$\begin{tikzcd}[column sep=small] & 1\times 0 \arrow[dl] \arrow[dr, "\mathrm{id}_1\times\Delta_0"] & & 1\times 0\times 0 \arrow[dl, "\mathrm{id}"'] \arrow[dr, "\mathrm{id}_1\times\beta_{\mathrm{univ}}\times\mathrm{id}_0"] & & 1\times 0 \arrow[dl, "\Delta_1\times\mathrm{id}_0"'] \arrow[dr] & \\ 1 & & 1\times 0\times 0 & & 1\times 1\times 0 & & 0. \end{tikzcd}$$

This recovers the unique span of the form

$$\begin{tikzcd} & 0 \arrow[dl] \arrow[dr] & \\ 1 & & 0 \end{tikzcd}$$

which is $\iota^R_{\mathcal{F}([1])}(\beta_{\mathrm{univ}})$, as desired. □

10.4 Beck-Chevalley conditions

We now discuss the universal property of the 2-category of correspondences.

Definition 10.4.1. *Let $\mathcal{D}$ be a 2-category. We say that a commutative diagram*

$$\begin{tikzcd} d' \arrow[r, "\alpha'"] \arrow[d, "\beta'"] & d \arrow[d, "\beta"] \\ e' \arrow[r, "\alpha"] & e \end{tikzcd}$$

in $\mathcal{D}$ is vertically right adjointable if the following conditions hold:

- *The maps β and β' admit right adjoints β^R and β'^R.*
- *The 2-cell*
$$\alpha'\beta'^R \to \beta^R\beta\alpha'\beta'^R = \beta^R\alpha\beta'\beta'^R \to \beta^R\alpha$$
built from the unit $\mathrm{id}_d \to \beta^R\beta$ and the counit $\beta'\beta^R \to \mathrm{id}_{e'}$, is an isomorphism.

We say that the above diagram is horizontally right adjointable if its transpose is vertically right adjointable. We say that it is right adjointable if it is both horizontally and vertically right adjointable. We say that it is (vertically / horizontally) left adjointable if it is (vertically / horizontally) right adjointable as a diagram in the 2-category $\mathcal{D}^{2\text{-op}}$ obtained from $\mathcal{D}$ by reversing the direction of the 2-cells.

Remark 10.4.2. Let $\mathcal{D}$ be a 2-category. A commutative square

$$\begin{tikzcd} d' \arrow[r, "\alpha'"] \arrow[d, "\beta'"] & d \arrow[d, "\beta"] \\ e' \arrow[r, "\alpha"] & e \end{tikzcd}$$

in $\mathcal{D}$ defines a morphism $\gamma : \alpha' \to \alpha$ in the arrow category $\mathrm{Funct}([1], \mathcal{D})$. It is shown in [Hau20] theorem 4.6 that β' and β admit right adjoints if and only if γ admits a right adjoint in the category $\mathrm{Funct}([1], \mathcal{D})_{\mathrm{lax}}$ of functors $[1] \to \mathcal{D}$ and lax natural transformations. In that case, the right adjoint to γ is given by the lax commutative square

$$\begin{tikzcd} d' \arrow[r, "\alpha'"] & d \\ e' \arrow[u, "\beta'^R"] \arrow[r, "\alpha"] \arrow[ur, phantom, "\Searrow\beta^R"] & e \arrow[u] \end{tikzcd}$$

where the 2-cell is the one described in definition 10.4.1. It follows that γ admits a right adjoint in $\mathrm{Funct}([1], \mathcal{D}) \subset \mathrm{Funct}([1], \mathcal{D})_{\mathrm{lax}}$ if and only if the original commutative square is vertically right adjointable. In this case, the morphism γ^R corresponds to a commutative square in $\mathcal{D}$; we say that this square arises from the original one by passage to right adjoints of vertical arrows.

Remark 10.4.3. Let $\mathcal{D}$ be a 2-category and let

$$\begin{tikzcd} d' \arrow[r, "\alpha'"] \arrow[d, "\beta'"] & d \arrow[d, "\beta"] \\ e' \arrow[r, "\alpha"] & e \end{tikzcd}$$

be a commutative diagram in $\mathcal{D}$. Assume that the vertical maps are right adjointable, and that the horizontal maps are left adjointable. We can then construct two lax commutative squares

$$\begin{tikzcd} d' \arrow[r, "\alpha'"] & d \\ e' \arrow[u, "\beta'^R"] \arrow[r, "\alpha"] \arrow[ur, phantom, "\Searrow\beta^R"] & e \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} d' \arrow[d, "\beta'"] & d \arrow[l, "\alpha'^L"'] \arrow[d, "\beta"] \arrow[dl, phantom, "\Nwarrow"] \\ e' & e. \arrow[l, "\alpha^L"] \end{tikzcd}$$

These two are related to each other by passage to left/right adjoints. In particular we have that our original square is vertically right adjointable if and only if it is horizontally left adjointable.

Remark 10.4.4. Denote by Adj the universal adjunction. This is a 2-category equipped with an epimorphism $L : [1] \to \mathrm{Adj}$ such that for every 2-category $\mathcal{D}$, precomposition with L induces an equivalence between the space of functors $\mathrm{Adj} \to \mathcal{D}$ and the space of maps $[1] \to \mathcal{D}$ which pick out a right adjointable arrow in $\mathcal{D}$. Let $\mathcal{U}_{\mathrm{lax}}$ be the pushout in 2Cat of the following diagram

$$\begin{tikzcd} \mathrm{Adj} & & [1]\times[1] & & \mathrm{Adj} \\ & {[1]} \arrow[ul, "L"] \arrow[ur, "\{0\}\times \mathrm{id}_{[1]}"] & & {[1]} \arrow[ul, "\{1\}\times \mathrm{id}_{[1]}"'] \arrow[ur, "L"'] & . \end{tikzcd}$$

This is the universal lax vertically right adjointable square. It contains a 2-cell μ_{univ} which is the universal instance of the 2-cell from definition 10.4.1.

Let $\mathcal{U}$ be the 2-category obtained from $\mathcal{U}_{\mathrm{lax}}$ by inverting the 2-cell μ_{univ}. The natural inclusion $i : [1] \times [1] \to \mathcal{U}$ is an epimorphism in 2Cat. For any 2-category $\mathcal{D}$, precomposition with i induces an equivalence between the space of functors $\mathcal{U} \to \mathcal{D}$ and the space of functors $[1] \times [1] \to \mathcal{D}$ which represent a vertically right adjointable square. By virtue of remark 10.4.2, we in fact have an equivalence between $i : [1] \times [1] \to \mathcal{U}$ and $L \times \mathrm{id}_{[1]} : [1] \times [1] \to \mathrm{Adj} \times [1]$.

As a consequence of the above we deduce the following fact: if $F : \mathcal{I} \to 2\mathrm{Cat}$ is a diagram in 2Cat with limit $\mathcal{D}$, then a commutative square in $\mathcal{D}$ is vertically right adjointable if and only if its image in $F(i)$ is vertically right adjointable, for all i in $\mathcal{I}$.

Definition 10.4.5. *Let $\mathcal{C}$ be a category admitting pullbacks and $\mathcal{D}$ be a 2-category. We say that a functor $F : \mathcal{C} \to \mathcal{D}$ satisfies the left Beck-Chevalley condition if for every cospan $x \to s \leftarrow y$ in $\mathcal{C}$, the induced commutative square in $\mathcal{D}$*

$$\begin{tikzcd} F(x \times_s y) \arrow[r] \arrow[d] & F(y) \arrow[d] \\ F(x) \arrow[r] & F(s) \end{tikzcd}$$

is right adjointable.

Proposition 10.4.6. *Let $\mathcal{C}$ be a category admitting pullbacks. Then the inclusion $\iota_{\mathcal{C}} : \mathcal{C} \to 2\mathrm{Corr}(\mathcal{C})$ satisfies the left Beck-Chevalley condition.*

Proof. Let $\mathcal{F} : \mathrm{Cat} \to \mathrm{Cat}_{\mathrm{pb}}$ be the left adjoint to the forgetful functor. Let $\mathcal{U}$ be universal cospan: this is the category with three objects $0, 1, 2$ and nontrivial arrows $\alpha : 0 \to 1 \leftarrow 2 : \beta$. Any cospan in $\mathcal{C}$ is the image of a pullback preserving morphism $\mathcal{F}(\mathcal{U}) \to \mathcal{C}$. It therefore suffices to prove that the image under $\iota_{\mathcal{F}(\mathcal{U})}$ of the universal cartesian square

$$\begin{tikzcd} 0 \times_1 2 \arrow[r, "\alpha'"] \arrow[d, "\beta'"] & 2 \arrow[d, "\beta"] \\ 0 \arrow[r, "\alpha"] & 1 \end{tikzcd}$$

is vertically right adjointable. Recall from proposition 10.3.1 that $\iota_{\mathcal{F}(\mathcal{U})}(\beta')$ is left adjoint to $\iota^R_{\mathcal{F}(\mathcal{U})}(\beta')$, and $\iota_{\mathcal{F}(\mathcal{U})}(\beta)$ is left adjoint to $\iota^R_{\mathcal{F}(\mathcal{U})}(\beta)$. We have that $\iota_{\mathcal{F}(\mathcal{U})}(\alpha')\iota^R_{\mathcal{F}(\mathcal{U})}(\beta')$ and $\iota^R_{\mathcal{F}(\mathcal{U})}(\beta)\iota_{\mathcal{F}(\mathcal{U})}(\alpha)$ are both given by the span

$$\begin{tikzcd} & 0 \times_1 2 \arrow[dl, "\beta'"'] \arrow[dr, "\alpha'"] & \\ 0 & & 2. \end{tikzcd}$$

Our claim now follows from the fact that $\mathrm{Hom}_{\mathcal{F}(\mathcal{U})}(0 \times_1 2, 0 \times_1 2) = [0]$, and thus any 2-cell

$$\iota_{\mathcal{F}(\mathcal{U})}(\alpha')\iota^R_{\mathcal{F}(\mathcal{U})}(\beta') \to \iota^R_{\mathcal{F}(\mathcal{U})}(\beta)\iota_{\mathcal{F}(\mathcal{U})}(\alpha)$$

is necessarily invertible. □

The rest of this chapter is devoted to showing that $2\mathrm{Corr}(\mathcal{C})$ is the universal 2-category equipped with a functor from $\mathcal{C}$ which satisfies the left Beck-Chevalley condition (theorem 10.4.18). A proof was given in [GR17] chapter 7 theorem 3.2.2. - here we present an alternative approach using the theory of two-sided fibrations, and in particular the universal property of the span fibration established in proposition 9.4.8. The proof will need a few preliminary lemmas.

Notation 10.4.7. Let $\mathcal{C}$ be a category. Recall the universal left adjointable arrow $L : [1] \to \mathrm{Adj}$ from remark 10.4.4. We let $\mathcal{C}^R$ be the 2-category defined by the pushout

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{Cat}}([1], \mathcal{C}) \times [1] \arrow[r, "\mathrm{ev}"] \arrow[d, "\mathrm{id} \times L"] & \mathcal{C} \arrow[d, "L_{\mathcal{C}}"] \\
\mathrm{Hom}_{\mathrm{Cat}}([1], \mathcal{C}) \times \mathrm{Adj} \arrow[r] & \mathcal{C}^R.
\end{tikzcd}$$

The functor $L_{\mathcal{C}}$ is an epimorphism in 2Cat. For every 2-category $\mathcal{D}$, precomposition with $L_{\mathcal{C}}$ induces an equivalence between the space $\mathrm{Hom}_{2\mathrm{Cat}}(\mathcal{C}^R, \mathcal{D})$ and the space of functors $F : \mathcal{C} \to \mathcal{D}$ such that for every arrow α in $\mathcal{C}$ the arrow $F(\alpha)$ admits a right adjoint in $\mathcal{D}$.

Lemma 10.4.8. *Let $\mathcal{C}$ be a category and let $\mathcal{D}$ be a 2-category. Let $\eta : F \to G$ be a morphism in $\mathrm{Funct}(\mathcal{C}, \mathcal{D})$. Then the morphism η is left adjointable if and only if for every morphism $\alpha : x \to y$ in $\mathcal{C}$, the commutative square*

$$\begin{tikzcd}
F(x) \arrow[r, "\eta(x)"] \arrow[d, "F(\alpha)"] & G(x) \arrow[d, "G(\alpha)"] \\
F(y) \arrow[r, "\eta(y)"] & G(y)
\end{tikzcd}$$

is horizontally left adjointable.

Proof. Let $\mathcal{S}$ be the full subcategory of Cat on those categories $\mathcal{C}$ for which the lemma holds. As discussed in remark 10.4.2, the walking arrow belongs to $\mathcal{S}$. To prove this lemma it suffices to show that $\mathcal{S}$ is closed under colimits in Cat. Assume given a diagram $C : \mathcal{I} \to \mathrm{Cat}$ with $C(i)$ in $\mathcal{S}$ for every i in $\mathcal{I}$. Then we have an equivalence

$$\mathrm{Funct}(\mathrm{colim}_{\mathcal{I}}\, C(i), \mathcal{D}) = \lim_{\mathcal{I}^{\mathrm{op}}} \mathrm{Funct}(C(i), \mathcal{D}).$$

A morphism $\eta : F \to G$ in $\mathrm{Funct}(\mathrm{colim}_{\mathcal{I}} C(i), \mathcal{D})$ is left adjointable if and only if its image in $\mathrm{Funct}(C(i), \mathcal{D})$ is left adjointable for every i in $\mathcal{I}$. This happens if and only if the square

$$\begin{tikzcd} F(x) \arrow[r, "\eta(x)"] \arrow[d, "F(\alpha)"] & G(x) \arrow[d, "G(\alpha)"] \\ F(y) \arrow[r, "\eta(y)"] & G(y) \end{tikzcd}$$

is horizontally left adjointable for every arrow $\alpha : x \to y$ in $\mathrm{colim}_{\mathcal{I}} C(i)$ which belongs to the image of the map $C(i) \to \mathrm{colim}_{\mathcal{I}} C(i)$ for some i in $\mathcal{I}$. Observe now that the family of arrows α in $\mathrm{colim}_{\mathcal{I}} C(i)$ for which the above square is horizontally left adjointable is closed under compositions (this follows from example from the characterization of adjointability of squares from remark 10.4.2). We conclude that η is left adjointable if and only if the above square is horizontally left adjointable for every arrow, which implies that $\mathrm{colim}_{\mathcal{I}} C(i)$ also belongs to $\mathcal{S}$, as desired. □

Lemma 10.4.9. *Let $\mathcal{C}$ be a category. Then the composition of the functor*

$$(L_{\mathcal{C}}^{\text{1-op}} \times \mathrm{id}_{\mathcal{C}})^* : \mathrm{Funct}((\mathcal{C}^R)^{\text{1-op}} \times \mathcal{C}, \mathscr{C}\!at)^{\leq 1} \to \mathrm{Funct}(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}, \mathrm{Cat})$$

and the two-sided Grothendieck construction [2]

$$\int_{\mathcal{C}^{\mathrm{op}} \times \mathcal{C}} : \mathrm{Funct}(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}, \mathrm{Cat}) \to \mathrm{Cat}_{/\mathcal{C} \times \mathcal{C}}$$

is a monomorphism, whose image is the category $\mathrm{Cat}_{/\mathcal{C}\times\mathcal{C}}^{\text{cocart,two-sided}} = \mathrm{Cat}_{/\mathcal{C}\times\mathcal{C}}^{\text{cocart}} \cap \mathrm{Cat}_{/\mathcal{C}\times\mathcal{C}}^{\text{two-sided}}$.

Proof. We note that since $L_{\mathcal{C}}$ is an epimorphism, we have that $L_{\mathcal{C}}^{\text{1-op}} \times \mathrm{id}_{\mathcal{C}}$ is also an epimorphism, and hence the induced functor

$$\mathrm{Funct}((\mathcal{C}^R)^{\text{1-op}} \times \mathcal{C}, \mathscr{C}\!at)^{\leq 1} \to \mathrm{Funct}(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}, \mathscr{C}\!at)^{\leq 1} = \mathrm{Funct}(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}, \mathrm{Cat})$$

is a indeed a monomorphism. We note that the above map is equivalent to the map

$$(L_{\mathcal{C}}^{\text{1-op}})^* : \mathrm{Funct}((\mathcal{C}^R)^{\text{1-op}}, \mathrm{Funct}(\mathcal{C}, \mathscr{C}\!at))^{\leq 1} \to \mathrm{Funct}(\mathcal{C}^{\mathrm{op}}, \mathrm{Funct}(\mathcal{C}, \mathrm{Cat})).$$

It follows from lemma 10.4.8 that an object belongs to the image of $(L_{\mathcal{C}}^{\text{1-op}})^*$ if and only if the associated functor $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Cat}$ is left adjointable in the $\mathcal{C}^{\mathrm{op}}$ coordinate. This happens if and only if the associated two-sided fibration is also a cocartesian fibration. We conclude that the lemma holds at the level of objects.

[2] In this chapter we use the convention where two-sided fibrations are cartesian over the first coordinate and cocartesian over the second coordinate, unless otherwise stated. Note that this differs from the convention used in chapter 9.

Consider now the functor

$$\mathrm{Funct}((\mathcal{C}^R)^{1\text{-op}}, \mathrm{Funct}(\mathcal{C} \times [1], \mathscr{C}at))^{\leq 1} \to \mathrm{Funct}(\mathcal{C}^{\mathrm{op}}, \mathrm{Funct}(\mathcal{C} \times [1], \mathrm{Cat}))$$

of precomposition with $L_{\mathcal{C}}^{1\text{-op}}$. Applying lemma 10.4.8 again we conclude that an object belongs to its image if and only if the associated functor $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \times [1] \to \mathrm{Cat}$ is left adjointable in the $\mathcal{C}^{\mathrm{op}}$ coordinate. Applying remark 9.3.19 we see that this happens if and only if the associated morphism in $\mathrm{Cat}^{\text{two-sided}}_{/\mathcal{C}\times\mathcal{C}}$ belongs also to $\mathrm{Cat}^{\mathrm{cocart}}_{/\mathcal{C}\times\mathcal{C}}$. This shows that the lemma holds also at the level of morphisms, which finishes the proof. □

Notation 10.4.10. Let $\mathcal{C}$ be a category admitting pullbacks. Recall the universal vertically right adjointable square $\mathrm{id}_{[1]} \times L : [1] \times [1] \to [1] \times \mathrm{Adj}$ from remark 10.4.4. Let S be the space of cartesian commutative squares in $\mathcal{C}$, and let $\mathrm{ev} : S \times ([1] \times [1]) \to \mathcal{C}$ be the evaluation functor. Let $2\mathrm{Corr}^{\mathrm{univ}}(\mathcal{C})$ be the 2-category defined by the pushout

$$\begin{tikzcd} S \times ([1] \times [1]) \arrow[r, "\mathrm{ev}"] \arrow[d, "\mathrm{id}_S \times (\mathrm{id}_{[1]} \times L)"] & \mathcal{C} \arrow[d, "\iota_{\mathcal{C}}^{\mathrm{univ}}"] \\ S \times ([1] \times \mathrm{Adj}) \arrow[r] & 2\mathrm{Corr}^{\mathrm{univ}}(\mathcal{C}) \end{tikzcd} .$$

The 2-category $2\mathrm{Corr}^{\mathrm{univ}}(\mathcal{C})$ is the universal 2-category equipped with the a functor from $\mathcal{C}$ which satisfies the left Beck-Chevalley condition. The map $\iota_{\mathcal{C}}^{\mathrm{univ}}$ is an epimorphism in 2Cat: for every 2-category $\mathcal{D}$, precomposition with $\iota_{\mathcal{C}}^{\mathrm{univ}}$ induces an equivalence between the space of functors $2\mathrm{Corr}^{\mathrm{univ}}(\mathcal{C}) \to \mathcal{D}$ and the space of functors $\mathcal{C} \to \mathcal{D}$ which satisfy the left Beck-Chevalley condition. It follows in particular that $\iota_{\mathcal{C}}^{\mathrm{univ}}$ factors through $\mathcal{C}^R$. We denote by $q : \mathcal{C}^R \to 2\mathrm{Corr}^{\mathrm{univ}}(\mathcal{C})$ the induced functor.

Lemma 10.4.11. *Let $\mathcal{C}$ be a category admitting pullbacks and let $\mathcal{D}$ be a 2-category. The functor*

$$q^* : \mathrm{Funct}(2\mathrm{Corr}^{\mathrm{univ}}(\mathcal{C}), \mathcal{D}) \to \mathrm{Funct}(\mathcal{C}^R, \mathcal{D})$$

induces equivalences at the level of Hom categories.

Proof. Observe that $2\mathrm{Corr}^{\mathrm{univ}}(\mathcal{C})$ is obtained out of $\mathcal{C}^R$ by inverting a family of 2-cells. It therefore suffices to show that if $q' : C_2 \to [1]$ is the projection from the walking 2-cell to the walking 1-cell, the induced functor

$$q'^* : \mathrm{Funct}([1], \mathcal{D}) \to \mathrm{Funct}(C_2, \mathcal{D})$$

induces equivalences at the level of Hom categories.

Let $\alpha, \beta : [1] \to \mathcal{D}$ be a pair of objects of $\mathrm{Funct}([1], \mathcal{D})$. We have a commutative cube of categories as follows:

$$\begin{tikzcd}
& \mathrm{Hom}_{\mathrm{Funct}(C_2,\mathcal{D})}(q'^*\alpha, q'^*\beta) \arrow[rr, "\mathrm{ev}_0"] \arrow[dd, "\mathrm{ev}_1" near start] & & \mathrm{Hom}_{\mathcal{D}}(\alpha(0), \beta(0)) \arrow[dd, "q'^*\beta"] \\
\mathrm{Hom}_{\mathrm{Funct}([1],\mathcal{D})}(\alpha, \beta) \arrow[ur, "q'^*"] \arrow[rr, "\mathrm{ev}_0" near end] \arrow[dd, "\mathrm{ev}_1"] & & \mathrm{Hom}_{\mathcal{D}}(\alpha(0), \beta(0)) \arrow[ur, "\mathrm{id}"] \arrow[dd, "\beta" near start] & \\
& \mathrm{Hom}_{\mathcal{D}}(\alpha(1), \beta(1)) \arrow[rr, "q'^*\alpha" near start] & & \mathrm{Funct}([1], \mathrm{Hom}_{\mathcal{D}}(\alpha(0), \beta(1))) \\
\mathrm{Hom}_{\mathcal{D}}(\alpha(1), \beta(1)) \arrow[ur, "\mathrm{id}"] \arrow[rr, "\alpha"] & & \mathrm{Hom}_{\mathcal{D}}(\alpha(0), \beta(1)) \arrow[ur] &
\end{tikzcd}$$

Here the front and back faces are cartesian. The bottom left and top right diagonal arrows are isomorphisms, and the bottom right diagonal arrow is the degeneracy map, which is fully faithful. It follows that the top left diagonal arrow is an isomorphism, as desired. □

Lemma 10.4.12. *Let $\mathcal{J}$ and $\mathcal{D}$ be 2-categories. Then a 2-cell in $\mathrm{Funct}(\mathcal{J}, \mathcal{D})$ is invertible if and only if its image under all evaluation functors is invertible.*

Proof. Let $\mathcal{S}$ be the full subcategory of 2Cat on those 2-categories $\mathcal{J}$ for which the lemma holds. We claim that $\mathcal{S}$ is closed under colimits in 2Cat. Assume given a diagram $J : \mathcal{I} \to \mathrm{2Cat}$, with $J(i)$ in $\mathcal{S}$ for every i in $\mathcal{I}$. Then we have an equivalence

$$\mathrm{Funct}(\mathrm{colim}_{\mathcal{I}}\, J(i), \mathcal{D}) = \lim_{\mathcal{I}^{\mathrm{op}}} \mathrm{Funct}(J(i), \mathcal{D}).$$

It follows that a 2-cell $\gamma : C_2 \to \mathrm{Funct}(\mathrm{colim}_{\mathcal{I}}\, J(i), \mathcal{D})$ is invertible if and only if its image in $\mathrm{Funct}(J(i), \mathcal{D})$ is invertible for all i in $\mathcal{I}$. Since $J(i)$ is assumed to belong to $\mathcal{S}$, this happens if and only if the 2-cell $\mathrm{ev}_j\, \gamma$ is invertible for all objects j in $\mathrm{colim}_{\mathcal{I}}\, J(i)$, which means that $\mathrm{colim}_{\mathcal{I}}\, J(i)$ also belongs to $\mathcal{S}$.

To prove the lemma it then suffices to show that $\mathcal{S}$ contains the walking 2-cell C_2. In other words, we have reduced to proving the lemma in the case $\mathcal{J} = C_2$. In this case, for every pair of objects μ, ν in $\mathrm{Funct}(C_2, \mathcal{D})$, we have a pullback diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{Funct}(C_2,\mathcal{D})}(\mu, \nu) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{D}}(\mu(0), \nu(0)) \arrow[d] \\
\mathrm{Hom}_{\mathcal{D}}(\mu(1), \nu(1)) \arrow[r] & \mathrm{Funct}([1], \mathrm{Hom}_{\mathcal{D}}(\mu(0), \nu(1)))
\end{tikzcd}$$

where the top and left arrows are the evaluation functors. A 2-cell in $\mathrm{Funct}(C_2, \mathcal{D})$ corresponds to an arrow in $\mathrm{Hom}_{\mathrm{Funct}(C_2,\mathcal{D})}(\mu, \nu)$, and this is invertible if and only if its image in $\mathrm{Hom}_{\mathcal{D}}(\mu(0), \nu(0))$ and $\mathrm{Hom}_{\mathcal{D}}(\mu(1), \nu(1))$ is invertible, as desired. □

Lemma 10.4.13. *Let $\mathcal{J}, \mathcal{D}$ be 2-categories, and let $\mathcal{C}$ be a category admitting pullbacks. Then a functor $F : \mathcal{C}^R \to \mathrm{Funct}(\mathcal{J}, \mathcal{D})$ extends to $2\mathrm{Corr}^{\mathrm{univ}}(\mathcal{C})$ if and only if for every object j in $\mathcal{J}$ the composite functor $\mathrm{ev}_j F : \mathcal{C}^R \to \mathcal{D}$ extends to $2\mathrm{Corr}^{\mathrm{univ}}(\mathcal{C})$.*

Proof. This follows directly from lemma 10.4.12 using the fact that $2\mathrm{Corr}^{\mathrm{univ}}(\mathcal{C})$ is obtained from $\mathcal{C}^R$ by inverting a family of 2-cells. □

Lemma 10.4.14. *Let $\mathcal{C}$ be a category admitting pullbacks. Then the composition of the functor*

$$((\iota_{\mathcal{C}}^{\mathrm{univ}})^{1\text{-op}} \times \mathrm{id}_{\mathcal{C}})^* : \mathrm{Funct}(2\mathrm{Corr}^{\mathrm{univ}}(\mathcal{C})^{1\text{-op}} \times \mathcal{C}, \mathscr{C}at)^{\leq 1} \to \mathrm{Funct}(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}, \mathrm{Cat})$$

and the two-sided Grothendieck construction

$$\int_{\mathcal{C}^{\mathrm{op}} \times \mathcal{C}} : \mathrm{Funct}(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}, \mathrm{Cat}) \to \mathrm{Cat}_{/\mathcal{C} \times \mathcal{C}}$$

is a monomorphism, whose image is the full subcategory of $\mathrm{Cat}_{/\mathcal{C}\times\mathcal{C}}^{\mathrm{cocart,two\text{-}sided}}$ on those fibrations which satisfy the Beck-Chevalley condition in the first coordinate.

Proof. The functor $q : \mathcal{C}^R \to 2\mathrm{Corr}^{\mathrm{univ}}(\mathcal{C})$ is an epimorphism, so we have a monomorphism

$$(q^{1\text{-op}} \times \mathrm{id}_{\mathcal{C}})^* : \mathrm{Funct}(2\mathrm{Corr}^{\mathrm{univ}}(\mathcal{C})^{1\text{-op}} \times \mathcal{C}, \mathscr{C}at)^{\leq 1} \to \mathrm{Funct}((\mathcal{C}^R)^{1\text{-op}} \times \mathcal{C}, \mathscr{C}at)^{\leq 1}.$$

The above is equivalent to the functor

$$\mathrm{Funct}(2\mathrm{Corr}^{\mathrm{univ}}(\mathcal{C})^{1\text{-op}}, \mathrm{Funct}(\mathcal{C}, \mathscr{C}at))^{\leq 1} \to \mathrm{Funct}((\mathcal{C}^R)^{1\text{-op}}, \mathrm{Funct}(\mathcal{C}, \mathscr{C}at))^{\leq 1}$$

obtained by precomposition with $q^{1\text{-op}}$. Applying lemma 10.4.11 we see that the above functor is fully faithful. Combining this fact with lemma 10.4.9 we conclude that precomposition with $(\iota_{\mathcal{C}}^{\mathrm{univ}})^{1\text{-op}} \times \mathrm{id}_{\mathcal{C}}$ followed by the two-sided Grothendieck construction embeds

$$\mathrm{Funct}(2\mathrm{Corr}^{\mathrm{univ}}(\mathcal{C})^{1\text{-op}} \times \mathcal{C}, \mathscr{C}at)^{\leq 1}$$

as a full subcategory of $\mathrm{Cat}_{/\mathcal{C}\times\mathcal{C}}^{\mathrm{cocart,two\text{-}sided}}$. Thanks to lemma 10.4.13, its image consists of those fibrations which satisfy the Beck-Chevalley condition in the first variable, as desired. □

The next three lemmas use the theory of algebroids and enriched categories as developed in [GH15] and [Hin20a], and in particular the approach to the Yoneda embedding via diagonal bimodules from [Hin20a]. We refer the reader to chapter 3 for our conventions regarding this subject.

Lemma 10.4.15. *Let $\mathcal{M}$ be a presentable monoidal category and let $f : \mathcal{A} \to \mathcal{B}$ be a morphism of $\mathcal{M}$-algebroids which is an equivalence on categories of objects. Let ${}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}$ be the diagonal $\mathcal{A}$-bimodule, and let ${}_{\mathcal{A}}\mathcal{B}_{\mathcal{B}}$ be the restriction of scalars of the diagonal $\mathcal{B}$-bimodule along the map $(f, \mathrm{id}_{\mathcal{B}})$. Then the induced morphism $f_* : {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} \to {}_{\mathcal{A}}\mathcal{B}_{\mathcal{B}}$ in $\mathrm{BMod}(\mathcal{M})$ is a cocartesian lift of the morphism $(\mathrm{id}_{\mathcal{A}}, f) : (\mathcal{A}, \mathcal{A}) \to (\mathcal{A}, \mathcal{B})$ in $\mathrm{Algbrd}(\mathcal{M}) \times \mathrm{Algbrd}(\mathcal{M})$.*

Proof. Let X be the category of objects of $\mathcal{A}$ and $\mathcal{B}$. By construction, the projection $p : \mathrm{BMod}(\mathcal{M}) \to \mathrm{Algbrd}(\mathcal{M}) \times \mathrm{Algbrd}(\mathcal{M})$ is a morphism of cartesian fibrations over $\mathrm{Cat} \times \mathrm{Cat}$. Thanks to [Lur09a] corollary 4.3.1.15, to show that f_* is p-cocartesian it suffices to show that it is cocartesian for the projection

$$p_{X,X} : \mathrm{BMod}_{X,X}(\mathcal{M}) \to \mathrm{Algbrd}_X(\mathcal{M}) \times \mathrm{Algbrd}_X(\mathcal{M}).$$

Recall from [Hin20a] proposition 3.3.6 that Assos_X is a flat associative operad. It follows that there is an associative operad $\mathcal{M}_X$ equipped with the universal map of associative operads $\mathrm{Assos}_X \times \mathcal{M}_X \to \mathcal{M}$. As discussed in [Hin20a] corollary 4.4.9, the associative operad $\mathcal{M}_X$ is a presentable monoidal category. The projection $p_{X,X}$ is equivalent to the projection

$$p'_{X,X} : \mathrm{Alg}_{\mathrm{BM}}(\mathcal{M}_X) \to \mathrm{Alg}_{\mathrm{Assos}}(\mathcal{M}_X) \times \mathrm{Alg}_{\mathrm{Assos}}(\mathcal{M}_X)$$

which sends each BM-algebra in $\mathcal{M}_X$ to its underlying associative algebras.

Let $\widetilde{\mathcal{A}}$ and $\widetilde{\mathcal{B}}$ be the associative algebras corresponding to the $\mathcal{M}$-algebroids $\mathcal{A}$ and $\mathcal{B}$,respectively. Then under the above equivalence, the map f_* corresponds to the map of bimodules $\widetilde{f_*} : {}_{\widetilde{\mathcal{A}}}\widetilde{\mathcal{A}}_{\widetilde{\mathcal{A}}} \to {}_{\widetilde{\mathcal{A}}}\widetilde{\mathcal{B}}_{\widetilde{\mathcal{B}}}$, which is $p'_{X,X}$-cocartesian, as desired. □

Lemma 10.4.16. *Let $\mathcal{M}$ be a presentable symmetric monoidal category and let $f : \mathcal{A} \to \mathcal{B}$ be a morphism of $\mathcal{M}$-algebroids which is an equivalence on categories of objects. Consider the morphisms*

$$\mathcal{H}_{\mathcal{A}} = \mathcal{A}(-,-) : \mathcal{A} \otimes \mathcal{A}^{\mathrm{op}} \to \overline{\mathcal{M}}$$

and

$$\mathcal{H}_{\mathcal{B}} = \mathcal{B}(-,-) : \mathcal{B} \otimes \mathcal{B}^{\mathrm{op}} \to \overline{\mathcal{M}}$$

where $\overline{\mathcal{M}}$ denotes the enhancement of $\mathcal{M}$ to an $\mathcal{M}$-enriched category. Let $f_* : \mathcal{H}_{\mathcal{A}} \to \mathcal{H}_{\mathcal{B}}|_{\mathcal{A}\otimes\mathcal{A}^{\mathrm{op}}}$ be the induced map. Then for every object G in $\mathrm{Funct}(\mathcal{A} \otimes \mathcal{B}^{\mathrm{op}}, \overline{\mathcal{M}})$, the composition of the restriction map

$$\tau_{\mathcal{M}} \mathrm{Hom}_{\mathrm{Funct}(\mathcal{A}\otimes\mathcal{B}^{\mathrm{op}},\overline{\mathcal{M}})}(\mathcal{H}_{\mathcal{B}}|_{\mathcal{A}\otimes\mathcal{B}^{\mathrm{op}}}, G) \to \tau_{\mathcal{M}} \mathrm{Hom}_{\mathrm{Funct}(\mathcal{A}\otimes\mathcal{A}^{\mathrm{op}},\overline{\mathcal{M}})}(\mathcal{H}_{\mathcal{B}}|_{\mathcal{A}\otimes\mathcal{A}^{\mathrm{op}}}, G|_{\mathcal{A}\otimes\mathcal{A}^{\mathrm{op}}})$$

with the precomposition with f_* map

$$\tau_{\mathcal{M}} \mathrm{Hom}_{\mathrm{Funct}(\mathcal{A}\otimes\mathcal{A}^{\mathrm{op}},\overline{\mathcal{M}})}(\mathcal{H}_{\mathcal{B}}|_{\mathcal{A}\otimes\mathcal{A}^{\mathrm{op}}}, G|_{\mathcal{A}\otimes\mathcal{A}^{\mathrm{op}}}) \to \tau_{\mathcal{M}} \mathrm{Hom}_{\mathrm{Funct}(\mathcal{A}\otimes\mathcal{A}^{\mathrm{op}},\overline{\mathcal{M}})}(\mathcal{H}_{\mathcal{A}}, G|_{\mathcal{A}\otimes\mathcal{A}^{\mathrm{op}}})$$

is an equivalence.

Proof. This is a translation of lemma 10.4.15 under the folding equivalence of [Hin20a] section 3.6, and the equivalence of [Hin20a] proposition 6.3.7. □

Lemma 10.4.17. *Let $\mathcal{C}$ be a category admitting pullbacks.*

(i) *The map $\iota^{\mathrm{univ}}_{\mathcal{C}}$ is surjective on objects.*

(ii) *The image of the natural transformation*

$$(\iota_{\mathcal{C}}^{\mathrm{univ}})_* : \mathrm{Hom}_{\mathcal{C}}(-,-) \to \mathrm{Hom}_{2\mathrm{Corr}^{\mathrm{univ}}(\mathcal{C})}(-,-)|_{\mathcal{C}^{\mathrm{op}}\times\mathcal{C}}$$

under the two-sided Grothendieck construction

$$\int_{\mathcal{C}^{\mathrm{op}}\times\mathcal{C}} : \mathrm{Funct}(\mathcal{C}^{\mathrm{op}}\times\mathcal{C}, \mathrm{Cat}) \to \mathrm{Cat}_{/\mathcal{C}\times\mathcal{C}}$$

is equivalent to the morphism of two-sided fibrations over $\mathcal{C}\times\mathcal{C}$

$$\begin{tikzcd} \mathrm{Funct}([1],\mathcal{C}) \arrow[rr, "\phi'"] \arrow[dr, "(\mathrm{ev}_0,\mathrm{ev}_1)"'] & & \mathrm{Funct}(\Lambda^2_0,\mathcal{C}) \arrow[dl, "(\mathrm{ev}_1,\mathrm{ev}_2)"] \\ & \mathcal{C}\times\mathcal{C} & \end{tikzcd}$$

where Λ^2_0 *is the category with objects* $0,1,2$ *and nontrivial morphisms* $1 \leftarrow 0 \rightarrow 2$*, and* ϕ' *is the functor of precomposition with the map* $\Lambda^2_0 \to [1]$ *which sends* $0,1,2$ *to* $0,0,1$*, respectively.*

Proof. Let S be the space of cartesian commutative squares in $\mathcal{C}$, and let $\mathrm{ev} : S\times([1]\times[1]) \to \mathcal{C}$ be the evaluation functor. Let $2\mathrm{Corr}^{\mathrm{univ}}_{\mathrm{algbrd}}(\mathcal{C})$ be the Cat-algebroid defined as the pushout

$$\begin{tikzcd} S\times([1]\times[1]) \arrow[r, "\mathrm{ev}"] \arrow[d, "\mathrm{id}_S\times(\mathrm{id}_{[1]}\times L)"] & \mathcal{C} \arrow[d, "\iota^{\mathrm{univ}}_{\mathcal{C},\mathrm{algbrd}}"] \\ S\times([1]\times\mathrm{Adj}) \arrow[r] & 2\mathrm{Corr}^{\mathrm{univ}}_{\mathrm{algbrd}}(\mathcal{C}) \end{tikzcd}$$

inside Algbrd(Cat). Note that the image of $\iota^{\mathrm{univ}}_{\mathcal{C},\mathrm{algbrd}}$ under the projection $\mathrm{Algbrd}(\mathrm{Cat})_{\mathrm{Spc}} \to 2\mathrm{Cat}$ recovers $\iota^{\mathrm{univ}}_{\mathcal{C}}$. It follows from proposition 3.3.12 that the map of algebroids $\iota^{\mathrm{univ}}_{\mathcal{C},\mathrm{algbrd}}$ is an equivalence at the level of objects. Item (i) now follows from this together with the fact that the map from any algebroid to its completion is surjective on objects.

Denote by $j : 2\mathrm{Corr}^{\mathrm{univ}}_{\mathrm{algbrd}}(\mathcal{C}) \to 2\mathrm{Corr}^{\mathrm{univ}}(\mathcal{C})$ the canonical map. As shown in [GH15] corollary 5.6.3, the map j is fully faithful. In other words, the natural transformation

$$j_* : \mathrm{Hom}_{2\mathrm{Corr}^{\mathrm{univ}}_{\mathrm{algbrd}}(\mathcal{C})}(-,-) \to \mathrm{Hom}_{2\mathrm{Corr}^{\mathrm{univ}}(\mathcal{C})}(-,-)|_{2\mathrm{Corr}^{\mathrm{univ}}_{\mathrm{algbrd}}(\mathcal{C})^{1\text{-op}}\times 2\mathrm{Corr}^{\mathrm{univ}}_{\mathrm{algbrd}}(\mathcal{C})}$$

is an equivalence. It follows that the image of the transformation

$$(\iota_{\mathcal{C}}^{\mathrm{univ}})_* : \mathrm{Hom}_{\mathcal{C}}(-,-) \to \mathrm{Hom}_{2\mathrm{Corr}^{\mathrm{univ}}(\mathcal{C})}(-,-)|_{\mathcal{C}^{\mathrm{op}}\times\mathcal{C}}$$

under the two-sided Grothendieck construction is equivalent to the image of

$$(\iota_{\mathcal{C},\mathrm{algbrd}}^{\mathrm{univ}})_* : \mathrm{Hom}_{\mathcal{C}}(-,-) \to \mathrm{Hom}_{2\mathrm{Corr}^{\mathrm{univ}}_{\mathrm{algbrd}}(\mathcal{C})}(-,-)|_{\mathcal{C}^{\mathrm{op}}\times\mathcal{C}}.$$

The fact that $\mathscr{C}at$ is complete as a Cat-algebroid implies that the restriction map

$$\mathrm{Funct}(2\mathrm{Corr}^{\mathrm{univ}}(\mathcal{C})^{1\text{-op}} \times \mathcal{C}, \mathscr{C}at) \to \mathrm{Funct}(2\mathrm{Corr}^{\mathrm{univ}}_{\mathrm{algbrd}}(\mathcal{C})^{1\text{-op}} \times \mathcal{C}, \mathscr{C}at)$$

is an equivalence, and therefore the conclusion of lemma 10.4.14 remains valid if we replace $2\mathrm{Corr}^{\mathrm{univ}}(\mathcal{C})$ with $2\mathrm{Corr}^{\mathrm{univ}}_{\mathrm{algbrd}}(\mathcal{C})$. Item (ii) now follows this together with a combination of lemma 10.4.16 and proposition 9.4.8. $\square$

Theorem 10.4.18. *Let $\mathcal{C}$ be a category admitting pullbacks, and $\mathcal{D}$ be a 2-category. Precomposition with the functor $\iota_{\mathcal{C}} : \mathcal{C} \to 2\mathrm{Corr}(\mathcal{C})$ induces an equivalence between the space $\mathrm{Hom}_{2\mathrm{Cat}}(2\mathrm{Corr}(\mathcal{C}), \mathcal{D})$ and the subspace of $\mathrm{Hom}_{2\mathrm{Cat}}(\mathcal{C}, \mathcal{D})$ consisting of those functors $F : \mathcal{C} \to \mathcal{D}$ which satisfy the left Beck-Chevalley condition.*

Proof. By virtue of proposition 10.4.6 the functor $\iota_{\mathcal{C}}$ factors through $2\mathrm{Corr}^{\mathrm{univ}}(\mathcal{C})$. Our goal is to show that the resulting functor $Q : 2\mathrm{Corr}^{\mathrm{univ}}(\mathcal{C}) \to 2\mathrm{Corr}(\mathcal{C})$ is an equivalence. Thanks to item (i) in lemma 10.4.17 it suffices show that for every pair of objects c, c' in $\mathcal{C}$, the induced functor

$$Q_* : \mathrm{Hom}_{2\mathrm{Corr}^{\mathrm{univ}}(\mathcal{C})}(\iota^{\mathrm{univ}}_{\mathcal{C}}(c), \iota^{\mathrm{univ}}_{\mathcal{C}}(c')) \to \mathrm{Hom}_{2\mathrm{Corr}(\mathcal{C})}(\iota_{\mathcal{C}}(c), \iota_{\mathcal{C}}(c'))$$

is an equivalence.

Denote by $\mathcal{R}$ the equivalence of item (ii) in lemma 10.4.17 between the two-sided fibration associated to the functor $\mathrm{Hom}_{2\mathrm{Corr}^{\mathrm{univ}}(\mathcal{C})}(-,-)|_{\mathcal{C}^{\mathrm{op}} \times \mathcal{C}}$ and $\mathrm{Funct}(\Lambda^2_0, \mathcal{C})$. In particular $\mathcal{R}$ gives for every pair of objects c, c' in $\mathcal{C}$ an isomorphism

$$\mathrm{Funct}(\Lambda^2_0, \mathcal{C})_{(c,c')} \xrightarrow{=} \mathrm{Hom}_{2\mathrm{Corr}^{\mathrm{univ}}(\mathcal{C})}(\iota^{\mathrm{univ}}_{\mathcal{C}}(c), \iota^{\mathrm{univ}}_{\mathcal{C}}(c')).$$

Note that the left hand side is also equivalent to $\mathrm{Hom}_{2\mathrm{Corr}(\mathcal{C})}(\iota_{\mathcal{C}}(c), \iota_{\mathcal{C}}(c'))$, so we have a (a priori non necessarily commutative) diagram of categories

$$\begin{tikzcd}
\mathrm{Hom}_{2\mathrm{Corr}^{\mathrm{univ}}(\mathcal{C})}(\iota^{\mathrm{univ}}_{\mathcal{C}}(c), \iota^{\mathrm{univ}}_{\mathcal{C}}(c')) \arrow[rr, "Q_*"] \arrow[dr, "="'] & & \mathrm{Hom}_{2\mathrm{Corr}(\mathcal{C})}(\iota_{\mathcal{C}}(c), \iota_{\mathcal{C}}(c')) \arrow[dl, "="] \\
& \mathrm{Funct}(\Lambda^2_0, \mathcal{C})_{(c,c')} &
\end{tikzcd}.$$

Since Cat is generated by the walking arrow, to show that Q_* is an equivalence, it suffices to show that the diagram of spaces obtained from the above by applying the functor $\mathrm{Hom}_{\mathrm{Cat}}([1], -) : \mathrm{Cat} \to \mathrm{Spc}$, can be made commutative. In other words, we have to show that Q_* is compatible with both equivalences above at the level of arrows.

Consider a morphism of spans $\eta : T \to S$ depicted as follows.

$$\begin{tikzcd}
& t \arrow[d, "\mu"] \arrow[ddl, "\alpha'"'] \arrow[ddr, "\beta'"] & \\
& s \arrow[dl, "\alpha"] \arrow[dr, "\beta"'] & \\
c & & c'
\end{tikzcd}$$

Recall that the projection $(\mathrm{ev}_1, \mathrm{ev}_2) : \mathrm{Funct}(\Lambda^2_0, \mathcal{C}) \to \mathcal{C} \times \mathcal{C}$ is both cartesian and cocartesian. Passing to cartesian lifts of α, α', μ and cocartesian lifts of μ, β, β' yields the following diagram of categories:

$$\begin{tikzcd}
 & \mathrm{Funct}(\Lambda^2_0, \mathcal{C})_{(c,t)} \arrow[d, shift left, "(\mathrm{id}_c,\mu)_*"] \arrow[drr, "(\mathrm{id}_c,\beta')_*"] & & \\
 & \mathrm{Funct}(\Lambda^2_0, \mathcal{C})_{(c,s)} \arrow[u, shift left, "(\mathrm{id}_c,\mu)^*"] \arrow[dr, "(\mathrm{id}_c,\beta)_*"] & & \\
\mathrm{Funct}(\Lambda^2_0, \mathcal{C})_{(c,c)} \arrow[uur, "(\mathrm{id}_c,\alpha')^*"] \arrow[ur, "(\mathrm{id}_c,\alpha)^*"] & & \mathrm{Funct}(\Lambda^2_0, \mathcal{C})_{(c,c')} &
\end{tikzcd}$$

Here $(\mathrm{id}_c, \mu)^*$ is right adjoint to $(\mathrm{id}_c, \mu)_*$. The morphism of spans η is the image of the identity span of c under the natural transformation

$$(\mathrm{id}_c, \beta')_*(\mathrm{id}_c, \alpha')^* = (\mathrm{id}_c, \beta)_*(\mathrm{id}_c, \mu)_*(\mathrm{id}_c, \mu)^*(\mathrm{id}_c, \alpha)^* \to (\mathrm{id}_c, \beta)_*(\mathrm{id}_c, \alpha)^*$$

induced by the adjunction $(\mathrm{id}_c, \mu)_* \dashv (\mathrm{id}_c, \mu)^*$. Under the equivalence $\mathcal{R}$, the above diagram becomes

$$\begin{tikzcd}
 & \mathrm{Hom}_{2\mathrm{Corr}^{\mathrm{univ}}(\mathcal{C})}(\iota^{\mathrm{univ}}_{\mathcal{C}}(c), \iota^{\mathrm{univ}}_{\mathcal{C}}(t)) \arrow[d, shift left, "\iota^{\mathrm{univ}}_{\mathcal{C}}(\mu)_*"] \arrow[drr, bend left, "\iota^{\mathrm{univ}}_{\mathcal{C}}(\beta')_*"] & & \\
 & \mathrm{Hom}_{2\mathrm{Corr}^{\mathrm{univ}}(\mathcal{C})}(\iota^{\mathrm{univ}}_{\mathcal{C}}(c), \iota^{\mathrm{univ}}_{\mathcal{C}}(s)) \arrow[u, shift left, "\iota^{\mathrm{univ}}_{\mathcal{C}}(\mu)^R_*"] \arrow[dr, "\iota^{\mathrm{univ}}_{\mathcal{C}}(\beta)_*"] & & \\
\mathrm{Hom}_{2\mathrm{Corr}^{\mathrm{univ}}(\mathcal{C})}(\iota^{\mathrm{univ}}_{\mathcal{C}}(c), \iota^{\mathrm{univ}}_{\mathcal{C}}(c)) \arrow[uur, bend left, "\iota^{\mathrm{univ}}_{\mathcal{C}}(\alpha')^R_*"] \arrow[ur, "\iota^{\mathrm{univ}}_{\mathcal{C}}(\alpha)^R_*"] & & \mathrm{Hom}_{2\mathrm{Corr}^{\mathrm{univ}}(\mathcal{C})}(\iota^{\mathrm{univ}}_{\mathcal{C}}(c), \iota^{\mathrm{univ}}_{\mathcal{C}}(c')). &
\end{tikzcd}$$

Furthermore, the identity span of c becomes $\mathrm{id}_{\iota^{\mathrm{univ}}_{\mathcal{C}}(c)}$. We thus see that η corresponds, under the equivalence $\mathcal{R}$, to the morphism

$$\iota^{\mathrm{univ}}_{\mathcal{C}}(\beta')\iota^{\mathrm{univ}}_{\mathcal{C}}(\alpha')^R = \iota^{\mathrm{univ}}_{\mathcal{C}}(\beta)\iota^{\mathrm{univ}}_{\mathcal{C}}(\mu)\iota^{\mathrm{univ}}_{\mathcal{C}}(\mu)^R\iota^{\mathrm{univ}}_{\mathcal{C}}(\alpha)^R \to \iota^{\mathrm{univ}}_{\mathcal{C}}(\beta)\iota^{\mathrm{univ}}_{\mathcal{C}}(\alpha)^R$$

induced by the counit of the adjunction $\iota^{\mathrm{univ}}_{\mathcal{C}}(\mu) \dashv \iota^{\mathrm{univ}}_{\mathcal{C}}(\mu)^R$. Applying the functor Q recovers the morphism

$$\iota_{\mathcal{C}}(\beta')\iota_{\mathcal{C}}(\alpha')^R = \iota_{\mathcal{C}}(\beta)\iota_{\mathcal{C}}(\mu)\iota_{\mathcal{C}}(\mu)^R\iota_{\mathcal{C}}(\alpha)^R \to \iota_{\mathcal{C}}(\beta)\iota_{\mathcal{C}}(\alpha)^R$$

induced by the counit of the adjunction $\iota_{\mathcal{C}}(\mu) \dashv \iota_{\mathcal{C}}(\mu)^R$. This agrees with the image of η under the usual isomorphism $\mathrm{Hom}_{2\mathrm{Corr}(\mathcal{C})}(\iota_{\mathcal{C}}(c), \iota_{\mathcal{C}}(c')) = \mathrm{Funct}(\Lambda^2_0, \mathcal{C})_{(c,c')}$, as we wanted. $\square$

Chapter 11

Higher categories of correspondences

Let $\mathcal{C}$ be a category admitting pullbacks. For each $n \geq 2$ one can construct an n-category $n\mathrm{Corr}(\mathcal{C})$ called the n-category of correspondences of $\mathcal{C}$. The case $n = 2$ of this construction was the focus of chapter 10. For $n > 2$ the n-category $n\mathrm{Corr}(\mathcal{C})$ is defined inductively so that its objects agree with the objects of $\mathcal{C}$, and for each pair of objects c, c' in $\mathcal{C}$, the hom $(n-1)$-category between them is $(n-1)\mathrm{Corr}(\mathrm{Hom}_{2\mathrm{Corr}(\mathcal{C})}(c, c'))$. Our goal in this chapter is to review the definition and main properties of the n-category of correspondences, and to establish two results (theorems 11.2.6 and 11.3.9) that provide ways of constructing functors from $n\mathrm{Corr}(\mathcal{C})$ into a target n-category $\mathcal{D}$.

We begin in 11.1 by reviewing the definition of the n-category of correspondences. Here we depart from previous approaches in the literature: rather than defining $n\mathrm{Corr}(\mathcal{C})$ as an $(n-1)$-fold simplicial category (as in [Hau18]), we use the language of enriched category theory to make sense of the fact that $n\mathrm{Corr}(\mathcal{C})$ is defined by applying the functor $(n-1)\mathrm{Corr}$ at the level of hom categories on $2\mathrm{Corr}(\mathcal{C})$. We also show that $n\mathrm{Corr}(\mathcal{C})$ enjoys strong adjointness properties that extend those of $2\mathrm{Corr}(\mathcal{C})$: every k-cell in $n\mathrm{Corr}(\mathcal{C})$ with $k < n-1$ is both left and right adjointable, and its left and right adjoints coincide. In particular, if $\mathcal{C}$ has a symmetric monoidal structure which is compatible with pullbacks, every object of $n\mathrm{Corr}(\mathcal{C})$ is fully dualizable in the $(n-1)$-category underlying $n\mathrm{Corr}(\mathcal{C})$.

We are ultimately interested in constructing functors out of $n\mathrm{Corr}(\mathcal{C})$: these are higher sheaf theories on $\mathcal{C}$. In the case $n = 2$, a way of constructing functors is provided by theorem 10.4.18: in order to construct a functor out of $2\mathrm{Corr}(\mathcal{C})$ it suffices to construct a functor out of $\mathcal{C}$ which satisfies the left Beck-Chevalley condition. In 11.2 we introduce higher analogs of the Beck-Chevalley condition. While the ordinary Beck-Chevalley condition for a functor $F : \mathcal{C} \to \mathcal{D}$ involves the adjointability of certain commutative squares in $\mathcal{D}$, the higher Beck-Chevalley condition is an inductively defined criterion that involves the adjointability of certain squares in $\mathcal{D}$, together with the adjointability of certain squares in $\mathrm{End}_{\mathcal{D}}(d)$ for objects d in the image of F, together with the adjointability of certain squares in $\mathrm{End}_{\mathrm{End}_{\mathcal{D}}(d)}(\mathrm{id}_d)$ for all such d, and so on.

The first main result of this chapter is theorem 11.2.6, which states that $n\mathrm{Corr}(\mathcal{C})$ is the universal n-category equipped with a functor from $\mathcal{C}$ satisfying the left $(n-1)$-fold Beck-

Chevalley condition. We also show that in the presence of a symmetric monoidal structure on a functor $F : \mathcal{C} \to \mathcal{D}$ which satisfies the left $(n-1)$-fold Beck-Chevalley condition, its extension to $n\mathrm{Corr}(\mathcal{C})$ also comes equipped with a symmetric monoidal structure.

Given a functor $F : \mathcal{C} \to \mathcal{D}$ into an n-category with colimits which is known to satisfy a higher Beck-Chevalley condition, one is sometimes interested in understanding whether the left Kan extension of F to the presheaf category $\mathcal{P}(\mathcal{C})$ also satisfies a higher Beck-Chevalley condition. This is instrumental in the study of higher sheaf theories in algebraic geometry, as these start out life as functors on the category of affine schemes which are then extended to prestacks. In 11.3 we apply the theory of conical colimits from chapter 5 to establish the second main result of this chapter, theorem 11.3.9: under appropriate conditions on $\mathcal{D}$, if a functor $F : \mathcal{C} \to \mathcal{D}$ is such that F and $F^{n\text{-op}}$ satisfy the $(n-1)$-fold Beck-Chevalley condition, then the left Kan extension $F' : \mathcal{P}(\mathcal{C}) \to \mathcal{D}$ satisfies the left $(n-1)$-fold Beck-Chevalley condition. These conditions are for instance verified in the case when $\mathcal{D}$ underlies a presentable n-category (see chapter 12).

11.1 The n-category of correspondences

We begin by giving a construction of the n-category of correspondences of a category with pullbacks. We will use the language of enriched categories as developed in [GH15] and [Hin20a] - we refer the reader to chapter 3 for our conventions.

Notation 11.1.1. For each $n \geq 1$ we denote by $n\mathrm{Cat}_{\mathrm{pb}}$ the category $(n-1)\mathrm{Cat}^{\mathrm{Cat}_{\mathrm{pb}}}$ of $(n-1)$-categories enriched in the cartesian symmetric monoidal category $\mathrm{Cat}_{\mathrm{pb}}$.

Remark 11.1.2. The inclusion $\mathrm{Cat}_{\mathrm{pb}} \to \mathrm{Cat}$ induces an inclusion $n\mathrm{Cat}_{\mathrm{pb}} \to n\mathrm{Cat}$ for each $n \geq 1$. The fact that the inclusion $\mathrm{Cat}_{\mathrm{pb}} \to \mathrm{Cat}$ creates limits implies that inclusion $n\mathrm{Cat}_{\mathrm{pb}} \to \mathrm{Cat}$ creates limits as well for all $n \geq 1$. Its image can be characterized inductively for $n \geq 2$ as follows:

- An n-category $\mathcal{C}$ belongs to $n\mathrm{Cat}_{\mathrm{pb}}$ if and only if for every pair of objects c, c' in $\mathcal{C}$ the $(n-1)$-category $\mathrm{Hom}_{\mathcal{C}}(c, c')$ belongs to $(n-1)\mathrm{Cat}_{\mathrm{pb}}$, and for every triple of objects c, c', c'' the composition map $\mathrm{Hom}_{\mathcal{C}}(c, c') \times \mathrm{Hom}_{\mathcal{C}}(c', c'') \to \mathrm{Hom}_{\mathcal{C}}(c, c'')$ belongs to $(n-1)\mathrm{Cat}_{\mathrm{pb}}$.

