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Intersection theory of the moduli space of elliptic K3 surfaces

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Intersection theory of the moduli space of elliptic K3 surfaces

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The Dissertation of Bochao Kong is approved, and it is acceptable in quality and form for publication on microfilm and electronically.

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EPIGRAPH

Pure mathematics is, in its way,
the poetry of logical ideas.

Albert Einstein

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ABSTRACT OF THE DISSERTATION

Intersection theory of the moduli space of elliptic K3 surfaces

by

Bochao Kong

Doctor of Philosophy in Mathematics

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Professor Dragos Oprea, Chair

Moduli spaces of K3 surfaces are fundamental objects in algebraic geometry. Elliptic K3 surfaces are K3 surfaces with elliptic fibration structure, and they are of particular interest due to their rich geometry. The moduli space of elliptic K3 surfaces can be studied using the theory of Weierstrass models. In this dissertation, we study the topology and intersection theory of the moduli space of elliptic K3 surfaces.

We compute the Poincaré polynomial of the moduli space of elliptic K3 surfaces. The main idea is constructing a compactification using the Weierstrass models, this compactification is a GIT quotient. We adapt Kirwan's blowup machinery to weighted projective space to compute the Poincaré polynomial. We find the cohomology is mostly

concentrated in the even degrees, but there is one odd degree class in degree 33.

We also study the Chow ring of the moduli space of elliptic surfaces of degree $N \geq 2$. We conclude that the Chow ring of the moduli space of elliptic surfaces is always generated by two classes. Furthermore, explicit relations between these classes are given, the Poincaré polynomial for the Chow ring is the same for any $N \geq 2$ and the ring is Gorenstein with socle in degree 16. When $N = 2$, we obtain the Chow ring for the moduli space of elliptic K3 surfaces, we conclude that the Chow ring in this case is tautological.

Finally, we present localization computations on the relative Quot scheme over the moduli space of elliptic K3 surfaces. Our calculations are sufficient to determine the divisorial κ-classes in terms of the Hodge class. We also represent one Noether-Lefschetz divisor in terms of the Hodge class, which agrees with the modularity nature of the Noether-Lefschetz divisors.

Chapter 1

Preliminaries

1.1 Moduli spaces of K3 surfaces

1.1.1 K3 surfaces

K3 surfaces are a class of complex surfaces that are of great interest in algebraic geometry, differential geometry, and mathematical physics. They were first defined by Andre Weil in 1958, and the name K3 comes from the names of the three mathematicians who first studied them: Kummer, Kähler, and Kodaira, as well as the mountain K2 in Kashmir. Algebraic K3 surfaces can be defined over any field, but we will only consider K3 surfaces over the complex numbers $\mathbb{C}$. Unless otherwise stated, we will assume that the K3 surfaces are smooth and projective.

Definition 1.1.1. A *K3 surface* is a smooth projective surface X over the complex numbers such that:

$$K_X \cong \mathcal{O}_X \text{ and } \dim\ H^1(X, \mathcal{O}_X) = 0.$$

The conditions we put on a K3 surface are quite strong, and it is a nontrivial fact that such surfaces exist. We present some examples of K3 surfaces below:

Example 1.1.2. 1. The most famous example of a K3 surface is the Fermat quartic surface in $\mathbb{P}^3$ given by the equation

$$x^4 + y^4 + z^4 + w^4 = 0.$$

More generally, any smooth quartic surface in $\mathbb{P}^3$ is a K3 surfaces.

2. Consider a smooth sextic curve C in $\mathbb{P}^2$. The double cover of $\mathbb{P}^2$ branched along C is a K3 surface.

3. Let A be an abelian surface. The involution map $[-1] : A \to A$ is an isogeny of degree 2. It will have 16 isolated fixed points. The minimal resolution $X \to A/\pm 1$ is a K3 surface.

The strong conditions in the definition of a K3 surface allow us to compute many of its invariants.

Proposition 1.1.3. *For a K3 surface X, we have:*

$$\dim H^0(X, \mathcal{O}_X) = 1 \text{ and } \dim H^2(X, \mathcal{O}_X) = 1.$$

The top Chern class integral $\int_X c_2(X)$ is equal to 24, we will also write $c_2(X) = 24$ if no confusion arises.

Proof. By Serre duality, we have

$$H^0(X, \mathcal{O}_X) \cong H^2(X, \mathcal{O}_X \otimes K_X)^\vee \cong H^2(X, \mathcal{O}_X)^\vee.$$

Since the surface is projective, we have $\dim H^0(X, \mathcal{O}_X) = \dim H^2(X, \mathcal{O}_X) = 1$. In particular, we have $\chi(X, \mathcal{O}_X) = 2$. The Hirzebruch-Riemann-Roch theorem gives

$$\chi(X, \mathcal{O}_X) = \int_X \mathrm{ch}(\mathcal{O}_X)\,\mathrm{td}(T_X) = \int_X \frac{c_1(T_X)^2 + c_2(T_X)}{12} = 2.$$

Note that $c_1(T_X) = -K_X$, so we have $c_1(T_X)^2 = 0$. Thus, $c_2(T_X) = 24$. $\square$

We can compute the Hodge diamond of a K3 surface using similar techniques. Recall the Hodge numbers are defined as $h^{p,q}(X) := \dim H^q(X, \Omega_X^p)$. We have the following result:

Proposition 1.1.4. *For a K3 surface X, the Hodge numbers are presented as follows:*

$$\begin{array}{ccccc} & & h^{0,0} & & \\ & h^{1,0} & & h^{0,1} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{2,1} & & h^{1,2} & \\ & & h^{2,2} & & \end{array} \qquad \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Proof. We have $h^{0,1}(X) = h^{1,2} = 0$ by definition, so $h^{1,0}(X) = 0$ as well. Now the only Hodge number that requires a computation is $h^{1,1}(X)$. We apply the Hirzebruch-Riemann-Roch theorem to the sheaf of Kähler differentials Ω_X:

$$\begin{aligned} \chi(X, \Omega_X) &= \int_X \mathrm{ch}(\Omega_X)\,\mathrm{td}(T_X) \\ &= \int_X \left(\mathrm{ch}_2(\Omega_X) + \mathrm{ch}_0(\Omega_X) \cdot \frac{c_1(T_X)^2 + c_2(T_X)}{12} \right) \\ &= c_2(T_X) + 2 \cdot 2 = -20. \\ &= h^{1,0} - h^{1,1} + h^{1,2}. \end{aligned}$$

Thus, $h^{1,1}(X) = 20$. $\square$

From the Hodge diamond, we see that the second Betti number $b_2(X)$ of a K3 surface is equal to 22. In fact, the integral cohomology ring of a K3 surface can be computed explicitly.

Theorem 1.1.5. *For a K3 surface X, the odd integral cohomology vanishes, and the integral cohomology group is given by:*

$$H^*(X,\mathbb{Z}) = H^0(X) \oplus H^2(X) \oplus H^4(X) \simeq \mathbb{Z} \oplus \mathbb{Z}^{22} \oplus \mathbb{Z}.$$

The ring structure is given by the lattice structure on $H^2(X,\mathbb{Z})$, which is isomorphic to the unique even unimodular lattice of signature $(3,19)$. More precisely, we have:

$$H^2(X,\mathbb{Z}) \simeq U^3 \oplus E_8(-1)^2,$$

where U is the hyperbolic plane and $E_8(-1)$ is the unique negative definite even unimodular lattice of rank 8.

Definition 1.1.6. The lattice $U^3 \oplus E_8(-1)^2$ is called the *K3 lattice*, we denote it by Λ_{K3}.

For a general surface X, the $\mathrm{Pic}(X)$, the $\mathrm{NS}(X)$, and the $\mathrm{Num}(X)$ are different. However, for a K3 surface, we have the following result:

Proposition 1.1.7. *[Huy16, 1.2.4] Let X be a K3 surface. The natural surjections:*

$$\mathrm{Pic}(X) \to \mathrm{NS}(X) \to \mathrm{Num}(X)$$

are all isomorphisms. Moreover, the intersection pairing on $\mathrm{NS}(X)$ is even, non-degenerate and of signature $(1, \rho(X)-1)$, where $\rho(X)$ is the rank of the Néron-Severi group $\mathrm{NS}(X)$.

1.1.2 Weight 2 Hodge structures and period map

In this section, we will study the weight 2 Hodge structures on a K3 surface. We will introduce the period domain and the Torelli theorem for (quasi-)polarized K3 surfaces. We follow the exposition in [DK07, Section 9]. In this section, we do not assume that our K3 surface is projective, we replace the projective assumption with the assumption that our K3 surface is Kähler. In fact, the Kähler assumption is not necessary due to Siu's Theorem [Siu83], but it simplifies the exposition.

The interesting part of the Hodge structure for K3 surfaces is of weight 2. Consider the intersection form on $H^2(X, \mathbb{C})$:

$$Q(\alpha, \beta) = \int_X \alpha \wedge \beta.$$

The intersection form Q is the complex extension of the usual intersection form in $H^2(X, \mathbb{Z}) \simeq \Lambda_{K3}$. We associate a Hermitian form on $H^2(X, \mathbb{C})$ by:

$$H(\alpha, \beta) = -Q(\alpha, \overline{\beta}).$$

The Hodge decomposition:

$$H^2(X, \mathbb{C}) = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X)$$

is orthogonal with respect to the Hermitian form H. Recall in Proposition 1.1.4, we have $(h^{2,0}, h^{1,1}, h^{0,2}) = (1, 20, 1)$. The Hodge decomposition is uniquely determined by the lowest part in the Hodge filtration $H^{2,0}(X) = F^2H^2(X, \mathbb{C})$. Furthermore, the Hermitian form H is negative definite on $H^{2,0}(X)$ by the Hodge index theorem. This motivates the following definition:

Definition 1.1.8. The *period domain* D is the open subset (in usual topology) of a quadric:

$$D = \{\omega \in \mathbb{P}(\Lambda_{K3} \otimes \mathbb{C}) \mid Q(\omega, \omega) = 0, Q(\omega, \overline{\omega}) > 0\}.$$

The period domain D is a complex manifold of dimension 20. It parametrizes all the Hodge structures of weight 2 on complex K3 surfaces. We would like to map a K3 surface to the period domain via its Hodge structure. To do this, we need to fix a marking of the K3 surface.

Definition 1.1.9. [DK07] A *marking* of a K3 surface X is an isomorphism $\phi : \Lambda_{K3} \to H^2(X, \mathbb{Z})$, where Λ_{K3} is the K3 lattice defined in Definition 1.1.6. A *marked K3 surface* is a pair (X, ϕ), where X is a K3 surface and ϕ is a marking of X. We will write $\phi_{\mathbb{R}}$ and $\phi_{\mathbb{C}}$ for the real and complex extensions of ϕ.

Two marked K3 surfaces (X, ϕ) and (X', ϕ') are *isomorphic* if there exists an isomorphism $f : X \to X'$ such that the following diagram:

$$\begin{array}{ccc} \Lambda_{K3} & \xrightarrow{\phi'} & H^2(X', \mathbb{Z}) \\ \big\downarrow{\scriptstyle \mathrm{id}} & & \big\downarrow{\scriptstyle f^*} \\ \Lambda_{K3} & \xrightarrow{\phi} & H^2(X, \mathbb{Z}) \end{array}$$

commutes.

For a marked K3 surface (X, ϕ), we can map the $H^{2,0}(X)$ to the period domain D via the marking ϕ:

$$H^{2,0}(X) \mapsto [\phi_{\mathbb{C}}^{-1}(H^{2,0}(X))] \in D. \tag{1.1.1}$$

Definition 1.1.10. The above association (1.1.1) is a well-defined map from the set of isomorphism classes of marked K3 surfaces to the period domain D. We call this map the *period map*:

$$\mathrm{Per} : \{\text{Marked complex K3 surfaces}\}/\sim \to D. \tag{1.1.2}$$

The weak Torelli theorem tells us a complex K3 surface is determined up to isomorphism by its Hodge structure. We can state it using marked K3 surfaces and the period map:

Theorem 1.1.11. *[Pvv71, LP81] The period map* Per *in* (1.1.2) *is injective.*

We would like to focus on the algebraic K3 surfaces, and we wish to construct a reasonable moduli space for them. The correct object to consider is the (quasi-)polarized

K3 surfaces.

Definition 1.1.12. A line bundle L on a K3 surface X is called *big and nef* if L is nef and $L^2 > 0$. A line bundle L is called *primitive* if L is not a multiple in the Picard group.

A *polarized K3 surface* is a pair (X, L), where X is a K3 surface and L is an ample primitive line bundle on X. A *quasi-polarized K3 surface* is a pair (X, L), where X is a K3 surface and L is a primitive, big and nef line bundle.

Two (quasi-)polarized K3 surfaces (X, L) and (X', L') are *isomorphic* if there exists an isomorphism $f : X \to X'$ such that $f^*L' \cong L$.

We call the intersection number L^2 the *degree* of the (quasi-)polarized K3 surface.

The coarse moduli space of degree $2d$ polarized K3 surfaces can be constructed using Hilbert scheme. The key ingredient is the following theorem due to Saint-Donat:

Theorem 1.1.13. *[SD74] Let L be any ample line bundle on a K3 surface X. Then L^2 is globally generated, and L^3 is very ample.*

Given any degree $2d$ polarized K3 surface (X, L), we can embed X into $\mathbb{P}^{9d+1}$ via L^3. The embedding is unique up to a projective transformation. The Hilbert polynomial of X in $\mathbb{P}^{9d+1}$ is $P(t) = 9dt^2 + 2$. All closed subvarieties in $\mathbb{P}^{9d+1}$ with a fixed Hilbert polynomial are parametrized by the Hilbert scheme, in our case we have the Hilbert scheme $\mathrm{Hilb}^{P(t)}(\mathbb{P}^{9d+1})$. The points in $\mathrm{Hilb}^{P(t)}(\mathbb{P}^{9d+1})$ that correspond to K3 surfaces actually form a quasi-projective variety, we denote it by Hilb_{2d}, the orbit space $\mathrm{Hilb}_{2d} / \mathrm{PGL}(9d + 2)$ exists as a quasi-projective variety, and it represents the coarse moduli space of polarized K3 surfaces of degree $2d$. We will denote the coarse moduli space by F°_{2d}, we have:

$$\mathsf{F}^{\circ}_{2d} \simeq \mathrm{Hilb}_{2d} / \mathrm{PGL}(9d + 2).$$

As we impose a polarization on a K3 surface, the possible domain for the holomorphic 2-forms is further restricted. We can model it by selecting a fixed primitive element $l \in \Lambda_{K3}$

and consider the period domain D_l:

$$D_l = \{\omega \in \mathbb{P}(l^{\perp}) \mid Q(\omega, \omega) = 0, Q(\omega, \overline{\omega}) > 0\},$$

where $l^{\perp}$ is the orthogonal complement of l in $\Lambda_{K3} \otimes \mathbb{C}$ with respect to the intersection form Q. The period domain D_l is a 19 dimensional complex manifold. We now define markings for (quasi-)polarized K3 surfaces:

Definition 1.1.14. [DK07] Given a (quasi-)polarized K3 surface (X, L), a *marking* of (X, L) is an isomorphism $\phi : \Lambda_{K3} \to H^2(X, \mathbb{Z})$ such that $\phi^{-1}(c_1(L)) = l$ for some primitive element $l \in \Lambda_{K3}$.

We say two marked (quasi-)polarized K3 surfaces (X, L, ϕ) and (X', L', ϕ') are *isomorphic* if there exists an isomorphism $f : X \to X'$ such that the following diagram commutes:

$$\begin{array}{ccc} \Lambda_{K3} & \xrightarrow{\phi'} & H^2(X', \mathbb{Z}) \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle f^*} \\ \Lambda_{K3} & \xrightarrow{\phi} & H^2(X, \mathbb{Z}) \end{array}$$

and $f^*L' \cong L$.

For any two primitive elements l and l' in Λ_{K3}, if $l^2 = l'^2 = 2d$, then they are isometric. Furthermore, any even integer $2d$ can be realized as the square of a primitive element in Λ_{K3}. Thus, for a (quasi-)polarized K3 surface (X, L), we can always find a marking $\phi : \Lambda_{K3} \to H^2(X, \mathbb{Z})$ such that $\phi^{-1}(c_1(L)) = l$ for some primitive element $l \in \Lambda_{K3}$ with $l^2 = 2d$. Sometimes we write D_{2d} for the period domain D_l with $l^2 = 2d$ if no confusion arises.

Similar to the case for complex K3 surfaces, we can define the period map for (quasi-)polarized K3 surfaces:

$$\{(X, L), \phi\} \mapsto [\phi_{\mathbb{C}}^{-1}(H^{2,0}(X))] \in D_l.$$

We wish to connect the period map for polarized K3 surfaces with the moduli space $\mathcal{F}_{2d}$. To do this, we need to remove the extra information coming from the marking. We have the following result known as the global Torelli theorem:

Theorem 1.1.15. *[Pvv71, LP81] Let (X, L) and (X', L') be two polarized K3 surfaces. If there exists an isometry of lattices $\psi : H^2(X, \mathbb{Z}) \to H^2(X', \mathbb{Z})$ such that $\psi(L) = L'$ and $\psi_{\mathbb{C}}(H^{2,0}(X)) = H^{2,0}(X')$. Then there exists a unique algebraic isomorphism $f : X \to X'$ such that $f^* = \psi$.*

Let Γ_l be the subgroup of $O(\Lambda_{K3})$ which stabilize the element l. We can remove the marking by consider the quotient:

$$\Gamma_l \backslash D_l.$$

The orbit space $\Gamma_l \backslash D_l$ is separated and can be endowed with a quasi-projective variety structure. Since the automorphism group is discrete, the dimension of $\Gamma_l \backslash D_l$ is 19.

Moreover, the Global Torelli theorem 1.1.15 implies the period map descends as an injective map:

$$\mathrm{Per} : \mathsf{F}^{\circ}_{2d} \to \Gamma_l \backslash D_l. \tag{1.1.3}$$

The above map is not surjective. In fact, the orbit space $\Gamma_l \backslash D_l$ is the coarse moduli space for quasi-polarized K3 surfaces. We denote it by F_{2d}, we have:

$$\mathsf{F}_{2d} \simeq \Gamma_l \backslash D_l.$$

Remark 1.1.16. We use the notation F_{2d} for the coarse moduli space of (quasi-)polarized K3 surfaces of degree $2d$. The notation $\mathcal{F}_{2d}$ is reserved for the fine moduli space.

The difference between a quasi-polarization and a polarization can be tested using (-2)-curves on the K3 surface. Let δ be an element in Λ_{K3} such that $\delta^2 = -2$ and $(\delta, l) = 0$, let H_δ be the hyperplane in D_l defined by $\{\omega \in D_l | (\delta, \omega) = 0\}$. We define the discriminant locus of D_l as:

$$\Delta_l = \bigcup_{\delta^2=-2,(\delta,l)=0} H_\delta.$$

Theorem 1.1.17. *[DK07, Theorem 9.4] Any point in D_l can be realized as the period point for a marked quasi-polarized K3 surface. The image of the period map* Per *in* (1.1.3) *is $\Gamma_l \backslash (D_l - \Delta_l)$.*

1.1.3 Moduli Spaces of lattice polarized K3 surfaces

In this section, we will generalize period domain constructions for (quasi-)polarized K3 surfaces to lattice polarized K3 surfaces. We follow the exposition in [DK07, Section 10]. We will fix a primitive sublattice:

$$\Lambda \hookrightarrow \Lambda_{K3},$$

we require the signature of Λ to be $(1, \rho - 1)$, where ρ is the rank of Λ. We start with some constructions motivated by (-2)-curves and nef classes on K3 surfaces.

Let $\Lambda_{-2} = \{m \in \Lambda | m^2 = -2\}$, consider the real cone:

$$\mathcal{V}_\Lambda = \{m \in \Lambda \otimes \mathbb{R} | m^2 \geq 0\}.$$

Let $\mathcal{V}_\Lambda^\circ$ be a connected component of $\mathcal{V}_\Lambda - \{0\}$. We fix a choice of a connected component $\mathcal{C}_\Lambda^+$ of the set:

$$\mathcal{V}_\Lambda^\circ \backslash \bigcup_{\delta \in \Lambda_{-2}} \delta^\perp.$$

Definition 1.1.18. [DK07] A *M-polarized K3 surface* is a pair (X, j). Where X is a K3 surface and $j : \Lambda \to \mathrm{Pic}(X)$ is a primitive lattice embedding such that $j(\mathcal{C}_\Lambda^+)$ contains a big and nef class. The polarization is called ample if $j(\mathcal{C}_\Lambda^+)$ contains an ample class.

We can construct the period domain for Λ-polarized K3 surfaces in a similar way. Let $N = \Lambda^\perp \subset \Lambda_{K3}$, we define:

$$D_\Lambda = \{\omega \in \mathbb{P}(N \otimes \mathbb{C}) \mid Q(\omega, \omega) = 0, Q(\omega, \overline{\omega}) > 0\}. \tag{1.1.4}$$

Consider the group $\Gamma_\Lambda \in O(\Lambda_{K3})$ that stabilizes the sublattice Λ. The group Γ_Λ acts on N thus acts on D_Λ. The orbit space $\Gamma_\Lambda \backslash D_\Lambda$ is again a quasi-projective variety, we denote it by F_Λ:

$$\mathsf{F}_\Lambda \simeq \Gamma_\Lambda \backslash D_\Lambda.$$

The automorphism group is discrete, so the dimension of F_Λ is $20 - \mathrm{rank}(\Lambda)$.

We can single out the coarse moduli space of K3 surfaces amplely polarized by Λ by considering the discriminant locus. Let $N_{-2} = \{\delta \in N | \delta^2 = -2\}$. For any $\delta \in N_{-2}$, we define $H_\delta = \{\omega \in D_\Lambda | (\delta, \omega) = 0\}$, and the discriminant locus:

$$\Delta_\Lambda = \bigcup_{\delta \in N_{-2}} H_\delta.$$

Let F°_Λ be the coarse moduli space of K3 surfaces amplely polarized by Λ, we have:

$$\mathsf{F}^\circ_\Lambda \simeq \Gamma_\Lambda \backslash (D_\Lambda - \Delta_\Lambda).$$

Remark 1.1.19. We use the notation F_Λ for the coarse moduli space of lattice polarized K3 surfaces. We will work with the moduli stack $\mathcal{F}_\Lambda$ in Section 4.3.1.

1.2 Elliptic surfaces over $\mathbb{P}^1$

Elliptic surfaces play central role in both complex algebraic geometry and number theory. In this section, we will survey the geometry of elliptic surfaces over $\mathbb{P}^1$, we focus on the relative minimal elliptic fibrations with a distinct section. We will start with the definition of elliptic surfaces and then study the fundamental line bundle and the numerical invariants of elliptic surfaces. We will introduce the Weierstrass models of elliptic surfaces and use them to study the moduli of elliptic surfaces. We follow the exposition in [Mir89] and [Mir81].

We begin with the definition of elliptic surfaces. Throughout this section, we will work over the complex numbers $\mathbb{C}$. An elliptic curve is a smooth projective curve of genus 1 with a distinguished point, the identity element under the group law.

Definition 1.2.1. An *elliptic surface over* $\mathbb{P}^1$ is a smooth projective surface X together with a surjective morphism $\pi : X \to \mathbb{P}^1$ such that the general fiber of π is an elliptic curve, and all fibers of π are connected.

A *minimal elliptic surface over* $\mathbb{P}^1$ is an elliptic surface X such that the fiber of π contains no (-1)-curve.

A section of an elliptic surface is a morphism $s : \mathbb{P}^1 \to X$ such that $\pi \circ s = \mathrm{id}_{\mathbb{P}^1}$.

Remark 1.2.2. The minimality condition above is *not* the same as the surface X is minimal. In fact, the above definition is relative minimality with respect to the fibration $\pi : X \to \mathbb{P}^1$.

We will only study minimal elliptic surfaces with a section in this Chapter. We will abbreviate a minimal elliptic surface over $\mathbb{P}^1$ with a section as an *elliptic surface* for simplicity.

1.2.1 Fundamental line bundle and numerical invariants

In this section, we will introduce the fundamental line bundle of an elliptic surface and study the numerical invariants of elliptic surfaces. The fundamental line bundle plays a central role in connecting elliptic surfaces with Weierstrass models.

We have the following basic results about various direct images of the sheaves on an elliptic surface. We will prove them under our setting for this chapter. For the general proof, we refer to [Mir89] and [Mir81].

Proposition 1.2.3. *[Mir81] Let $\pi : X \to \mathbb{P}^1$ be a minimal elliptic surface with a section $s : \mathbb{P}^1 \to X$. Let $S \subset X$ be the image of the section s. Then:*

1. $\pi_\star \mathcal{O}_X = \mathcal{O}_{\mathbb{P}^1}$,

2. $R^1\pi_\star \mathcal{O}_X$ *is invertible on* $\mathbb{P}^1$,

3. $\dim H^1(X, \mathcal{O}_X) = \dim H^0(\mathbb{P}^1, R^1\pi_\star \mathcal{O}_X)$,

4. $\dim H^2(X, \mathcal{O}_X) = \dim H^1(\mathbb{P}^1, R^1\pi_\star \mathcal{O}_X)$,

5. $R^1\pi_\star \mathcal{O}_X \cong \pi_\star (\mathcal{O}_X(S)/\mathcal{O}_X) \cong s^\star \mathcal{O}_X(S)$,

6. $S \cdot S = \deg R^1\pi_\star \mathcal{O}_X$,

7. *the canonical class K_X is an integral multiple of the fiber class F of the fibration $\pi : X \to \mathbb{P}^1$.*

Proof. For any closed point $t \in \mathbb{P}^1$, the fiber $X_t \in X$ define a same divisor class F in X. In particular the morphism $\pi : X \to \mathbb{P}^1$ is flat due to equi-dimensinality of the fibers. By Zariski's Lemma, we know $F = rF'$ for some $r \in \mathbb{Q}$ and F' effective divisor supported on fiber components of π. In our case $r = 1$ due to the existence of the section. Recall we require the fibers to be connected, and we know no multiple fiber can occur in this case, we have $\dim H^0(X_t, \mathcal{O}_{X_t}) = 1$ for all $t \in \mathbb{P}^1$. The first statement follows from the flat base change theorem.

For any $t \in \mathbb{P}^1$, we have the exact sequence:

$$0 \to \mathcal{O}(-X_t) \to \mathcal{O}_X \to \mathcal{O}_{X_t} \to 0.$$

So we have $\chi(X_t, \mathcal{O}_{X_t}) = \chi(X, \mathcal{O}_X) - \chi(X, \mathcal{O}(-X_t))$. Since $\mathcal{O}(-X_t)$ are isomorphic for all $t \in \mathbb{P}^1$, we know that:

$$\chi(X_t, \mathcal{O}_{X_t}) = \dim H^0(X_t, \mathcal{O}_{X_t}) - \dim H^1(X_t, \mathcal{O}_{X_t}) = 1 - \dim H^1(X_t, \mathcal{O}_{X_t})$$

is the constant 0 for all $t \in \mathbb{P}^1$. So we have $\dim H^1(X_t, \mathcal{O}_{X_t}) = 1$ for all $t \in \mathbb{P}^1$. Again by the flat base change theorem, we have $R^1\pi_\star\mathcal{O}_X$ is a locally free sheaf of rank 1 on $\mathbb{P}^1$.

Now we consider the Leray spectral sequence for the morphism $\pi : X \to \mathbb{P}^1$:

$$E_2^{p,q} = H^p(\mathbb{P}^1, R^q\pi_\star\mathcal{O}_X) \Rightarrow H^{p+q}(X, \mathcal{O}_X).$$

Taking the $p + q = 1$ terms, we have:

$$0 \to H^1(\mathbb{P}^1, R^0\pi_\star\mathcal{O}_X) \to H^1(X, \mathcal{O}_X) \to H^0(\mathbb{P}^1, R^1\pi_\star\mathcal{O}_X) \to 0.$$

The third statement follows from $H^1(\mathbb{P}^1, R^0\pi_\star\mathcal{O}_X) \cong H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = 0$. The fourth statement follows from the $p + q = 2$ terms of the Leray spectral sequence, we notice only the $p = q = 1$ term is non-zero.

Next we consider the exact sequence:

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(S) \to \mathcal{O}_X(S)/\mathcal{O}_X \to 0.$$

We apply the functor $\pi_\star$ to the above exact sequence, we have:

$$0 \to \pi_\star\mathcal{O}_X \xrightarrow{f} \pi_\star\mathcal{O}_X(S) \to s^\star\mathcal{O}_X(S) \to R^1\pi_\star\mathcal{O}_X \to R^1\pi_\star\mathcal{O}_X(S) \to 0.$$

By base-change the sequence to the fibers of π, we have f is an isomorphism and $R^1\pi_\star\mathcal{O}_X(S) = 0$. So we have $R^1\pi_\star\mathcal{O}_X \cong s^\star\mathcal{O}_X(S) \cong \pi_\star\left(\mathcal{O}_X(S)/\mathcal{O}_X\right)$.

The sixth statement follows from direct computation based on the previous statements:

$$S \cdot S = \deg(O(S)|_S) = \deg(\mathcal{O}(S)/\mathcal{O}_X) = \deg R^1\pi_\star\mathcal{O}_X.$$

The last statement follows from the adjunction formula applied to a general fiber

F of $\pi : X \to \mathbb{P}^1$:

$$K_F = (K_X + F)|_F = K_X \cdot F + F^2 = K_X \cdot F.$$

So we have $K_X \cdot F = 0$. By Zariski's Lemma, we know K_X is an integral multiple of F.

$\square$

We are now ready to introduce the fundamental line bundle of an elliptic surface:

Definition 1.2.4. [Mir89] Let $\pi : X \to \mathbb{P}^1$ be a minimal elliptic surface with a section $s : \mathbb{P}^1 \to X$. The *fundamental line bundle* of X is the invertible sheaf $L := (R^1\pi_\star\mathcal{O}_X)^{-1}$ on $\mathbb{P}^1$. The *fundamental invariant* of X is the degree of the fundamental line bundle, denoted by $N := \deg L$.

Based on the previous Proposition 1.2.3 and our definition of the fundamental invariant N, we have the following corollary:

Corollary 1.2.5. *[Mir81] Let $\pi : X \to \mathbb{P}^1$ be a minimal elliptic surface with a section $s : \mathbb{P}^1 \to X$. Let N be the fundamental invariant of the elliptic fibration, $S \subset X$ be the image of the section s. Assume $N > 0$, then the following statements hold:*

1. $\dim H^1(X, \mathcal{O}_X) = 0$, $\dim H^2(X, \mathcal{O}_X) = N - 1$ *and* $S \cdot S = -N$*,*

2. $K_X = (N-2)F$*, where F is the fiber class of the fibration $\pi : X \to \mathbb{P}^1$.*

Proof. The first statement follows from the previous Proposition 1.2.3. The second statement follows from the adjunction formula applied to the section S, assume $K_X = rF$, we have:

$$\mathcal{O}_{\mathbb{P}^1}(2) \cong K_S = (K_X + S)|_S = (rF + S)|_S.$$

Compare the degree of the above equation, we have $r = N - 2$.

$\square$

Remark 1.2.6. The fundamental invariant N is always a non-negative integer. When $N = 0$, the elliptic surface is a product of two elliptic curves. When $N = 1$, the elliptic surface is a rational elliptic surface. When $N = 2$, the elliptic surface is a K3 surface.

1.2.2 Weierstrass fibrations

In this section, we will introduce the Weierstrass fibrations. Weierstrass fibrations can have singularities. We can construct a Weierstrass fibration from any elliptic surface by contracting some fiber components. We will later show this construction is reversible, but not all Weierstrass fibrations are coming from elliptic surfaces.

Definition 1.2.7. [Mir89] A *Weierstrass fibration* over $\mathbb{P}^1$ is a reduced irreducible projective surface $\bar{X}$ together with a flat proper morphism $\bar{\pi} : \bar{X} \to \mathbb{P}^1$ such that every fiber is one of the following types:

1. a smooth elliptic curve,
2. a rational curve with a single node,
3. a rational curve with a single cusp.

A section of a Weierstrass model is a morphism $\bar{s} : \mathbb{P}^1 \to \bar{X}$ such that $\bar{\pi} \circ \bar{s} = \mathrm{id}_{\mathbb{P}^1}$, and the image of $\bar{s}$ does not pass through the singular points of the fibers.

Similar to the case of elliptic surfaces, we will only study Weierstrass fibrations over $\mathbb{P}^1$ with a section in this Chapter. We will abbreviate a Weierstrass fibration over $\mathbb{P}^1$ with a section as a *Weierstrass fibration* for simplicity.

A Weierstrass fibration over $\mathbb{P}^1$ is a family of elliptic curves over $\mathbb{P}^1$ with mild singularities. We recall some basic results about cohomology of elliptic curves.

Proposition 1.2.8. *Let E be a smooth elliptic curve or a rational curve with a single node or a single cusp. We have:*

$$H^0(E, \mathcal{O}_E) \cong \mathbb{C}, \quad H^1(E, \mathcal{O}_E) \cong \mathbb{C}.$$

Furthermore, let $p \in E$ be a smooth point, we have for any $n \geq 1$:

$$H^0(E, \mathcal{O}_E(np)) \cong \mathbb{C}^n, \quad H^1(E, \mathcal{O}_E(np)) = 0.$$

We review the construction of the *Weierstrass Basis* for the elliptic curve (E, p).

Proposition 1.2.9. *[Mir89] Let $S(n) := H^0(E, \mathcal{O}_E(np))$ for any $n \geq 0$. We have:*

1. *There is a nonzero element $y \in S(3) \backslash S(2)$ such that $y^2 \in \operatorname{Sym}^3 S(2)$,*
2. *There is a nonzero element $x \in S(2) \backslash S(1)$ such that:*

$$y^2 = x^3 + Ax + B, \text{ for some } A, B \in \mathbb{C}.$$

3. *If $(x_1, y_1)and(x_2, y_2)$ are two pairs of elements satisfying the above conditions, then there exists a constant $\lambda \in \mathbb{C}^\star$ such that: $x_2 = \lambda^2 x_1$ and $y_2 = \lambda^3 y_1$.*

The resulting pair (x, y) is called a Weierstrass basis *for the elliptic curve (E, p), they're unique up to the action in the third statement.*

Proof. We view the elements in $S(n)$ as rational functions on E. We have the constant function 1 as basis for $S(0)$ and $S(1)$. We take any $f \in S(2) \backslash S(1)$, then take any $g \in S(3) \backslash S(2)$. Then we can construct basis for $S(6)$ as $\{1, f, g, f^2, fg, f^3\}$, the linear independence is guaranteed by the vanishing orders of the functions at the point p. Now we note $g^2 \in S(6)$, so we can write:

$$g^2 = a_6 f^3 + a_5 f^2 g + a_4 f g^2 + a_3 f^2 + a_2 g + a_1.$$

The leading coefficient a_6 is nonzero since g^2 is not $S(5)$. By replace $f \to a_6 f$ and $g \to a_6^2 g$, we can assume $a_6 = 1$. Then we can replace $g \to g - \frac{a_5}{2} f - \frac{a_3}{2}$ to eliminate the linear term of g in the above equation. We can then replace $f \to f + \frac{a_2}{3}$ to eliminate the quadratic term of f in the above equation. We use x and y to denote the new f and g respectively, then we have:

$$y^2 = x^3 + Ax + B.$$

So we have addressed the first two statements.

For the last uniqueness statement, assume we have (x_1, y_1) and (x_2, y_2) satisfying the above conditions. Let $y_1^2 = x_1^3 + A_1x_1 + B_1$ and $y_2^2 = x_2^3 + A_2x_2 + B_2$ be the Weierstrass equations for the two pairs. Let $y_2 = \alpha y_1 + \beta x_1 + \gamma$ with $\alpha \neq 0$, then we have:

$$y_2^2 = \alpha^2 y_1^2 + 2(\beta x_1 + \gamma)\alpha y_1 + (\beta x_1 + \gamma)^2 = \alpha^2(x_1^3 + A_1x_1 + B_1) + 2(\beta x_1 + \gamma)\alpha y_1 + (\beta x_1 + \gamma)^2.$$

The rational function $y_2^2 \in \operatorname{Sym}^3 S(2)$, so in coefficients of x_1y_1 and y_1 must be zero. We have $\beta = 0$ and $\gamma = 0$. We have $y_2 = \alpha y_1$. Now let $x_2 = ax_1 + b$ with $a \neq 0$, substitute in $y_2^2 = x_2^3 + A_2x_2 + B_2$ we have:

$$\alpha^2 y_1^2 = (ax_1 + b)^3 + A_2(ax_1 + b) + B_2 = \alpha^2(x_1^3 + A_1x_1 + B_1).$$

We have $b = 0$ and thus $x_2 = ax_1$. Then we have $\alpha^2 = a^3$, there exists a constant $\lambda \in \mathbb{C}^\star$ such that $\alpha = \lambda^3$ and $a = \lambda^2$. $\square$

We can define the fundamental line bundle and the fundamental invariant of a Weierstrass fibration in the same way as for elliptic surfaces. The strong restriction on the singular fibers of a Weierstrass fibration will give us the following result:

Proposition 1.2.10. *[Mir89] Let $\bar{\pi} : \bar{X} \to \mathbb{P}^1$ be a Weierstrass fibration with a section $\bar{s} : \mathbb{P}^1 \to \bar{X}$. Let $S \subset \bar{X}$ be the image of the section $\bar{s}$. Then:*

1. *$\bar{\pi}_\star \mathcal{O}_{\bar{X}} = \mathcal{O}_{\mathbb{P}^1}$ and $R^1\bar{\pi}_\star \mathcal{O}_{\bar{X}}$ is invertible on $\mathbb{P}^1$,*

2. *for any $n \geq 1$, $\bar{\pi}_\star \mathcal{O}_{\bar{X}}(nS)$ is a locally free sheaf of rank n on $\mathbb{P}^1$ and $R^1\bar{\pi}_\star \mathcal{O}_{\bar{X}}(nS) = 0$.*

Proof. The Weierstrass fibration has all fibers to be smooth elliptic curve or rational curves with a single node or a single cusp. Both the first and second statements follow from Proposition 1.2.8 and flat base change theorem. □

Definition 1.2.11. [Mir89] Let $\bar{\pi} : \bar{X} \to \mathbb{P}^1$ be a Weierstrass fibration with a section $\bar{s} : \mathbb{P}^1 \to \bar{X}$. The *fundamental line bundle* of $\bar{X}$ is the invertible sheaf $\bar{L} := (R^1\bar{\pi}_\star \mathcal{O}_{\bar{X}})^{-1}$ on $\mathbb{P}^1$. The *fundamental invariant* of $\bar{X}$ is the degree of the fundamental line bundle, denoted by $N := \deg \bar{L}$.

We have the following splitting results for the direct images of the sheaves on a Weierstrass fibration:

Lemma 1.2.12. *[Mir89] Under the setting of Proposition 1.2.10. For any $n \geq 2$, there exists a short exact sequence:*

$$0 \to \bar{\pi}_\star \mathcal{O}_{\bar{X}}\left((n-1)S\right) \to \bar{\pi}_\star \mathcal{O}_{\bar{X}}(nS) \to \bar{L}^{-n} \to 0.$$

Moreover, we have a splitting:

$$\bar{\pi}_\star \mathcal{O}_{\bar{X}}(nS) \cong \mathcal{O}_{\bar{X}} \oplus \bar{L}^{-2} \oplus \bar{L}^{-3} \oplus \cdots \oplus \bar{L}^{-n}.$$

Proof. Consider the exact sequence:

$$0 \to \mathcal{O}_{\bar{X}}\left((n-1)S\right) \to \mathcal{O}_{\bar{X}}(nS) \to \mathcal{O}_S(nS) \to 0.$$

We apply the functor $\bar{\pi}_\star$ to the above exact sequence and use the previous Proposition 1.2.10:

$$0 \to \bar{\pi}_\star \mathcal{O}_{\bar{X}}\left((n-1)S\right) \to \bar{\pi}_\star \mathcal{O}_{\bar{X}}(nS) \to \mathcal{O}_S(nS) \to R^1\bar{\pi}_\star \mathcal{O}_{\bar{X}}\left((n-1)S\right) = 0.$$

The first statement follows from $\mathcal{O}_S(nS) \cong \bar{L}^{-n}$.

We have shown in the previous Proposition 1.2.10 that $\bar{\pi}_\star \mathcal{O}_{\bar{X}}(nS)$ is vector bundle. Now note any vector bundle on $\mathbb{P}^1$ is a direct sum of line bundles, we have the second statement. $\square$

Remark 1.2.13. The splitting in Lemma 1.2.12 actually holds for any base curve, not just $\mathbb{P}^1$. The idea is to use Weierstrass basis (x, y) fiberwise. Using Proposition 1.2.9, we know the $(1, x, y)$ will define canonical directions on the fibers of $\bar{\pi}_\star \mathcal{O}_{\bar{X}}(3S)$. For general n, let $k = \lfloor \frac{n}{2} \rfloor$ and $l = \lfloor \frac{n-3}{2} \rfloor$, we can construct:

$$\{1, x, x^2, ..., x^k, y, xy, x^2 y, ..., x^l y\}$$

as a basis for fiber of $\bar{\pi}_\star \mathcal{O}_{\bar{X}}(nS)$. The linear independence again follows from the vanishing orders. The basis defines canonical directions every fiber, and each direction can be identified with a line bundle by restrict the exact sequence in Lemma 1.2.12 to the fiber.

1.2.3 Weierstrass equations

In this section, we will introduce the Weierstrass equations of Weierstrass fibrations. We begin with the local description. Let $\{U_i\}$ be a open affine cover of $\mathbb{P}^1$, so the fundamental line bundle $\bar{L}$ is trivial on each U_i. We choose a basis t_i for the local section space $\Gamma(U_i, \bar{L})$. Then t_i^n will be a basis for the local section space $\Gamma(U_i, \bar{L}^n)$ for any $n \in \mathbb{Z}$.

We can apply the standard argument to the local sections to build the Weierstrass equation. We first pick an x_i in $\Gamma(U_i, \bar{\pi}_\star(\mathcal{O}_{\bar{X}}(2S)))$ such that x_i projects onto t_i^2 under the isomorphism in Lemma 1.2.12. Then we pick a y_i in $\Gamma(U_i, \bar{\pi}_\star(\mathcal{O}_{\bar{X}}(3S)))$ such that y_i projects onto t_i^3 under the isomorphism in Lemma 1.2.12. We know $\{x_i^3, x_i y_i, x_i^2, y_i, x_i, 1\}$ will be a basis for the local section space $\Gamma(U_i, \bar{\pi}_\star \mathcal{O}_{\bar{X}}(6S))$. Now y_i^2 will also be a local section of $\bar{\pi}_\star \mathcal{O}_{\bar{X}}(6S)$, so we can write:

$$y_i^2 = a_6 x_i^3 + a_5 x_i y_i + a_4 x_i^2 + a_3 y_i + a_2 x_i + a_1,$$

where $a_1, a_2, a_3, a_4, a_5, a_6$ are regular functions on U_i. By compare the leading terms, we know $a_6 = 1$. Now we can replace $y_i \to y_i - \frac{a_5}{2} x_i - \frac{a_3}{2}$ to eliminate the linear term of y_i in the above equation. We can then replace $x_i \to x_i + \frac{a_2}{3}$ to eliminate the quadratic term of x_i in the above equation. So we get the following equation:

$$y_i^2 = x_i^3 + A x_i + B.$$

We can glue the local descriptions to get a global description of the Weierstrass fibration. We will need to understand the transition functions for x_i, y_i, A, B. Let $a_{ij} \in \mathbb{C}^\star$ be the transition functions for t_i and t_j, then we have:

Lemma 1.2.14. *[Mir89] The transition maps for x_i, y_i, A, B are given by:*

$$x_j = a_{ij}^{-2} x_i \qquad y_j = a_{ij}^{-3} y_i$$
$$A_j = a_{ij}^{-4} A_i \qquad B_j = a_{ij}^{-6} B_i.$$

From the above transition rules, we can glue the local functions $\{A_i\}$ and $\{B_i\}$ to global sections of line bundles. Consider local sections $\{A_i \cdot t_i^4\}$ and $\{B_i \cdot t_i^6\}$ on the open affine cover $\{U_i\}$, Lemma 1.2.14 tells us that they glue to a global section of $\bar{L}^4$ and $\bar{L}^6$ respectively.

Remark 1.2.15. Recall in Definition 1.2.7, we require the fibration has general fibers to be smooth elliptic curves. This translates to the condition that the discriminant:

$$\Delta := 4A^3 + 27B^2$$

is not identically zero as a global section of $\bar{L}^{12}$.

Lemma 1.2.16. *[Mir89] Fix a fundamental line bundle $\bar{L}$ over $\mathbb{P}^1$. Then the two pairs (A_1, B_1) and (A_2, B_2) together with $\bar{L}$ induce isomorphic Weierstrass fibrations if and only if there exists a constant $\lambda \in \mathbb{C}^\star$ such that:*

$$A_2 = \lambda^4 A_1, \quad B_2 = \lambda^6 B_1.$$

Proof. We will only show the only if direction, that is if the two pairs (A_1, B_1) and (A_2, B_2) together with $\bar{L}$ induce isomorphic Weierstrass fibrations, then there exists a constant $\lambda \in \mathbb{C}^\star$ such that $A_2 = \lambda^4 A_1$ and $B_2 = \lambda^6 B_1$.

Pick an affine open set $U \in \mathbb{P}^1$, let $y_1^2 = x_1^3 + A_1 x_1 + B_1$ and $y_2^2 = x_2^3 + A_2 x_2 + B_2$ be the local Weierstrass equations for the two pairs. Let $y_2 = \alpha y_1 + \beta x_1 + \gamma$ with $\alpha \neq 0$, Using the same technique in the proof of Proposition 1.2.9, we have $x_2 = \lambda^2 x_1$ and $y_2 = \lambda^3 y_1$ in the local Weierstrass equation, we have:

$$x_2^3 + A_2 x_2 + B_2 = y_2^2 = \lambda^6 y_1^2 = \lambda^6 (x_1^3 + A_1 x_1 + B_1) = x_1^3 + \lambda^4 A_1 x_1 + \lambda^6 B_1.$$

So we have $A_2 = \lambda^4 A_1$ and $B_2 = \lambda^6 B_1$. It's not hard to see the above argument glue to the global level. $\square$

We can further globalize the discussion, consider the natural map:

$$\phi : \bar{\pi}^\star \bar{\pi}_\star \mathcal{O}_{\bar{X}}(3S) \to \mathcal{O}_{\bar{X}}(3S).$$

We have seen above morphism is surjective on all fibers, so it's a surjection between $\mathcal{O}_{\bar{X}}$-modules, hence induces a morphism:

$$f : X \to \underline{\mathrm{Proj}}_{\mathbb{P}^1}(\bar{\pi}_\star \mathcal{O}_{\bar{X}}(3S)).$$

Recall $\bar{\pi}_\star \mathcal{O}_{\bar{X}}(3S) \cong \mathcal{O}_{\mathbb{P}^1} \oplus \bar{L}^{-2} \oplus \bar{L}^{-3}$, so we have:

$$\mathbb{P} := \underline{\mathrm{Proj}}_{\mathbb{P}^1}(\bar{\pi}_\star \mathcal{O}_{\bar{X}}(3S))$$

is a $\mathbb{P}^2$-bundle over $\mathbb{P}^1$. Let $p : \mathbb{P} \to \mathbb{P}^1$ be the natural projection. Then we have:

$$X \xrightarrow{p \circ f} \mathbb{P}^1$$

recovers the Weierstrass fibration map $\bar{\pi}$.

Let (A, B) be the global sections of $\bar{L}^4$ and $\bar{L}^6$ that induce the Weierstrass fibration $\bar{\pi} : \bar{X} \to \mathbb{P}^1$. The local Weierstrass equation $y_i^2 = x_i^3 + A_i x_i + B_i$ over $U_i \in \mathbb{P}^1$ can be globalized to a global Weierstrass equation:

$$Y^2 Z = X^3 + AXZ^2 + BZ^3.$$

The variables X, Y, Z in the above equation formally correspond to $O_{\mathbb{P}^1}, \bar{L}^2, \bar{L}^3$ respectively. For more detail, consult the discussion in [MS72].

Definition 1.2.17. [MS72, Mir89] A *Weierstrass equation* over $\mathbb{P}^1$ is a choice of a line bundle $\bar{L}$ over $\mathbb{P}^1$, together with an equation:

$$Y^2 Z = X^3 + AXZ^2 + BZ^3,$$

where A, B are global sections of $\bar{L}^4$ and $\bar{L}^6$ such that the discriminant $\Delta := 4A^3 + 27B^2$ is not identically zero. The variables Z, X, Y are global coordinates corresponding to the projectivization of $\bar{\pi}_\star \mathcal{O}_{\bar{X}}(3S) \cong \mathcal{O}_{\mathbb{P}^1} \oplus \bar{L}^{-2} \oplus \bar{L}^{-3}$ respectively.

The section $\bar{s}$ factor through the natural section s_0 of the $\mathbb{P}^2$-bundle $\mathbb{P}$ over $\mathbb{P}^1$ corresponds to: $\mathcal{O}_{\mathbb{P}^1} \oplus \bar{L}^{-2} \oplus \bar{L}^{-3} \twoheadrightarrow \bar{L}^{-2}$. More directly, the image of the section s_0 is given by $X = Z = 0$ in $\mathbb{P}$. We have the following commutative diagram:

$$\begin{tikzcd}
\bar{X} \arrow[r, "f"] \arrow[dr, shift left, "\bar{\pi}"] & \mathbb{P} \arrow[d, shift left, "p"] \\
& \mathbb{P}^1 \arrow[ul, shift left, "\bar{s}"] \arrow[u, shift left, "s_0"]
\end{tikzcd}$$

We have established the following 1:1 correspondence:

$$\{\text{Weierstrass Equations}/\sim\} \longleftrightarrow \{\text{Weierstrass fibrations}/\sim\}$$

1.2.4 Weierstrass models of elliptic surfaces

In this section, we will connect the Weierstrass fibrations with the elliptic surfaces. Unlike in Section 1.2.3, we will not get a one-to-one correspondence. We will specify the conditions for a Weierstrass fibration to be a Weierstrass model of an elliptic surface with a section.

Let $\pi : X \to \mathbb{P}^1$ be an elliptic surface with a section $s : \mathbb{P}^1 \to X$. Kodaira [Kod63] has classified the singular fibers of an elliptic surface, we have the following table:

Table 1.1. Kodaira's classification of singular fibers.

Name	Fiber	Reducible
I_0	smooth elliptic curve	No
I_1	nodal rational curve	No
I_2	two smooth rational curves meeting at two points	Yes
I_3	three smooth rational curves meeting in a cycle; a triangle	Yes
$I_{N\geq 4}$	N smooth rational curves meeting in a cycle, dual graph $\widetilde{A}_N$	Yes
I_N^* $N \geq 0$	$N+5$ smooth rational curves meeting with dual graph $\widetilde{D}_{N+4}$	Yes
II	a cuspidal rational curve	No
III	two smooth rational curves meeting at one point to order 2	Yes
IV	three smooth rational curves all meeting at one point	Yes
IV^*	7 smooth rational curves meeting with dual graph $\widetilde{E}_6$	Yes
III^*	8 smooth rational curves meeting with dual graph $\widetilde{E}_7$	Yes
II^*	9 smooth rational curves meeting with dual graph $\widetilde{E}_8$	Yes
$I_{N,M}^*$, $N \geq 0$	topologically an I_N, but each curve has multiplicity M	Yes

All components of the reducible fibers of an elliptic surface are smooth rational curves with self-intersection -2.