- A functor of n-categories $F : \mathcal{C} \to \mathcal{D}$ belongs to $n\mathrm{Cat}_{\mathrm{pb}}$ if and only if for every pair of objects c, c' in $\mathcal{C}$ the induced functor of $(n-1)$-categories $\mathrm{Hom}_{\mathcal{C}}(c, c') \to \mathrm{Hom}_{\mathcal{D}}(Fc, Fc')$ belongs to $(n-1)\mathrm{Cat}_{\mathrm{pb}}$.

In particular, the subcategory $\mathrm{Cat} \subset n\mathrm{Cat}$ is contained in $n\mathrm{Cat}_{\mathrm{pb}}$ for all $n > 1$. Moreover, if $\mathcal{C}$ is a category and $\mathcal{D}$ is an object of $n\mathrm{Cat}_{\mathrm{pb}}$ for $n > 1$, any functor $F : \mathcal{C} \to \mathcal{D}$ belongs to $n\mathrm{Cat}_{\mathrm{pb}}$.

Proposition 11.1.3. *The functor* $2\mathrm{Corr} : \mathrm{Cat}_{\mathrm{pb}} \to 2\mathrm{Cat}$ *factors through* $2\mathrm{Cat}_{\mathrm{pb}}$.

Proof. Let $\mathcal{C}$ be a category admitting pullbacks. We first show that $2\mathrm{Corr}(\mathcal{C})$ belongs to $2\mathrm{Cat}_{\mathrm{pb}}$. Let c, c' be objects in $\mathcal{C}$ and recall from remark 10.1.10 that we have an equivalence $\mathrm{Hom}_{2\mathrm{Corr}(\mathcal{C})}(c, c') = \mathcal{C}_{/c,c'}$. Let $S = (c \leftarrow s \rightarrow c')$ be an object in $\mathrm{Hom}_{2\mathrm{Corr}(\mathcal{C})}(c, c')$. The forgetful functor $\mathcal{C}_{/c,c'} \rightarrow \mathcal{C}$ induces an equivalence

$$\mathrm{Hom}_{2\mathrm{Corr}(\mathcal{C})}(c, c')_{/S} = \mathcal{C}_{/s}.$$

Since $\mathcal{C}$ admits pullbacks, we conclude that the category $\mathrm{Hom}_{2\mathrm{Corr}(\mathcal{C})}(c, c')_{/S}$ admits products for all S, which means that $\mathrm{Hom}_{2\mathrm{Corr}(\mathcal{C})}(c, c')$ has pullbacks.

Let c, c', c'' be a triple of objects in $\mathcal{C}$. Recall from remark 10.1.10 that the composition map for the Segal category $2\mathrm{Corr}'(\mathcal{C})$ is equivalent to the composition of the right Kan extension functor

$$\mathrm{Funct}(\mathrm{Tw}([2])_{\mathrm{el}}, \mathcal{C}) \rightarrow \mathrm{Funct}(\mathrm{Tw}([2]), \mathcal{C})$$

with the functor

$$\mathrm{Funct}(\mathrm{Tw}([2]), \mathcal{C}) \rightarrow \mathrm{Funct}(\mathrm{Tw}([1]), \mathcal{C})$$

of precomposition with the functor $\mathrm{Tw}([1]) \rightarrow \mathrm{Tw}([2])$ induced by the active arrow $[1] \rightarrow [2]$. Both of these preserve pullbacks, and therefore the composition map for $2\mathrm{Corr}'(\mathcal{C})$ preserves pullbacks. The composition map

$$\mathrm{Hom}_{2\mathrm{Corr}(\mathcal{C})}(c, c') \times \mathrm{Hom}_{2\mathrm{Corr}(\mathcal{C})}(c', c'') \rightarrow \mathrm{Hom}_{2\mathrm{Corr}(\mathcal{C})}(c, c'')$$

is obtained from the above by passing to fibers over (c, c', c'') and therefore it also preserves pullbacks. It follows that $\mathcal{C}$ satisfies the criteria of remark 11.1.2, so it belongs to $2\mathrm{Cat}_{\mathrm{pb}}$, as desired.

Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a morphism in $\mathrm{Cat}_{\mathrm{pb}}$ and let c, c' be objects in $\mathcal{C}$. Let $S = (c \leftarrow s \rightarrow c')$ be an object in $\mathrm{Hom}_{2\mathrm{Corr}(\mathcal{C})}(c, c')$. We have a commutative diagram of categories

$$\begin{tikzcd}
\mathrm{Hom}_{2\mathrm{Corr}(\mathcal{C})}(c, c')_{/S} \arrow[r] \arrow[d] & \mathcal{C}_{/s} \arrow[d] \\
\mathrm{Hom}_{2\mathrm{Corr}(\mathcal{D})}(Fc, Fc')_{/FS} \arrow[r] & \mathcal{D}_{/Fs}
\end{tikzcd}$$

whose horizontal arrows are isomorphisms. Since F preserves pullbacks, the right vertical arrow preserves products. Hence the left vertical arrow reserves products, and it follows that the morphism $\mathrm{Hom}_{2\mathrm{Corr}(\mathcal{C})}(c, c') \rightarrow \mathrm{Hom}_{2\mathrm{Corr}(\mathcal{D})}(Fc, Fc')$ induced by F preserves pullbacks. Our result now follows from remark 11.1.2. $\square$

Construction 11.1.4. Let $n \geq 3$ and assume given a limit preserving functor

$$(n-1)\mathrm{Corr} : \mathrm{Cat}_{\mathrm{pb}} \rightarrow (n-1)\mathrm{Cat}$$

which factors through $(n-1)\mathrm{Cat}_{\mathrm{pb}}$, and such that its composition with the truncation functor $(-)^{\leq 0} : (n-1)\mathrm{Cat} \to \mathrm{Spc}$ is equivalent to the truncation functor $(-)^{\leq 0} : \mathrm{Cat}_{\mathrm{pb}} \to \mathrm{Spc}$. Consider the composite functor

$$\mathrm{Cat}_{\mathrm{pb}} \xrightarrow{2\mathrm{Corr}} 2\mathrm{Cat}_{\mathrm{pb}} = \mathrm{Cat}^{\mathrm{Cat}_{\mathrm{pb}}} \xrightarrow{(n-1)\mathrm{Corr}_!} \mathrm{Algbrd}((n-1)\mathrm{Cat}).$$

For each object $\mathcal{C}$ in $\mathrm{Cat}_{\mathrm{pb}}$, the underlying Segal space to $(n-1)\,\mathrm{Corr}_!\, 2\mathrm{Corr}(\mathcal{C})$ is given by

$$(((n-1)\,\mathrm{Corr})^{\leq 0})_! 2\mathrm{Corr}(\mathcal{C}) = ((-)^{\leq 0})_! 2\mathrm{Corr}(\mathcal{C})$$

which recovers the underlying Segal space of $2\mathrm{Corr}(\mathcal{C})$. Since $2\mathrm{Corr}(\mathcal{C})$ is a 2-category, we conclude that $(n-1)\,\mathrm{Corr}_!\, 2\mathrm{Corr}(\mathcal{C})$ is an n-category. We denote by $n\mathrm{Corr}$ the resulting functor $\mathrm{Cat}_{\mathrm{pb}} \to n\mathrm{Cat}$.

In the setting of construction 11.1.4, the resulting functor $n\mathrm{Corr}$ is again limit preserving and factors through $n\mathrm{Cat}_{\mathrm{pb}}$. Moreover, the functor $(n\mathrm{Corr})^{\leq 0}$ is equivalent to $(2\mathrm{Corr})^{\leq 0}$ which is in turn equivalent to the truncation functor $(-)^{\leq 0} : \mathrm{Cat}_{\mathrm{pb}} \to \mathrm{Spc}$. We thus see that $n\mathrm{Corr}$ satisfies the hypothesis of construction 11.1.4 for $n+1$. Starting with the functor $2\mathrm{Corr}$ we can thus produce functors $n\mathrm{Corr} : \mathrm{Cat}_{\mathrm{pb}} \to n\mathrm{Cat}$ for every $n \geq 3$. In what follows, it will be convenient to allow n to be 1 as well, by setting $1\mathrm{Corr} : \mathrm{Cat}_{\mathrm{pb}} \to \mathrm{Cat}$ to be the forgetful functor.

Definition 11.1.5. *Let $\mathcal{C}$ be a category admitting pullbacks and let $n \geq 1$. We call $n\mathrm{Corr}(\mathcal{C})$ the n-category of correspondences of $\mathcal{C}$.*

We now study the relationship of the n-category of correspondences for different values of n. In what follows, we leave implicit the inclusions $n\mathrm{Cat} \to (n+1)\mathrm{Cat}$. In other words, we work in the category $\omega\mathrm{Cat} = \mathrm{colim}_{n\geq 0}\, n\mathrm{Cat}$, so that all the functors $n\mathrm{Corr}$ for different values of n can be considered to have the same target.

Construction 11.1.6. Let $n \geq 3$ and assume given a natural transformation

$$\iota^{n-2,n-1} : (n-2)\mathrm{Corr} \to (n-1)\mathrm{Corr}\,.$$

Consider the induced natural transformation $((n-2)\mathrm{Corr})_! \to ((n-1)\mathrm{Corr})_!$. Composition with the functor $2\mathrm{Corr}$ yields a natural transformation

$$\iota^{n-1,n} : (n-1)\mathrm{Corr} \to n\mathrm{Corr}$$

which we continue denoting by ι. Applying this inductively starting with the natural transformation of construction 10.2.4 we obtain a sequence of functors $\mathrm{Cat}_{\mathrm{pb}} \to \omega\mathrm{Cat}$ and natural transformations between them

$$1\mathrm{Corr} \to 2\mathrm{Corr} \to 3\mathrm{Corr} \to \dots.$$

For each $m \leq n$ we let $\iota^{m,n} : m\mathrm{Corr} \to n\mathrm{Corr}$ the associated natural transformation. In the case $m = 1$ we will simply write $\iota^n = \iota^{1,n}$.

Remark 11.1.7. Let $n \geq m \geq 1$. Then the natural transformation $\iota^{m,n}$ is a monomorphism. Moreover, its composition with the truncation functor $(-)^{\leq m} : n\mathrm{Cat} \to m\mathrm{Cat}$ is an equivalence.

Remark 11.1.8. We can think about the sequence of functors and transformations of construction 11.1.6 as a functor $-\mathrm{Corr} : \mathbb{N} \to \omega\mathrm{Cat}$, where $\mathbb{N}$ denotes the poset of natural numbers. Since $n\mathrm{Corr}$ is limit preserving for all values of n, we have that $-\mathrm{Corr}$ is limit preserving.

In particular, if $\mathcal{C}$ is a symmetric monoidal category admitting pullbacks and such that the functor $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves pullbacks, then $n\mathrm{Corr}(\mathcal{C})$ inherits a symmetric monoidal structure for all n, and the functors $\iota^{m,n}_{\mathcal{C}}$ also inherit symmetric monoidal structures. In the case when $\mathcal{C}$ is a category with finite limits equipped with its cartesian symmetric monoidal structure, it follows from proposition 10.3.3 that every object of $n\mathrm{Corr}(\mathcal{C})$ is dualizable.

Proposition 11.1.9. *Let $\mathcal{C}$ be a category admitting pullbacks. Let $n \geq 3$ and $1 \leq k < n-1$. Then every k-cell μ in $n\mathrm{Corr}(\mathcal{C})$ admits both a right adjoint μ^R and a left adjoint μ^L, and moreover there is an equivalence $\mu^L = \mu^R$.*

Proof. We argue by induction on k. Consider first the case $k = 1$. By remark 11.1.7 we have that any arrow in $n\mathrm{Corr}(\mathcal{C})$ belongs to the image of the functor $\iota^{3,n}_{\mathcal{C}} : 3\mathrm{Corr}(\mathcal{C}) \to n\mathrm{Corr}(\mathcal{C})$. It therefore suffices to consider the case $n = 3$. Recall from remark 10.2.6 that there is a natural equivalence $2\mathrm{Corr} = 2\mathrm{Corr}^{1\text{-op}}$ that restricts to the identity on objects. It follows that there is an equivalence $3\mathrm{Corr}(\mathcal{C}) = 3\mathrm{Corr}(\mathcal{C})^{2\text{-op}}$ which restricts to the identity on objects and arrows. Therefore for every arrow μ in $3\mathrm{Corr}(\mathcal{C})$, we have that μ admits a right adjoint if and only if it admits a left adjoint, and moreover in that case there is an equivalence $\mu^R = \mu^L$.

It now suffices to show that every arrow in $3\mathrm{Corr}(\mathcal{C})$ can be written as a composition of arrows which admit either a left or right adjoint. Since the functor $\iota^{2,3}_{\mathcal{C}} : 2\mathrm{Corr}(\mathcal{C}) \to 3\mathrm{Corr}(\mathcal{C})$ is surjective on arrows, it suffices to show that this is the case in $2\mathrm{Corr}(\mathcal{C})$. Indeed, any morphism in $2\mathrm{Corr}(\mathcal{C})$ is represented by a span

$$\begin{tikzcd} & s \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ c & & c' \end{tikzcd}$$

and can thus be written as the composition $\iota_{\mathcal{C}}(\beta)\iota^R_{\mathcal{C}}(\alpha)$, which admit adjoints thanks to proposition 10.3.1.

Assume now that $k > 1$. Let c, c' be objects in $\mathcal{C}$ such that μ is a k-cell with source and target objects $\iota^n_{\mathcal{C}}(c)$ and $\iota^n_{\mathcal{C}}(c')$, respectively. Then μ can be thought of as a $(k-1)$-cell in

$$\mathrm{Hom}_{n\mathrm{Corr}(\mathcal{C})}(\iota^n_{\mathcal{C}}(c), \iota^n_{\mathcal{C}}(c')) = (n-1)\mathrm{Corr}(\mathcal{C})(\mathrm{Hom}_{2\mathrm{Corr}(\mathcal{C})}(\iota_{\mathcal{C}}(c), \iota_{\mathcal{C}}(c'))).$$

Our result now follows from the inductive hypothesis. $\square$

The following result appears previously in [Hau18] corollary 12.5.

Corollary 11.1.10. *Let $\mathcal{C}$ be a category admitting finite limits equipped with its cartesian symmetric monoidal structure, and let $n \geq 2$. Then every object of the $(n-1)$-category $n\mathrm{Corr}(\mathcal{C})^{\leq n-1}$ is fully dualizable.*

Proof. Combine remark 11.1.8 with proposition 11.1.9. □

11.2 Higher Beck-Chevalley conditions

Our next goal is to generalize the universal property of theorem 10.4.18 to the case of higher categories of correspondences.

Construction 11.2.1. Let $\mathcal{D}$ be a 2-category. For each right adjointable arrow $\alpha : d \to e$ in $\mathcal{D}$ we denote by α^R its right adjoint, and by $\epsilon_\alpha : \alpha\alpha^R \to \mathrm{id}_d$ the counit of the adjunction.

Consider a commutative diagram

$$\begin{tikzcd} d' \arrow[r, "\alpha'"] \arrow[d, "\beta'"] & d \arrow[d, "\beta"] \\ e' \arrow[r, "\alpha"] & e \end{tikzcd}$$

in $\mathcal{D}$ such that all four arrows admit right adjoints. The diagonal map $\gamma : d' \to e$ has a right adjoint which can be computed as in two ways as

$$\beta'^R\alpha^R = \gamma^R = \alpha'^R\beta^R$$

and the counit of the adjunction can be described as

$$(\alpha\epsilon_{\beta'}\alpha^R) \circ \epsilon_\alpha = \epsilon_\gamma = (\beta\epsilon_{\alpha'}\beta^R) \circ \epsilon_\beta.$$

The above equivalence exhibits the following square in $\mathrm{End}_{\mathcal{C}}(e)$ as a commutative square:

$$\begin{tikzcd} \gamma\gamma^R \arrow[r, "\beta\epsilon_{\alpha'}\beta^R"] \arrow[d, "\alpha\epsilon_{\beta'}\alpha^R"] & \beta\beta^R \arrow[d, "\epsilon_\beta"] \\ \alpha\alpha^R \arrow[r, "\epsilon_\alpha"] & \mathrm{id}_e\,. \end{tikzcd}$$

Definition 11.2.2. *Let $\mathcal{D}$ be an n-category. We say that a commutative diagram*

$$\begin{tikzcd} d' \arrow[r, "\beta'"] \arrow[d, "\alpha'"] & d \arrow[d, "\alpha"] \\ e' \arrow[r, "\beta"] & e \end{tikzcd}$$

in $\mathcal{D}$ is 1-fold (vertically / horizontally) right adjointable if it is right adjointable in the 2-category underlying $\mathcal{D}$. For each $k \geq 2$ we say that the above diagram is k-fold (vertically

/ horizontally) right adjointable if it is (vertically / horizontally) right adjointable, all its arrows admit right adjoints, and the commutative diagram square

$$\begin{tikzcd} \gamma\gamma^R \arrow[r, "\beta\epsilon_{\alpha'}\beta^R"] \arrow[d, "\alpha\epsilon_{\beta'}\alpha^R"] & \beta\beta^R \arrow[d, "\epsilon_\beta"] \\ \alpha\alpha^R \arrow[r, "\epsilon_\alpha"] & \mathrm{id}_e . \end{tikzcd}$$

in $\mathrm{End}_{\mathcal{D}}(e)$ *defined in construction 11.2.1, is* $(k-1)$*-fold (vertically/ horizontally) right adjointable.*

Construction 11.2.3. Recall the universal left adjointable arrow $L : [1] \to \mathrm{Adj}$ and the universal vertically right adjointable square

$$\mathrm{id}_{[1]} \times L : [1] \times [1] \to [1] \times \mathrm{Adj}$$

from remark 10.4.4. Let $\mathcal{U}^+$ be the colimit in 2Cat of the following diagram:

$$\begin{tikzcd} \mathrm{Adj} & & [1] \times \mathrm{Adj} & & \mathrm{Adj} \\ & & {[1] \times [1]} \arrow[u, "\mathrm{id}_{[1]} \times L"] & & \\ & {[1]} \arrow[uul, "L"] \arrow[ur, "\mathrm{id}_{[1]} \times \{0\}"] & & {[1]} \arrow[uur, "L"'] \arrow[ul, "\{1\} \times \mathrm{id}_{[1]}"'] & \end{tikzcd}$$

In other words, $\mathcal{U}^+$ is the universal vertically right adjointable square such that the horizontal arrows are also right adjointable. Construction 11.2.1 provides a functor

$$C_{\mathrm{univ}} : \Sigma([1] \times [1]) \to \mathcal{U}^+$$

where Σ denotes the functor which associates to each n-category $\mathcal{T}$ an $(n+1)$-category $\Sigma(\mathcal{T})$ with objects s, t and such that its only nontrivial Hom category is $\mathrm{Hom}_{\Sigma(\mathcal{T})}(s,t) = \mathcal{T}$.

We define for each $n \geq 2$ an n-category $\mathcal{U}_n$ equipped with a functor $i_n : [1] \times [1] \to \mathcal{U}_n$ as follows:

- When $n = 2$ we set $i_1 = \mathrm{id}_{[1]} \times L : [1] \times [1] \to [1] \times \mathrm{Adj}$.
- Assume that $n > 2$. Then we let $\mathcal{U}_n$ be the pushout of the following diagram:

$$\begin{tikzcd} \mathcal{U}^+ & & \Sigma(\mathcal{U}_{n-1}) \\ & \Sigma([1] \times [1]) \arrow[ul, "C_{\mathrm{univ}}"] \arrow[ur, "\Sigma(i_{n-1})"'] & \end{tikzcd}$$

.

The functor i_n is the universal $(n-1)$-fold vertically right adjointable commutative square. The map i_n an epimorphism - for each n-category $\mathcal{D}$ precomposition with i_n induces an equivalence between the space of functors $\mathcal{U}_n \to \mathcal{D}$ and the space of functors $[1] \times [1] \to \mathcal{D}$ which correspond to $(n-1)$-fold right adjointable squares.

Remark 11.2.4. As a consequence of the existence of a universal $(n-1)$-fold vertically right adjointable commutative square we deduce the following fact: if $F : \mathcal{I} \to n\mathrm{Cat}$ is a diagram with limit $\mathcal{D}$, then a commutative square in $\mathcal{D}$ is $(n-1)$-fold vertically right adjointable if and only if its image in $F(i)$ is $(n-1)$-fold vertically right adjointable for all i in $\mathcal{I}$.

Definition 11.2.5. *Let $\mathcal{C}$ be a category admitting pullbacks and $\mathcal{D}$ be an n-category. We say that a functor $F : \mathcal{C} \to \mathcal{D}$ satisfies the left n-fold Beck-Chevalley condition if for every cospan $x \to s \leftarrow y$ in $\mathcal{C}$, the induced commutative square*

$$\begin{array}{ccc} F(x \times_s y) & \longrightarrow & F(y) \\ \downarrow & & \downarrow \\ F(x) & \longrightarrow & F(s) \end{array}$$

is n-fold right adjointable.

We are now ready to state the universal property of the n-category of correspondences.

Theorem 11.2.6. *Let $\mathcal{C}$ be a category admitting pullbacks and $\mathcal{D}$ be an n-category. Restriction along the inclusion $\iota^n_{\mathcal{C}} : \mathcal{C} \to n\mathrm{Corr}(\mathcal{C})$ induces an identification of $\mathrm{Hom}_{n\mathrm{Cat}}(n\mathrm{Corr}(\mathcal{C}), \mathcal{D})$ with the subspace of $\mathrm{Hom}_{n\mathrm{Cat}}(\mathcal{C}, \mathcal{D})$ consisting of functors satisfying the left $(n-1)$-fold Beck-Chevalley condition.*

The proof of theorem 11.2.6 needs a few lemmas.

Lemma 11.2.7. *Let $\mathcal{O}$ be an operad* [1] *and let $\mathcal{M}$ be an $\mathcal{O}$-monoidal category. Let $F : \mathcal{A} \to \mathcal{B}$ be a morphism of $\mathcal{O}$-algebras in $\mathcal{M}$. Assume that for every operation μ in $\mathcal{O}$ with source $\{X_s\}_{s\in S}$ and target X the induced map*

$$F(\mu) : \mu(\{\mathcal{A}(X_s)\}_{s\in S}) \to \mu(\{\mathcal{B}(X_s)\}_{s\in S})$$

is an epimorphism in $\mathcal{M}(X)$. Then

(i) *The morphism F is an epimorphism in $\mathrm{Alg}_{\mathcal{O}}(\mathcal{M})$.*

[1] In this thesis we use a language for speaking about operads which is close in spirit to the classical language in terms of objects and operations which satisfy a composition rule. Namely, given an operad $\mathcal{O}$ with associated category of operators $p : \mathcal{O}^{\otimes} \to \mathrm{Fin}_*$, we call $p^{-1}(\langle 1 \rangle)$ the category of objects of $\mathcal{O}$, and arrows in $\mathcal{O}^{\otimes}$ lying above an active arrow of the form $\langle n \rangle \to \langle 1 \rangle$ are called operations of $\mathcal{O}$.

(ii) *Let $\mathcal{B}'$ be another $\mathcal{O}$-algebra in $\mathcal{M}$. Then a morphism $F' : \mathcal{A} \to \mathcal{B}'$ factors through $\mathcal{B}$ if and only if for every object X in $\mathcal{O}$ the map $F'(X) : \mathcal{A}(X) \to \mathcal{B}'(X)$ factors through $\mathcal{B}(X)$.*

Proof. Denote by $p : \mathcal{O}^\otimes \to \mathrm{Fin}_*$ and $q : \mathcal{M}^\otimes \to \mathcal{O}^\otimes$ the categories of operators associated to $\mathcal{O}$ and $\mathcal{M}$. Recall that $\mathrm{Alg}_{\mathcal{O}}(\mathcal{M})$ is the full subcategory of

$$\mathrm{Funct}_{\mathcal{O}^\otimes}(\mathcal{O}^\otimes, \mathcal{M}^\otimes) = \mathrm{Funct}(\mathcal{O}^\otimes, \mathcal{M}^\otimes) \times_{\mathrm{Funct}(\mathcal{O}^\otimes, \mathcal{O}^\otimes)} [0]$$

on those functors that map inert arrows in $\mathcal{O}^\otimes$ to inert arrows in $\mathcal{M}^\otimes$. The algebras $\mathcal{A}, \mathcal{B}$ determine functors $\mathcal{A}, \mathcal{B} : \mathcal{O}^\otimes \to \mathcal{M}^\otimes$, and F is a natural transformation $\mathcal{A} \to \mathcal{B}$.

We will show that F is an epimorphism by showing that the diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r, "F"] \arrow[d, "F"] & \mathcal{B} \arrow[d, "\mathrm{id}"] \\ \mathcal{B} \arrow[r, "\mathrm{id}"] & \mathcal{B} \end{tikzcd}$$

is a pushout diagram in $\mathrm{Alg}_{\mathcal{O}}(\mathcal{M})$. This would follow if we are able to show that the above is a pushout in $\mathrm{Funct}_{\mathcal{O}^\otimes}(\mathcal{O}^\otimes, \mathcal{M}^\otimes)$. Note that the square which is constant $\mathrm{id}_{\mathcal{O}^\otimes}$ is a pushout square in $\mathrm{Funct}(\mathcal{O}^\otimes, \mathcal{O}^\otimes)$. Therefore we may in fact restrict to showing that the above square defines a pushout square in $\mathrm{Funct}(\mathcal{O}^\otimes, \mathcal{M}^\otimes)$. This can be done pointwise. Let $\{X_s\}_{s\in S}$ be an object of $\mathcal{O}^\otimes$, corresponding to a finite collection of objects of $\mathcal{O}$. We must show that the square

$$\begin{tikzcd} \{\mathcal{A}(X_s)\}_{s\in S} \arrow[r, "\{F(X_s)\}_{s\in S}"] \arrow[d, "\{F(X_s)\}_{s\in S}"] & \{\mathcal{B}(X_s)\}_{s\in S} \arrow[d, "\mathrm{id}"] \\ \{\mathcal{B}(X_s)\}_{s\in S} \arrow[r, "\mathrm{id}"] & \{\mathcal{B}(X_s)\}_{s\in S} \end{tikzcd}$$

taking place in $q^{-1}(\{X_s\}_{s\in S}) = \prod_{s\in S} \mathcal{M}(X_s) \subset \mathcal{M}^\otimes$, defines a pushout square in $\mathcal{M}^\otimes$. Since the image under q of the above square is constant (and thus a pushout), suffices to show that the above is in fact a q-pushout. Using [Lur09a] proposition 4.3.1.10, we reduce to showing that for every arrow $\alpha : \{X_s\}_{s\in S} \to \{X_t\}_{t\in T}$ in $\mathcal{O}^\otimes$, the image of the above square under the functor

$$\alpha_! : q^{-1}(\{X_s\}_{s\in S}) \to q^{-1}(\{X_t\}_{t\in T})$$

is a pushout. It suffices to do this in the case when α is either inert or active, and this follows readily from our assumptions on F. This proves item (i).

We now prove item (ii). Assume given a morphism $F' : \mathcal{A} \to \mathcal{B}'$ such that for every object X in $\mathcal{O}$ the map $F'(X) : \mathcal{A}(X) \to \mathcal{B}'(X)$ factors through $\mathcal{B}(X)$. We have to show that F' factors through $\mathcal{B}$. Let $\{X_s\}_{s\in S}$ be an object in $\mathcal{O}^\otimes$. Note that the map

$$F'(\{X_s\}_{s\in S}) : \{\mathcal{A}(X_s)\}_{s\in S} \to \{\mathcal{B}'(X_s)\}_{s\in S}$$

factors through $\{\mathcal{B}(X_s)\}_{s\in S}$. It follows that we have a pushout diagram

$$\begin{tikzcd}
\{\mathcal{A}(X_s)\}_{s\in S} \arrow[r, "\{F(X_s)\}_{s\in S}"] \arrow[d, "\{F'(X_s)\}_{s\in S}"] & \{\mathcal{B}(X_s)\}_{s\in S} \arrow[d] \\
\{\mathcal{B}'(X_s)\}_{s\in S} \arrow[r, "\mathrm{id}"] & \{\mathcal{B}'(X_s)\}_{s\in S}
\end{tikzcd}$$

in $q^{-1}(\{X_s\}_{s\in S})$. We claim that the above is a q-colimit diagram. As before, by [Lur09a] proposition 4.3.1.10, we reduce to showing that for every arrow $\alpha : \{X_s\}_{s\in S} \to \{X_t\}_{t\in T}$ in $\mathcal{O}^\otimes$, the image of the above square under $\alpha_!$ is a pushout. The case when α is inert is clear; the case when α is active follows from our assumption on F.

Using [Lur17] lemma 3.2.2.9 we see that there is a square

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "F"] \arrow[d, "F'"] & \mathcal{B} \arrow[d] \\
\mathcal{B}' \arrow[r, "G"] & \mathcal{B}''
\end{tikzcd}$$

in $\mathrm{Funct}_{\mathcal{O}^\otimes}(\mathcal{O}^\otimes, \mathcal{M}^\otimes)$ with the property that, for every object $\{X_s\}_{s\in S}$ in $\mathcal{O}^\otimes$, the induced square

$$\begin{tikzcd}
\{\mathcal{A}(X_s)\}_{s\in S} \arrow[r, "F(\{X_s\}_{s\in S})"] \arrow[d, "F'(\{X_s\}_{s\in S})"] & \{\mathcal{B}(X_s)\}_{s\in S} \arrow[d] \\
\{\mathcal{B}'(X_s)\}_{s\in S} \arrow[r, "G(\{X_s\}_{s\in S})"] & \{\mathcal{B}''(X_s)\}_{s\in S}
\end{tikzcd}$$

is a q-colimit diagram. It follows that $G(\{X_s\}_{s\in S})$ is an isomorphism, and hence G is an isomorphism, which implies that F' factors through $\mathcal{B}$, as desired. $\square$

Lemma 11.2.8. *Let $\mathcal{C}$ be a category admitting pullbacks and let*

$$\begin{tikzcd}
x \times_s y \arrow[r, "\alpha'"] \arrow[d, "\beta'"] & y \arrow[d, "\beta"] \\
x \arrow[r, "\alpha"] & s
\end{tikzcd} \qquad (*)$$

be a cartesian square in $\mathcal{C}$. Then the commutative square

$$\begin{tikzcd}
\iota_{\mathcal{C}}(\gamma)\iota_{\mathcal{C}}^R(\gamma) \arrow[r, "\iota_{\mathcal{C}}(\beta)\epsilon_{\iota_{\mathcal{C}}(\alpha')}\iota_{\mathcal{C}}^R(\beta)"] \arrow[d, "\iota_{\mathcal{C}}(\alpha)\epsilon_{\iota_{\mathcal{C}}(\beta')}\iota_{\mathcal{C}}^R(\alpha)"] & \iota_{\mathcal{C}}(\beta)\iota_{\mathcal{C}}^R(\beta) \arrow[d, "\epsilon_{\iota_{\mathcal{C}}(\beta)}"] \\
\iota_{\mathcal{C}}(\alpha)\iota_{\mathcal{C}}(\alpha)^R \arrow[r, "\epsilon_{\iota_{\mathcal{C}}(\alpha)}"] & \mathrm{id}_{\iota_{\mathcal{C}}(s)}
\end{tikzcd}$$

in $\mathrm{End}_{2\mathrm{Corr}(\mathcal{C})}(\iota_{\mathcal{C}}(s))$ *that results from construction 11.2.1 applied to the image of* $(*)$ *under* $\iota_{\mathcal{C}}$ *is equivalent to the image under the canonical map*

$$\mathcal{C}_{/s} \to \mathcal{C}_{/s,s} = \mathrm{End}_{2\mathrm{Corr}(\mathcal{C})}(\iota_{\mathcal{C}}(s))$$

of the commutative square

$$\begin{tikzcd} \gamma \arrow[r, "\overline{\beta'}"] \arrow[d, "\overline{\alpha'}"] & \alpha \arrow[d, "\overline{\alpha}"] \\ \beta \arrow[r, "\overline{\beta}"] & \mathrm{id}_s \end{tikzcd}$$

whose image under the forgetful functor $\mathcal{C}_{/s} \to \mathcal{C}$ *recovers* $(*)$*.*

Proof. Let $\mathcal{U}$ be the universal cospan: this is the category with objects $0, 1, 2$ and nontrivial arrows $0 \xrightarrow{\alpha_u} 1 \xleftarrow{\beta_u} 2$. Everything in the statement is functorial in $\mathcal{C}$, so we may reduce to the case when $\mathcal{C} = \mathcal{F}(\mathcal{U})$ is the free category with pullbacks on $\mathcal{U}$, and the cartesian square under consideration is

$$\begin{tikzcd} 0 \times_1 2 \arrow[r, "\alpha'_u"] \arrow[d, "\beta'_u"] & 2 \arrow[d, "\beta_u"] \\ 0 \arrow[r, "\alpha_u"] & 1. \end{tikzcd}$$

The two squares that we have to show are equivalent can easily be seen to have the same vertices. Our result now follows from the fact that in $\mathrm{End}_{2\mathrm{Corr}(\mathcal{F}(\mathcal{U}))}(\mathrm{id}_1)$ there is a unique commutative square with those vertices. □

Lemma 11.2.9. *Let* $\mathcal{C}$ *be a category admitting pullbacks and let* $\mathcal{D}$ *be an* n*-category for* $n \geq 3$*. Let* $F : \mathcal{C} \to \mathcal{D}$ *be a functor satisfying the left Beck-Chevalley condition. Then* F *satisfies the left* $(n-1)$*-fold Beck-Chevalley condition if and only if for every pair of objects* z, w *in* $\mathcal{C}$ *the induced functor*

$$\mathrm{Hom}_{2\mathrm{Corr}(\mathcal{C})}(\iota_{\mathcal{C}}(z), \iota_{\mathcal{C}}(w)) \to \mathrm{Hom}_{\mathcal{D}}(F(z), F(w))$$

satisfies the left $(n-2)$*-fold Beck-Chevalley condition.*

Proof. Assume first that for every pair of objects z, w in $\mathcal{C}$ the functor

$$\mathrm{Hom}_{2\mathrm{Corr}(\mathcal{C})}(\iota_{\mathcal{C}}(z), \iota_{\mathcal{C}}(w)) \to \mathrm{Hom}_{\mathcal{D}}(F(z), F(w))$$

satisfies the left $(n-2)$-fold Beck-Chevalley condition. Consider a cartesian square

$$\begin{tikzcd} x \times_s y \arrow[r, "\alpha'"] \arrow[d, "\beta'"] & y \arrow[d, "\beta"] \\ x \arrow[r, "\alpha"] & s \end{tikzcd}$$

in $\mathcal{C}$ and denote by $\gamma : x \times_s y \to s$ the induced map. Since F is assumed to satisfy the left Beck-Chevalley condition, we know that the induced commutative square

$$\begin{tikzcd} F(x \times_s y) \arrow[r, "F(\alpha')"] \arrow[d, "F(\beta')"] & F(y) \arrow[d, "F(\beta)"] \\ F(x) \arrow[r, "F(\alpha)"] & F(s) \end{tikzcd}$$

is right adjointable. We need to show that the commutative square

$$\begin{tikzcd} F(\gamma)F(\gamma)^R \arrow[r, "F(\beta)\epsilon_{F(\alpha')}F(\beta)^R"] \arrow[d, "F(\alpha)\epsilon_{F(\beta')}F(\alpha)^R"] & F(\beta)F(\beta)^R \arrow[d, "\epsilon_{F(\beta)}"] \\ F(\alpha)F(\alpha)^R \arrow[r, "\epsilon_{F(\alpha)}"] & \mathrm{id}_{F(s)} \end{tikzcd}$$

in $\mathrm{End}_{\mathcal{D}}(F(s))$ arising from construction 11.2.1 is $(n-2)$-fold right adjointable. This square is the image under the functor $2\mathrm{Corr}(\mathcal{C}) \to \mathcal{D}$ of the commutative square

$$\begin{tikzcd} \iota_{\mathcal{C}}(\gamma)\iota_{\mathcal{C}}^R(\gamma) \arrow[r, "\iota_{\mathcal{C}}(\beta)\epsilon_{\iota_{\mathcal{C}}(\alpha')}\iota_{\mathcal{C}}^R(\beta)"] \arrow[d, "\iota_{\mathcal{C}}(\alpha)\epsilon_{\iota_{\mathcal{C}}(\beta')}\iota_{\mathcal{C}}^R(\alpha)"] & \iota_{\mathcal{C}}(\beta)\iota_{\mathcal{C}}^R(\beta) \arrow[d, "\epsilon_{\iota_{\mathcal{C}}(\beta)}"] \\ \iota_{\mathcal{C}}(\alpha)\iota_{\mathcal{C}}(\alpha)^R \arrow[r, "\epsilon_{\iota_{\mathcal{C}}(\alpha)}"] & \mathrm{id}_{\iota_{\mathcal{C}}(s)} . \end{tikzcd}$$

This commutative square is described by lemma 11.2.8. In particular, note that the image of it under the canonical map

$$\mathrm{End}_{2\mathrm{Corr}(\mathcal{C})}(\mathrm{id}_s) = \mathcal{C}_{/s,s} \to \mathcal{C}$$

is a cartesian square. Since the projection $\mathcal{C}_{/s,s} \to \mathcal{C}$ creates pullbacks, we conclude that the above is in fact a cartesian square, and therefore its image inside $\mathrm{End}_{\mathcal{D}}(F(s))$ is $(n-2)$-fold right adjointable, as we wanted.

Assume now that F satisfies the left $(n-1)$-fold Beck-Chevalley condition, and let z, w be a pair of objects of $\mathcal{C}$. Let

$$\begin{tikzcd} x^\sharp \times_{s^\sharp} y^\sharp \arrow[r, "\beta'^\sharp"] \arrow[d, "\alpha'^\sharp"] & y^\sharp \arrow[d, "\alpha^\sharp"] \\ x^\sharp \arrow[r, "\beta^\sharp"] & s^\sharp \end{tikzcd} \qquad (\star)$$

be a cartesian square in $\mathrm{Hom}_{2\mathrm{Corr}(\mathcal{C})}(\iota_{\mathcal{C}}(z), \iota_{\mathcal{C}}(w))$, whose image under the forgetful functor $\mathrm{Hom}_{2\mathrm{Corr}(\iota_{\mathcal{C}}(z),\iota_{\mathcal{C}}(w))} = \mathcal{C}_{/z,w} \to \mathcal{C}$ is a cartesian square in $\mathcal{C}$ which we denote as follows:

$$\begin{tikzcd} x \times_s y \arrow[r, "\beta'"] \arrow[d, "\alpha'"] & y \arrow[d, "\alpha"] \\ x \arrow[r, "\beta"] & s \end{tikzcd}$$

Let $z \leftarrow s : \sigma$ and $\tau : s \xrightarrow{\tau} w$ be the legs of the span $s^\sharp$. Then the square $(\star)$ is equivalent to the image of the square

$$\begin{tikzcd} \gamma \arrow[r, "\overline{\beta'}"] \arrow[d, "\overline{\alpha'}"] & \alpha \arrow[d, "\overline{\alpha}"] \\ \beta \arrow[r, "\overline{\beta}"] & \mathrm{id}_s \end{tikzcd} \qquad (\star\star)$$

under the composite map

$$\mathcal{C}_{/s} \to \mathcal{C}_{/s,s} = \mathrm{End}_{2\mathrm{Corr}(\mathcal{C})}(\iota_{\mathcal{C}}(s)) \xrightarrow{\iota^R_{\mathcal{C}}(\sigma)^*} \mathrm{Hom}_{2\mathrm{Corr}(\mathcal{C})}(\iota_{\mathcal{C}}(z), \iota_{\mathcal{C}}(s)) \xrightarrow{\iota_{\mathcal{C}}(\tau)_*} \mathrm{Hom}_{2\mathrm{Corr}(\mathcal{C})}(\iota_{\mathcal{C}}(z), \iota_{\mathcal{C}}(w)).$$

It therefore suffices to show that the image of the square $(\star\star)$ under the composite map

$$\mathcal{C}_{/s} \to \mathcal{C}_{/s,s} = \mathrm{End}_{2\mathrm{Corr}(\mathcal{C})}(\iota_{\mathcal{C}}(s)) \to \mathrm{End}_{\mathcal{D}}(F(s))$$

is $(n-2)$-fold right adjointable. This follows from the fact that F satisfies the left $(n-1)$-fold Beck-Chevalley condition, combined with lemma 11.2.8. □

Proof of theorem 11.2.6. We argue by induction on n. The case $n = 2$ is theorem 10.4.18, so we assume $n \geq 3$. The inclusion $\iota^{2,n}_{\mathcal{C}} : 2\mathrm{Corr}(\mathcal{C}) \to n\mathrm{Corr}(\mathcal{C})$ defines a morphism in $\mathrm{Algbrd}_{\mathcal{C}^{\leq 0}}((n-1)\mathrm{Cat})$ which is an epimorphism by item (i) in lemma 11.2.7 combined with our inductive hypothesis. The second part of lemma 11.2.7 guarantees that for any other $\mathrm{Assos}_{\mathcal{C}^{\leq 0}}$-algebra $\mathcal{B}'$ in $(n-1)\mathrm{Cat}$ precomposition with $\iota^{2,n}_{\mathcal{C}}$ induces an equivalence between the space of maps $n\mathrm{Corr}(\mathcal{C}) \to \mathcal{B}'$ and the space of maps $2\mathrm{Corr}(\mathcal{C}) \to \mathcal{B}'$ such that for every pair of objects z, w in $\mathcal{C}$ the induced map

$$\mathrm{Hom}_{2\mathrm{Corr}(\mathcal{C})}(\iota_{\mathcal{C}}(z), \iota_{\mathcal{C}}(w)) \to \mathcal{B}'(z, w)$$

satisfies the left $(n-2)$-fold Beck-Chevalley condition. Since the projection

$$\mathrm{Algbrd}((n-1)\mathrm{Cat}) \to \mathrm{Cat}$$

is a cartesian fibration, we conclude that $\iota^{2,n}_{\mathcal{C}}$ is in fact an epimorphism in $\mathrm{Algbrd}((n-1)\mathrm{Cat})$, and in particular it is an epimorphism in $n\mathrm{Cat}$. Moreover, a morphism $G : 2\mathrm{Corr}(\mathcal{C}) \to \mathcal{D}$ factors through $n\mathrm{Corr}(\mathcal{C})$ if and only if for every pair of objects z, w in $\mathcal{C}$ the induced functor

$$\mathrm{Hom}_{2\mathrm{Corr}(\mathcal{C})}(\iota_{\mathcal{C}}(z), \iota_{\mathcal{C}}(w)) \to \mathrm{Hom}_{\mathcal{D}}(G(\iota_{\mathcal{C}}(z)), G(\iota_{\mathcal{C}}(w)))$$

satisfies the left $(n-2)$-fold Beck-Chevalley condition. Our result now follows from a combination of theorem 10.4.18 and lemma 11.2.9. □

Remark 11.2.10. Let $\mathcal{C}$ be a symmetric monoidal category which admits pullbacks and assume such that the tensor product functor $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves pullbacks. Combining theorem 11.2.6 with lemma 11.2.7 we see that for every $n \geq 2$ the functor $\iota^n_{\mathcal{C}} : \mathcal{C} \to n\mathrm{Corr}(\mathcal{C})$ is an epimorphism of symmetric monoidal n-categories. Given a symmetric monoidal n-category $\mathcal{D}$, precomposition with $\iota_{\mathcal{C}}$ yields an equivalence between the space of symmetric monoidal functors $n\mathrm{Corr}(\mathcal{C}) \to \mathcal{D}$ and the space of symmetric monoidal functors $\mathcal{C} \to \mathcal{D}$ which satisfy the left $(n-1)$-fold Beck-Chevalley condition.

11.3 Extension along the Yoneda embedding

Our next goal is to show that under suitable conditions, a functor out of $n\mathrm{Corr}(\mathcal{C})$ can be extended to a functor on $(n-1)\mathrm{Corr}(\mathcal{P}(\mathcal{C}))$. We begin by recalling the passage to adjoints property from chapter 12. The definition requires the notion of conical colimits and limits in n-categories - we refer the reader to chapter 5 for a general discussion of this concept in the setting of enriched categories.

Notation 11.3.1. Let $\mathcal{D}$ be an n-category. We denote by $\mathcal{D}^{\leq 1}$ the 1-category underlying $\mathcal{D}$, and by $(\mathcal{D}^{\leq 1})^{\mathrm{radj}}$ (resp. $\mathcal{D}^{\leq 1})^{\mathrm{ladj}}$ the subcategory of $\mathcal{D}^{\leq 1}$ containing all objects, and only those morphisms which are right (resp. left) adjointable in $\mathcal{D}$.

Definition 11.3.2. *Let $\mathcal{D}$ be an n-category. We say that $\mathcal{D}$ satisfies the passage to adjoints property if the following conditions are satisfied:*

- *The category $(\mathcal{D}^{\leq 1})^{\mathrm{radj}}$ has all colimits, and the inclusion $(\mathcal{D}^{\leq 1})^{\mathrm{radj}} \to \mathcal{D}$ preserves conical colimits.*
- *The category $(\mathcal{D}^{\leq 1})^{\mathrm{ladj}}$ has all limits, and the inclusion $(\mathcal{D}^{\leq 1})^{\mathrm{ladj}} \to \mathcal{D}$ preserves conical limits.*

Remark 11.3.3. Let $\mathcal{D}$ be an n-category. Then passage to adjoints induces an equivalence between the categories $(\mathcal{D}^{\leq 1})^{\mathrm{radj}}$ and $(\mathcal{D}^{\leq 1})^{\mathrm{ladj}}$. It follows that if $\mathcal{D}$ satisfies the passage to adjoints property, then a right adjointable diagram $F : \mathcal{I}^{\rhd} \to \mathcal{D}$ in $\mathcal{D}$ is a conical colimit if and only if the diagram $F^R : (\mathcal{I}^{\mathrm{op}})^{\lhd} \to \mathcal{D}$ is a conical limit.

Definition 11.3.4. *Let $\mathcal{D}$ be an n-category. We say that $\mathcal{D}$ satisfies the 1-fold passage to adjoints property if it satisfies the passage to adjoints property. For each $(n-1) \geq k \geq 2$ we say that $\mathcal{D}$ satisfies the k-fold passage to adjoints property if it satisfies the passage to adjoints property and for every pair of objects d, e in $\mathcal{D}$ the $(n-1)$-category $\mathrm{Hom}_{\mathcal{D}}(d, e)$ satisfies the $(k-1)$-fold passage to adjoints property.*

Definition 11.3.5. *Let $\mathcal{D}$ be an n-category. We say that $\mathcal{D}$ is 1-fold conically cocomplete if it admits all small conical colimits. For each $n \geq k \geq 2$ we say that $\mathcal{D}$ is k-fold conically cocomplete if it is 1-fold conically cocomplete and for every pair of objects d, e in $\mathcal{D}$ the $(n-1)$-category $\mathrm{Hom}_{\mathcal{D}}(d, e)$ is $(k-1)$-fold conically cocomplete.*

Example 11.3.6. Recall the category $n\mathrm{Pr}^L$ of presentable n-categories from chapter 12. Each object D in $n\mathrm{Pr}^L$ has an underlying n-category $\psi_n(D)$. It follows from theorems 12.4.6 and 12.5.14 together with remark 12.3.3 that $\psi_n(D)$ is n-fold conically cocomplete and satisfies the $(n-1)$-fold passage to adjoints property.

Definition 11.3.7. *Let $\mathcal{C}$ be a category admitting pullbacks. We say that a map $f : x \to y$ in $\mathcal{P}(\mathcal{C})$ is representable if for every map $c \to y$ with c in $\mathcal{C}$, the presheaf $x \times_y c$ is representable.*

Remark 11.3.8. Let $\mathcal{C}$ be a category admitting pullbacks. The class of representable morphisms defines a subcategory $\mathcal{P}(\mathcal{C})_{\mathrm{rep}}$ of $\mathcal{P}(\mathcal{C})$. The category $\mathcal{P}(\mathcal{C})_{\mathrm{rep}}$ contains all pullbacks, and these are preserved by the inclusion into $\mathcal{P}(\mathcal{C})$.

We are now ready to state our extension theorem.

Theorem 11.3.9. *Let $n \geq 2$. Let $\mathcal{C}$ be a category admitting pullbacks and $\mathcal{D}$ be an $(n-1)$-fold conically cocomplete n-category satisfying the $(n-1)$-fold passage to adjoints property. Let $F : \mathcal{P}(\mathcal{C}) \to \mathcal{D}$ be a conical colimit preserving functor.*

(i) *If $F|_{\mathcal{C}}$ satisfies the left $(n-1)$-fold Beck-Chevalley condition, then for every pair of maps $\alpha : x \to s$ and $\beta : y \to s$ in $\mathcal{P}(\mathcal{C})$ where β is representable the commutative square*

$$\begin{tikzcd} F(x \times_s y) \arrow[r] \arrow[d] & F(y) \arrow[d, "F(\beta)"] \\ F(x) \arrow[r, "F(\alpha)"] & F(s) \end{tikzcd}$$

is $(n-1)$-fold vertically right adjointable. In particular, $F|_{\mathcal{P}(\mathcal{C})_{\mathrm{rep}}}$ satisfies the left $(n-1)$-fold Beck-Chevalley condition.

(ii) *If $F|_{\mathcal{C}} : \mathcal{C} \to \mathcal{D}$ and $(F|_{\mathcal{C}})^{n\text{-op}} : \mathcal{C} \to \mathcal{D}^{n\text{-op}}$ satisfy the left $(n-1)$-fold Beck-Chevalley conditions, then F satisfies the left $(n-1)$-fold beck-Chevalley condition.*

Before giving the proof of theorem 11.3.9, we study a few consequences.

Corollary 11.3.10. *Let $n \geq 3$. Let $\mathcal{C}$ be a category admitting pullbacks and let $\mathcal{D}$ be an $(n-2)$-fold conically cocomplete n-category satisfying the $(n-2)$-fold passage to adjoints property. Let $F : \mathcal{P}(\mathcal{C}) \to \mathcal{D}$ be a conical colimit preserving functor such that $F|_{\mathcal{C}}$ satisfies the left $(n-1)$-fold Beck-Chevalley condition. Then F satisfies the left $(n-2)$-fold Beck-Chevalley condition.*

Proof. By theorem 11.2.6, F admits an extension to $n\mathrm{Corr}(\mathcal{C})$. Recall from remark 10.2.6 that there is a natural equivalence $2\mathrm{Corr} = 2\mathrm{Corr}^{1\text{-op}}$ which restricts to the identity on underlying spaces. It follows by induction that for each $m \geq 3$ there is a natural equivalence between the functor $m\mathrm{Corr}$ and the functor $m\mathrm{Corr}^{(m-1)\text{-op}}$, which is an equivalence upon passage to underlying categories. In particular, we have an equivalence between $n\mathrm{Corr}(\mathcal{C})^{(n-1)\text{-op}}$ and $n\mathrm{Corr}(\mathcal{C})$ which is the identity on the underlying category. It follows that $F^{(n-1)\text{-op}}$ admits an extension to a functor from $n\mathrm{Corr}(\mathcal{C})$ into $\mathcal{D}^{(n-1)\text{-op}}$, and therefore $F^{(n-1)\text{-op}}$ satisfies the $(n-1)$-fold Beck-Chevalley condition. Our result now follows from theorem 11.3.9. $\square$

Remark 11.3.11. In the setting of theorem 11.3.9 item (i), it is not true in general that F will satisfy the full left $(n-1)$-fold Beck-Chevalley condition, although it does satisfy the left $(n-2)$-fold Beck-Chevalley condition thanks to corollary 11.3.10. In other words, the statement of part (i) in theorem 11.3.9 does not hold if we remove the representability condition. For example, the functor $\mathrm{mod} : \mathrm{CAlg}(\mathrm{Sp})^{\mathrm{op}} \to (\mathscr{P}r^L)^{1\text{-op}}$ that assigns to each commutative ring spectrum its category of modules satisfies the left Beck-Chevalley condition, but its left Kan extension along $\mathrm{CAlg}(\mathrm{Sp})^{\mathrm{op}} \to \mathcal{P}(\mathrm{CAlg}(\mathrm{Sp}))^{\mathrm{op}}$ is the functor that assigns to each (nonconnective) prestack its category of quasicoherent sheaves, which does not satisfy base change in all generality.

Definition 11.3.12. *Let $\mathcal{D}$ be an n-category. For each $0 \leq k < n$ we denote by $\mathcal{D}^{k\text{-adj}}$ the largest $(k+1)$-category contained inside $\mathcal{D}$ and such that all cells of dimension at most k admit both left and right adjoints. We say that an arrow α in $\mathcal{D}$ is k-fold adjointable if it belongs to $\mathcal{D}^{k\text{-adj}}$.*

Corollary 11.3.13. *Let $n \geq 3$. Let $\mathcal{C}$ be a category admitting pullbacks and let $\mathcal{D}$ be an $(n-1)$-fold conically cocomplete n-category satisfying the $(n-1)$-fold passage to adjoints property. Let $F : \mathcal{P}(\mathcal{C}) \to \mathcal{D}$ be a conical colimit preserving functor such that $F|_{\mathcal{C}}$ and $(F|_{\mathcal{C}})^{n\text{-op}}$ satisfy the left $(n-1)$-fold Beck-Chevalley condition. Let $\beta : x \to y$ be a morphism in $\mathcal{P}(\mathcal{C})$. Then $F(\beta)$ is $(n-2)$-fold adjointable, and its left and right adjoints are equivalent.*

Proof. Combine theorems 11.2.6 and 11.3.9 with proposition 11.1.9. □

Corollary 11.3.14. *Let $n \geq 2$. Let $\mathcal{C}$ be a category admitting pullbacks. Let $\mathcal{D}$ be an $(n-1)$-fold conically cocomplete symmetric monoidal n-category satisfying the $(n-1)$-fold passage to adjoints property. Equip $\mathcal{P}(\mathcal{C})$ with its cartesian symmetric monoidal structure. Let $F : \mathcal{P}(\mathcal{C}) \to \mathcal{D}$ be a symmetric monoidal conical colimit preserving functor such that $F|_{\mathcal{C}}$ and $(F|_{\mathcal{C}})^{n\text{-op}}$ satisfy the left $(n-1)$-fold Beck-Chevalley condition. Let x be an object in $\mathcal{P}(\mathcal{C})$. Then*

(i) *The object $F(x)$ in $\mathcal{D}$ is fully dualizable in $\mathcal{D}^{\leq n-1}$.*

(ii) *Let $\Delta : x \to x \times x$ be the diagonal map and $\pi : x \to *$ be the projection to the final object of $\mathcal{P}(\mathcal{C})$. If $F(\Delta)$ and $F(\pi)$ are $(n-1)$-fold adjointable then x is a fully dualizable object of $\mathcal{D}$.*

Proof. Combine theorem 11.3.9, remark 11.2.10 and corollary 11.1.10, with the description of the self duality of x in $2\mathrm{Corr}(\mathcal{P}(\mathcal{C}))$ from proposition 10.3.3 and remark 10.3.4. □

The proof of theorem 11.3.9 needs a few lemmas.

Lemma 11.3.15. *Let $\mathcal{C}$ be a category admitting pullbacks and $\mathcal{D}$ be a 2-category. Let $F : \mathcal{P}(\mathcal{C}) \to \mathcal{D}$ be a conical colimit preserving functor. Let*

$$\begin{tikzcd} x \times_s y \arrow[r, "\alpha'"] \arrow[d, "\beta'"] & y \arrow[d, "\beta"] \\ x \arrow[r, "\alpha"] & s \end{tikzcd} \qquad (\star)$$

be a cartesian square in $\mathcal{P}(\mathcal{C})$. Assume that for every arrow $c \to c'$ in $\mathcal{C}_{/s}$ the induced square

$$\begin{tikzcd} F(c \times_s y) \arrow[r] \arrow[d, "F(\beta_c)"] & F(c' \times_s y) \arrow[d, "F(\beta_{c'})"] \\ F(c) \arrow[r] & F(c') \end{tikzcd}$$

is vertically right adjointable. Then the image of $(\star)$ under F is vertically right adjointable.

Proof. Let $X^{\triangleright} : \mathcal{I}^{\triangleright} \to \mathcal{P}(\mathcal{C})$ be a colimit diagram such that its value on the cone point $X(*)$ recovers x, and such that $X = X^{\triangleright}|_{\mathcal{I}}$ factors through $\mathcal{C}$. Let

$$(\mathcal{I}^+)^{\triangleright} = (\mathcal{I} \sqcup [0])^{\triangleright} = \mathcal{I}^{\triangleright} \bigcup_{[0]} [1]$$

be the category obtained by adjoining a final object to $\mathcal{I} \sqcup [0]$. There is an evident inclusion

$$i : (\mathcal{I}^+)^{\triangleright} \to \mathcal{I}^{\triangleright} \times [1]$$

defined by the fact that it preserves final objects, and extends the inclusions $\mathcal{I} \xrightarrow{\mathrm{id}_{\mathcal{I}} \times 1} \mathcal{I} \times [1]$ and $[0] \xrightarrow{* \times 0} \mathcal{I} \times [1]$. Consider the functor $(X^{\triangleright})^+ : (\mathcal{I}^+)^{\triangleright} \to \mathcal{P}(\mathcal{C})$ which extends $X^{\triangleright}$ and such that $(X^{\triangleright})^+|_{[0]^{\triangleright}}$ recovers the arrow β'.

Let $Y^{\triangleright} : \mathcal{I}^{\triangleright} \times [1] \to \mathcal{P}(\mathcal{C})$ be the right Kan extension of $(X^{\triangleright})^+$ along i, and let $Y = Y^{\triangleright}|_{\mathcal{I} \times [1]}$. In other words, $Y^{\triangleright}$ is such that the restriction $Y^{\triangleright}|_{* \times [1]}$ recovers the arrow β', the restriction $Y^{\triangleright}|_{\mathcal{I} \times \{1\}}$ recovers $X^{\triangleright}$, and for every index i in $\mathcal{I}$ the induced commutative square

$$\begin{tikzcd} (Y(i), 0) \arrow[r] \arrow[d] & x \times_s y \arrow[d, "\beta'"] \\ (Y(i), 1) \arrow[r] & x \end{tikzcd}$$

is a pullback square. Since pullbacks distribute over colimits in presheaf categories, we have that $Y^{\triangleright}|_{\mathcal{I} \times \{j\}}$ is a colimit diagram in $\mathcal{P}(\mathcal{C})$ for $j = 0, 1$.

Let $Z^{\triangleright} : \mathcal{I}^{\triangleright} \to \mathrm{Funct}([1], \mathcal{C})$ be the functor associated to $Y^{\triangleright}$, and $Z = Z^{\triangleright}|_{\mathcal{I}}$. Note that the functor $FZ^{\triangleright} : \mathcal{I}^{\triangleright} \to \mathrm{Funct}([1], \mathcal{D})$ is a conical colimit diagram for FZ. Our hypothesis on F together with remark 10.4.2 imply that the functor FZ factors through $\mathrm{Funct}(\mathrm{Adj}, \mathcal{D})$. Since $\mathcal{D}$ is conically cocomplete, we conclude that $FZ^{\triangleright}$ also factors through $\mathrm{Funct}(\mathrm{Adj}, \mathcal{D})$, and in fact defines a conical colimit diagram in $\mathrm{Funct}(\mathrm{Adj}, \mathcal{D})$. In particular, for all objects i in $\mathcal{I}$, the square

$$\begin{tikzcd} F(X(i) \times_x x \times_s y) \arrow[r] \arrow[d, "F(\beta_{X(i)})"] & F(x \times_s y) \arrow[d, "F(\beta')"] \\ F(X(i)) \arrow[r] & F(x) \end{tikzcd}$$

is vertically right adjointable.

Repeating the above argument with the conical colimit diagram $\mathcal{C}_{/s}^{\triangleright} \to \mathcal{P}(\mathcal{C})$ shows that for every i in $\mathcal{I}$ the square

$$\begin{tikzcd} F(X(i) \times_x x \times_s y) \arrow[r] \arrow[d, "F(\beta_{X(i)})"] & F(y) \arrow[d, "F(\beta)"] \\ F(X(i)) \arrow[r] & F(s) \end{tikzcd}$$

is vertically right adjointable. In other words, the morphism $F(\beta') \to F(\beta)$ in $\mathrm{Funct}([1], \mathcal{D})$ induced by the square $(\star)$ is such that its composition with the map $F(\beta_X(i)) \to F(\beta')$ factors through the subcategory $\mathrm{Funct}(\mathrm{Adj}, \mathcal{D})$. Since $FZ^{\triangleright}$ is a conical colimit diagram we conclude that the map $F(\beta') \to F(\beta)$ factors through $\mathrm{Funct}(\mathrm{Adj}, \mathcal{D})$, which means that $(\star)$ is vertically right adjointable, as desired. $\square$

Lemma 11.3.16. *Let $\mathcal{C}$ be a category admitting pullbacks, and $\mathcal{D}$ be a 2-category. Assume that $\mathcal{D}$ is satisfies the passage to adjoints property. Let $F : \mathcal{P}(\mathcal{C}) \to \mathcal{D}$ be a conical colimit preserving functor and assume that $F|_{\mathcal{C}}$ satisfies the left and right Beck-Chevalley conditions. Then F satisfies the left Beck-Chevalley condition.*

Proof. Let

$$\begin{tikzcd} x \times_s y \arrow[r, "\alpha'"] \arrow[d, "\beta'"] & y \arrow[d, "\beta"] \\ x \arrow[r, "\alpha"] & s \end{tikzcd}$$

be a cartesian square in $\mathcal{P}(\mathcal{C})$. We have to show that its image under F is vertically right adjointable. By virtue of lemma 11.3.15, we may assume that x and s are representable.

Let $X^{\triangleright} : \mathcal{I}^{\triangleright} \to \mathcal{P}(\mathcal{C})$ be a colimit diagram such that $X = X^{\triangleright}|_{\mathcal{I}}$ factors through $\mathcal{C}$, and its value on the cone point $X^{\triangleright}(*)$ recovers y. Since F preserves conical colimits we have that $FX^{\triangleright}$ is a conical colimit diagram.

Note that the composite morphism

$$F(X(i)) \to F(y) \xrightarrow{F(\beta)} F(s)$$

admits a right adjoint for every i in $\mathcal{I}$. Since $\mathcal{D}$ satisfies the passage to adjoints property and FX factors through $(\mathcal{D}^{\leq 1})^{\mathrm{radj}}$, we conclude that $F(\beta)$ admits a right adjoint in $\mathcal{D}$. A similar argument guarantees that $F(\beta')$ admits a right adjoint.

Using remark 10.4.3 we now see that in order to show that the image of our square under F is vertically right adjointable it suffices to show that it is horizontally left adjointable. Applying the dual version of lemma 11.3.15 we may assume that our square takes place in $\mathcal{C}$, and in this case the conclusion follows from the fact that $F|_{\mathcal{C}}$ satisfies the right Beck-Chevalley condition. $\square$

Lemma 11.3.17. *Let $\mathcal{D}$ be an n-category for $n \geq 3$ and let*

$$\begin{tikzcd} \tau_{d'} \arrow[r, "\tau_{\alpha'}"] \arrow[d, "\tau_{\beta'}"] & \tau_d \arrow[d, "\tau_\beta"] \\ \tau_{e'} \arrow[r, "\tau_\alpha"] & \tau_e \end{tikzcd} \qquad (\star)$$

be a commutative square in $\mathrm{Funct}([1], \mathcal{D})$*, corresponding to a commutative cube as follows:*

$$\begin{tikzcd} & d'_1 \arrow[rr, "\alpha'_1"] \arrow[dd, "\beta'_1"] & & d_1 \arrow[dd, "\beta_1"] \\ d'_0 \arrow[ur, "\tau_{d'}"] \arrow[rr, "\alpha'_0"] \arrow[dd, "\beta'_0"] & & d_0 \arrow[ur, "\tau_d"] \arrow[dd, "\beta_0"] & \\ & e'_1 \arrow[rr, "\alpha_1"] & & e_1 \\ e'_0 \arrow[ur, "\tau_{e'}"] \arrow[rr, "\alpha_0"] & & e_0 \arrow[ur, "\tau_e"] & \end{tikzcd}$$

Then the square $(\star)$ is $(n-1)$-fold vertically right adjointable if and only if the front and back faces of the cube are $(n-1)$-fold vertically right adjointable, the left and right faces are vertically right adjointable, and the top and bottom faces are horizontally right adjointable.

Proof. It follows from remark 10.4.2 that:

- The morphisms $\tau_{\alpha'}$ and τ_α are right adjointable if and only if the top and bottom faces of the cube are horizontally right adjointable.
- The morphisms $\tau_{\beta'}$ and τ_β are right adjointable if and only if the left and right faces of the cube are vertically right adjointable.

Assume that both both of the items above hold. It follows from lemma 10.4.12 that the square $(\star)$ is vertically right adjointable if and only if the front and back faces of the cube are vertically right adjointable. Consider now the pullback square

$$\begin{tikzcd} \mathrm{End}_{\mathrm{Funct}([1],\mathcal{D})}(\tau_e) \arrow[r, "\mathrm{ev}_0"] \arrow[d, "\mathrm{ev}_1"] & \mathrm{End}_{\mathcal{D}}(e_0) \arrow[d, "(\tau_e)_*"] \\ \mathrm{End}_{\mathcal{D}}(e_1) \arrow[r, "(\tau_e)^*"] & \mathrm{Hom}_{\mathcal{D}}(e_0, e_1). \end{tikzcd}$$

Using remark 11.2.4 we see that a commutative square in $\mathrm{End}_{\mathrm{Funct}([1],\mathcal{D})}(\tau_e)$ is $(n-2)$-fold vertically right adjointable if and only if its image under ev_0 and ev_1 is $(n-2)$-fold vertically right adjointable. The lemma now follows by applying this fact to the square associated to $(\star)$ under construction 11.2.1. $\square$

Lemma 11.3.18. *Let $\mathcal{C}$ be a category admitting pullbacks and $\mathcal{D}$ be a 2-category. Assume that $\mathcal{D}$ satisfies the passage to adjoints property. Let $F : \mathcal{P}(\mathcal{C}) \to \mathcal{D}$ be a conical colimit preserving functor satisfying the left Beck-Chevalley condition. Let c, c' be objects of $\mathcal{P}(\mathcal{C})$. Then the induced functor*

$$F_* : \mathrm{Hom}_{2\mathrm{Corr}(\mathcal{P}(\mathcal{C}))}(\iota_{\mathcal{P}(\mathcal{C})}(c), \iota_{\mathcal{P}(\mathcal{C})}(c')) \to \mathrm{Hom}_{\mathcal{D}}(F(c), F(c'))$$

is colimit preserving.

Proof. We first show that F_* preserves (infinite) coproducts. Consider a family of spans

$$\begin{tikzcd} & x_i \arrow[dl, "a_i"'] \arrow[dr, "b_i"] & \\ c & & c' \end{tikzcd}$$

in $\mathcal{P}(\mathcal{C})$, indexed by a set $\mathcal{I}$. Let

$$\begin{tikzcd} & x \arrow[dl, "a"'] \arrow[dr, "b"] & \\ c & & c' \end{tikzcd}$$

be their coproduct, and for each i denote by $j_i : x_i \to x$ the induced map. We must show that the family of morphisms

$$F(b_i)F(a_i)^R = F(b)F(j_i)F(j_i)^R F(a)^R \to F(b)F(a)^R$$

induced from the counit maps $F(j_i)F(j_i)^R \to \mathrm{id}_{F(x)}$ exhibit $F(b)F(a)^R$ as the coproduct of the family $\{F(b_i)F(a_i)^R\}_{i \in \mathcal{I}}$ inside $\mathrm{Hom}_{\mathcal{D}}(F(c), F(c'))$.

Since $\mathcal{D}$ satisfies the passage to adjoints property, we have that the functors

$$(F(j_i)^R)_* : \mathrm{Hom}_{\mathcal{D}}(F(c), F(x)) \to \mathrm{Hom}_{\mathcal{D}}(F(c), F(x_i))$$

exhibit the left hand side as the product of the categories on the right hand side. It follows that the natural transformations $(F(j_i)F(j_i)^R)_* \to \mathrm{id}_{\mathrm{Hom}_{\mathcal{D}}(F(c),F(x))}$ induced from the counits of the adjunctions $F(j_i) \dashv F(j_i)^R$ exhibit $\mathrm{id}_{\mathrm{Hom}_{\mathcal{D}}(F(c),F(x))}$ as the coproduct of the endofunctors $(F(j_i)F(j_i)^R)_*$. Since $F(b)_*$ is right adjointable we see that the induced natural transformations $(F(b)F(j_i)F(j_i)^R)_* \to F(b)_*$ exhibit $F(b)_*$ as the coproduct of the functors $F(b)F(j_i)F(j_i)^R = F(b_i)F(j_i)^R$. Our claim now follows from this by evaluation at $F(a)^R$.

We now show that F_* preserves pushouts. Consider a cospan of spans

$$\begin{tikzcd} & x_0 \arrow[dl, "\mu"'] \arrow[dr, "\nu"] \arrow[ddl, "a_0"] \arrow[ddr, "b_0"'] & \\ x_1 \arrow[d, "a_1"'] \arrow[drr, "b_1"] & & x_2 \arrow[d, "b_2"] \arrow[dll, "a_2"'] \\ c & & c' \end{tikzcd}$$

and let

$$\begin{tikzcd} & x \arrow[dl, "a"'] \arrow[dr, "b"] & \\ c & & c' \end{tikzcd}$$

be its pushout. For each i denote by $j_i : x_i \to x$ the associated map. We must show that the induced square

$$\begin{tikzcd} F(b_0)F(a_0)^R \arrow[r] \arrow[d] & F(b_2)F(a_2)^R \arrow[d] \\ F(b_1)F(a_1)^R \arrow[r] & F(b)F(a)^R \end{tikzcd}$$

in $\mathrm{Hom}_{\mathcal{D}}(F(c), F(c'))$ is a pushout square. As before, since $\mathcal{D}$ satisfies the passage to adjoints property, we have that the induced commutative square of categories

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{D}}(F(c), F(x)) \arrow[r, "(F(j_2)^R)_*"] \arrow[d, "(F(j_1)^R)_*"] & \mathrm{Hom}_{\mathcal{D}}(F(c), F(x_2)) \arrow[d, "F(\nu)^R_*"] \\ \mathrm{Hom}_{\mathcal{D}}(F(c), F(x_1)) \arrow[r, "F(\mu)^R_*"] & \mathrm{Hom}_{\mathcal{D}}(F(c), F(x_0)) \end{tikzcd}$$

is a pullback square. It follows that the commutative square

$$\begin{tikzcd} (F(j_0)F(j_0)^R)_* \arrow[r] \arrow[d] & (F(j_2)F(j_2)^R)_* \arrow[d] \\ (F(j_1)F(j_1)^R)_* \arrow[r] & \mathrm{id}_{\mathrm{Hom}_{\mathcal{D}}(F(c),F(x))} \end{tikzcd}$$

induced from the counit of the adjunctions $F(j_i) \dashv F(j_i)^R$, is a pushout square. Since $F(b)_*$ is a left adjoint, we obtain that the following commutative square in the category $\mathrm{Funct}(\mathrm{Hom}_{\mathcal{D}}(F(c), F(x)), \mathrm{Hom}_{\mathcal{D}}(F(c), F(c')))$ is a pushout square:

$$\begin{tikzcd} (F(b)F(j_0)F(j_0)^R)_* \arrow[r] \arrow[d] & (F(b)F(j_2)F(j_2)^R)_* \arrow[d] \\ (F(b)F(j_1)F(j_1)^R)_* \arrow[r] & F(b)_* \end{tikzcd}$$

Our result now follows from this by evaluation at $F(a)^R$. □

Proof of theorem 11.3.9. We begin by proving item (ii). We argue by induction on n. The case $n = 2$ is proven in lemma 11.3.16, so assume $n > 2$.

It follows from our inductive hypothesis (in the form of corollary 11.3.10) that F satisfies the left Beck-Chevalley condition. Let

$$\begin{tikzcd}
x \times_s y \arrow[r, "\alpha'"] \arrow[d, "\beta'"] & y \arrow[d, "\beta"] \\
x \arrow[r, "\alpha"] & s
\end{tikzcd}$$

be a cartesian square in $\mathcal{P}(\mathcal{C})$. Let $S^{\triangleright} : \mathcal{I}^{\triangleright} \to \mathcal{P}(\mathcal{C})$ be a colimit diagram such that its value on the cone point $S(*)$ recovers s, and $S = S^{\triangleright}|_{\mathcal{I}}$ factors through $\mathcal{C}$. Arguing as in the proof of lemma 11.3.15, we may construct a functor $T^{\triangleright} : \mathcal{I}^{\triangleright} \times ([1] \times [1])$ such that $T^{\triangleright}|_{\mathcal{I}^{\triangleright} \times \{(1,1)\}}$ recovers $S^{\triangleright}$, the functor $T^{\triangleright}|_{* \times ([1] \times [1])}$ is given by the above square, and moreover for every object i in $\mathcal{I}$ the cube

$$\begin{tikzcd}
& x \times_s y \arrow[rr, "\alpha'"] \arrow[dd, "\beta'"] & & y \arrow[dd, "\beta"] \\
T^{\triangleright}(i,(0,0)) \arrow[ur] \arrow[rr] \arrow[dd] & & T^{\triangleright}(i,(1,0)) \arrow[ur] \arrow[dd] & \\
& x \arrow[rr, "\alpha"] & & s \\
T^{\triangleright}(i,(0,1)) \arrow[ur] \arrow[rr] & & T^{\triangleright}(i,(1,1)) \arrow[ur] &
\end{tikzcd}$$

has all its faces cartesian. In particular, for every object (j,k) in $[1] \times [1]$ the restriction $T^{\triangleright}|_{\mathcal{I}^{\triangleright} \times \{(j,k)\}}$ is a colimit diagram in $\mathcal{P}(\mathcal{C})$.

Let $R^{\triangleright} : \mathcal{I}^{\triangleright} \to \mathrm{Funct}([1] \times [1], \mathcal{P}(\mathcal{C}))$ be the induced functor. The composite functor $FR^{\triangleright} : \mathcal{I}^{\triangleright} \to \mathrm{Funct}([1], [1], \mathcal{D})$ is a conical colimit diagram, and its value on the cone point $*$ recovers the image under F of our original square.

Recall from construction 11.2.3 the universal $(n-1)$-fold vertically right adjointable square $i_n : [1] \times [1] \to \mathcal{U}_n$. Our task is to show that $FR^{\triangleright}(*)$ belongs to $\mathrm{Funct}(\mathcal{U}_n, \mathcal{D})$. This would follow if we are able to show that $FR^{\triangleright}|_{\mathcal{I}}$ factors through $\mathrm{Funct}(\mathcal{U}_n, \mathcal{D})$. Using lemma 11.3.17 together with the fact that F satisfies the left Beck-Chevalley condition, we reduce to showing that $FR^{\triangleright}(i)$ belongs to $\mathrm{Funct}(\mathcal{U}_n, \mathcal{D})$ for every i in $\mathcal{I}$. In other words, we must show that the front face of the above cube is $(n-1)$-fold vertically right adjointable. This face is a cartesian square in $\mathcal{P}(\mathcal{C})$, and its lower right corner $c = T^{\triangleright}(i,(1,1))$ is representable.

It follows from lemma 11.2.8 that the commutative square in $\mathrm{End}_{2\mathrm{Corr}(\mathcal{P}(\mathcal{C}))}(\iota_{\mathcal{P}(\mathcal{C})}(c))$ associated to the front face of the cube by construction 11.2.1 is cartesian. It therefore suffices to show that the functor

$$\mathrm{End}_{2\mathrm{Corr}(\mathcal{P}(\mathcal{C}))}(\iota_{\mathcal{P}(\mathcal{C})}(c)) \to \mathrm{End}_{\mathcal{D}}(F(c))$$

induced from F satisfies the left $(n-2)$-fold Beck-Chevalley condition. Using [GHN17] corollary 9.9, we obtain equivalences

$$\mathcal{P}(\mathrm{End}_{2\mathrm{Corr}(\mathcal{C})}(\iota_{\mathcal{C}}(c))) = \mathcal{P}(\mathcal{C}_{/c,c}) = \mathcal{P}(\mathcal{C})_{/c,c} = \mathrm{End}_{2\mathrm{Corr}(\mathcal{P}(\mathcal{C}))}(\iota_{\mathcal{P}(\mathcal{C})}(c)).$$

Our result now follows from our inductive hypothesis together with lemmas 11.3.18 and 11.2.9.

We now give a proof of item (i). We proceed along the same lines as the proof of item (ii) carried out above, with suitable modifications. Again we argue by induction on n. The case $n = 2$ follows directly from lemma 11.3.15. Assume now that $n > 2$. Using our inductive hypothesis (in the form of corollary 11.3.10) we see that F satisfies the left Beck-Chevalley condition. Let

$$\begin{tikzcd} x \times_s y \arrow[r, "\alpha'"] \arrow[d, "\beta'"] & y \arrow[d, "\beta"] \\ x \arrow[r, "\alpha"] & s \end{tikzcd}$$

be a cartesian square in $\mathcal{P}(\mathcal{C})$, where β is representable. We have to show that its image under F is $(n-1)$-fold vertically right adjointable. Arguing in the same way as in the proof of item (ii), we may reduce to the case when s belongs to $\mathcal{C}$. Since β is representable, we also have that y belongs to $\mathcal{C}$.

As before, the square in $\mathrm{End}_{2\mathrm{Corr}(\mathcal{P}(\mathcal{C}))}(\iota_{\mathcal{P}(\mathcal{C})}(s))$ associated to the above square via construction 11.2.1, is cartesian. It follows from lemma 11.2.8 that the right vertical side of that square corresponds to the morphism of spans

$$\begin{tikzcd} & y \arrow[d, "\beta"] \arrow[ddl, "\beta"'] \arrow[ddr, "\beta"] & \\ & s \arrow[dl, "\mathrm{id}"] \arrow[dr, "\mathrm{id}"'] & \\ s & & s. \end{tikzcd}$$

Our result now follows again from our inductive hypothesis combined with lemmas 11.3.18 and 11.2.9, using the fact that the above diagram defines a representable morphism in $\mathcal{P}(\mathcal{C}_{/s,s})$. $\square$

Chapter 12

Presentable n-categories

Let $\mathcal{M}$ be a presentable symmetric monoidal category. Then any presentable category $\mathcal{C}$ with an action of $\mathcal{M}$ compatible with colimits has a natural structure of $\mathcal{M}$-enriched category. As we discussed in chapter 4, this procedure gives rise to a lax symmetric monoidal functor from the symmetric monoidal category $\mathcal{M}\text{-mod}_{\text{pr}}$ of $\mathcal{M}$-modules in Pr^L to the category $\text{Cat}^{\mathcal{M}}$ of $\mathcal{M}$-enriched categories. Under certain conditions, a category equipped with an action of $\mathcal{M}\text{-mod}_{\text{pr}}$ inherits the structure of an $\mathcal{M}\text{-mod}_{\text{pr}}$-enriched category, and it therefore also has the structure of a category enriched in $\text{Cat}^{\mathcal{M}}$ - in other words, a 2-category enriched in $\mathcal{M}$. Our goal in this chapter is to introduce higher versions of this procedure that allow one to produce n-categories enriched in $\mathcal{M}$ for all values of n.

A naive iteration of the enrichment procedure described above runs into set theoretical difficulties: even though $\mathcal{M}$ is presentable, the category $\mathcal{M}\text{-mod}_{\text{pr}}$ is no longer presentable in general as it is generally not locally small. It therefore does not make sense to consider its category of modules in Pr^L. However, we still have that $\mathcal{M}\text{-mod}_{\text{pr}}$ admits all small colimits and its tensor product structure preserves all colimits, so one may consider the symmetric monoidal category $\mathcal{M}\text{-mod}^2$ of $\mathcal{M}\text{-mod}_{\text{pr}}$-modules in the symmetric monoidal category $\widehat{\text{Cat}}_{\text{cocompl}}$ of large cocomplete categories. One can then define a functor from $\mathcal{M}\text{-mod}^2$ to the category of 2-categories enriched in a suitable completion of $\mathcal{M}$.

If one attempts to repeat the above to obtain enriched 3-categories one runs into a similar problem than in the 2-categorical case. Again the category $\mathcal{M}\text{-mod}^2$ is not presentable. However, $\mathcal{M}\text{-mod}^2$ is now not large, but very large. It therefore does not make sense to consider its category of modules in $\widehat{\text{Cat}}_{\text{cocompl}}$. One way around this problem is to consider instead its category of modules in the category $\text{CAT}_{\text{cocompl}}$ of very large categories admitting all large colimits. This can be iterated to yield a theory that works for all values of n, however this requires one to work with an infinite sequence of nested universes and keep careful track of the relative sizes of various objects.

In this thesis we pursue a different approach that is based on the observation that although $\mathcal{M}\text{-mod}^2$ is very large, it is controlled by a large cocomplete category $\mathcal{M}\text{-mod}^2_{\text{pr}}$ of so called presentable $\mathcal{M}\text{-mod}_{\text{pr}}$-modules. It therefore makes sense to consider the symmetric monoidal category $\mathcal{M}\text{-mod}^3$ of modules for $\mathcal{M}\text{-mod}^2_{\text{pr}}$ inside $\widehat{\text{Cat}}_{\text{cocompl}}$, which is again controlled by a subcategory $\mathcal{M}\text{-mod}^3_{\mathrm{pr}}$ of presentable $\mathcal{M}\text{-mod}^2_{\mathrm{pr}}$-modules. Iterating this reasoning yields symmetric monoidal categories $\mathcal{M}\text{-mod}^n$ and $\mathcal{M}\text{-mod}^n_{\mathrm{pr}}$ for all $n \geq 1$, which admit functors into the category of n-categories enriched in a suitable completion of $\mathcal{M}$.

We begin in 12.1 by introducing the notion of presentable module over an arbitrary monoidal cocomplete category $\mathcal{E}$. This is obtained as a special case of the notion of κ-compactness in presentable categories in the case when $\kappa = \kappa_0$ is the smallest large cardinal, applied to the very large presentable category $\mathcal{E}\text{-mod}(\widehat{\mathrm{Cat}}_{\mathrm{cocompl}})$. We show that if A is an algebra object in $\mathcal{E}$ then the category of right A-modules is a presentable left $\mathcal{E}$-module. In the case when $\mathcal{E}$ is a presentable monoidal category, we show that the category $\mathcal{E}\text{-mod}_{\mathrm{pr}}$ of presentable $\mathcal{E}$-modules agrees with the category $\mathcal{E}\text{-mod}(\mathrm{Pr}^L)$ of $\mathcal{E}$-modules in Pr^L. When $\mathcal{E} = \mathrm{Spc}$ this yields a characterization of $\mathrm{Spc}\text{-mod}_{\mathrm{pr}} = \mathrm{Pr}^L$ as the full subcategory of $\widehat{\mathrm{Cat}}_{\mathrm{cocompl}}$ on the κ_0-compact objects.

In 12.2 we iterate the above to obtain symmetric monoidal categories $\mathcal{E}\text{-mod}^n$ and $\mathcal{E}\text{-mod}^n_{\mathrm{pr}}$ attached to each symmetric monoidal cocomplete category $\mathcal{E}$. Specializing to the case $\mathcal{E} = \mathrm{Spc}$, we obtain a symmetric monoidal category $n\mathrm{Pr}^L = \mathrm{Spc}\text{-mod}^n_{\mathrm{pr}}$ whose objects we call presentable n-categories. In the case when $\mathcal{E}$ is the category of spectra, this yields a full subcategory $n\mathrm{Pr}^L_{\mathrm{St}}$ of $n\mathrm{Pr}^L$ whose objects we call presentable stable n-categories. Applying the results from 12.1 inductively, we are able to conclude that for every commutative ring spectrum A there is an associated presentable stable n-category $A\text{-mod}^n$ of A-linear presentable stable $(n-1)$-categories.

In 12.3 we use the theory from chapter 4 to construct a lax symmetric monoidal functor $\psi_n : \mathrm{Spc}\text{-mod}^n \to \widehat{n\mathrm{Cat}}$. In other words, even though our approach to presentable n-categories is 1-categorical in nature, we are able to upgrade presentable n-categories to honest n-categories. We use the realization functor ψ_n to obtain $(n+1)$-categorical enhancements $n\mathscr{P}r^L$ and $n\mathscr{C}at^L$ for $n\mathrm{Pr}^L$ and $\mathrm{Spc}\text{-mod}^n$, respectively.

In 12.4 we establish our first main result regarding the theory of presentable n-categories, theorem 12.4.6: if $\mathcal{C}$ belongs to $\mathrm{Spc}\text{-mod}^n$ then $\psi_n(\mathcal{C})$ is a conically cocomplete n-category, and moreover any limit that exists in $\mathcal{C}$ is a conical limit in $\psi_n(\mathcal{C})$. In particular, we conclude that $n\mathscr{P}r^L$ is a conically cocomplete n-category.

A fundamental feature of the 2-category $\mathscr{P}r^L$ is that colimits of right adjointable diagrams can be computed as limits after passage to right adjoints. In 12.5 we establish a generalization of this statement (theorem 12.5.14) : we show that for every object $\mathcal{C}$ in $\mathrm{Spc}\text{-mod}^n$ the n-category $\psi_n(\mathcal{C})$ satisfies the so-called passage to adjoints property. In particular, we are able to conclude that $\psi_n(\mathcal{C})$ admits limits of left adjointable diagrams.

12.1 Presentable modules over cocomplete monoidal categories

We begin by discussing the notion of smallness for objects of a category with large colimits. This is a special case of the notion of κ-compactness from [Lur09a] section 5.3.4, in the case when κ is the smallest large cardinal.

Notation 12.1.1. Denote by κ_0 the smallest large cardinal. In other words, κ_0 is such that κ_0-small spaces are what we usually call small spaces. We denote by $\widehat{\mathrm{Cat}}_{\mathrm{cocompl}}$ the category of large categories admitting all small colimits, and functors which preserve small colimits. As usual, we denote by Pr^L the full subcategory of $\widehat{\mathrm{Cat}}_{\mathrm{cocompl}}$ on the presentable categories.

Definition 12.1.2. *Let $\mathcal{C}$ be a very large, locally large category admitting all large colimits. An object c in $\mathcal{C}$ is said to be κ_0-compact if the functor $\mathcal{C} \to \widehat{\mathrm{Spc}}$ corepresented by c preserves large κ_0-filtered colimits. We say that $\mathcal{C}$ is κ_0-compactly generated if it is generated under large colimits by its κ_0-compact objects, and the space of κ_0-compact objects in $\mathcal{C}$ is large.*

Remark 12.1.3. Let $\mathcal{C}$ be a very large, locally large category admitting all large colimits. Then the collection of κ_0-compact objects is closed under small colimits in $\mathcal{C}$ ([Lur09a] corollary 5.3.4.15). Moreover, if $\mathcal{C}$ is κ_0-compactly generated then it can be recovered from its full subcategory of κ_0-compact objects by freely adjoining large κ_0-filtered colimits.

Proposition 12.1.4. *The category $\widehat{\mathrm{Cat}}_{\mathrm{cocompl}}$ is κ_0-compactly generated. Moreover, an object of $\widehat{\mathrm{Cat}}_{\mathrm{cocompl}}$ is κ_0-compact if and only if it is a presentable category.*

Proof. The fact that $\widehat{\mathrm{Cat}}_{\mathrm{cocompl}}$ is presentable as a very large category follows from [Lur17] lemma 4.8.4.2. Let $U : \widehat{\mathrm{Cat}}_{\mathrm{cocompl}} \to \widehat{\mathrm{Cat}}$ be the forgetful functor. It follows from [Lur09a] proposition 5.5.7.11 that U preserves κ_0-filtered colimits, and therefore its left adjoint $\mathcal{P} : \widehat{\mathrm{Cat}} \to \widehat{\mathrm{Cat}}_{\mathrm{cocompl}}$ sends κ_0-compact objects to κ_0-compact objects. Note that $\widehat{\mathrm{Cat}}$ is κ_0-compactly generated by the small categories. Since U is furthermore conservative we conclude that $\widehat{\mathrm{Cat}}_{\mathrm{cocompl}}$ is generated under large colimits by the objects of the form $\mathcal{P}(\mathcal{I})$ where $\mathcal{I}$ is a small category. In particular we conclude that $\widehat{\mathrm{Cat}}_{\mathrm{cocompl}}$ is κ_0-compactly generated.

We now show that Pr^L is closed under small colimits inside $\widehat{\mathrm{Cat}}_{\mathrm{cocompl}}$. Recall from [Lur09a] section 5.5.7 that Pr^L is the union of its subcategories Pr^L_κ of κ-compactly generated categories and functors that preserve κ-compact objects, for κ a small regular cardinal. The functor $\mathrm{Pr}^L_\kappa \to \mathrm{Cat}$ sending each object $\mathcal{C}$ to its subcategory of κ-compact objects induces an equivalence between Pr^L_κ and the subcategory $\mathrm{Cat}^{\mathrm{rex}(\kappa),\mathrm{id}}$ of Cat consisting of idempotent complete categories with κ-small colimits and functors that preserve those colimits. It follows from [Lur17] lemma 4.8.4.2 that Pr^L_κ is presentable and in particular it has all small colimits.

For $\tau > \kappa$ a pair of regular cardinals, the inclusion $\mathrm{Pr}^L_\kappa \to \mathrm{Pr}^L_\tau$ is equivalent to the functor $\mathrm{Cat}^{\mathrm{rex}(\kappa),\mathrm{id}} \to \mathrm{Cat}^{\mathrm{rex}(\tau),\mathrm{id}}$ which freely adjoins τ-small κ-filtered colimits. This is a left adjoint, and so it follows that the inclusion $\mathrm{Pr}^L_\kappa \to \mathrm{Pr}^L_\tau$ preserves small colimits. Similarly, for each κ the inclusion $\mathrm{Pr}^L_\kappa \to \widehat{\mathrm{Cat}}_{\mathrm{cocompl}}$ is equivalent to the functor $\mathrm{Cat}^{\mathrm{rex}(\kappa),\mathrm{id}} \to \widehat{\mathrm{Cat}}_{\mathrm{cocompl}}$ that freely adjoins small κ-filtered colimits, and so it preserves small colimits.

Note that Pr^L is the colimit of the categories Pr^L_κ in $\widehat{\mathrm{Cat}}$, and in particular also in the category of very large categories. Moreover, the inclusion $\mathrm{Pr}^L \to \widehat{\mathrm{Cat}}_{\mathrm{cocompl}}$ is induced by passing to the colimit the inclusions $\mathrm{Pr}^L_\kappa \to \widehat{\mathrm{Cat}}_{\mathrm{cocompl}}$. It follows from [Lur09a] proposition 5.5.7.11 that Pr^L admits small colimits and the inclusion $\mathrm{Pr}^L \to \widehat{\mathrm{Cat}}_{\mathrm{cocompl}}$ preserves small colimits. In other words, Pr^L is closed under small colimits in $\widehat{\mathrm{Cat}}_{\mathrm{cocompl}}$, as we claimed.

It remains to show that Pr^L is generated under small colimits by the objects $\mathcal{P}(\mathcal{I})$ for $\mathcal{I}$ in Cat. Indeed, for each small regular cardinal κ the forgetful functor $\mathrm{Pr}^L_\kappa \to \mathrm{Cat}$ is conservative and preserves κ-filtered colimits, which implies that Pr^L_κ is generated under colimits by the image of $\mathcal{P}|_{\mathrm{Cat}}$. Our claim now follows from the fact that Pr^L is the union of the subcategories Pr^L_κ, and that the inclusions $\mathrm{Pr}^L_\kappa \to \mathrm{Pr}^L$ preserve small colimits. $\square$

Recall from [Lur17] section 4.8 that $\widehat{\mathrm{Cat}}_{\mathrm{cocompl}}$ has a symmetric monoidal structure $\otimes$ such that for each triple of objects $\mathcal{C}, \mathcal{D}, \mathcal{E}$, the space of morphisms $\mathcal{C} \otimes \mathcal{D} \to \mathcal{E}$ is equivalent to the space of functors $\mathcal{C} \times \mathcal{D} \to \mathcal{E}$ which are colimit preserving in each coordinate. The unit of this symmetric monoidal structure is the category Spc of spaces. This symmetric monoidal structure is compatible with large colimits, in the sense that the functor

$$\otimes : \widehat{\mathrm{Cat}}_{\mathrm{cocompl}} \times \widehat{\mathrm{Cat}}_{\mathrm{cocompl}} \to \widehat{\mathrm{Cat}}_{\mathrm{cocompl}}$$

preserves large colimits in each variable. Moreover, the subcategory Pr^L is closed under tensor product and therefore Pr^L also inherits a symmetric monoidal structure which is compatible with small colimits.

Notation 12.1.5. Let $\mathcal{E}$ be an algebra object in $\widehat{\mathrm{Cat}}_{\mathrm{cocompl}}$ (in other words, $\mathcal{E}$ is a large cocomplete category equipped with a monoidal structure compatible with colimits). We denote by $\mathcal{E}\text{-mod}^l$ (resp. $\mathcal{E}\text{-mod}^r$) the category of left (resp. right) $\mathcal{E}$-modules in $\widehat{\mathrm{Cat}}_{\mathrm{cocompl}}$.

For each algebra A in $\mathcal{E}$ we denote by $A\text{-mod}^l$ (resp. $A\text{-mod}^r$) the category of left (resp. right) A-modules, thought of as an object of $\mathcal{E}\text{-mod}^r$ (resp. $\mathcal{E}\text{-mod}^l$).

If $\mathcal{E}$ is a commutative algebra, we will use the notation $\mathcal{E}\text{-mod}$ for $\mathcal{E}\text{-mod}^l = \mathcal{E}\text{-mod}^r$. If A is a commutative algebra in $\mathcal{E}$, we will use the notation $A\text{-mod}$ instead of $A\text{-mod}^l = A\text{-mod}^r$.

Warning 12.1.6. Let $\mathcal{E}$ be an algebra object in $\widehat{\mathrm{Cat}}_{\mathrm{cocompl}}$, and assume that $\mathcal{E}$ is furthermore presentable. Then $\mathcal{E}\text{-mod}^l$ has two different meanings. If we think about $\mathcal{E}$ as an algebra in $\widehat{\mathrm{Cat}}_{\mathrm{cocompl}}$ then $\mathcal{E}\text{-mod}^l$ denotes the category of cocomplete categories with an action of $\mathcal{E}$. However if we think about $\mathcal{E}$ as an algebra in Pr^L (which is itself a commutative algebra in $\widehat{\mathrm{Cat}}_{\mathrm{cocompl}}$) then $\mathcal{E}\text{-mod}^l$ can be taken to mean the category of presentable categories with an action of $\mathcal{E}$. We will consider modules in $\widehat{\mathrm{Cat}}_{\mathrm{cocompl}}$ by default, unless it is clear from context that $\mathcal{E}$ is to be considered specifically as an algebra in Pr^L. When in doubt, we will make the context clear in our notation by writing $\mathcal{E}\text{-mod}^l(\widehat{\mathrm{Cat}}_{\mathrm{cocompl}})$ or $\mathcal{E}\text{-mod}^l_{\mathrm{pr}} = \mathcal{E}\text{-mod}^l(\mathrm{Pr}^L)$.

Proposition 12.1.7. *Let $\mathcal{E}$ be an algebra object in $\widehat{\mathrm{Cat}}_{\mathrm{cocompl}}$. Then the category $\mathcal{E}\text{-mod}^l$ is κ_0-compactly generated. Moreover, the collection of free left $\mathcal{E}$-modules $\mathcal{E} \otimes \mathcal{C}$ where $\mathcal{C}$ is a presentable category is a collection of κ_0-compact generators for $\mathcal{E}\text{-mod}^l$.*

Proof. The forgetful functor $\mathcal{E}\text{-mod}^l \to \widehat{\mathrm{Cat}}_{\mathrm{cocompl}}$ preserves large colimits by [Lur17] corollary 4.2.3.5, and therefore its left adjoint preserves κ_0-compact objects. Combining this with proposition 12.1.4 we conclude that the free left $\mathcal{E}$-module $\mathcal{E} \otimes \mathcal{C}$ is κ_0-compact for every presentable category $\mathcal{C}$. The fact that these objects generate $\mathcal{E}\text{-mod}^l$ under large colimits follows by observing that $\mathcal{E}\text{-mod}^l$ is generated under colimits by free modules ([Lur17] proposition 4.7.3.14), and that free modules live in the colimit-closure of the modules $\mathcal{E}\otimes\mathcal{C}$ with $\mathcal{C}$ presentable. Using [Lur17] 4.2.3.7 we conclude that $\mathcal{E}\text{-mod}^l$ is κ_0-compactly generated. $\square$

Definition 12.1.8. *Let $\mathcal{E}$ be an algebra in $\widehat{\mathrm{Cat}}_{\mathrm{cocompl}}$. We say that a left $\mathcal{E}$-module is presentable if it is a κ_0-compact object in $\mathcal{E}\text{-mod}^l$. We denote by $\mathcal{E}\text{-mod}^l_{\mathrm{pr}}$ the full subcategory of $\mathcal{E}\text{-mod}^l$ on the presentable $\mathcal{E}$-modules. If $\mathcal{E}$ comes equipped with the structure of a commutative algebra, we will use the notation $\mathcal{E}\text{-mod}_{\mathrm{pr}}$ instead of $\mathcal{E}\text{-mod}^l_{\mathrm{pr}}$.*

Remark 12.1.9. Recall from the proof of proposition 12.1.4 that Pr^L is generated under small colimits by presheaf categories. Since the category of small categories is generated under small colimits by $[1]$, we in fact have that Pr^L is generated under small colimits by $\mathcal{P}([1])$. Using proposition 12.1.7 we see that for any algebra $\mathcal{E}$ in $\widehat{\mathrm{Cat}}_{\mathrm{cocompl}}$ the category $\mathcal{E}\text{-mod}_{\mathrm{pr}}$ is the smallest category of left $\mathcal{E}$-modules containing $\mathcal{E}\otimes\mathcal{P}([1]) = \mathrm{Funct}([1],\mathcal{E})$ and closed under small colimits.

The following proposition provides an abundant source of presentable $\mathcal{E}$-modules.

Proposition 12.1.10. *Let $\mathcal{E}$ be an algebra in $\widehat{\mathrm{Cat}}_{\mathrm{cocompl}}$, and let A be an algebra in $\mathcal{E}$. Then the category $A\text{-mod}^r$ of right A-modules equipped with its natural left $\mathcal{E}$-module structure, is a presentable $\mathcal{E}$-module.*

Proof. Recall from [Lur17] remark 4.8.4.8 that $A\text{-mod}^r$ is left dualizable as a left $\mathcal{E}$-module - its left dual is the category $A\text{-mod}^l$ of left A-modules with its natural right $\mathcal{E}$-module structure. We thus have an equivalence

$$\mathrm{Hom}_{\mathcal{E}\text{-mod}^l}(A\text{-mod}^r, -) = \mathrm{Hom}_{\widehat{\mathrm{Cat}}_{\mathrm{cocompl}}}(\mathrm{Spc}, A\text{-mod}^l\otimes_{\mathcal{E}} -) : \mathcal{E}\text{-mod}^l \to \widehat{\mathrm{Spc}}.$$

It follows from a combination of [Lur17] proposition 4.4.2.14 and [Lur09a] proposition 5.5.7.11 that the right hand side preserves κ_0-filtered colimits, and thus $A\text{-mod}^r$ is κ_0-compact. $\square$

Remark 12.1.11. Let $\varphi : \mathcal{E}\to\mathcal{E}'$ be a morphism of algebras in $\widehat{\mathrm{Cat}}_{\mathrm{cocompl}}$. Then the extension of scalars functor

$$\varphi_* = \mathcal{E}'\otimes_{\mathcal{E}} - : \mathcal{E}\text{-mod}^l \to \mathcal{E}'\text{-mod}^l$$

preserves large colimits, and maps each free $\mathcal{E}$-module $\mathcal{E}\otimes\mathcal{C}$ to the free $\mathcal{E}'$-module $\mathcal{E}'\otimes\mathcal{C}$. It follows from proposition 12.1.7 that φ_* maps presentable $\mathcal{E}$-modules to presentable $\mathcal{E}'$-modules.

Similarly, given a pair of algebras $\mathcal{E},\mathcal{E}'$ in $\widehat{\mathrm{Cat}}_{\mathrm{cocompl}}$, the exterior product functor

$$\boxtimes : \mathcal{E}\text{-mod}^l \times \mathcal{E}'\text{-mod}^l \to (\mathcal{E}\otimes\mathcal{E}')\text{-mod}^l$$

preserves colimits in each variable, and maps the pair $(\mathcal{E}\otimes\mathcal{C},\mathcal{E}'\otimes\mathcal{D})$ to $\mathcal{E}\otimes\mathcal{E}'\otimes\mathcal{C}\otimes\mathcal{D}$. Using proposition 12.1.7 we conclude that $\boxtimes$ restricts to the subcategories of presentable objects.

We therefore conclude that the assignment $\mathcal{E} \mapsto \mathcal{E}\text{-mod}^l_{\mathrm{pr}}$ has a natural enhancement to a lax symmetric monoidal functor

$$\text{-mod}^l_{\mathrm{pr}} : \mathrm{Alg}(\widehat{\mathrm{Cat}}_{\mathrm{cocompl}}) \to \widehat{\mathrm{Cat}}_{\mathrm{cocompl}}.$$

In particular, passing to commutative algebra objects yields an endofunctor

$$\text{-mod}_{\mathrm{pr}} : \mathrm{CAlg}(\widehat{\mathrm{Cat}}_{\mathrm{cocompl}}) \to \mathrm{CAlg}(\widehat{\mathrm{Cat}}_{\mathrm{cocompl}}).$$

In other words, if $\mathcal{E}$ is a commutative algebra in $\widehat{\mathrm{Cat}}_{\mathrm{cocompl}}$, then the natural symmetric monoidal structure on $\mathcal{E}$-mod restricts to a symmetric monoidal structure on $\mathcal{E}\text{-mod}_{\mathrm{pr}}$.

Example 12.1.12. Let $\mathcal{E} = \mathrm{Spc}$ with the cartesian symmetric monoidal structure. This is the trivial commutative algebra in $\widehat{\mathrm{Cat}}_{\mathrm{cocompl}}$ and therefore $\mathcal{E}\text{-mod}(\widehat{\mathrm{Cat}}_{\mathrm{cocompl}}) = \widehat{\mathrm{Cat}}_{\mathrm{cocompl}}$. Using proposition 12.1.4 we see that $\mathcal{E}\text{-mod}_{\mathrm{pr}}$ coincides with the category Pr^L of presentable categories. The symmetric monoidal structure on $\mathcal{E}\text{-mod}_{\mathrm{pr}}$ agrees with the usual symmetric monoidal structure on Pr^L.

Remark 12.1.13. Let $\mathcal{E}$ be a commutative algebra in $\widehat{\mathrm{Cat}}_{\mathrm{cocompl}}$. Recall from [Lur17] section 4.8.5 that the functor $\text{-mod}^r : \mathrm{Alg}(\mathcal{E}) \to \mathcal{E}\text{-mod}$ has a natural symmetric monoidal structure. Combining propositions 12.1.10 and remark 12.1.11 we see that the above restricts to a symmetric monoidal functor $\mathrm{Alg}(\mathcal{E}) \to \mathcal{E}\text{-mod}_{\mathrm{pr}}$. In particular, we obtain a symmetric monoidal functor

$$\text{-mod} : \mathrm{CAlg}(\mathcal{E}) \to \mathrm{CAlg}(\mathcal{E}\text{-mod}_{\mathrm{pr}}).$$

We finish by studying the interaction of the notion of presentability with extensions of commutative algebras.

Proposition 12.1.14. *Let $\varphi : \mathcal{E} \to \mathcal{E}'$ be a morphism of commutative algebras in $\widehat{\mathrm{Cat}}_{\mathrm{cocompl}}$, so that $\mathcal{E}'$ can alternatively be thought of as a commutative algebra in $\mathcal{E}$-mod. Assume that $\mathcal{E}'$ is presentable as an $\mathcal{E}$-module. Then the canonical equivalence of symmetric monoidal categories*

$$\mathcal{E}'\text{-mod}(\widehat{\mathrm{Cat}}_{\mathrm{cocompl}}) = \mathcal{E}'\text{-mod}(\mathcal{E}\text{-mod})$$

restricts to an equivalence

$$\mathcal{E}'\text{-mod}_{\mathrm{pr}} = \mathcal{E}'\text{-mod}(\mathcal{E}\text{-mod}_{\mathrm{pr}}).$$

Proof. By proposition 12.1.7, the subcategory $\mathcal{E}'\text{-mod}_{\mathrm{pr}}$ of $\mathcal{E}'\text{-mod}(\widehat{\mathrm{Cat}}_{\mathrm{cocompl}})$ is generated under small colimits by the free modules $\mathcal{E}' \otimes \mathcal{C}$ with $\mathcal{C}$ presentable. Similarly, $\mathcal{E}\text{-mod}_{\mathrm{pr}}$ is generated under small colimits by the free modules $\mathcal{E} \otimes \mathcal{C}$ with $\mathcal{C}$ presentable, and thus $\mathcal{E}'\text{-mod}(\mathcal{E}\text{-mod}_{\mathrm{pr}})$ is generated under small colimits by the objects of the form

$$\mathcal{E}' \otimes_{\mathcal{E}} (\mathcal{E} \otimes \mathcal{C}) = \mathcal{E}' \otimes \mathcal{C}.$$

We thus see that $\mathcal{E}'\text{-mod}_{\mathrm{pr}}$ and $\mathcal{E}'\text{-mod}(\mathcal{E}\text{-mod}_{\mathrm{pr}})$ are generated under small colimits by the same collection of objects, and they therefore agree. □

Corollary 12.1.15. *Let $\mathcal{E}$ be a commutative algebra in Pr^L. Then $\mathcal{E}\text{-mod}_{\mathrm{pr}}$ is equivalent to the category $\mathcal{E}\text{-mod}(\mathrm{Pr}^L)$ of presentable categories equipped with an $\mathcal{E}$-module structure.*

Proof. Apply proposition 12.1.14 in the case of the unit $\mathrm{Spc} \to \mathcal{E}$. ☐

12.2 Higher module categories

We now iterate the functor $\text{-mod}_{\mathrm{pr}}$ to arrive at a notion of higher presentable modules over a cocomplete symmetric monoidal category.

Notation 12.2.1. For each $n \geq 0$ we let

$$\text{-mod}^n_{\mathrm{pr}} : \mathrm{CAlg}(\widehat{\mathrm{Cat}}_{\mathrm{cocompl}}) \to \mathrm{CAlg}(\widehat{\mathrm{Cat}}_{\mathrm{cocompl}})$$

be the endofunctor obtained by composing the endofunctor $\text{-mod}_{\mathrm{pr}}$ of remark 12.1.11 with itself n times (in particular, if $n = 0$ we set $\text{-mod}^0_{\mathrm{pr}}$ to be the identity endofunctor).

Denote by

$$\text{-mod} : \mathrm{CAlg}(\widehat{\mathrm{Cat}}_{\mathrm{cocompl}}) \to \mathrm{CAlg}(\mathrm{CAT})$$

the functor that assigns to each commutative algebra $\mathcal{E}$ in $\widehat{\mathrm{Cat}}_{\mathrm{cocompl}}$ the (very large) symmetric monoidal category $\mathcal{E}\text{-mod}$. For each $n \geq 1$ we let -mod^n be the composite functor

$$\mathrm{CAlg}(\widehat{\mathrm{Cat}}_{\mathrm{cocompl}}) \xrightarrow{\text{-mod}^{n-1}_{\mathrm{pr}}} \mathrm{CAlg}(\widehat{\mathrm{Cat}}_{\mathrm{cocompl}}) \xrightarrow{\text{-mod}} \mathrm{CAlg}(\mathrm{CAT}).$$

Let $\mathcal{E}$ be a commutative algebra in $\widehat{\mathrm{Cat}}_{\mathrm{cocompl}}$. We inductively define a symmetric monoidal functor $\text{-mod}^n : \mathrm{CAlg}(\mathcal{E}) \to \mathrm{CAlg}(\mathcal{E}\text{-mod}^n_{\mathrm{pr}})$ as follows:

- When $n = 0$ we let -mod^0 be the identity functor.
- When $n > 1$ we let -mod^n be the composite functor

  $$\mathrm{CAlg}(\mathcal{E}) \xrightarrow{\text{-mod}^{n-1}} \mathrm{CAlg}(\mathcal{E}\text{-mod}^{n-1}_{\mathrm{pr}}) \xrightarrow{\text{-mod}} \mathrm{CAlg}(\mathcal{E}\text{-mod}^n_{\mathrm{pr}}).$$

  where -mod denotes the functor of remark 12.1.13.

Definition 12.2.2. *Let $\mathcal{E}$ be a commutative algebra in $\widehat{\mathrm{Cat}}_{\mathrm{cocompl}}$ and let $n \geq 0$. We call $\mathcal{E}\text{-mod}^n_{\mathrm{pr}}$ the category of presentable $\mathcal{E}$-linear n-categories. In the case $\mathcal{E} = \mathrm{Spc}$ we use the notation $n\mathrm{Pr}^L = \mathrm{Spc}\text{-mod}^n_{\mathrm{pr}}$, and call it the category of presentable n-categories.*

Warning 12.2.3. Let $\mathcal{E}$ be a presentable symmetric monoidal category and let $n \geq 1$. By virtue of being a commutative algebra in $\widehat{\mathrm{Cat}}_{\mathrm{cocompl}}$, we have associated categories $\mathcal{E}\text{-mod}^n$ and $\mathcal{E}\text{-mod}^n_{\mathrm{pr}}$ which are related to each other by passage to κ_0-compact objects, and Ind-κ_0-completion. We can also think about $\mathcal{E}$ as a commutative algebra in the cocomplete symmetric monoidal category Pr^L, and attach to it the object $\mathcal{E}\text{-mod}^n$ in $\mathrm{Pr}^L\text{-mod}^n_{\mathrm{pr}}$. These two notations are in conflict: as explained in remark 12.2.4 below, the underlying symmetric monoidal category to the latter agrees in fact with $\mathcal{E}\text{-mod}^n_{\mathrm{pr}}$. We hope it will be clear from context which version of the construction we are using at each time, and in case where it may be ambiguous we will use the notation $\mathcal{E}\text{-mod}^n(\widehat{\mathrm{Cat}}_{\mathrm{cocompl}})$ or $\mathcal{E}\text{-mod}^n(\mathrm{Pr}^L)$ to specify which version of the two constructions we are using.

Remark 12.2.4. Let $\varphi : \mathcal{E} \to \mathcal{E}'$ be a morphism of commutative algebras in $\widehat{\mathrm{Cat}}_{\mathrm{cocompl}}$, and assume that $\mathcal{E}'$ is a presentable $\mathcal{E}$-module. Denote by $\overline{\mathcal{E}'}$ the associated commutative algebra in $\mathcal{E}\text{-mod}_{\mathrm{pr}}$. An inductive application of proposition 12.1.14 shows that for every $n \geq 1$ the symmetric monoidal category $\mathcal{E}'\text{-mod}^n_{\mathrm{pr}}$ is equivalent to the symmetric monoidal category underlying $\overline{\mathcal{E}'}\text{-mod}^n$. In particular, $\mathcal{E}'\text{-mod}^n_{\mathrm{pr}}$ can be enhanced to a commutative algebra in presentable $\mathcal{E}$-linear $(n+1)$-categories.

Specializing the above to the case when $\mathcal{E} = \mathrm{Spc}$ and $\mathcal{E}'$ is a presentable symmetric monoidal category, we see that $\mathcal{E}'\text{-mod}^n_{\mathrm{pr}}$ has a natural enhancement to a commutative algebra in presentable $(n+1)$-categories. Moreover, if $n > 1$ we have equivalences

$$\mathcal{E}'\text{-mod}^n_{\mathrm{pr}} = \mathcal{E}'\text{-mod}^n(\mathrm{Pr}^L) = (\mathcal{E}'\text{-mod}^{n-1}(\mathrm{Pr}^L))\text{-mod}(n\mathrm{Pr}^L) = \mathcal{E}'\text{-mod}^{n-1}_{\mathrm{pr}}\text{-mod}(n\mathrm{Pr}^L).$$

In other words, a presentable $\mathcal{E}$-linear n-category is the same data as a presentable n-category equipped with the action of the presentable symmetric monoidal n-category of $\mathcal{E}$-linear $(n-1)$-categories.

Specializing to the case when $\mathcal{E}$ is the category of spectra we obtain a notion of presentable stable n-category.

Definition 12.2.5. *Denote by* Sp *the symmetric monoidal category of spectra. For each* $n \geq 0$ *we denote by* $n\mathrm{Pr}^L_{\mathrm{St}}$ *the category* $\mathrm{Sp}\text{-mod}^n_{\mathrm{pr}} = \mathrm{Sp}\text{-mod}^n(\mathrm{Pr}^L)$*, and call it the category of presentable stable n-categories. For each commutative ring spectrum A we call* $A\text{-mod}^n$ *the presentable n-category of presentable A-linear* $(n-1)$*-categories.*

Remark 12.2.6. Recall from [Lur17] section 4.8.2 that Sp is an idempotent commutative algebra in Pr^L. It follows by induction that $n\mathrm{Pr}^L_{\mathrm{St}}$ is an idempotent commutative algebra in $(n+1)\,\mathrm{Pr}^L$ for every $n \geq 0$. We conclude that for every $n \geq 1$ the functor

$$- \otimes (n-1)\mathrm{Pr}^L_{\mathrm{St}} : n\mathrm{Pr}^L \to n\mathrm{Pr}^L_{\mathrm{St}}$$

is a localization functor. We think about the above as the stabilization functor for presentable n-categories. The fact that it is a localization implies that being stable is a property of a presentable n-category.

12.3 The n-categorical structure

Our next goal is to enhance the category $\mathrm{Spc}\text{-mod}^n$ to an $(n+1)$-category. We will first need to construct a lax symmetric monoidal functor

$$\psi_n : \mathrm{Spc}\text{-mod}^n \to \widehat{n\mathrm{Cat}}.$$

This is accomplished by applying in an inductive way the functor that turns modules over presentable monoidal categories into enriched categories.

Notation 12.3.1. For each $n \geq 1$ denote by

$$\upsilon_{\mathrm{Spc}\text{-}\mathrm{mod}^n} : \mathrm{Spc}\text{-}\mathrm{mod}^n \to \widehat{\mathrm{Cat}}$$

the lax symmetric monoidal functor given by the composition

$$\mathrm{Spc}\text{-}\mathrm{mod}^n = (n-1)\mathrm{Pr}^L\text{-}\mathrm{mod}(\widehat{\mathrm{Cat}}_{\mathrm{cocompl}}) \to \widehat{\mathrm{Cat}}_{\mathrm{cocompl}} \to \widehat{\mathrm{Cat}}$$

where the second and third arrow are the forgetful functors. The composition of $\upsilon_{\mathrm{Spc}\text{-}\mathrm{mod}^n}$ with the symmetric monoidal functor $(-)^{\leq 0} : \widehat{\mathrm{Cat}} \to \widehat{\mathrm{Spc}}$ is the lax symmetric monoidal functor $\tau_{\mathrm{Spc}\text{-}\mathrm{mod}^n}$ that maps each object of $\mathrm{Spc}\text{-}\mathrm{mod}^n$ to its underlying space.

Construction 12.3.2. Let

$$\psi_1 : \mathrm{Spc}\text{-}\mathrm{mod} = \widehat{\mathrm{Cat}}_{\mathrm{cocompl}} \to \widehat{\mathrm{Cat}}$$

be the forgetful functor, equipped with its canonical lax symmetric monoidal structure. In other words, we have $\psi_1 = \upsilon_1$. Let $n > 1$ and assume given a lax symmetric monoidal functor

$$\psi_{n-1} : \mathrm{Spc}\text{-}\mathrm{mod}^{n-1} \to \widehat{(n-1)\mathrm{Cat}}$$

such that the composite lax symmetric monoidal functor

$$\mathrm{Spc}\text{-}\mathrm{mod}^{n-1} \xrightarrow{\psi_{n-1}} \widehat{(n-1)\mathrm{Cat}} \xrightarrow{(-)^{\leq 1}} \widehat{\mathrm{Cat}}$$

is equivalent to $\upsilon_{\mathrm{Spc}\text{-}\mathrm{mod}^{n-1}}$.