Using Kodaira's classification of singular fibers, we can contract the components of the singular fibers that are not meeting S to get a surface with only rational double points. We have the following result:

Proposition 1.2.18. *[Mir89] Let $F_0 \subset X$ be a reducible singular fiber of an elliptic surface $\pi : X \to \mathbb{P}^1$ with a section $s : \mathbb{P}^1 \to X$. Let $S \subset X$ be the image of the section s. Then the following statements hold:*

1. *F_0 is a reducible fiber of type $I_{N\geq 2}, I_N^*, III, IV, IV^*, III^*, II^*, I_{N,M}^*$,*

2. *the map $p : X \to \bar{X}$ that contracts all components of singular fibers that do not meet S produces a surface $\bar{X}$ with at worst rational double points. The surface $\bar{X}$ is a Weierstrass fibration over $\mathbb{P}^1$ with a section.*

Proof. Since $F_0 \cdot S = 1$, we know the component of F_0 meeting S must have multiplicity 1. So by checking the Kodaira's Classification Table 1.1, we have F_0 is a reducible fiber of type $I_{N\geq 2}, I_N^*, III, IV, IV^*, III^*, II^*, I_{N,M}^*$.

For the second statement, note all the types of reducible fibers in the first statement consist of smooth rational curves with self-intersection -2. The components that do not meet the section S form a Dynkin diagram, so the contraction $\bar{X}$ is a surface with at worst rational double points. Now the contraction is fiberwise, so the map $\pi : X \to \mathbb{P}^1$ factors through the contraction map $p : X \to \bar{X}$, so we have a map $\bar{\pi} : \bar{X} \to \mathbb{P}^1$. The fibers of $\bar{\pi}$ clearly satisfy the conditions in Definition 1.2.7. The section $\bar{s} : \mathbb{P}^1 \to \bar{X}$ is the composition of the section $s : \mathbb{P}^1 \to X$ and the contraction map $p : X \to \bar{X}$. $\square$

We have constructed a Weierstrass fibration from a minimal elliptic surface. Conversely, if we have a Weierstrass fibration, we can take its minimal resolution to get a minimal elliptic surface. We have the following two maps:

$$\{\text{minimal elliptic surfaces w/section}\} \underset{j}{\overset{\iota}{\rightleftarrows}} \{\text{Weierstrass fibrations}\}.$$

The composition $j \circ \iota$ is the identity map, but the composition $\iota \circ j$ is not. In fact, the map j is surjective, the map ι is strictly injective. Based on the classification of the singular fibers Table 1.1, we know $Im(\iota)$ are those Weierstrass fibrations with at worst rational double points.

We want to specify the conditions on Weierstrass equation for the corresponding fibration has at worst rational double points. The key idea is using the double induced by $\bar{\pi}_\star \mathcal{O}_{\bar{X}}(2S)$ to detect the singularities. Recall the natural map:

$$\phi : \bar{\pi}^\star \bar{\pi}_\star \mathcal{O}_{\bar{X}}(2S) \to \mathcal{O}_{\bar{X}}(2S).$$

Examine on the fibers, we know the morphism ϕ is surjective on all fibers, hence induces a morphism:

$$g : \bar{X} \to \underline{\mathrm{Proj}}_{\mathbb{P}^1}(\bar{\pi}_\star \mathcal{O}_{\bar{X}}(2S)) = \underline{\mathrm{Proj}}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \bar{L}^{-2}) := \mathbb{F}.$$

The map g is a double-covering, we can compute the branch locus:

Proposition 1.2.19. *[Mir89] Let $\bar{\pi} : \bar{X} \to \mathbb{P}^1$ be a Weierstrass fibration and $Y^2 Z = X^3 + AXZ^2 + BZ^3$ be its Weierstrass equation. Let $g : \bar{X} \to \mathbb{F}$ be the double-covering produced by above process. Then the branch locus of g is a disjoint union of the following divisors:*

1. *the divisor R corresponds to $Z = 0$,*

2. *the divisor T corresponds to $X^3 + AXZ^2 + BZ^2 = 0$.*

Proof. The global coordinates Z, X, Y correspond to the projectivization of $\mathcal{O}_{\bar{X}}, \bar{L}^{-2}, \bar{L}^{-3}$ respectively. If $Z \neq 0$, the map g is given by the projection: $[X : Y : Z] \to [Z : X]$. The infinity section $[0 : 1 : 0]$ can be mapped to $[0 : 1]$ on $\mathbb{F}$.

It's easy to verify that the above description for g agrees with its construction. The branching happens if $Z = 0$ or $Y = 0$, the corresponding locus consists of two components: R given by $Z = 0$ and T given by $X^3 + AXZ^2 + BZ^2 = 0$. $\square$

The classical result of double coverings tells us the double cover is smooth if and only if the branch locus is smooth, the double cover has at worst rational double points if and only if the branch locus has simple singularities. We can translate the condition to the Weierstrass equation:

Lemma 1.2.20. *[Mir89] Let $\bar{\pi} : \bar{X} \to \mathbb{P}^1$ be a Weierstrass fibration and $Y^2Z = X^3 + AXZ^2 + BZ^3$ be its Weierstrass equation. For point $p \in \mathbb{P}^1$ and any section s of line bundle on $\mathbb{P}^1$, the integer $v_p(s)$ is the order of the zero of s at p. Then the Weierstrass fibration has at worst rational double points if and only if the following conditions hold:*

- *there is no point $p \in \mathbb{P}^1$ such that $v_p(A) \geq 4$ and $v_p(B) \geq 6$.*

Proof. The two components of the branch locus R and T are clearly disjoint. The divisor R is clearly smooth. Thus, the Weierstrass fibration has at worst rational double points if and only if the divisor T has simple singularities. The local equation of T has degree 3, so we need to guarantee there will be no triple tacnodes. Equivalently, we need to guarantee after at any point $c \in T$, we will not have triple point again.

We take t as the local coordinates on the base $\mathbb{P}^1$, and x as the local coordinates on the fiber. We have the local equation of T is given by: $x^3 + A(t)x + B(t) = 0$. We may assume the point $c \in T$ has coordinates $(x_0, 0)$. If point c is a triple tacnode, in particular $(x_0, 0)$ must be a triple root for $x^3 + A(0)x + B(0) = 0$. So we have $A(0) = 0$ and $B(0) = 0$ and $x_0 = 0$. Now we know the triple tacnode must be $(0, 0)$. The equation $x^3 + A(t)x + B(t) = 0$ has a triple root at $(0, 0)$ if and only if $t^2 | A(t)$ and $t^3 | B(t)$. To make the triple root $(0, 0)$ a triple tacnode, we blow up the point $(0, 0)$ and get the local equation:

$$t^3 \left(x^3 + (A(t)/t^2)x + (B(t)/t^3)\right) = 0.$$

Repeat the argument for $x^3 + (A(t)/t^2)x + (B(t)/t^3)$ has a triple root at $(0, 0)$, we have $t^4 | A(t)$ and $t^6 | B(t)$.

Conversely, if $t^4|A(t)$ and $t^6|B(t)$, it's straightforward to check $(0,0)$ is a triple tacnode. $\square$

1.2.5 Moduli spaces of elliptic surfaces

In this section, we construct the moduli space of elliptic surfaces. We have seen in the previous section that the elliptic surfaces are in 1:1 correspondence with the Weierstrass equations subject to certain conditions. The following Proposition summarizes the correspondence:

Proposition 1.2.21. *[Mir89] Let $\pi : X \to \mathbb{P}^1$ be a minimal elliptic surface with a section $s : \mathbb{P}^1 \to X$. It is in 1:1 correspondence with a Weierstrass fibration $\bar{\pi} : \bar{X} \to \mathbb{P}^1$ with a section $\bar{s} : \mathbb{P}^1 \to \bar{X}$ such that $\bar{X}$ has at worst rational double points. Furthermore, the Weierstrass fibration is in 1:1 correspondence with the Weierstrass equations $Y^2Z = X^3 + AXZ^2 + BZ^3$ subject to the following conditions:*

1. *the discriminant $\Delta := 4A^3 + 27B^2$ is not identically zero (see Remark 1.2.15),*
2. *there is no point $p \in \mathbb{P}^1$ such that $v_p(A) \geq 4$ and $v_p(B) \geq 6$ (see Lemma 1.2.20).*

To form the moduli space of elliptic surfaces, we need to fix the fundamental invariant N, then fundamental line bundles for the elliptic surface and the induced Weierstrass fibration will be $L \cong \bar{L} \cong \mathcal{O}_{\mathbb{P}^1}(N)$. The total parameter space for the Weierstrass equations is the space of global sections of $\mathcal{O}_{\mathbb{P}^1}(4N)$ and $\mathcal{O}_{\mathbb{P}^1}(6N)$. Let $V_N := \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(N))$, define $T_N \subset V_{4N} \oplus V_{6N}$ to be the subset of pairs (A, B) subject to the conditions in Proposition 1.2.21. The isomorphism of the elliptic surfaces coming from two sources. The first source is the isomorphism of the Weierstrass equations, recall Lemma 1.2.16, we have $\mathbb{C}^\star$ acts on $V_{4N} \oplus V_{6N}$ by:

$$\lambda \cdot (A, B) = (\lambda^4 A, \lambda^6 B).$$

The second source is the automorphism of the base $\mathbb{P}^1$. Let T_0, T_1 be the standard coordinates on $\mathbb{P}^1$, the group $\mathrm{SL}_2(\mathbb{C})$ on coordinates by:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} T_0 \\ T_1 \end{pmatrix} = \begin{pmatrix} aT_0 + bT_1 \\ cT_0 + dT_1 \end{pmatrix}. \tag{1.2.1}$$

Identify the V_{4N} and V_{6N} with the binary forms of degree $4N$ and $6N$ respectively, we have induced the action of $\mathrm{SL}_2(\mathbb{C})$ on $V_{4N} \oplus V_{6N}$.

Remark 1.2.22. The action of $\mathbb{C}^\star$ on $V_{4N} \oplus V_{6N}$ is not effective. We prefer to work with the induced effective action:

$$\lambda \cdot (A, B) = (\lambda^2 A, \lambda^3 B).$$

We will work with the reduced effective action from now on.

Proposition 1.2.23. *Set theoretically, the moduli space of elliptic surfaces with fundamental invariant N is the quotient space:*

$$\mathsf{E}_N := T_N/(\mathbb{C}^\star \times \mathrm{SL}_2(\mathbb{C})).$$

Remark 1.2.24. We use the notion E_N to denote the coarse moduli space of elliptic surfaces with fundamental invariant N. We will work with the corresponding quotient stack $\mathcal{E}_N := [T_N/\mathbb{C}^\star \times \mathrm{SL}_2(\mathbb{C})]$ in Section 4.3.1.

Now we want to make the set E_N an algebraic variety. Which is in fact the coarse moduli space of the elliptic surfaces. The main idea is to use the geometric invariant theory (GIT). We can view the quotient process as a three-step process:

1. We take the group $\mathbb{C}^\star$ action on $V_{4N} \oplus V_{6N} - \{0\}$. This results in the weighted

projective space:

$$V_{4N} \oplus V_{6N} - \{0\}/\mathbb{C}^\star = W\mathbb{P}(\underbrace{2, ..., 2}_{4N+1}, \underbrace{3, ..., 3}_{6N+1}).$$

2. Find a line bundle $\mathcal{L}$ on $W\mathbb{P}$ such that the action of $\mathrm{SL}_2(\mathbb{C})$ is linear. Examine the stability condition of above GIT quotient, make sure the image of T_N is contained in the stable locus.

3. We can take the GIT quotient:

$$\overline{\mathsf{E}_N} := W\mathbb{P}(\underbrace{2, ..., 2}_{4N+1}, \underbrace{3, ..., 3}_{6N+1}) /\!\!/_{\mathcal{L}} \mathrm{SL}_2(\mathbb{C}).$$

The previous step assures the moduli space of elliptic surfaces E_N will be a quasi-projective variety sits inside $\overline{\mathsf{E}_N}$.

To execute the above plan, the non-trivial part is to find the line bundle $\mathcal{L}$ and verify the stability condition. The following Proposition gives the answer [Mir81, Proposition 5.1]:

Proposition 1.2.25. *[Mir81] Consider the Veronese embedding $\iota : W\mathbb{P} \to \mathbb{P}$ given by:*

$$[A : B] \mapsto [A^3 : B^2],$$

let $\mathcal{L} := \iota^\star \mathcal{O}_{\mathbb{P}}(1)$. The line bundle $\mathcal{L}$ is ample and has a natural $\mathrm{SL}_2(\mathbb{C})$-linearization. Furthermore, the pair (A, B) is not semistable if and only if there is a point $p \in \mathbb{P}$ such that:

$$v_p(A) > 2N \text{ and } v_p(B) > 3N.$$

A pair (A, B) is not stable if and only if there is a point $p \in \mathbb{P}$ such that:

$$v_p(A) \geq 2N \text{ and } v_p(B) \geq 3N.$$

Proof. The line bundle $\mathcal{L}$ is clearly ample, it is isomorphic to $\mathcal{O}_{W\mathbb{P}}(6)$. The $\mathrm{SL}_2(\mathbb{C})$-linearization is given by identifying the A^3 and B^2 with the binary forms of degree $12N$. We now use Hilbert-Mumford criterion to check the stability condition. We take the one-parameter subgroup:

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \subset \mathrm{SL}_2(\mathbb{C}).$$

Let T_0, T_1 be the homogeneous coordinates on $\mathbb{P}$, the action of λ on T_0 and T_1 is given by: $\lambda \cdot T_0 = \lambda T_0$ and $\lambda \cdot T_1 = \lambda^{-1} T_1$. We can expand the binary form or equivalently the polynomial A, B in terms of T_0, T_1:

$$A = \sum_{i=0}^{4N} a_i T_0^i T_1^{4N-i}, \quad B = \sum_{l=0}^{6N} b_l T_0^l T_1^{6N-l}.$$

Clearly the action will send coordinates a_i to $\lambda^{2i-4N} a_i$ and b_l to $\lambda^{2l-6N} b_l$. The polynomial A^3 will have coefficients $a_i a_j a_k$, and B^2 will have coefficients $b_l b_m$. Coordinate-wise, the action of λ on the coefficients of A^3 and B^2 is given by:

$$\lambda^{2i+2j+2k-12N} a_i a_j a_k, \quad \lambda^{2l+2m-12N} b_l b_m.$$

The Hilbert-Mumford criterion tells us the pair (A, B) is not semistable for the selected one-parameter subgroup if and only if the following conditions are met:

- If $i + j + k \le 6N$, then $a_i a_j a_k = 0$.
- If $l + m \le 6N$, then $b_l b_m = 0$.

In particular, we have $a_n^3 = 0$ if $n \le 2N$ and $b_m^2 = 0$ if $m \le 3N$. This implies for the point $p = [0 : 1] \in \mathbb{P}$, we have $v_p(A) > 2N$ and $v_p(B) > 3N$.

Conversely, if we have a point $p \in \mathbb{P}$ such that $v_p(A) > 2N$ and $v_p(B) > 3N$, we can a suitable coordinate system such that $p = [0 : 1]$. Then we take the corresponding one-parameter subgroup, the above two conditions will be met. So we conclude the pair (A, B) is not semistable if and only if there is a point $p \in \mathbb{P}$ such that $v_p(A) > 2N$ and $v_p(B) > 3N$.

The argument for the stable condition is completely analogous. $\square$

Corollary 1.2.26. *The moduli space of elliptic surfaces with fundamental invariant $N \geq 2$ is a quasi-projective variety with at worst finite quotient singularities.*

Proof. Let $f : V_{4N} \oplus V_{6N} - \{0\} \to W\mathbb{P}$ be the quotient map induced by the $\mathbb{C}^\star$-action. Let $W\mathbb{P}^s$ denote the stable locus of the GIT quotient. For $N \geq 2$, compare the conditions in Proposition 1.2.21 and Proposition 1.2.25, we have $f(T_N) \subset W\mathbb{P}^s$. $\square$

1.2.6 Moduli space of elliptic K3 surfaces

In this section, we will realize elliptic K3 surfaces with a section as lattice polarized K3 surfaces. We will fix a primitive sublattice:

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \subset \Lambda_{K3}.$$

Instead of using the standard basis of the hyperbolic lattice U, it is useful to consider the basis σ, f that corresponds to the section and fiber class of the elliptic surface. The basis σ, f satisfies the following intersection matrix:

$$\begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}.$$

Definition 1.2.27. An elliptic K3 surface with a section is a K3 surface X together with an elliptic fibration $\pi : X \to \mathbb{P}^1$ such that the general fiber of π is an elliptic curve, and a section $s : \mathbb{P}^1 \to X$.

Before we show the one-to-one correspondence between elliptic K3 surfaces and U-polarized K3 surfaces, we will need some basic results about nef line bundles on K3 surfaces.

Proposition 1.2.28. *Let L be a nontrivial line bundle on a K3 surface X such that $L^2 \geq -2$. Then either L is effective or L^{-1} is effective.*

Proof. We apply the Riemann-Roch theorem to the line bundle L, note $K_X = \mathcal{O}_X$, we have

$$\chi(X, L) = \frac{L^2}{2} + \chi(X, \mathcal{O}_X) = \frac{L^2}{2} + 2 \geq 0.$$

Since $L^2 \geq -2$, we have $\dim H^0(X, L) + \dim H^2(X, L) \geq 1$. By Serre duality, $H^2(X, L) \simeq H^0(X, L^{-1})^\vee$. Moreover, if $\dim H^0(X, L) > 0$ and $\dim H^0(X, L) > 0$, then L must be trivial. So we have either $\dim H^0(X, L) > 0$ or $\dim H^0(X, L^{-1}) > 0$, which means either L is effective or L^{-1} is effective.

□

Definition 1.2.29. [Huy16] For a smooth rational curve C on a K3 surface X, we define the reflection $s_{[C]} : \mathrm{Pic}(X) \to \mathrm{Pic}(X)$ by:

$$s_{[C]}(L) = L + (L \cdot C)C.$$

A reflection $s_{[C]}$ will be an isometry of the Picard lattice.

Proposition 1.2.30. *[Huy16] Let L be a line bundle on a K3 surface X such that $L^2 \geq 0$. Then there exists a sequence of smooth rational curves $C_1, \ldots, C_n$ on X such that:*

$$s_{[C_1]} \circ \cdots \circ s_{[C_n]}(L)$$

is a nef line bundle.

Proposition 1.2.31. *[Huy16] Let L be a nef line bundle on a K3 surface X such that $L^2 \geq 0$. Then there exists an elliptic fibration $\pi : X \to \mathbb{P}^1$ such that $L \cong \mathcal{O}_X(mf)$, where f is the fiber class of the fibration.*

From the elliptic surface point of view, the elliptic K3 surfaces are elliptic surfaces with a section and the fundamental invariant $N = 2$.

Proposition 1.2.32. *There is a one-to-one correspondence between elliptic surfaces with a section and the fundamental invariant $N = 2$ and U-polarized K3 surfaces.*

Proof. Suppose we have an elliptic surface with the fundamental invariant $N = 2$. Let $\pi : X \to \mathbb{P}^1$ be the elliptic surface, and $s : \mathbb{P}^1 \to X$ be the section. From Corollary 1.2.5, we know $K_X \cong \mathcal{O}_X$ and $H^1(X, \mathcal{O}_X) = 0$. Use Corollary 1.2.5 again, the section class S will have self-intersection $S^2 = -2$. So the section class S together with the fiber class F will generate the desired lattice $\mathbb{L}$, with $H = S + 2F$ being a big and nef class.

Conversely, suppose we have a U-polarized K3 surface X. Let D_1 and D_2 be the basis of the lattice U such that $D_1^2 = 0$, $D_2^2 = -2$ and $D_1 \cdot D_2 = 1$. Using Proposition 1.2.30, we can find a nef class f with self-intersection $f^2 = 0$ using successive reflections on D_1. Let σ be the corresponding class of D_2 after reflections. From Proposition 1.2.31, we know there is an elliptic fibration $\pi : X \to \mathbb{P}^1$ such that f is a multiple of the fiber class of π. Notice $\sigma \cdot f = 1$, we know f must be the fiber class itself. Now since $\sigma^2 = -2$, from Proposition 1.2.28 and nefness of f, we know σ is effective. Consider the fixed part of the base locus in the linear system $\mathrm{BS}|\sigma|$, we know the fixed part must contain a component S which is a section of the elliptic fibration. □

Corollary 1.2.33. *The coarse moduli space of elliptic K3 surfaces is isomorphic to E_2.*

Remark 1.2.34. From the construction of the moduli spaces of K3 surfaces, we expect the moduli space of elliptic K3 surfaces to be a quasi-projective variety with at worst finite quotient singularities. We expect the dimension of the moduli space to be 18, it's easy to check the dimension of E_2:

$$\dim \mathsf{E}_2 = \dim V_8 + \dim V_{12} - \dim \mathrm{SL}_2(\mathbb{C}) \times \mathbb{C}^\star = 9 + 13 - 4 = 18.$$

Chapter 2

Summary of results

2.1 The cohomology of the moduli space of elliptic K3 surfaces

In Chapter 1, we constructed the coarse moduli space of elliptic K3 surfaces, from both the K3 surface point of view and the elliptic surface point of view. The coarse moduli space is a quasi-projective variety with at worst finite quotient singularities. One of the most interesting aspects is the topology of these spaces. For F_2, the coarse moduli space of degree 2 quasi-polarized K3 surfaces, Kirwan and Lee computed the Poincaré polynomial in [KL89]. Their main tools are equivariant perfect stratification (cf. [Kir84]) and Kirwan's blowup (cf. [Kir85]). Recently, the same machinery has been used to obtain data for the moduli of Enriques surfaces [For23], the moduli of non-hyperelliptic genus four curves [For21], and the moduli of cubic fourfolds [Si23].

In Chapter 3, we will study the topology of the moduli space of elliptic K3 surfaces. From the lattice polarization approach, these are K3 surfaces equipped with a lattice polarization:

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \subset \mathrm{Pic}(X).$$

To study the topology of F_U, we will use the theory of Weierstrass fibration over $\mathbb{P}^1$. The basic setup can be found in Chapter 3 or [Mir81]. We have demonstrated that the relevant moduli space is E_2, the moduli space of elliptic K3 surfaces with the fundamental invariant $N = 2$. Theorem 1.2.32 shows $\mathsf{F}_U \simeq \mathsf{E}_2$. We have constructed the moduli space E_2 from the geometric invariant theory in Chapter 3. The stability conditions are given in Proposition 1.2.25. The coarse moduli space of Weierstrass fibrations E_2 has a natural compactification $\overline{\mathsf{E}_2}$:

$$\mathsf{E}_2 \subset \overline{\mathsf{E}_2} = W\mathbb{P}(\underbrace{2, ..., 2}_{9}, \underbrace{3, ..., 3}_{13}) /\!\!/_{\mathcal{L}} \mathrm{SL}_2(\mathbb{C}).$$

So we can use Kirwan's machinery to study the cohomology of the GIT quotient, and then remove the compactification boundary using results in [OO21]. The main result of the chapter is the Poincaré polynomial of E_2.

Theorem 2.1.1. *The Poincaré polynomial of* E_2 *or equivalently* F_U *is:*

$$\begin{aligned} P_t(\mathsf{E}_2) = P_t(\mathsf{F}_U) = 1 &+ t^2 + 2t^4 + 2t^6 + 3t^8 + 3t^{10} + 4t^{12} + 4t^{14} + 5t^{16} \\ &+4t^{18} + 4t^{20} + 3t^{22} + 3t^{24} + 2t^{26} + 2t^{28} + t^{30} + t^{32} + t^{33}. \end{aligned}$$

The tautological cohomology $RH^*(\mathsf{F}_U) \subset H^*(\mathsf{F}_U)$ would be a more interesting object to study (cf. [MP13, MOP17, PY20, BLMM17]). Theorem 2.1.1 shows the tautological cohomology is *not* the full cohomology due to the odd generator in degree 33. But $RH^*(\mathsf{F}_U)$ alone satisfies the vanishing result proved by Peterson [Pet19].

Corollary 2.1.2. *The tautological cohomology of* F_U *vanishes in the top two degree.*

$$RH^{34}(\mathsf{F}_U) = RH^{36}(\mathsf{F}_U) = 0.$$

The Chow ring of F_U is completely understood in [CK23] using the Weierstrass fibration approach. Although we have odd cohomology in degree 33, in even degrees, we can show all cohomology comes from algebraic classes.

Theorem 2.1.3. *The cycle maps of F_U are isomorphisms:*

$$\mathrm{cl} : A^i(\mathsf{F}_U) \xrightarrow{\sim} H^{2i}(\mathsf{F}_U).$$

2.2 Chow ring of the moduli space of elliptic K3 surfaces

Given a smooth stack X that is the solution to a moduli problem, there are often natural algebraic cycles called tautological classes in $A^*(X)$, the Chow ring of X with rational coefficients. For example, when $X = \mathcal{M}_g$, the moduli space of smooth curves of genus g, there is the tautological subring $R^*(\mathcal{M}_g) \subset A^*(\mathcal{M}_g)$ generated by the κ-classes. Faber [Fab99] gave a series of conjectures on the structure of $R^*(\mathcal{M}_g)$, which assert that $R^*(\mathcal{M}_g)$ behaves like the algebraic cohomology ring of a smooth projective variety of dimension $g-2$, even though $\mathcal{M}_g$ is neither projective nor of dimension $g-2$. Looijenga [Loo95] proved that $R^i(\mathcal{M}_g) = 0$ for $i > g-2$ and that $R^{g-2}(\mathcal{M}_g) \cong \mathbb{Q}$, settling one of Faber's conjectures. Looijenga's theorem gives a new proof of Diaz's result [Dia84] that the maximal dimension of a complete subvariety of $\mathcal{M}_g$ is $g-2$. Faber further conjectured that $R^*(\mathcal{M}_g)$ should be a Gorenstein ring with socle in codimension $g-2$, meaning that the intersection product is a perfect pairing

$$R^i(\mathcal{M}_g) \times R^{g-2-i}(\mathcal{M}_g) \to R^{g-2}(\mathcal{M}_g) \cong \mathbb{Q}.$$

Faber [Fab99] and Faber–Zagier proved this conjecture for $g \leq 23$ by producing relations in the tautological ring and showing computationally that the resulting quotient is Gorenstein.