Denote by $\widehat{\mathrm{Pr}}^L$ the category of very large presentable categories and (large) colimit preserving functors, and by CAT the category of very large categories. We let

$$\varphi_n : \mathrm{Spc}\text{-}\mathrm{mod}^n \to \mathrm{CAT}^{\mathrm{Spc}\text{-}\mathrm{mod}^{n-1}}$$

be the lax symmetric monoidal functor given by the following composition:

$$\mathrm{Spc}\text{-}\mathrm{mod}^n = (n-1)\mathrm{Pr}^L\text{-}\mathrm{mod}(\widehat{\mathrm{Cat}}_{\mathrm{cocompl}}) \xrightarrow{\mathrm{Ind}_{\kappa_0}} (\mathrm{Spc}\text{-}\mathrm{mod}^{n-1})\text{-}\mathrm{mod}(\widehat{\mathrm{Pr}}^L) \to \mathrm{CAT}^{\mathrm{Spc}\text{-}\mathrm{mod}^{n-1}}$$

Here the second arrow is the lax symmetric monoidal functor $\theta_{\mathrm{Spc}\text{-}\mathrm{mod}^{n-1}}$ which sends each very large presentable module over $\mathrm{Spc}\text{-}\mathrm{mod}^{n-1}$ to its underlying $\mathrm{Spc}\text{-}\mathrm{mod}^{n-1}$-enriched category.

Note that the lax symmetric monoidal functor $(\tau_{\mathrm{Spc}\text{-}\mathrm{mod}^{n-1}})_!\varphi_n$ is equivalent to the composite

$$\mathrm{Spc}\text{-}\mathrm{mod}^n \xrightarrow{\upsilon_{\mathrm{Spc}\text{-}\mathrm{mod}^n}} \widehat{\mathrm{Cat}} \xrightarrow{\mathrm{Ind}_{\kappa_0}} \mathrm{CAT}\,.$$

In particular, we observe that for each morphism $f : \mathcal{C} \to \mathcal{D}$ in $\mathrm{Spc}\text{-}\mathrm{mod}^n$, the induced morphism $\varphi_n(f) : \varphi_n(\mathcal{C}) \to \varphi_n(\mathcal{D})$ restricts to the full enriched-subcategories of $\varphi_n(\mathcal{C})$ and

$\varphi_n(\mathcal{D})$ on those objects which are κ_0-compact, when thought of as an object in $\mathcal{C}$ or $\mathcal{D}$. Furthermore, for every pair of objects $\mathcal{C}, \mathcal{D}$ in $\mathrm{Spc}\text{-}\mathrm{mod}^n$, the morphism

$$\varphi_n(\mathcal{C}) \otimes \varphi_n(\mathcal{D}) \to \varphi_n(\mathcal{C} \otimes \mathcal{D})$$

arising from the lax symmetric monoidal structure on φ_n, restricts to the subcategories of κ_0-compact objects. We thus have a well defined lax symmetric monoidal functor

$$\varphi'_n : \mathrm{Spc}\text{-}\mathrm{mod}^n \to \widehat{\mathrm{Cat}}^{\mathrm{Spc}\text{-}\mathrm{mod}^{n-1}}$$

equipped with a monomorphism into φ_n, and which maps every object $\mathcal{C}$ in $\mathrm{Spc}\text{-}\mathrm{mod}^n$ to the full enriched subcategory of $\varphi_n(\mathcal{C})$ on those objects which are κ_0-compact when thought of as objects of $\mathcal{C}$.

Let ψ'_n be the composite lax symmetric monoidal functor

$$\mathrm{Spc}\text{-}\mathrm{mod}^n \xrightarrow{\varphi'_n} \widehat{\mathrm{Cat}}^{\mathrm{Spc}\text{-}\mathrm{mod}^{n-1}} \xrightarrow{(\psi_{n-1})_!} \widehat{\mathrm{Algbrd}}(\widehat{(n-1)\mathrm{Cat}}).$$

Note that for every object $\mathcal{C}$ in $\mathrm{Spc}\text{-}\mathrm{mod}^n$, the Segal space underlying $\psi'_n(\mathcal{C})$ is given by $(\psi_{n-1}^{\leq 0})_!\varphi'_n(\mathcal{C}) = (\tau_{\mathrm{Spc}\text{-}\mathrm{mod}^{n-1}})_!\varphi'_n(\mathcal{C})$ which is a complete Segal space since $\varphi'_n(\mathcal{C})$ is an enriched category. It follows that $\psi'_n(\mathcal{C})$ is in fact also an enriched category. Hence ψ'_n defines, by corestriction, a lax symmetric monoidal functor

$$\psi_n : \mathrm{Spc}\text{-}\mathrm{mod}^n \to \widehat{\mathrm{Cat}}^{\widehat{(n-1)\mathrm{Cat}}} = \widehat{n\mathrm{Cat}}.$$

Observe that we have equivalences of lax symmetric monoidal functors

$$(\psi_n)^{\leq 1} = (-)^{\leq 0}_! \psi_n = ((\psi_{n-1})^{\leq 0})_! \varphi'_n = (\tau_{\mathrm{Spc}\text{-}\mathrm{mod}^{n-1}})_! \varphi'_n$$

and the latter is obtained from $(\tau_{\mathrm{Spc}\text{-}\mathrm{mod}^{n-1}})_!\varphi_n = \mathrm{Ind}_{\kappa_0} \upsilon_{\mathrm{Spc}\text{-}\mathrm{mod}^n}$ by restricting to κ_0-compact objects. It follows that $(\psi_n)^{\leq 1}$ is equivalent, as a lax symmetric monoidal functor, to $\upsilon_{\mathrm{Spc}\text{-}\mathrm{mod}^n}$. We conclude that this construction may be iterated to yield lax symmetric monoidal functors ψ_n for all $n \geq 1$.

Remark 12.3.3. Let $n \geq 2$ and let $\mathcal{C}$ be an object of $\mathrm{Spc}\text{-}\mathrm{mod}^n$. Unwinding construction 12.3.2 reveals that $\psi_n(\mathcal{C})$ is an n-category whose underlying category is the category underlying $\mathcal{C}$. For each pair of objects x, y of $\mathcal{C}$ we have an equivalence

$$\mathrm{Hom}_{\psi_n(\mathcal{C})}(x, y) = \psi_{(n-1)}(\mathscr{H}om_{\mathrm{Ind}_{\kappa_0}(\mathcal{C})}(x, y))$$

where the right hand side denotes the Hom object between x and y, where we consider $\mathrm{Ind}_{\kappa_0}(\mathcal{C})$ as a module over $\mathrm{Spc}\text{-}\mathrm{mod}^{n-1}$.

We now use the functors ψ_n to obtain the desired $(n+1)$-categorical enhancement of $\mathrm{Spc}\text{-}\mathrm{mod}^n$.

Notation 12.3.4. Let $n \geq 1$. We denote by $n\mathscr{C}at^L$ the (very large) symmetric monoidal $(n+1)$-category obtained by applying the composite lax symmetric monoidal functor

$$(\mathrm{Spc}\text{-}\mathrm{mod}^n)\text{-}\mathrm{mod}(\widehat{\mathrm{Pr}^L}) \xrightarrow{\theta_{\mathrm{Spc}\text{-}\mathrm{mod}^n}} \mathrm{CAT}^{\mathrm{Spc}\text{-}\mathrm{mod}^n} \xrightarrow{(\psi_n)_!} \mathrm{CAT}^{\widehat{n\mathrm{Cat}}} \hookrightarrow (n+1)\mathrm{CAT}$$

to the unit object of $(\mathrm{Spc}\text{-}\mathrm{mod}^n)\text{-}\mathrm{mod}(\widehat{\mathrm{Pr}^L})$, where the first map is the map that turns presentable modules into enriched categories.

Remark 12.3.5. The symmetric monoidal category underlying $n\mathscr{C}at^L$ is $\mathrm{Spc}\text{-}\mathrm{mod}^n$. Given two objects $\mathcal{C}, \mathcal{D}$ in $\mathrm{Spc}\text{-}\mathrm{mod}^n$, we have an equivalence

$$\mathrm{Hom}_{n\mathscr{C}at^L}(\mathcal{C}, \mathcal{D}) = \psi_n \mathscr{H}om_{\mathrm{Spc}\text{-}\mathrm{mod}^n}(\mathcal{C}, \mathcal{D}).$$

Given a third object $\mathcal{E}$ in $\mathrm{Spc}\text{-}\mathrm{mod}^n$, the composition map

$$\mathrm{Hom}_{n\mathscr{C}at^L}(\mathcal{C}, \mathcal{D}) \times \mathrm{Hom}_{n\mathscr{C}at^L}(\mathcal{D}, \mathcal{E}) \to \mathrm{Hom}_{n\mathscr{C}at^L}(\mathcal{C}, \mathcal{E})$$

is obtained by applying the lax symmetric monoidal functor ψ_n to the morphism

$$\mathscr{H}om_{\mathrm{Spc}\text{-}\mathrm{mod}^n}(\mathcal{C}, \mathcal{D}) \otimes \mathscr{H}om_{\mathrm{Spc}\text{-}\mathrm{mod}^n}(\mathcal{D}, \mathcal{E}) \to \mathscr{H}om_{\mathrm{Spc}\text{-}\mathrm{mod}^n}(\mathcal{C}, \mathcal{E})$$

which is associated to the composite map

$$\mathcal{C} \otimes \mathscr{H}om_{\mathrm{Spc}\text{-}\mathrm{mod}^n}(\mathcal{C}, \mathcal{D}) \otimes \mathscr{H}om_{\mathrm{Spc}\text{-}\mathrm{mod}^n}(\mathcal{D}, \mathcal{E}) \to \mathcal{D} \otimes \mathscr{H}om_{\mathrm{Spc}\text{-}\mathrm{mod}^n}(\mathcal{D}, \mathcal{E}) \to \mathcal{E}.$$

Definition 12.3.6. *We denote by $n\mathscr{P}r^L$ the full $(n+1)$-subcategory of $n\mathscr{C}at^L$ on the presentable n-categories. We call $n\mathscr{P}r^L$ the $(n+1)$-category of presentable n-categories. We let $n\mathscr{P}r^L_{\mathrm{St}}$ be the full $(n+1)$-subcategory of $n\mathscr{P}r^L$ on the presentable stable n-categories. We call $n\mathscr{P}r^L_{\mathrm{St}}$ the $(n+1)$-category of presentable stable n-categories*

Remark 12.3.7. The categories underlying $n\mathscr{P}r^L$ and $n\mathscr{P}r^L_{\mathrm{St}}$ are $n\mathrm{Pr}^L$ and $n\mathrm{Pr}^L_{\mathrm{St}}$, respectively. Since these are closed under tensor products inside $\mathrm{Spc}\text{-}\mathrm{mod}^n$, we conclude that $n\mathscr{P}r^L$ and $n\mathscr{P}r^L_{\mathrm{St}}$ inherit symmetric monoidal structures from $n\mathscr{C}at^L$.

12.4 Conical colimits in presentable n-categories

Our next goal is to show that the realization functor ψ_n takes value in conically cocomplete n-categories. This will be accomplished by combining proposition 5.4.8 with a generalization of proposition 5.3.6, which depends on a variant of proposition 3.5.27.

Lemma 12.4.1. *Let $F : \mathcal{M} \to \mathcal{M}'$ be a lax symmetric monoidal functor between presentable symmetric monoidal categories. Assume that F is accessible and preserves limits, and that the lax symmetric monoidal functor $\tau_{\mathcal{M}'}F$ is equivalent to $\tau_{\mathcal{M}}$. Then the induced functors*

$$F_! : \mathrm{Algbrd}(\mathcal{M}) \to \mathrm{Algbrd}(\mathcal{M}')$$

and

$$F_! : \mathrm{Cat}^{\mathcal{M}} \to \mathrm{Cat}^{\mathcal{M}'}$$

are accessible and preserve limits.

Proof. The fact that $\tau_{\mathcal{M}'}F$ is equivalent to $\tau_{\mathcal{M}}$ implies that the functor

$$F_! : \mathrm{Algbrd}(\mathcal{M}) \to \mathrm{Algbrd}(\mathcal{M}')$$

restricts to the full subcategories of enriched categories. Since the inclusion of enriched categories inside algebroids is creates limits and sufficiently filtered colimits, to prove the lemma it suffices to show that the above functor is accessible and limit preserving.

Recall that $F_!$ is a morphism of cartesian fibrations over Cat. For every category X, the functor

$$(F_!)_X : \mathrm{Algbrd}_X(\mathcal{M}) \to \mathrm{Algbrd}_X(\mathcal{M}')$$

is accessible and limit preserving thanks to [Lur17] corollaries 3.2.2.4 and 3.2.3.1. We know that $\mathrm{Algbrd}_X(\mathcal{M})$ and $\mathrm{Algbrd}_X(\mathcal{M}')$ are presentable by [Lur17] corollary 3.2.3.5, and therefore by the adjoint functor theorem we see that $(F_!)_X$ admits a left adjoint. Using [Lur17] proposition 7.3.2.6 we see that $F_!$ itself admits a left adjoint, and the lemma follows. □

Lemma 12.4.2. *Let $F : \mathcal{M} \to \mathcal{M}'$ be a lax symmetric monoidal functor between presentable symmetric monoidal categories. Assume that F is accessible and preserves limits, and that the lax symmetric monoidal functor $\tau_{\mathcal{M}'}F$ is equivalent to $\tau_{\mathcal{M}}$. Let $\mathcal{I}$ be a category and let $\mathcal{D}$ be an $\mathcal{M}$-enriched category. Then there is an equivalence of $\mathcal{M}'$-enriched categories*

$$F_! \,\mathrm{Funct}(\mathcal{I}_{\mathcal{M}}, \mathcal{D}) = \mathrm{Funct}(\mathcal{I}_{\mathcal{M}'}, F_!(\mathcal{D}))$$

which is natural in $\mathcal{I}$, and which at the level of objects enhances the canonical equivalence

$$\mathrm{Hom}_{\mathrm{Cat}^{\mathcal{M}}}(\mathcal{I}_{\mathcal{M}}, \mathcal{D}) = \mathrm{Hom}_{\mathrm{Cat}}(\mathcal{I}, (\tau_{\mathcal{M}})_!\mathcal{D}) = \mathrm{Hom}_{\mathrm{Cat}^{\mathcal{M}'}}(\mathcal{I}_{\mathcal{M}'}, F_!\mathcal{D}).$$

Proof. Denote by $(F_!)^L$ the left adjoint to $F_! : \mathrm{Cat}^{\mathcal{M}} \to \mathrm{Cat}^{\mathcal{M}'}$, which is guaranteed to exist by lemma 12.4.1. Let $\mathcal{E}$ be an $\mathcal{M}'$-algebroid. We have

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Cat}^{\mathcal{M}'}}(\mathcal{E}, \mathrm{Funct}(\mathcal{I}_{\mathcal{M}'}, F_!(\mathcal{D}))) &= \mathrm{Hom}_{\mathrm{Cat}^{\mathcal{M}'}}(\mathcal{E} \otimes \mathcal{I}_{\mathcal{M}'}, F_!(\mathcal{D})) \\ &= \mathrm{Hom}_{\mathrm{Cat}^{\mathcal{M}}}((F_!)^L(\mathcal{E} \otimes \mathcal{I}_{\mathcal{M}'}), \mathcal{D}).\end{aligned}$$

Similarly, we have

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Cat}^{\mathcal{M}'}}(\mathcal{E}, F_! \,\mathrm{Funct}(\mathcal{I}_{\mathcal{M}}, \mathcal{D})) &= \mathrm{Hom}_{\mathrm{Cat}^{\mathcal{M}}}((F_!)^L(\mathcal{E}), \mathrm{Funct}(\mathcal{I}_{\mathcal{M}}, \mathcal{D})) \\ &= \mathrm{Hom}_{\mathrm{Cat}^{\mathcal{M}}}((F_!)^L(\mathcal{E}) \otimes \mathcal{I}_{\mathcal{M}}, \mathcal{D}).\end{aligned}$$

To construct the desired equivalence it suffices to construct a functorial equivalence between $(F_!)^L(\mathcal{E} \otimes \mathcal{I}_{\mathcal{M}'})$ and $(F_!)^L(\mathcal{E}) \otimes \mathcal{I}_{\mathcal{M}}$. Note that since $(\tau_{\mathcal{M}'})_!F_!$ is equivalent to $(\tau_{\mathcal{M}})_!$, we can rewrite the latter as $(F_!)^L(\mathcal{E}) \otimes (F_!)^L(\mathcal{I}_{\mathcal{M}'})$. We now observe that there is a natural morphism

$$\eta_{\mathcal{E},\mathcal{I}} : (F_!)^L(\mathcal{E} \otimes \mathcal{I}_{\mathcal{M}'}) \to (F_!)^L(\mathcal{E}) \otimes (F_!)^L(\mathcal{I}_{\mathcal{M}'})$$

obtained by adjunction from the composite map

$$\mathcal{E} \otimes \mathcal{I}_{\mathcal{M}'} \to F_!(F_!)^L(\mathcal{E}) \otimes F_!(F_!)^L(\mathcal{I}_{\mathcal{M}'}) \to F_!((F_!)^L(\mathcal{E}) \otimes (F_!)^L(\mathcal{I}_{\mathcal{M}'}))$$

where the first arrow is induced from the unit of the adjunction $(F_!)^L \dashv F_!$, and the second map is induced from the lax symmetric monoidal structure on $F_!$.

Since the assignment $(\mathcal{E}, \mathcal{I}) \mapsto \eta_{\mathcal{E},\mathcal{I}}$ is colimit preserving, to show that $\eta_{\mathcal{E},\mathcal{I}}$ is an isomorphism it suffices to consider the case $\mathcal{I} = [1]$ and $\mathcal{E} = \overline{C}_m$ is the enriched category induced from the cell C_m for some m in $\mathcal{M}'$. Note that we have $(F_!)^L(\overline{C}_m) = \overline{C}_{F^L m}$ where F^L denotes the left adjoint to F. Using proposition 3.5.25 together with the fact that $(F_!)^L$ preserves colimits, we obtain a pushout square of $\mathcal{M}$-algebroids

$$\begin{tikzcd}
\overline{C}_{F^L m} \arrow[r] \arrow[d] & \overline{C}_{F^L m, 1_{\mathcal{M}}} \arrow[d] \\
\overline{C}_{1_{\mathcal{M}}, F^L m} \arrow[r] & (F_!)^L(\overline{C}_m \otimes \overline{C}_{1_{\mathcal{M}'}}).
\end{tikzcd}$$

The fact that $\eta_{\overline{C}_m,[1]}$ is an isomorphism follows now from another application of proposition 3.5.25. □

Lemma 12.4.3. *Let $F : \mathcal{M} \to \mathcal{M}'$ be a lax symmetric monoidal functor between presentable symmetric monoidal categories. Assume that the lax symmetric monoidal functor $\tau_{\mathcal{M}'}F$ is equivalent to $\tau_{\mathcal{M}}$. Let $\mathcal{C}$ be an $\mathcal{M}$-enriched category. Let $G : \mathcal{C} \to \mathcal{D}$ be a functor between $\mathcal{M}$-enriched categories. Let d be an object in $\mathcal{D}$ and let (c, ϵ) be right adjoint to G at d. Then (c, ϵ) is also right adjoint to $F_!G$ at d.*

Proof. We have to show that for every object e in $(\tau_{\mathcal{M}'})_!F_!\mathcal{D} = (\tau_{\mathcal{M}})_!\mathcal{D}$ the composite map

$$\mathrm{Hom}_{F_!\mathcal{C}}(e, c) \xrightarrow{(F_!G)_*} \mathrm{Hom}_{F_!\mathcal{D}}(G(e), G(c)) \xrightarrow{\epsilon} \mathrm{Hom}_{F_!\mathcal{D}}(G(e), d)$$

is an isomorphism. This is equivalent to the image under F of the composite map

$$\mathrm{Hom}_{\mathcal{C}}(e, c) \xrightarrow{G_*} \mathrm{Hom}_{\mathcal{D}}(G(e), G(c)) \xrightarrow{\epsilon} \mathrm{Hom}_{\mathcal{D}}(G(e), d).$$

Our claim now follows from the fact that the above composite map is an isomorphism since (c, ϵ) is right adjoint to G at d. □

Lemma 12.4.4. *Let $F : \mathcal{M} \to \mathcal{M}'$ be a lax symmetric monoidal functor between presentable symmetric monoidal categories. Assume that F is accessible and preserves limits, and that the lax symmetric monoidal functor $\tau_{\mathcal{M}'}F$ is equivalent to $\tau_{\mathcal{M}}$. Let $\mathcal{C}$ be an $\mathcal{M}$-enriched category. Let $\mathcal{I}$ be a category and $X^{\lhd} : \mathcal{I}^{\lhd} \to (\tau_{\mathcal{M}})_!\mathcal{C}$ be a conical limit diagram in $\mathcal{C}$. Then $X^{\lhd}$ defines also a conical limit diagram in $F_!\mathcal{C}$.*

Proof. Let $X = X^{\triangleleft}|_{\mathcal{I}}$. The diagram $X^{\triangleleft}$ defines a pair (c, ϵ) right adjoint to $\Delta : \mathcal{C} \to \operatorname{Funct}(\mathcal{I}_{\mathcal{M}}, \mathcal{C})$ at X. By lemma 12.4.3 we have that (c, ϵ) is also right adjoint to $F_!\Delta : F_!\mathcal{C} \to F_! \operatorname{Funct}(\mathcal{I}_{\mathcal{M}}, \mathcal{C})$ at X. Using lemma 12.4.2 we may identify this with the diagonal functor $F_!\mathcal{C} \to \operatorname{Funct}(\mathcal{I}_{\mathcal{M}'}, \mathcal{C})$. This identification is the identity on objects, so we conclude that X also has a conical limit in $F_!\mathcal{C}$. The lemma now follows from corollary 5.3.7. □

Lemma 12.4.5. *Let $n \geq 1$. Then the functor $\psi_n : \operatorname{Spc-mod}^n \to \widehat{n\mathrm{Cat}}$ is accessible and preserves large limits.*

Proof. We argue by induction. The case $n = 1$ is a direct consequence of the fact that the forgetful functor $\widehat{\mathrm{Cat}}_{\mathrm{cocompl}} \to \widehat{\mathrm{Cat}}$ preserves limits and κ_0-filtered colimits.

Assume now that $n > 1$. We continue with the notation from construction 12.3.2. Note that we have an equivalence

$$\operatorname{Spc-mod}^n = (n-1)\mathrm{Pr}^L\text{-mod} = (\operatorname{Spc-mod}^{n-1})\text{-mod}(\widehat{\mathrm{Pr}}^L_{\kappa_0})$$

given by ind-κ_0-completion. From this point of view, the functor φ'_n agrees with the functor $\theta^{\kappa_0}_{\operatorname{Spc-mod}^{n-1}}$ from notation 4.2.21. It follows from proposition 4.2.24 that the functor φ'_n is accessible and preserves large limits. Using lemma 12.4.1 together with our inductive hypothesis we conclude that ψ'_n is accessible and preserves large limits. The lemma now follows from the fact that the inclusion of $\widehat{n\mathrm{Cat}}$ inside $\widehat{\mathrm{Algbrd}}(\widehat{(n-1)\mathrm{Cat}})$ creates large limits and filtered colimits. □

Theorem 12.4.6. *Let $n \geq 1$ and let $\mathcal{C}$ be an object in $\operatorname{Spc-mod}^n$. Then the n-category $\psi_n(\mathcal{C})$ is admits all small conical colimits. Furthermore, the inclusion $\mathcal{C} \to \psi_n(\mathcal{C})$ maps limits in $\mathcal{C}$ to conical limits in $\psi_n(\mathcal{C})$.*

Proof. The case $n = 1$ is clear, so assume $n > 1$. We continue with the notation from construction 12.3.2. By proposition 5.4.8, the enriched category $\varphi_n(\mathcal{C})$ admits all large conical limits and colimits. It follows that its full subcategory $\varphi'_n(\mathcal{C})$ admits all small conical colimits, and that any small limit in $\mathcal{C}$ defines a conical limit in $\varphi'_n(\mathcal{C})$. Our result now follows from a combination of lemmas 12.4.4 and 12.4.5. □

Corollary 12.4.7. *The $(n+1)$-category $n\mathscr{P}r^L$ is conically cocomplete for each $n \geq 1$.*

Proof. Apply theorem 12.4.6 to $\mathcal{C} = n\mathrm{Pr}^L$ inside $\operatorname{Spc-mod}^{n+1}$. □

12.5 The passage to adjoints property

A fundamental feature of presentable 1-categories is that they are stable under many of the usual constructions of category theory. The following would imply that the world of presentable n-categories enjoys similar closure properties:

Conjecture 12.5.1. *The category $n\mathrm{Pr}^L$ has all small limits for all $n \geq 0$.*

Note that for every $n \geq 1$ the inclusion

$$n\mathrm{Pr}^L \to \mathrm{Spc}\text{-}\mathrm{mod}^n(\widehat{\mathrm{Cat}}_{\mathrm{cocompl}}) = \mathrm{Ind}_{\kappa_0}(n\mathrm{Pr}^L)$$

preserves all small limits that exist in $n\mathrm{Pr}^L$. Therefore conjecture 12.5.1 is equivalent to the claim that the κ_0-small objects in $\mathrm{Spc}\text{-}\mathrm{mod}^n(\widehat{\mathrm{Cat}}_{\mathrm{cocompl}})$ are closed under small limits.

Our next goal is to prove a weak form of conjecture 12.5.1 (stated below as proposition 12.5.10) which guarantees the existence of limits of diagrams of right adjoints.

Definition 12.5.2. *Let $\mathcal{E}$ be a commutative algebra in $\widehat{\mathrm{Cat}}_{\mathrm{cocompl}}$ and let $F : \mathcal{C} \to \mathcal{D}$ be a morphism in $\mathcal{E}$-mod. We say that F is left adjointable if its underlying functor admits a left adjoint $F^L : \mathcal{D} \to \mathcal{C}$, and the canonical structure of oplax morphism of $\mathcal{E}$-modules on F^L is strict. We say that F is right adjointable if its underlying functor admits a colimit preserving right adjoint $F^R : \mathcal{C} \to \mathcal{D}$, and the canonical structure of lax morphism of $\mathcal{E}$-modules on F^R is strict.*

Let $X : \mathcal{I} \to \mathcal{E}$-mod be a diagram. We say that X is left (resp. right) adjointable if for every arrow α in $\mathcal{I}$ the morphism $X(\alpha)$ is left (resp. right) adjointable. In this case, the induced diagram $X^L : \mathcal{I}^{\mathrm{op}} \to \mathcal{E}$-mod (resp. $X^R : \mathcal{I}^{\mathrm{op}} \to \mathcal{E}$-mod) is said to arise from X by passage to left (resp. right) adjoints.

Proposition 12.5.3. *Let $\mathcal{E}$ be a commutative algebra in $\widehat{\mathrm{Cat}}_{\mathrm{cocompl}}$ which is generated under small colimits by its dualizable objects. Let $F : \mathcal{C} \to \mathcal{D}$ be a morphism in $\mathcal{E}$-mod. Then*

(i) *F is left adjointable if and only if its underlying functor is left adjointable.*

(ii) *F is right adjointable if and only if its underlying functor has a colimit preserving right adjoint.*

Proof. We give a proof of item (i) - the proof of (ii) is completely analogous. The structure of morphism of $\mathcal{E}$-modules on F induces a commutative square

$$\begin{tikzcd} \mathcal{E} \times \mathcal{C} \arrow[r, "\otimes"] \arrow[d, "\mathrm{id}_{\mathcal{E}} \times F"] & \mathcal{C} \arrow[d, "F"] \\ \mathcal{E} \times \mathcal{D} \arrow[r, "\otimes"] & \mathcal{D} \end{tikzcd}$$

which we have to show is vertically left adjointable. Since the horizontal arrows preserve colimits in the $\mathcal{E}$-variable and F^L preserves colimits, we may restrict to showing that for every dualizable object e in $\mathcal{E}$ the induced commutative diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "e\otimes -"] \arrow[d, "F"] & \mathcal{C} \arrow[d, "F"] \\ \mathcal{D} \arrow[r, "e\otimes -"] & \mathcal{D} \end{tikzcd}$$

is vertically left adjointable. Note that the horizontal arrows have right adjoints given by $e^\vee \otimes -$, and the fact that F is a morphism of $\mathcal{E}$-modules implies that the square is in fact horizontally right adjointable. Since the vertical arrows admit left adjoints we conclude that the above square is also vertically left adjointable, as desired. $\square$

Notation 12.5.4. Let κ be an uncountable regular cardinal. Denote by Pr^L_κ the full subcategory of Pr^L on the κ-compactly generated categories and functors which preserve κ-compact objects. Denote by $\mathrm{Cat}^{\mathrm{rex}(\kappa)}$ the full subcategory of Cat on those categories admitting κ-small colimits, and functors which preserve κ-small colimits. For each presentable category $\mathcal{C}$ we denote by $\mathcal{C}^\kappa$ the full subcategory of $\mathcal{C}$ on the κ-compact objects. Recall from [Lur09a] proposition 5.5.7.10 that passage to κ-compact objects and ind-κ-completion are inverse equivalences between Pr^L_κ and $\mathrm{Cat}^{\mathrm{rex}(\kappa)}$.

Lemma 12.5.5. *Let κ be an uncountable regular cardinal. Then the inclusion $\mathrm{Pr}^L_\kappa \to \mathrm{Pr}^L$ creates κ-small limits.*

Proof. Let $X^\triangleleft : \mathcal{I}^\triangleleft \to \mathrm{Pr}^L$ be a κ-small limit diagram, and assume that $X = X^\triangleleft|_\mathcal{I}$ factors through Pr^L_κ. Denote by $*$ the initial object of $\mathcal{I}^\triangleleft$. We have to show that $X^\triangleleft(*)$ is κ-compactly generated, and that an object in $X^\triangleleft(*)$ is κ-compact if and only if its projection to $X(i)$ is κ-compact for every i in $\mathcal{I}$. It suffices moreover to consider the case of κ-small products, and pullbacks.

We begin with the case of κ-small products, so that $\mathcal{I}$ is a κ-small set, and we have $X^\triangleleft(*) = \prod_{i\in\mathcal{I}} X(i)$. For each i in $\mathcal{I}$ the projection $X^\triangleleft(*) \to X(i)$ has a right adjoint, which is induced by the identity map $X(i) \to X(i)$ together with the map $X(i) \to X(j)$ that picks out the final object of $X(j)$ for all $j \neq i$. This preserves colimits indexed by contractible categories, and in particular κ-filtered colimits, so we conclude that the projection $X^\triangleleft(*) \to X(i)$ preserves κ-compact objects. Combining this with [Lur09a] lemma 5.3.4.10 we see that an object in $X^\triangleleft(*)$ is κ-compact if and only if its projection to $X(i)$ is κ-compact for all i in $\mathcal{I}$.

It remains to show that $X^\triangleleft(*)$ is κ-compactly generated. Observe that the projections $X^\triangleleft(*) \to X(i)$ are jointly conservative, admit left adjoints, and preserve κ-filtered colimits. It follows that $X^\triangleleft(*)$ is generated under colimits by the sequences $(c_i)_{i\in\mathcal{I}}$ where c_i is a κ-compact object of $X(i)$ for all i and c_i is initial for all but one index i - and moreover these objects are κ-compact in $X^\triangleleft(*)$. This shows that $X^\triangleleft(*)$ is κ-compactly generated.

We now deal with the case of pullbacks, so that $\mathcal{I}^\triangleleft = [1] \times [1]$, and X corresponds to a pullback diagram which we depict as follows:

$$\begin{tikzcd} \mathcal{C}' \arrow[r, "q'"] \arrow[d, "p'"] & \mathcal{C} \arrow[d, "p"] \\ \mathcal{D}' \arrow[r, "q"] & \mathcal{D} \end{tikzcd}$$

It follows from [Lur09a] lemma 5.4.5.7 that if c' is an object of $\mathcal{C}'$ such that $q'c'$ and $p'c'$ are κ-compact then c' is κ-compact. It therefore remains to show that $\mathcal{C}'$ is generated under colimits by those κ-compact objects c' such that $q'c'$ and $p'c'$ are κ-compact. This is essentially a consequence of the proof of [Lur09a] proposition 5.4.6.6. We repeat the relevant part of the argument below, suitably adapted to our context.

Denote by $\mathcal{C}''$ the full subcategory of $\mathcal{C}'$ on those objects whose projections to $\mathcal{C}$ and $\mathcal{D}'$ are κ-compact. Let c' be an arbitrary object of $\mathcal{C}'$, and let $c = q'c'$, $d' = p'c'$ and $d = pq'c'$. We have a pullback diagram

$$\begin{tikzcd} \mathcal{C}''_{/c'} \arrow[r, "f'"] \arrow[d, "g'"] & \mathcal{C}^{\kappa}_{/c} \arrow[d, "g"] \\ (\mathcal{D}')^{\kappa}_{/d'} \arrow[r, "f"] & \mathcal{D}^{\kappa}_{/d}. \end{tikzcd}$$

It follows from [Lur09a] lemma 5.4.6.1 (applied to the cardinals $\omega << \kappa$) that f and g are ω-cofinal (see [Lur09a] definition 5.4.5.8). Applying [Lur09a] lemma 5.4.6.5 we see that $\mathcal{C}''_{/c'}$ is κ-filtered that f' and g' are ω-cofinal, and therefore from [Lur09a] lemma 5.4.5.12 we have that f' and g' are cofinal. Denote by c'' the colimit of the natural map $\mathcal{C}''_{/c'} \to \mathcal{C}'$. Since f' and g' are cofinal and $\mathcal{C}, \mathcal{D}'$ are κ-compactly generated we see that the canonical map $c'' \to c'$ becomes an isomorphism upon composition with q' and p'. Therefore c' is a colimit of objects of $\mathcal{C}''$, as desired. $\square$

Lemma 12.5.6. *Let κ be an uncountable regular cardinal and let $X : \mathcal{I} \to \mathrm{Cat}^{\mathrm{rex}(\kappa)}$ be a κ-small diagram. Let $X^{\triangleright} : \mathcal{I}^{\triangleright} \to \mathrm{Cat}^{\mathrm{rex}(\kappa)}$ be a colimit diagram for X, and denote by $*$ the final object of $\mathcal{I}^{\triangleright}$. Assume that for every arrow α in $\mathcal{I}$ the induced functor $X(\alpha)$ admits a right adjoint which preserves κ-small colimits. Then*

(i) *For every i in $\mathcal{I}$ the induced functor $X(i) \to X(*)$ admits a right adjoint which preserves κ-small colimits.*

(ii) *The induced diagram $(X^{\triangleright})^R : (\mathcal{I}^{\mathrm{op}})^{\triangleleft} \to \mathrm{Cat}^{\mathrm{rex}(\kappa)}$ is a limit diagram.*

Proof. Consider the functor $\mathrm{Ind}_\kappa X^{\triangleright} : \mathcal{I}^{\triangleright} \to \mathrm{Pr}^L$. This is a colimit diagram since Ind_κ preserves colimits. Note that we have a natural monomorphism $X \to X^{\triangleright}$. The adjoint functor theorem guarantees that for every arrow α in $\mathcal{I}^{\triangleright}$ the functor $\mathrm{Ind}_\kappa X^{\triangleright}(\alpha)$ has a right adjoint. Moreover, results from [Lur09a] section 5.5.3 guarantee that the induced diagram

$$(\mathrm{Ind}_\kappa X^{\triangleright})^R : (\mathcal{I}^{\mathrm{op}})^{\triangleleft} \to \widehat{\mathrm{Cat}}$$

is a limit diagram.

Let α be an arrow in $\mathcal{I}$. By virtue of being right adjoint to a functor of κ-compactly generated categories which preserves κ-compact objects we have that $(\mathrm{Ind}_\kappa X^{\triangleright})^R(\alpha)$ preserves κ-filtered colimits. Since $X(\alpha)$ is itself right adjointable we see that $(\mathrm{Ind}_\kappa X^{\triangleright})^R(\alpha)$ preserves κ-compact objects. Moreover, the fact that the right adjoint to $X(\alpha)$ preserves κ-small colimits implies that $(\mathrm{Ind}_\kappa X^{\triangleright})^R(\alpha)$ is in fact colimit preserving.

Since the forgetful functor $\mathrm{Pr}^L \to \widehat{\mathrm{Cat}}$ creates small limits, we conclude that $(\mathrm{Ind}_\kappa X^{\triangleright})^R$ factors through Pr^L. Since $(\mathrm{Ind}_\kappa X^{\triangleright})^R|_{\mathcal{I}^{\mathrm{op}}}$ factors through Pr^L_κ and $\mathcal{I}$ is κ-small we conclude using lemma 12.5.5 that $(\operatorname{Ind}_\kappa X^\rhd)^R$ factors through Pr^L_κ, and in fact defines a limit diagram in Pr^L_κ. In particular, we have that for every i in $\mathcal{I}$ the right adjoint to the functor $\operatorname{Ind}_\kappa X(i) \to \operatorname{Ind}_\kappa X(*)$ preserves κ-compact objects and is colimit preserving, which establishes item (i). Item (ii) follows from the fact that passage to κ-compact objects provides an equivalence $\mathrm{Pr}^L_\kappa = \mathrm{Cat}^{\mathrm{rex}(\kappa)}$. □

Corollary 12.5.7. *Let $\mathcal{E}$ be a commutative algebra in $\widehat{\mathrm{Cat}}_{\mathrm{cocompl}}$, generated under colimits by its dualizable objects. Let $X : \mathcal{I} \to \mathcal{E}\text{-mod}$ be a right adjointable diagram, with $\mathcal{I}$ small. Let $X^\rhd : \mathcal{I}^\rhd \to \mathcal{E}\text{-mod}$ be a colimit diagram for X. Then $X^\rhd$ is right adjointable, and the induced diagram $(X^\rhd)^R : (\mathcal{I}^{\mathrm{op}})^\lhd \to \mathcal{E}\text{-mod}$ is a limit diagram.*

Proof. This is a direct consequence of proposition 12.5.3 and lemma 12.5.6 applied to $\kappa = \kappa_0$ after enlarging the universe. □

We now specialize the above discussion to the case when $\mathcal{E}$ is $(n-1)\mathrm{Pr}^L$.

Remark 12.5.8. Let $n > 1$. Recall from remark 12.1.9 that $(n-1)\mathrm{Pr}^L$ is generated under small colimits by the free modules of the form $(n-2)\mathrm{Pr}^L \otimes \mathcal{P}(\mathcal{C})$, where $\mathcal{C}$ is a small category. Since $\mathcal{P}(\mathcal{C})$ is dualizable in $\widehat{\mathrm{Cat}}_{\mathrm{cocompl}}$ we conclude that $(n-2)\mathrm{Pr}^L \otimes \mathcal{P}(\mathcal{C})$ is dualizable in $(n-1)\mathrm{Pr}^L$. Hence $(n-1)\mathrm{Pr}^L$ is generated under small colimits by its dualizable objects.

Proposition 12.5.9. *Let $n \geq 1$ and let $F : \mathcal{C} \to \mathcal{D}$ be a morphism in $\mathrm{Spc}\text{-mod}^n$. Then the functor of n-categories $\psi_n(F)$ admits a left adjoint if and only if F is left adjointable.*

Proof. The case $n = 1$ is clear, so assume that $n > 1$. Recall from our construction that the functor of 1-categories underlying $\psi_n(F)$ is equivalent to F. It follows that if $\psi_n(F)$ is left adjointable then F is left adjointable.

Assume now that F is left adjointable. We continue with the notation from construction 12.3.2. Note that the functor of very large presentable categories $\operatorname{Ind}_{\kappa_0} F : \operatorname{Ind}_{\kappa_0} \mathcal{C} \to \operatorname{Ind}_{\kappa_0} \mathcal{D}$ is left adjointable. It follows from an application of proposition 12.5.3 that $\operatorname{Ind}_{\kappa_0} F$ is a left adjointable morphism of $\mathrm{Spc}\text{-mod}^{n-1}$-modules. By proposition 5.4.1, we have that $\varphi_n(F)$ is a left adjointable functor of $\mathrm{Spc}\text{-mod}^{n-1}$-enriched categories. It follows from corollary 5.4.4, together with the fact that $\operatorname{Ind}_{\kappa_0} F$ and its left adjoint restrict to an adjunction on the subcategories of κ_0-compact objects, that $\varphi'_n(F)$ is also left adjointable. The fact that $\psi_n(F)$ is left adjointable follows now from an application of lemma 12.4.3. □

Proposition 12.5.10. *Let $n \geq 1$ and let $X : \mathcal{I} \to n\mathrm{Pr}^L$ be a diagram with $\mathcal{I}$ small, such that for every arrow α in $\mathcal{I}$ the functor underlying $X(\alpha)$ admits a left adjoint. Then*

(i) *The diagram X is left adjointable, when thought of as a functor into $(n-1)\mathrm{Pr}^L\text{-mod}$.*

(ii) *Let $(X^L)^\rhd : (\mathcal{I}^{\mathrm{op}})^\rhd \to n\mathrm{Pr}^L$ be a colimit diagram for X^L. Then $(X^L)^\rhd$ is right adjointable, and the induced diagram $((X^L)^\rhd)^R : \mathcal{I}^\lhd \to n\mathrm{Pr}^L$ is a limit diagram for X.*

Proof. The case $n = 1$ follows from the results of [Lur09a] section 5.5, so we assume that $n > 1$. Item (i) then follows from an application of proposition 12.5.3 and (ii) is a direct consequence of corollary 12.5.7. □

Our next goal is to recast proposition 12.5.10 in a language intrinsic to the $(n+1)$-categorical structure on $n\mathrm{Pr}^L$. We will in fact be able to obtain a variant of it which works for any object in $\mathrm{Spc}\text{-mod}^n$.

Notation 12.5.11. Let $\mathcal{D}$ be an n-category. We denote by $\mathcal{D}^{\leq 1}$ the 1-category underlying $\mathcal{D}$, and by $(\mathcal{D}^{\leq 1})^{\mathrm{radj}}$ (resp. $\mathcal{D}^{\leq 1})^{\mathrm{ladj}}$ the subcategory of $\mathcal{D}^{\leq 1}$ containing all objects, and only those morphisms which are right (resp. left) adjointable in $\mathcal{D}$.

Definition 12.5.12. *Let $\mathcal{D}$ be an n-category. We say that $\mathcal{D}$ satisfies the passage to adjoints property if the following conditions are satisfied:*

- *The category $(\mathcal{D}^{\leq 1})^{\mathrm{radj}}$ has all colimits, and the inclusion $(\mathcal{D}^{\leq 1})^{\mathrm{radj}} \to \mathcal{D}$ preserves conical colimits.*
- *The category $(\mathcal{D}^{\leq 1})^{\mathrm{ladj}}$ has all limits, and the inclusion $(\mathcal{D}^{\leq 1})^{\mathrm{ladj}} \to \mathcal{D}$ preserves conical limits.*

Remark 12.5.13. Let $\mathcal{D}$ be an n-category. Then the passage to adjoints equivalence[1] from [GR17] induces an equivalence between the categories $(\mathcal{D}^{\leq 1})^{\mathrm{radj}}$ and $(\mathcal{D}^{\leq 1})^{\mathrm{ladj}}$. It follows that if $\mathcal{D}$ satisfies the passage to adjoints property, then a right adjointable diagram $F : \mathcal{I}^{\triangleright} \to \mathcal{D}$ in $\mathcal{D}$ is a conical colimit if and only if the diagram $F^R : (\mathcal{I}^{\mathrm{op}})^{\triangleleft} \to \mathcal{D}$ is a conical limit.

Theorem 12.5.14. *Let $n \geq 2$ and let $\mathcal{C}$ be an object in* $\mathrm{Spc}\text{-mod}^n$. *Then the n-category $\psi_n(\mathcal{C})$ satisfies the passage to adjoints property.*

Our proof of theorem 12.5.14 requires a few lemmas.

Lemma 12.5.15. *Let $n \geq 2$ and let $\mathcal{C}$ be an object in* $\mathrm{Spc}\text{-mod}^n$. *Let $\alpha : c \to d$ be a morphism in $\mathcal{C}$. Denote by $\mathscr{H}om_{\mathcal{C}}$ the functor of Hom objects for $\mathcal{C}$ as a module over* $(n-1)\mathrm{Pr}^L$. *Then*

(i) *The morphism α is left adjointable in $\psi_n(\mathcal{C})$ if and only if for every morphism $\gamma : e \to e'$ in $\mathcal{C}$, the commutative square of categories underlying the commutative square of* $(n-2)\mathrm{Pr}^L$*-modules*

$$\begin{array}{ccc} \mathscr{H}om_{\mathcal{C}}(e, c) & \xrightarrow{\alpha_*} & \mathscr{H}om_{\mathcal{C}}(e, d) \\ \gamma^* \uparrow & & \gamma^* \uparrow \\ \mathscr{H}om_{\mathcal{C}}(e', c) & \xrightarrow{\alpha_*} & \mathscr{H}om_{\mathcal{C}}(e', d) \end{array}$$

is horizontally left adjointable.

[1]The proof of our result does not rely on the existence of such an equivalence in the general case - however our choice of name for the passage to adjoints property is motivated by it.

(ii) *The morphism α is right adjointable in $\psi_n(\mathcal{C})$ if and only if for every morphism $\gamma : e \to e'$ in $\mathcal{C}$, the commutative square of categories underlying the commutative square of $(n-2)\mathrm{Pr}^L$-modules*

$$\begin{tikzcd} \mathscr{H}om_{\mathcal{C}}(d,e') \arrow[r, "\alpha^*"] & \mathscr{H}om_{\mathcal{C}}(c,e') \\ \mathscr{H}om_{\mathcal{C}}(d,e) \arrow[r, "\alpha^*"] \arrow[u, "\gamma_*"] & \mathscr{H}om_{\mathcal{C}}(c,e) \arrow[u, "\gamma_*"] \end{tikzcd}$$

is horizontally left adjointable.

Proof. This is a direct consequence of lemma 5.2.9 applied to the Yoneda embedding, as in the proof of proposition 5.2.10. □

Lemma 12.5.16. *Let $n \geq 2$. Let $\mathcal{I}$ be a small category and let $X^{\lhd}, X'^{\lhd} : \mathcal{I}^{\lhd} \to \mathrm{Spc}\text{-}\mathrm{mod}^{n-1}$ be two small limit diagrams. Denote by $*$ the initial object of $\mathcal{I}^{\lhd}$, and let $X = X^{\lhd}|_{\mathcal{I}}$ and $X' = X'^{\lhd}|_{\mathcal{I}}$. Let $\eta : X^{\lhd} \to X'^{\lhd}$ be a natural transformation, and assume that for every arrow $\alpha : i \to j$ in $\mathcal{I}$ the commutative square of categories*

$$\begin{tikzcd} X(i) \arrow[r, "X(\alpha)"] \arrow[d, "\eta_i"] & X(j) \arrow[d, "\eta_j"] \\ X'(i) \arrow[r, "X'(\alpha)"] & X'(j) \end{tikzcd}$$

is horizontally left adjointable. Then

(i) *For every index j in $\mathcal{I}$ the commutative square of categories*

$$\begin{tikzcd} X^{\lhd}(*) \arrow[r] \arrow[d, "\eta_*"] & X(j) \arrow[d, "\eta_j"] \\ X'^{\lhd}(*) \arrow[r] & X'(j) \end{tikzcd}$$

is horizontally left adjointable.

(ii) *Assume given an extension of $\eta|_{\mathcal{I}}$ to a natural transformation μ between functors $Y^{\lhd}, Y'^{\lhd} : \mathcal{I}^{\lhd} \to \mathrm{Spc}\text{-}\mathrm{mod}^{n-1}$, and that for every index j in $\mathcal{I}$ the commutative square of categories*

$$\begin{tikzcd} Y^{\lhd}(*) \arrow[r] \arrow[d, "\mu_*"] & X(j) \arrow[d, "\eta_j"] \\ Y'^{\lhd}(*) \arrow[r] & X'(j) \end{tikzcd}$$

is horizontally left adjointable. Then the induced commutative square of categories

$$\begin{tikzcd} Y^{\lhd}(*) \arrow[r] \arrow[d, "\mu_*"] & X^{\lhd}(*) \arrow[d, "\eta_*"] \\ Y'^{\lhd}(*) \arrow[r] & X'^{\lhd}(*) \end{tikzcd}$$

is horizontally left adjointable.

Proof. We first establish item (i). It follows from corollary 12.5.7 that for every j in $\mathcal{I}$ the functors $X^{\lhd}(*) \to X(j)$ and $X'^{\lhd}(*) \to X'(j)$ are left adjointable. Consider now the commutative square of categories

$$\begin{tikzcd} \mathrm{Ind}_{\kappa_0} X^{\lhd}(*) \arrow[r] \arrow[d, "\mathrm{Ind}_{\kappa_0} \eta_*"] & \mathrm{Ind}_{\kappa_0} X(j) \arrow[d, "\mathrm{Ind}_{\kappa_0} \eta_j"] \\ \mathrm{Ind}_{\kappa_0} X'^{\lhd}(*) \arrow[r] & \mathrm{Ind}_{\kappa_0} X'(j). \end{tikzcd}$$

Our claim will follow if we are able to show that this is horizontally left adjointable. Since the horizontal arrows admit left adjoints, it suffices in fact to show that this is vertically right adjointable. By virtue of lemma 5.2.9, it suffices to show that the diagram

$$G^{\lhd} : \mathcal{I}^{\lhd} \to \mathrm{Funct}([1], \mathrm{CAT}^{\text{2-cat}})$$

induced from the natural transformation

$$\mathrm{Ind}_{\kappa_0} \eta : \mathrm{Ind}_{\kappa_0} X^{\lhd} \to \mathrm{Ind}_{\kappa_0} X'^{\lhd}$$

factors through $\mathrm{Funct}(\mathrm{Adj}, \mathrm{CAT}^{\text{2-cat}})$, where $\mathrm{CAT}^{\text{2-cat}}$ denotes the 2-category of very large categories. Combining propositions 5.4.8 and 5.3.13 with lemma 12.5.5 we see that $G^{\lhd}$ is in fact a conical limit diagram.

For each arrow $\alpha : i \to j$ in $\mathcal{I}$, the commutative square of categories

$$\begin{tikzcd} \mathrm{Ind}_{\kappa_0} X(i) \arrow[r, "\mathrm{Ind}_{\kappa_0} X(\alpha)"] \arrow[d, "\mathrm{Ind}_{\kappa_0} \eta_i"] & \mathrm{Ind}_{\kappa_0} X(j) \arrow[d, "\mathrm{Ind}_{\kappa_0} \eta_j"] \\ \mathrm{Ind}_{\kappa_0} X'(i) \arrow[r, "\mathrm{Ind}_{\kappa_0} X'(\alpha)"] & \mathrm{Ind}_{\kappa_0} X'(j) \end{tikzcd}$$

is horizontally left adjointable thanks to our hypothesis on η. By the adjoint functor theorem, we also know that the vertical arrows are right adjointable. It follows that the above square is also vertically right adjointable. Another application of lemma 5.2.9 shows that $G^{\lhd}|_{\mathcal{I}}$ factors through $\mathrm{Funct}(\mathrm{Adj}, \mathrm{CAT}^{\text{2-cat}})$. Applying proposition 5.3.13 we conclude that $G^{\lhd}$ factors through $\mathrm{Funct}(\mathrm{Adj}, \mathrm{CAT}^{\text{2-cat}})$, as we claimed.

We now prove item (ii). Applying corollary 12.5.7 we see that $(X^{\lhd})^L$ is a colimit diagram in $\mathrm{Spc}\text{-}\mathrm{mod}^{n-1}$ and hence we have a morphism $(X^{\lhd})^L \to (Y^{\lhd})^L$ which restricts to the identity on $\mathcal{I}^{\mathrm{op}}$. Consider now the induced morphism

$$\mathrm{Ind}_{\kappa_0}(X^{\lhd})^L \to \mathrm{Ind}_{\kappa_0}(Y^{\lhd})^L.$$

Passing to right adjoints, we obtain a natural transformation $\mathrm{Ind}_{\kappa_0} Y^{\lhd} \to \mathrm{Ind}_{\kappa_0} X^{\lhd}$ which is the identity on $\mathcal{I}$. Since $\mathrm{Ind}_{\kappa_0} X^{\lhd}$ is a limit diagram by lemma 12.5.5, we conclude that the induced functor

$$\mathrm{Ind}_{\kappa_0} Y^{\lhd}(*) \to \mathrm{Ind}_{\kappa_0} X^{\lhd}(*)$$

is obtained by ind-κ_0-completion of the map $Y^{\lhd}(*) \to X^{\lhd}(*)$ induced from μ. It follows that the functor $Y^{\lhd}(*) \to X^{\lhd}(*)$ is left adjointable. Similarly, we have that the functor $Y'^{\lhd}(*) \to X'^{\lhd}(*)$ is horizontally left adjointable.

As before, our claim would follow if we are able to show that the commutative square of categories

$$\begin{tikzcd}
\mathrm{Ind}_{\kappa_0} Y^{\lhd}(*) \arrow[r] \arrow[d, "\mathrm{Ind}_{\kappa_0} \mu_*"] & \mathrm{Ind}_{\kappa_0} X^{\lhd}(*) \arrow[d, "\mathrm{Ind}_{\kappa_0} \eta_*"] \\
\mathrm{Ind}_{\kappa_0} Y'^{\lhd}(*) \arrow[r] & \mathrm{Ind}_{\kappa_0} X'^{\lhd}(*)
\end{tikzcd}$$

is horizontally left adjointable. Since the horizontal arrows admit left adjoints, it suffices to show that the square is in fact vertically right adjointable. Consider the diagram

$$H^{\lhd} : \mathcal{I}^{\lhd} \to \mathrm{Funct}([1], \mathrm{CAT}^{\text{2-cat}})$$

induced from the natural transformation

$$\mathrm{Ind}_{\kappa_0} \mu : \mathrm{Ind}_{\kappa_0} Y^{\lhd} \to \mathrm{Ind}_{\kappa_0} Y'^{\lhd}.$$

This extends the functor $G^{\lhd}|_{\mathcal{I}}$. By lemma 5.2.9, we may reduce to showing that the induced natural transformation $H^{\lhd} \to G^{\lhd}$ factors through $\mathrm{Funct}(\mathrm{Adj}, \mathrm{CAT}^{\text{2-cat}})$. This is a consequence of the fact that $G^{\lhd}$ is a conical limit diagram in $\mathrm{Funct}(\mathrm{Adj}, \mathrm{CAT}^{\text{2-cat}})$. □

Proof of theorem 12.5.14. To simplify notation, we set $\mathcal{C}^{\mathrm{radj}} = (\psi_n(\mathcal{C})^{\leq 1})^{\mathrm{radj}}$ and $\mathcal{C}^{\mathrm{ladj}} = (\psi_n(\mathcal{C})^{\leq 1})^{\mathrm{ladj}}$.

Let $X : \mathcal{I} \to \mathcal{C}^{\mathrm{radj}}$ be a diagram, and let $X^{\rhd} : \mathcal{I}^{\rhd} \to \mathcal{C}$ be an extension of X to a colimit diagram in $\mathcal{C}$. We claim that $X^{\rhd}$ factors through $\mathcal{C}^{\mathrm{radj}}$. Let $\gamma : e \to e'$ be a morphism in $\mathcal{C}$. We have limit diagrams

$$\mathscr{H}om_{\mathcal{C}}(X^{\rhd}, e) : (\mathcal{I}^{\rhd})^{\mathrm{op}} \to \mathrm{Spc}\text{-}\mathrm{mod}^{n-1}$$

and

$$\mathscr{H}om_{\mathcal{C}}(X^{\rhd}, e') : (\mathcal{I}^{\rhd})^{\mathrm{op}} \to \mathrm{Spc}\text{-}\mathrm{mod}^{n-1}.$$

The morphism γ induces a natural transformation

$$\mathscr{H}om_{\mathcal{C}}(X^{\triangleright}, \gamma) : \mathscr{H}om_{\mathcal{C}}(X^{\triangleright}, e) \to \mathscr{H}om_{\mathcal{C}}(X^{\triangleright}, e').$$

Combining lemma 12.5.15 and part (i) of lemma 12.5.16 we see that for every i in $\mathcal{I}$ the commutative square of categories

$$\begin{tikzcd} \mathscr{H}om_{\mathcal{C}}(X^{\triangleright}(*), e) \arrow[r] \arrow[d, "\gamma_*"] & \mathscr{H}om_{\mathcal{C}}(X(i), e) \arrow[d, "\gamma_*"] \\ \mathscr{H}om_{\mathcal{C}}(X^{\triangleright}(*), e') \arrow[r] & \mathscr{H}om_{\mathcal{C}}(X(i), e') \end{tikzcd}$$

is horizontally left adjointable. Using lemma 12.5.15 once more we conclude that $X^{\triangleright}$ factors through $\mathcal{C}^{\mathrm{radj}}$.

Similarly, combining lemma 12.5.15 with part (ii) of lemma 12.5.16 we see that if $Y^{\triangleright}$ is another extension of X which factors through $\mathcal{C}^{\mathrm{radj}}$, then the induced morphism $X^{\triangleright}(*) \to Y^{\triangleright}(*)$ belongs to $\mathcal{C}^{\mathrm{radj}}$. This means that $X^{\triangleright}$ is in fact also a colimit diagram in $\mathcal{C}^{\mathrm{radj}}$, and therefore we see that the inclusion $\mathcal{C}^{\mathrm{radj}} \to \mathcal{C}$ creates colimits. It now follows from theorem 12.4.6 that $\psi_n(\mathcal{C})$ satisfies the first condition in definition 12.5.12.

We now show that $\psi_n(\mathcal{C})$ satisfies the second condition in definition 12.5.12. Thanks to theorem 12.4.6, it suffices to show that $\mathcal{C}^{\mathrm{ladj}}$ has arbitrary products and fiber products, and that these are preserved by its inclusion into $\mathcal{C}$. An argument analogous to the case of colimits reduces one to showing that $\mathcal{C}$ has arbitrary products and fiber products of diagrams in $\mathcal{C}^{\mathrm{ladj}}$.

We consider the case of pullbacks - the proof for arbitrary products is analogous. Let $\mathcal{I}$ be the category with objects $0, 1, 2$ and nontrivial arrows $0 \to 2 \leftarrow 1$. Let $X : \mathcal{I} \to \mathcal{C}^{\mathrm{ladj}}$ be a diagram. Extend X to a limit diagram $X^{\triangleleft} : \mathcal{I}^{\triangleleft} \to \mathrm{Ind}_{\kappa_0} \mathcal{C}$. Repeating our argument for the case of colimits shows that $X^{\triangleleft}$ defines in fact a limit diagram in the wide subcategory $(\mathrm{Ind}_{\kappa_0} \mathcal{C})^{\mathrm{ladj}}$ of $\mathrm{Ind}_{\kappa_0} \mathcal{C}$ containing those arrows which are left adjointable when considered inside the n-category $(\psi_{n-1})_! \varphi_n(\mathcal{C})$.