Recently, there has been significant interest in the tautological rings $R^*(\mathcal{F}_\Lambda)$ of the moduli stacks $\mathcal{F}_\Lambda$ of lattice polarized K3 surfaces [MP13, MOP17, PY20, BLMM17, BL19]. In [MOP17], the tautological rings are defined as the subrings of $A^*(\mathcal{F}_\Lambda)$ generated by the fundamental classes of Noether–Lefschetz loci together with push forwards of κ-classes from all Noether-Lefschetz loci. There are natural analogues of Faber's conjectures for $R^*(\mathcal{F}_\Lambda)$.[1]

Conjecture 2.2.1 (Oprea–Pandharipande)**.** *Let $d = \dim \mathcal{F}_\Lambda$.*

1. *For $i > d-2$, $R^i(\mathcal{F}_\Lambda) = 0$.*
2. *There is an isomorphism $R^{d-2}(\mathcal{F}_\Lambda) \cong \mathbb{Q}$.*

The primary evidence for part (1) of this conjecture is a theorem of Petersen [Pet19, Theorem 2.2], which says that the image $RH^{2*}(\mathcal{F}_\Lambda)$ of $R^*(\mathcal{F}_\Lambda)$ in cohomology under the cycle class map vanishes above cohomology degree $2(d-2)$. If Conjecture 2.2.1 holds, then one can further ask for the analogue of Faber's Gorenstein conjecture: is there a perfect pairing

$$R^i(\mathcal{F}_\Lambda) \times R^{d-2-i}(\mathcal{F}_\Lambda) \to R^{d-2}(\mathcal{F}_\Lambda) \cong \mathbb{Q}?$$

In this paper, we study the Chow rings of moduli spaces E_N of elliptic surfaces Y fibered over $\mathbb{P}^1$ with section $s : \mathbb{P}^1 \to Y$ and fundamental invariant N (see Section 2 for definitions). The main result is that natural analogues of Faber's vanishing and Gorenstein conjectures hold for the entire Chow ring $A^*(E_N)$ for each $N \geq 2$.

Theorem 2.2.2. *Let $N \geq 2$ be an integer.*

1. *The Chow ring has the form*

$$A^*(E_N) = \mathbb{Q}[a_1, c_2]/I_N$$

where $a_1 \in A^1(E_N), c_2 \in A^2(E_N)$, and I_N is the ideal generated by the two relations from Proposition 4.2.4.

[1] The author learned about these analogues from a lecture given by Rahul Pandharipande in the algebraic geometry seminar at UCSD and from a course on K3 surfaces given by Dragos Oprea.

2. The Poincaré polynomial collecting dimensions of the Chow groups is given by

$$\begin{aligned} p_N(t) &= \sum \dim A^i(E_N) t^i \\ &= 1 + t + 2t^2 + 2t^3 + 3t^4 + 3t^5 + 4t^6 + 4t^7 + 5t^8 + \\ &\quad + 4t^9 + 4t^{10} + 3t^{11} + 3t^{12} + 2t^{13} + 2t^{14} + t^{15} + t^{16}. \end{aligned}$$

3. The Chow ring $A^*(E_N)$ is Gorenstein with socle in codimension 16.

We also have similar partial results for Poincaré polynomial for the cohomology ring when $N = 2$ that will appear in future work.

A notable property is that the dimensions of the Chow groups are independent of N. In particular, the Chow groups $A^i(E_N)$ are only nonzero in codimension $0 \leq i \leq 16$, despite the fact that the dimensions of the moduli spaces E_N go to infinity with N. Moreover, the ring structure depends in a simple and explicit way on N coming from the relations in Proposition 4.2.4. As a consequence of Theorem 2.2.2, we obtain an analogue of Diaz's theorem [Dia84] on the maximal dimension of a complete subvariety of $\mathcal{M}_g$. In our case, the bound is independent of N.

Corollary 2.2.3. *Let $N \geq 2$ be an integer. The maximal dimension of a complete subvariety of E_N is* 16.

When $N = 2$, the corresponding elliptic surfaces are K3 surfaces polarized by a hyperbolic lattice U with intersection matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We show that the generators a_1 and c_2 of $A^*(E_2)$ have natural interpretations as tautological classes in $R^*(\mathcal{F}_U)$, where $\mathcal{F}_U$ is the moduli space of U-polarized K3 surfaces.

Theorem 2.2.4. *Under the identification of $A^*(E_2)$ with $A^*(\mathcal{F}_U)$, the classes a_1 and c_2 lie in $R^*(\mathcal{F}_U)$. Therefore, $A^*(\mathcal{F}_U) = R^*(\mathcal{F}_U)$ is a Gorenstein ring with socle in codimension 16.*

We view Theorem 2.2.4 as providing a piece of evidence toward Conjecture 2.2.1. Since $\dim(A^1(\mathcal{F}_U)) = 1$, we can determine the κ classes in terms of the Hodge class $\lambda \in R^1(\mathcal{F}_U)$.

Proposition 2.2.5. *The following four linear combinations of κ-classes are independent of the choice of universal line bundles. Moreover, they are all multiples of the Hodge class λ.*

$$\kappa_{3,0,0} + \frac{1}{4}\kappa_{1,0,1} = \frac{7}{2}\lambda, \quad 3\kappa_{2,1,0} - \frac{1}{4}\kappa_{1,0,1} + \frac{1}{4}\kappa_{0,1,1} = \frac{1}{2}\lambda,$$

$$3\kappa_{1,2,0} - \frac{1}{4}\kappa_{0,1,1} = -3\lambda, \quad \kappa_{0,3,0} = 0.$$

where $\kappa_{i,j,k} := \pi_*\left(c_1(\mathcal{O}(\sigma))^i \cdot c_1(\mathcal{O}(f))^j \cdot c_2(T_\pi)^k\right)$.

2.3 Tautological relations in the moduli space of elliptic K3 surfaces

For the moduli stack $\mathcal{F}_\Lambda$, its tautological ring $R^\star(\mathcal{F}_\Lambda)$ encodes natural geometric cycles that capture the geometry of the moduli space. Understanding the generators and relations in $R^\star(\mathcal{F}_\Lambda)$ is perhaps just as important as understanding the entire Chow ring. Let

$$\pi : \mathcal{X}_\Lambda \to \mathcal{F}_\Lambda; \quad \mathcal{H}_1, \cdots, \mathcal{H}_r \in \mathrm{Pic}(\mathcal{X}_\Lambda)$$

be the universal surface and line bundles obtained by fixing a basis for lattice polarization.

Definition 2.3.1. We have the following tautological classes arising from tautological constructions:

- *Hodge classes*: Let ω_π be the relative cotangent sheaf of the universal surface π. The Hodge class is defined as: $\lambda = c_1(\pi_\star\omega_\pi)$.

- *Noether-Lefschetz classes*: Let $\Lambda \subset \Lambda'$ be a lattice embedding within the K3 lattice. We define the Noether-Lefschetz class as the image of the induced map: $\iota : \mathcal{F}_{\Lambda'} \to \mathcal{F}_{\Lambda}$.

- *Kappa classes*: The enriched kappa classes are defined in [MOP17]:

$$\kappa_{a_1,\ldots,a_r,b} = \pi_* \left(c_1 \left(\mathcal{H}_1\right)^{a_1} \cdots c_1 \left(\mathcal{H}_r\right)^{a_r} \cdot c_2 \left(T_\pi\right)^b \right).$$

In [MOP17], it was conjectured that the tautological ring (see Definition 5.0.1) is the same subring generated solely by the Noether-Lefschetz loci. This conjecture was proven in [PY20]:

Theorem 2.3.2. *[PY20, Theorem 1]* $NL^\star(\mathcal{F}_\Lambda) = R^\star(\mathcal{F}_\Lambda)$.

The proof uses the relative moduli space of stable maps over the universal surface. The key idea is to employ the WDVV equations and Getzler's relation.

In the original work [MOP17], computations on the low degree $\mathcal{F}_{2\ell}$ are conducted by localization over the relative Quot scheme of the universal surface. Similarly, we can perform these computations over $\mathcal{F}_U$. Let f and σ be the universal line bundles corresponding to the fiber and the section of the universal elliptic fibration.

Proposition 2.3.3. *Let the* $\kappa_{i,j} = \pi_\star(c_1(\mathcal{O}(\mathcal{H}))^i \cdot c_2(T_\pi)^j)$. *Let* S^{red} *be the reduced Noether-Lefschetz divisor associated to the following lattice:*

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

We have the following relations.

1. *For* $\mathcal{H} = \sigma + 2f, \chi = 0$, *we have:* $\kappa_{1,1} - 4\kappa_{3,0} - 30\lambda = 0$.

2. *For* $\mathcal{H} = \sigma + 2f, \chi = 1$, *we have:* $\frac{40}{3}(\kappa_{1,1} - 4\kappa_{3,0}) + 128\lambda - 4S^{\mathrm{red}} = 0$.

3. *For* $\mathcal{H} = \sigma + 3f, \chi = 0$, *we have:* $10(\kappa_{1,1} - 2\kappa_{3,0}) - 44\lambda - 3S^{\mathrm{red}} = 0$.

4. *For* $\mathcal{H} = \sigma + 4f, \chi = -1$, *we have:* $3\kappa_{1,1} - 4\kappa_{3,0} - 38\lambda - S^{\mathrm{red}} = 0$.

5. *For* $\mathcal{H} = \sigma + 5f, \chi = -2$, *we have:* $\frac{8}{3}(\kappa_{1,1} - \kappa_{3,0}) - 52\lambda - S^{\mathrm{red}} = 0$.

Using the first two relations in Proposition 2.3.3, we have obtained $S^{\mathrm{red}} = 132\lambda$. This result aligns with the modularity theorem in [MP13]. Where we expect $S = 2S^{red} = 264\lambda$. The coefficient 264 is the second coefficient of modular form $E_4(\tau)E_{10}(\tau)$. In fact, we can prove:

Theorem 2.3.4.

$$D_{h,(d_1,d_2)} = -E_4(q)E_6(q)\left[\frac{\det(\Lambda)}{2}\right] \cdot \lambda$$

where $E_4(q)$ *and* $E_6(q)$ *are the Eisenstein series, and the matrix* Λ *is:*

$$\begin{pmatrix} 0 & 1 & d_1 \\ 1 & 0 & d_2 \\ d_1 & d_2 & 2h-2 \end{pmatrix}.$$

We can substitute S^{red} with 132λ in the remaining relations, and they will yield a set of linearly independent relations that is equivalent to Proposition 2.2.5.

This chapter is, in part, adapted from the material as it appears in

- Samir Canning and Bochao Kong, *"The Chow rings of moduli spaces of elliptic surfaces over* $\mathbb{P}^1$*"*, Algebraic Geometry 10.4 (2023).

The dissertation author was the co-primary investigator and author of this paper.

Chapter 3

Cohomology of the moduli space of elliptic K3 surfaces

3.1 Notations and conventions

1. Varieties and groups are over the complex number field $\mathbb{C}$. We abbreviate $\mathrm{SL}_2(\mathbb{C})$ as SL_2 in this chapter. All homologies, cohomologies, and Chow rings are with rational coefficients.

2. Let G be a connected topological group, BG is a classifying space of G, and EG is the universal bundle over BG.

3. For a topological space Y, its Poincaré polynomial is:

$$P_t(Y) := \sum_{i\geq 0} t^i \dim H^i(Y).$$

4. Let G be a connected topological group. For a topological space Y with G-action, its equivariant Poincaré polynomial is:

$$P_t^G(Y) := \sum_{i\geq 0} t^i \dim H^i_G(Y) = \sum_{i\geq 0} t^i H^i(Y \times_G EG).$$

5. The vector space V_n is the space $H^0(\mathbb{P}^1, \mathcal{O}(n))$, which can be identified with the

space of homogeneous polynomials of degree n. The SL_2-action on V_n is given by the standard change of variables, for more details, see Equation (1.2.1).

6. The stand-along symbol $W\mathbb{P}$ is the weighted projective space:

$$W\mathbb{P} := V_8 \oplus V_{12} - \{0\}/\mathbb{C}^\star = W\mathbb{P}(\underbrace{2,...,2}_{9},\underbrace{3,...,3}_{13}).$$

   The $\mathbb{C}^\star$-action is given by: $\lambda \cdot (A, B) = (\lambda^2 A, \lambda^3 B)$. We denote the semistable locus by $W\mathbb{P}^{ss}$, and the stable locus by $W\mathbb{P}^s$.

7. The subset $T_2 \subset V_8 \oplus V_{12}$ is the subset of pairs (A, B) subject to the conditions in Proposition 1.2.21.

8. We denote the coarse moduli space of elliptic K3 surfaces with fundamental as W, i.e. $\mathsf{W} \simeq \mathsf{F}_U \simeq \mathsf{E}_2 = T_2/\mathbb{C}^\star \times \mathrm{SL}_2$.

9. The GIT compactification of W is denoted by W^{GIT}, i.e. $\mathsf{W}^{GIT} = W\mathbb{P} /\!\!/_{\mathcal{L}} \mathrm{SL}_2$. The definition of the linearization $\mathcal{L}$ is given in Proposition 1.2.25.

3.2 GIT compactification boundaries

In this section, we describe the boundary components in W^{GIT}. The boundary only consists of only dimension 1 and dimension 0 components. Furthermore, the GIT compactification W^{GIT} agrees with the e Satake-Baily-Borel compactification for the moduli space of elliptic K3 surface (cf. [OO21, Theorem 7.9]). We have a concrete description of the boundary.

Proposition 3.2.1. *[OO21, Theorem 7.4] The boundary $\mathsf{W}^{GIT} - \mathsf{W}$ consists of two curves $C_{ss} \cup C_{nn}$.*

1. *C_{ss} parametrizes the strictly semistable Weierstrass equations.*

2. *C_{nn} parametrizes Weierstrass equations which fail the condition (1) in Proposition 1.2.21.*

Moreover, $C_{nn} \simeq \mathbb{P}^1$, C_{ss} and C_{nn} intersect at a single point in W^{GIT}.

3.3 Orbifold structure on $W\mathbb{P}$

The main technical difficulty in applying Kirwan's machinery is the orbifold structure on the $W\mathbb{P}$. In this section, we describe the orbifold structure on $W\mathbb{P}$. There are two different types of orbifold structures for weighted projective spaces. The first type is induced by the global quotient of the usual projective space by a finite group action:

$$\mathbb{P}^n/(\mathbb{Z}_{w_0} \times \cdots \times \mathbb{Z}_{w_n}),$$

where the group action is given by:

$$(\zeta_{w_0}^{k_0}, ..., \zeta_{w_n}^{k_n}) \cdot [z_0 : ... : z_n] = [\zeta_{w_0}^{k_0} z_0 : ... : \zeta_{w_n}^{k_n} z_n].$$

The second type is the orbifold structure induced by the $\mathbb{C}^\star$-action on the vector spaces:

$$W\mathbb{P}(w_0, ..., w_n) := V(w_0, ..., w_n) - \{0\}/\mathbb{C}^\star,$$

where $V(w_0, ..., w_n)$ is a representation of $\mathbb{C}^\star$ with weights $w_0, ..., w_n$. Clearly, the relevant orbifold structure for us is of the second type. We can write down orbifold charts for weighted projective space of the second type concretely:

Definition 3.3.1. Let $V(w_0, ..., w_n)$ be a representation of $\mathbb{C}^\star$ with weights $w_0, ..., w_n$, i.e.

$$\lambda \cdot (z_0, ..., z_n) = (\lambda^{w_0} z_0, ..., \lambda^{w_n} z_n).$$

The weighted projective space $W\mathbb{P}(w_0, ..., w_n)$ is the quotient of $V(w_0, ..., w_n) - \{0\}$ by the $\mathbb{C}^\star$-action. The orbifold structure on $W\mathbb{P}(w_0, ..., w_n)$ is given by the following charts $(U_i, \mathbb{Z}_{w_i}, \phi_i)$:

$$U_i \simeq \mathbb{C}^n, \quad \mathbb{Z}_{w_i} = \{\zeta_{w_i}^k\}_{k\in\mathbb{Z}}, \quad \phi_i(\widetilde{z_0}, ..., 1_i, ..., \widetilde{z_n}) = [\widetilde{z_0} : ... : 1_i : ... : \widetilde{z_n}],$$

where 1_i is viewed as a placeholder for the i-th coordinate on the left-hand side. The group action is given by:

$$\zeta_{w_i} \cdot (\widetilde{z_0}, ..., 1_i, ..., \widetilde{z_n}) = (\zeta_{w_i}^{w_0} \widetilde{z_0}, ..., 1_i, ..., \zeta_{w_i}^{w_n} \widetilde{z_n}).$$

This orbifold structure can be described as a translation Lie groupoid. We sketch the relevant notions.

Definition 3.3.2. A Lie groupoid $\mathcal{G}$ consists of the following data:

- A pair of manifolds G_0 and G_1.
- Two submersions $s, t : G_1 \to G_0$ called the source and target maps.
- A smooth composition map $m : G_1 \times G_1 \to G_1$.
- An smooth inversion map $i : G_1 \to G_1$.
- A smooth unit map $u : G_0 \to G_1$.

The maps satisfy the following axioms:

- The composition map is associative.
- The unit map is a two-sided identity for the composition.
- The inversion map is a two-sided inverse for the composition.

- The source and target maps are compatible with the unit map.

A proper foliation Lie groupoid will give us an orbifold structure. The precise definitions are:

Definition 3.3.3. Let $\mathcal{G}$ be a Lie groupoid. For any point $x \in G_0$, the isotropy group at x is the group:

$$G_x := s^{-1}(x) \cap t^{-1}(x).$$

The groupoid $\mathcal{G}$ is proper if the map $(s,t) : G_1 \to G_0 \times G_0$ is proper. The groupoid $\mathcal{G}$ is a foliation groupoid if every isotropy group G_x is discrete. The groupoid $\mathcal{G}$ is an orbifold groupoid if it is proper and a foliation groupoid.

The groupoid gives us a way to study orbifolds globally. The orbifold structure on $W\mathbb{P}(w_0, ..., w_n)$ can be described by the translation groupoid. The groupoid is given by:

Definition 3.3.4. Let $V(w_0, ..., w_n)$ be a representation of $\mathbb{C}^\star$ with weights $w_0, ..., w_n$. The translation groupoid $\mathcal{W}$ for $W\mathbb{P}(w_0, ..., w_n)$ is given by:

- The objects $G_0 = V(w_0, ..., w_n) - \{0\}$.
- The arrows $G_1 = G_0 \times \mathbb{C}^\star$.
- The source map $s : G_1 \to G_0$ is given by $s(z, \lambda) = z$.
- The target map $t : G_1 \to G_0$ is given by $t(z, \lambda) = \lambda \cdot z$. Recall the action is given by:

$$\lambda \cdot z = \lambda \cdot (z_0, ..., z_n) = (\lambda^{w_0} z_0, ..., \lambda^{w_n} z_n).$$

If we realize $\mathbb{C}^\star \simeq S^1 \times \mathbb{R}^+$ and $V(w_0, ..., w_n) - \{0\} \simeq S^{2n+1} \times \mathbb{R}^+$, the action of $\mathbb{C}^\star$ on $V(w_0, ..., w_n) - \{0\}$ is a product of two actions. The S^1-action on S^{2n+1} is proper and the $\mathbb{R}^+$-action on $\mathbb{R}^+$ is free and proper. Furthermore, the isotropy groups of $\mathbb{C}^\star$ on

$V(w_0, ..., w_n) - \{0\}$ are all finite. Thus, the translation groupoid $\mathcal{W}$ is an orbifold groupoid. We can construct the orbifold charts from $\mathcal{W}$ using the classical slice theorem.

Theorem 3.3.5. *Let a Lie group G act on a manifold M smoothly and properly. Let $x \in M$ be a point with isotropy group G_x. Then there exists a G-equivariant diffeomorphism from a disk bundle $G \times_{G_x} D$ onto a G-invariant neighborhood of the orbit $G \cdot x$, whose restriction to the zero section is the orbit map:*

$$G \times_{G_x} \{0\} \simeq G/G_x \to G \cdot x.$$

Applying the slice theorem to the translation groupoid $\mathcal{W}$, for any point $z \in V(w_0, ..., w_n) - \{0\}$, we have a G-equivariant diffeomorphism from a disk bundle $G \times_{G_z} D$ onto a G-invariant neighborhood U of the orbit $G \cdot z$. The restriction orbifold $\mathcal{W}|_U$ is Morita equivalent to the etale groupoid $G_z \times D \rightrightarrows D$. One can check the etale groupoid gives orbifold charts that are equivalent (up to refinement) to the charts in Definition 3.3.1.

To prepare for the equivariant cohomology computation, we need to work with a good notion of sub-orbifolds. We will not go into the details of the general theory of sub-orbifolds. Since we will only be working with translation groupoids, the discussion can be reduced to equivariant immersions of manifolds. We will follow the method introduced in [CHS13].

Definition 3.3.6. Let N, M be manifolds with a smooth action of a Lie group G. We assume the induced translation groupoids are orbifolds. A smooth map $f : N \to M$ is a strong equivariant immersion if:

- For any $x \in N$, the differential $df_x : T_x N \to T_{f(x)} M$ is injective.
- The map f is G-equivariant.

- For any $p \in M$, the isotropy group G_p acts transitively on the set of points $x \in N$ such that $f(x) = p$.

The strong equivariant immersion will induce an orbifold embedding map.

Theorem 3.3.7. *Under the same assumptions as in Definition 3.3.6. Let $\mathcal{N}, \mathcal{M}$ be the translation groupoids of $[G \times N \rightrightarrows N]$ and $[G \times M \rightrightarrows M]$ respectively. If $f : N \to M$ is a strong equivariant immersion, then the map f induces an orbifold embedding map:*

$$f : \mathcal{N} \to \mathcal{M}.$$

An orbifold embedding map will induce maps between orbifold tangent bundles, we can define orbifold normal bundle and orbifold normal Euler class. We will always work with holomorphic orbifold embeddings, so the induced bundles will be holomorphic as well. Unless otherwise stated, we will only consider holomorphic orbifold tangent bundle and holomorphic orbifold normal bundle.

3.4 Equivariant stratification

In this section, we summarize Kirwan's equivariant perfect stratification results in the fundamental work [Kir84]. We will apply the theory to $W\mathbb{P} /\!\!/ \mathrm{SL}_2$ to compute the equivariant cohomology of the semistable locus. In [Kir84], the machinery of equivariant perfect stratification is developed for smooth projective varieties with linear reductive group action. However, the theory can be applied to the weighted projective space $W\mathbb{P}$ as well. In [KL89], the authors applied the equivariant perfect stratification to a weighted projective space produced by weighted blowups. We will define the HKKN stratification for $W\mathbb{P}$ concretely, and prove the equivariant perfectness holds for our case.

3.4.1 The HKKN stratification

There are symplectic and algebraic approaches to the Hesselink-Kempf-Kirwan-Ness (HKKN) stratification. We start with the algebraic setting that fits better in our case.

We set $X \subset \mathbb{P}^n$ to be a projective variety (not necessarily smooth), and let G be a reductive group that acts linearly on $\mathbb{P}^n$. To describe the stratification, we pick $T \subset G$ to be a maximal torus and $K \subset G$ to be a maximal compact subgroup. Then equip the real Lie algebra $\mathfrak{t}^\star := \mathrm{Lie}\,(K \cap T)^\star$ with an invariant inner product, and let $\|.\|$ be the corresponding norm.

Now we can construct the index set $\mathcal{B}$ of the HKKN stratification. Let:

$$\{\alpha_0, \ldots, \alpha_n\} \subset \mathfrak{t}^\star$$

be the weights of the representation of T on $\mathbb{C}^{n+1}$, using the inner product we identify them as subset of $\mathfrak{t}$. To form the set $\mathcal{B}$, we select a positive Weyl chamber $\mathfrak{t}_+$, and then collect all the points $\beta \in \overline{\mathfrak{t}_+}$ such that β is the closest point to the origin for a convex hull formed by some non-empty subset of $\{\alpha_0, \ldots, \alpha_n\}$. The positive weights and the origin 0 are included in $\mathcal{B}$.

Theorem 3.4.1. *[Kir84, Theorem 12.26] In the above setting, there exists a G-invariant stratification on X:*

$$X = \bigsqcup_{\beta \in \mathcal{B}} S_\beta.$$

The partial order on $\mathcal{B}$ is given by the norm $\|.\|$. Moreover, the dominant stratum S_0 coincides with the semistable locus X^{ss} defined by GIT.

If the variety X is smooth, the stratification admits a Morse theoretic description. In the above setting, assume the G-action on X is Hamiltonian. Let $\mu : X \to \mathrm{Lie}\,(K)^\star$ be a moment map. Let's take an invariant inner product on $\mathrm{Lie}\,(K)$, Kirwan constructs the stratification by taking $\|\mu\|^2$ as a Morse function. Furthermore, the stratification enjoys the following good property.

Theorem 3.4.2. *[Kir84, Theorem 13.5] If the variety X is smooth, then the stratification stated in Theorem 3.4.1 is equivariant-perfect, i.e. we have:*

$$P_t^G(X) = \sum_{\beta \in \mathcal{B}} t^{2\,\mathrm{codim}\left(S_\beta\right)} P_t^G\left(S_\beta\right) \tag{3.4.1}$$

or equivalently:

$$P_t^G\left(X^{ss}\right) = P_t^G(X) - \sum_{0 \neq \beta \in \mathcal{B}} t^{2\,\mathrm{codim}\left(S_\beta\right)} P_t^G\left(S_\beta\right). \tag{3.4.2}$$

The above result tells us that on the level of equivariant Poincaré polynomials, we can remove the unstable strata to get the result for the semistable locus. The equivariant perfectness has many corollaries, we list the most relevant ones below.

Corollary 3.4.3. *[Kir84] If the stratification $X = \bigsqcup_{\beta \in \mathcal{B}} S_\beta$ is equivariantly perfect. Then we have a surjection:*

$$H_G^*(X) \twoheadrightarrow H_G^*(X^{ss}). \tag{3.4.3}$$

If X is smooth, we further have:

$$H_G^*(X) \simeq H^*(X) \otimes H^*(BG). \tag{3.4.4}$$

Our goal is to apply the above theory to the weighted projective space $W\mathbb{P}$ with SL_2 action. Then examine the stratification to show we have equivariant perfectness even though $W\mathbb{P}$ has mild singularities. To achieve this, we need to introduce the construction of the stratification in more detail.

For each weight $\beta \in \mathcal{B} \backslash \{0\}$, we construct the linear section $Z_\beta \subset X$:

$$Z_\beta := \left\{(x_0 : \ldots : x_n) \in X : x_i = 0 \text{ if } \alpha_i \cdot \beta \neq \|\beta\|^2\right\}.$$

In [Kir84, 8.11], a suitable linear action of the stabilizer group $\operatorname{Stab}\beta$ on Z_β^{ss} is defined. We write Z_β^{ss} as the semistable locus for this action. We need another set $Y_\beta \subset X$:

$$Y_\beta := \left\{(x_0 : \ldots : x_n) \in X : x_i = 0 \text{ if } \alpha_i \cdot \beta < \|\beta\|^2, \text{ at least one } x_j \neq 0 \text{ if } \alpha_j \cdot \beta = \|\beta\|^2\right\}.$$

The set Y_β carries a similar action of the group $\operatorname{Stab}\beta$ as well. We denote Y_β^{ss} as the semistable locus for this action. The stratification of X is given by:

$$S_\beta := GY_\beta^{ss}, \quad X = \bigsqcup_{\beta \in \mathcal{B}} S_\beta.$$

If X is smooth, then the set Z_β is smooth. In fact, Z_β is fixed by the torus $T_\beta : \exp(\mathbb{R}\beta) \subset \operatorname{Stab}\beta$. Let μ_β be the moment map for the action of T_β on X. Then Y_β is the Morse stratum for Z_β under the gradient flow of μ_β. Kirwan has shown that we can compute $P_t^G(S_\beta)$ based on the action of $\operatorname{Stab}\beta$ on Z_β^{ss} and thus achieve an inductive algorithm:

Theorem 3.4.4. *[Kir84] If X is smooth, then we have the following inductive formula:*

$$P_t^G(S_\beta) = P_t^{\operatorname{Stab}\beta}(Z_\beta^{ss}).$$

3.4.2 Stratification for $W\mathbb{P}$ with SL_2 action

We now explain the HKKN stratification for the SL_2 action on $W\mathbb{P}$. The index set for the stratification is easy to describe. Note that the maximal torus

$$\mathbb{T} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \subset \mathrm{SL}_2$$

is one-dimensional. We take the positive chamber and identify it with $\mathbb{R}_{\geq 0}$, the index set $\mathcal{B}$ of the stratification is a subset of $\mathbb{R}_{\geq 0}$. Recall the action is given by Equation (1.2.1). If we ignore the weighted structure on $W\mathbb{P}$, the weights of the $\mathbb{T}$-action on $\mathbb{C}^{22}$ are given by:

$$\{2m | -4 \leq m \leq 4\} \text{ on } V_8, \quad \{2n | -6 \leq n \leq 6\} \text{ on } V_{12}. \tag{3.4.5}$$

As explained in Proposition 1.2.25, the correct stratification for $W\mathbb{P}$ is given by the realization of $W\mathbb{P}$ as a projective variety:

$$\iota : W\mathbb{P} \hookrightarrow \mathbb{P}^{49}, \quad [A : B] \mapsto [A^3 : B^2].$$

The weights on the projective space $\mathbb{P}^{49} = \mathbb{P}\left(V_{24} \oplus V_{24}'\right)$ are given by:

$$\{2m | -12 \leq n \leq 12\} \text{ on } V_{24}, \quad \{3n | -18 \leq n \leq 18\} \text{ on } V_{24}'. \tag{3.4.6}$$

According to the construction in Theorem 3.4.1, we should construct the stratification using the weights in Equation (3.4.6). However, many of the weights in Equation (3.4.6) are irrelevant for our case, we have the following proposition:

Proposition 3.4.5. *Let $\beta \in \mathbb{Z}_+$ be a weight of $\mathbb{T}$ on $\mathbb{P}^{49}$. We have:*

1. *If $3 \nmid \beta$, then any pair $(A, B) \in Z_\beta$ satisfies $A = 0$,*

2. *If $2 \nmid \beta$, then any pair $(A, B) \in Z_\beta$ satisfies $B = 0$.*

Proof. We will prove the first statement, the second statement can be proved similarly. Let $A = \sum_{i=0}^{8} a_i T_0^i T_1^{8-i}$, we expand A^3 in terms of coordinates:

$$A^3 = \sum_{i,j,k} a_i a_j a_k T_0^{i+j+k} T_1^{24-i-j-k}.$$

The weight of $\mathbb{T}$ on the coordinate a_i is $2i - 8$, the weight on the coordinate $a_i a_j a_k$ is $2i + 2j + 2k - 24$. In particular, the weight of a_i^3 is $6i - 24$. If $3 \nmid \beta$, then for all $0 \leq i \leq 8$, we have $6i - 24 \neq \beta$. Thus, for all $0 \leq i \leq 8$, we have $a_i = 0$, which implies $A = 0$. $\square$

The above proposition and its proof tell us we can only consider the weights corresponding to Equation (3.4.5) to form the index set $\mathcal{B}$. However, we need to normalize them to cancel the degree factors coming from the embedding $\iota : W\mathbb{P} \hookrightarrow \mathbb{P}^{49}$. The normalized weights are:

$$\{m| -4 \leq m \leq 4\} \text{ on } V_8, \quad \{\frac{2n}{3}| -6 \leq n \leq 6\} \text{ on } V_{12}. \tag{3.4.7}$$

We denote the index set $\hat{\mathcal{B}}$ as the subset of $\mathbb{R}_{\geq 0}$ formed by these normalized weights. We can view the normalized weights from the perspective of fixed points. For any $\hat{\beta} = m = \frac{2n}{3} \in \hat{\mathcal{B}}$, the torus $\mathbb{T}$ acts with weights $(2m, 2n)$ on corresponding coordinates in $V_8 \oplus V_{12}$ respectively. Since $m/n = 2/3$, the torus $\mathbb{T}$ will fix the points $[a_{m+4}T_0^{m+4}T_1^{4-m} : b_{n+6}T_0^{n+6}T_1^{6-n}] \in W\mathbb{P}$. Now we describe the $Z_{\hat{\beta}}$ and $Y_{\hat{\beta}}$ for each $\hat{\beta} \in \hat{\mathcal{B}}$ concretely.

Definition 3.4.6. Let (A, B) be a pair in $V_8 \oplus V_{12}$, we expand them in terms of coordinates:

$$A = \sum_{i=0}^{8} a_i T_0^i T_1^{8-i}, \quad B = \sum_{j=0}^{12} b_j T_0^j T_1^{12-j}.$$

Let $\hat{\mathcal{B}} := \{m|0 \leq m \leq 4\} \cup \{\frac{2n}{3}|0 \leq n \leq 6\}$. For each $\hat{\beta} \in \hat{\mathcal{B}}$, we define:

$$Z_{\hat{\beta}} := \left\{[A : B] \in W\mathbb{P}| a_i = 0 \text{ if } i \neq \hat{\beta} + 4, b_j = 0 \text{ if } \frac{2j}{3} \neq \hat{\beta} + 4\right\},$$

$$Y_{\hat{\beta}} := \left\{ [A : B] \in W\mathbb{P} \,\middle|\, \begin{array}{l} a_i = 0 \text{ if } i < \hat{\beta} + 4, \quad b_j = 0 \text{ if } \dfrac{2j}{3} < \hat{\beta} + 4, \\ a_i \neq 0 \text{ or } b_j \neq 0 \text{ if } i = \dfrac{2j}{3} = \hat{\beta} + 4 \end{array} \right\}.$$

The set $Z_{\hat{\beta}}$ is closed sub-orbifold of $W\mathbb{P}$ and $Y_{\hat{\beta}}$ is a locally closed sub-orbifold of $W\mathbb{P}$. For $\hat{\beta} \neq 0$, we define:

$$S_{\hat{\beta}} := \mathrm{SL}_2 \cdot Y_{\hat{\beta}}.$$

For $\hat{\beta} = 0$, we define $S_0 = W\mathbb{P}^{ss}$, the semistable locus for the SL_2-action, see Proposition

1.2.25. Then we have a stratification:

$$W\mathbb{P} = \bigsqcup_{\hat{\beta}\in\hat{\mathcal{B}}} S_{\hat{\beta}}.$$

Remark 3.4.7. In our case, the group $\operatorname{Stab}\beta$ is the standard maximal torus $\mathbb{T}$ in SL_2. It acts on $Z_{\hat{\beta}}$ and $Y_{\hat{\beta}}$ trivially. The semistable loci $Z^{ss}_{\hat{\beta}}$ and $Y^{ss}_{\hat{\beta}}$ are the same as $Z_{\hat{\beta}}$ and $Y_{\hat{\beta}}$ respectively. We will not distinguish them in the following discussion.

Proposition 3.4.8. *For any $\hat{\beta} \in \hat{\mathcal{B}}$, the coordinate-wise projection map:*

$$p : Y_{\hat{\beta}} \to Z_{\hat{\beta}}$$

is a fibration, where the fibers are affine space quotients by a finite group action.

Proof. Let $\pi : V_8 \oplus V_{12} - \{0\} \to W\mathbb{P}$ be quotient map induced by the $\mathbb{C}^\star$-action. Clearly $\pi^{-1}(Y_{\hat{\beta}}) = \pi^{-1}(Z_{\hat{\beta}}) \times \mathbb{A}^m$. The map p is induced by the projection onto the first factor in $\pi^{-1}(Z_{\hat{\beta}}) \times \mathbb{A}^m$. For any point $z \in Z_{\hat{\beta}}$, the fiber $p^{-1}(z)$ is the affine space $\mathbb{A}^m$ quotient $\operatorname{Stab} z \subset \mathbb{C}^\star$. The group $\operatorname{Stab} z$ is a finite group. □

Our next step is to show the stratum $S_{\hat{\beta}}$ is actually a locally closed sub-orbifold of $W\mathbb{P}$. We will work on the vector spaces and use Theorem 3.3.7 to show we get sub-orbifolds. Let $C(Y_{\hat{\beta}})$ be the cone over $Y_{\hat{\beta}}$ in $V_8 \oplus V_{12} \backslash \{0\}$, and $C(\mathrm{SL}_2 \cdot Y_{\hat{\beta}})$ be the cone over $\mathrm{SL}_2 \cdot Y_{\hat{\beta}}$ in $V_8 \oplus V_{12} \backslash \{0\}$. Recall the action of SL_2 and $\mathbb{C}^\star$ commutes, so we have:

$$C(\mathrm{SL}_2 \cdot Y_{\hat{\beta}}) = \mathrm{SL}_2 \cdot C(Y_{\hat{\beta}}).$$

We begin with the following propositions are parallel to [Kir84, Lemma 13.4] and [Kir84, Theorem 13.5].

Proposition 3.4.9. *Let P be the following parabolic subgroup in SL_2 that contains $\mathbb{T}$:*

$$P = \left\{ \begin{pmatrix} \lambda & 0 \\ \mu & \lambda^{-1} \end{pmatrix} \in \mathrm{SL}_2 \right\}.$$

Let $\mathfrak{p}$ be the Lie algebra of P, $\mathfrak{sl}_2$ be the Lie algebra of SL_2. For any point $y \in C(Y_{\hat{\beta}})$, we have:

1. $\{g \in \mathrm{SL}_2 \,| g \cdot y \in C(Y_{\hat{\beta}})\} = P$,

2. $\{\xi \in \mathfrak{sl}_2 | \xi_y \in T_y(C(Y_{\hat{\beta}}))\} = \mathfrak{p}$, *where ξ_y is the induced vector field of ξ at y.*

Proof. Recall we can represent an element in $V_8 \oplus V_{12}$ as a pair (A, B), where $A = \sum_{i=0}^{8} a_i T_0^i T_1^{8-i}$ and $B = \sum_{j=0}^{12} b_j T_0^j T_1^{12-j}$. A pair (A, B) is in $C(Y_{\hat{\beta}})$ if and only if the point $[T_0 : T_1] = [0 : 1]$ is a zero of the polynomial A with degree at least $\hat{\beta} + 4$ and is a zero of the polynomial B with degree at least $\frac{3\hat{\beta}}{2} + 6$. Moreover, at least one of the degree lower bounds is sharp. Note that any element $g \in \mathrm{SL}_2$, g act on $\mathbb{P}^1$ linearly, and both A and B the point $[0 : 1]$ has vanishing degree more than half of the total degree. So $g \cdot (A, B) \in C(Y_{\hat{\beta}})$ if and only if g fixes the point $[0 : 1]$, which implies $g \in P$. This proves the first statement.

For the second statement, recall we have the standard basis

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

for $\mathfrak{sl}_2$. We can compute the induced vector fields explicitly. For example, the induced vector field of E at y is given by:

$$E_y = \frac{d}{dt}\Big|_{t=0} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \cdot (A, B).$$

We consider the two coordinates of E_y: $i \cdot a_i \cdot \partial T_0^{i-1} T_1^{7-i}$ and $j \cdot b_j \cdot \partial T_0^{j-1} T_1^{11-i}$. Suppose $E_y \in T_y(C(Y_{\hat{\beta}}))$, then we must have $a_i = b_j = 0$, but this contradicts the definition of $C(Y_{\hat{\beta}})$. So we know $E_y \notin \{\xi \in \mathfrak{sl}_2 | \xi_y \in T_y(C(Y_{\hat{\beta}}))\}$. Using similar arguments, we can show $F_y, H_y \in \{\xi \in \mathfrak{sl}_2 | \xi_y \in T_y(C(Y_{\hat{\beta}}))\}$. Since the subset is linear, we conclude:

$$\{\xi \in \mathfrak{sl}_2 | \xi_y \in T_y(C(Y_{\hat{\beta}}))\} = \langle F_y, H_y \rangle = \mathfrak{p}.$$

$\square$

Proposition 3.4.10. *For any $\hat{\beta} \in \hat{\mathcal{B}} \backslash \{0\}$, the stratum $S_{\hat{\beta}}$ is isomorphic to a mixed quotient:*

$$S_{\hat{\beta}} := \mathrm{SL}_2 \cdot Y_{\hat{\beta}} \cong \mathrm{SL}_2 \times_P Y_{\hat{\beta}}.$$

In particular, the stratum $S_{\hat{\beta}}$ is a locally closed sub-orbifold of $W\mathbb{P}$.

Proof. We consider the maps in $V_8 \oplus V_{12} \backslash \{0\}$:

$$\mathrm{SL}_2 \times_P C(Y_{\hat{\beta}}) \xrightarrow{f} \mathrm{SL}_2 \cdot C(Y_{\hat{\beta}}) \xrightarrow{h} V_8 \oplus V_{12} \backslash \{0\}.$$

The cone $C(Y_{\hat{\beta}})$ is P-invariant, so the map f is well-defined. Furthermore, set theoretically, the map f and h descend to a map:

$$\mathrm{SL}_2 \times_P Y_{\hat{\beta}} \xrightarrow{\bar{f}} \mathrm{SL}_2 \cdot Y_{\hat{\beta}} \xrightarrow{\bar{h}} W\mathbb{P}.$$

We need to show the map $\bar{h} \circ \bar{f}$ is an orbifold embedding. According to Theorem 3.3.7, it's sufficient to show $h \circ f$ is a strong equivariant immersion. The map $h \circ f$ is clearly $\mathbb{C}^\star$-equivariant, we will check the following.

- The map $h \circ f$ is injective.
- The map $d(h \circ f)$ is an injective map at each point.

The first item is equivalent to the first statement in Proposition 3.4.9. For the second item, let $d(h\circ f): T_{(gP,y)}\left(\mathrm{SL}_2\times_P C(Y_{\hat\beta})\right)\to T_y(V_8\oplus V_{12}\backslash\{0\})$ be the differential at (gP,y).

It is enough to check for (P,y), $d(h\circ f)$ is injective. Let

$$(\xi+\mathfrak{p},\eta)\in T_{(P,y)}\left(\mathrm{SL}_2\times_P C(Y_{\hat\beta})\right)$$

be a tangent vector, where $\xi+\mathfrak{p}\in\mathfrak{sl}_2/\mathfrak{p}$ and $\eta\in T_y(C(Y_{\hat\beta}))$. Furthermore, we have $-\xi_y+\eta\in T_y(C(Y_{\hat\beta}))$. Now assume we have:

$$d(h\circ f)(\xi+\mathfrak{p},\eta)=\eta=0.$$

Then $\xi_y\in T_y(C(Y_{\hat\beta}))$, which implies $\xi\in\mathfrak{p}$ by the second statement in Proposition 3.4.9. We have $d(h\circ f)$ is injective at (P,y). So we conclude on the level of cones in $V_8\oplus V_{12}\backslash\{0\}$, the map $h\circ f$ is a strong equivariant immersion. So the map $\bar h\circ\bar f$ is an orbifold embedding. The stratum $S_{\hat\beta}$ is a locally closed sub-orbifold of $W\mathbb{P}$, and its closure is given by $\mathrm{SL}_2\cdot\overline{Y_{\hat\beta}}$, where

$$\overline{Y_{\hat\beta}}:=\left\{[A:B]\in W\mathbb{P}|a_i=0\text{ if }i<\hat\beta+4, b_j=0\text{ if }\frac{2j}{3}<\hat\beta+4\right\}.$$

□

Lemma 3.4.11. *The stratification $W\mathbb{P}=\bigsqcup_{\hat\beta\in\hat{\mathcal{B}}}S_{\hat\beta}$ is equivariant-perfect.*

Proof. From Proposition 3.4.10, we know the stratification $W\mathbb{P}=\bigsqcup_{\hat\beta\in\hat{\mathcal{B}}}S_{\hat\beta}$ formed by locally closed sub-orbifolds. We need to show the equivalent Thom-Gysin sequence for the stratification:

$$\cdots\to H^{k-d(\hat\beta)}_{\mathrm{SL}_2}(S_{\hat\beta})\to H^k_{\mathrm{SL}_2}(\bigsqcup_{\gamma\le\hat\beta}S_\gamma)\to H^k_{\mathrm{SL}_2}(\bigsqcup_{\gamma<\hat\beta}S_\gamma)\to\cdots$$

splits into short exact sequences. It is enough to show the first map is injective:

$$H^{k-d(\hat{\beta})}_{\mathrm{SL}_2}(S_{\hat{\beta}}) \to H^k_{\mathrm{SL}_2}(\bigsqcup_{\gamma \leq \hat{\beta}} S_\gamma).$$

We can compose the above map with the restriction and obtain:

$$H^{k-d(\hat{\beta})}_{\mathrm{SL}_2}(S_{\hat{\beta}}) \to H^k_{\mathrm{SL}_2}(S_{\hat{\beta}}).$$

We will show the above map is injective for any $\hat{\beta}$ and k. The map is multiplication by the equivariant Euler class of the orbifold normal bundle $\mathcal{N}$ of $S_{\hat{\beta}}$ in $\bigsqcup_{\gamma \leq \hat{\beta}} S_\gamma$. We will show $e_{\mathrm{SL}_2}(\mathcal{N})$ is a non-zero divisor in $H^*_{\mathrm{SL}_2}(S_{\hat{\beta}})$. Thanks to Proposition 3.4.10 and Proposition 3.4.8, we have the following isomorphisms:

$$H^*_{\mathrm{SL}_2}(S_{\hat{\beta}}) \simeq H^*_P(Y_{\hat{\beta}}) \simeq H^*_{\mathrm{Stab}\,\hat{\beta}}(Y_{\hat{\beta}}) \simeq H^*_{\mathrm{Stab}\,\hat{\beta}}(Z_{\hat{\beta}}).$$

Consider the restriction of the normal bundle $\mathcal{N}|_{Z_{\hat{\beta}}}$, it will be a quotient of the normal bundle of $Z_{\hat{\beta}}$ in $W\mathbb{P}$. Recall $Z_{\hat{\beta}}$ is a fixed component of $\mathrm{Stab}\,\hat{\beta}$, so $\mathrm{Stab}\,\hat{\beta}$ must have non-trivial weights on $\mathcal{N}|_{Z_{\hat{\beta}}}$. Then $e_{\mathrm{Stab}\,\hat{\beta}}(\mathcal{N}|_{Z_{\hat{\beta}}})$ must be a non-zero divisor in $H^*_{\mathrm{Stab}\,\hat{\beta}}(Z_{\hat{\beta}})$ by the Theorem [AB83, Proposition 13.4]. □

Now we are almost ready to compute the equivariant cohomology of the semistable locus $W\mathbb{P}^{ss}$. We need to compute the codimension of each stratum $S_{\hat{\beta}}$ and examine the shape of $Z^{ss}_{\hat{\beta}}$. Let $n(\hat{\beta})$ be the number of weights (counting with multiplicity) α_i in (3.4.7) satisfying $\alpha_i.\beta < \|\beta\|^2$. It can be used to compute the codimension of each stratum (see [Kir84, 3.1]):

$$\mathrm{codim}\,(S_{\hat{\beta}}) = n(\hat{\beta}) - \dim G/P.$$

In our case, P will always be the parabolic group in Proposition 3.4.9, so $\dim G/P = 1$. Furthermore, $Z^{ss}_{\hat{\beta}}$ all agrees with $Z_{\hat{\beta}}$. We do the codimension count and summarize the data for unstable strata in the following table.

Table 3.1. Cohomology of the unstable strata.

normalized weights $\hat{\beta}$	Stab $\hat{\beta}$	codim $(S_{\hat{\beta}})$	$Z^{ss}_{\hat{\beta}}$	$P^G_t(S_{\hat{\beta}})$
2/3	$\mathbb{T}$	11	pt	$(1-t^2)^{-1}$
1	$\mathbb{T}$	12	pt	$(1-t^2)^{-1}$
4/3	$\mathbb{T}$	13	pt	$(1-t^2)^{-1}$
2	$\mathbb{T}$	14	$\mathbb{P}^1$	$(1+t^2)(1-t^2)^{-1}$
8/3	$\mathbb{T}$	16	pt	$(1-t^2)^{-1}$
3	$\mathbb{T}$	17	pt	$(1-t^2)^{-1}$
10/3	$\mathbb{T}$	18	pt	$(1-t^2)^{-1}$
4	$\mathbb{T}$	19	$\mathbb{P}^1$	$(1+t^2)(1-t^2)^{-1}$

We also need to state one fact about the equivariant cohomology of the total space $W\mathbb{P}$. The isomorphism (3.4.4) extends to our case, we suspect Kirwan's isomorphism is true for any compact symplectic orbifold. We will only prove the following proposition which is sufficient for our purpose.