Consider now the diagram $X^L : \mathcal{I}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{radj}}$ obtained from X by passage to left adjoints. As in the case of X, we may extend X^L to a diagram $(X^L)^{\triangleright} : (\mathcal{I}^{\mathrm{op}})^{\triangleright} \to (\mathrm{Ind}_{\kappa_0} \mathcal{C})^{\mathrm{radj}}$ which is both a colimit diagram in $(\mathrm{Ind}_{\kappa_0} \mathcal{C})^{\mathrm{radj}}$ and $\mathrm{Ind}_{\kappa_0} \mathcal{C}$. Since pushout squares in $(\mathrm{Ind}_{\kappa_0} \mathcal{C})^{\mathrm{radj}}$ and pullback squares in $(\mathrm{Ind}_{\kappa_0} \mathcal{C})^{\mathrm{ladj}}$ are in one to one correspondence by passing to adjoints of all arrows involved, we obtain an equivalence between $(X^L)^{\triangleright}(*)$ and $X^{\triangleleft}(*)$. Since $\mathcal{C}$ is closed under small colimits inside $\mathrm{Ind}_{\kappa_0} \mathcal{C}$ we conclude that $X^{\triangleleft}(*)$ in fact belongs to $\mathcal{C}$. Hence $X^{\triangleleft}$ gives the desired extension of X to a limit diagram in $\mathcal{C}$. $\square$

Chapter 13

Categorical spectra

Consider the following series of analogies between concepts in homotopy theory and concepts in higher category theory:

Homotopy theory	*Higher category theory*
Space	ω-category
n-truncated space	n-category
Pointed space	Pointed ω-category
Based loopspace	ω-category of endomorphisms
Grouplike monoid in spaces	Monoidal ω-category
Grouplike commutative algebra in spaces	Symmetric monoidal ω-category

The goal of this chapter is to enhance the above dictionary by studying an analogue in higher category theory of the notion of spectrum from homotopy theory. In the same way that the category of spectra is obtained by formally inverting the based loopspace endofunctor $\Omega : \mathrm{Spc}_* \to \mathrm{Spc}_*$, we introduce a category CatSp obtained by formally inverting the endofunctor $\Omega_{\omega\mathrm{Cat}} : \omega\mathrm{Cat}_* \to \omega\mathrm{Cat}_*$ that sends each pointed ω-category $(\mathcal{C}, x)$ to the pointed ω-category $(\mathrm{End}_{\mathcal{C}}(x), \mathrm{id}_x)$. In other words, an object of CatSp is a sequence of pointed ω-categories $\mathcal{C}_n$ and identifications $\Omega_{\omega\mathrm{Cat}}(\mathcal{C}_{n+1}) = \mathcal{C}_n$ for all $n \geq 0$.

We call CatSp the category of categorical spectra. In the same way that the passage from spaces to spectra amounts to allowing negative homotopy groups, we may think about the passage from ω-categories to categorical spectra as allowing cells of negative dimensions.

The theory of categorical spectra may be considered as a joint generalization of the theory of symmetric monoidal ω-categories and the theory of spectra. This is justified by the fact that CatSp sits in a commutative square of categories and fully faithful functors

$$\begin{tikzcd}
\mathrm{CAlg}_{\mathrm{grpllike}}(\mathrm{Spc}) \arrow[r] \arrow[d] & \mathrm{Sp} \arrow[d] \\
\mathrm{CAlg}(\omega\mathrm{Cat}) \arrow[r] & \mathrm{CatSp}\,.
\end{tikzcd}$$

A fundamental observation that leads to the notion of spectra is that one may induce a grouplike commutative algebra structure on a space by giving a sequence of deloopings of it. In the same way, the point of view of categorical spectra is that sometimes symmetric monoidal structures on an ω-category are more naturally studied by studying a sequence of deloopings of it.

We now describe the contents of this chapter in more detail. We begin in 13.1 by studying, for each monoidal category $\mathcal{M}$, the functor $\Omega_{\mathcal{M}} : \mathrm{Cat}_*^{\mathcal{M}} \to \mathrm{Alg}(\mathcal{M})$ which sends each pointed $\mathcal{M}$-enriched category to the algebra of endomorphisms of its basepoint. We show that this has a fully faithful left adjoint $B_{\mathcal{M}} : \mathrm{Alg}(\mathcal{M}) \to \mathrm{Cat}_*^{\mathcal{M}}$ which identifies the category of algebras in $\mathcal{M}$ with the category of pointed $\mathcal{M}$-enriched categories with a connected space of objects. This allows us to define for each symmetric monoidal category $\mathcal{M}$ the category $\mathrm{CatSp}_{\mathcal{M}}$ of $\mathcal{M}$-enriched categorical spectra as the limit of the sequence of categories

$$\mathcal{M}_* \xleftarrow{\Omega_{\mathcal{M}}} \mathrm{Cat}_*^{\mathcal{M}} \xleftarrow{\Omega_{\mathrm{Cat}^{\mathcal{M}}}} 2\mathrm{Cat}_*^{\mathcal{M}} \xleftarrow{\Omega_{2\mathrm{Cat}^{\mathcal{M}}}} 3\mathrm{Cat}_*^{\mathcal{M}} \dots .$$

In 13.2 we specialize the above to the case $\mathcal{M} = \omega\mathrm{Cat}$, to obtain the category $\mathrm{CatSp} = \mathrm{CatSp}_{\omega\mathrm{Cat}}$ of categorical spectra. We show that this may be identified with the limit of the sequence of categories

$$\omega\mathrm{Cat}_* \xleftarrow{\Omega_{\omega\mathrm{Cat}}} \omega\mathrm{Cat}_* \xleftarrow{\Omega_{\omega\mathrm{Cat}}} \omega\mathrm{Cat}_* \xleftarrow{\Omega_{\omega\mathrm{Cat}}} \omega\mathrm{Cat}_* \dots$$

and in particular, admits a description as the category obtained from $\omega\mathrm{Cat}_*$ by formally adding an inverse $B_{\omega\mathrm{Cat}}$ to the endomorphism $\Omega_{\omega\mathrm{Cat}}$.

We explore here the idea that categorical spectra behave like ω-categories with cells of arbitrary integer dimension. We introduce for each categorical spectrum spaces of cells of dimension i for every integer i, and show that these satisfy familiar composition and unitaly laws. In the same way that $\omega\mathrm{Cat}$ comes equipped with an infinite family of order reversing commuting involutions indexed by positive integers, we show that CatSp admits an infinite family of order reversing commuting involutions indexed by arbitrary integers.

We also study for each integer n the subcategory $n\mathrm{CatSp}$ of CatSp consisting of categorical spectra for which all cells of dimension greater than n are invertible. We show that the inclusion of $n\mathrm{CatSp}$ inside CatSp admits both left and right adjoints, which allow one to universally invert cells above dimension n, or discard non-invertible cells above dimension n.

In 13.3 we discuss a variety of examples of categorical spectra:

- We show that any spectrum (in the sense of homotopy theory) gives an example of a categorical spectrum. This gives an embedding $\mathrm{Sp} \to \mathrm{CatSp}$, which we show admits both left and right adjoints.

- Any grouplike commutative algebra in spaces gives an example of a spectrum. We provide here a generalization of this by attaching to each commutative algebra A in a symmetric monoidal category $\mathcal{M}$ an $\mathcal{M}$-categorical spectrum which we call the Eilenberg-MacLane $\mathcal{M}$-categorical spectrum of A.

- Given a presentable symmetric monoidal category $\mathcal{E}$, we construct a categorical spectrum which organizes the ω-categories of presentable n-categories tensored over $\mathcal{E}$ for all values of n.
- Given a category with finite limits $\mathcal{C}$, we construct a categorical spectrum which organizes the ω-categories $n\mathrm{Corr}(\mathcal{C})$ of correspondences of $\mathcal{C}$ for all values of n.
- Given a symmetric monoidal category with good relative tensor products, we discuss the Morita categorical spectrum of $\mathcal{C}$, which organizes the higher Morita categories from [Hau17].

In 13.4 we study the connection between the theory of categorical spectra and the theory of symmetric monoidal ω-categories. We show that for a symmetric monoidal category $\mathcal{M}$, the category $\mathrm{CatSp}_{\mathcal{M}}$ can alternately be defined as the limit of a sequence of functors

$$\mathrm{CAlg}(\mathcal{M}) \xleftarrow{\Omega_{\mathcal{M}}} \mathrm{CAlg}(\mathrm{Cat}^{\mathcal{M}}) \xleftarrow{\Omega_{\mathrm{Cat}^{\mathcal{M}}}} \mathrm{CAlg}(2\mathrm{Cat}^{\mathcal{M}}) \xleftarrow{\Omega_{2\mathrm{Cat}^{\mathcal{M}}}} \mathrm{CAlg}(3\mathrm{Cat}^{\mathcal{M}}) \ldots.$$

In other words, an $\mathcal{M}$-categorical spectrum is the same data as a sequence of symmetric monoidal $\mathcal{M}$-enriched higher categories compatible under looping. We use the above fact to show that the symmetric monoidal structure on $\mathrm{CatSp}_{\mathcal{M}}$ inherited from $\mathcal{M}$ is cocartesian. In the case $\mathcal{M} = \omega\mathrm{Cat}$ this implies that CatSp admits biproducts.

From the above sequence we obtain a projection $\Omega^{\infty}_{\mathcal{M}} : \mathrm{CatSp}_{\mathcal{M}} \to \mathrm{CAlg}(\mathcal{M})$. In other words, each $\mathcal{M}$-categorical spectrum has an underlying commutative algebra in $\mathcal{M}$. We show that this admits a fully faithful left adjoint, which sends each commutative algebra in $\mathcal{M}$ to its Eilenberg-MacLane $\mathcal{M}$-categorical spectrum. This identifies $\mathrm{CAlg}(\mathcal{M})$ with a certain full subcategory of $\mathrm{CatSp}_{\mathcal{M}}$ on objects that we call connective.

Specializing to the case $\mathcal{M} = \mathrm{CatSp}$ we obtain close connections between the theory of categorical spectra and the theory of symmetric monoidal ω-categories, which run parallel to the connections between spectra and grouplike commutative algebras in spaces. We may summarize this series of analogies as follows:

Homotopy theory	*Higher category theory*
Spectrum	Categorical spectrum
Inverse automorphisms B, Ω on Sp	Inverse automorphisms $B_{\omega\mathrm{Cat}}, \Omega_{\omega\mathrm{Cat}}$ on CatSp
$\Omega^{\infty} : \mathrm{Sp} \to \mathrm{CAlg}_{\mathrm{grplike}}(\mathrm{Spc})$	$\Omega^{\infty}_{\omega\mathrm{Cat}} : \mathrm{CatSp} \to \mathrm{CAlg}(\omega\mathrm{Cat})$
Connective spectrum	Connective categorical spectrum
Embedding $\mathrm{CAlg}_{\mathrm{grplike}}(\mathrm{Spc}) \to \mathrm{Sp}$	Embedding $\mathrm{CAlg}(\omega\mathrm{Cat}) \to \mathrm{CatSp}$

13.1 Basic notions

Recall that the category of spectra is obtained by formally inverting the loop space functor $\Omega : \mathrm{Spc}_* \to \mathrm{Spc}_*$. We begin this section by studying the categorical analogue of this functor.

Notation 13.1.1. Let $\mathcal{M}$ be an associative operad and recall the projection $\mathrm{Algbrd}(\mathcal{M}) \to \mathrm{Cat}$ which sends each $\mathcal{M}$-algebroid to its category of objects. We denote by $\mathrm{Algbrd}_*(\mathcal{M})$ the pullback $\mathrm{Algbrd}(\mathcal{M}) \times_{\mathrm{Cat}} \mathrm{Cat}_*$. We call this the category of pointed $\mathcal{M}$-algebroids.

We let $\mathrm{Algbrd}_*(\mathcal{M})_{\mathrm{Spc}}$ (resp. $\mathrm{Cat}_*^{\mathcal{M}}$) denote the full subcategory of $\mathrm{Algbrd}_*(\mathcal{M})$ on those pointed $\mathcal{M}$-algebroids whose underlying $\mathcal{M}$-algebroid has a space of objects (resp. is an $\mathcal{M}$-enriched category). We call $\mathrm{Cat}_*^{\mathcal{M}}$ the category of pointed $\mathcal{M}$-enriched categories.

Warning 13.1.2. It is common in the literature to use the terminology pointed category to refer to a category with a zero object. In notation 13.1.1 we are using this terminology to refer instead to pointed objects in the category of $\mathcal{M}$-enriched categories.

Remark 13.1.3. Let $\mathcal{M}$ be a monoidal category. Then the category $\mathrm{Algbrd}_*(\mathcal{M})$ (resp. $\mathrm{Cat}_*(\mathcal{M})$) can be identified with the undercategory $\mathrm{Algbrd}(\mathcal{M})_{1_{\mathcal{M}}/}$ (resp. $\mathrm{Cat}(\mathcal{M})_{1_{\mathcal{M}}/}$).

Notation 13.1.4. Let $\mathcal{M}$ be a monoidal category. Since the projection $\mathrm{Algbrd}_*(\mathcal{M}) \to \mathrm{Cat}_*$ is a cartesian fibration, the inclusion of the fiber over $[0]$ admits a right adjoint. We denote by

$$\Omega_{\mathcal{M}} : \mathrm{Algbrd}_*(\mathcal{M}) \to \mathrm{Algbrd}_{[0]}(\mathcal{M}) = \mathrm{Alg}(\mathcal{M})$$

the resulting functor. In other words, this is the unique functor extending the identity on $\mathrm{Algbrd}_{[0]}(\mathcal{M})$, and which sends arrows cartesian for the projection $\mathrm{Algbrd}_*(\mathcal{M}) \to \mathrm{Cat}_*$ (that is, fully faithful morphisms of pointed algebroids) to isomorphisms in $\mathrm{Alg}(\mathcal{M})$.

We will sometimes abuse notation and continue denoting by $\Omega_{\mathcal{M}}$ the restriction of the above functor to $\mathrm{Cat}_*^{\mathcal{M}}$. We will also use the same notion for the functors obtained from these by composition with the forgetful functor

$$\mathrm{Alg}(\mathcal{M}) \to \mathcal{M}_*$$

where $\mathcal{M}_*$ denotes the category of E_0-algebras in $\mathcal{M}$.

We denote by $B_{\mathcal{M}}$ the composite functor

$$\mathrm{Alg}(\mathcal{M}) \to \mathrm{Algbrd}_*(\mathcal{M})_{\mathrm{Spc}} \to \mathrm{Cat}_*^{\mathcal{M}}$$

where the first arrow is the inclusion, and the second arrow is the localization functor. In other words, $B_{\mathcal{M}}$ is the left adjoint to $\Omega_{\mathcal{M}} : \mathrm{Cat}_*^{\mathcal{M}} \to \mathrm{Alg}(\mathcal{M})$.

Remark 13.1.5. Let $\mathcal{M}$ be a monoidal category. Then the functor $\Omega_{\mathcal{M}}$ sends an $\mathcal{M}$-algebroid $\mathcal{A}$ equipped with a basepoint x to an algebra in $\mathcal{M}$ whose underlying object is $\mathcal{A}(x, x)$. On the other hand, the functor $B_{\mathcal{M}}$ sends an associative algebra A in $\mathcal{M}$ to a pointed $\mathcal{M}$-enriched category $B_{\mathcal{M}}A$ with a connected space of objects, and equipped with an identification $\Omega_{\mathcal{M}} B_{\mathcal{M}} A = A$.

Remark 13.1.6. Let $\mathcal{M}$ be a monoidal category. Recall from remark 3.4.13 that morphisms which are local for the localization $\mathrm{Algbrd}(\mathcal{M})_{\mathrm{Spc}} \to \mathrm{Cat}^{\mathcal{M}}$ are cartesian for the projection $\mathrm{Algbrd}(\mathcal{M})_{\mathrm{Spc}} \to \mathrm{Spc}$. Therefore their images under the morphism $\Omega_{\mathcal{M}} : \mathrm{Algbrd}_*(\mathcal{M})_{\mathrm{Spc}} \to \mathrm{Alg}(\mathcal{M})$ are invertible.

It follows that the functor $B_{\mathcal{M}} : \mathrm{Alg}(\mathcal{M}) \to \mathrm{Cat}_*^{\mathcal{M}}$ is fully faithful. This gives an identification of $\mathrm{Alg}(\mathcal{M})$ with the full subcategory of $\mathrm{Cat}_*^{\mathcal{M}}$ on the pointed $\mathcal{M}$-enriched categories with a connected space of objects.

Remark 13.1.7. Let $F : \mathcal{M} \to \mathcal{M}'$ be a monoidal functor between monoidal categories. Consider the commutative square

$$\begin{tikzcd}
\mathrm{Alg}(\mathcal{M}) \arrow[r] \arrow[d, "F_!"] & \mathrm{Algbrd}_*(\mathcal{M})_{\mathrm{Spc}} \arrow[r] \arrow[d, "F_!"] & \mathrm{Cat}_*^{\mathcal{M}} \arrow[d, "F_!"] \\
\mathrm{Alg}(\mathcal{M}') \arrow[r] & \mathrm{Algbrd}_*(\mathcal{M}')_{\mathrm{Spc}} \arrow[r] & \mathrm{Cat}_*^{\mathcal{M}'}
\end{tikzcd}$$

where the left horizontal arrows are the inclusions, and the right horizontal arrows are the projections.

The compositions of the horizontal rows recover the functors $B_{\mathcal{M}}$ and $B_{\mathcal{M}'}$. For each pointed $\mathcal{M}$-enriched category $\mathcal{A}$ the unit map $u : B_{\mathcal{M}}\Omega_{\mathcal{M}}\mathcal{A} \to \mathcal{A}$ is fully faithful. The map $F_!u$ is also fully faithful, and it therefore presents $F_!\Omega_{\mathcal{M}}\mathcal{A}$ as right adjoint to $B_{\mathcal{M}}$ at $F_!\mathcal{A}$.

We conclude that the outer commutative square is horizontally right adjointable. It follows that there is a natural transformation

$$\Omega_{(-)} : \mathrm{Cat}_*^{(-)} \to \mathrm{Alg}(-)$$

of functors $\mathrm{Alg}(\mathrm{Cat}) \to \widehat{\mathrm{Cat}}$, which specializes upon evaluation on a monoidal category $\mathcal{M}$ to the functor $\Omega_{\mathcal{M}} : \mathrm{Cat}_*^{\mathcal{M}} \to \mathrm{Alg}(\mathcal{M})$ of notation 13.1.4.

Note that the natural transformation $\Omega_{(-)}$ is characterized uniquely by the fact that for each monoidal category $\mathcal{M}$ the map $\Omega_{\mathcal{M}}$ inverts fully faithful functors, and the fact that the composite natural transformation $\Omega_{(-)}B_{(-)}$ comes equipped with an identification to $\mathrm{id}_{\mathrm{Alg}(-)}$. Indeed, if $\Omega'_{(-)}$ is another such natural transformation, then the fact that $\Omega'_{\mathcal{M}} = \Omega_{\mathcal{M}}$ for all $\mathcal{M}$ implies that $\Omega'_{(-)}$ factors as $\eta\Omega_{(-)}$ for some natural transformation $\eta : \mathrm{Alg}(-) \to \mathrm{Alg}(-)$, and the fact that $\Omega'_{(-)}B_{(-)}$ is the identity implies that η is the identity.

We are now ready to give the definition of categorical spectrum.

Notation 13.1.8. Let $\mathcal{M}$ be a symmetric monoidal category and let $n \geq 1$. We denote by $n\mathrm{Cat}^{\mathcal{M}}$ the symmetric monoidal category of $\mathcal{M}$-enriched n-categories. We let $n\mathrm{Cat}_*^{\mathcal{M}}$ be the category of pointed $(n-1)\mathrm{Cat}^{\mathcal{M}}$-enriched categories (where in the case $n = 1$ we use the convention $0\mathrm{Cat}^{\mathcal{M}} = \mathcal{M}$). We call $n\mathrm{Cat}_*^{\mathcal{M}}$ the category of pointed $\mathcal{M}$-enriched n-categories.

Definition 13.1.9. *Let $\mathcal{M}$ be a symmetric monoidal category. We denote by $\mathrm{CatSp}_{\mathcal{M}}$ limit of the sequence of categories*

$$\mathcal{M}_* \xleftarrow{\Omega_{\mathcal{M}}} \mathrm{Cat}_*^{\mathcal{M}} \xleftarrow{\Omega_{\mathrm{Cat}^{\mathcal{M}}}} 2\mathrm{Cat}_*^{\mathcal{M}} \xleftarrow{\Omega_{2\mathrm{Cat}^{\mathcal{M}}}} 3\mathrm{Cat}_*^{\mathcal{M}} \dots.$$

We call $\mathrm{CatSp}_{\mathcal{M}}$ the category of $\mathcal{M}$-categorical spectra.

Notation 13.1.10. Let $\mathcal{M}$ be a symmetric monoidal category. We denote by

$$\Omega^\infty_{\mathcal{M}} : \mathrm{CatSp}_{\mathcal{M}} \to \mathcal{M}_*$$

the canonical projection. Given an $\mathcal{M}$-categorical spectrum $\mathcal{C}$, we call $\Omega^\infty_{\mathcal{M}}\mathcal{C}$ the pointed $\mathcal{M}$-enriched category underlying $\mathcal{C}$.

Remark 13.1.11. Using remark 13.1.7 we may construct a sequence of functors from $\mathrm{CAlg}(\mathrm{Cat})$ into $\widehat{\mathrm{Cat}}$

$$(-)_* \xleftarrow{\Omega_{(-)}} \mathrm{Cat}_*^{(-)} \xleftarrow{\Omega_{\mathrm{Cat}^{(-)}}} 2\mathrm{Cat}_*^{(-)} \xleftarrow{\Omega_{2\mathrm{Cat}^{(-)}}} 3\mathrm{Cat}_*^{(-)} \dots$$

which specializes to the sequence of definition 13.1.9 upon evaluation at a symmetric monoidal category $\mathcal{M}$. The limit of the above tower is a functor

$$\mathrm{CatSp}_{(-)} : \mathrm{CAlg}(\mathrm{Cat}) \to \widehat{\mathrm{Cat}}$$

which sends each symmetric monoidal category $\mathcal{M}$ to $\mathrm{CatSp}_{\mathcal{M}}$. Note that there is a natural transformation

$$\Omega^\infty_{(-)} : \mathrm{CatSp}_{(-)} \to (-)_*$$

of functors $\mathrm{CAlg}(\mathrm{Cat}) \to \mathrm{Cat}$, which recovers upon evaluation at a symmetric monoidal category $\mathcal{M}$ the functor $\Omega^\infty_{\mathcal{M}}$ from notation 13.1.10.

Remark 13.1.12. Let $F : \mathcal{M} \to \mathcal{M}'$ be a monoidal functor between monoidal categories. Consider the commutative square

$$\begin{tikzcd}
\mathrm{Alg}(\mathcal{M}) \arrow[r, "B_{\mathcal{M}}"] \arrow[d, "F_!"] & \mathrm{Cat}_*(\mathcal{M}) \arrow[d, "F_!"] \\
\mathrm{Alg}(\mathcal{M}') \arrow[r, "B_{\mathcal{M}'}"] & \mathrm{Cat}_*(\mathcal{M}').
\end{tikzcd} \qquad (*)$$

As discussed in remark 13.1.7, this square is horizontally right adjointable. Passing to right adjoints to the horizontal arrows we obtain the commutative square

$$\begin{tikzcd}
\mathrm{Alg}(\mathcal{M}) \arrow[d, "F_!"] & \mathrm{Cat}_*(\mathcal{M}) \arrow[l, "\Omega_{\mathcal{M}}"'] \arrow[d, "F_!"] \\
\mathrm{Alg}(\mathcal{M}') & \mathrm{Cat}_*(\mathcal{M}') \arrow[l, "\Omega_{\mathcal{M}'}"']
\end{tikzcd} \qquad (**)$$

which features in the functoriality of enriched categorical spectra discussed in remark 13.1.11.

Assume now that F has a strictly monoidal right adjoint F^R. In this case we have that the right vertical arrows in the square $(*)$ are right adjointable, and furthermore this adjunction restricts to the full subcategories from the left column. It follows from this that $(*)$ is also vertically right adjointable. Passing to right adjoints to the vertical arrows we obtain the commutative square

$$\begin{tikzcd} \mathrm{Alg}(\mathcal{M}) \arrow[r, "B_{\mathcal{M}}"] & \mathrm{Cat}_*(\mathcal{M}) \\ \mathrm{Alg}(\mathcal{M}') \arrow[u, "(F^R)_!"] \arrow[r, "B_{\mathcal{M}'}"] & \mathrm{Cat}_*(\mathcal{M}'). \arrow[u, "(F^R)_!"'] \end{tikzcd} \qquad (***)$$

This is horizontally right adjointable, and the square obtained from it by passing to right adjoints to horizontal arrows is equivalent to the square obtained from $(**)$ by passing to right adjoints to vertical arrows.

Assume now that $\mathcal{M}$ is symmetric monoidal and that F and F^R are strictly symmetric monoidal. Consider the commutative diagram

$$\begin{tikzcd} \mathcal{M}_* \arrow[d, "F"] & \mathrm{Cat}^{\mathcal{M}}_* \arrow[l, "\Omega_{\mathcal{M}}"'] \arrow[d, "F_!"] & 2\mathrm{Cat}^{\mathcal{M}}_* \dots \arrow[l, "\Omega_{\mathrm{Cat}\mathcal{M}}"'] \arrow[d, "F_!"] \\ \mathcal{M}'_* & \mathrm{Cat}^{\mathcal{M}'}_* \arrow[l, "\Omega_{\mathcal{M}'}"'] & 2\mathrm{Cat}^{\mathcal{M}'}_* \dots \arrow[l, "\Omega_{\mathrm{Cat}\mathcal{M}'}"'] \end{tikzcd}$$

whose limit recovers the functor $F_! : \mathrm{CatSp}_{\mathcal{M}} \to \mathrm{CatSp}_{\mathcal{M}'}$ induced from F. By the previous discussion, the above diagram is vertically right adjointable, and passing to right adjoints of the vertical arrows we recover the commutative diagram

$$\begin{tikzcd} \mathcal{M}_* & \mathrm{Cat}^{\mathcal{M}}_* \arrow[l, "\Omega_{\mathcal{M}}"'] & 2\mathrm{Cat}^{\mathcal{M}}_* \dots \arrow[l, "\Omega_{\mathrm{Cat}\mathcal{M}}"'] \\ \mathcal{M}'_* \arrow[u, "F^R"] & \mathrm{Cat}^{\mathcal{M}'}_* \arrow[l, "\Omega_{\mathcal{M}'}"'] \arrow[u, "F^R_!"] & 2\mathrm{Cat}^{\mathcal{M}'}_* \dots \arrow[l, "\Omega_{\mathrm{Cat}\mathcal{M}'}"'] \arrow[u, "F^R_!"] \end{tikzcd}$$

whose limit is the functor $(F^R)_! : \mathrm{CatSp}_{\mathcal{M}'} \to \mathrm{CatSp}_{\mathcal{M}}$ induced from F^R. It then follows from an application of proposition 5.3.17 that we have an adjunction

$$F_! : \mathrm{CatSp}_{\mathcal{M}} \rightleftarrows \mathrm{CatSp}_{\mathcal{M}'} : (F^R)_!.$$

Remark 13.1.13. Let $\mathcal{M}$ be a presentable symmetric monoidal category. Then it follows from propositions 3.3.12 and 3.4.12 that the category $\mathrm{Algbrd}_*(\mathcal{M})_{\mathrm{Spc}}$ of pointed $\mathcal{M}$-algebroids with a space of objects is presentable, and $\mathrm{Cat}^{\mathcal{M}}_*$ is an accessible localization of it. The functor $\Omega_{\mathcal{M}} : \mathrm{Cat}^{\mathcal{M}}_* \to \mathcal{M}_*$ admits a left adjoint, given by the composite functor

$$\mathcal{M}_* \to \mathrm{Alg}(\mathcal{M}) \xrightarrow{B_{\mathcal{M}}} \mathrm{Cat}^{\mathcal{M}}_*$$

where the first arrow is given by operadic left Kan extension along the inclusion $E_0 \to E_1$.

In particular, it follows from the above that the sequence of categories from definition 13.1.9 takes place in Pr^R, and hence its limit $\mathrm{CatSp}_{\mathcal{M}}$ is a presentable category.

Remark 13.1.14. Recall from remarks 3.3.8 and 3.4.11 that there is an involution $(-)^{\mathrm{op}}$ on Algbrd and a $\mathbb{Z}/2\mathbb{Z}$-equivariant structure on the projection Algbrd $\to \mathrm{Op}_{\mathrm{Assos}} \times \mathrm{Cat}$, which restricts to an involution on the full subcategory of Algbrd on the enriched categories. Observe moreover that the map of operads $E_0 \to E_1$ admits a canonical $\mathbb{Z}/2\mathbb{Z}$ equivariant structure, where we equip E_0 with the trivial involution. It follows that there is a $\mathbb{Z}/2\mathbb{Z}$ equivariant structure on the natural transformation

$$\Omega_{(-)} : \mathrm{Cat}_*^{(-)} \to (-)_*$$

of functors $\mathrm{Alg}(\mathrm{Cat}) \to \widehat{\mathrm{Cat}}$. For each monoidal category $\mathcal{M}$, this induces a $\mathbb{Z}/2\mathbb{Z}$ equivariant structure on the map $\Omega_{\mathcal{M}} : \mathrm{Cat}_*^{\mathcal{M}} \to \mathcal{M}_*$, where $\mathcal{M}_*$ is equipped with the trivial involution.

We conclude that the sequence of functors from remark 13.1.11 comes equipped with an infinite family of commuting involutions indexed by nonpositive integers, where the i-th involution acts on $n\mathrm{Cat}_*^{(-)}$ as $(-)^{(-i+1)\text{-op}}$ when $0 \geq i \geq n-1$, and as the identity otherwise.

Passing to the limit, we obtain an infinite family of commuting involutions $(-)^{i\text{-op}}$ on the functor $\mathrm{CatSp}_{(-)}$, where i ranges over all nonpositive integers. We think about $(-)^{i\text{-op}}$ as the involution which reverses the direction of cells of dimension i.

Recall that the category of spectra comes equipped with inverse automorphisms

$$B : \mathrm{Sp} \rightleftarrows \mathrm{Sp} : \Omega.$$

We now discuss the analogue functors in the setting of $\mathcal{M}$-categorical spectra.

Notation 13.1.15. Let $\mathcal{M}$ be a symmetric monoidal category. Then the category $\mathrm{CatSp}_{\mathcal{M}}$ is equivalent to the limit of the sequence of categories

$$\mathrm{Cat}_*^{\mathcal{M}} \xleftarrow{\Omega_{\mathrm{Cat}^{\mathcal{M}}}} 2\mathrm{Cat}_*^{\mathcal{M}} \xleftarrow{\Omega_{2\mathrm{Cat}^{\mathcal{M}}}} 3\mathrm{Cat}_*^{\mathcal{M}} \dots$$

obtained from the sequence of definition 13.1.9 by removing the first category. This is the same sequence which arises in the definition of $\mathrm{CatSp}_{\mathrm{Cat}^{\mathcal{M}}}$. We therefore have an equivalence $\mathrm{CatSp}_{\mathrm{Cat}^{\mathcal{M}}} = \mathrm{CatSp}_{\mathcal{M}}$, which is evidently functorial in $\mathcal{M}$. We will denote by

$$B_{\mathcal{M}} : \mathrm{CatSp}_{\mathcal{M}} \leftrightarrows \mathrm{CatSp}_{\mathrm{Cat}^{\mathcal{M}}} : \Omega_{\mathcal{M}}$$

the resulting functors.

Warning 13.1.16. The functors $B_{\mathcal{M}}, \Omega_{\mathcal{M}}$ from notation 13.1.15 are different (but related) to the functors $B_{\mathcal{M}}, \Omega_{\mathcal{M}}$ from notation 13.1.4. It will usually be clear from context which variant of the functors we are using.

Remark 13.1.17. Let $\mathcal{M}$ be a symmetric monoidal category. The equivalence $\mathrm{CatSp}_{\mathcal{M}} = \mathrm{CatSp}_{\mathrm{Cat}^{\mathcal{M}}}$ from notation 13.1.15 exchanges the involution $(-)^{i\text{-op}}$ on $\mathrm{CatSp}_{\mathcal{M}}$ with the involution $(-)^{(i+1)\text{-op}}$ on $\mathrm{CatSp}_{\mathrm{Cat}^{\mathcal{M}}}$ for each $i \leq 0$ (where in the case $i = 0$ we set $(-)^{1\text{-op}}$ to be the involution on $\mathrm{CatSp}_{\mathrm{Cat}^{\mathcal{M}}}$ which arises from the order reversing involution on $\mathrm{Cat}^{\mathcal{M}}$). Furthermore, there is an equivalence

$$\Omega^{\infty}_{\mathcal{M}}\Omega_{\mathcal{M}} = \Omega_{\mathcal{M}}\Omega^{\infty}_{\mathrm{Cat}^{\mathcal{M}}}$$

of functors $\mathrm{CatSp}_{\mathrm{Cat}^{\mathcal{M}}} \to \mathcal{M}_*$.

We finish this section by discussing the notion of cells in $\mathcal{M}$-categorical spectra.

Notation 13.1.18. Let $\mathcal{M}$ be a monoidal category. For each object m in $\mathcal{M}$ we denote by

$$\mathrm{cells}_m(-) : \mathcal{M} \to \mathrm{Spc}$$

the functor corepresented by m.

Assume now that $\mathcal{M}$ has an inital object which is compatible with the monoidal structure. We denote by

$$\mathrm{cells}_{C_m} : \mathrm{Cat}^{\mathcal{M}} \to \mathrm{Spc}$$

the functor corepresented by (the enriched category underlying) the walking m-cell C_m from example 3.3.6. In other words, cells_{C_m} is the functor that assigns to each $\mathcal{M}$-enriched category $\mathcal{C}$ the space of pairs of objects x, y in $\mathcal{C}$ together with a map $m \to \mathrm{Hom}_{\mathcal{C}}(x, y)$.

Remark 13.1.19. Let $\mathcal{M}$ be a monoidal category with compatible initial object and let m be an object of $\mathcal{M}$. Then for each pointed $\mathcal{M}$-enriched monoidal category $\mathcal{C}$ we have an equivalence

$$\mathrm{Hom}_{\mathrm{Algbrd}(\mathcal{M})}(C_m, \Omega_{\mathcal{M}}(\mathcal{C})) = \mathrm{cells}_m(\Omega_{\mathcal{M}}(\mathcal{C}))$$

where the two occurrences of $\Omega_{\mathcal{M}}(\mathcal{C})$ in the above formula are first as an associative algebra, and second as an object of $\mathcal{M}$. Composing with the morphism of algebroids $\Omega_{\mathcal{M}}(\mathcal{C}) \to \mathcal{C}$ we obtain a map of spaces

$$\mathrm{cells}_m(\Omega_{\mathcal{M}}(\mathcal{C})) \to \mathrm{cells}_{C_m}(\mathcal{C}).$$

Observe that both sides are functorial in m and $\mathcal{C}$, and the above maps can be assembled into a natural transformation

$$\mathrm{cells}_- |_{\mathcal{M}_*}(\Omega_{\mathcal{M}}(-)) \to \mathrm{cells}_{C_-} |_{\mathrm{Cat}^{\mathcal{M}}_*}$$

of functors $\mathcal{M}^{\mathrm{op}} \times \mathrm{Cat}^{\mathcal{M}}_* \to \mathrm{Spc}$.

Definition 13.1.20. *Let $\mathcal{M}$ be a symmetric monoidal category with compatible initial object. Let $\mathcal{C}$ be an $\mathcal{M}$-categorical spectrum, corresponding to a sequence of pointed $\mathcal{M}$-enriched n-categories $\mathcal{C}_n$ indexed by nonnegative integers, and equivalences $\Omega_{n\mathrm{Cat}^{\mathcal{M}}}\mathcal{C}_{n+1} = \mathcal{C}_n$ for all $n \geq 0$. For each object m in $\mathcal{M}$ we define the space $\mathrm{cells}_m(\mathcal{C})$ to be the colimit of the diagram*

$$\mathrm{cells}_m(\mathcal{C}_0) \to \mathrm{cells}_{C_m}(\mathcal{C}_1) \to \mathrm{cells}_{C_{C_m}}(\mathcal{C}_2) \to \mathrm{cells}_{C_{C_{C_m}}}(\mathcal{C}_3) \dots$$

where the transition maps are as in remark 13.1.19.

Warning 13.1.21. We use the notation cells_m for two different purposes, in notation 13.1.18 and definition 13.1.20. If $\mathcal{C}$ is an $\mathcal{M}$-categorical spectrum, there is a map

$$\mathrm{cells}_m(\Omega^\infty_{\mathcal{M}}(\mathcal{C})) \to \mathrm{cells}_m(\mathcal{C})$$

which is in general not an equivalence. It will generally be clear from context which version of cells_m we are using.

Remark 13.1.22. Let $\mathcal{M}$ be a symmetric monoidal category with compatible initial object and let m be an object in $\mathcal{M}$. Definition 13.1.20 can be upgraded to a functor

$$\mathrm{cells}_m : \mathrm{CatSp}_{\mathcal{M}} \to \mathrm{Spc}\,.$$

As opposed to the analogous functor in the setting of $\mathcal{M}$-categories from notation 13.1.18, the above functor is generally not corepresented.

Assume now that $\mathcal{M}$ is presentable symmetric monoidal. Then each of the terms in the sequences from definition 13.1.20 is limit preserving in $\mathcal{C}$, and hence we have that the functor $\mathrm{cells}_m : \mathrm{CatSp}_{\mathcal{M}} \to \mathrm{Spc}$ preserves finite limits. In other words, we may think about the functor cells_m as being corepresented by a pro-object in $\mathrm{CatSp}_{\mathcal{M}}$.

Remark 13.1.23. Let $\mathcal{M}$ be a symmetric monoidal category with compatible initial object. Let m be an object in $\mathcal{M}$ and let $\mathcal{C}$ be a $\mathrm{Cat}^{\mathcal{M}}$-categorical spectrum, corresponding to a sequence of pointed $\mathrm{Cat}^{\mathcal{M}}$-enriched n-categories $\mathcal{C}_n$ indexed by nonnegative integers, and equivalences $\Omega_{(n+1)\mathrm{Cat}^{\mathcal{M}}}(\mathcal{C}_{n+1}) = \mathcal{C}_n$ for all $n \geq 0$. Then the sequence

$$\mathrm{cells}_{C_m}(\mathcal{C}_0) \to \mathrm{cells}_{C_{C_m}}(\mathcal{C}_1) \to \mathrm{cells}_{C_{C_{C_m}}}(\mathcal{C}_2) \to \mathrm{cells}_{C_{C_{C_{C_m}}}}(\mathcal{C}_3) \dots$$

which features in the definition of $\mathrm{cells}_{C_m}(\mathcal{C})$ is a shift of the sequence which features in the definition of $\mathrm{cells}_m(\Omega_{\mathcal{M}}(\mathcal{C}))$. We therefore have an equivalence $\mathrm{cells}_{C_m}(\mathcal{C}) = \mathrm{cells}_m(\Omega_{\mathcal{M}}(\mathcal{C}))$. This equivalence admits an evident upgrade to a commutative triangle

$$\begin{tikzcd}
\mathrm{CatSp}_{\mathrm{Cat}^{\mathcal{M}}} \arrow[r, "\Omega_{\mathcal{M}}"] \arrow[dr, swap, "\mathrm{cells}_{C_m}"] & \mathrm{CatSp}_{\mathcal{M}} \arrow[d, "\mathrm{cells}_m"] \\
& \mathrm{Spc}
\end{tikzcd}$$

which is furthermore functorial in m.

13.2 The case $\mathcal{M} = \omega\mathrm{Cat}$

We now specialize the above discussion to the case of $\omega\mathrm{Cat}$-categorical spectra. We refer to section 3.6 for general background on the theory of ω-categories.

Definition 13.2.1. *We let* CatSp *be the category* $\mathrm{CatSp}_{\omega\mathrm{Cat}}$*, where* $\omega\mathrm{Cat}$ *is equipped with its cartesian symmetric monoidal structure. We call* CatSp *the category of categorical spectra.*

Remark 13.2.2. Recall from remark 3.6.12 that there is an equivalence $\mathrm{Cat}^{\omega\mathrm{Cat}} = \omega\mathrm{Cat}$. It follows that we have equivalences $n\mathrm{Cat}^{\omega\mathrm{Cat}} = (n-1)\mathrm{Cat}^{\omega\mathrm{Cat}}$ for each $n \geq 1$. Composing these we obtain an equivalence $n\mathrm{Cat}^{\omega\mathrm{Cat}} = \omega\mathrm{Cat}$ for each $n \geq 1$.

Concretely, we may understand the above equivalence as follows. Consider the diagram of categories

$$\mathrm{Spc} \xleftarrow{(-)^{\leq 0}} \mathrm{Cat} \xleftarrow{(-)^{\leq 1}} 2\mathrm{Cat} \xleftarrow{(-)^{\leq 2}} 3\mathrm{Cat} \dots$$

and recall that its limit recovers $\omega\mathrm{Cat}$. This induces for each $n \geq 1$ a sequence of categories

$$n\mathrm{Cat}^{\mathrm{Spc}} \xleftarrow{(-)^{\leq 0}_!} n\mathrm{Cat}^{\mathrm{Cat}} \xleftarrow{(-)^{\leq 1}_!} n\mathrm{Cat}^{2\mathrm{Cat}} \xleftarrow{(-)^{\leq 2}_!} n\mathrm{Cat}^{3\mathrm{Cat}} \dots$$

whose limit is $n\mathrm{Cat}^{\omega\mathrm{Cat}}$. The above two sequences are equivalent up to a shift - the induced equivalence between their limits recovers our equivalence $n\mathrm{Cat}^{\omega\mathrm{Cat}} = \omega\mathrm{Cat}$.

Consider now for each $n \geq 1$ the diagram

$$\begin{tikzcd}
(n+1)\mathrm{Cat}^{\mathrm{Spc}}_* \arrow[d, "\Omega_{n\mathrm{Cat}^{\mathrm{Spc}}}"] & (n+1)\mathrm{Cat}^{\mathrm{Cat}}_* \arrow[l, "(-)^{\leq 0}_!"'] \arrow[d, "\Omega_{n\mathrm{Cat}^{\mathrm{Cat}}}"] & (n+1)\mathrm{Cat}^{2\mathrm{Cat}}_* \arrow[l, "(-)^{\leq 1}_!"'] \arrow[d, "\Omega_{n\mathrm{Cat}^{2\mathrm{Cat}}}"] & (n+1)\mathrm{Cat}^{3\mathrm{Cat}}_* \dots \arrow[l, "(-)^{\leq 2}_!"'] \arrow[d, "\Omega_{n\mathrm{Cat}^{3\mathrm{Cat}}}"] \\
n\mathrm{Cat}^{\mathrm{Spc}}_* & n\mathrm{Cat}^{\mathrm{Cat}}_* \arrow[l, "(-)^{\leq 0}_!"'] & n\mathrm{Cat}^{2\mathrm{Cat}}_* \arrow[l, "(-)^{\leq 1}_!"'] & n\mathrm{Cat}^{3\mathrm{Cat}}_* \dots . \arrow[l, "(-)^{\leq 2}_!"']
\end{tikzcd}$$

Its limit recovers the functor $\Omega_{n\mathrm{Cat}^{\omega\mathrm{Cat}}} : (n+1)\mathrm{Cat}^{\omega\mathrm{Cat}}_* \to n\mathrm{Cat}^{\omega\mathrm{Cat}}_*$. The above diagram is equivalent to a shift of the diagram

$$\begin{tikzcd}
\mathrm{Cat}^{\mathrm{Spc}}_* \arrow[d, "\Omega_{\mathrm{Spc}}"] & \mathrm{Cat}^{\mathrm{Cat}}_* \arrow[l, "(-)^{\leq 0}_!"'] \arrow[d, "\Omega_{\mathrm{Cat}}"] & \mathrm{Cat}^{2\mathrm{Cat}}_* \arrow[l, "(-)^{\leq 1}_!"'] \arrow[d, "\Omega_{2\mathrm{Cat}}"] & \mathrm{Cat}^{3\mathrm{Cat}}_* \dots \arrow[l, "(-)^{\leq 2}_!"'] \arrow[d, "\Omega_{3\mathrm{Cat}}"] \\
\mathrm{Spc}_* & \mathrm{Cat}_* \arrow[l, "(-)^{\leq 0}"'] & 2\mathrm{Cat}_* \arrow[l, "(-)^{\leq 1}"'] & 3\mathrm{Cat}_* \dots . \arrow[l, "(-)^{\leq 2}"']
\end{tikzcd}$$

The limit of the latter recovers the functor $\Omega_{\omega\mathrm{Cat}} : \mathrm{Cat}^{\omega\mathrm{Cat}}_* \to \omega\mathrm{Cat}_*$.

The functor $\Omega_{\omega\mathrm{Cat}} : \mathrm{Cat}^{\omega\mathrm{Cat}}_* \to \omega\mathrm{Cat}_*$ induces, via the equivalence $\mathrm{Cat}^{\omega\mathrm{Cat}} = \omega\mathrm{Cat}$ an endofunctor of $\omega\mathrm{Cat}_*$, which we will continue denoting by $\Omega_{\omega\mathrm{Cat}}$. It follows from the above that for each $n \geq 1$ there is a commutative square of categories

$$\begin{tikzcd}
(n+1)\mathrm{Cat}^{\omega\mathrm{Cat}}_* \arrow[r] \arrow[d, "\Omega_{n\mathrm{Cat}^{\omega\mathrm{Cat}}}"] & \omega\mathrm{Cat}_* \arrow[d, "\Omega_{\omega\mathrm{Cat}}"] \\
n\mathrm{Cat}^{\omega\mathrm{Cat}}_* \arrow[r] & \omega\mathrm{Cat}_*
\end{tikzcd}$$

whose horizontal arrows are the equivalences discussed above.

In particular, we have that the category CatSp is equivalent to the limit of the sequence

$$\omega\mathrm{Cat}_* \xleftarrow{\Omega_{\omega\mathrm{Cat}}} \omega\mathrm{Cat}_* \xleftarrow{\Omega_{\omega\mathrm{Cat}}} \omega\mathrm{Cat}_* \xleftarrow{\Omega_{\omega\mathrm{Cat}}} \omega\mathrm{Cat}_* \dots . \qquad (*)$$

Remark 13.2.3. Recall that $\omega\mathrm{Cat}$ comes equipped with an infinite family of commuting involutions indexed by positive integers, where the i-th involution reverses the direction of i-dimensional cells. It follows from remark 13.1.14 that the category CatSp admits an infinite family of commuting involutions $(-)^i$ indexed by arbitrary integers. From the point of view of the sequence $(*)$ from remark 13.2.2, these are induced by the fact that the endofunctor $\Omega_{\omega\mathrm{Cat}}$ of $\omega\mathrm{Cat}_*$ exchanges the involution $(-)^{(i+1)\text{-op}}$ with $(-)^{i\text{-op}}$ for each $i \geq 0$.

Thinking about a categorical spectrum $\mathcal{C}$ as a category whose cells may have any integer dimension, we may interpret the categorical spectrum $\mathcal{C}^{i\text{-op}}$ as being obtained from $\mathcal{C}$ by reversing the direction of i-dimensional cells.

We now specialize the shift automorphisms from notation 13.1.15 to the case $\mathcal{M} = \omega\mathrm{Cat}$.

Remark 13.2.4. The equivalence of categories $\mathrm{Cat}^{\omega\mathrm{Cat}} = \omega\mathrm{Cat}$ induces an equivalence between $\mathrm{CatSp}_{\mathrm{Cat}^{\omega\mathrm{Cat}}}$ and CatSp. Under this equivalence, the equivalence

$$B_{\omega\mathrm{Cat}} : \mathrm{CatSp}_{\omega\mathrm{Cat}} \xrightleftharpoons{} \mathrm{CatSp}_{\mathrm{Cat}^{\omega\mathrm{Cat}}} : \Omega_{\omega\mathrm{Cat}}$$

from notation 13.1.15 induces inverse automorphisms on CatSp, which we will continue denoting by $B_{\omega\mathrm{Cat}}$ and $\Omega_{\omega\mathrm{Cat}}$. These automorphisms are induced from the symmetry under shifts of the sequence $(*)$ from remark 13.2.2. Note that the inverse automorphisms $B_{\omega\mathrm{Cat}}$ and $\Omega_{\omega\mathrm{Cat}}$ exchange the involutions $(-)^{i\text{-op}}$ and $(-)^{(i+1)\text{-op}}$ on CatSp for each integer i.

Observe that we have a commutative square

$$\begin{tikzcd}
\mathrm{CatSp} \arrow[r, "\Omega^\infty_{\omega\mathrm{Cat}}"] \arrow[d, "\Omega_{\omega\mathrm{Cat}}"] & \omega\mathrm{Cat}_* \arrow[d, "\Omega_{\omega\mathrm{Cat}}"] \\
\mathrm{CatSp} \arrow[r, "\Omega^\infty_{\omega\mathrm{Cat}}"] & \omega\mathrm{Cat}_* .
\end{tikzcd}$$

In other words, the projection $\Omega^\infty_{\omega\mathrm{Cat}} : \mathrm{CatSp} \to \omega\mathrm{Cat}_*$ commutes with the endofunctors $\Omega_{\omega\mathrm{Cat}}$ on both sides.

Our next goal is to show that CatSp and its shift automorphism arise formally from $\omega\mathrm{Cat}_*$ by inverting the endofunctor $\Omega_{\omega\mathrm{Cat}}$.

Notation 13.2.5. Let $\mathcal{C}$ be a category. Denote by $\mathcal{C}_{\mathrm{end}}$ the pullback

$$\begin{tikzcd}
\mathcal{C}_{\mathrm{end}} \arrow[r] \arrow[d] & \mathcal{C} \arrow[d, "\Delta"] \\
\mathrm{Funct}([1], \mathcal{C}) \arrow[r, "(\mathrm{ev}_0, \mathrm{ev}_1)"] & \mathcal{C} \times \mathcal{C}.
\end{tikzcd}$$

We let $\mathcal{C}_{\mathrm{aut}}$ be the full subcategory of $\mathcal{C}_{\mathrm{end}}$ on those objects whose projection to $\mathrm{Funct}([1], \mathcal{C})$ corresponds to an invertible arrow in $\mathcal{C}$.

Remark 13.2.6. Let $\mathcal{C}$ be a category. Then $\mathrm{Cat}_{\mathrm{end}}$ is a category whose objects are objects of $\mathcal{C}$ equipped with an endomorphism, and morphisms from (c, α) to (d, β) are commutative squares

$$\begin{tikzcd} c \arrow[r, "f"] \arrow[d, "\alpha"] & d \arrow[d, "\alpha"] \\ c \arrow[r, "f"] & d. \end{tikzcd}$$

Remark 13.2.7. Denote by $\mathbb{N}$ the free monoid on one object. Recall that we have a pushout square of categories

$$\begin{tikzcd} {[0] \sqcup [0]} \arrow[r, "{(0,1)}"] \arrow[d] & {[1]} \arrow[d] \\ {[0]} \arrow[r] & B\mathbb{N} \end{tikzcd}$$

where $B\mathbb{N}$ denotes the category with a single object, with endomorphisms given by $\mathbb{N}$. It follows that for every category $\mathcal{C}$ we have an equivalence $\mathcal{C}_{\mathrm{end}} = \mathrm{Funct}(B\mathbb{N}, \mathcal{C})$.

Consider now the free abelian group on one object $\mathbb{Z}$, and its classifying space $B\mathbb{Z}$. The map $B\mathbb{N} \to B\mathbb{Z}$ is an epimorphism of categories. For every category $\mathcal{C}$, a functor $B\mathbb{N} \to \mathcal{C}$ factors through $B\mathbb{Z}$ if and only if it corresponds to an object equipped with an automorphism.

It follows from this that for every category $\mathcal{C}$ we have a commutative square

$$\begin{tikzcd} \mathcal{C}_{\mathrm{aut}} \arrow[r] \arrow[d] & \mathcal{C}_{\mathrm{end}} \arrow[d] \\ \mathrm{Funct}(B\mathbb{Z}, \mathcal{C}) \arrow[r] & \mathrm{Funct}(B\mathbb{N}, \mathcal{C}) \end{tikzcd}$$

where the vertical arrows are equivalences, the top horizontal arrow is the inclusion, and the bottom horizontal arrow is given by restriction along the map $B\mathbb{N} \to B\mathbb{Z}$.

If $\mathcal{C}$ admits colimits indexed by the poset of natural numbers, then the bottom horizontal arrow in the above diagram admits a left adjoint, given by left Kan extension along $B\mathbb{N} \to B\mathbb{Z}$. In this case, each object (c, α) of $\mathcal{C}_{\mathrm{end}}$ admits a universal map to an object (d, β) of $\mathcal{C}_{\mathrm{aut}}$. Here the object d is given by the colimit of the diagram

$$c \xrightarrow{\alpha} c \xrightarrow{\alpha} c \dots$$

and is equipped with the shift automorphism.

Proposition 13.2.8. *The inclusion* $\widehat{\mathrm{Cat}}_{\mathrm{aut}} \to \widehat{\mathrm{Cat}}_{\mathrm{end}}$ *admits a right adjoint, which maps* $(\omega\mathrm{Cat}_*, \Omega_{\omega\mathrm{Cat}})$ *to* $(\mathrm{CatSp}, \Omega_{\omega\mathrm{Cat}})$.

Proof. This is a direct consequence of the description of CatSp from remark 13.2.2, together with remark 13.2.7. $\square$

We now specialize the notion of cell from definition 13.1.20 to the case $\mathcal{M} = \omega\mathrm{Cat}$. This will make precise the idea that a categorical spectrum has cells whose dimensions are indexed by arbitrary integers.

Notation 13.2.9. Let $i \geq 0$. We denote by

$$\mathrm{cells}_i(-) : \omega\mathrm{Cat} \to \mathrm{Spc}$$

the functor corepresented by the universal i-dimensional cell C_i from example 3.6.3. In other words, this is the functor that assigns to each ω-category $\mathcal{C}$ the space $\mathrm{cells}_i(\mathcal{C})$ of i-dimensional cells in $\mathcal{C}$. Note that this is a special case of the functors from notation 13.1.18.

We will also use the notation $\mathrm{cells}_i(-)$ for the functor $\mathrm{cells}_{C_i} : \mathrm{CatSp} \to \mathrm{Spc}$ obtained from definition 13.1.20 by setting $\mathcal{M} = \omega\mathrm{Cat}$ and $m = C_i$.

Warning 13.2.10. Let $i \geq 0$. We use the notation cells_i for two different purposes. If $\mathcal{C}$ is a categorical spectrum, there is a map

$$\mathrm{cells}_i(\Omega^\infty_{\omega\mathrm{Cat}}(\mathcal{C})) \to \mathrm{cells}_i(\mathcal{C})$$

which is in general not an equivalence. It will generally be clear from context which version of cells_i we are using (see also warning 13.1.21).

Remark 13.2.11. Let $i \geq 0$. Then it follows from remark 13.1.23 that there is a commutative triangle

$$\begin{tikzcd} \mathrm{CatSp} \arrow[r, "\Omega_{\omega\mathrm{Cat}}"] \arrow[dr, "\mathrm{cells}_{i+1}"'] & \mathrm{CatSp} \arrow[d, "\mathrm{cells}_i"] \\ & \mathrm{Spc} \end{tikzcd}$$

In other words, the inverse equivalences $B_{\omega\mathrm{Cat}}$ and $\Omega_{\omega\mathrm{Cat}}$ shift the dimension of cells by one.

This allows us to extend the functors cells_i to the case $i < 0$. Namely, we inductively define for each $i < 0$ a functor $\mathrm{cells}_i : \mathrm{CatSp} \to \mathrm{Spc}$ by setting $\mathrm{cells}_{i-1} = \mathrm{cells}_i B_{\omega\mathrm{Cat}}$.

Definition 13.2.12. *Let $\mathcal{C}$ be a categorical spectrum. For each integer i we call* $\mathrm{cells}_i(\mathcal{C})$ *the space of i-dimensional cells in $\mathcal{C}$.*

Notation 13.2.13. We define for each $i \geq 0$ a functor

$$C_{i,-} : \omega\mathrm{Cat} \to \omega\mathrm{Cat}$$

inductively by setting $C_{0,-} = \mathrm{id}_{\omega\mathrm{Cat}}$, and $C_{i+1,-} = C_{C_{i,-}}$. Note in particular that $C_{i,-}$ maps $[0]$ to the walking i-dimensional cell C_i.

Remark 13.2.14. The functoriality of the commutative triangle from 13.1.23 implies that for every $i \geq 0$ we have a natural isomorphism

$$\mathrm{cells}_{C_{i,-}} \Omega_{\omega\mathrm{Cat}} = \mathrm{cells}_{C_{i+1,-}}$$

of functors $\omega\mathrm{Cat}^{\mathrm{op}} \times \mathrm{CatSp} \to \mathrm{Spc}$. This allows us to define the functors $\mathrm{cells}_{C_{i,-}}$ for negative values of i by inductively setting $\mathrm{cells}_{C_{i-1,-}} = \mathrm{cells}_{C_{i,-}} B_{\omega\mathrm{Cat}}$.

Remark 13.2.15. Let i be an integer. Consider the map $[0] \to [1]$ which picks out the source (resp. target) object. Thanks to remark 13.2.14 this induces a natural transformation $\mathrm{cells}_{i+1} \to \mathrm{cells}_i$ of functors $\mathrm{CatSp} \to \mathrm{Spc}$, which we call the source (resp. target) map. For each categorical spectrum $\mathcal{C}$ this produces a map $\mathrm{cells}_{i+1}(\mathcal{C}) \to \mathrm{cells}_i(\mathcal{C})$ which we think about as sending each i-dimensional cell in $\mathcal{C}$ to its source (resp. target) cell.

Similarly, using the projection $[1] \to [0]$ we obtain a natural transformation $\mathrm{cells}_i \to \mathrm{cells}_{i+1}$ which we think about as sending each i-dimensional cell in a categorical spectrum to the corresponding degenerate $(i+1)$-dimensional cell.