Proposition 3.4.12. *The equivariant Poincaré polynomial of $W\mathbb{P}$ is:*

$$P_t^{\mathrm{SL}_2}(W\mathbb{P}) = \frac{\sum_{i=0}^{21} t^{2i}}{1-t^4}.$$

Proof. We replace the groups by the maximal real compact subgroups:

$$S^1 \subset \mathbb{C}^\star; \quad \mathrm{SU}(2) \subset \mathrm{SL}_2\,.$$

It's enough to do the computation for $P_t^{\mathrm{SU}(2)}(W\mathbb{P})$. Recall by [Kaw73, Theorem 1] we know the cohomology of $W\mathbb{P}$ is free and isomorphic to the cohomology of a usual projective space $\mathbb{P}$. Consider the fibration:

$$W\mathbb{P} \to W\mathbb{P} \times_{S^1} ES^1 \to BS^1.$$

Since $W\mathbb{P}$ is a free and no odd degree cohomology, we know the Serre spectral sequence degenerates at E_2 page. In fact, the fibration satisfies the hypothesis of the Leray-Hirsch theorem, so we have:

$$H^*_{S^1}(W\mathbb{P}) \simeq H^*(W\mathbb{P}) \otimes H^*(BS^1).$$

Now we consider the fibration:

$$\mathrm{SU}(2)/S^1 \simeq S^2 \to W\mathbb{P} \times_{S^1} ES^1 \to W\mathbb{P} \times_{\mathrm{SU}(2)} \mathrm{ESU}(2),$$

again, the Serre spectral sequence degenerates at E_2-page and the Leray-Hirsch theorem applies. We have:

$$H^*_{S^1}(W\mathbb{P}) \simeq H^*_{\mathrm{SU}(2)}(W\mathbb{P}) \otimes H^*(S^2).$$

Combining the above two isomorphisms and counting dimensions, we have:

$$P_t^{\mathrm{SU}(2)}(W\mathbb{P}) = \frac{\sum_{i=0}^{21} t^{2i}}{1-t^2} \cdot \frac{1}{1+t^2} = \frac{\sum_{i=0}^{21} t^{2i}}{1-t^4}.$$

□

Lemma 3.4.13. *The equivariant Poincaré polynomial of $W\mathbb{P}^{ss}$ is:*

$$\begin{aligned} P_t^{\mathrm{SL}_2}(W\mathbb{P}^{ss}) &= 1 + t^2 + 2t^4 + 2t^6 + 3t^8 + 3t^{10} + 4t^{12} + 4t^{14} + 5t^{16} + 5t^{18} \\ &+6t^{20} + 5t^{22} + 5t^{24} + 4t^{26} + 4t^{28} + 3t^{30} + 3t^{32} + 2t^{34} + 2t^{36} + \sum_{i=19}^{\infty} t^{2i} \end{aligned}$$

Proof. Using Equation 3.4.2 in Theorem 3.4.2, Proposition 3.4.12 and the datum in the Table 3.1, we compute:

$$P_t^{\mathrm{SL}_2}(W\mathbb{P}^{ss}) = \frac{\sum_{i=0}^{21} t^{2i}}{1-t^4} - \frac{(t^{38}+t^{28})(t^2+1)}{1-t^2} - \frac{t^{22}+t^{24}+t^{26}+t^{32}+t^{34}+t^{36}}{1-t^2}$$

which expands to the series in the Lemma. □

Remark 3.4.14. The equivariant Poincaré polynomial $P_t^{\mathrm{SL}_2}(W\mathbb{P}^{ss})$ is not a finite polynomial, this is due to the $W\mathbb{P}^s \neq W\mathbb{P}^{ss}$. For the same reason, $H^*_{\mathrm{SL}_2}(W\mathbb{P}^{ss})$ is different from $H^*(W\mathbb{P} /\!\!/ \mathrm{SL}_2)$. In the next sections, we'll get around this problem using Kirwan's partial desingularisation.

3.5 Kirwan blowup

In this section, we will study the Kirwan's partial desingularisation W^K for $\mathsf{W}^{GIT} = W\mathbb{P} /\!\!/ \mathrm{SL}_2$. In [Kir85], Kirwan developed a general theory for constructing the canonical partial desingularisation for GIT quotient of smooth projective varieties. The process consists of a sequence of blowups centered at loci with positive dimensional stabilizers. The change of Poincaré polynomial in the partial desingularisation process is understood. Later, in [ER21], the partial desingularisation theory is extended to the case for irreducible Artin stack $\mathcal{X}$ with a good moduli space. The main construction is saturated blowup, which will maintain a good moduli space. Since our space $W\mathbb{P}$ has singularities, we should construct the partial desingularisation using saturated blowup. Moreover, we still wish to keep track of the change of Poincaré polynomial in the partial desingularisation process. The strategy is to construct an equivariantly perfect stratification for the desingularisation explicitly. Finally, we note that the result is essentially repeating the computation in [Kir85] for a space with finite quotient singularities.

3.5.1 Kirwan blowup for W^{GIT}

As explained in Remark 3.4.14, the SL_2 action on $W\mathbb{P}$ will have strictly semistable points. They form the set $\mathrm{SL}_2 \cdot Y_0$ using our notation in (3.4.6). There are points in $W\mathbb{P}^{ss}$ with positive dimensional stabilizers as well. More precisely, they are $\mathrm{SL}_2 \cdot Z_{\mathbb{T}}$, where

$$Z_{\mathbb{T}} := \{x \in W\mathbb{P}^{ss} |\ \mathbb{T} \text{ fixes } x\} = \{[\lambda x^4 y^4 + \mu x^6 y^6] \in \mathbb{P}^{ss}\} \simeq W\mathbb{P}(2,3).$$

In the notation of [Kir85], $Z_{\mathbb{T}}^{ss}$ is used to denote the semistable locus for an induced $\mathbb{T}$-action on $Z_{\mathbb{T}}$. In our case, $Z_{\mathbb{T}}^{ss}$ and $Z_{\mathbb{T}}$ will be the same, and we will not distinguish them.

In the smooth case, we need to blowup $\mathrm{SL}_2 \cdot Z_{\mathbb{T}} \subset W\mathbb{P}^{ss}$. Then we lift the SL_2-action on the blowup appropriately such that the action will have no strictly semistable points. But our space $W\mathbb{P}$ has quotient singularities. To cope the difficulty, we will do the blowup on the corresponding smooth affine cones.

Definition 3.5.1. Consider $q : V_8 \oplus V_{12} \backslash \{0\} \to W\mathbb{P}$ be the quotient map induced by the $\mathbb{C}^\star$-action. For a subset $Z \subset W\mathbb{P}$, let $C(Z)$ be the cone over Z in $V_8 \oplus V_{12} \backslash \{0\}$. We write $C(W\mathbb{P}^{ss})$ as V^{ss}, and $C(\mathrm{SL}_2 \cdot Z_{\mathbb{T}})$ as C.

The cone C is covered by the following two spaces:

$$V_2 \backslash \{0\} \times \mathbb{C} \to V^{ss}; \quad (F, x) \mapsto [x \cdot F^4 : F^6],$$

$$V_2 \backslash \{0\} \times \mathbb{C} \to V^{ss}; \quad (F, y) \mapsto [F^4 : y \cdot F^6].$$

It is straightforward to check that the cone C is smooth in V^{ss}. We consider the blowup of V^{ss} along C. Let $\mathcal{E} \subset \mathrm{Bl}_C(V^{ss})$ be the exceptional divisor. We have the following commutative diagram:

$$\begin{tikzcd} \mathcal{E} \arrow[r] \arrow[d, "/\mathbb{C}^\star"] & \mathrm{Bl}_C(V^{ss}) \arrow[d, "/\mathbb{C}^\star"] \\ E \arrow[r] & \widetilde{X} \end{tikzcd}$$

The existence of the vertical quotients can be explained using saturated blowup or equivalently the equivariant Reichstein transform in [ER21, Definition 4.4]. Consider the closed substack $[C/\mathbb{C}^\star] \subset [\mathrm{Bl}_C(V^{ss})/\mathbb{C}^\star]$. The map $p : [V^{ss}/\mathbb{C}^\star] \to W\mathbb{P}^{ss}$ is a good moduli space. The equivariant Reichstein transform $\mathrm{Bl}_C^p V^{ss}$ is obtained by deleting the strict transform of the saturation $p^{-1}p(C)$ in $\mathrm{Bl}_C(V^{ss})$. However, $p^{-1}p(C)$ is still C in our case, so we have $\mathrm{Bl}^p_C V^{ss} = \mathrm{Bl}_C(V^{ss})$. Then we have the good moduli space map $[\mathrm{Bl}_C(V^{ss})/\mathbb{C}^\star] \to \widetilde{X}$, which is in fact still the coarse moduli space map of a Deligne-Mumford stack.

Now we consider the good moduli space map $\pi : [V^{ss}/\mathbb{C}^\star \times \mathrm{SL}_2] \to \mathsf{W}^{GIT}$. The Kirwan desingularisation is obtained by considering the substack:

$$[C/\mathbb{C}^\star \times \mathrm{SL}_2] \subset [V^{ss}/\mathbb{C}^\star \times \mathrm{SL}_2].$$

Let $\mathrm{Bl}^\pi_C V^{ss}$ be the equivariant Reichstein transform, then Kirwan's partial desingularisation is the coarse moduli space:

$$[\mathrm{Bl}^\pi_C V^{ss}/\mathbb{C}^\star \times \mathrm{SL}_2] \to \mathsf{W}^K.$$

So we need to understand the difference between $\mathrm{Bl}_C(V^{ss})$ and $\mathrm{Bl}^\pi_C V^{ss}$. By the construction, we need to remove the strict transform of the saturation $\pi^{-1}\pi(C)$ in $\mathrm{Bl}_C(V^{ss})$. Recall C is the cone over $\mathrm{SL}_2 \cdot Z_{\mathbb{T}}$, we can identify the set $\pi^{-1}\pi(C)$ as:

$$\pi^{-1}\pi(C) = C(\mathrm{SL}_2 \cdot Y_0) \subset V^{ss},$$

where Y_0 is the subset of $W\mathbb{P}$ defined in (3.4.6). Clearly, these correspond to the strictly semistable points in $W\mathbb{P}$ with respect to the SL_2-action. If we remove the strict transform of $C(\mathrm{SL}_2 \cdot Y_0)$ in $\mathrm{Bl}_C(V^{ss})$, we will obtain a space with $\mathbb{C}^\star \times \mathrm{SL}_2$-action and no strictly semistable points. The coarse moduli space of the quotient stack is of finite quotient singularities.

3.5.2 The cohomology of W^K

The Kirwan partial desingularisation for a GIT quotient is intrinsic, and the Betti numbers of the partial desingularisation can be viewed as invariants for the GIT quotient.

In [Kir85], a concrete formula is developed to compute Betti numbers of the partial desingularisation. We would like to adapt the computation to $\mathrm{Bl}_C^\pi V^{ss} \subset \mathrm{Bl}_C(V^{ss})$. To simplify the process and make it more explicit, we will work with the coarse moduli spaces of $[\mathrm{Bl}_C^\pi V^{ss}/\mathbb{C}^\star]$ and $[\mathrm{Bl}_C(V^{ss})/\mathbb{C}^\star]$, we define:

$$\widetilde{X}^{ss} := \mathrm{Bl}_C^\pi V^{ss}/\mathbb{C}^\star; \quad \widetilde{X} := \mathrm{Bl}_C(V^{ss})/\mathbb{C}^\star.$$

The goal for this section is to compute the Poincaré polynomial for W^K, which is identical to the SL_2-equivariant Poincaré polynomial of $\widetilde{X}^{ss}$. The main strategy is to mimic the stratification construction in [Kir85], and construct the equivariantly perfect stratification for $\widetilde{X}$ explicitly:

$$\widetilde{X} = \bigsqcup_{\alpha \in \mathcal{A}} S_\alpha.$$

The subset $\widetilde{X}^{ss}$ will be the biggest stratum in the stratification. We begin with computing the equivariant cohomology of the entire space $\widetilde{X}$.

Recall the cone $C \subset V^{ss}$ is smooth. So the equivariant cohomology of the blowup $\mathrm{Bl}_C(V^{ss})$ can be computed using the following formula:

$$H_G^*(\mathrm{Bl}_C(V^{ss})) = H_G^*(V^{ss}) \oplus H_G^*(\mathcal{E})/H_G^*(C), \tag{3.5.1}$$

where $G = \mathbb{C}^\star \times \mathrm{SL}_2$. The exceptional divisor $\mathcal{E}$ is a projective bundle over C. More precisely, let $\mathcal{N}_C$ be the normal bundle of the embedding $C \subset V^{ss}$, and let $r = \mathrm{rank}\,(\mathcal{N}_C)$ and h be the hyperplane class on projectivization of $\mathcal{N}_C$, we have:

$$H_G^*(\mathcal{E}) = H_G^*(C)(1 + h + \cdots + h^{r-1}).$$

Notice the group $\mathbb{C}^\star$ acts with finite isotropy on V^{ss} and on $\mathcal{N}_C$, so we have the corresponding formulas for SL_2-equivariant cohomology (see e.g. [Bri98, Remarks 2]):

$$H^*_{\mathrm{SL}_2}(\widetilde{X}) = H^*_{\mathrm{SL}_2}(W\mathbb{P}^{ss}) \oplus H^*_{\mathrm{SL}_2}(E)/H^*_{\mathrm{SL}_2}(\mathrm{SL}_2 \cdot Z_{\mathbb{T}}), \tag{3.5.2}$$

and

$$H^*_{\mathrm{SL}_2}(E) = H^*_{\mathrm{SL}_2}(\mathrm{SL}_2 \cdot Z_{\mathbb{T}})(1 + h + \cdots + h^{r-1}). \tag{3.5.3}$$

The rank r of the normal bundle $\mathcal{N}_C$ in our case will be 18.

It's easy to verify $\mathrm{SL}_2 \cdot Z_{\mathbb{T}}$ is homeomorphic to $\mathrm{SL}_2 \times_{N(\mathbb{T})} Z_{\mathbb{T}}$. Here $N(\mathbb{T}) \subset \mathrm{SL}_2$ is the normaliser of $\mathbb{T}$, $N(\mathbb{T})$ is generated by diagonal and anti-diagonal matrices in SL_2:

$$N(\mathbb{T}) = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \begin{pmatrix} 0 & \mu^{-1} \\ -\mu & 0 \end{pmatrix} \right\} \subset \mathrm{SL}_2 .$$

Now note that $N(\mathbb{T})$ acts trivially on $Z_{\mathbb{T}}$. The equivariant cohomology of $\mathrm{SL}_2 \cdot Z_{\mathbb{T}}$ can be identified with $H^*_{\mathrm{SL}_2}(\mathrm{SL}_2 \cdot Z_{\mathbb{T}}) \simeq H^*_{N(\mathbb{T})}(Z_{\mathbb{T}})$, hence we have:

$$P_t^{\mathrm{SL}_2}(\mathrm{SL}_2 \cdot Z_{\mathbb{T}}^{ss}) = P_t^{N(\mathbb{T})}(Z_{\mathbb{T}}^{ss}) = P_t(\mathbb{P}^1)P_t(BN(\mathbb{T})) = (1+t^2)(1-t^4)^{-1}.$$

Based on Lemma 3.4.13 and Equation 3.5.2, we can already solve $P_t^{\mathrm{SL}_2}(\widetilde{X})$. But the polynomial we're looking for is $P_t(\mathsf{W}^K) = P_t^{\mathrm{SL}_2}(\widetilde{X}^{ss})$. Kirwan developed the concrete formula for removing these contributions. For the reader's convenience, we state the simplified formula which closely related to our case.

Theorem 3.5.2. *[Kir85] Let $X \subset \mathbb{P}^N$ be a nonsingular projective variety, and a reductive group G acts on it linearly. Assume Kirwan's desingularisation $\widetilde{X} /\!\!/ G$ is obtained by* *one-step blowup of X^{ss} along $G \cdot Z_T^{ss}$, then we have:*

$$P_t(\widetilde{X} /\!\!/ G) = P_t(\widetilde{X}^{ss} /\!\!/ G) = P_t^G(\widetilde{X}) - \sum_{0 \neq \beta' \in \mathcal{B}(\rho)} \frac{1}{w(\beta', T, G)} t^{2d(\mathbb{P}\mathcal{N}_p, \beta')} P_t^{N(T) \cap \operatorname{Stab} \beta'}(Z_{\beta', T}^{ss}). \tag{3.5.4}$$

Unfortunately, the formula 3.5.4 is not directly applicable to our case. The main reason is that the weighted projective space $W\mathbb{P}$ is singular. The construction of our $\widetilde{X}^{ss}$ is by the equivariant Reichstein transform. Furthermore, we have identified:

$$\widetilde{X} \backslash \widetilde{X}^{ss} = \widetilde{\mathrm{SL}_2 \cdot Y_0},$$

where $\widetilde{\mathrm{SL}_2 \cdot Y_0}$ is the strict transform of $\mathrm{SL}_2 \cdot Y_0$ in $\widetilde{X} = \mathrm{Bl}_C(V^{ss})/\mathbb{C}^\star$.

Definition 3.5.3. Let $\mathcal{A} = \{2, 4, 6, 8, 10, 12\}$. Let $Y_{0,\alpha} \subset Y_0$ be the subset defined by the following condition:

$$Y_{0,\alpha} := \left\{ [A : B] \in W\mathbb{P} \;\middle|\; \begin{array}{l} a_i = 0 \text{ if } i < \alpha/2 + 4, \quad b_j = 0 \text{ if } j < \alpha/2 + 6, \\ a_{\alpha/2+4} \neq 0 \text{ or } b_{\alpha/2+6} \neq 0 \end{array} \right\}.$$

Definition 3.5.4. Let $S_\alpha \subset \widetilde{X}$ be the strict transform of $\mathrm{SL}_2 \cdot Y_{0,\alpha}$ in $\widetilde{X}$. Take $S_0 := \widetilde{X}^{ss}$. We get a stratification:

$$\widetilde{X} = S_0 \cup \bigsqcup_{\alpha \in \mathcal{A}} S_\alpha.$$

In $W\mathbb{P}^{ss}$, the set $Y_{0,\alpha}$ is only fixed by the standard torus $\mathbb{T} \subset \mathrm{SL}_2$. We can check that $\mathrm{SL}_2 \cdot Y_{0,\alpha} \simeq \mathrm{SL}_2 \times_{\mathbb{T}} Y_{0,\alpha}$.

Proposition 3.5.5. *The stratification S_α in Definition 3.5.4 is SL_2-equivariantly perfect.*

Proof. Consider the projection map in $W\mathbb{P}$:

$$p : Y_{0,\alpha} \to W\mathbb{P}(2, 3) \times Z_{\mathbb{T}}, \quad [A : B] \mapsto ([a_{\alpha/2+4} : b_{\alpha/2+6}], [a_4 : b_6])$$

if $\alpha \leq 8$, and

$$p : Y_{0,\alpha} \to Z_{\mathbb{T}}, \quad [A : B] \mapsto [a_4 : b_6]$$

if $\alpha > 8$. The map p is a locally trivial fibration, with affine fibers up to a finite group action. We lift the map p to the blowup $\widetilde{X}$:

$$\widetilde{p} : Y_{0,\alpha} \to Z_{\alpha},$$

where Z_α a subset of the exceptional locus E in $\widetilde{X}$. The map $\widetilde{p}$ is also a locally trivial fibration with affine fibers up to a finite group action. Using the same argument in 3.4.11, it's sufficient to show the equivariant Euler class of the normal bundle of S_α in $\widetilde{X}$ is not a zero divisor in $H^*_{\mathrm{SL}_2}(S_\alpha)$. Notice we have:

$$H^*_{\mathrm{SL}_2}(S_\alpha) \simeq H^*_{\mathrm{SL}_2}(\mathrm{SL}_2 \times_{\mathbb{T}} Y_{0,\alpha}) \simeq H^*_{\mathbb{T}}(Y_{0,\alpha}) \simeq H^*_{\mathbb{T}}(Z_\alpha).$$

Again, the subspace Z_α is fixed by the standard torus $\mathbb{T} \subset \mathrm{SL}_2$. Moreover, the normal bundle of S_α in $\widetilde{X}$ is a quotient of the normal bundle of Z_α in $\widetilde{X}$. We conclude by the Theorem [AB83, Proposition 13.4]. □

To proceed with our computation, we need to know the topology of the fixed locus $Z_\alpha \subset E$. We pick a point $x \in Z_{\mathbb{T}}$. Up to a finite group $\mathbb{Z}_2$ or $\mathbb{Z}_3$, the fiber E_x is a projective space $\mathbb{P}(\mathcal{N}_x)$. The isotropy group $\mathbb{Z}_2$ or $\mathbb{Z}_3$ action is homotopic to a trivial action, and the map:

$$Z_\alpha \to Z_{\mathbb{T}}$$

is locally trivial fibration in the orbifold sense, see [PS03, Definition 2.1]. Furthermore, the orbi-bundle $\mathcal{O}(1)$ over Z_α gives the generators for every fiber. The orbifold Leray-Hirsch

Theorem [PS03, Theorem 2] applies to our case:

$$H^*(Z_\alpha) \simeq H^*(Z_{\mathbb{T}}) \otimes H^*(\mathbb{P}(\mathcal{N}_x)).$$

Since $\mathcal{N}_x$ is the normal bundle of C in V^{ss} at a point x, the subspace $Z_\alpha|_x$ is a hyperplane section of E_x. The fixed locus Z_α will be a projective bundle fibration over $Z_{\mathbb{T}}$ up to a finite isotropy group, and the fibers are homeomorphic to $\mathbb{P}^1$ or a point. We collect the data in the following table.

Table 3.2. SL_2-stratification on $\mathbb{P}\mathcal{N}_x$.

weights $\alpha \in \mathcal{A}$	$d(\mathbb{P}\mathcal{N}_x, \alpha)$	$Z_\alpha\vert_x$
2	9	pt
4	10	$\mathbb{P}^1$
6	12	$\mathbb{P}^1$
8	14	$\mathbb{P}^1$
10	16	pt
12	17	pt

Lemma 3.5.6. *The Poincaré polynomial of* W^K *is:*

$$P_t(\mathsf{W}^K) = 1 + 2t^2 + 4t^4 + 5t^6 + 7t^8 + 8t^{10} + 10t^{12} + 11t^{14} + 13t^{16} + 13t^{18}$$
$$+13t^{20} + 11t^{22} + 10t^{24} + 8t^{26} + 7t^{28} + 5t^{30} + 4t^{32} + 2t^{34} + t^{36}.$$

Proof. Using Equation 3.5.2, we can compute:

$$P_t^{\mathrm{SL}_2}(\widetilde{X}) = P_t^{\mathrm{SL}_2}(W\mathbb{P}^{ss}) + P_t^{N(\mathbb{T})}(Z_{\mathbb{T}})(\sum_{i=1}^{17} t^{2i}).$$

The term $P_t^{\mathrm{SL}_2}(W\mathbb{P}^{ss})$ has been computed in Lemma 3.4.13. Recall that we have:

$$P_t^{N(\mathbb{T})}(Z_{\mathbb{T}}^{ss}) = P_t(\mathbb{P}^1)P_t(BN(\mathbb{T})) = (1+t^2)(1-t^4)^{-1}.$$

We can compute the equivariant Poincaré polynomial of $\widetilde{X} \backslash \widetilde{X}^{ss}$ using the perfect stratification in Definition 3.5.4 and the data in Table 3.2:

$$\begin{aligned}\sum_{\alpha \in \mathcal{A}} P_t^{\mathrm{SL}_2}(S_\alpha) &= \sum_{\alpha \in \mathcal{A}} P_t^{\mathbb{T}}(Z_\alpha) \\ &= P_t(\mathbb{T}) \cdot P_t(Z_\alpha|_x) \cdot P_t(\mathrm{BS}^1) \\ &= \frac{1+t^2}{1-t^2}\left(t^{18}+t^{32}+t^{34}+(t^{20}+t^{24}+t^{28})(1+t^2)\right).\end{aligned}$$

Putting everything together using:

$$P_t(\mathsf{W}^K) = P_t^{\mathrm{SL}_2}(\widetilde{X}) - \sum_{\alpha \in \mathcal{A}} t^{2d(\mathbb{P}\mathcal{N}_x,\alpha)} P_t^{\mathrm{SL}_2}(S_\alpha)$$

gives us the desired result. □

Remark 3.5.7. The Kirwan partial desingularisation W^K should be a compact orbifold, and thus is subject to Poincaré duality. Our computation result is indeed a finite polynomial with the Poincaré symmetry.

Although in principle not applicable, one can still try to use the formula in Theorem 3.5.2 to compute the equivariant Poincaré polynomial of $\widetilde{X}^{ss}$. The computation will give the identical result as that in Lemma 3.5.6.

Let E be the exceptional locus $\mathcal{E}/\mathbb{C}^\star$ in $\widetilde{X}$. We finish this section by computing the Poincaré polynomial of the exceptional locus of the Kirwan blowup $\mathsf{W}^K \to \mathsf{W}^{GIT}$. The locus is a quotient $F := E^{ss}/\,\mathrm{SL}_2$, where E^{ss} is the part inside of the equivariant Reichstein transformation:

$$E^{ss} := E \cap \widetilde{X}^{ss}.$$

We have $H^*(F) = H^*_{\mathrm{SL}_2}(E^{ss})$. We need a suitable stratification for E^{ss} to compute the cohomology. Note that we have the following locally trivial fibration in the orbifold sense [PS03, Section 2]:

$$\mathbb{P}(\mathcal{N}_x)/\operatorname{Aut}(x) \to E \to \mathrm{SL}_2 \cdot Z_{\mathbb{T}},$$

where $\operatorname{Aut}(x)$ is either $\mathbb{Z}_2$ or $\mathbb{Z}_3$. The $\operatorname{Aut}(x)$ action is always homotoptic to a trivial action. Recall the group $N(\mathbb{T})$ acts trivially on $Z_{\mathbb{T}}$, and $\mathrm{SL}_2 \cdot Z_{\mathbb{T}}$ is isomorphic to $\mathrm{SL}_2 \times_{N(\mathbb{T})} Z_{\mathbb{T}}$. Let $E_Z := E|_{Z_{\mathbb{T}}}$ and $E_Z^{ss} := E^{ss}|_{Z_{\mathbb{T}}}$. We have:

$$H^*_{\mathrm{SL}_2}(SL_2 \cdot Z_{\mathbb{T}}) \simeq H^*_{N(\mathbb{T})}(Z_{\mathbb{T}}), \quad H^*_{\mathrm{SL}_2}(E) \simeq H^*_{N(\mathbb{T})}(E_Z), \quad H^*_{\mathrm{SL}_2}(E^{ss}) \simeq H^*_{N(\mathbb{T})}(E_Z^{ss}).$$

Let ρ be the linear action of $\mathbb{T} \subset \mathrm{SL}_2$ on $\mathcal{N}_x$, relative to this action, the classical $\mathbb{T}$-equivariant stratification construction in [Kir85] applies to $\mathbb{P}(\mathcal{N}_x)$. We have:

Proposition 3.5.8. *Let $\mathcal{B}(\rho)$ be the weights of $\mathbb{T}$-action ρ, we have the following $\mathbb{T}$-perfect stratification on $\mathbb{P}(\mathcal{N}_x)$:*

$$\mathbb{P}(\mathcal{N}_x) = S_0 \cup \bigsqcup_{\beta' \in \mathcal{B}(\rho)} S_{\beta'},$$

where S_0 is the semistable locus, and the equivariant Poincaré polynomial of $S_{\beta'}$ is:

$$P_t^{\mathbb{T}}(S_{\beta'}) = P_t^{\operatorname{Stab}\beta'}(Z^{ss}_{\beta',\rho}).$$

We can summarize the data in Table 3.3:

Table 3.3. $\mathbb{T}$-stratification on $\mathbb{P}\mathcal{N}_x$.

weights $\beta' \in \mathcal{B}(\rho)$	$\operatorname{Stab}\beta'$	$d(\mathbb{P}\mathcal{N}_x, \beta')$	$Z^{ss}_{\beta',\rho}$
± 2	$\mathbb{T}$	9	pt
± 4	$\mathbb{T}$	10	$\mathbb{P}^1$
± 6	$\mathbb{T}$	12	$\mathbb{P}^1$
± 8	$\mathbb{T}$	14	$\mathbb{P}^1$
± 10	$\mathbb{T}$	16	pt
± 12	$\mathbb{T}$	17	pt

The stratification will include negative weights, this is because the Weyl group of $\mathbb{T}$ is trivial. From our construction, we know:

$$E^{ss}|_x = S_0/\operatorname{Aut}(x) \subset \mathbb{P}(\mathcal{N}_x)/\operatorname{Aut}(x).$$

All the strata $S_{\beta'}$ in the stratification of $\mathbb{P}(\mathcal{N}_x)$ will be $\operatorname{Aut}(x)$-invariant. The action of $\operatorname{Aut}(x)$ on each stratum is homotopic to a trivial action. So we can ignore the isotropy groups.