The pushout square of categories

$$\begin{tikzcd} {[0]} \arrow[r, "0"] \arrow[d, "1"] & {[1]} \arrow[d] \\ {[1]} \arrow[r] & {[2]} \end{tikzcd}$$

induces a pullback square of functors

$$\begin{tikzcd} \mathrm{cells}_{C_{i,[2]}} \arrow[r] \arrow[d] & \mathrm{cells}_{i+1} \arrow[d] \\ \mathrm{cells}_{i+1} \arrow[r] & \mathrm{cells}_i \end{tikzcd}$$

where the right vertical and bottom horizontal maps are the sources and target. The active map $[1] \to [2]$ induces a natural transformation $\mathrm{cells}_{C_{i,[2]}} \to \mathrm{cells}_{i+1}$. If $\mathcal{C}$ is a categorical spectrum and α, β are a pair of i-dimensional cells in $\mathcal{C}$ such that the source of β is identified with the target of α, we can use the above to construct a new i-dimensional cell $\beta \circ \alpha$.

In other words, each cell in a categorical spectrum has a source and a target, and there are composition and unit maps. A standard argument shows that composition is associative and unital up to homotopy.

Remark 13.2.16. For each $i \geq 0$ the functor $C_{i,-} : \omega\mathrm{Cat} \to \omega\mathrm{Cat}$ from notation 13.2.13 exchanges the involutions $(-)^{j\text{-op}}$ and $(-)^{(j+i)\text{-op}}$ for $j \geq 1$, and is a fixed point for the involutions $(-)^{k\text{-op}}$ on the target for $k \leq i$. Hence the functor

$$\mathrm{cells}_{C_{i,-}} : \omega\mathrm{Cat}^{\mathrm{op}} \times \mathrm{CatSp} \to \mathrm{Spc}$$

is a fixed point for the involutions $(\mathrm{id}, (-)^{k\text{-op}})$ with $k \leq i$, and $((-)^{j\text{-op}}, (-)^{(j+i)\text{-op}})$ for all $j \geq 1$. This fixed point structure is compatible with shifts, so we may extend it by induction to all integers i.

It follows from the above that for every pair of integers i, j, the functor $\mathrm{cells}_i : \mathrm{CatSp} \to \mathrm{Spc}$ is a fixed point for the involution $(-)^{j\text{-op}}$, and furthermore the source and target maps are invariant except in the case when $i = j$, in which case they are exchanged. This makes precise the idea that the involutions $(-)^{j\text{-op}}$ invert the direction of j-dimensional cells.

When working with ω-categories, one is in many cases interested in those which only have non-invertible cells below a certain dimension. We now explore the analogous notion in the setting of categorical spectra.

Definition 13.2.17. *Let $\mathcal{C}$ be a categorical spectrum and let i be an integer. We say that an i-dimensional cell in $\mathcal{C}$ is invertible if it lies in the image of the degeneracy map* $\mathrm{cells}_{i-1}(\mathcal{C}) \to \mathrm{cells}_i(\mathcal{C})$.

Remark 13.2.18. Let $\mathcal{C}$ be a categorical spectrum, corresponding to a sequence $\mathcal{C}_n$ of pointed ω-categories indexed by nonnegative integers, and equivalences $\Omega_{\omega\mathrm{Cat}}(\mathcal{C}_{n+1}) = \mathcal{C}_n$ for every $n \geq 0$. For every positive integer i the degeneracy map $\mathrm{cells}_{i-1}(\mathcal{C}) \to \mathrm{cells}_i(\mathcal{C})$ is obtained by passing to the colimit the degeneracy map $\mathrm{cells}_{i-1+n}(\mathcal{C}_n) \to \mathrm{cells}_{i+n}(\mathcal{C}_n)$. Since the projection $C_{i+n} \to C_{i-1+n}$ which picks out the degenerate $(i+n)$-dimensional cell is an epimorphism, we wee that the maps $\mathrm{cells}_{i-1+n}(\mathcal{C}_n) \to \mathrm{cells}_{i+n}(\mathcal{C}_n)$ are monomorphisms. Therefore the degeneracy map $\mathrm{cells}_{i-1}(\mathcal{C}) \to \mathrm{cells}_i(\mathcal{C})$ itself is a monomorphism.

In other words, the space of invertible i-dimensional cells in $\mathcal{C}$ is equivalent to the space of $(i-1)$-dimensional cells in $\mathcal{C}$.

Proposition 13.2.19. *Let $\mathcal{C}$ be a categorical spectrum, corresponding to a sequence $\mathcal{C}_n$ of pointed ω-categories indexed by nonnegative integers, and equivalences $\Omega_{\omega\mathrm{Cat}}(\mathcal{C}_{n+1}) = \mathcal{C}_n$ for every $n \geq 0$. Let i be an integer, and let $n \geq -i+1$. Then an $(n+i)$-dimensional cell in $\mathcal{C}_n$ is invertible if and only if its image in* $\mathrm{cells}_i(\mathcal{C})$ *is invertible.*

Proof. If an $(n+i)$-dimensional cell in $\mathcal{C}_n$ is invertible then its image in $\mathrm{cells}_i(\mathcal{C})$ is clearly invertible. It remains to prove the converse. Let α be an $(n+i)$-dimensional cell in $\mathcal{C}_n$ whose induced i-dimensional cell in $\mathcal{C}$ is invertible. This implies that the image of α in $\mathrm{cells}_{N+i}(\mathcal{C}_N)$ is invertible for some $N > n$. Working by induction, we may assume that $N = n+1$.

The fact that the $(n+1+i)$-dimensional cell induced by α in $\mathcal{C}_{n+1}$ is invertible means that the composite map

$$C_{C_{n+i}} \to \Omega_{\omega\mathrm{Cat}}\mathcal{C}_{n+1} \to \mathcal{C}_{n+1}$$

of $\omega\mathrm{Cat}$-algebroids factors through the degeneracy map $C_{C_{n+i}} \to C_{C_{n+i-1}}$, where the first arrow above is induced from α, and the second arrow is the counit map. Since the second arrow is fully faithful, we conclude that the first arrow in fact factors through $C_{C_{n+i-1}}$. This means that α itself factors through C_{n+i-1}, as desired. $\square$

Proposition 13.2.20. *Let $n \geq 0$. Then the inclusion $n\mathrm{Cat} \to \omega\mathrm{Cat}$ induces an equivalence between $\mathrm{CatSp}_{n\mathrm{Cat}}$ and the full subcategory of CatSp on those categorical spectra for which every cell of dimension greater than n is invertible.*

Proof. Note that the functor $\mathrm{CatSp}_{n\mathrm{Cat}} \to \mathrm{CatSp}$ is obtained by passing to the limit the inclusions $(n+m)\mathrm{Cat}_* \to \omega\mathrm{Cat}_*$. These are fully faithful, and therefore the map $\mathrm{CatSp}_{n\mathrm{Cat}} \to \mathrm{CatSp}$ is indeed fully faithful. Let $\mathcal{C}$ be a categorical spectrum, corresponding to a sequence $\mathcal{C}_n$ of pointed ω-categories indexed by nonnegative integers, and equivalences $\Omega_{\omega\mathrm{Cat}}(\mathcal{C}_{n+1}) = \mathcal{C}_n$

for every $n \geq 0$. Then $\mathcal{C}$ belongs to $\mathrm{CatSp}_{n\mathrm{Cat}}$ if and only all cells in $\mathcal{C}_m$ of dimension greater than $(n+m)$ are invertible. Our result now follows from an application of proposition 13.2.19. □

Notation 13.2.21. For each integer n we denote by nCatSp the full subcategory of CatSp on those categorical spectra for which every cell of dimension greater than n is invertible.

Warning 13.2.22. Contrary to our conventions with categories, the category 1CatSp is different from CatSp. In other words, we chose to reserve the simplest notation CatSp for the most general notion.

Remark 13.2.23. Let $\mathcal{C}$ be a categorical spectrum. Recall that for every integer i we have an equivalence $\mathrm{cells}_i(\Omega_{\omega\mathrm{Cat}}\mathcal{C}) = \mathrm{cells}_{i+1}\mathcal{C}$. This equivalence preserves the invertibility of cells. It follows that for each integer n the image of the subcategory nCatSp of CatSp under the automorphism $\Omega_{\omega\mathrm{Cat}}$ agrees with $(n-1)$CatSp. In particular, the categories nCatSp for different values of n are all equivalent.

Remark 13.2.24. It follows from remark 13.1.12 that for every $n \geq 0$ the inclusion of nCatSp inside CatSp admits both left and right adjoints, induced by the functors $\omega\mathrm{Cat} \to n\mathrm{Cat}$ which are left and right adjoints to the inclusion $n\mathrm{Cat} \to \omega\mathrm{Cat}$.

By remark 13.2.23, the inclusion $n\mathrm{CatSp} \to \mathrm{CatSp}$ admits both left and right adjoints for every integer n. We think about its left adjoint as the functor that universally inverts all cells of dimension greater than n, and its right adjoint as the functor that universally discards all non-invertible cells of dimension greater than n.

13.3 Examples of categorical spectra

We now discuss a variety of examples of categorical spectra. We begin by observing that ordinary spectra provide examples of categorical spectra.

Example 13.3.1. Observe that there is a commutative square of categories

$$\begin{tikzcd} \mathrm{Spc}_* \arrow[d, "\mathrm{id}"] & \mathrm{Spc}_* \arrow[l, "\Omega"'] \arrow[d] \\ \mathrm{Spc}_* & \mathrm{Cat}_* \arrow[l, "\Omega_{\mathrm{Spc}}"'] \end{tikzcd}$$

where the right vertical arrow is the canonical inclusion. It follows from this that we have a commutative square

$$\begin{tikzcd} \mathrm{Spc}_* \arrow[d] & \mathrm{Spc}_* \arrow[l, "\Omega"'] \arrow[d] \\ \omega\mathrm{Cat}_* & \omega\mathrm{Cat}_* \arrow[l, "\Omega_{\omega\mathrm{Cat}}"'] \end{tikzcd}$$

where the vertical arrows are the canonical inclusions. We therefore have a commutative diagram

$$\begin{tikzcd}
\mathrm{Spc}_* \arrow[d] & \mathrm{Spc}_* \arrow[l, "\Omega"'] \arrow[d] & \mathrm{Spc}_* \arrow[l, "\Omega"'] \arrow[d] & \mathrm{Spc}_* \ldots \arrow[l, "\Omega"'] \arrow[d] \\
\omega\mathrm{Cat}_* & \omega\mathrm{Cat}_* \arrow[l, "\Omega_{\omega\mathrm{Cat}}"'] & \omega\mathrm{Cat}_* \arrow[l, "\Omega_{\omega\mathrm{Cat}}"'] & \omega\mathrm{Cat}_* \ldots . \arrow[l, "\Omega_{\omega\mathrm{Cat}}"']
\end{tikzcd}$$

Passing to the limit, this induces a fully faithful functor $\mathrm{Sp} \to \mathrm{CatSp}$. In other words, any spectrum can be thought of as a categorical spectrum in a canonical way. Furthermore, this assignment exchanges the automorphisms B, Ω on Sp and the functor $\Omega^\infty : \mathrm{Sp} \to \mathrm{Spc}_*$ with the automorphisms $B_{\omega\mathrm{Cat}}, \Omega_{\omega\mathrm{Cat}}$ on CatSp and the functor $\Omega^\infty_{\omega\mathrm{Cat}} : \mathrm{CatSp} \to \omega\mathrm{Cat}_*$.

Remark 13.3.2. It follows from proposition 13.2.19 that the inclusion $\mathrm{Sp} \to \mathrm{CatSp}$ from example 13.3.1 identifies Sp with the full subcategory of CatSp on those categorical spectra for which all arrows are invertible.

In other words, this is the intersection of the subcategories nCatSp as n ranges over all integers. It follows from remark 13.2.24 that each of this subcategories is closed under limits and colimits inside CatSp. We conclude that Sp is also closed under limits and colimits inside CatSp. Therefore the inclusion admits both left and right adjoints.

The left adjoint $\mathrm{CatSp} \to \mathrm{Sp}$ is the functor that universally inverts all cells. We may think about this as a stable version of the geometric realization functor from categories to spaces. The right adjoint $\mathrm{CatSp} \to \mathrm{Sp}$ is the functor that universally discards all non-invertible cells. For each categorical spectrum $\mathcal{C}$ we call its image under this right adjoint the spectrum underlying $\mathcal{C}$.

Any grouplike commutative algebra in spaces provides an example of a spectrum, its Eilenberg-MacLane spectrum. The following example generalizes this to the context of categorical spectra.

Example 13.3.3. Let $\mathcal{M}$ be a symmetric monoidal category. The canonical symmetric monoidal structure on the functor $B_{\mathcal{M}} : \mathrm{Alg}(\mathcal{M}) \to \mathrm{Cat}_*^{\mathcal{M}}$ induces a functor

$$\mathrm{CAlg}(\mathcal{M}) = \mathrm{CAlg}(\mathrm{Alg}(\mathcal{M})) \to \mathrm{CAlg}(\mathrm{Cat}_*^{\mathcal{M}}) = \mathrm{CAlg}(\mathrm{Cat}^{\mathcal{M}})$$

which maps each commutative algebra A in $\mathcal{M}$ to a symmetric monoidal $\mathcal{M}$-enriched category whose underlying $\mathcal{M}$-enriched category is $B_{\mathcal{M}}A$.

Iterating the above, given a commutative algebra A in $\mathcal{M}$ we may construct for every $n \geq 1$ a symmetric monoidal $\mathcal{M}$-enriched n-category $B^n_{\mathcal{M}}A$. The underlying pointed $\mathcal{M}$-enriched higher categories are characterized by the fact that they have a connected space of objects and they come equipped with equivalences

$$\Omega_{(n-1)\mathrm{Cat}^{\mathcal{M}}} B^n_{\mathcal{M}} A = B^{n-1}_{\mathcal{M}} A$$

where we use the convention $B^0_{\mathcal{M}}A = A$.

We conclude that the higher $\mathcal{M}$-enriched categories $B^n_{\mathcal{M}}A$ can be put together into an $\mathcal{M}$-categorical spectrum $\underline{A}$. We call this the Eilenberg-MacLane $\mathcal{M}$-categorical spectrum of A. In the case when A is a symmetric monoidal ω-category, we simply call this the Eilenberg-MacLane categorical spectrum of A.

Remark 13.3.4. Let A be a commutative algebra in spaces. Thinking about A as a symmetric monoidal ω-category, we may form the symmetric monoidal ω-category $B_{\omega\mathrm{Cat}}A$ as in example 13.3.3. This has a connected space of objects, and moreover the ω-category of endomorphisms of the basepoint is the space A. It follows that $B_{\omega\mathrm{Cat}}A$ is in fact a symmetric monoidal 1-category.

If A is grouplike then every endomorphism of the unit in $B_{\omega\mathrm{Cat}}A$ is invertible, so we have that $B_{\omega\mathrm{Cat}}A$ is again a commutative algebra in spaces. Note that it comes equipped with an identification

$$\Omega B_{\omega\mathrm{Cat}}A = A$$

as pointed spaces.

Since $B_{\omega\mathrm{Cat}}A$ is connected, we have that it is in fact a grouplike commutative algebra in spaces. Iterating this, we see that for every $n \geq 1$ we have a connected grouplike commutative algebra in spaces $B^n_{\omega\mathrm{Cat}}A$, and equivalences of pointed spaces

$$\Omega B^n_{\omega\mathrm{Cat}}A = B^{n-1}A$$

We thus see that the Eilenberg-MacLane categorical spectrum of a grouplike commutative algebra in spaces A lies in the image of the embedding $\mathrm{Sp} \to \mathrm{CatSp}$ of example 13.3.1. Furthermore, the associated spectrum is connective. As we shall see, the image of $\underline{A}$ under the usual equivalence between grouplike commutative algebras in spaces and connective spectra is equivalent to A.

For later purposes, we provide a functorial enhancement of example 13.3.3.

Construction 13.3.5. Let $\mathcal{M}$ be a symmetric monoidal category. Consider the sequence of categories

$$\mathrm{CAlg}(\mathcal{M}) \xrightarrow{B_{\mathcal{M}}} \mathrm{CAlg}(\mathrm{Cat}^{\mathcal{M}}) \xrightarrow{B_{\mathrm{Cat}^{\mathcal{M}}}} \mathrm{CAlg}(2\mathrm{Cat}^{\mathcal{M}}) \xrightarrow{B_{2\mathrm{Cat}^{\mathcal{M}}}} \dots \qquad (*)$$

where the transition maps are induced from the symmetric monoidal structure on the functors $B_{n\mathrm{Cat}^{\mathcal{M}}} : \mathrm{Alg}(n\mathrm{Cat}^{\mathcal{M}}) \to (n+1)\mathrm{Cat}^{\mathcal{M}}_*$, as in example 13.3.3.

For each $n \geq 0$ the functor $B_{n\mathrm{Cat}^{\mathcal{M}}}$ in the sequence $(*)$ is fully faithful and admits a right adjoint

$$\Omega_{n\mathrm{Cat}^{\mathcal{M}}} : \mathrm{CAlg}((n+1)\mathrm{Cat}^{\mathcal{M}}) \to \mathrm{CAlg}(n\mathrm{Cat}^{\mathcal{M}})$$

which is compatible with the similarly notated functor from notation 13.1.4 in that there is a commutative diagram

$$\begin{tikzcd}
\mathrm{CAlg}(n\mathrm{Cat}^{\mathcal{M}}) \arrow[d] & \mathrm{CAlg}((n+1)\mathrm{Cat}^{\mathcal{M}}) \arrow[l, "\Omega_{n\mathrm{Cat}^{\mathcal{M}}}"'] \arrow[d] \\
n\mathrm{Cat}^{\mathcal{M}}_* & (n+1)\mathrm{Cat}^{\mathcal{M}}_* \arrow[l, "\Omega_{n\mathrm{Cat}^{\mathcal{M}}}"']
\end{tikzcd}$$

where the vertical arrows are the projections.

For each $n \geq 0$ there is a functor

$$B^n_{\mathcal{M}} : \mathrm{CAlg}(\mathcal{M}) \to \mathrm{CAlg}(n\mathrm{Cat}^{\mathcal{M}})$$

obtained by composing the first n functors in the sequence $(*)$. These come equipped with identifications $\Omega_{n\mathrm{Cat}^{\mathcal{M}}} B^{n+1}_{\mathcal{M}} = B^n_{\mathcal{M}}$ for all $n \geq 0$, and can therefore be put together into a functor from $\mathrm{CAlg}(\mathcal{M})$ into the limit of the sequence $(*)$. This limit maps into the limit of the sequence from definition 13.1.9. Composing these two maps we obtain a functor

$$\mathrm{EM} : \mathrm{CAlg}(\mathcal{M}) \to \mathrm{CatSp}_{\mathcal{M}}$$

which sends each commutative algebra in $\mathcal{M}$ to its associated Eilenberg-MacLane $\mathcal{M}$-categorical spectrum. Observe that there is a canonical identification between $\Omega^\infty_{\mathcal{M}} \mathrm{EM}$ and the forgetful functor $\mathrm{CAlg}(\mathcal{M}) \to \mathcal{M}_*$.

The theory of higher presentable categories provides an abundant source of categorical spectra.

Example 13.3.6. Let $\mathcal{M}$ be a presentable symmetric monoidal category, thought of as a commutative algebra in Pr^L. Recall from section 12.2 that for each $n \geq 0$ there is a presentable symmetric monoidal $(n+1)$-category $\mathcal{M}\text{-mod}^n$, where in the case $n = 0$ we set $\mathcal{M}\text{-mod}^0 = \mathcal{M}$. We can therefore define for each $n \geq 0$ a symmetric monoidal $(n+1)$-category $\psi_{n+1}(\mathcal{M}\text{-mod}^n)$.

For each $n \geq 0$ the unit of the symmetric monoidal structure on $\mathcal{M}\text{-mod}^{n+1}$ is given by $\mathcal{M}\text{-mod}^n$. Furthermore, we have an equivalence of $(n+1)$-categories

$$\mathrm{End}_{\psi_{n+2}(\mathcal{M}\text{-mod}^{n+1})}(\mathcal{M}\text{-mod}^n) = \psi_{n+1}(\mathcal{M}\text{-mod}^n)$$

which respects units. It follows that the ω-categories $\psi_{n+1}(\mathcal{M}\text{-mod}^n)$ assemble into a categorical spectrum, which we denote by $\underline{\mathcal{M}}$. Observe that by definition we have an equivalence $\Omega^\infty_{\omega\mathrm{Cat}}(\underline{\mathcal{M}}) = \mathcal{M}$.

We now specialize example 13.3.6 to obtain a number of categorical spectra of interest.

Example 13.3.7. Consider the case where $\mathcal{M} = \mathrm{Spc}$ is the unit in Pr^L. Then the categorical spectrum $\underline{\mathrm{Spc}}$ has as its n-th ω-category the $(n+1)$-category $n\mathscr{P}r^L$ of presentable n-categories.

Example 13.3.8. Consider the case where $\mathcal{M} = \mathrm{Sp}$ is the category of spectra with the smash product symmetric monoidal structure. Then the categorical spectrum $\underline{\mathrm{Sp}}$ has as its n-th ω-category the $(n+1)$-category $n\mathscr{P}r^L_{\mathrm{St}}$ of presentable stable n-categories.

Example 13.3.9. Let A be a commutative ring spectrum. Then the category $A\text{-mod}$ of A-module spectra has a canonical structure of symmetric monoidal presentable category. We may therefore consider the categorical spectrum $\underline{A\text{-mod}}$. We will usually denote this by $A\text{-}\underline{\mathrm{mod}}$. We observe that its image under the automorphism $\Omega_{\omega\mathrm{Cat}} : \mathrm{CatSp} \to \mathrm{CatSp}$ is a categorical spectrum whose underlying pointed ω-category is the pointed space $\Omega^\infty(A)$.

Note that we may recover the categorical spectrum of example 13.3.8 by specializing the above to the case $A = \mathbb{S}$.

Higher categories of correspondences provide another source of categorical spectra.

Example 13.3.10. Let $\mathcal{C}$ be a category admitting finite limits, with final object $1_{\mathcal{C}}$. For each $n \geq 1$ we have an isomorphism

$$\mathrm{End}_{(n+1)\mathrm{Corr}(\mathcal{C})}(1_{\mathcal{C}}) = n\mathrm{Corr}(\mathcal{C}).$$

It follows that the pointed ω-categories $(n\mathrm{Corr}(\mathcal{C}), 1_{\mathcal{C}})$ assemble into a categorical spectrum, which we will denote by $\underline{\mathrm{Corr}}(\mathcal{C})$.

For later purposes, it will be convenient to have a version of example 13.3.10 which is functorial in $\mathcal{C}$.

Construction 13.3.11. Recall from section 10.2 that the functor $2\mathrm{Corr} : \mathrm{Cat}_{\mathrm{pb}} \to 2\mathrm{Cat}$ is by definition a subfunctor of the functor

$$\overline{2\mathrm{Corr}} : \mathrm{Cat}_{\mathrm{pb}} \to \mathrm{Funct}(\Delta^{\mathrm{op}}, \mathrm{Cat})$$

which sends each category with pullbacks $\mathcal{C}$ to the simplicial category whose value on $[n]$ is $\mathrm{Funct}(\mathrm{Tw}([n]), \mathcal{C})$.

Denote by $\mathrm{Cat}_{\mathrm{lex}}$ the category of categories with finite limits and limit preserving functors. Observe that there is a subfunctor F of $\overline{2\mathrm{Corr}}|_{\mathrm{Cat}_{\mathrm{lex}}}$ that maps each category with finite limits $\mathcal{C}$ to the subobject of $2\mathrm{Corr}(\mathcal{C})$ such that $F(\mathcal{C})([n])$ is the full subcategory of $2\mathrm{Corr}(\mathcal{C})([n])$ on those objects whose image under the evaluation maps $2\mathrm{Corr}(\mathcal{C})([n]) \to 2\mathrm{Corr}(\mathcal{C})([0]) = \mathcal{C}^{\leq 0}$ is equivalent to the final object of $\mathcal{C}$. Note that F is also a subfunctor of $2\mathrm{Corr}$ (with values in simplicial categories). Moreover, since $2\mathrm{Corr}$ takes values in Segal categories, we also have that F factors through the full subcategory of $\mathrm{Funct}(\Delta^{\mathrm{op}}, \mathrm{Cat})$ on the Segal categories.

Recall from [Hin20a] section 5 that there is an equivalence between Segal categories with a space of objects and algebroids in Cat with a space of objects. We can therefore interpret the inclusion $F \to 2\mathrm{Corr}$ as a natural transformation of functors $\mathrm{Cat}_{\mathrm{lex}} \to \mathrm{Algbrd}(\mathrm{Cat})_{\mathrm{Spc}}$. It follows from the definition of F that it factors through $\mathrm{Algbrd}_{[0]}(\mathrm{Cat})$, and that for each category with finite limits $\mathcal{C}$ the map $F(\mathcal{C}) \to 2\mathrm{Corr}(\mathcal{C})$ is cartesian for the projection $\mathrm{Algbrd}(\mathrm{Cat})_{\mathrm{Spc}} \to \mathrm{Spc}$.

It follows from the above that the functor

$$\mathrm{Cat}_{\mathrm{lex}} \xrightarrow{2\mathrm{Corr}} 2\mathrm{Cat}_* \xrightarrow{\Omega_{\mathrm{Cat}}} \mathrm{Algbrd}_{[0]}(\mathrm{Cat})$$

is in fact equivalent to F. Therefore the composite functor

$$\mathrm{Cat}_{\mathrm{lex}} \xrightarrow{2\mathrm{Corr}} 2\mathrm{Cat}_* \xrightarrow{\Omega_{\mathrm{Cat}}} \mathrm{Cat}_*$$

is equivalent to the composite functor

$$\mathrm{Cat}_{\mathrm{lex}} \xrightarrow{F} \mathrm{Algbrd}_{[0]}(\mathrm{Cat}) \hookrightarrow \mathrm{Funct}(\Delta^{\mathrm{op}}, \mathrm{Cat}) \xrightarrow{\mathrm{ev}_{[1]}} \mathrm{Cat}_* \,.$$

Inspecting the definition of F, we see that the above is equivalent to the forgetful functor $\mathrm{Cat}_{\mathrm{lex}} \to \mathrm{Cat}_*$.

Consider now for each $n \geq 1$ the commutative diagram

$$\begin{tikzcd}
\mathrm{Cat}_{\mathrm{lex}} \arrow[r, "2\mathrm{Corr}"] \arrow[dr, "(n+1)\mathrm{Corr}"'] & (2\mathrm{Cat}_{\mathrm{pb}})_* \arrow[r, "\Omega_{\mathrm{Cat}_{\mathrm{pb}}}"] \arrow[d, "n\mathrm{Corr}_!"] & (\mathrm{Cat}_{\mathrm{pb}})_* \arrow[d, "n\mathrm{Corr}"] \\
 & (n+1)\mathrm{Cat}_* \arrow[r, "\Omega_{n\mathrm{Cat}}"] & n\mathrm{Cat}_*.
\end{tikzcd}$$

The outer commutative square yields an equivalence $\Omega_{n\mathrm{Cat}}(n+1)\mathrm{Corr}\,|_{\mathrm{Cat}_{\mathrm{lex}}} = n\mathrm{Corr}|_{\mathrm{Cat}_{\mathrm{lex}}}$ of functors $\mathrm{Cat}_{\mathrm{lex}} \to n\mathrm{Cat}_*$. We conclude that the functors

$$n\mathrm{Corr}\,|_{\mathrm{Cat}_{\mathrm{lex}}} : \mathrm{Cat}_{\mathrm{lex}} \to n\mathrm{Cat}_*$$

fit together into a functor

$$\underline{\mathrm{Corr}} : \mathrm{Cat}_{\mathrm{lex}} \to \mathrm{CatSp}_{\mathrm{Cat}}$$

such that $\Omega^\infty_{\mathrm{Cat}}\underline{\mathrm{Corr}} : \mathrm{Cat}_{\mathrm{lex}} \to \mathrm{Cat}_*$ is the forgetful functor. For each category with finite limits $\mathcal{C}$, the image of $\mathcal{C}$ under $\underline{\mathrm{Corr}}$ recovers the categorical spectrum from example 13.3.10.

We finish by mentioning one last source of examples, closely related to both the theory of correspondences, and to higher presentable categories.

Example 13.3.12. Let $\mathcal{C}$ be a symmetric monoidal category with good relative tensor products, in the sense of [Hau17] definition 4.18 (for instance, this is satisfied if $\mathcal{C}$ has geometric realizations which are preserved by the tensor product functor, or if the symmetric monoidal structure on $\mathcal{C}$ is cocartesian and $\mathcal{C}$ admits pushouts). Then [Hau17] constructs for each $n \geq 0$ an $(n+1)$-category $\mathrm{Morita}_n(\mathcal{C})$ called the Morita $(n+1)$-category of E_n-algebras in $\mathcal{C}$. The objects of $\mathrm{Morita}_n(\mathcal{C})$ are E_n-algebras in $\mathcal{C}$, morphisms between a pair of E_n-algebras A, B are E_{n-1}-algebras in $A-B$-bimodules, and so on (where at the last step we use morphisms of unpointed E_0-algebras).

We equip $\mathrm{Morita}_n(\mathcal{C})$ with the pointing arising from the unit E_n-algebra $1_{\mathcal{C}}$. Then [Hau17] corollary 5.51 shows that for all $n \geq 0$ there is an equivalence

$$\mathrm{End}_{\mathrm{Morita}_{n+1}(\mathcal{C})}(1_{\mathcal{C}}) = \mathrm{Morita}_n(\mathcal{C}).$$

The above equivalence is in fact an equivalence of pointed $(n+1)$-categories. It follows that the pointed ω-categories $(\mathrm{Morita}_n(\mathcal{C}), 1_{\mathcal{C}})$ fit into a categorical spectrum, which we denote by $\underline{\mathrm{Morita}}(\mathcal{C})$. We call this the Morita categorical spectrum of $\mathcal{C}$. Observe that the category $\Omega^\infty_{\omega\mathrm{Cat}}(\underline{\mathrm{Morita}}(\mathcal{C}))$ is equivalent to $\mathcal{C}$.

Remark 13.3.13. Let $\mathcal{C}$ be a category admitting finite limits, and equip $\mathcal{C}^{\mathrm{op}}$ with its cocartesian symmetric monoidal structure. It was shown in [HMS20] that for each $n \geq 0$ the Morita $(n+1)$-category $\mathrm{Morita}_n(\mathcal{C}^{\mathrm{op}})$ is equivalent to the $(n+1)$-category of cospans in $\mathcal{C}^{\mathrm{op}}$ (in its incarnation as an $(n+1)$-fold complete Segal space). It seems plausible that these equivalences can be combined into an equivalence of categorical spectra $\underline{\mathrm{Corr}}(\mathcal{C}) = \underline{\mathrm{Morita}}(\mathcal{C}^{\mathrm{op}})^{1\text{-op}}$, which can furthermore be upgraded to an equivalence of functors $\mathrm{Cat}_{\mathrm{lex}} \to \mathrm{CatSp}$.

Let $\mathcal{M}$ be a presentable symmetric monoidal category, thought of as a commutative algebra in Pr^L. Then for each $n \geq 0$ one may relate E_n-algebras in $\mathcal{M}$ and $\mathcal{M}$-linear presentable n-categories: there is a functor $\text{-mod}^n : \mathrm{Alg}_{E_n}(\mathcal{M}) \to \mathcal{M}\text{-mod}^n$. The following conjecture would provide a stronger link between the Morita theory of $\mathcal{M}$ and the theory of $\mathcal{M}$-linear higher presentable categories, generalizing the fact that bimodules between algebras provide functors between categories of modules.

Conjecture 13.3.14. *Let $\mathcal{M}$ be a presentable symmetric monoidal category. Then there is a morphism of categorical spectra*

$$\underline{\mathrm{Morita}}(\mathcal{M}) \to \underline{\mathcal{M}}$$

which sends each E_n-algebra A in $\mathcal{M}$ to the object $A\text{-mod}^n$ in $\mathcal{M}\text{-mod}^n$.

13.4 Relation to symmetric monoidal categories

The category of spectra can be defined formally inverting the loop space functor on either the category of pointed spaces, or the category of grouplike commutative algebras in spaces. Our approach to categorical spectra has so far focused on an analogue of the first definition of spectra. Our next goal is to discuss the analogue of the second perspective.

Notation 13.4.1. Let $\mathcal{M}$ be a symmetric monoidal structure. We equip the functor $B_{\mathcal{M}} : \mathrm{Alg}(\mathcal{M}) \to \mathrm{Cat}_*^{\mathcal{M}}$ from notation 13.1.4 with its canonical symmetric monoidal structure. We continue denoting by $B_{\mathcal{M}}$ the induced functor

$$B_{\mathcal{M}} : \mathrm{CAlg}(\mathcal{M}) = \mathrm{CAlg}(\mathrm{Alg}(\mathcal{M})) \to \mathrm{CAlg}(\mathrm{Cat}_*^{\mathcal{M}}) = \mathrm{CAlg}(\mathrm{Cat}^{\mathcal{M}}).$$

We endow the functor $\Omega_{\mathcal{M}} : \mathrm{Cat}_*^{\mathcal{M}} \to \mathrm{Alg}(\mathcal{M})$ with the induced lax symmetric monoidal structure. We continue denoting by $\Omega_{\mathcal{M}}$ the induced functor

$$\Omega_{\mathcal{M}} : \mathrm{CAlg}(\mathrm{Cat}^{\mathcal{M}}) = \mathrm{CAlg}(\mathrm{Cat}_*^{\mathcal{M}}) \to \mathrm{CAlg}(\mathrm{Alg}(\mathcal{M})) = \mathrm{CAlg}(\mathcal{M}).$$

Note that is is right adjoint to the functor $B_{\mathcal{M}}$.

Proposition 13.4.2. *Let $\mathcal{M}$ be a symmetric monoidal category. Then the lax symmetric monoidal structure on $\Omega_{\mathcal{M}} : \mathrm{Cat}_*^{\mathcal{M}} \to \mathrm{Alg}(\mathcal{M})$ is strict.*

Proof. Let $\mathcal{C}, \mathcal{D}$ be two pointed $\mathcal{M}$-enriched categories. The lax symmetric monoidal structure on $\Omega_{\mathcal{M}}$ induces a morphism of algebras $\Omega_{\mathcal{M}}(\mathcal{C}) \otimes \Omega_{\mathcal{M}}(\mathcal{D}) \to \Omega_{\mathcal{M}}(\mathcal{C} \otimes \mathcal{D})$ which we have to show is an isomorphism. Since $B_{\mathcal{M}}$ is fully faithful, we have that the above is the image under $\Omega_{\mathcal{M}}$ of the map

$$\alpha : B_{\mathcal{M}}(\Omega_{\mathcal{M}}(\mathcal{C}) \otimes \Omega_{\mathcal{M}}(\mathcal{D})) = B_{\mathcal{M}}(\Omega_{\mathcal{M}}(\mathcal{C})) \otimes B_{\mathcal{M}}(\Omega_{\mathcal{M}}(\mathcal{D}))) \to \mathcal{C} \otimes \mathcal{D}$$

obtained by tensoring the unit morphisms $B_{\mathcal{M}}\Omega_{\mathcal{M}}(\mathcal{C}) \to \mathcal{C}$ and $B_{\mathcal{M}}\Omega_{\mathcal{M}}(\mathcal{D}) \to \mathcal{D}$. The map α is equivalent to the image under the localization functor $\mathrm{Algbrd}_*(\mathcal{M})_{\mathrm{Spc}} \to \mathrm{Cat}_*^{\mathcal{M}}$ of the morphism of algebroids

$$\beta : \Omega_{\mathcal{M}}(\mathcal{C}) \otimes \Omega_{\mathcal{M}}(\mathcal{D}) \to \mathcal{C} \otimes \mathcal{D}$$

obtained by tensoring the unit morphisms $\Omega_{\mathcal{M}}(\mathcal{C}) \to \mathcal{C}$ and $\Omega_{\mathcal{M}}(\mathcal{D}) \to \mathcal{D}$.

Since the localization functor $\mathrm{Algbrd}_*(\mathcal{M})_{\mathrm{Spc}} \to \mathrm{Cat}_*^{\mathcal{M}}$ maps fully faithful functors to fully faithful functors, and the colocalization map $\Omega_{\mathcal{M}} : \mathrm{Cat}_*^{\mathcal{M}} \to \mathrm{Alg}(\mathcal{M})$ inverts fully faithful functor, we reduce to showing that β is fully faithful.

It follows from remark 3.5.7 that β is the image under the map $m_! : \mathrm{Algbrd}(\mathcal{M} \times \mathcal{M}) \to \mathrm{Algbrd}(\mathcal{M})$ of the morphism of $\mathcal{M} \times \mathcal{M}$-algebroids

$$\Omega_{\mathcal{M}}(\mathcal{C}) \boxtimes \Omega_{\mathcal{M}}(\mathcal{D}) \to \mathcal{C} \boxtimes \mathcal{D}$$

obtained as the exterior tensor product of the unit morphisms $\Omega_{\mathcal{M}}(\mathcal{C}) \to \mathcal{C}$ and $\Omega_{\mathcal{M}}(\mathcal{D}) \to \mathcal{D}$. Our result now follows from the fact that exterior tensor products of fully faithful functors are fully faithful, which is a consequence of part (i) of proposition 3.5.5. □

Corollary 13.4.3. *Let $\mathcal{M}$ be a symmetric monoidal category. Then there is an induced symmetric monoidal structure on $\mathrm{CatSp}_{\mathcal{M}}$, which makes it into the limit of the sequence of symmetric monoidal categories and symmetric monoidal functors*

$$\mathcal{M}_* \xleftarrow{\Omega_{\mathcal{M}}} \mathrm{Cat}_*^{\mathcal{M}} \xleftarrow{\Omega_{\mathrm{Cat}^{\mathcal{M}}}} 2\mathrm{Cat}_*^{\mathcal{M}} \xleftarrow{\Omega_{2\mathrm{Cat}^{\mathcal{M}}}} 3\mathrm{Cat}_*^{\mathcal{M}} \dots .$$

Proof. This is a direct consequence of the fact that the forgetful functor $\mathrm{CAlg}(\mathrm{Cat}) \to \mathrm{Cat}$ creates limits. □

Proposition 13.4.4. *Let $\mathcal{M}$ be a symmetric monoidal category. Then there is an equivalence of symmetric monoidal categories between $\mathrm{CatSp}_{\mathcal{M}}$ and the limit of the sequence*

$$\mathrm{CAlg}(\mathcal{M}) \xleftarrow{\Omega_{\mathcal{M}}} \mathrm{CAlg}(\mathrm{Cat}^{\mathcal{M}}) \xleftarrow{\Omega_{\mathrm{Cat}^{\mathcal{M}}}} \mathrm{CAlg}(2\mathrm{Cat}^{\mathcal{M}}) \xleftarrow{\Omega_{2\mathrm{Cat}^{\mathcal{M}}}} \mathrm{CAlg}(3\mathrm{Cat}^{\mathcal{M}}) \dots .$$

Proof. Consider the following commutative diagram of symmetric monoidal categories and symmetric monoidal functors:

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
\mathrm{Alg}(\mathrm{Alg}(\mathrm{Alg}(\mathcal{M}_*))) \arrow[d] & \mathrm{Alg}(\mathrm{Alg}(\mathrm{Cat}_*^{\mathcal{M}})) \arrow[l, "\Omega_{\mathcal{M}}"'] \arrow[d] & \mathrm{Alg}(2\mathrm{Cat}_*^{\mathcal{M}}) \arrow[l, "\Omega_{\mathrm{Cat}^{\mathcal{M}}}"'] \arrow[d] & 3\mathrm{Cat}_*^{\mathcal{M}} \dots \arrow[l, "\Omega_{2\mathrm{Cat}^{\mathcal{M}}}"'] \arrow[d] \\
\mathrm{Alg}(\mathrm{Alg}(\mathcal{M}_*)) \arrow[d] & \mathrm{Alg}(\mathrm{Cat}_*^{\mathcal{M}}) \arrow[l, "\Omega_{\mathcal{M}}"'] \arrow[d] & 2\mathrm{Cat}_*^{\mathcal{M}} \arrow[l, "\Omega_{\mathrm{Cat}^{\mathcal{M}}}"'] \arrow[d] & 3\mathrm{Cat}_*^{\mathcal{M}} \dots \arrow[l, "\Omega_{2\mathrm{Cat}^{\mathcal{M}}}"'] \arrow[d] \\
\mathrm{Alg}(\mathcal{M}_*) \arrow[d] & \mathrm{Cat}_*^{\mathcal{M}} \arrow[l, "\Omega_{\mathcal{M}}"'] \arrow[d] & 2\mathrm{Cat}_*^{\mathcal{M}} \arrow[l, "\Omega_{\mathrm{Cat}^{\mathcal{M}}}"'] \arrow[d] & 3\mathrm{Cat}_*^{\mathcal{M}} \dots \arrow[l, "\Omega_{2\mathrm{Cat}^{\mathcal{M}}}"'] \arrow[d] \\
\mathcal{M}_* & \mathrm{Cat}_*^{\mathcal{M}} \arrow[l, "\Omega_{\mathcal{M}}"'] & 2\mathrm{Cat}_*^{\mathcal{M}} \arrow[l, "\Omega_{\mathrm{Cat}^{\mathcal{M}}}"'] & 3\mathrm{Cat}_*^{\mathcal{M}} \dots \arrow[l, "\Omega_{2\mathrm{Cat}^{\mathcal{M}}}"']
\end{tikzcd}$$

Here the downwards vertical arrows are forgetful functors (or identities). Every row has limit $\mathrm{CatSp}_{\mathcal{M}}$, with identity transition maps between them. Hence the limit of the above diagram is the symmetric monoidal category $\mathrm{CatSp}_{\mathcal{M}}$ from corollary 13.4.3.

We can compute this limit alternatively by first passing to limits of the columns, and then taking the limit of the resulting sequence. Observe that the limit row receives a map from the following sequence of symmetric monoidal categories:

$$\mathrm{CAlg}(\mathcal{M}_*) \xleftarrow{\Omega_{\mathcal{M}}} \mathrm{CAlg}(\mathrm{Cat}_*^{\mathcal{M}}) \xleftarrow{\Omega_{\mathrm{Cat}^{\mathcal{M}}}} \mathrm{CAlg}(2\mathrm{Cat}_*^{\mathcal{M}}) \xleftarrow{\Omega_{2\mathrm{Cat}^{\mathcal{M}}}} \mathrm{CAlg}(3\mathrm{Cat}_*^{\mathcal{M}}) \dots .$$

Our result will follow if we are able to show that this map is an isomorphism. It suffices then to show that for each $n \geq 0$ the symmetric monoidal category $\mathrm{CAlg}(n\mathrm{Cat}_*^{\mathcal{M}})$ is the limit of the symmetric monoidal categories $\mathrm{Alg}(\mathrm{Alg}(\dots(\mathrm{Alg}(n\mathrm{Cat}_*^{\mathcal{M}}))\dots))$. This is a direct consequence of the fact that the commutative operad is the colimit of the operads $\mathrm{Assos}^{\otimes n}$ (see [Lur17] corollary 5.1.1.5). □

Corollary 13.4.5. *Let $\mathcal{M}$ be a symmetric monoidal category. Then the induced symmetric monoidal structure on* $\mathrm{CatSp}_{\mathcal{M}}$ *is cocartesian.*

Proof. This follows directly from proposition 13.4.4, using the fact that the symmetric monoidal structure on the category of commutative algebras in a symmetric monoidal category is always cocartesian. □

Remark 13.4.6. Let $\mathcal{M}$ be a cartesian symmetric monoidal category. Then, as discussed in proposition 3.5.8, the symmetric monoidal structures on $n\mathrm{Cat}_{\mathcal{M}}$ are also cartesian. It follows that the induced symmetric monoidal structure on $\mathrm{CatSp}_{\mathcal{M}}$ is both cartesian and cocartesian.

Notation 13.4.7. Let $\mathcal{M}$ be a symmetric monoidal category. The symmetric monoidal functor $\Omega^\infty_{\mathcal{M}} : \mathrm{CatSp}_{\mathcal{M}} \to \mathcal{M}$ induces a symmetric monoidal functor

$$\mathrm{CatSp}_{\mathcal{M}} = \mathrm{CAlg}(\mathrm{CatSp}_{\mathcal{M}}) \to \mathrm{CAlg}(\mathcal{M})$$

which we will continue denoting by $\Omega^\infty_{\mathcal{M}}$. For each $\mathcal{M}$-categorical spectrum $\mathcal{C}$, we call $\Omega^\infty_{\mathcal{M}}\mathcal{C}$ the commutative algebra underlying $\mathcal{M}$.

Remark 13.4.8. Let $\mathcal{M}$ be a symmetric monoidal category. It follows directly from the definition of $\mathrm{CatSp}_{\mathcal{M}}$ that the limit of the sequence of symmetric monoidal categories and symmetric monoidal functors

$$\mathrm{CAlg}(\mathcal{M}) \xleftarrow{\Omega_{\mathcal{M}}} \mathrm{CAlg}(\mathrm{Cat}^{\mathcal{M}}) \xleftarrow{\Omega_{\mathrm{Cat}^{\mathcal{M}}}} \mathrm{CAlg}(2\mathrm{Cat}^{\mathcal{M}}) \xleftarrow{\Omega_{2\mathrm{Cat}^{\mathcal{M}}}} \mathrm{CAlg}(3\mathrm{Cat}^{\mathcal{M}}) \dots$$

from proposition 13.4.4 recovers $\mathrm{CAlg}(\mathrm{CatSp}_{\mathcal{M}})$. Corollary 13.4.5 implies that the forgetful functor $\mathrm{CAlg}(\mathrm{CatSp}_{\mathcal{M}}) \to \mathrm{CatSp}_{\mathcal{M}}$ is an equivalence of symmetric monoidal categories. This equivalence agrees with the one produced in the proof of proposition 13.4.4.

In particular, we see that the functor $\Omega^\infty_{\mathcal{M}} : \mathrm{CatSp}_{\mathcal{M}} \to \mathrm{CAlg}(\mathcal{M})$ is equivalent to the projection arising from proposition 13.4.4.

Remark 13.4.9. Let $\mathcal{M}$ be a symmetric monoidal category. Then the sequence

$$\mathrm{CAlg}(\mathrm{Cat}^{\mathcal{M}}) \xleftarrow{\Omega_{\mathrm{Cat}^{\mathcal{M}}}} \mathrm{CAlg}(2\mathrm{Cat}^{\mathcal{M}}) \xleftarrow{\Omega_{2\mathrm{Cat}^{\mathcal{M}}}} \mathrm{CAlg}(3\mathrm{Cat}^{\mathcal{M}}) \xleftarrow{\Omega_{3\mathrm{Cat}^{\mathcal{M}}}} \mathrm{CAlg}(4\mathrm{Cat}^{\mathcal{M}}) \dots$$

computing $\mathrm{CatSp}_{\mathrm{Cat}^{\mathcal{M}}}$ is a shift of the sequence from proposition 13.4.4 computing $\mathrm{CatSp}_{\mathrm{Cat}^{\mathcal{M}}}$. The induced equivalence $\mathrm{CatSp}_{\mathrm{Cat}^{\mathcal{M}}} = \mathrm{CatSp}$ agrees with the one given by the shift functors from notation 13.1.15.

In particular, we have commutative a commutative square

$$\begin{tikzcd}
\mathrm{CatSp}_{\mathrm{Cat}^{\mathcal{M}}} \arrow[r, "\Omega_{\mathcal{M}}"] \arrow[d, "\Omega^\infty_{\mathrm{Cat}^{\mathcal{M}}}"] & \mathrm{CatSp}_{\mathcal{M}} \arrow[d, "\Omega^\infty_{\mathcal{M}}"] \\
\mathrm{CAlg}(\mathrm{Cat}^{\mathcal{M}}) \arrow[r, "\Omega_{\mathcal{M}}"] & \mathrm{CAlg}(\mathcal{M})
\end{tikzcd}$$

which upgrades the equivalence $\Omega^\infty_{\mathcal{M}}\Omega_{\mathcal{M}} = \Omega_{\mathcal{M}}\Omega^\infty_{\mathrm{Cat}^{\mathcal{M}}}$ from remark 13.1.17.

Remark 13.4.10. Consider the case $\mathcal{M} = \omega\mathrm{Cat}$, equipped with its cartesian symmetric monoidal structure. Then the induced symmetric monoidal structure on CatSp is both cartesian and cocartesian. We will continue denoting by $B_{\omega\mathrm{Cat}}$ and $\Omega_{\omega\mathrm{Cat}}$ the adjoint functors

$$B_{\omega\mathrm{Cat}} : \mathrm{CAlg}(\omega\mathrm{Cat}) \rightleftarrows \mathrm{CAlg}(\omega\mathrm{Cat}) : \Omega_{\omega\mathrm{Cat}}$$

obtained from the adjunction from notation 13.4.1 by using the equivalence $\mathrm{Cat}^{\omega\mathrm{Cat}} = \omega\mathrm{Cat}$. Note that the right adjoint $\Omega_{\omega\mathrm{Cat}}$ is induced from the product preserving functor $\Omega_{\omega\mathrm{Cat}} : \omega\mathrm{Cat}_* \to \omega\mathrm{Cat}_*$ from remark 13.2.2.

Consider the sequence of categories

$$\mathrm{CAlg}(\omega\mathrm{Cat}) \xleftarrow{\Omega_{\omega\mathrm{Cat}}} \mathrm{CAlg}(\omega\mathrm{Cat}) \xleftarrow{\Omega_{\omega\mathrm{Cat}}} \mathrm{CAlg}(\omega\mathrm{Cat}) \xleftarrow{\Omega_{\omega\mathrm{Cat}}} \mathrm{CAlg}(\omega\mathrm{Cat}) \dots.$$

The forgetful functors $\mathrm{CAlg}(\omega\mathrm{Cat}) \to \omega\mathrm{Cat}_*$ induce a morphism from the above sequence to the sequence of remark 13.2.2. Passing to the limit we obtain an equivalence $\mathrm{CAlg}(\mathrm{CatSp}) \to \mathrm{CatSp}$. This agrees with the equivalence obtained from the proof of proposition 13.4.4 by using the usual equivalences $n\mathrm{Cat}^{\omega\mathrm{Cat}} = \omega\mathrm{Cat}$.

Specializing remark 13.4.9 we obtain a commutative square

$$\begin{tikzcd}
\mathrm{CatSp} \arrow[r, "\Omega_{\omega\mathrm{Cat}}"] \arrow[d, "\Omega^{\infty}_{\omega\mathrm{Cat}}"] & \mathrm{CatSp} \arrow[d, "\Omega^{\infty}_{\omega\mathrm{Cat}}"] \\
\mathrm{CAlg}(\omega\mathrm{Cat}) \arrow[r, "\Omega_{\omega\mathrm{Cat}}"] & \mathrm{CAlg}(\omega\mathrm{Cat})
\end{tikzcd}$$

which enhances the commutative square from remark 13.2.4.

Example 13.4.11. Recall the functor $\underline{\mathrm{Corr}} : \mathrm{Cat}_{\mathrm{lex}} \to \mathrm{CatSp}$ from construction 13.3.11 which sends each category with finite limits $\mathcal{C}$ to its categorical spectrum of correspondences. For each $n \geq 0$ the composition of $\underline{\mathrm{Corr}}$ with the n-th projection $\Omega^{\infty-n}_{\omega\mathrm{Cat}} : \mathrm{CatSp} \to \omega\mathrm{Cat}$ recovers the functor $n\mathrm{Corr}$, which is limit preserving. Hence $\underline{\mathrm{Corr}}$ is limit preserving.

Consider for each $n \geq 0$ the commutative triangle of categories

$$\begin{tikzcd}
\mathrm{Cat}_{\mathrm{lex}} \arrow[r, "\underline{\mathrm{Corr}}"] \arrow[dr, "n\mathrm{Corr}"'] & \mathrm{CatSp} \arrow[d, "\Omega^{\infty-n}_{\omega\mathrm{Cat}}"] \\
& \omega\mathrm{Cat}\,.
\end{tikzcd}$$

This can be upgraded to a commutative triangle of cartesian symmetric monoidal categories and symmetric monoidal functors. Passing to categories of commutative algebras we obtain a commutative triangle of categories

$$\begin{tikzcd}
\mathrm{CAlg}(\mathrm{Cat}_{\mathrm{lex}}) \arrow[r, "\underline{\mathrm{Corr}}"] \arrow[dr, "n\mathrm{Corr}"'] & \mathrm{CAlg}(\mathrm{CatSp}) \arrow[d, "\Omega^{\infty-n}_{\omega\mathrm{Cat}}"] \\
& \mathrm{CAlg}(\omega\mathrm{Cat}).
\end{tikzcd}$$

We enhance the above commutative triangle as follows:

$$\begin{tikzcd}
\mathrm{Cat}_{\mathrm{lex}} \arrow[r, "\underline{\mathrm{Corr}}"] & \mathrm{CatSp} \\
\mathrm{CAlg}(\mathrm{Cat}_{\mathrm{lex}}) \arrow[u] \arrow[r, "\underline{\mathrm{Corr}}"] \arrow[dr, "n\mathrm{Corr}"'] & \mathrm{CAlg}(\mathrm{CatSp}) \arrow[u] \arrow[d, "\Omega^{\infty-n}_{\omega\mathrm{Cat}}"] \\
& \mathrm{CAlg}(\omega\mathrm{Cat})
\end{tikzcd}$$

Here the upward arrows are the forgetful functors, and in particular the right upward arrow is an isomorphism.

Note that there is a fully faithful section $\mathrm{Cat}_{\mathrm{lex}} \to \mathrm{CAlg}(\mathrm{Cat}_{\mathrm{lex}})$ to the forgetful functor, which maps each category with finite limits $\mathcal{C}$ to itself equipped with the cartesian symmetric monoidal structure. We thus obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{Cat}_{\mathrm{lex}} \arrow[r, "\underline{\mathrm{Corr}}"] \arrow[d] & \mathrm{CatSp} \arrow[d] \\
\mathrm{CAlg}(\mathrm{Cat}_{\mathrm{lex}}) \arrow[r, "\underline{\mathrm{Corr}}"] \arrow[dr, "n\mathrm{Corr}"'] & \mathrm{CAlg}(\mathrm{CatSp}) \arrow[d, "\Omega^{\infty-n}_{\omega\mathrm{Cat}}"] \\
 & \mathrm{CAlg}(\omega\mathrm{Cat}).
\end{tikzcd}$$

Looking at the outer commutative square, we conclude that the enhancement of $n\mathrm{Corr} : \mathrm{Cat}_{\mathrm{lex}} \to \omega\mathrm{Cat}$ to a functor into $\mathrm{CAlg}(\omega\mathrm{Cat})$ obtained by virtue of the fact that $n\mathrm{Corr}$ is part of a functor into CatSp, agrees with its enhancement arising from the fact that $n\mathrm{Corr}$ preserves products. In particular, for every category with finite limits $\mathcal{C}$, the symmetric monoidal structure on $n\mathrm{Corr}(\mathcal{C})$ arising by virtue of its presentation as $\Omega^{\infty-n}_{\omega\mathrm{Cat}}(\underline{\mathrm{Corr}}(\mathcal{C}))$ agrees with the symmetric monoidal structure used in chapter 11.

Recall that in the setting of spectra there is a functor $\Omega^\infty : \mathrm{Sp} \to \mathrm{CAlg}_{\mathrm{grplike}}(\mathrm{Sp})$ that sends each spectrum to its underlying grouplike commutative algebra in spaces. This is known to admit a fully faithful left adjoint which identifies $\mathrm{CAlg}_{\mathrm{grplike}}(\mathrm{Sp})$ with the category of connective spectra. We now explore the analogue of this in the setting of $\mathcal{M}$-categorical spectra.

Definition 13.4.12. *Let $\mathcal{M}$ be a symmetric monoidal category and let $\mathcal{C}$ be an $\mathcal{M}$-categorical spectrum, corresponding to a sequence of $\mathcal{M}$-enriched higher categories $\mathcal{C}_n$ and equivalences $\Omega_{n\mathrm{Cat}^{\mathcal{M}}}(\mathcal{C}_{n+1}) = \mathcal{C}_n$ for all $n \geq 0$. We say that $\mathcal{C}$ is connective if the space of objects of $\mathcal{C}_n$ is connected for all $n \geq 1$.*

Example 13.4.13. Let S be a spectrum. Then S is connective if and only if the associated categorical spectrum from example 13.3.1 is connective.

Proposition 13.4.14. *Let $\mathcal{M}$ be a symmetric monoidal category. Then the functor*

$$\Omega^\infty_{\mathcal{M}} : \mathrm{CAlg}(\mathrm{CatSp}_{\mathcal{M}}) \to \mathrm{CAlg}(\mathcal{M})$$

from notation 13.4.7 admits a fully faithful left adjoint, given by the functor

$$\mathrm{EM} : \mathrm{CAlg}(\mathcal{M}) \to \mathrm{CatSp}$$

from construction 13.3.5, which sends each commutative algebra in $\mathcal{M}$ to its Eilenberg-MacLane $\mathcal{M}$-categorical spectrum. Furthermore, this identifies $\mathrm{CAlg}(\mathcal{M})$ with the full subcategory of $\mathrm{CatSp}_{\mathcal{M}}$ on the connective $\mathcal{M}$-categorical spectra.

The proof of proposition 13.4.14 will need some preliminary lemmas.

Lemma 13.4.15. *Let $\mathcal{I}$ be a category admitting an initial object 0, and let $F : \mathcal{I} \to \mathrm{Cat}$ be a functor. Assume that for all arrows α in $\mathcal{I}$, the functor $F(\alpha)$ is fully faithful and admits a right adjoint. Denote by $F^R : \mathcal{I}^{\mathrm{op}} \to \mathrm{Cat}$ the functor obtained from F by passing to right adjoints. Then the projection $q : \lim_{\mathcal{I}^{\mathrm{op}}} F^R \to F(0)$ admits a fully faithful left adjoint.*

Proof. Let $G : \mathcal{I}^{\mathrm{op}} \to \mathrm{Cat}$ be the constant functor $F(0)$. The projection q is the limit of the natural transformation $\eta : F^R \to G$ that arises from the fact that G is the right Kan extension along the inclusion $\{0\} \to \mathcal{I}^{\mathrm{op}}$ of the constant functor $F(0)$.