Lemma 3.5.9. *The Poincaré polynomial of the exceptional locus $F \subset \mathsf{W}^K$ is:*

$$\begin{aligned} P_t(F) = 1 + 2t^2 + 3t^4 + 4t^6 + 5t^8 + 6t^{10} + 7t^{12} + 8t^{14} + 9t^{16} + 9t^{18} \\ +8t^{20} + 7t^{22} + 6t^{24} + 5t^{26} + 4t^{28} + 3t^{30} + 2t^{32} + t^{34}. \end{aligned}$$

Proof. To compute the Poincaré polynomial of F, it is sufficient to compute the $N(\mathbb{T})$-equivariant Poincaré polynomial of E_Z^{ss}. Consider the fibration:

$$\mathbb{P}(\mathcal{N}_x)/\operatorname{Aut}(x) \to E_Z \to Z_{\mathbb{T}}.$$

Note the $\operatorname{Aut}(x)$ action is homotopic to trivial action for any $g \in \operatorname{Aut}(x)$. Thus, every fiber has the same $N(\mathbb{T})$-equivariant cohomology concentrated on even degrees. The base is simply connected. The Leray spectral sequence degenerates, and we have:

$$H^*_{N(\mathbb{T})}(E_Z) \simeq H^*(Z_{\mathbb{T}}) \otimes H^*_{N(\mathbb{T})}(\mathbb{P}(\mathcal{N}_x)).$$

The Weyl group $N(\mathbb{T})/\mathbb{T} \simeq \mathbb{Z}_2$, we have:

$$H^*_{N(\mathbb{T})}(E_Z) \simeq H^*(Z_{\mathbb{T}}) \otimes [H^*_{\mathbb{T}}(\mathbb{P}(\mathcal{N}_x))]^{\mathbb{Z}_2}.$$

Recall we have the fibration:

$$S_0/\operatorname{Aut}(x) \to E_Z^{ss} \to Z_{\mathbb{T}}.$$

We know $P_t^{\mathbb{T}}(S_0/\operatorname{Aut}(x)) = P_t^{\mathbb{T}}(S_0) = P_t^{\mathbb{T}}(\mathbb{P}(\mathcal{N}_x)) - \sum_{0\neq\beta'\in\mathcal{B}(\rho)} P_t^{\mathbb{T}}(S_{\beta'})$, which does not depend on the choice of the fiber and concentrated on even degrees. So we have:

$$H^*_{N(\mathbb{T})}(E_Z^{ss}) \simeq H^*(Z_{\mathbb{T}}) \otimes [H^*_{\mathbb{T}}(S_0)]^{\mathbb{Z}_2}.$$

We can compute:

$$P_t^{\mathbb{T}}(S_0) = P_t(\mathbb{P}(\mathcal{N}_x))P_t(B\mathbb{T}) - \sum_{0\neq\beta'\in\mathcal{B}(\rho)} t^{2d(\mathbb{P}\mathcal{N}_x,\beta')} \cdot P_t^{\operatorname{Stab}\beta'}(Z^{ss}_{\beta',\mathbb{T}})$$

where the $Z^{ss}_{\beta',\mathbb{T}}$ and $t^{2d(\mathbb{P}\mathcal{N}_x,\beta')}$ are given in Table 3.3. To get the invariant part $[H^*_{\mathbb{T}}(S_0)]^{\mathbb{Z}_2}$, we notice the Weyl group $\mathbb{Z}_2$ will change the sign of the generator of $H^*(B\mathbb{T})$ and permute the strata $S_{\beta'}$ and $S_{-\beta'}$ for any $\beta' \in \mathcal{B}(\rho)$. So we have:

$$\begin{aligned}\sum_{i=0}^{\infty} t^i \cdot \dim [H^i_{\mathbb{T}}(S_0)]^{\mathbb{Z}_2} &= P_t(\mathbb{P}(\mathcal{N}_x))(1-t^4)^{-1} - \frac{1}{2}\sum_{0\neq\beta'\in\mathcal{B}'} t^{2d(\mathbb{P}\mathcal{N}_x,\beta')} \cdot P_t^{\operatorname{Stab}(\beta')}(Z^{ss}_{\beta',T}) \\ &= (\sum_{i=0}^{17} t^{2i})(1-t^4)^{-1} - (1-t^2)^{-1}(t^{18}+t^{32}+t^{34}+(t^{20}+t^{24}+t^{28})(1+t^2)).\end{aligned}$$

The Lemma now follows from $P_t(F) = (1+t^2)\cdot\sum_{i=0}^{\infty} t^i \cdot \dim [H^i_{\mathbb{T}}(S_0)]^{\mathbb{Z}_2}$.

$\square$

3.6 The cohomology of W^s

In this section, we compute the cohomology of stable locus quotient $\mathsf{W}^s := W\mathbb{P}^s/\operatorname{SL}_2$ following the ideas presented in [KL89].

For simplicity of notation, we write $G = \operatorname{SL}_2$ in this section.

Lemma 3.6.1. *The natural maps*

$$H_i(\mathsf{W}^K) \to H_i(\mathsf{W}^K, \mathsf{W}^s)$$

are surjections for all $0 \le i \le 36$.

Proof. Since W^K only has quotient singularities, we have the following commutative diagram by Poincaré duality:

$$\begin{tikzcd} H^{36-i}(\mathsf{W}^K) \arrow[r, "\chi^{36-i}"] \arrow[d, "\wr"] & H^{36-i}(E /\!\!/ G) \arrow[d, "\wr"] \\ H_i(\mathsf{W}^K) \arrow[r, "\alpha_i"] & H_i(\mathsf{W}^K, \mathsf{W}^s) \end{tikzcd}$$

To prove α_i are surjections, it's sufficient to show all χ^i are surjections. Note the SL_2 actions on $\widetilde{X}$ and E have no strictly semistable points, we can further lift χ^i to equivariant cohomologies:

$$\begin{tikzcd} H^i_G(\widetilde{X}^{ss}) \arrow[r, "\chi^i"] \arrow[d, "\wr"] & H^i_G(E^{ss}) \arrow[d, "\wr"] \\ H^i(\mathsf{W}^K) \arrow[r, "\chi^i"] & H^i(E /\!\!/ G) \end{tikzcd}$$

According to Kirwan's surjectivity, the equivariant cohomologies $H^i_G(\widetilde{X}^{ss})$ and $H^i_G(E^{ss})$ receive surjections from $H^i_G(\widetilde{X})$ and $H^i_G(E)$ respectively. We have the commutative diagram.

$$\begin{tikzcd} H^i_G(\widetilde{X}) \arrow[r, "res^i"] \arrow[d, two heads] & H^i_G(E) \arrow[d, two heads] \\ H^i_G(\widetilde{X}^{ss}) \arrow[r, "\chi^i"] & H^i_G(E^{ss}) \end{tikzcd}$$

It is now sufficient to demonstrate that the restriction maps res^i are all surjections.

We have explained in (3.5.2) and (3.5.3), for cohomology groups, the map $\pi : \widetilde{X} \to W\mathbb{P}^{ss}$ behaves like a smooth blowup. Let $\iota : E \subset \widetilde{X}$ be the exceptional locus, r be the rank of the normal $\mathcal{N}_C$, and h be the hyperplane class on E. We have one more commutative diagram:

$$\begin{array}{ccc} A \oplus H^i_G(W\mathbb{P}^{ss}) & \xrightarrow{\phi=\phi_1\oplus\phi_2} & B \\ {\scriptstyle \iota_!\oplus\pi^*}\downarrow\wr & & \downarrow\wr \\ H^i_G(\widetilde{X}) & \xrightarrow{res^i} & H^i_G(E) \end{array}$$

where $A = \bigoplus_{j=1}^{r-1} H_G^{i-2j}(G\cdot Z_{\mathbb{T}})\cdot h^{j-1}$ and $B = \bigoplus_{j=0}^{r-1} H_G^{i-2j}(G\cdot Z_{\mathbb{T}})\cdot h^{j}$.

On the summand A, the map ϕ_1 is simply the multiplication by h. Therefore, it is sufficient to show

$$\phi_2 : H^i_G(W\mathbb{P}^{ss}) \to H^i_G(G\cdot Z_{\mathbb{T}})$$

is a surjection. We may compute:

$$H^*_G(G\cdot Z_{\mathbb{T}}) = H^*_{N(\mathbb{T})}(Z_{\mathbb{T}}) = H^*(Z_{\mathbb{T}})\otimes H^*(BN(\mathbb{T})).$$

Now recall that:

$$Z_{\mathbb{T}} \simeq \mathbb{P}^1 \subset W\mathbb{P}^{ss} \subset W\mathbb{P}.$$

So we have a surjection:

$$H^*(W\mathbb{P}) \twoheadrightarrow H^*(Z_{\mathbb{T}}).$$

The map $BN(\mathbb{T}) \to BG$ induces an isomorphism on cohomologies in our case. Thus, the composition of the following restriction maps is a surjection

$$H^*_G(W\mathbb{P}) \to H^i_G(W\mathbb{P}^{ss}) \xrightarrow{\phi_2} H^*_G(G\cdot Z_{\mathbb{T}}),$$

and so is ϕ_2. $\square$

Remark 3.6.2. In the proof, we claimed Kirwan surjectivity holds for $H^i_G(\widetilde{X}) \to H^i_G(\widetilde{X}^{ss})$ and $H^i_G(E) \to H^i_G(E^{ss})$. Note that although $\widetilde{X}$ and E are not compact, it has been shown that the G-action on the two spaces will induce equivariantly perfect stratifications (see [Kir85, 7.5]). So the surjectivity still holds.

Lemma 3.6.3. *The Poincaré polynomial of* W^s *is:*

$$\begin{aligned} P_t(\mathsf{W}^s) = 1 + t^2 + 2t^4 + 2t^6 + 3t^8 + 3t^{10} + 4t^{12} + 4t^{14} + 5t^{16} \\ +4t^{18} + 4t^{20} + 3t^{22} + 3t^{24} + 2t^{26} + 2t^{28} + t^{30} + t^{32}. \end{aligned}$$

Proof. Consider the excision sequence for homology:

$$\cdots \to H_{i+1}(\mathsf{W}^K, \mathsf{W}^s) \to H_i(\mathsf{W}^s) \to H_i(\mathsf{W}^K) \xrightarrow{\alpha_i} H_i(\mathsf{W}^K, \mathsf{W}^s) \to \cdots$$

Recall from Lemma 3.5.9 and Poincaré duality that $H_{2i+1}(\mathsf{W}^K, \mathsf{W}^s) \simeq H^{35-2i}(F) = 0$. The maps α_i are all surjections as established by Lemma 3.6.1. The long exact sequence thus splits into short exact sequences:

$$0 \to H_i(\mathsf{W}^s) \to H_i(\mathsf{W}^K) \to H_i(\mathsf{W}^K, \mathsf{W}^s) \to 0.$$

Passing to the cohomology of W^s we have:

$$\dim H^i(\mathsf{W}^s) = \dim H_i(\mathsf{W}^s) = \dim H^{36-i}(\mathsf{W}^K) - \dim H^{36-i}(F).$$

Now the Poincaré polynomial $P_t(\mathsf{W}^s)$ follows immediately from the data in Lemma 3.5.6 and Lemma 3.5.9. □

3.7 The cohomology of W

In this section, we compute Betti numbers of W. As explained in Proposition 3.2.1, the space W is obtained by removing two curves from W^{GIT}, and the space W^s appears in the intermediate step:

$$\mathsf{W}^{GIT} - C_{ss} = \mathsf{W}^s, \ \mathsf{W}^{GIT} - \{C_{ss} \cup C_{nn}\} = \mathsf{W}.$$

Furthermore, we completely understand the topology of $C_{nn} \simeq \mathbb{P}^1$ and thus:

$$P_t(C_{nn}) = 1 + t^2. \tag{3.7.1}$$

We can now complete the proofs of Theorems 2.1.1 and 2.1.3.

Proof of Theorem 2.1.1. Using the universal coefficient theorem, it's sufficient to compute the dimension of $H_i(\mathsf{W})$. Consider the excision sequence:

$$\cdots \to H_i(\mathsf{W}) \to H_i(\mathsf{W}^s) \to H_i(\mathsf{W}^s, \mathsf{W}) \to H_{i-1}(\mathsf{W}) \to \cdots .$$

Now for the homology $H_i(\mathsf{W}^s, \mathsf{W})$, we view $\mathsf{W} \subset \mathsf{W}^s$ as subset of W^K. Then by duality and excision, we have:

$$H_i(\mathsf{W}^s, \mathsf{W}) \simeq H^{36-i}(C_{ss} \cup C_{nn}, C_{ss}) \simeq H^{36-i}(C_{nn}, pt).$$

By equation 3.7.1, $H^2(C_{nn}, pt) \simeq \mathbb{Q}$ and $H^i(C_{nn}, pt) = 0$ if $i \neq 2$. Putting them into the long exact sequence we conclude:

$$H_i(\mathsf{W}) \simeq H_i(\mathsf{W}^s) \text{ if } i \neq 33; \;\; H_{33}(\mathsf{W}) \simeq H^2(C_{nn}, pt) \simeq \mathbb{Q}.$$

□

Proof of Theorem 2.1.3. In the proof of Theorem 2.1.1, we have shown $H_{2i}(\mathsf{W}) \simeq H_{2i}(\mathsf{W}^s)$. Taking a dual we get:

$$H^{2i}(\mathsf{W}^s) \simeq H^{2i}(\mathsf{W}).$$

Recall in the proof of Lemma 3.6.3, we have shown the generators of $H^{2i}(\mathsf{W}^s)$ can be lifted to generators of $H^*(W\mathbb{P}) \otimes H^*(\mathrm{BSL}_2)$.

Now under the cycle map, the hyperplane class on $W\mathbb{P}$ corresponds to the Chow class a_1 as defined in [CK23], and the generator of $H^*(\mathrm{BSL}_2)$ corresponds to Chow class c_2 as defined in [CK23]. A direct comparison of dimensions concludes that the cycle maps must be isomorphisms. $\square$

Chapter 4

The Chow ring of the moduli space of elliptic K3 surfaces

4.1 Elliptic surfaces and Weierstrass fibrations

In this section, we recall the necessary background on elliptic surfaces and Weierstrass fibrations. A detailed survey following [Mir81, Mir89] can be found in Section 1.2. The main objects of interest in this section will be moduli stacks of minimal elliptic surfaces over $\mathbb{P}^1$ with section.

Definition 4.1.1. A minimal elliptic surface over $\mathbb{P}^1$ with section consists of the following data:

1. a smooth projective surface Y,
2. a proper morphism $\pi : Y \to \mathbb{P}^1$ whose general fiber is a smooth connected curve of genus 1 and such that none of the fibers contain any (-1)-curves,
3. a section $s : \mathbb{P}^1 \to Y$ of π.

Remark 4.1.2. Note that the minimality condition is different from the usual one given in the birational geometry of surfaces. There can be (-1)-curves on the surface Y, but they must not lie in the fibers of p.

We will study moduli spaces of minimal elliptic surfaces by studying the closely related notion of Weierstrass fibrations. We refer the reader to Section 1.2 for a detailed survey.

Definition 4.1.3. A Weierstrass fibration over $\mathbb{P}^1$ consists of the following data:

1. a projective surface X,
2. a flat proper morphism $p : X \to \mathbb{P}^1$ such that every fiber is an irreducible curve of arithmetic genus 1 and the general fiber is smooth,
3. a section $s : \mathbb{P}^1 \to X$ of p whose image does not intersect the singular points of any fiber.

Weierstrass fibrations $X \to \mathbb{P}^1$ have a natural invariant associated to them that governs aspects of the geometry of X and the associated moduli spaces.

Definition 4.1.4. Let $p : X \to \mathbb{P}^1$ be a Weierstrass fibration.

1. The fundamental line bundle associated to $p : X \to \mathbb{P}^1$ is the line bundle
$$\mathbb{L} = (R^1 p_* \mathcal{O}_X)^\vee.$$
2. The fundamental invariant associated to $p : X \to \mathbb{P}^1$ is the integer
$$N = \deg \mathbb{L}.$$

Because $\mathbb{L}$ is a line bundle on $\mathbb{P}^1$, it is of the form $\mathcal{O}(N)$ where N is the fundamental invariant. By [Mir81, Corollary 2.4], the fundamental invariant is always nonnegative.

There is a one-to-one correspondence between minimal elliptic surfaces with a section and Weierstrass fibrations with at worst rational double points as singularities.

Given a minimal elliptic surface $\pi : Y \to \mathbb{P}^1$, we obtain a Weierstrass fibration with at worst rational double points $\mathbb{P} : X \to \mathbb{P}^1$ by contracting any rational components in the fibers that do not meet the section. Conversely, given a Weierstrass fibration $p : X \to \mathbb{P}^1$ with at worst rational double points as singularities, resolving the singularities and blowing down (-1)-curves in the fibers yields a minimal elliptic surface $\pi : Y \to \mathbb{P}^1$. We say that Y contracts to X and X resolves to Y. We surveyed the details of this correspondence in Proposition 1.2.18 in Section 1.2.4.

Weierstrass fibrations have a representation as divisors on a $\mathbb{P}^2$-bundle over $\mathbb{P}^1$. This representation can be further described by Weierstrass equations. For details of the correspondence, see Section 1.2.3. Miranda [Mir81] used Weierstrass equations to construct coarse moduli spaces for Weierstrass fibrations, and hence elliptic surfaces. The construction is based on Geometric Invariant Theory (GIT), we surveyed the details in Section 1.2.5, we will state the results that we need in this chapter.

Lemma 4.1.5 (Corollary 2.5 of [Mir81]). *Let $\pi : Y \to \mathbb{P}^1$ be a minimal elliptic surface with section contracting to a Weierstrass fibration $p : X \to \mathbb{P}^1$ with fundamental invariant N. Then X is isomorphic to the closed subscheme of $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(2N) \oplus \mathcal{O}(3N))$ defined by*

$$y^2 z = x^3 + Axz^2 + Bz^3,$$

where $A \in H^0(\mathbb{P}^1, \mathcal{O}(4N))$ and $B \in H^0(\mathbb{P}^1, \mathcal{O}(6N))$. Moreover,

1. *$4A^3 + 27B^2$ is not identically zero. If it vanishes at $q \in \mathbb{P}^1$, the fiber of X over q is singular.*

2. *For every $q \in \mathbb{P}^1$, $v_q(A) \leq 3$ or $v_q(B) \leq 5$, where v_q is the order of vanishing at q.*

We have proven the Lemma 4.1.5 in an equivalent form in Lemma 1.2.20 and Proposition 1.2.21. Set $V_{4N} := H^0(\mathbb{P}^1, \mathcal{O}(4N))$ and $V_{6N} := H^0(\mathbb{P}^1, \mathcal{O}(6N))$. Let $T_N \subset$

$V_{4N} \oplus V_{6N}$ denote the open subspace satisfying conditions (1) and (2) from Lemma 4.1.5. The following is [Mir81, Corollary 2.8].

Corollary 4.1.6. *The set of isomorphism classes of minimal elliptic surfaces* $\pi : Y \to \mathbb{P}^1$ *with* $\deg R^1 p_* \mathcal{O}_X = -N$ *and with fixed section (equivalently, Weierstrass fibrations with only rational double points) is in* $1-1$ *correspondence with the set of orbits of* $\mathrm{SL}_2 \times \mathbb{G}_m$ *on* T_N*.*

In order to give the set of orbits a geometric structure, Miranda analyzes the stability of the action of $\mathrm{SL}_2 \times \mathbb{G}_m$ on T_N.

Proposition 4.1.7. *Let* $(A, B) \in V_{4N} \oplus V_{6N}$ *be a pair of forms.*

1. *The point corresponding to* (A, B) *is not semistable if and only if there is a point* $q \in \mathbb{P}^1$ *such that*
$$v_q(A) > 2N \text{ and } v_q(B) > 3N.$$

2. *The point corresponding to* (A, B) *is not stable if and only if there is a point* $q \in \mathbb{P}^1$ *such that*
$$v_q(A) \geq 2N \text{ and } v_q(B) \geq 3N.$$

The proof of Proposition 4.1.7 is about finding the appropriate linearization of the action and use the Hilbert-Mumford criterion. We have given a detailed proof in Proposition 1.2.25. From Lemma 4.1.5 and Proposition 4.1.7, we see that as long as $N \geq 2$, points in T_N are stable, and thus $\mathsf{E}_N := T_N // \mathrm{SL}_2 \times \mathbb{G}_m$ is a coarse moduli space for Weierstrass fibrations with fundamental invariant N. In particular, the natural morphism

$$\mathcal{E}_N := [T_N / \mathrm{SL}_2 \times \mathbb{G}_m] \to \mathsf{E}_N$$

from the quotient stack to the GIT quotient is a coarse moduli space morphism.

In Section 4.3, it will be useful for us to work on a stack $\mathcal{W}_N$ of Weierstrass fibrations with fundamental invariant N, not just the coarse moduli space constructed by Miranda. This stack is not the stack $\mathcal{E}_N$ defined above, but it is closely related as we will now explain. The stack $\mathcal{W}_N$ was recently defined in work of Park–Schmitt [PS21], and we will briefly recall their construction.

Definition 4.1.8. Let S be a scheme. A family of Weierstrass fibrations over S is given by the data

$$\mathcal{X} \xrightarrow{p} \mathcal{P} \xrightarrow{\gamma} S, \mathcal{P} \xrightarrow{s} \mathcal{X}$$

where

1. γ is a smooth, proper morphism locally of finite type, with geometric fibers isomorphic to $\mathbb{P}^1$,
2. p is a proper map with section s,
3. the fibers $(\mathcal{X}_t \to \mathcal{P}_t, \mathcal{P}_t \to \mathcal{X}_t)$ on geometric points $t \in S$ are Weierstrass fibrations.

Park–Schmitt [PS21] define $\mathcal{W}$ to be the moduli stack whose objects over S are families of Weierstrass fibrations over S with morphisms over $T \to S$ given by fiber diagrams. The stack $\mathcal{W}_N$ is the open and closed substack parametrizing Weierstrass fibrations with fundamental invariant N. Finally, we consider the open substacks $\mathcal{W}_{\min,N} \subset \mathcal{W}_N$ of Weierstrass fibrations satisfying the two conditions from Lemma 4.1.5. These stacks parametrize the Weierstrass fibrations with fundamental invariant N that resolve to minimal elliptic surfaces. By [PS21, Theorem 1.2], the stacks $\mathcal{W}_{\min,N}$ are smooth, separated Deligne-Mumford stacks for $N \geq 2$, and by [PS21, Theorem 1.4], E_N is a coarse moduli space for $\mathcal{W}_{\min,N}$

We now have three spaces of interest: $\mathcal{E}_N$, $\mathcal{W}_{\min,N}$ and E_N. We want to compare their Chow rings.

Proposition 4.1.9. *The Chow rings of $\mathcal{E}_N$, $\mathcal{W}_{\min,N}$ and E_N are isomorphic.*

Proof. The space E_N is a coarse moduli space for both stacks $\mathcal{E}_N$ and $\mathcal{W}_{\min,n}$. Therefore, since we are using rational coefficients, all three Chow rings are isomorphic by a result of Vistoli [Vis89, Proposition 6.1]. □

Remark 4.1.10. The difference between the stacks $\mathcal{W}_{\min,N}$ and $\mathcal{E}_N$ is that $\mathcal{E}_N$ is a μ_2-banded gerbe over $\mathcal{W}_{\min,N}$. The gerbe structure arises from the map $\mathrm{BSL}_2 \to \mathrm{BPGSp}_2$.

Notations and conventions

1. Schemes are over a fixed algebraically closed field k of characteristic not 2 or 3. All stacks are fibered over the category of schemes over k.

2. We denote the Chow ring of a space X with *rational coefficients* by $A^*(X)$.

3. We use the subspace (classical) convention for projective bundles.

4.2 Computing the Chow ring

By Proposition 4.1.9, it suffices to compute $A^*(\mathcal{E}_N)$ in order to prove Theorem 2.2.2. Let $\Delta_N \subset V_{4N} \oplus V_{6N}$ denote the complement of T_N. We have the excision exact sequence

$$A_*([\Delta_N / \mathrm{SL}_2 \times \mathbb{G}_m]) \to A^*([V_{4N} \oplus V_{6N} / \mathrm{SL}_2 \times \mathbb{G}_m]) \to A^*(\mathcal{E}_N) \to 0. \tag{4.2.1}$$

We want to study the image of $A_*([\Delta_N / \mathrm{SL}_2 \times \mathbb{G}_m])$ in $A^*([V_{4N} \oplus V_{6N} / \mathrm{SL}_2 \times \mathbb{G}_m])$.