To show that q admits a left adjoint it suffices to show that for every arrow $\alpha : i \to j$ in $\mathcal{I}$ the resulting commutative square of categories

$$\begin{tikzcd} F(j) \arrow[r, "F^R(\alpha)"] \arrow[d, "\eta(j)"] & F(i) \arrow[d, "\eta(i)"] \\ G(j) \arrow[r, "G(\alpha)"] & G(i) \end{tikzcd}$$

is vertically left adjointable. Observe that the above square has the form

$$\begin{tikzcd} F(j) \arrow[r, "F^R(\alpha)"] \arrow[d, "F^R(\mu)"] & F(i) \arrow[d, "F^R(\nu)"] \\ F(0) \arrow[r, "\mathrm{id}"] & F(0) \end{tikzcd}$$

where μ, ν are the unique morphisms $0 \to j$ and $0 \to i$. Since the vertical arrows are colocalizations, to show that the above square is vertically left adjointable, it suffices to show that the image of $F^R(\alpha)F(\mu)$ is contained in the image of $F(\nu)$. This follows from the equivalence

$$F^R(\alpha)F(\mu) = F^R(\alpha)F(\alpha)F(\nu) = F(\nu).$$

This shows that the limit of the natural transformation η is indeed left adjointable. The fully faithfulness of the left adjoint follows from the fact that for every i in $\mathcal{I}$ the left adjoint to $\eta(i)$ is fully faithful. $\square$

Lemma 13.4.16. *Let*

$$\mathcal{C}_0 \xleftarrow{q_0} \mathcal{C}_1 \xleftarrow{q_1} \mathcal{C}_2 \xleftarrow{q_2} \dots$$

be a sequence of categories, and let $\mathcal{C}$ be its limit. Assume that for every $n \geq 0$ the functor q_n admits a fully faithful left adjoint i_n. Then the projection $q : \mathcal{C} \to \mathcal{C}_0$ admits a fully faithful left adjoint, whose image is the full subcategory of $\mathcal{C}$ on those objects whose projection to $\mathcal{C}_{n+1}$ belongs to the image of i_n for all $n \geq 0$.

Proof. The existence of a fully faithful left adjoint for q is a direct consequence of lemma 13.4.15, applied to the functor $F : \mathbb{N} \to \mathrm{Cat}$ associated to the sequence obtained from the one in the statement by passing to left adjoints of all arrows, where $\mathbb{N}$ is the poset of nonnegative integers. Inspecting the proof of lemma 13.4.15, we see that an object x in $\mathcal{C}$ belongs to the image of the left adjoint to q if and only if for all $n \geq 0$ its projection x_n to $\mathcal{C}_n$ belongs to the image of the inclusion $\mathcal{C}_0 \to \mathcal{C}_n$. This happens if and only if x_{n+1} belongs to the image of i_n for all $n \geq 0$, as desired. □

Proof of proposition 13.4.14. For each $n \geq 0$ the functor

$$\Omega_{n\mathrm{Cat}^{\mathcal{M}}} : \mathrm{CAlg}((n+1)\mathrm{Cat}^{\mathcal{M}}) \to \mathrm{CAlg}(n\mathrm{Cat}^{\mathcal{M}})$$

has a left adjoint which is induced from the canonical symmetric monoidal structure on the functor

$$B_{\mathcal{M}} : \mathrm{Alg}(n\mathrm{Cat}^{\mathcal{M}}) \to (n+1)\mathrm{Cat}^{\mathcal{M}}_* .$$

As discussed in remark 13.1.6, the latter is fully faithful, and identifies $\mathrm{Alg}(n\mathrm{Cat}^{\mathcal{M}})$ with the full subcategory of $(n+1)\mathrm{Cat}^{\mathcal{M}}_*$ on those pointed $\mathcal{M}$-enriched $(n+1)$-categories with a connected space of objects.

Applying lemma 13.4.16 to the sequence of categories from proposition 13.4.4 we conclude that $\Omega^\infty_{\mathcal{M}} : \mathrm{CatSp}_{\mathcal{M}} \to \mathrm{CAlg}(\mathcal{M})$ admits a fully faithful left adjoint whose image is the full subcategory of $\mathrm{CatSp}_{\mathcal{M}}$ on the connective $\mathcal{M}$-categorical spectra.

It remains to show that the left adjoint to $\Omega^\infty_{\mathcal{M}}$ recovers the functor EM from construction 13.3.5. Observe that for every commutative algebra A in $\mathcal{M}$ the $\mathcal{M}$-categorical spectrum EM is indeed connective. Hence we can write $\mathrm{EM} = (\Omega^\infty_{\mathcal{M}})^L F$ for some functor $F : \mathrm{CAlg}(\mathcal{M}) \to \mathrm{CAlg}(\mathcal{M})$, where $(\Omega^\infty_{\mathcal{M}})^L$ denotes the left adjoint to $\Omega^\infty_{\mathcal{M}}$.

Inspecting the construction, we see that the composite functor

$$\mathrm{CAlg}(\mathcal{M}) \xrightarrow{\mathrm{EM}} \mathrm{CatSp} \xrightarrow{\Omega^\infty_{\mathcal{M}}} \mathrm{CAlg}(\mathcal{M})$$

is equivalent to the identity. Since $\Omega^\infty_{\mathcal{M}}$ is a colocalization, we conclude that F is equivalent to the identity. Therefore EM is equivalent to $(\Omega^\infty_{\mathcal{M}})^L$, as desired. □

Example 13.4.17. Recall the inclusion $\mathrm{Sp} \to \mathrm{CatSp}$ from example 13.3.1. This fits into a commutative square of presentable categories and right adjoints

$$\begin{tikzcd} \mathrm{Spc}_* \arrow[d] & \mathrm{Sp} \arrow[l, "\Omega^\infty"'] \arrow[d] \\ \omega\mathrm{Cat}_* & \mathrm{CatSp} . \arrow[l, "\Omega^\infty_{\omega\mathrm{Cat}}"'] \end{tikzcd}$$

The cartesian symmetric monoidal structures on CatSp and Sp are also cocartesian. Passing to categories of commutative algebras and restricting to grouplike commutative algebras in spaces we obtain a commutative square of categories

$$\begin{tikzcd}
\mathrm{CAlg}_{\mathrm{grplike}}(\mathrm{Spc}) \arrow[d] & \mathrm{Sp} \arrow[l, "\Omega^\infty"'] \arrow[d] \\
\mathrm{CAlg}(\omega\mathrm{Cat}) & \mathrm{CatSp}\,. \arrow[l, "\Omega^\infty_{\omega\mathrm{Cat}}"']
\end{tikzcd}$$

It follows from a combination of remark 13.3.4 and proposition 13.4.14 that the above commutative square is horizontally left adjointable. Passing to left adjoints of the horizontal arrows we obtain a commutative square

$$\begin{tikzcd}
\mathrm{CAlg}_{\mathrm{grplike}}(\mathrm{Spc}) \arrow[r] \arrow[d] & \mathrm{Sp} \arrow[d] \\
\mathrm{CAlg}(\omega\mathrm{Cat}) \arrow[r, "\mathrm{EM}"] & \mathrm{CatSp}\,.
\end{tikzcd}$$

The top horizontal arrow in the above diagram is the standard inclusion of grouplike commutative algebras in spaces into spectra, as the full subcategory of connective spectra. The commutativity of the above square shows that this is compatible with our functor EM.

The next result provides a more concrete description of the functor that sends each categorical spectrum to its underlying spectrum.

Proposition 13.4.18. *The commutative square of categories*

$$\begin{tikzcd}
\mathrm{CAlg}_{\mathrm{grplike}}(\mathrm{Spc}) \arrow[d] & \mathrm{Sp} \arrow[l, "\Omega^\infty"'] \arrow[d] \\
\mathrm{CAlg}(\omega\mathrm{Cat}) & \mathrm{CatSp}\,. \arrow[l, "\Omega^\infty_{\omega\mathrm{Cat}}"']
\end{tikzcd}$$

from example 13.4.17, is vertically right adjointable.

Proof. The left vertical arrow is the composition of the inclusions

$$\mathrm{CAlg}_{\mathrm{grplike}}(\mathrm{Spc}) \to \mathrm{CAlg}(\mathrm{Spc}) \to \mathrm{CAlg}(\omega\mathrm{Cat}).$$

The first arrow admits a right adjoint, which sends a commutative algebra in spaces to its subspace of invertible elements with its induced commutative algebra structure. The second arrow admits a right adjoint induced by the truncation functor $(-)^{\leq 0} : \omega\mathrm{Cat} \to \mathrm{Spc}$. Hence we see that the left vertical arrow in the square of the statement is right adjointable.

As observed in remark 13.3.2, the right vertical arrow in the above square is also right adjointable. Our result now follows from the fact, observed in example 13.4.17, that this square is horizontally left adjointable. $\square$

Corollary 13.4.19. *Let $\mathcal{C}$ be a categorical spectrum and let $\mathcal{C}^{\leq -\infty}$ be the spectrum underlying $\mathcal{C}$. Then $\Omega^\infty(\mathcal{C}^{\leq -\infty})$ is the image of $\Omega^\infty_{\omega\mathrm{Cat}}(\mathcal{C})$ under the right adjoint to the inclusion $\mathrm{CAlg}_{\mathrm{grplike}}(\mathrm{Spc}) \to \mathrm{CAlg}(\omega\mathrm{Cat})$.*

Remark 13.4.20. Let $\mathcal{C}$ be a categorical spectrum, corresponding to a sequence of symmetric monoidal ω-categories $\mathcal{C}_n$ and equivalences $\Omega_{\omega\mathrm{Cat}}(\mathcal{C}_{n+1}) = \mathcal{C}_n$ for all $n \geq 0$. Recall that the inclusion $\mathrm{Sp} \to \mathrm{CatSp}$ exchanges the shift automorphisms B, Ω on Sp with the automorphisms $B_{\omega\mathrm{Cat}}, \Omega_{\omega\mathrm{Cat}}$ on CatSp. It follows from corollary 13.4.19 that for every $n \geq 0$ the grouplike commutative algebra in spaces $\Omega^{\infty-n}(\mathcal{C}^{\leq -\infty})$ is the image of the symmetric monoidal ω-category $\mathcal{C}_n$ under the right adjoint to the inclusion $\mathrm{CAlg}_{\mathrm{grouplike}}(\mathrm{Spc}) \to \mathrm{CAlg}(\omega\mathrm{Cat})$.

In other words, the n-th space of the spectrum $\mathcal{C}^{\leq -\infty}$ is obtained from $\mathcal{C}_n$ by discarding all non-invertible cells and objects.

Remark 13.4.21. The commutative square from proposition 13.4.18 is not vertically left adjointable. Given a categorical spectrum $\mathcal{C}$, corresponding to a sequence of symmetric monoidal ω-categories $\mathcal{C}_n$ and equivalences $\Omega_{\omega\mathrm{Cat}}(\mathcal{C}_{n+1}) = \mathcal{C}_n$ for all $n \geq 0$, it is not true in general that the pointed spaces obtained from $\mathcal{C}_n$ by inverting all objects and morphisms assemble into a spectrum.

For instance, take $\mathcal{C} = \Omega_{\omega\mathrm{Cat}}(\mathrm{EM}(F))$, where F is the free symmetric monoidal category on one object (in other words, F is the category of finite sets and bijections with the disjoint union symmetric monoidal structure). Then $\mathcal{C}_0$ is the terminal symmetric monoidal category, which is already a grouplike commutative algebra object in spaces. Meanwhile, $\mathcal{C}_1 = F$ and the space obtained by inverting all objects in $\mathcal{C}_1$ is the space underlying the sphere spectrum, whose loopspace is nontrivial.

Chapter 14

Higher quasicoherent sheaves

Let X be a prestack. Then one may attach to X a presentable stable category $\mathrm{QCoh}(X)$, defined as the limit over all affine schemes $\mathrm{Spec}(A)$ over X of the category A-mod. We call $\mathrm{QCoh}(X)$ the category of quasicoherent sheaves on X. The goal of this chapter is to introduce higher categorical generalizations of $\mathrm{QCoh}(X)$, and study the functoriality, descent, and affineness properties of the resulting sheaf theories.

We begin in 14.1 by introducing the notion of quasicoherent sheaf of higher categories on affine schemes. We then prove the first main result of this chapter (theorem 14.1.4) which states that the theory $n\mathrm{QCoh}$ of quasicoherent sheaves of n-categories gives rise to a representation of the $(n+1)$-category of correspondences of affine schemes, and these representations are compatible as we change the parameter n: in other words, they give rise to a representation of the categorical spectrum of correspondences of affine schemes. As a consequence, we obtain very strong functoriality properties: the theory $n\mathrm{QCoh}$ on affine schemes has pullbacks and pushforwards which are adjoints to each other to both sides, as soon as $n \geq 2$.

In 14.2 we introduce the notion of quasicoherent sheaf of higher categories on arbitrary prestacks, by right Kan extension from the affine case. We then prove the second main result of this chapter (theorem 14.2.9) which states that the theory $n\mathrm{QCoh}$ gives rise to a representation of the n-category of correspondences of prestacks, and the $(n+1)$-category of correspondences of prestacks and affine-schematic morphisms. These representations are once again compatible as we change the parameter n, so they give rise to a representation of the categorical spectrum of correspondences of prestacks. We use this theorem to show that the theory $n\mathrm{QCoh}$ on prestacks has pullbacks and pushforwards which are adjoints to each other to both sides as soon as $n \geq 3$, and in the case $n \geq 2$ it comes equipped with a strictly symmetric monoidal structure.

In 14.3 we study the class of morphisms for which the pullback (resp. pushforward) functor on $n\mathrm{QCoh}$ is comonadic (resp. monadic). In the case of pullbacks, we show that comonadicity reduces to a descent statement. On the other hand, monadicity of pushforwards reduces to an affineness assertion, as in [Gai15]. We prove here two fundamental results which allow one to reduce descent and affineness questions to the case $n = 2$. In particular, we conclude that the theory $n\mathrm{QCoh}$ satisfies étale descent, and that prestacks with quasicompact, quasiseparated schematic diagonal are n-affine for all $n \geq 2$.

14.1 Quasicoherent sheaves of n-categories on affine schemes

We begin by reinterpreting the higher module categories from definition 12.2.5 from the point of view of spectral algebraic geometry.

Notation 14.1.1. Let SchAff be the opposite of the category of connective commutative ring spectra. We call its objects affine schemes. For each commutative ring spectrum A we denote by $\mathrm{Spec}(A)$ the corresponding affine scheme. For each $n \geq 0$ we denote by $n\mathrm{QCoh}$ the composite symmetric monoidal functor

$$\mathrm{SchAff}^{\mathrm{op}} = \mathrm{CAlg}(\mathrm{Sp})_{\mathrm{cn}} \xrightarrow{\text{-mod}^n} n\mathrm{Pr}^L_{\mathrm{St}}\,.$$

For each morphism of affine schemes $f : X \to Y$ we denote by $f^* : n\mathrm{QCoh}(Y) \to n\mathrm{QCoh}(X)$ the induced pullback functor.

In the case $n = 0$ we will sometimes use the notation $\mathcal{O}(X)$ instead of $0\mathrm{QCoh}(X)$. In the case $n = 1$ we will use the notation $\mathrm{QCoh}(X)$ instead of $1\mathrm{QCoh}(X)$.

Definition 14.1.2. *Let X be an affine scheme. For each $n \geq 2$ we call $n\mathrm{QCoh}(X)$ the presentable stable n-category of quasicoherent sheaves of $(n-1)$-categories on X.*

We now state our main result concerning the functoriality of the theory of higher quasicoherent sheaves on affine schemes.

Notation 14.1.3. Let $\mathcal{C}$ be a category admitting pullbacks and let $n \geq 2$. We denote by $(\iota^n_{\mathcal{C}})^R$ the composite inclusion

$$\mathcal{C}^{\mathrm{op}} \xrightarrow{\iota^R_{\mathcal{C}}} 2\mathrm{Corr}(\mathcal{C}) \xrightarrow{\iota^{2,n}_{\mathcal{C}}} n\mathrm{Corr}(\mathcal{C}).$$

Theorem 14.1.4. *Let $n \geq 1$. There exists a unique extension of the symmetric monoidal functor $n\mathrm{QCoh}$ from notation 14.1.1 along the inclusion $(\iota^{n+1}_{\mathrm{SchAff}})^R$ to a symmetric monoidal functor*

$$n\mathrm{QCoh}_{(n+1)\mathrm{Corr}(\mathrm{SchAff})} : (n+1)\mathrm{Corr}(\mathrm{SchAff}) \to (n\mathscr{P}r^L_{\mathrm{St}})^{(n+1)\text{-op}}.$$

Furthermore, for each $n \geq 2$ the square of ω-categories

$$\begin{array}{ccc}
n\mathrm{Corr}(\mathrm{SchAff}) & \xrightarrow{(n-1)\mathrm{QCoh}_{n\mathrm{Corr}(\mathrm{SchAff})}} & ((n-1)\mathscr{P}r^L_{\mathrm{St}})^{n\text{-op}} \\
\downarrow{=} & & \downarrow{=} \\
\mathrm{End}_{(n+1)\mathrm{Corr}(\mathrm{SchAff})}(\mathrm{Spec}(\mathbb{S})) & \xrightarrow{(n\mathrm{QCoh}_{(n+1)\mathrm{Corr}(\mathrm{SchAff})})_*} & \mathrm{End}_{(n\mathscr{P}r^L_{\mathrm{St}})^{(n+1)\text{-op}}}((n-1)\mathrm{Pr}^L_{\mathrm{St}})
\end{array}$$

commutes.

Before giving a proof of theorem 14.1.4, we study a few consequences.

Corollary 14.1.5. *Let $f : X \to Y$ be a morphism of affine schemes and let $n \geq 2$. Then the morphism $f^* : n\mathrm{QCoh}(Y) \to n\mathrm{QCoh}(X)$ admits both right and left adjoints, and these are equivalent.*

Proof. Combine theorem 14.1.4 with proposition 11.1.9. $\square$

Remark 14.1.6. Let $n \geq 1$ and let $\mathcal{M}$ be a commutative algebra object in $\mathrm{Spc}\text{-}\mathrm{mod}^n$. Let $F : \mathcal{C} \to \mathcal{D}$ be a morphism of $\mathcal{M}$-modules. Then F is right adjointable in $\psi_{n+1}(\mathcal{M}\text{-}\mathrm{mod}(\mathrm{Spc}\text{-}\mathrm{mod}^n))$ if and only if the functor of categories underlying F admits a colimit preserving right adjoint $G : \mathcal{D} \to \mathcal{C}$, and the canonical structure of lax morphism of $\mathcal{M}$-modules on G is strict. In this case, the morphism of $\mathcal{M}$-modules G is right adjoint to F in $\psi_{n+1}(\mathcal{M}\text{-}\mathrm{mod}(\mathrm{Spc}\text{-}\mathrm{mod}^n))$. Furthermore, a commutative square

$$\begin{tikzcd} \mathcal{C}' \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ \mathcal{D}' \arrow[r] & \mathcal{D} \end{tikzcd}$$

in $\psi_{n+1}(\mathcal{M}\text{-}\mathrm{mod}(n\mathrm{Pr}^L))$ is vertically right adjointable if and only if the vertical arrows admit right adjoints, and the underlying commutative square of categories is vertically right adjointable.

It follows from the above that there is no ambiguity in the statement of 14.1.5 as to where the right and left adjoints are considered (that is, as morphisms in $n\mathrm{Pr}^L_{\mathrm{St}}$, or as functors of categories). Furthermore, the base change properties that are guaranteed to hold by theorem 14.1.4 also hold at the level of functors of categories.

Notation 14.1.7. Let $f : X \to Y$ be a morphism of affine schemes and let $n \geq 2$. We will usually denote by $f_* : n\mathrm{QCoh}(X) \to n\mathrm{QCoh}(Y)$ the (right and left) adjoint to f^*. We call this the functor of pushforward along f. In the case when $Y = \mathrm{Spec}(\mathbb{S})$, this recovers a morphism

$$\Gamma(X, -) : n\mathrm{QCoh}(X) \to (n-1)\mathrm{Pr}^L_{\mathrm{St}}$$

which we call the global sections functor for X.

Corollary 14.1.8. *The sequence of functors $n\mathrm{QCoh}_{(n+1)\mathrm{Corr}(\mathrm{SchAff})}$ may be assembled into a morphism of categorical spectra*

$$\underline{\mathcal{O}} : \underline{\mathrm{Corr}}(\mathrm{SchAff}) \to \underline{\mathrm{Sp}}^{1\text{-}\mathrm{op}}.$$

Proof. This is a restatement of the compatibility between the functors for different values of n that appears in theorem 14.1.4. $\square$

Remark 14.1.9. Let $n \geq 1$. Then the morphism of categorical spectra $\underline{\mathcal{O}}$ induces a symmetric monoidal functor

$$(n+1)\underline{\mathcal{O}} : (n+1)\mathrm{Corr}(\mathrm{SchAff}) \to (n\mathscr{P}r^L_{\mathrm{St}})^{n+1\text{-op}}$$

whose underlying functor is $n\mathrm{QCoh}_{(n+1)\mathrm{Corr}(\mathrm{SchAff})}$. By the uniqueness part of theorem 14.1.4, we see that this agrees with the usual symmetric monoidal structure on $n\mathrm{QCoh}_{(n+1)\mathrm{Corr}(\mathrm{SchAff})}$. In particular, we have that the equivalence of functors

$$(n-1)\mathrm{QCoh}_{n\mathrm{Corr}(\mathrm{SchAff})} = \Omega_{\omega\mathrm{Cat}} n\mathrm{QCoh}_{(n+1)\mathrm{Corr}(\mathrm{SchAff})}$$

from theorem 14.1.4 can be upgraded to an equivalence of symmetric monoidal functors.

Our proof of theorem 14.1.4 requires a few lemmas.

Lemma 14.1.10. *Let $\mathcal{M}$ be a symmetric monoidal category with symmetric monoidal structure compatible with colimits, and let*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"] & A' \arrow[d, "g'"] \\ B \arrow[r, "f'"] & B' \end{tikzcd}$$

be a pushout square of commutative algebras in $\mathcal{M}$. Then the induced commutative square of categories

$$\begin{tikzcd} A\text{-mod} \arrow[r, "f_*"] \arrow[d, "g_*"] & A'\text{-mod} \arrow[d, "g'_*"] \\ B\text{-mod} \arrow[r, "f'_*"] & B'\text{-mod} \end{tikzcd}$$

is horizontally right adjointable.

Proof. Denote by $\mathrm{CAlgMod}(\mathcal{M})$ the symmetric monoidal category whose objects are pairs (A, M) of a commutative algebra A in $\mathcal{M}$ and an A-module M. This comes equipped with a symmetric monoidal functor $p : \mathrm{CAlgMod}(\mathcal{M}) \to \mathrm{CAlg}(\mathcal{M})$ which is a cocartesian fibration of operads, and a cartesian fibration. There is moreover a symmetric monoidal forgetful functor $q : \mathrm{CAlgMod}(\mathcal{M}) \to \mathcal{M}$ (taking the underlying module), and an arrow in $\mathrm{CAlgMod}(\mathcal{M})$ is p-cartesian if and only if its image under q is an isomorphism.

We assume first that A is the trivial commutative algebra in $\mathcal{M}$. We interpret the first square in the statement as a functor $C : [1] \times [1] \to \mathrm{CAlg}(\mathcal{M})$. Our goal is to show that the base change of p along C is a bivariant fibration. This is evidently both a cocartesian and cartesian fibration. Assume now given a p-cartesian lift

$$\overline{f} : (A, M) \to (A', M')$$

for f. Tensoring with the p-cocartesian morphism $(A, A) \to (B, B)$ yields a commutative square

$$\begin{tikzcd} (A, M) \arrow[r, "\overline{f}"] \arrow[d, "\overline{g}"] & (A', M') \arrow[d] \\ (B, N) \arrow[r] & (A' \otimes B, N') \end{tikzcd}$$

where the horizontal arrows are p-cartesian, and the vertical arrows are p-cocartesian. The projection of the above square under p recovers the commutative square of commutative algebras

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"] & A' \arrow[d] \\ B \arrow[r] & A' \otimes B \end{tikzcd}$$

where the bottom horizontal arrow is obtained by tensoring f with B, and the right vertical arrow is obtained by tensoring g with A'. Since the symmetric monoidal structure on $\mathrm{CAlg}(\mathcal{M})$ is cocartesian, we see that the above square is equivalent to our original square C. We conclude that there exists a lift of C with p-cartesian horizontal arrows and p-cocartesian vertical arrows, whose top-right entry is (A', M'). Since (A', M') is arbitrary, we conclude that the base change of p along C is a two-sided fibration. By symmetry, we have that this is in fact a bivariant fibration, as desired.

Assume now that A is arbitrary. Recall that there is an equivalence $\mathrm{CAlg}(\mathcal{M})_{A/} = \mathrm{CAlg}(A\text{-mod}(\mathcal{M}))$. Base change along this equivalence sends $\mathrm{CAlgMod}(A\text{-mod}(\mathcal{M}))$ to $\mathrm{CAlgMod}(\mathcal{M}) \times_{\mathrm{CAlg}(\mathcal{M})} \mathrm{CAlg}(\mathcal{M})_{A/}$. We can thus interpret the first square in the statement as a pushout square in $\mathrm{CAlg}(A\text{-mod}(\mathcal{M}))$, and the second square is in turn equivalent to the commutative square of categories

$$\begin{tikzcd} A\text{-mod}(A\text{-mod}(\mathcal{M})) \arrow[r, "f_*"] \arrow[d, "g_*"] & A'\text{-mod}(A\text{-mod}(\mathcal{M})) \arrow[d, "g'_*"] \\ B\text{-mod}(A\text{-mod}(\mathcal{M})) \arrow[r, "f'_*"] & B'\text{-mod}(A\text{-mod}(\mathcal{M})). \end{tikzcd}$$

Our lemma now follows from the fact that A is the trivial commutative algebra in $A\text{-mod}(\mathcal{M})$. $\square$

Lemma 14.1.11. *Let $\mathcal{M}$ be a symmetric monoidal category with symmetric monoidal structure compatible with colimits , and let*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"] & A' \arrow[d, "g'"] \\ B \arrow[r, "f'"] & B' \end{tikzcd}$$

be a pushout square of commutative algebras in $\mathcal{M}$. Then the induced commutative square

$$\begin{tikzcd} A\text{-mod} \arrow[r, "f_*"] \arrow[d, "g_*"] & A'\text{-mod} \arrow[d, "g'_*"] \\ B\text{-mod} \arrow[r, "f'_*"] & B'\text{-mod} \end{tikzcd}$$

of $\mathcal{M}$-commutative algebras in $\widehat{\mathrm{Cat}}_{\mathrm{cocompl}}$, is a pushout square.

Proof. Since the category of $\mathcal{M}$-commutative algebras in $\widehat{\mathrm{Cat}}_{\mathrm{cocompl}}$ is equivalent to the undercategory $\mathrm{CAlg}(\widehat{\mathrm{Cat}}_{\mathrm{cocompl}})_{\mathcal{M}/}$, the second square in the statement is a pushout of $\mathcal{M}$-commutative algebras if and only if it is a pushout square of commutative algebras in $\widehat{\mathrm{Cat}}_{\mathrm{cocompl}}$. As in the proof of lemma 14.1.10, replacing $\mathcal{M}$ with $A\text{-mod}(\mathcal{M})$ if necessary, we may assume that A is the trivial commutative algebra in $\mathcal{M}$.

We can think about the second square in the statement as arising by applying the symmetric monoidal functor

$$\text{-mod} : \mathrm{CAlg}(\mathcal{M}) \to \mathcal{M}\text{-mod}(\widehat{\mathrm{Cat}}_{\mathrm{cocompl}})$$

to the first square. In other words, it can be obtained by applying the (symmetric monoidal) functor

$$\mathrm{CAlg}(\text{-mod}) : \mathrm{CAlg}(\mathcal{M}) = \mathrm{CAlg}(\mathrm{CAlg}(\mathcal{M})) \to \mathrm{CAlg}(\mathcal{M}\text{-mod}(\widehat{\mathrm{Cat}}_{\mathrm{cocompl}})).$$

Our claim now follows from the fact that the symmetric monoidal structures above are cocartesian, and therefore CAlg(-mod) preserves finite coproducts. □

Lemma 14.1.12. *Let $n \geq 1$. Then the functor $n\mathrm{QCoh}^{\mathrm{op}} : \mathrm{SchAff} \to (n\mathscr{P}r^L_{\mathrm{St}})^{1\text{-op}}$ satisfies the right Beck-Chevalley condition.*

Proof. Since the inclusion $\mathrm{SchAff}^{\mathrm{op}} = \mathrm{CAlg}_{\mathrm{cn}}(\mathrm{Sp}) \to \mathrm{CAlg}(\mathrm{Sp})$ preserves colimits, we see that it suffices to show that the functor

$$\text{-mod}^n : \mathrm{CAlg}(\mathrm{Sp}) \to n\mathrm{Pr}^L_{\mathrm{St}}$$

maps pushout squares of commutative algebras to squares which are right adjointable in $n\mathscr{P}r^L_{\mathrm{St}}$. By remark 14.1.6, it suffices to check that the composition of -mod^n with the forgetful functor $n\mathrm{Pr}^L_{\mathrm{St}} \to \widehat{\mathrm{Cat}}$ maps pushout squares of commutative algebras to right adjointable squares of categories. We can factor this as the composition

$$\mathrm{CAlg}(\mathrm{Sp}) \xrightarrow{\text{-mod}^{n-1}} \mathrm{CAlg}((n-1)\mathrm{Pr}^L_{\mathrm{St}}) \xrightarrow{\text{-mod}} \widehat{\mathrm{Cat}}.$$

The second functor sends pushout squares to right adjointable squares of categories by virtue of lemma 14.1.10. Our lemma is now a consequence of the fact that the first functor in the above composition preserves pushouts, which follows inductively from lemma 14.1.11. □

Construction 14.1.13. Let $\mathcal{C}$ be a category and $\mathcal{D}$ be a 2-category. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor such that $F(\alpha)$ is right adjointable for every arrow α in $\mathcal{C}$. Let c, c' be a pair of objects of $\mathcal{C}$. Denote by $p : E \to \mathcal{C}$ the cartesian fibration associated to the composite functor

$$\mathcal{C}^{\mathrm{op}} \xrightarrow{F^{1\text{-op}}} \mathcal{D}^{1\text{-op}} \xrightarrow{\mathrm{Hom}_{\mathcal{D}}(-,F(c'))} \mathscr{C}at.$$

Let s be the unique cartesian section of $p|_{\mathcal{C}_{/c'}}$ such that $s(c') = \mathrm{id}_{F(c')}$. Observe that the projection $E \times_{\mathcal{C}} \mathcal{C}_{/c} \to \mathcal{C}_{/c}$ is a cocartesian fibration, and hence the inclusion of the fiber E_c over the final object c admits a left adjoint $q : E \times_{\mathcal{C}} \mathcal{C}_{/c} \to E_c$. We denote by $F_{c,c'}$ the composite functor

$$\mathcal{C}_{/c,c'} \xrightarrow{s|_{\mathcal{C}_{/c,c'}}} E \times_{\mathcal{C}} \mathcal{C}_{/c,c'} \to E \times_{\mathcal{C}} \mathcal{C}_{/c} \xrightarrow{q} E_c = \mathrm{Hom}_{\mathcal{D}}(F(c), F(c')).$$

where the middle arrow is the canonical projection.

Example 14.1.14. Let $\mathcal{D}$ be a 2-category, and let

$$\begin{array}{ccc} d' & \longrightarrow & d \\ \downarrow & & \downarrow \\ e' & \longrightarrow & e \end{array}$$

be a commutative square in $\mathcal{D}$, such that all arrows admit right adjoints. We can think about the above as a functor $F : [1] \times [1] \to \mathcal{D}$. The functor

$$F_{(1,1),(1,1)} : [1] \times [1] \to \mathrm{End}_{\mathcal{D}}(e)$$

defines a commutative square in the category $\mathrm{End}_{\mathcal{D}}(e)$, which agrees with the commutative square from construction 11.2.1.

Lemma 14.1.15. *Let $\mathcal{C}$ be a category admitting pullbacks. Let $\mathcal{D}$ be a 2-category and let $F : 2\mathrm{Corr}(\mathcal{C}) \to \mathcal{D}$ be a functor. Let c, c' be objects of $\mathcal{C}$. Then the functor*

$$(F|_{\mathcal{C}})_{c,c'} : \mathcal{C}_{/c,c'} \to \mathrm{Hom}_{\mathcal{D}}(F(c), F(c'))$$

is equivalent to the composite functor

$$\mathcal{C}_{/c,c'} = \mathrm{Hom}_{2\mathrm{Corr}(\mathcal{C})}(c, c') \xrightarrow{F_*} \mathrm{Hom}_{\mathcal{D}}(F(c), F(c')).$$

Proof. We have natural transformations of functors $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Cat}$

$$\mathrm{Hom}_{\mathcal{C}}(-,-) \xrightarrow{(\iota_{\mathcal{C}})_*} \mathrm{Hom}_{2\mathrm{Corr}(\mathcal{C})}(-,-)|_{\mathcal{C}^{\mathrm{op}}\times\mathcal{C}} \xrightarrow{(F_*)|_{\mathcal{C}^{\mathrm{op}}\times\mathcal{C}}} \mathrm{Hom}_{\mathcal{D}}(F(-), F(-))|_{\mathcal{C}^{\mathrm{op}}\times\mathcal{C}}.$$

Applying the two-sided Grothendieck construction

$$\int_{\mathcal{C}^{\mathrm{op}}\times\mathcal{C}} : \mathrm{Funct}(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}, \mathrm{Cat}) \to \mathrm{Cat}_{/\mathcal{C}\times\mathcal{C}}$$

we obtain morphisms

$$\int_{\mathcal{C}^{\mathrm{op}}\times\mathcal{C}} \mathrm{Hom}_{\mathcal{C}}(-,-) \to \int_{\mathcal{C}^{\mathrm{op}}\times\mathcal{C}} \mathrm{Hom}_{2\mathrm{Corr}(\mathcal{C})}(-,-)|_{\mathcal{C}^{\mathrm{op}}\times\mathcal{C}} \to \int_{\mathcal{C}^{\mathrm{op}}\times\mathcal{C}} \mathrm{Hom}_{\mathcal{D}}(F(-),F(-))|_{\mathcal{C}^{\mathrm{op}}\times\mathcal{C}}$$

of two-sided fibrations over $\mathcal{C}\times\mathcal{C}$.

Let Λ^2_0 be the category with objects $0,1,2$ and nontrivial morphisms $1 \leftarrow 0 \rightarrow 2$. Thanks to lemma 10.4.17, the first morphism above is equivalent to the morphism

$$\begin{tikzcd} \mathrm{Funct}([1],\mathcal{C}) \arrow[rr, "\phi'"] \arrow[dr, "(\mathrm{ev}_0,\mathrm{ev}_1)"'] & & \mathrm{Funct}(\Lambda^2_0,\mathcal{C}) \arrow[dl, "(\mathrm{ev}_1,\mathrm{ev}_2)"] \\ & \mathcal{C}\times\mathcal{C} & \end{tikzcd}$$

where ϕ' is the functor of precomposition with the map $\Lambda^2_0 \to [1]$ which sends $0,1,2$ to $0,0,1$, respectively.

We now base change the above series of maps along the functor

$$\mathrm{id}_{\mathcal{C}} \times \{c'\} : \mathcal{C} \to \mathcal{C}\times\mathcal{C}.$$

to obtain morphisms

$$\int_{\mathcal{C}^{\mathrm{op}}} \mathrm{Hom}_{\mathcal{C}}(-,c') \to \int_{\mathcal{C}^{\mathrm{op}}} \mathrm{Hom}_{2\mathrm{Corr}(\mathcal{C})}(-,c')|_{\mathcal{C}^{\mathrm{op}}} \to \int_{\mathcal{C}^{\mathrm{op}}} \mathrm{Hom}_{\mathcal{D}}(F(-),F(c'))|_{\mathcal{C}^{\mathrm{op}}}$$

of cartesian fibrations over $\mathcal{C}$. It follows from lemma 10.4.14 that the second and third fibrations are in fact also cocartesian fibrations, and the morphism between them is also a morphism of cocartesian fibrations. Furthermore, we may identify the first morphism above as the map

$$\xi : \mathcal{C}_{/c'} \to \mathrm{Funct}(\Lambda^2_0,\mathcal{C})|_{\mathcal{C}\times\{c'\}}$$

induced by base change from ϕ'.

Denote by $p : E \to \mathcal{C}$ the cartesian-cocartesian fibration $\int_{\mathcal{C}^{\mathrm{op}}} \mathrm{Hom}_{\mathcal{D}}(F(-),F(c'))|_{\mathcal{C}^{\mathrm{op}}}$, and by $G : \mathrm{Funct}(\Lambda^2_0,\mathcal{C})|_{\mathcal{C}\times\{c'\}} \to E$ the induced morphism of cartesian and cocartesian fibrations over $\mathcal{C}$.

Note that the diagonal map $\Delta : \mathcal{C}_{/c'} \to \mathcal{C}_{/c'} \times_{\mathcal{C}} \mathcal{C}_{/c'}$ is a cartesian section, since the projection $\mathcal{C}_{/c'} \to \mathcal{C}$ is a right fibration. Its image under (the base change of) the morphism $G\xi$ yields a cartesian section s for $p|_{\mathcal{C}_{/c'}}$ whose value on c' recovers $\mathrm{id}_{F(c')}$.

Consider now the composite functor

$$J : \mathcal{C}_{/c,c'} \xrightarrow{\Delta|_{\mathcal{C}_{/c,c'}}} \mathcal{C}_{/c,c'} \times_{\mathcal{C}} \mathcal{C}_{/c'} \to \mathcal{C}_{/c} \times_{\mathcal{C}} \mathcal{C}_{/c'} \xrightarrow{\xi|_{\mathcal{C}_{/c}}} \mathcal{C}_{/c} \times_{\mathcal{C}} \mathrm{Funct}(\Lambda^2_0,\mathcal{C})|_{\mathcal{C}\times\{c'\}}$$

where the middle arrow is the canonical projection. Observe that the right side category fits into a pullback diagram

$$\begin{tikzcd} \mathcal{C}_{/c} \times_{\mathcal{C}} \mathrm{Funct}(\Lambda^2_0,\mathcal{C})|_{\mathcal{C}\times\{c'\}} \arrow[r] \arrow[d] & \mathrm{Funct}(\mathcal{I},\mathcal{C}) \arrow[d, "(\mathrm{ev}_3,\mathrm{ev}_2)"] \\ {[0]} \arrow[r, "(c,c')"] & \mathcal{C}\times\mathcal{C} \end{tikzcd}$$

where $\mathcal{I}$ is the category with objects $0, 1, 2, 3$ and morphisms generated by arrows

$$3 \leftarrow 1 \leftarrow 0 \rightarrow 2.$$

The functor J is then induced from the functor $\rho : \mathcal{I} \to \Lambda^2_0$ which maps $0, 1, 2, 3$ to $0, 0, 2, 1$, respectively.

Note that there is a unique natural transformation $\eta : \rho \to \rho'$ where $\rho' : \mathcal{I} \to \Lambda^2_0$ which maps $0, 1, 2, 3$ to $0, 1, 2, 1$. This induces a natural transformation $\mu : J \to J'$ of functors $\mathcal{C}_{/c,c'} \to \mathcal{C}_{/c} \times_{\mathcal{C}} \mathrm{Funct}(\Lambda^2_0, \mathcal{C})|_{\mathcal{C}\times\{c'\}}$. Concretely, for each span $c \xleftarrow{\alpha} s \xrightarrow{\beta} y$ in $\mathcal{C}_{/c,c'}$, we have

$$J(s) = (s, s \xleftarrow{\mathrm{id}_s} s \xrightarrow{\beta} y)$$

and

$$J'(s) = (c, c \xleftarrow{\alpha} s \xrightarrow{\beta} y),$$

and the morphism $\mu(s)$ has components α and

$$\begin{tikzcd}
 & s \arrow[dl, "\mathrm{id}_s"'] \arrow[d, "\mathrm{id}_s"] \arrow[dr, "\beta"] & \\
s \arrow[d, "\alpha"] & s \arrow[dl, "\mathrm{id}_s"] \arrow[dr, "\beta"] & y \arrow[d, "\mathrm{id}_y"] \\
c & & y.
\end{tikzcd}$$

It follows that $\mu(s)$ is a cocartesian arrow for the projection $\mathcal{C}_{/c} \times_{\mathcal{C}} \mathrm{Funct}(\Lambda^2_0, \mathcal{C})$. Furthermore, observe that the functor J' is equivalent to the canonical inclusion of $\mathcal{C}_{/c,c'}$ as the fiber of $\mathcal{C}_{/c} \times_{\mathcal{C}} \mathrm{Funct}(\Lambda^2_0, \mathcal{C})|_{\mathcal{C}\times\{c'\}}$ over c.

Observe that the composite functor

$$\mathcal{C}_{/c,c'} \xrightarrow{J} \mathcal{C}_{/c} \times_{\mathcal{C}} \mathrm{Funct}(\Lambda^2_0, \mathcal{C})|_{\mathcal{C}\times\{c'\}} \xrightarrow{G|_{\mathcal{C}_{/c}}} \mathcal{C}_{/c} \times_{\mathcal{C}} E$$

is equivalent to the composite functor

$$\mathcal{C}_{/c,c'} \xrightarrow{s|_{\mathcal{C}_{c,c'}}} \mathcal{C}_{/c,c'} \times_{\mathcal{C}} E \to \mathcal{C}_{/c} \times_{\mathcal{C}} E$$

where the second arrow is the canonical projection. Since G is a morphism of cocartesian fibrations over $\mathcal{C}$, we see that $G|_{\mathcal{C}_{/c}}\mu$ presents GJ' as the composition of $G|_{\mathcal{C}_{/c}}J$ under the localization $\mathcal{C}_{/c} \times_{\mathcal{C}} E \to E_c$. Unwinding the definition, we see that the corestriction of GJ' to the fiber E_c over c is equivalent to $(F|_{\mathcal{C}})_{c,c'}$. The lemma now follows from the observation that GJ' is equivalent the functor $F_* : \mathcal{C}_{/c,c'} \to \mathrm{Hom}_{\mathcal{D}}(c, c')$ induced by F. □

Notation 14.1.16. Let $n \geq 1$. We denote by

$$n\mathrm{QCoh}'_{2\mathrm{Corr}(\mathrm{SchAff})} : 2\mathrm{Corr}(\mathrm{SchAff}) \to (n\mathscr{P}r^L_{\mathrm{St}})^{1\text{-op},2\text{-op}}$$

the unique extension of $n\mathrm{QCoh}^{\mathrm{op}}$ along $\iota^2_{\mathrm{SchAff}}$ (which exists thanks to a combination of lemma 14.1.12 and theorem 11.2.6).

Lemma 14.1.17. *Let $n \geq 1$. The functor*

$$\mathrm{SchAff} = \mathrm{End}_{2\mathrm{Corr}(\mathrm{SchAff})}(\mathrm{Spec}(\mathbb{S})) \to \mathrm{End}_{(n\mathscr{P}r^L_{\mathrm{St}})^{1\text{-op},2\text{-op}}}((n-1)\mathrm{Pr}^L_{\mathrm{St}}) = ((n-1)\mathscr{P}r^L_{\mathrm{St}})^{1\text{-op}}$$

induced from $n\mathrm{QCoh}'_{2\mathrm{Corr}(\mathrm{SchAff})}$ is equivalent to $(n-1)\mathrm{QCoh}^{\mathrm{op}}$.

Proof. Thanks to lemma 14.1.15, we have that the functor in the statement is equivalent to the functor

$$(n\mathrm{QCoh}^{\mathrm{op}})_{\mathrm{Spec}(\mathbb{S}),\mathrm{Spec}(\mathbb{S})} : \mathrm{SchAff} \to \mathrm{End}_{(n\mathscr{P}r^L_{\mathrm{St}})^{1\text{-op},2\text{-op}}}((n-1)\mathrm{Pr}^L_{\mathrm{St}}) = ((n-1)\mathscr{P}r^L_{\mathrm{St}})^{1\text{-op}}.$$

To compute this, we need to understand the cartesian fibration associated to the composite functor

$$\mathrm{SchAff}^{\mathrm{op}} \xrightarrow{n\mathrm{QCoh}} ((n\mathscr{P}r^L_{\mathrm{St}})^{2\text{-op}})^{\leq 2} \xrightarrow{\mathrm{Hom}_{((n\mathscr{P}r^L_{\mathrm{St}})^{1\text{-op},2\text{-op}})^{\leq 2}}(-,(n-1)\mathrm{Pr}^L_{\mathrm{St}}))} \widehat{\mathscr{C}at}.$$

The above is induced by the composite functor

$$\mathrm{SchAff}^{\mathrm{op}} \xrightarrow{n\mathrm{QCoh}} n\mathrm{Pr}^L_{\mathrm{St}} \xrightarrow{\mathscr{H}om_{n\mathrm{Pr}^L_{\mathrm{St}}}((n-1)\mathrm{Pr}^L_{\mathrm{St}},-)} n\mathrm{Pr}^L_{\mathrm{St}} \to \widehat{\mathrm{Cat}} \xrightarrow{(-)^{\mathrm{op}}} \widehat{\mathrm{Cat}}$$

where the second to last arrow is the forgetful functor. The cartesian fibration associated to the above is the opposite of the cocartesian fibration $p : E \to \mathrm{CAlg}_{\mathrm{cn}}(\mathrm{Sp})$ associated to the functor $\text{-mod}^n : \mathrm{CAlg}_{\mathrm{cn}}(\mathrm{Sp}) \xrightarrow{\text{-mod}^n} \widehat{\mathrm{Cat}}$. The functor $(n\mathrm{QCoh}^{\mathrm{op}})_{\mathrm{Spec}(\mathbb{S}),\mathrm{Spec}(\mathbb{S})}$ is thus the opposite of the composite functor

$$\mathrm{CAlg}_{\mathrm{cn}}(\mathrm{Sp}) \to E \to (n-1)\mathrm{Pr}^L_{\mathrm{St}}$$

where the first arrow is the unique cocartesian section that maps $\mathbb{S}$ to $(n-2)\mathrm{Pr}^L_{\mathrm{St}}$, and the second arrow is the right adjoint to the inclusion of the fiber over $\mathbb{S}$. The lemma now follows from the fact that the above composite map is equivalent to $\text{-mod}^{n-1} : \mathrm{CAlg}_{\mathrm{cn}}(\mathrm{Sp}) \to (n-1)\mathrm{Pr}^L_{\mathrm{St}}$. □

Notation 14.1.18. Let $\mathcal{D}$ be an ω-category. For each set S of positive integers we denote by $\mathcal{D}^{S\text{-op}}$ the ω-category obtained from $\mathcal{D}$ by inverting the direction of all cells whose dimension belongs to S. If $i \leq j$ are positive integers, we denote by $[i,j]$ the set of all integers k such that $i \leq k \leq j$, so that $\mathcal{D}^{[i,j]\text{-op}}$ is the ω-category obtained from $\mathcal{D}$ by inverting all cells of dimension between i and j.

Lemma 14.1.19. *Let $n \geq 1$. Then the functor*

$$n\mathrm{QCoh}^{\mathrm{op}} : \mathrm{SchAff} \to (n\mathscr{P}r^L_{\mathrm{St}})^{[1,n+1]\text{-op}}$$

satisfies the left n-fold Beck-Chevalley condition.

Proof. We argue by induction on n. When $n = 1$ this is the content of lemma 14.1.12, so we assume $n > 1$. By lemma 11.2.9, it suffices to show that for each pair of affine schemes X, Y the functor

$$\mathrm{Hom}_{2\mathrm{Corr}(\mathrm{SchAff})}(X, Y) \to \mathrm{Hom}_{(n\mathscr{Pr}^L_{\mathrm{St}})^{[1,n+1]\text{-op}}}(n\mathrm{QCoh}(X), n\mathrm{QCoh}(Y))$$

induced by $n\mathrm{QCoh}'_{2\mathrm{Corr}(\mathrm{SchAff})}$, satisfies the left $(n-1)$-fold Beck-Chevalley condition.

Let $\pi_Y : Y \to \mathrm{Spec}(\mathbb{S})$ be the projection. The morphism $\iota_{\mathrm{SchAff}}(\pi_Y)$ induces a commutative square of n-categories

$$\begin{array}{ccc} \mathrm{Hom}_{2\mathrm{Corr}(\mathrm{SchAff})}(X, Y) & \longrightarrow & \mathrm{Hom}_{(n\mathscr{Pr}^L_{\mathrm{St}})^{[1,n+1]\text{-op}}}(n\mathrm{QCoh}(X), n\mathrm{QCoh}(Y)) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{2\mathrm{Corr}(\mathrm{SchAff})}(X, \mathrm{Spec}(\mathbb{S})) & \longrightarrow & \mathrm{Hom}_{(n\mathscr{Pr}^L_{\mathrm{St}})^{[1,n+1]\text{-op}}}(n\mathrm{QCoh}(X), n\mathrm{QCoh}(\mathrm{Spec}(\mathbb{S}))). \end{array}$$

Here the left vertical arrow is the canonical projection, the horizontal arrows are induced by $n\mathrm{QCoh}'_{2\mathrm{Corr}(\mathrm{SchAff})}$. The right vertical arrow is equivalent to the functor

$$\mathrm{Hom}_{n\mathscr{Pr}^L_{\mathrm{St}}}(n\mathrm{QCoh}(Y), n\mathrm{QCoh}(X))^{[1,n]\text{-op}} \to \mathrm{Hom}_{n\mathscr{Pr}^L_{\mathrm{St}}}(n\mathrm{QCoh}(\mathrm{Spec}(\mathbb{S})), n\mathrm{QCoh}(X))^{[1,n]\text{-op}}$$

induced by precomposition with $n\mathrm{QCoh}(\pi_Y)$. This functor is the image under the composite functor

$$n\mathrm{Pr}^L_{\mathrm{St}} \xrightarrow{\psi_n} n\mathrm{Cat} \xrightarrow{(-)^{[1,n]\text{-op}}} n\mathrm{Cat}$$

of the morphism $n\mathrm{QCoh}(\pi_Y)^*$ of presentable stable n-categories

$$\mathscr{H}om_{n\mathrm{Pr}^L_{\mathrm{St}}}(n\mathrm{QCoh}(Y), n\mathrm{QCoh}(X)) \to \mathscr{H}om_{n\mathrm{Pr}^L_{\mathrm{St}}}(n\mathrm{QCoh}(\mathrm{Spec}(\mathbb{S})), n\mathrm{QCoh}(X))$$

induced by precomposition with $n\mathrm{QCoh}(\pi_Y)$.

Observe that the objects $n\mathrm{QCoh}(Y)$ and $n\mathrm{QCoh}(\mathrm{Spec}(\mathbb{S}))$ in $n\,\mathrm{Pr}^L_{\mathrm{St}} = (n-1)\,\mathrm{Pr}^L_{\mathrm{St}}\text{-mod}_{\mathrm{pr}}$ are modules over commutative algebra objects in $(n-1)\,\mathrm{Pr}^L_{\mathrm{St}}$. By a combination of [Lur17] remark 4.8.4.8 and proposition 4.6.2.19, we see that $n\mathrm{QCoh}(Y)$ and $n\mathrm{QCoh}(\mathrm{Spec}(\mathbb{S}))$ are self dual objects of the symmetric monoidal category $n\,\mathrm{Pr}^L_{\mathrm{St}}$. Hence we see that $n\mathrm{QCoh}(\pi_Y)^*$ is given by tensoring with $n\mathrm{QCoh}(X)$ the morphism $n\mathrm{QCoh}(Y) \to n\mathrm{QCoh}(\mathrm{Spec}(\mathbb{S}))$ dual to $n\mathrm{QCoh}(\pi_Y)$. Since $n\mathrm{QCoh}(X)$ is also self dual and the functor $n\mathrm{QCoh}$ is symmetric monoidal, we see that $n\mathrm{QCoh}(\pi_Y)^*$ is equivalent to the dual of the morphism

$$n\mathrm{QCoh}(\pi_Y \times \mathrm{id}_X) : n\mathrm{QCoh}(X) \to n\mathrm{QCoh}(Y \times X).$$

It follows that $n\mathrm{QCoh}(\pi_Y)^*$ is the morphism of restriction of scalars along the morphism of commutative algebras

$$(n-1)\mathrm{QCoh}(\pi_Y \times \mathrm{id}_X) : (n-1)\mathrm{QCoh}(X) \to (n-1)\mathrm{QCoh}(Y \times X).$$

It now follows from remark 14.1.6 that a commutative square in the n-category

$$\psi_n(\mathscr{H}om_{n\mathrm{Pr}^L_{\mathrm{St}}}(n\mathrm{QCoh}(Y), n\mathrm{QCoh}(X)))$$

is vertically right adjointable if and only if its image under $\psi_n(n\mathrm{QCoh}(\pi_Y)^*)$ is vertically right adjointable. The same remains true after reversing the directions of all cells. Hence we see that a commutative square in

$$\mathrm{Hom}_{(n\mathscr{P}r^L_{\mathrm{St}})^{[1,n+1]\text{-op}}}(n\mathrm{QCoh}(X), n\mathrm{QCoh}(Y))$$

is vertically right adjointable if and only if its image in

$$\mathrm{Hom}_{(n\mathscr{P}r^L_{\mathrm{St}})^{[1,n+1]\text{-op}}}(n\mathrm{QCoh}(X), n\mathrm{QCoh}(\mathrm{Spec}(\mathbb{S})))$$

is vertically right adjointable. Since the map

$$(\iota_{\mathrm{SchAff}}(\pi_Y))_* : \mathrm{Hom}_{2\mathrm{Corr}(\mathrm{SchAff})}(X, Y) \to \mathrm{Hom}_{2\mathrm{Corr}(\mathrm{SchAff})}(X, \mathrm{Spec}(\mathbb{S}))$$

preserves pullbacks, we see that the functor

$$\mathrm{Hom}_{2\mathrm{Corr}(\mathrm{SchAff})}(X, Y) \to \mathrm{Hom}_{(n\mathscr{P}r^L_{\mathrm{St}})^{[1,n+1]\text{-op}}}(n\mathrm{QCoh}(X), n\mathrm{QCoh}(Y))$$

induced by $n\mathrm{QCoh}'_{2\mathrm{Corr}(\mathrm{SchAff})}$ satisfies the left $(n-1)$-fold Beck-Chevalley condition if the functor

$$\mathrm{Hom}_{2\mathrm{Corr}(\mathrm{SchAff})}(X, \mathrm{Spec}(\mathbb{S})) \to \mathrm{Hom}_{(n\mathscr{P}r^L_{\mathrm{St}})^{[1,n+1]\text{-op}}}(n\mathrm{QCoh}(X), \mathrm{Spec}(\mathbb{S}))$$

satisfies the left $(n-1)$-fold Beck-Chevalley condition. In other words, we may now assume that $Y = \mathrm{Spec}(\mathbb{S})$.

Let $\pi_X : X \to \mathrm{Spec}(\mathbb{S})$ be the projection. The morphism $\iota^R_{\mathrm{SchAff}}(\pi_X)$ induces a commutative square of n-categories as follows:

$$\begin{tikzcd}
\mathrm{Hom}_{2\mathrm{Corr}(\mathrm{SchAff})}(X, \mathrm{Spec}(\mathbb{S})) \arrow[r] \arrow[d] & \mathrm{Hom}_{(n\mathscr{P}r^L_{\mathrm{St}})^{[1,n+1]\text{-op}}}(n\mathrm{QCoh}(X), n\mathrm{QCoh}(\mathrm{Spec}(\mathbb{S}))) \arrow[d] \\
\mathrm{Hom}_{2\mathrm{Corr}(\mathrm{SchAff})}(\mathrm{Spec}(\mathbb{S}), \mathrm{Spec}(\mathbb{S})) \arrow[r] & \mathrm{Hom}_{(n\mathscr{P}r^L_{\mathrm{St}})^{[1,n+1]\text{-op}}}(n\mathrm{QCoh}(\mathrm{Spec}(\mathbb{S})), n\mathrm{QCoh}(\mathrm{Spec}(\mathbb{S})))
\end{tikzcd}$$

Here the left vertical arrow is the canonical projection, the horizontal arrows are induced by $n\mathrm{QCoh}'_{2\mathrm{Corr}(\mathrm{SchAff})}$. The right vertical arrow is equivalent, after passing to opposites of arrows of dimension at most n, to the functor

$$\mathrm{Hom}_{n\mathscr{P}r^L_{\mathrm{St}}}(n\mathrm{QCoh}(\mathrm{Spec}(\mathbb{S})), n\mathrm{QCoh}(X)) \to \mathrm{Hom}_{n\mathscr{P}r^L_{\mathrm{St}}}(n\mathrm{QCoh}(\mathrm{Spec}(\mathbb{S})), n\mathrm{QCoh}(\mathrm{Spec}(\mathbb{S})))$$

induced by composition with the right adjoint to $n\mathrm{QCoh}(\pi_X)$. It follows that the right vertical arrow is the image under the composite functor

$$n\mathrm{Pr}^L_{\mathrm{St}} \xrightarrow{\psi_n} n\mathrm{Cat} \xrightarrow{(-)^{[1,n]\text{-op}}} n\mathrm{Cat}$$

of the morphism of presentable stable n-categories

$$\mathscr{H}om_{n\mathrm{Pr}^L_{\mathrm{St}}}(n\mathrm{QCoh}(\mathrm{Spec}(\mathbb{S})), n\mathrm{QCoh}(X)) \to \mathscr{H}om_{n\mathrm{Pr}^L_{\mathrm{St}}}(n\mathrm{QCoh}(\mathrm{Spec}(\mathbb{S})), n\mathrm{QCoh}(\mathrm{Spec}(\mathbb{S})))$$

induced by composition with the right adjoint to $n\mathrm{QCoh}(\pi_X)$. Since $n\mathrm{QCoh}(\mathrm{Spec}(\mathbb{S}))$ is the unit in $n\mathrm{Pr}^L_{\mathrm{St}}$, the above is in fact equivalent to the right adjoint to $n\mathrm{QCoh}(\pi_X)$. Using remark 14.1.6 we see that the above morphism is given by restriction of scalars along the morphism of commutative algebras $(n-1)\mathrm{QCoh}(\pi_X)$. Furthermore, we see that a commutative square in the n-category

$$\psi_n(\mathscr{H}om_{n\mathrm{Pr}^L_{\mathrm{St}}}(n\mathrm{QCoh}(\mathrm{Spec}(\mathbb{S})), n\mathrm{QCoh}(X)))$$

is vertically right adjointable if and only if its image in

$$\psi_n(\mathscr{H}om_{n\mathrm{Pr}^L_{\mathrm{St}}}(n\mathrm{QCoh}(\mathrm{Spec}(\mathbb{S})), n\mathrm{QCoh}(\mathrm{Spec}(\mathbb{S}))))$$

is vertically right adjointable. The same remains true after reversing the directions of all cells. Hence we see that a commutative square in

$$\mathrm{Hom}_{(n\mathscr{P}\!r^L_{\mathrm{St}})^{[1,n+1]\text{-op}}}(n\mathrm{QCoh}(X), n\mathrm{QCoh}(\mathrm{Spec}(\mathbb{S})))$$

is vertically right adjointable if and only if its image in

$$\mathrm{Hom}_{(n\mathscr{P}\!r^L_{\mathrm{St}})^{[1,n+1]\text{-op}}}(n\mathrm{QCoh}(\mathrm{Spec}(\mathbb{S})), n\mathrm{QCoh}(\mathrm{Spec}(\mathbb{S})))$$

is vertically right adjointable. Since the map

$$\mathrm{Hom}_{2\mathrm{Corr}(\mathrm{SchAff})}(X, \mathrm{Spec}(\mathbb{S})) \to \mathrm{Hom}_{2\mathrm{Corr}(\mathrm{SchAff})}(\mathrm{Spec}(\mathbb{S}), \mathrm{Spec}(\mathbb{S}))$$

of precomposition with $\iota^R_{\mathrm{SchAff}}(\pi_X)$ preserves pullbacks, we reduce to showing that the functor

$$\mathrm{Hom}_{2\mathrm{Corr}(\mathrm{SchAff})}(\mathrm{Spec}(\mathbb{S}), \mathrm{Spec}(\mathbb{S})) \to \mathrm{Hom}_{(n\mathscr{P}\!r^L_{\mathrm{St}})^{[1,n+1]\text{-op}}}(n\mathrm{QCoh}(\mathrm{Spec}(\mathbb{S})), n\mathrm{QCoh}(\mathrm{Spec}(\mathbb{S})))$$

induced by $n\mathrm{QCoh}'_{2\mathrm{Corr}(\mathrm{SchAff})}$ satisfies the left $(n-1)$-fold Beck-Chevalley condition. In other words, we have now reduced to checking the case $X = Y = \mathrm{Spec}(\mathbb{S})$. This follows from our inductive hypothesis, by using lemma 14.1.17. □

Notation 14.1.20. Let $\mathrm{Cat}_{\mathrm{pb}}$ be the subcategory of Cat on those categories admitting pullbacks, and pullback preserving functors. We denote by $i_{\mathrm{Cat}_{\mathrm{pb}}} : \mathrm{Cat}_{\mathrm{pb}} \to \omega\mathrm{Cat}$ the canonical inclusion.

Lemma 14.1.21. *Let $n \geq 1$ and let S be a subset of $[1, n]$ not containing 1. Then there is a commutative square*

$$\begin{tikzcd} i_{\mathrm{Cat}_{\mathrm{pb}}} \arrow[r, "\iota^{n+1}"] \arrow[d, "\mathrm{id}"] & (n+1)\mathrm{Corr} \arrow[d] \\ i_{\mathrm{Cat}_{\mathrm{pb}}} \arrow[r, "(\iota^{n+1})^{S\text{-op}}"] & (n+1)\mathrm{Corr}^{S\text{-op}} \end{tikzcd}$$

of functors $\mathrm{Cat}_{\mathrm{pb}} \to (n+1)\mathrm{Cat}$, where the vertical arrows are isomorphisms.

Proof. We work by induction on n. Note that the result clearly holds when S is empty, and in particular it holds when $n = 1$. Assume now that $n > 1$ and that the result holds for $n - 1$.

Consider first the case when S does not contain 2. Let S^* be the set of integers i such that $i + 1$ belongs to S. By inductive hypothesis, we have a commutative square

$$\begin{tikzcd} i_{\mathrm{Cat}_{\mathrm{pb}}} \arrow[r, "\iota^n"] \arrow[d, "\mathrm{id}"] & n\mathrm{Corr} \arrow[d] \\ i_{\mathrm{Cat}_{\mathrm{pb}}} \arrow[r, "(\iota^n)^{S^*\text{-op}}"] & n\mathrm{Corr}^{S^*\text{-op}} \end{tikzcd}$$

of functors $\mathrm{Cat}_{\mathrm{pb}} \to n\mathrm{Cat}$, where the vertical arrows are isomorphisms.

The above square admits a unique enhancement to a commutative square of symmetric monoidal functors $\mathrm{Cat}_{\mathrm{pb}} \to n\mathrm{Cat}$ and symmetric monoidal natural transformations, where we equip $\mathrm{Cat}_{\mathrm{pb}}$ and $n\mathrm{Cat}$ with their cartesian symmetric monoidal structures. In particular, it induces a commutative square of functors $2\mathrm{Cat}_{\mathrm{pb}} \to (n+1)\mathrm{Cat}$, where $2\mathrm{Cat}_{\mathrm{pb}}$ is the category of categories enriched in $\mathrm{Cat}_{\mathrm{pb}}$. Composing with the functor $2\mathrm{Corr} : \mathrm{Cat}_{\mathrm{pb}} \to 2\mathrm{Cat}_{\mathrm{pb}}$ yields a commutative square

$$\begin{tikzcd} 2\mathrm{Corr} \arrow[r, "\iota^{2,n+1}"] \arrow[d, "\mathrm{id}"] & (n+1)\mathrm{Corr} \arrow[d] \\ 2\mathrm{Corr} \arrow[r, "(\iota^{2,n+1})^{S\text{-op}}"] & (n+1)\mathrm{Corr}^{S\text{-op}} \end{tikzcd}$$

of functors $\mathrm{Cat}_{\mathrm{pb}} \to (n+1)\mathrm{Cat}$, with invertible vertical arrows. Then the result follows by considering the outer commutative square in the following commutative diagram:

$$\begin{tikzcd} i_{\mathrm{Cat}_{\mathrm{pb}}} \arrow[r, "\iota^2"] \arrow[d, "\mathrm{id}"] & 2\mathrm{Corr} \arrow[r, "\iota^{2,n+1}"] \arrow[d, "\mathrm{id}"] & (n+1)\mathrm{Corr} \arrow[d] \\ i_{\mathrm{Cat}_{\mathrm{pb}}} \arrow[r, "\iota^2"] & 2\mathrm{Corr} \arrow[r, "(\iota^{2,n+1})^{S\text{-op}}"] & (n+1)\mathrm{Corr}^{S\text{-op}} \end{tikzcd}$$

Consider now the case $S = \{2\}$. Denote by $i_{\mathrm{Spc}}(-)^{\leq 0}$ the composition of the truncation functor $\mathrm{Cat}_{\mathrm{pb}} \to \mathrm{Spc}$ and the inclusion $\mathrm{Spc} \to \omega\text{-}\mathrm{Cat}$. It follows from remark 10.2.6 that there is a commutative square

$$\begin{tikzcd} i_{\mathrm{Spc}}(-)^{\leq 0} \arrow[r] \arrow[d, "\mathrm{id}"] & 2\mathrm{Corr} \arrow[d] \\ i_{\mathrm{Spc}}(-)^{\leq 0} \arrow[r] & 2\mathrm{Corr}^{1\text{-op}} \end{tikzcd}$$

of functors $\mathrm{Cat}_{\mathrm{pb}} \to 2\mathrm{Cat}$, where the horizontal arrows are the canonical inclusions, and the vertical arrows are isomorphisms. The above can be enhanced to a commutative square of symmetric monoidal functors and symmetric monoidal natural transformations, and it therefore induces a commutative square of functors $2\mathrm{Cat}_{\mathrm{pb}} \to 3\mathrm{Cat}$. Composing with the functor 2Corr yields a commutative square

$$\begin{tikzcd}
2\mathrm{Corr}^{\leq 1} \arrow[r] \arrow[d, "\mathrm{id}"] & 3\mathrm{Corr} \arrow[d] \\
2\mathrm{Corr}^{\leq 1} \arrow[r] & 3\mathrm{Corr}^{2\text{-op}}
\end{tikzcd}$$

of functors $\mathrm{Cat}_{\mathrm{pb}} \to 3\mathrm{Cat}_{\mathrm{pb}}$, where the horizontal arrows are the canonical inclusions, and the vertical arrows are isomorphisms.

Applying the natural transformation $(\iota^{n-2})_! : (i_{\mathrm{Cat}_{\mathrm{pb}}})_! \to (n-2)\mathrm{Corr}_!$ to the right vertical arrow yields a commutative square

$$\begin{tikzcd}
3\mathrm{Corr} \arrow[r, "i^{3,n+1}"] \arrow[d] & (n+1)\mathrm{Corr} \arrow[d] \\
3\mathrm{Corr}^{2\text{-op}} \arrow[r, "(i^{3,n+1})^{2\text{-op}}"] & (n+1)\mathrm{Corr}^{2\text{-op}}
\end{tikzcd}$$

with invertible vertical arrows. We now have a commutative diagram

$$\begin{tikzcd}
i_{\mathrm{Cat}_{\mathrm{pb}}} \arrow[r] \arrow[d, "\mathrm{id}"] & 2\mathrm{Corr}^{\leq 1} \arrow[r] \arrow[d, "\mathrm{id}"] & 3\mathrm{Corr} \arrow[r, "i^{3,n+1}"] \arrow[d] & (n+1)\mathrm{Corr} \arrow[d] \\
i_{\mathrm{Cat}_{\mathrm{pb}}} \arrow[r] & 2\mathrm{Corr}^{\leq 1} \arrow[r] & 3\mathrm{Corr}^{2\text{-op}} \arrow[r, "(i^{3,n+1})^{2\text{-op}}"] & (n+1)\mathrm{Corr}^{2\text{-op}}
\end{tikzcd}$$

of functors $\mathrm{Cat}_{\mathrm{pb}} \to (n+1)\mathrm{Cat}$, with invertible vertical arrows. The case $S = \{2\}$ of the lemma now follows by looking at the outer commutative square in the above diagram.

Assume now S arbitrary, containing 2. Let $S' = S - \{2\}$. Using the previous two cases we can construct a commutative diagram

$$\begin{tikzcd}
i_{\mathrm{Cat}_{\mathrm{pb}}} \arrow[r, "\iota^{n+1}"] \arrow[d, "\mathrm{id}"] & (n+1)\mathrm{Corr} \arrow[d] \\
i_{\mathrm{Cat}_{\mathrm{pb}}} \arrow[r, "(\iota^{n+1})^{S'\text{-op}}"] \arrow[d, "\mathrm{id}"] & (n+1)\mathrm{Corr}^{S'\text{-op}} \arrow[d] \\
i_{\mathrm{Cat}_{\mathrm{pb}}} \arrow[r, "(\iota^{n+1})^{S\text{-op}}"] & (n+1)\mathrm{Corr}^{S\text{-op}}
\end{tikzcd}$$

of functors $\mathrm{Cat}_{\mathrm{pb}} \to (n+1)\mathrm{Cat}$, with invertible vertical arrows. Our lemma now follows by looking at the outer commutative square in the above diagram. $\square$

Lemma 14.1.22. *Let $n \geq 1$ and let S be a subset of $[1, n]$ containing 1. Then there is a commutative square*

$$\begin{tikzcd} i^{1\text{-op}}_{\mathrm{Cat}_{\mathrm{pb}}} \arrow[r, "(\iota^{n+1})^R"] \arrow[d, "\mathrm{id}"] & (n+1)\mathrm{Corr} \arrow[d] \\ i^{1\text{-op}}_{\mathrm{Cat}_{\mathrm{pb}}} \arrow[r, "(\iota^{n+1})^{S\text{-op}}"] & (n+1)\mathrm{Corr}^{S\text{-op}} \end{tikzcd}$$

of functors $\mathrm{Cat}_{\mathrm{pb}} \to (n+1)\mathrm{Cat}$, where the vertical arrows are isomorphisms.

Proof. Consider first the case $S = \{1\}$. If $n = 1$ our result follows from remark 10.2.6, so we do indeed have a commutative square

$$\begin{tikzcd} i^{1\text{-op}}_{\mathrm{Cat}_{\mathrm{pb}}} \arrow[r, "(\iota^{2})^R"] \arrow[d, "\mathrm{id}"] & 2\mathrm{Corr} \arrow[d] \\ i^{1\text{-op}}_{\mathrm{Cat}_{\mathrm{pb}}} \arrow[r, "(\iota^{2})^{1\text{-op}}"] & 2\mathrm{Corr}^{1\text{-op}} \end{tikzcd}$$

of functors $\mathrm{Cat}_{\mathrm{pb}} \to 2\mathrm{Cat}$ with invertible vertical arrows. Applying the natural transformation $\iota^n : (\iota_{\mathrm{Cat}_{\mathrm{pb}}})_! \to n\mathrm{Corr}_!$ to the right vertical arrow in the above diagram yields a commutative square

$$\begin{tikzcd} 2\mathrm{Corr} \arrow[r, "\iota^{2,n}"] \arrow[d] & (n+1)\mathrm{Corr} \arrow[d] \\ 2\mathrm{Corr}^{1\text{-op}} \arrow[r, "(\iota^{2,n})^{1\text{-op}}"] & (n+1)\mathrm{Corr}^{1\text{-op}} \end{tikzcd}.$$

of functors $\mathrm{Cat}_{\mathrm{pb}} \to (n+1)\mathrm{Cat}$ with invertible vertical arrows.

Pasting the previous two commutative squares yields a commutative diagram

$$\begin{tikzcd} i^{1\text{-op}}_{\mathrm{Cat}_{\mathrm{pb}}} \arrow[r, "(\iota^{2})^R"] \arrow[d, "\mathrm{id}"] & 2\mathrm{Corr} \arrow[r, "\iota^{2,n}"] \arrow[d] & (n+1)\mathrm{Corr} \arrow[d] \\ i^{1\text{-op}}_{\mathrm{Cat}_{\mathrm{pb}}} \arrow[r, "(\iota^{2})^{1\text{-op}}"] & 2\mathrm{Corr}^{1\text{-op}} \arrow[r, "(\iota^{2,n})^{1\text{-op}}"] & (n+1)\mathrm{Corr}^{1\text{-op}} \end{tikzcd}$$

with invertible vertical arrows, and the outer commutative square proves our result in the case $S = \{1\}$.

We now assume that S is arbitrary. Applying lemma 14.1.21 with $S-\{1\}$ we obtain a commutative diagram

$$\begin{tikzcd}
i^{1\text{-op}}_{\mathrm{Cat_{pb}}} \arrow[r, "(\iota^{n+1})^R"] \arrow[d, "\mathrm{id}"] & (n+1)\mathrm{Corr} \arrow[d] \\
i^{1\text{-op}}_{\mathrm{Cat_{pb}}} \arrow[r, "(\iota^{n+1})^{1\text{-op}}"] \arrow[d, "\mathrm{id}"] & (n+1)\mathrm{Corr}^{1\text{-op}} \arrow[d] \\
i^{1\text{-op}}_{\mathrm{Cat_{pb}}} \arrow[r, "(\iota^{n+1})^{S\text{-op}}"] & (n+1)\mathrm{Corr}^{S\text{-op}}
\end{tikzcd}$$

with invertible vertical arrows. Our result now follows by considering the outer commutative square. □

Lemma 14.1.23. *Let $\mathcal{C}$ be a category admitting finite limits and let $n \geq 1$. Let S be a subset of $[1,n]$ containing 1. Then there is a commutative square of symmetric monoidal $(n+1)$-categories*

$$\begin{tikzcd}
\mathcal{C}^{\mathrm{op}} \arrow[r, "(\iota^{n+1}_{\mathcal{C}})^R"] \arrow[d, "\mathrm{id}"] & (n+1)\mathrm{Corr}(\mathcal{C}) \arrow[d] \\
\mathcal{C}^{\mathrm{op}} \arrow[r, "(\iota^{n+1}_{\mathcal{C}})^{S\text{-op}}"] & (n+1)\mathrm{Corr}(\mathcal{C})^{S\text{-op}}
\end{tikzcd}$$

where the vertical arrows are isomorphisms.

Proof. Equipping $\mathrm{Cat_{pb}}$ and $\omega\text{-}\mathrm{Cat}$ with their cartesian symmetric monoidal structures, we see that the commutative square from the statement of 14.1.22 can be enhanced to a commutative square of symmetric monoidal functors and symmetric monoidal natural transformations. Our lemma now follows by evaluating this square on $\mathcal{C}$ (thought of as a commutative algebra in $\mathrm{Cat_{pb}}$). □

Proof of theorem 14.1.4. We first prove that the extension $n\mathrm{QCoh}_{(n+1)\mathrm{Corr}(\mathrm{SchAff})}$ exists and is unique. Combining lemma 14.1.19 together with theorem 11.2.6 (in its symmetric monoidal incarnation, see remark 11.2.10) we see that there exists a unique symmetric monoidal functor

$$(n+1)\mathrm{Corr}(\mathrm{SchAff}) \to (n\mathscr{P}r^L_{\mathrm{St}})^{[1,n+1]\text{-op}}$$

whose restriction along $\iota^{n+1}_{\mathrm{SchAff}}$ recovers the symmetric monoidal functor

$$n\mathrm{QCoh}^{\mathrm{op}} : \mathrm{SchAff} \to (n\mathscr{P}r^L_{\mathrm{St}})^{[1,n+1]\text{-op}}.$$

Passing to opposites of cells of dimension 1 to n we see that there exists a unique symmetric monoidal functor

$$(n+1)\mathrm{Corr}(\mathrm{SchAff})^{[1,n]\text{-op}} \to (n\mathscr{P}r^L_{\mathrm{St}})^{(n+1)\text{-op}}$$

whose restriction along $(\iota^{n+1}_{\mathrm{SchAff}})^{[1,n]\text{-op}}$ recovers the symmetric monoidal functor $n\mathrm{QCoh}$. Our result now follows from an application of lemma 14.1.23.

We now prove the second half of the theorem, concerning the compatibility between $n\mathrm{QCoh}_{(n+1)\mathrm{Corr}(\mathrm{SchAff})}$ and $(n-1)\mathrm{QCoh}_{n\mathrm{Corr}(\mathrm{SchAff})}$. It suffices to show that the restriction along $(i^n_{\mathrm{SchAff}})^R$ of the map $n\mathrm{Corr}(\mathrm{SchAff}) \to (n-1)\mathscr{P}r^L_{\mathrm{St}}$ induced by $n\mathrm{QCoh}_{(n+1)\mathrm{Corr}(\mathrm{SchAff})}$ is equivalent to $(n-1)\mathrm{QCoh}$.

Consider the functor

$$n\mathrm{QCoh}^{\{1,2\}\text{-op}}_{(n+1)\mathrm{Corr}(\mathrm{SchAff})} : (n+1)\mathrm{Corr}(\mathrm{SchAff})^{\{1,2\}\text{-op}} \to (n\mathscr{P}r^L_{\mathrm{St}})^{\{1,2,n\}\text{-op}}.$$

By lemma 14.1.23 we see that the functor of 2-categories underlying $n\mathrm{QCoh}^{\{1,2\}\text{-op}}_{(n+1)\mathrm{Corr}(\mathrm{SchAff})}$ is equivalent to the functor of 2-categories underlying the functor

$$n\mathrm{QCoh}'_{2\mathrm{Corr}(\mathrm{SchAff})} : 2\mathrm{Corr}(\mathrm{SchAff}) \to (n\mathscr{P}r^L_{\mathrm{St}})^{1\text{-op},2\text{-op}}$$

from notation 14.1.16. Under this equivalence, the composite functor

$$\mathrm{SchAff} \xrightarrow{(\iota^n_{\mathrm{SchAff}})^R} n\mathrm{Corr}(\mathrm{SchAff})^{1\text{-op}} \xrightarrow{(n\mathrm{QCoh}^{\{1,2\}\text{-op}}_{(n+1)\mathrm{Corr}(\mathrm{SchAff})})_*} ((n-1)\mathscr{P}r^L_{\mathrm{St}})^{\{1,n\}\text{-op}}$$

gets exchanged with the functor $(n\mathrm{QCoh}'_{2\mathrm{Corr}(\mathrm{SchAff})})_* : \mathrm{SchAff} \to ((n-1)\mathscr{P}r^L_{\mathrm{St}})^{1\text{-op}}$. Our claim now follows from an application of lemma 14.1.17. $\square$

14.2 Extension to prestacks

We now study the theory of higher quasicoherent sheaves on prestacks.

Notation 14.2.1. We denote by PreStk the category of accessible presheaves on SchAff. In other words, PreStk is the smallest full subcategory of $\mathrm{Funct}(\mathrm{SchAff}^{\mathrm{op}}, \mathrm{Spc})$ containing the representable presheaves and closed under small colimits. We call PreStk the category of prestacks.

Proposition 14.2.2. *Let $n \geq 1$. Then the functor $n\mathrm{QCoh} : \mathrm{SchAff}^{\mathrm{op}} \to n\mathrm{Pr}^L_{\mathrm{St}}$ admits a right Kan extension along the inclusion $\mathrm{SchAff}^{\mathrm{op}} \to \mathrm{PreStk}^{\mathrm{op}}$.*

Proof. This is a consequence of corollary 14.1.5, together with the fact that $n\mathrm{Pr}^L$ admits limits of left adjointable diagrams (see corollary 12.5.7). $\square$

Notation 14.2.3. Let $n \geq 0$. We denote by $n\mathrm{QCoh}_{\mathrm{PreStk}}$ the right Kan extension of $n\mathrm{QCoh}$ along the inclusion $\mathrm{SchAff}^{\mathrm{op}} \to \mathrm{PreStk}^{\mathrm{op}}$. For each prestack X we will continue denoting by $n\mathrm{QCoh}(X)$ the value of $n\mathrm{QCoh}_{\mathrm{PreStk}}$ on X. As before, in the special cases $n = 0$ and $n = 1$ we will use the notation $\mathcal{O}(X)$ and $\mathrm{QCoh}(X)$ instead. Given a morphism of prestacks $f : X \to Y$, we will denote by $f^* : n\mathrm{QCoh}(Y) \to n\mathrm{QCoh}(X)$ the induced pullback functor.

Definition 14.2.4. *Let X be a prestack and let $n \geq 2$. We call $n\mathrm{QCoh}(X)$ the presentable stable n-category of quasicoherent sheaves of $(n-1)$-categories on X.*

Remark 14.2.5. Let X be a prestack and let $n \geq 1$. Then there is an equivalence

$$n\mathrm{QCoh}(X) = \lim_{S \in \mathrm{SchAff}_{/X}} n\mathrm{QCoh}(S).$$

In other words, a higher quasicoherent sheaf on X is a compatible family of higher quasicoherent sheaves on all affine schemes equipped with a map to X.

Remark 14.2.6. In the case $n = 2$, definition 14.1.2 recovers the notion of sheaf of categories from [Gai15], and the notion of stable quasicoherent stack from [Lur18] definition 10.1.2.1.

We now state our main result concerning the functoriality of the theory of higher quasicoherent sheaves on prestacks.

Notation 14.2.7. Let $n \geq 2$ and denote by $n\mathrm{QCoh}^{\mathrm{cov}} : \mathrm{SchAff} \to n\mathrm{Pr}^L_{\mathrm{St}}$ the restriction of $n\mathrm{QCoh}_{(n+1)\mathrm{Corr}(\mathrm{SchAff})}$ along the inclusion $\iota^{n+1}_{\mathrm{SchAff}}$. We let $n\mathrm{QCoh}^{\mathrm{cov}}_{\mathrm{PreStk}}$ be the left Kan extension of $n\mathrm{QCoh}^{\mathrm{cov}}$ along the inclusion $\mathrm{SchAff} \to \mathrm{PreStk}$.

Notation 14.2.8. Denote by $\mathrm{PreStk}_{\mathrm{rep}}$ the wide subcategory of PreStk whose morphisms are the affine schematic morphisms of prestacks.

Theorem 14.2.9. *Let $n \geq 2$. Then there is a unique extension of $n\mathrm{QCoh}_{\mathrm{PreStk}}$ along $(i^n_{\mathrm{PreStk}})^R$ to a functor*

$$n\mathrm{QCoh}_{n\mathrm{Corr}(\mathrm{PreStk})} : n\mathrm{Corr}(\mathrm{PreStk}) \to n\mathscr{P}\!r^L_{\mathrm{St}}.$$

This satisfies the following properties:

(i) *The restriction of $n\mathrm{QCoh}_{n\mathrm{Corr}(\mathrm{PreStk})}$ along the inclusion $\iota^n_{\mathrm{PreStk}}$ recovers the functor $n\mathrm{QCoh}^{\mathrm{cov}}_{\mathrm{PreStk}}$ from notation 14.2.7.*

(ii) *The restriction of $n\mathrm{QCoh}_{n\mathrm{Corr}(\mathrm{PreStk})}$ to $n\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{rep}})$ admits a unique extension to a functor*

$$n\mathrm{QCoh}_{(n+1)\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{rep}})} : (n+1)\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{rep}}) \to (n\mathscr{P}\!r^L_{\mathrm{St}})^{(n+1)\text{-op}}$$

(iii) *In the case $n \geq 3$, the square of ω-categories*

$$\begin{tikzcd} (n-1)\mathrm{Corr}(\mathrm{PreStk}) \arrow[r, "(n-1)\mathrm{QCoh}_{(n-1)\mathrm{Corr}(\mathrm{PreStk})}"] \arrow[d, "="] & (n-1)\mathscr{P}\!r^L_{\mathrm{St}} \arrow[d, "="] \\ \mathrm{End}_{n\mathrm{Corr}(\mathrm{PreStk})}(\mathrm{Spec}(\mathbb{S})) \arrow[r, "(n\mathrm{QCoh}_{n\mathrm{Corr}(\mathrm{PreStk})})_*"] & \mathrm{End}_{n\mathscr{P}\!r^L_{\mathrm{St}}}((n-1)\mathrm{Pr}^L_{\mathrm{St}}) \end{tikzcd}$$

commutes.

Proof. The uniqueness part follows from a combination of lemma 14.1.23 and theorem 11.2.6. Thanks to theorems 11.2.6 and 11.3.9, the functor $n\mathrm{QCoh}^{\mathrm{cov}}_{\mathrm{PreStk}}$ admits a unique extension along $\iota^n_{\mathrm{PreStk}}$ to a functor

$$n\mathrm{QCoh}^{\mathrm{cov}}_{n\mathrm{Corr}(\mathrm{PreStk})} : n\mathrm{Corr}(\mathrm{PreStk}) \to n\mathscr{P}r^L_{\mathrm{St}}.$$

Denote by Q the restriction of $n\mathrm{QCoh}^{\mathrm{cov}}_{n\mathrm{Corr}(\mathrm{PreStk})}$ along $(i^n_{\mathrm{PreStk}})^R$. To show that the functor $n\mathrm{QCoh}_{n\mathrm{Corr}(\mathrm{PreStk})}$ in the statement exists, and that item (i) holds, it suffices to show that Q is equivalent to $n\mathrm{QCoh}_{\mathrm{PreStk}}$. To see this, it suffices to show that Q preserves limits. Indeed, this follows from the fact that $n\mathrm{QCoh}^{\mathrm{cov}}_{\mathrm{PreStk}}$ preserves colimits, together with the fact that $n\mathscr{P}r^L_{\mathrm{St}}$ satisfies the passage to adjoints property.

Item (ii) now follows from theorem 11.3.9. Item (iii) is a consequence of the compatibility part of theorem 14.1.4, together with lemma 11.3.18. □

We now study a few consequences of theorem 14.2.9.

Corollary 14.2.10. *Let $f : X \to Y$ be a morphism of prestacks and let $n \geq 2$. Then the morphism $f^* : n\mathrm{QCoh}(Y) \to n\mathrm{QCoh}(X)$ admits a left adjoint. In the case $n \geq 3$, it also admits a right adjoint, and the left and right adjoints are equivalent. The same holds in the case $n = 2$ if f is assumed to be affine schematic.*

Proof. Combine theorem 14.2.9 with propositions 10.3.1 and 11.1.9. □

Notation 14.2.11. Let $f : X \to Y$ be a morphism of prestacks and let $n \geq 3$ (or $n \geq 2$ in the case when f is affine schematic). We denote by $f_* : n\mathrm{QCoh}(X) \to n\mathrm{QCoh}(Y)$ the left and right adjoint to f^*. We call this the functor of pushforward along f. In the case when $Y = \mathrm{Spec}(\mathbb{S})$, this recovers a morphism

$$\Gamma(X, -) : n\mathrm{QCoh}(X) \to (n-1)\mathrm{Pr}^L_{\mathrm{St}}$$

which we call the global sections functor for X.

Remark 14.2.12. Throughout this chapter we have worked with the sphere spectrum as our coefficients. This is not essential: given any base connective commutative ring spectrum A, one may define variants of the functors $n\mathrm{QCoh}$ and $n\mathrm{QCoh}_{\mathrm{PreStk}}$ which are defined over $\mathrm{Spec}(A)$, and take values in A-linear higher presentable stable categories. Theorems 14.1.4 and 14.2.9 still work in this context, with the same proofs (although note that the case of a general base is in fact implied by the spectral case: higher categories of correspondences over a base $\mathrm{Spec}(A)$ are Hom-categories in the higher categories of correspondences over $\mathrm{Spec}(\mathbb{S})$).

Corollary 14.2.13. *The sequence of functors $n\mathrm{QCoh}_{n\mathrm{Corr}(\mathrm{PreStk})}$ for $n \geq 2$ may be assembled into a morphism of categorical spectra*

$$\underline{\mathrm{QCoh}} : \underline{\mathrm{Corr}}(\mathrm{PreStk}) \to B_{\omega\mathrm{Cat}}\underline{\mathrm{Sp}}.$$

Proof. This is a restatement of item (iii) in theorem 14.2.9. □

Remark 14.2.14. Let $\mathcal{C}$ be a category admitting coproducts and let $\mathcal{D}$ be a symmetric monoidal category. Equip $\mathcal{C}$ with its cocartesian symmetric monoidal structure. Recall from [Lur17] proposition 3.2.4.6 that the morphism of operads $\mathcal{C} \to \mathcal{C} \otimes \mathrm{Comm}$ induced by tensoring with the unique map $[0] \to \mathrm{Comm}$, is an equivalence. It follows from this that composition with the forgetful functor $\mathrm{CAlg}(\mathcal{D}) \to \mathcal{D}$ induces an equivalence between the space of lax symmetric monoidal functors $\mathcal{C} \to \mathrm{CAlg}(\mathcal{D})$ and the space of lax symmetric monoidal functors $\mathcal{C} \to \mathcal{D}$. By a combination of [Lur17] proposition 2.4.3.16 and proposition 3.2.4.7, the data of a lax symmetric monoidal functor $\mathcal{C} \to \mathrm{CAlg}(\mathcal{D})$ is in fact equivalent to the data of a functor $\mathcal{C} \to \mathrm{CAlg}(\mathcal{D})$.

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. It follows from the above that the space of lax symmetric monoidal structures on F is equivalent to the space of lifts of F along the projection $\mathrm{CAlg}(\mathcal{D}) \to \mathcal{D}$. Given such a lift F^{enh}, one recovers a lax symmetric monoidal structure on F by composing the lax symmetric monoidal forgetful functor $\mathrm{CAlg}(\mathcal{D}) \to \mathcal{D}$ with the unique lax symmetric monoidal enhancement of F^{enh}. Conversely, given a lax symmetric monoidal structure on F, the induced functor

$$\mathcal{C} = \mathrm{CAlg}(\mathcal{C}) \xrightarrow{\mathrm{CAlg}(F)} \mathrm{CAlg}(\mathcal{D})$$

provides a lift for F.

Proposition 14.2.15. *Let $n \geq 1$. Equip* PreStk *with its cartesian symmetric monoidal structure, so that* $\mathrm{SchAff}^{\mathrm{op}}$ *inherits a cocartesian symmetric monoidal structure. Then there is a unique lax symmetric monoidal structure on the functor* $n\mathrm{QCoh}_{\mathrm{PreStk}}$ *extending the symmetric monoidal structure on* $n\mathrm{QCoh}$. *Furthermore, in the case $n \geq 2$ this lax symmetric monoidal structure is strict.*

Proof. It follows from corollary 14.2.13 that in the case $n \geq 2$, the functor $n\mathrm{QCoh}_{n\mathrm{Corr}(\mathrm{PreStk})}$ (and therefore $n\mathrm{QCoh}_{\mathrm{PreStk}}$) admits a symmetric monoidal structure. The restriction of the morphism of categorical spectra $\underline{\mathrm{QCoh}}$ from corollary 14.2.13 to $\underline{\mathrm{Corr}}(\mathrm{SchAff})$ recovers the restriction along the inclusion $\underline{\mathrm{Corr}}(\mathrm{SchAff}) \to B_{\omega\mathrm{Cat}}(\underline{\mathrm{Corr}}(\mathrm{SchAff}))^{2\text{-op}}$ of the morphism of categorical spectra $B_{\omega\mathrm{Cat}}(\underline{\mathcal{O}})^{2\text{-op}}$, where $\underline{\mathcal{O}}$ is as in corollary 14.1.8. By remark 14.1.9 we conclude that the above symmetric monoidal structure on $n\mathrm{QCoh}_{\mathrm{PreStk}}$ extends the one on $n\mathrm{QCoh}$.

It remains to prove the uniqueness part of the statement. By virtue of remark 14.2.14, we need to show that $n\mathrm{QCoh}_{\mathrm{PreStk}}$ admits a unique lift to a functor $\mathrm{PreStk}^{\mathrm{op}} \to \mathrm{CAlg}(n\mathrm{Pr}^L_{\mathrm{St}})$ extending the functor

$$\mathrm{SchAff}^{\mathrm{op}} = \mathrm{CAlg}(\mathrm{SchAff}^{\mathrm{op}}) \xrightarrow{\mathrm{CAlg}(n\mathrm{QCoh})} \mathrm{CAlg}(n\mathrm{Pr}^L_{\mathrm{St}}).$$

Indeed, it follows from [Lur17] corollary 3.2.2.4 that the space of such lifts is equivalent to the space of right Kan extensions of the above along the inclusion $\mathrm{SchAff}^{\mathrm{op}} \to \mathrm{PreStk}^{\mathrm{op}}$, which is contractible thanks to proposition 14.2.2. $\square$

Warning 14.2.16. In the case $n = 1$, the lax symmetric monoidal structure on $\mathrm{QCoh}_{\mathrm{PreStk}}$ from proposition 14.2.15 is not strict. In other words, given a pair of prestacks X, Y, the external tensor product functor $\mathrm{QCoh}(X) \otimes \mathrm{QCoh}(Y) \to \mathrm{QCoh}(X \times Y)$ is not an equivalence in general (although it becomes an equivalence if one assumes that at least one of $\mathrm{QCoh}(X)$ or $\mathrm{QCoh}(Y)$ is a dualizable object in $\mathrm{Pr}^L_{\mathrm{St}}$).

Remark 14.2.17. Let $n \geq 1$ and let X be a prestack. We usually endow $n\mathrm{QCoh}(X)$ with the structure of commutative algebra in $n\mathrm{Pr}^L_{\mathrm{St}}$ arising from the unique structure of commutative algebra in $\mathrm{PreStk}^{\mathrm{op}}$ on X, and the lax symmetric monoidal structure on $n\mathrm{QCoh}_{\mathrm{PreStk}}$ from proposition 14.2.15. Note that if $X = \mathrm{Spec}(A)$ is an affine scheme, then this coincides with the standard commutative algebra structure on $A\text{-mod}^n$.

14.3 Descent and affineness

Our next goal is to study the descent and affineness properties of the theory of higher quasicoherent sheaves.

Proposition 14.3.1. *Let $f : X \to Y$ be a morphism of affine schemes and let $n \geq 1$. The following conditions are equivalent:*

(i) *The map $f^* : n\mathrm{QCoh}(Y) \to n\mathrm{QCoh}(X)$ is monadic as a morphism in the ω-category $(n\mathscr{P}r^L_{\mathrm{St}})^{[2,n+1]\text{-op}}$.*

(ii) *The augmented cosimplicial object $n\mathrm{QCoh}(X^\bullet/Y)$ obtained by applying the contravariant functor $n\mathrm{QCoh}$ to the augmented Čech nerve of f, is a conical limit diagram in $n\mathscr{P}r^L_{\mathrm{St}}$.*

(iii) *The augmented cosimplicial object $n\mathrm{QCoh}(X^\bullet/Y)$ obtained by applying the contravariant functor $n\mathrm{QCoh}$ to the augmented Čech nerve of f, is a conical limit diagram of categories.*

Proof. The equivalence between items (ii) and (iii) follows from theorem 12.4.6. It now follows from [Lur17] corollary 4.7.5.3 that item (iii) is equivalent to the functor of categories underlying f^* being comonadic. A combination of theorems 12.4.6 and 7.4.10 shows that this happens if and only if the functor $\psi_n(n\mathrm{QCoh}(Y))^{[1,n]\text{-op}} \to \psi_n(n\mathrm{QCoh}(X))^{[1,n]\text{-op}}$ induced by f, is monadic. This functor is equivalent to the functor

$$\mathrm{Hom}_{(n\mathscr{P}r^L_{\mathrm{St}})^{[2,n+1]\text{-op}}}((n-1)\mathrm{Pr}^L_{\mathrm{St}}, n\mathrm{QCoh}(Y)) \to \mathrm{Hom}_{(n\mathscr{P}r^L_{\mathrm{St}})^{[2,n+1]\text{-op}}}((n-1)\mathrm{Pr}^L_{\mathrm{St}}, n\mathrm{QCoh}(X))$$

of composition with f^*. This shows that condition (i) implies condition (iii).

It remains to see that (iii) implies (i). This amounts to showing that for every object $\mathcal{C}$ in $n\mathrm{Pr}^L_{\mathrm{St}}$, the functor

$$\mathrm{Hom}_{(n\mathscr{P}r^L_{\mathrm{St}})^{[2,n+1]\text{-op}}}(\mathcal{C}, n\mathrm{QCoh}(Y)) \to \mathrm{Hom}_{(n\mathscr{P}r^L_{\mathrm{St}})^{[2,n+1]\text{-op}}}(\mathcal{C}, n\mathrm{QCoh}(X))$$

of composition with f^* is monadic. It follows from proposition 7.3.10 that the collection of objects $\mathcal{C}$ for which this holds is closed under colimits. Hence it suffices to show that this holds in the case $\mathcal{C} = (n-1)\mathrm{Pr}^L_{\mathrm{St}} \otimes \mathcal{I}$ for a small category $\mathcal{I}$. In this case, the above functor becomes the functor

$$\psi_n(\mathrm{Funct}(\mathcal{I}, n\mathrm{QCoh}(Y))^{[1,n]\text{-op}} \to \psi_n(\mathrm{Funct}(\mathcal{I}, n\mathrm{QCoh}(X))^{[1,n]\text{-op}}$$

induced by f^*. The fact that the above is monadic follows from another application of theorems 12.4.6 and 7.4.10. □

Definition 14.3.2. *Let $f : X \to Y$ be a morphism of affine schemes and let $n \geq 1$. We say that $n\mathrm{QCoh}$ satisfies descent along f if the equivalent conditions of proposition 14.3.1 are satisfied.*

Proposition 14.3.3. *Let $n \geq 3$ and let $f : X \to Y$ be a morphism of affine schemes. Assume that $(n-1)\mathrm{QCoh}$ satisfies descent along f. Then $n\mathrm{QCoh}$ also satisfies descent along f.*

Proof. By corollary 14.1.5, the functor of categories underlying $f^* : n\mathrm{QCoh}(Y) \to n\mathrm{QCoh}(X)$ admits adjoints to both sides. It suffices then to show that f^* is conservative. Equivalently, we have to show that the image of $f_* : n\mathrm{QCoh}(X) \to n\mathrm{QCoh}(Y)$ generates $n\mathrm{QCoh}(Y)$ under colimits. Since f_* is a morphism of $n\mathrm{QCoh}(Y)$-module categories, it suffices to show that the unit $(n-1)\mathrm{QCoh}(Y)$ is a colimit of objects in the image of f_*.

Since $(n-1)\mathrm{QCoh}$ satisfies descent along f, we have that the augmented cosimplicial object $(n-1)\mathrm{QCoh}(X^\bullet/Y)$ is a limit diagram in $(n-1)\mathrm{QCoh}(Y)\text{-mod}(\widehat{\mathrm{Cat}}_{\mathrm{cocompl}})$. The result now follows from an application of proposition 12.5.7. □

Corollary 14.3.4. *Let $n \geq 1$. Then $n\mathrm{QCoh}$ satisfies descent along étale covers of affine schemes.*

Proof. The case $n = 1$ follows for instance from [Lur11a] proposition 2.7.14. The case $n = 2$ follows from [Lur11b] theorem 5.4. The case $n \geq 3$ follows inductively from this, by virtue of proposition 14.3.3. □

Corollary 14.3.5. *Let $n \geq 1$. Then the functor $n\mathrm{QCoh} : \mathrm{SchAff}^{\mathrm{op}} \to n\,\mathrm{Pr}^L_{\mathrm{St}}$ is a sheaf for the étale topology.*

We finish by studying the notion of higher affineness.

Proposition 14.3.6. *Let $n \geq 2$ and let X be a prestack. Then the following are equivalent:*

(i) *The global sections functor $\Gamma(X, -) : (n+1)\mathrm{QCoh}(X) \to n\mathrm{Pr}^L_{\mathrm{St}}$ is a monadic morphism in $(n+1)\mathscr{P}r^L_{\mathrm{St}}$.*

(ii) *The global sections functor $\Gamma(X, -) : (n+1)\mathrm{QCoh}(X) \to n\mathrm{Pr}^L_{\mathrm{St}}$ is a monadic functor of categories.*

Proof. Assume first that item (i) holds. Then the functor of $(n+1)$-categories $\psi_{n+1}((n+1)\mathrm{QCoh}(X)) \to \psi_{n+1}(n\mathrm{Pr}^L_{\mathrm{St}})$ induced by $\Gamma(X,-)$ is monadic. This implies that the functor of categories underlying $\Gamma(X,-)$ is monadic, which means that item (ii) holds.

Assume now that item (ii) holds. We have to show that for every $\mathcal{C}$ in $(n+1)\mathscr{P}\!r^L_{\mathrm{St}}$ the functor

$$\mathrm{Hom}_{(n+1)\mathscr{P}\!r^L_{\mathrm{St}}}(\mathcal{C},(n+1)\mathrm{QCoh}(X)) \to \mathrm{Hom}_{(n+1)\mathscr{P}\!r^L_{\mathrm{St}}}(\mathcal{C}, n\mathrm{Pr}^L_{\mathrm{St}}))$$

is monadic. It follows from proposition 7.3.10 that the collection of objects $\mathcal{C}$ for which this holds is closed under colimits. Hence it suffices to show that this holds in the case $\mathcal{C} = (n-1)\mathrm{Pr}^L_{\mathrm{St}} \otimes \mathcal{I}$ for a small category $\mathcal{I}$. In this case, the above functor becomes the functor

$$\psi_{n+1}(\mathrm{Funct}(\mathcal{I},(n+1)\mathrm{QCoh}(X))) \to \psi_{n+1}(\mathrm{Funct}(\mathcal{I}, n\mathrm{Pr}^L_{\mathrm{St}}))$$

induced by $\Gamma(X,-)$. The fact that the above is monadic follows from a combination of theorems 12.4.6 and 7.4.10. $\square$

Definition 14.3.7. *Let $n \geq 2$ and let X be a prestack. We say that X is n-affine if the equivalent conditions of proposition 14.3.6 are satisfied. We say that a morphism of prestacks $f : X \to Y$ is n-affine if for every affine scheme S equipped with a map to Y, the prestack $X \times_Y S$ is n-affine.*

Remark 14.3.8. One may extend definition 14.3.7 in the case $n = 1$ as follows: a prestack X is 1-affine if the pullback morphism

$$\pi^* : \mathrm{Pr}^L_{\mathrm{St}} = 2\mathrm{QCoh}(\mathrm{Spec}(\mathbb{S})) \to 2\mathrm{QCoh}(X)$$

admits a monadic right adjoint in $2\mathscr{P}\!r^L_{\mathrm{St}}$. A variant of the argument in proposition 14.3.6 shows that this happens if and only if the functor of categories underlying π^* admits a colimit preserving monadic right adjoint. This agrees with the notion of 1-affineness studied in [Gai15].

If X is perfect and 1-affine, then the functor π^* forms part of an ambidextrous adjunction. In particular, we see that in this case the left adjoint to π^* is also monadic.

Theorem 14.3.9. *Let $n \geq 2$ and let X be a prestack. Assume that the diagonal of X is $(n-1)$-affine. Then X is n-affine.*

Remark 14.3.10. A consequence of theorem 14.3.9 is that (quasicompact quasiseparated) schemes are n-affine for all $n \geq 1$. Using this fact one can show that in the statement of theorem 14.1.4 one may replace the categories of correspondences of affine schemes with categories of correspondences of schemes, with a similar proof.

Our proof of theorem 14.3.9 will need some preliminaries.

Definition 14.3.11. *We say that an ω-category $\mathcal{C}$ is locally conically cocomplete if $\mathrm{Hom}_{\mathcal{C}}(c,c')$ is conically cocomplete for all pair of objects c, c' in $\mathcal{C}$, and moreover the composition maps in $\mathcal{C}$ preserve conical colimits in each coordinate.*

Lemma 14.3.12. *Let $\mathcal{C}$ be a locally conically cocomplete ω-category. Let $g : x \to y$ be a morphism in $\mathcal{C}$ admitting a left adjoint, and denote by $S^{\triangleright}$ the augmented simplicial object in* $\mathrm{End}_{\mathcal{C}}(x)$ *given informally as follows:*

$$\ldots \substack{\to\\\to\\\to} g^L g g^L g \rightrightarrows g^L g \to \mathrm{id}_x \,.$$

In other words, $S^{\triangleright}$ is the Bar construction for the endomorphism monad of g. Then g is monadic if and only if $S^{\triangleright}$ is a conical colimit diagram.

Proof. Assume first that that g is monadic. Then we have that the functor of ω-categories

$$g_* : \mathrm{Hom}_{\mathcal{C}}(x, x) \to \mathrm{Hom}_{\mathcal{C}}(x, y)$$

is monadic, and therefore the functor of categories

$$g_*^{\leq 1} : \mathrm{Hom}_{\mathcal{C}}(x, x)^{\leq 1} \to \mathrm{Hom}_{\mathcal{C}}(x, y)^{\leq 1}$$

is also monadic. The augmented simplicial object $S^{\triangleright}$ is the Bar construction of $g_*^{\leq 1}$ and is therefore a (conical) colimit diagram.

Assume now that $S^{\triangleright}$ is a conical colimit diagram. We need to show that for every z in $\mathcal{C}$ the morphism

$$g_* : \mathrm{Hom}_{\mathcal{C}}(z, x) \to \mathrm{Hom}_{\mathcal{C}}(z, y)$$

is a monadic functor of ω-categories. Since both ω-categories above admit all conical colimits which are preserved by g_* and moreover g admits a left adjoint, we see that g_* is monadic if and only if it is conservative. Assume given a morphism α in $\mathrm{Hom}_{\mathcal{C}}(z, x)$ such that $g_*\alpha$ is invertible. Then $S^{\triangleright}\alpha$ is a conical colimit diagram, which expresses α as a conical colimit of morphisms of the form $(g^L g)^n \alpha$. It follows that α is a conical colimit of invertible arrows, and therefore α is invertible. ☐

Lemma 14.3.13. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of locally conically cocomplete ω-categories such that for every pair of objects c, c' in $\mathcal{C}$, the induced functor*

$$F_* : \mathrm{Hom}_{\mathcal{C}}(c, c') \to \mathrm{Hom}_{\mathcal{D}}(Fc, Fc')$$

preserves conical colimits. Then F maps monadic morphisms in $\mathcal{C}$ to monadic morphisms in $\mathcal{D}$.

Proof. This is a direct consequence of lemma 14.3.12. ☐

Notation 14.3.14. Let $f : X \to Y$ be a morphism of prestacks and let $n \geq 2$. We denote by $n\mathrm{QCoh}_{X/Y}$ the object in $(n+1)\mathrm{QCoh}(Y)$ obtained as the image of the span $\mathrm{Spec}(\mathbb{S}) \leftarrow X \xrightarrow{f} Y$ under the functor $(n+1)\mathrm{QCoh}_{(n+1)\mathrm{Corr}_{\mathrm{PreStk}}}$.

Lemma 14.3.15. *Let $n \geq 2$ and let $f : X \to Y$ be an $(n-1)$-affine morphism of prestacks. Then there exists a monadic morphism $n\mathrm{QCoh}_{X/Y} \to n\mathrm{QCoh}_{Y/Y}$ in $\psi_{n+1}((n+1)\mathrm{QCoh}(Y))$.*

Proof. Consider the morphism $f^* : n\mathrm{QCoh}_{X/Y} \to n\mathrm{QCoh}_{Y/Y}$ obtained as the image under $(n+1)\mathrm{QCoh}_{(n+1)\mathrm{Corr}_{\mathrm{PreStk}}}$ of the morphism of spans

$$\begin{tikzcd}
 & X \arrow[dl] \arrow[d, "f"] \arrow[dr, "f"] & \\
 & Y \arrow[dl] \arrow[dr, "\mathrm{id}"'] & \\
\mathrm{Spec}(\mathbb{S}) & & Y.
\end{tikzcd}$$

We claim that f^* admits a monadic right adjoint. It follows from lemma 14.3.12 that a morphism in $\psi_{n+1}((n+1)\mathrm{QCoh}(Y))$ is monadic if and only if its image in $\psi_{n+1}((n+1)\mathrm{QCoh}(S))$ is monadic for every affine scheme S over Y. We may therefore assume that Y is affine. In this case, f^* is a morphism in $\psi_{n+1}(n\mathrm{QCoh}(Y)\text{-mod}((n-1)\mathrm{Pr}^L_{\mathrm{St}}))$. As in the proof of proposition 14.3.6, to show that it admits a monadic right adjoint it suffices to show that the functor of categories underlying f^* admits a colimit preserving monadic right adjoint. Let f_* be the (possibly discontinuous) right adjoint for f^*. To show that f_* is colimit preserving and monadic it suffices to show that the functor $\Gamma(Y, f_*) : n\mathrm{QCoh}(X) \to (n-1)\mathrm{Pr}^L_{\mathrm{St}}$ is colimit preserving and monadic. This follows from our hypothesis. $\square$

Proof of theorem 14.3.9. Let $\pi : X \to \mathrm{Spec}(\mathbb{S})$ be the projection. Using corollary 14.2.10 we reduce to showing that $\pi_* : (n+1)\mathrm{QCoh}(X) \to (n+1)\mathrm{QCoh}(\mathrm{Spec}(\mathbb{S}))$ is conservative. Let $\alpha : \mathcal{C} \to \mathcal{D}$ be a morphism in $(n+1)\mathrm{QCoh}(X)$ which is inverted by π_*. We need to see that α is an isomorphism.

Let $p_1, p_2 : X \times X \to X$ be the projection maps. Then for every object $\mathcal{E}$ in $(n+1)\mathrm{QCoh}(X)$ we have an equivalence $\pi^*\pi_*\mathcal{E} = (p_1)_*(p_2)^*\mathcal{E}$. Since α is inverted by π_*, we have that the induced map

$$\alpha^* : \mathrm{Hom}_{\psi_{n+1}((n+1)\mathrm{QCoh}(X))}(\mathcal{D}, (p_1)_*(p_2)^*\mathcal{E}) \to \mathrm{Hom}_{\psi_{n+1}((n+1)\mathrm{QCoh}(X))}(\mathcal{C}, (p_1)_*(p_2)^*\mathcal{E})$$

is an equivalence.

Let $\Delta : X \to X \times X$ be the diagonal map. By lemma 14.3.15, we have that there exists a monadic morphism $n\mathrm{QCoh}_{X/X\times X} \to n\mathrm{QCoh}_{X\times X/X\times X}$ in $\psi_{n+1}((n+1)\mathrm{QCoh}(X \times X))$. Tensoring with $(p_2)^*\mathcal{E}$ we obtain a monadic morphism $\beta : \mathcal{F} \to (p_2)^*\mathcal{E}$, where $\mathcal{F}$ is equivalent to

$$(\Delta_* n\mathrm{QCoh}_{X/X}) \otimes (p_2)^*\mathcal{E} = \Delta_*\Delta^*(p_2)^*\mathcal{E} = \Delta_*\mathcal{E}.$$

Composing with $(p_1)_*$ we obtain a monadic morphism

$$(p_1)_*\beta : \mathcal{E} \to (p_1)_*(p_2)^*\mathcal{E}.$$

It follows from this that the map

$$\alpha^* : \mathrm{Hom}_{\psi_{n+1}((n+1)\mathrm{QCoh}(X))}(\mathcal{D}, \mathcal{E}) \to \mathrm{Hom}_{\psi_{n+1}((n+1)\mathrm{QCoh}(X))}(\mathcal{C}, \mathcal{E})$$

is an isomorphism. Since $\mathcal{E}$ was arbitrary, we conclude that α is an isomorphism, as desired. $\square$

Bibliography

[Bar18] C. Barwick. *From operator categories to higher operads.* In: *Geom. Topol.* 22.4 (2018), pp. 1893–1959.

[BGN18] C. Barwick, S. Glasman, and D. Nardin. *Dualizing cartesian and cocartesian fibrations.* In: *Theory Appl. Categ.* 33 (2018), Paper No. 4, 67–94.

[BN12] D. Ben-Zvi and D. Nadler. *Loop spaces and connections.* In: *J. Topol.* 5.2 (2012), pp. 377–430.

[BN13] D. Ben-Zvi and D. Nadler. *Nonlinear Traces.* arXiv:1305.7175. 2013.

[BN18] D. Ben-Zvi and D. Nadler. *Betti geometric Langlands.* In: *Algebraic geometry: Salt Lake City 2015.* Vol. 97. Proc. Sympos. Pure Math. Amer. Math. Soc., Providence, RI, 2018, pp. 3–41.

[CH20] H. Chu and R. Haugseng. *Enriched ∞-operads.* In: *Adv. Math.* 361 (2020), pp. 106913, 85.

[CS19] D. Calaque and C. Scheimbauer. *A note on the (∞, n)-category of cobordisms.* In: *Algebr. Geom. Topol.* 19.2 (2019), pp. 533–655.

[Dub70] E. J. Dubuc. *Kan extensions in enriched category theory.* Lecture Notes in Mathematics, Vol. 145. Springer-Verlag, Berlin-New York, 1970, pp. xvi+173.

[Gai15] D. Gaitsgory. *Sheaves of categories and the notion of 1-affineness.* In: *Stacks and categories in geometry, topology, and algebra.* Vol. 643. Contemp. Math. Amer. Math. Soc., 2015, pp. 127–225.

[GH15] D. Gepner and R. Haugseng. *Enriched ∞-categories via non-symmetric ∞-operads.* In: *Adv. Math.* 279 (2015), pp. 575–716.

[GHN17] D. Gepner, R. Haugseng, and T. Nikolaus. *Lax colimits and free fibrations in ∞-categories.* In: *Doc. Math.* 22 (2017), pp. 1225–1266.

[GR17] D. Gaitsgory and N. Rozenblyum. *A study in derived algebraic geometry. Vol. I. Correspondences and duality.* Vol. 221. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2017, xl+533pp.

[Hau15] R. Haugseng. *Rectification of enriched ∞-categories.* In: *Algebr. Geom. Topol.* 15.4 (2015), pp. 1931–1982.

[Hau17] R. Haugseng. *The higher Morita category of $\mathbb{E}_n$-algebras*. In: *Geom. Topol.* 21.3 (2017), pp. 1631–1730.

[Hau18] R. Haugseng. *Iterated spans and classical topological field theories*. In: *Math. Z.* 289.3-4 (2018), pp. 1427–1488.

[Hau20] R. Haugseng. *On lax transformations, adjunctions, and monads in $(\infty, 2)$-categories*. arxiv:2002.01037. 2020.

[Hei20] H. Heine. *An equivalence between enriched ∞-categories and ∞-categories with weak action*. arXiv:2009.02428. 2020.

[Hin20a] V. Hinich. *Yoneda lemma for enriched ∞-categories*. In: *Adv. Math.* 367 (2020), pp. 107129, 119.

[Hin20b] V. Hinich. *So, what is a derived functor?* In: *Homology Homotopy Appl.* 22.2 (2020), pp. 279–293.

[Hin21] V. Hinich. *Colimits in enriched ∞-categories and Day convolution*. arXiv:2101.09538. 2021.

[HMS20] R. Haugseng, V. Melani, and P. Safronov. *Shifted coisotropic correspondences*. arXiv:2009.02428. 2020.

[KW07] A. Kapustin and E. Witten. *Electric-magnetic duality and the geometric Langlands program*. In: *Commun. Number Theory Phys.* 1.1 (2007), pp. 1–236.

[Lur09a] J. Lurie. *Higher topos theory*. Vol. 170. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2009, pp. xviii+925.

[Lur09b] J. Lurie. *On the classification of topological field theories*. In: *Current developments in mathematics, 2008*. Int. Press, Somerville, MA, 2009, pp. 129–280.

[Lur11a] J. Lurie. *Derived Algebraic Geometry VIII: Quasi-Coherent Sheaves and Tannaka Duality Theorems*. Available at https://www.math.ias.edu/~lurie/. 2011.

[Lur11b] J. Lurie. *Derived Algebraic Geometry XI: Descent Theorems*. Available from the author's webpage. 2011.

[Lur17] J. Lurie. *Higher Algebra*. Available at https://www.math.ias.edu/~lurie/. 2017.

[Lur18] J. Lurie. *Spectral Algebraic Geometry*. Available from the author's webpage. 2018.

[Mac20] A. W. Macpherson. *A bivariant Yoneda lemma and $(\infty, 2)$-categories of correspondences*. arXiv:2005.10496. 2020.

[Mac21] A. W. Macpherson. *The operad that co-represents enrichment*. In: *Homology Homotopy Appl.* 23.1 (2021), pp. 387–401.

[Pre15] A. Preygel. *Ind-coherent complexes on loop spaces and connections*. In: *Stacks and categories in geometry, topology, and algebra*. Vol. 643. Contemp. Math. Amer. Math. Soc., Providence, RI, 2015, pp. 289–323.

[RV16] E. Riehl and D. Verity. *Homotopy coherent adjunctions and the formal theory of monads*. In: *Adv. Math.* 286 (2016), pp. 802–888.

[Sch14] C. Scheimbauer. *Factorization Homology as a Fully Extended Topological Field Theory*. PhD Thesis. Available at http://www.scheimbauer.at/ScheimbauerThesis.pdf. 2014.

[Ste20a] G. Stefanich. *Higher sheaf theory I: Correspondences*. arXiv:2011.03027. 2020.

[Ste20b] G. Stefanich. *Presentable* (∞, n)*-categories*. arXiv:2011.03035. 2020.

[TV09] B. Toën and G. Vezzosi. *Chern character, loop spaces and derived algebraic geometry*. In: *Algebraic topology*. Vol. 4. Abel Symp. Springer, Berlin, 2009, pp. 331–354.