We begin with background information on the stack $[V_{4N} \oplus V_{6N} / \mathrm{SL}_2 \times \mathbb{G}_m]$. The stack BSL_2 is the classifying stack for rank 2 vector bundles with trivial first Chern class. Let $\mathcal{V}$ denote the universal rank 2 vector bundle with trivial first Chern class over BSL_2. Set $c_2 := c_2(\mathcal{V})$. Similarly, the stack $\mathrm{B}\mathbb{G}_\mathrm{m}$ is the classifying stack for line bundles. Let $\mathcal{M}$ denote the universal line bundle over $\mathrm{B}\mathbb{G}_\mathrm{m}$. Set $a_1 := c_1(\mathcal{M})$. By abuse of notation, we will not distinguish between $\mathcal{V}$, $\mathcal{M}$, c_2, and a_1 and their pullbacks to the product $\mathrm{BSL}_2 \times \mathrm{B}\mathbb{G}_\mathrm{m}$

under the natural projection maps. We will interpret the stack $\mathrm{BSL}_2 \times \mathrm{B}\mathbb{G}_{\mathrm{m}}$ as the stack of line bundles of relative degree N on $\mathbb{P}^1$-bundles as in [Lar23] as follows. Consider the universal $\mathbb{P}^1$-bundle

$$\gamma : \mathbb{P}(\mathcal{V}) \to \mathrm{BSL}_2 \times \mathrm{B}\mathbb{G}_{\mathrm{m}}\,.$$

Fix $N \geq 0$ and set $\mathcal{L} := \gamma^*\mathcal{M}(N)$, the universal relative degree N line bundle on $\mathbb{P}(\mathcal{V})$.

Lemma 4.2.1.

1. *The stack* $[V_{4N} \oplus V_{6N} / \mathrm{SL}_2 \times \mathbb{G}_m]$ *is the total space of the vector bundle* $\gamma_*(\mathcal{L}^{\otimes 4} \oplus \mathcal{L}^{\otimes 6})$ *on* $\mathrm{BSL}_2 \times \mathrm{B}\mathbb{G}_{\mathrm{m}}$.

2. *There is an isomorphism of graded rings*

$$A^*([V_{4N} \oplus V_{6N} / \mathrm{SL}_2 \times \mathbb{G}_m]) \cong \mathbb{Q}[a_1, c_2],$$

with a_1 *in degree* 1 *and* c_2 *in degree* 2.

Proof. Part (1) follows from cohomology and base change. Indeed, the fibers of $\gamma_*(\mathcal{L}^{\otimes 4} \oplus \mathcal{L}^{\otimes 6})$ are canonically identified with $V_{4N} \oplus V_{6N}$, and the higher cohomology vanishes. For part (2), we note that by part (1) and the homotopy property for Chow rings, there is an isomorphism

$$A^*([V_{4N} \oplus V_{6N} / \mathrm{SL}_2 \times \mathbb{G}_m]) \cong A^*(\mathrm{BSL}_2 \times \mathrm{B}\mathbb{G}_{\mathrm{m}}).$$

A standard calculation in equivariant intersection theory [Tot99, Section 15] shows that

$$A^*(\mathrm{BSL}_2 \times \mathrm{B}\mathbb{G}_{\mathrm{m}}) \cong \mathbb{Q}[a_1, c_2]$$

as graded rings. $\square$

4.2.1 Computing the ideal of relations

By Lemma 4.2.1, the exact sequence (4.2.1) can be rewritten as

$$A_*([\Delta_N/\operatorname{SL}_2\times\mathbb{G}_m]) \to \mathbb{Q}[a_1,c_2] \to A^*(\mathcal{E}_N) \to 0. \tag{4.2.2}$$

It follows that $A^*(\mathcal{E}_N)$, and hence $A^*(\mathsf{E}_N)$, is a quotient of $\mathbb{Q}[a_1,c_2]$ by the ideal I_N generated by the image of $A_*([\Delta_N/\operatorname{SL}_2\times\mathbb{G}_m])$.

Lemma 4.1.5 tells us exactly when a pair $(A,B)\in V_{4N}\oplus V_{6N}$ is contained in Δ_N. We write $\Delta_N=\Delta_N^1\cup\Delta_N^2$, where Δ_N^1 parametrizes the pairs of forms (A,B) such that $4A^3+27B^2$ is identically zero (corresponding to Lemma 4.1.5 part (1)), and Δ_N^2 parametrizing pairs of forms (A,B) such that $v_q(A)\geq 4$ or $v_q(B)\geq 6$ for some point $p\in\mathbb{P}^1$ (corresponding to Lemma 4.1.5 part (2)). First, we will determine the relations obtained from excising the pairs $(A,B)\in\Delta_N^2$. To do so, we need to introduce bundles of principal parts. We will follow the treatment in [EH16].

Let $b:Y\to Z$ be a smooth proper morphism. Let $\Delta_{Y/Z}\subset Y\times_Z Y$ be the relative diagonal. With p_1 and p_2 the projection maps, we obtain the following commutative diagram:

$$\begin{tikzcd}
\Delta_{Y/Z} \arrow[dr] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& Y\times_Z Y \arrow[r, "p_1"] \arrow[d, "p_2"'] & Y \arrow[d, "b"] \\
& Y \arrow[r, "b"'] & Z.
\end{tikzcd}$$

Definition 4.2.2. Let $\mathcal{F}$ be a vector bundle on Y and let $\mathcal{I}_{\Delta_{Y/Z}}$ denote the ideal sheaf of the diagonal in $Y\times_Z Y$. The bundle of relative m^{th} order principal parts $P_b^m(\mathcal{F})$ is defined as

$$P_b^m(\mathcal{F})=p_{2*}(p_1^*\mathcal{F}\otimes\mathcal{O}_{Y\times_Z Y}/\mathcal{I}_{\Delta_{Y/Z}}^{m+1}).$$

The following explains all the basic properties of bundles of principal parts that we need.

Proposition 4.2.3 (Theorem 11.2 in [EH16]). *With notation as above,*

1. *There is an isomorphism* $b^*b_*\mathcal{F} \xrightarrow{\sim} p_{2*}p_1^*\mathcal{F}$.

2. *The quotient map* $p_1^*\mathcal{F} \to p_1^*\mathcal{F} \otimes \mathcal{O}_{Y\times_Z Y}/\mathcal{I}_{\Delta_{Y/Z}}^{m+1}$ *pushes forward to a map*

$$b^*b_*\mathcal{F} \cong p_{2*}p_1^*\mathcal{F} \to P_b^m(\mathcal{F}),$$

   which, fiber by fiber, associates to a global section δ *of* $\mathcal{F}$ *a section* δ' *whose value at* $z \in Z$ *is the restriction of* δ *to an* m^{th} *order neighborhood of* z *in the fiber* $b^{-1}b(z)$.

3. $P_b^0(\mathcal{F}) = \mathcal{F}$. *For* $m > 1$, *the filtration of the fibers* $P_b^m(\mathcal{F})_y$ *by order of vanishing at* y *gives a filtration of* $P_b^m(\mathcal{F})$ *by subbundles that are kernels of the natural surjections* $P_b^m(\mathcal{F}) \to P_b^k(\mathcal{F})$ *for* $k < m$. *The graded pieces of the filtration are identified by the exact sequences*

$$0 \to \mathcal{F} \otimes \mathrm{Sym}^m(\Omega_{Y/Z}) \to P_b^m(\mathcal{F}) \to P_b^{m-1}(\mathcal{F}) \to 0.$$

By (2) of Proposition 4.2.3, there is a morphism

$$\psi : \gamma^*\gamma_*(\mathcal{L}^{\otimes 4} \oplus \mathcal{L}^{\otimes 6}) \to P_\gamma^3(\mathcal{L}^{\otimes 4}) \oplus P_\gamma^5(\mathcal{L}^{\otimes 6})$$

which, along points in the $\mathbb{P}^1$ fibers, sends A (respectively, B) to a third (respectively, fifth) order neighborhood. The kernel of this map therefore parametrizes the triples (A, B, q) such that $v_q(A) \geq 4$ and $v_q(B) \geq 6$. Looking fiber-by-fiber, one sees that the map ψ is surjective. Therefore, the kernel K of ψ is a vector bundle. We obtain the following commutative diagram where ϕ, ϕ' and ϕ'' are vector bundle morphisms.

$$\begin{tikzcd} K \arrow[r, hook, "i"] \arrow[dr, "\phi''"'] & \gamma^*\gamma_*(\mathcal{L}^{\otimes 4} \oplus \mathcal{L}^{\otimes 6}) \arrow[r, "\gamma'"] \arrow[d, "\phi'"'] & \gamma_*(\mathcal{L}^{\otimes 4} \oplus \mathcal{L}^{\otimes 6}) \arrow[d, "\phi"] \\ & \mathbb{P}(\mathcal{V}) \arrow[r, "\gamma"] & \mathrm{BSL}_2 \times \mathrm{B}\mathbb{G}_{\mathrm{m}} . \end{tikzcd} \tag{4.2.3}$$

By construction, K maps properly and surjectively onto $[\Delta_N^2/\operatorname{SL}_2 \times \mathbb{G}_m]$ under the identification of $\gamma_*(\mathcal{L}^{\otimes 4} \oplus \mathcal{L}^{\otimes 6})$ with $[V_{4N} \oplus V_{6N}/\operatorname{SL}_2 \times \mathbb{G}_m]$ from Lemma 4.2.1. Consequently, the images of the push forward maps

$$\gamma'_* i_* : A_*(K) \to A^*(\gamma_*(\mathcal{L}^{\otimes 4} \oplus \mathcal{L}^{\otimes 6})) = A^*([V_{4N} \oplus V_{6N}/\operatorname{SL}_2 \times \mathbb{G}_m])$$

and

$$A_*([\Delta_N^2/\operatorname{SL}_2 \times \mathbb{G}_m]) \to A^*([V_{4N} \oplus V_{6N}/\operatorname{SL}_2 \times \mathbb{G}_m])$$

are the same.

Proposition 4.2.4. *Let z denote the hyperplane class of $\mathbb{P}(\mathcal{V})$. The image of the push forward map $\gamma'_* i_* : A^*(K) \to A^*(\gamma_*(\mathcal{L}^{\otimes 4} \oplus \mathcal{L}^{\otimes 6}))$ is the ideal generated by the two classes*

1. $\phi^*\gamma_*(c_{top}(P^3_\gamma(\mathcal{L}^{\otimes 4}) \oplus P^5_\gamma(\mathcal{L}^{\otimes 6})))$, *and*

2. $\phi^*\gamma_*(c_{top}(P^3_\gamma(\mathcal{L}^{\otimes 4}) \oplus P^5_\gamma(\mathcal{L}^{\otimes 6})) \cdot z)$.

Proof. Let $\alpha \in A^*(K)$. Then because K is a vector bundle over $\mathbb{P}(\mathcal{V})$, we see that $\alpha = \phi''^*(\beta)$ for some class $\beta \in A^*(\mathbb{P}(\mathcal{V}))$, so we have

$$\alpha = \phi''^*(\beta) = i^*\phi'^*(\beta).$$

Pushing forward, we obtain

$$\gamma'_* i_* \alpha = \gamma'_* i_* i^* \phi'^*(\beta) = \gamma'_*([K] \cdot \phi'^* \beta).$$

Because K is the kernel of the vector bundle morphism

$$\psi : \gamma^*\gamma_*(\mathcal{L}^{\otimes 4} \oplus \mathcal{L}^{\otimes 6}) \to P^3_\gamma(\mathcal{L}^{\otimes 4}) \oplus P^5_\gamma(\mathcal{L}^{\otimes 6}),$$

the fundamental class $[K]$ is given by $\phi'^*(c_{\mathrm{top}}(P^3_\gamma(\mathcal{L}^{\otimes 4}) \oplus P^5_\gamma(\mathcal{L}^{\otimes 6})))$. Because the square in the commutative diagram (4.2.3) is Cartesian, $\gamma'_*\phi'^* = \phi^*\gamma_*$, so

$$\gamma'_* i_* \alpha = \phi^*\gamma_*(c_{\mathrm{top}}(P^3_\gamma(\mathcal{L}^{\otimes 4}) \oplus P^5_\gamma(\mathcal{L}^{\otimes 6})) \cdot \beta).$$

Because $\mathbb{P}(\mathcal{V})$ is a projective bundle, β can be written as

$$\beta = \gamma^*\beta_1 + \gamma^*\beta_2 z,$$

where β_1 and β_2 are classes in $A^*(\mathrm{BSL}_2 \times \mathrm{B}\mathbb{G}_{\mathrm{m}})$. The statement of the proposition follows. $\square$

Remark 4.2.5. The relations from Proposition 4.2.4 can be computed explicitly as polynomials of a_1, c_2, and N using the splitting principle and Proposition 4.2.3. We carried out this computation in Macaulay2 [GS] using the package Schubert2 [GSS+].

$$\begin{aligned}
\phi^*\gamma_*(c_{\mathrm{top}}(P^3_\gamma(\mathcal{L}^{\otimes 4}) \oplus P^5_\gamma(\mathcal{L}^{\otimes 6}))) = {} & 119439360N^9c_2^4a_1 - 859963392N^8c_2^4a_1 - 1433272320N^7c_2^3a_1^3 \\
& + 2598469632N^7c_2^4a_1 + 8026324992N^6c_2^3a_1^3 + 3009871872N^5c_2^2a_1^5 \\
& - 4277919744N^6c_2^4a_1 - 18189287424N^5c_2^3a_1^3 - 12039487488N^4c_2^2a_1^5 \\
& - 1433272320N^3c_2a_1^7 + 4164009984N^5c_2^4a_1 + 21389598720N^4c_2^3a_1^3 \\
& + 18189287424N^3c_2^2a_1^5 + 3439853568N^2c_2a_1^7 + 119439360Na_1^9 \\
& - 2427125760N^4c_2^4a_1 - 13880033280N^3c_2^3a_1^3 - 12833759232N^2c_2^2a_1^5 \\
& - 2598469632Nc_2a_1^7 - 95551488a_1^9 + 813809664N^3c_2^4a_1 \\
& + 4854251520N^2c_2^3a_1^3 + 4164009984Nc_2^2a_1^5 + 611131392c_2a_1^7 \\
& - 139567104N^2c_2^4a_1 - 813809664Nc_2^3a_1^3 - 485425152c_2^2a_1^5 \\
& + 8847360Nc_2^4a_1 + 46522368c_2^3a_1^3.
\end{aligned}$$

$$
\begin{aligned}
\phi^*\gamma_*(c_{\mathrm{top}}(P^3_\gamma(\mathcal{L}^{\otimes 4}) \oplus P^5_\gamma(\mathcal{L}^{\otimes 6})) \cdot z) = & -11943936N^{10}c_2^5 + 95551488N^9c_2^5 + 537477120N^8c_2^4a_1^2 \\
& - 324808704N^8c_2^5 - 3439853568N^7c_2^4a_1^2 - 2508226560N^6c_2^3a_1^4 \\
& + 611131392N^7c_2^5 + 9094643712N^6c_2^4a_1^2 + 12039487488N^5c_2^3a_1^4 \\
& + 2508226560N^4c_2^2a_1^6 - 694001664N^6c_2^5 - 12833759232N^5c_2^4a_1^2 \\
& - 22736609280N^4c_2^3a_1^4 - 8026324992N^3c_2^2a_1^6 - 537477120N^2c_2a_1^8 \\
& + 485425152N^5c_2^5 + 10410024960N^4c_2^4a_1^2 + 21389598720N^3c_2^3a_1^4 \\
& + 9094643712N^2c_2^2a_1^6 + 859963392Nc_2a_1^8 + 11943936a_1^{10} \\
& - 203452416N^4c_2^5 - 4854251520N^3c_2^4a_1^2 - 10410024960N^2c_2^3a_1^4 \\
& - 4277919744Nc_2^2a_1^6 - 324808704c_2a_1^8 + 46522368N^3c_2^5 \\
& + 1220714496N^2c_2^4a_1^2 + 2427125760Nc_2^3a_1^4 + 694001664c_2^2a_1^6 \\
& - 4423680N^2c_2^5 - 139567104Nc_2^4a_1^2 - 203452416c_2^3a_1^4 + 4423680c_2^4a_1^2.
\end{aligned}
$$

Simplifying, we have that the ideal that these two classes generate is the ideal generated by the following two polynomials, p_1 and p_2.

$$
\begin{aligned}
p_1 = & (1620N - 1296)a_1^9 + (-19440N^3 + 46656N^2 - 35244N + 8289)a_1^7c_2 \\
& + (40824N^5 - 163296N^4 + 246708N^3 - 174069N^2 + 56478N - 6584)a_1^5c_2^2 \\
& + (-19440N^7 + 108864N^6 - 246708N^5 + 290115N^4 - 188260N^3 + 65840N^2 - 11038N + 631)a_1^3c_2^3 \\
& + (1620N^9 - 11664N^8 + 35244N^7 - 58023N^6 + 56478N^5 - 32920N^4 + 11038N^3 - 1893N^2 + 120N)a_1c_2^4.
\end{aligned}
$$

$$
\begin{aligned}
p_2 = & 324a_1^{10} + (-14580N^2 + 23328N - 8811)a_1^8c_2 \\
& + (68040N^4 - 217728N^3 + 246708N^2 - 116046N + 18826)a_1^6c_2^2 \\
& + (-68040N^6 + 326592N^5 - 616770N^4 + 580230N^3 - 282390N^2 + 65840N - 5519)a_1^4c_2^3 \\
& + (14580N^8 - 93312N^7 + 246708N^6 - 348138N^5 + 282390N^4 - 131680N^3 + 33114N^2 - 3786N + 120)a_1^2c_2^4 \\
& + (-324N^{10} + 2592N^9 - 8811N^8 + 16578N^7 - 18826N^6 + 13168N^5 - 5519N^4 + 1262N^3 - 120N^2)c_2^5.
\end{aligned}
$$

Lemma 4.2.6. *The codimension of Δ^1_N in $V_{4N} \oplus V_{6N}$ is $8N+1$.*

Proof. Let t be an affine coordinate on $\mathbb{P}^1$. Then we can factor $A(t)$ and $B(t)$ into linear factors as

$$A(t) = a\prod_{i=1}^{4N}(t - c_i) \text{ and } B(t) = b\prod_{i=1}^{6N}(t - d_i).$$

Because $4A^3 + 27B^2$ is identically zero, we have the equation

$$4a^3 \prod_{i=1}^{4N} (t - c_i)^3 = -27b^2 \prod_{i=1}^{6N} (t - d_i)^2.$$

By comparing the orders of vanishing of each side, we see that $A(t) = aG(t)^2$ and $B(t) = bG(t)^3$, where G is a polynomial of degree $2N$ and $4a^3 + 27b^2 = 0$. It follows that the codimension of Δ_N^1 is given by

$$\dim(V_{4N} \oplus V_{6N}) - \dim V_{2N} = 10N + 2 - 2N - 1 = 8N + 1.$$

□

We can now complete the proof of Theorem 2.2.2.

Proof of Theorem 2.2.2. By a calculation in Macaulay2 [GS], the graded ring $\mathbb{Q}[a_1, c_2]/I_N$ vanishes in degree 17 and higher, where I_N is the ideal generated by the relations from Proposition 4.2.4. We have the excision exact sequence

$$A_*([\Delta_N^1/\operatorname{SL}_2 \times \mathbb{G}_m]) \to \mathbb{Q}[a_1, c_2]/I_N \to A^*(\mathcal{E}_N) \to 0.$$

By Lemma 4.2.6, the image of

$$A_*([\Delta_N^1/\operatorname{SL}_2 \times \mathbb{G}_m]) \to \mathbb{Q}[a_1, c_2]/I_N$$

lies in codimension 17 or higher, so it is identically zero. Therefore,

$$\mathbb{Q}[a_1, c_2]/I_N \cong A^*(\mathcal{E}_N).$$

This completes the proof of Theorem 2.2.2 part (1). Parts (2) and (3) are consequences of part (1) together with a computation in Macaulay2 [GS] that computes the Hilbert Series of the ring $\mathbb{Q}[a_1, c_2]/I_N$ and verifies that the intersection pairing is perfect. $\square$

Proof of Corollary 2.2.3. Miranda's construction of E_N by geometric invariant theory [Mir81] shows that E_N is a quasi-projective variety. It thus admits an ample line bundle L. If S is a complete subvariety of dimension d, then, because L is ample,

$$c_1(L)^d \cdot S > 0.$$

Hence, $c_1(L)^d$ is numerically nonzero. By Theorem 2.2.2, it follows that $d \leq 16$. $\square$

4.3 The Tautological ring

4.3.1 Stacks of lattice polarized K3 surfaces

Let $\Lambda \subset U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$ be a fixed rank r primitive sublattice with signature $(1, r-1)$, and let $v_1, \ldots, v_r$ be an integral basis of Λ. A Λ-polarization on a K3 surface X is a primitive embedding

$$j : \Lambda \hookrightarrow \mathrm{Pic}(X)$$

such that

1. The lattices $H^2(X, \mathbb{Z})$ and $U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$ are isomorphic via an isometry restricting to the identity on Λ, where we view Λ as sitting inside $H^2(X, \mathbb{Z})$ via $\Lambda \hookrightarrow \mathrm{Pic}(X) \hookrightarrow H^2(X, \mathbb{Z})$.

2. The image of j contains the class of a quasi-polarization.

Beauville [Bea04] constructed moduli stacks $\mathcal{F}_\Lambda$ of Λ-polarized K3 surfaces, and showed that they are smooth Deligne–Mumford stacks of dimension $19 - r$. Using the surjectivity of the period map, one can construct coarse moduli spaces F_Λ for $\mathcal{F}_\Lambda$ [Dol96].

We think of the stacks $\mathcal{F}_\Lambda$ as parametrizing families of K3 surfaces

$$\pi : X \to S$$

together with r line bundles $H_1, \ldots, H_r$ on X corresponding to the basis $v_1, \ldots, v_r$ of Λ, well-defined up to pullbacks from $\operatorname{Pic}(S)$. Technically, these bundles exist only étale locally, as they are defined as sections of the sheaf $\operatorname{Pic}_{X/S}$, which is the étale sheafification of the presheaf on the category of schemes over S

$$T \mapsto \operatorname{Pic}(X_T)/\operatorname{Pic}(T).$$

We will generally suppress this detail, but we will remark when it is important. There are forgetful morphisms

$$\mathcal{F}_{\Lambda'} \hookrightarrow \mathcal{F}_\Lambda$$

for any lattice $\Lambda \subset \Lambda'$. When Λ is strictly contained in Λ', we call the subvarieties $\mathcal{F}_{\Lambda'}$ *Noether-Lefschetz loci* of $\mathcal{F}_\Lambda$.

4.3.2 The tautological ring of $\mathcal{F}_\Lambda$

The stack $\mathcal{F}_\Lambda$ comes equipped with a universal K3 surface

$$\pi_\Lambda : \mathcal{X}_\Lambda \to \mathcal{F}_\Lambda.$$

and universal bundles $\mathcal{H}_1, \ldots \mathcal{H}_r$, well-defined up to pullbacks from $\mathcal{F}_\Lambda$. Let $\mathbb{T}_{\pi_\Lambda}$ denote the relative tangent bundle. Following [MOP17], we define the κ-classes

$$\kappa^\Lambda_{a_1,\ldots,a_r,b} := \pi_{\Lambda *}\left(c_1(\mathcal{H}_1)^{a_1} \cdots c_1(\mathcal{H}_r)^{a_r} \cdot c_2(\mathbb{T}_{\pi_\Lambda})^b\right).$$

Definition 4.3.1. The tautological ring $R^*(\mathcal{F}_\Lambda)$ is the subring of $A^*(\mathcal{F}_\Lambda)$ generated by pushforwards from the Noether–Lefschetz loci of all κ-classes.

By [Bor99] or [FR20], the Hodge class $\lambda := c_1(\pi_{\Lambda *}\omega_{\pi_\Lambda})$ lies in the tautological ring $R^*(\mathcal{F}_\Lambda)$ for all Λ, as it is supported on Noether–Lefschetz divisors.

4.3.3 Moduli of elliptic K3 surfaces and Weierstrass fibrations

Let $p : X \to \mathbb{P}^1$ be a minimal elliptic surface over $\mathbb{P}^1$ with fundamental invariant 2. Then X is a K3 surface, and the class of the fiber f and section σ form a primitively embedded lattice $U \subset \mathrm{Pic}(X)$ equivalent to a hyperbolic lattice, whose image contains a quasi-polarization $\sigma + 2f$. Conversely, given a K3 surface X, a primitive embedding of a hyperbolic lattice $U \hookrightarrow \mathrm{Pic}(X)$ whose image contains a quasi-polarization allows one to define a morphism $p : X \to \mathbb{P}^1$ with section $s : \mathbb{P}^1 \to X$ with fundamental invariant 2 [CD07, Theorem 2.3]. Because of this, we call the stack $\mathcal{F}_U$ the stack parametrizing elliptic K3 surfaces with section. By [OO21, Theorem 7.9], the coarse moduli space F_U is isomorphic to E_2. By the discussion in subsection 4.3.1, $\mathcal{F}_U$ comes equipped with a universal K3 surface and two universal line bundles

$$\pi_U : \mathcal{X}_U \to \mathcal{F}_U, \quad \mathcal{O}(f) \to \mathcal{X}_U, \quad \mathcal{O}(\sigma) \to \mathcal{X}_U.$$

The intersection matrix of $\mathcal{O}(\sigma)$ and $\mathcal{O}(f)$ is

$$\begin{bmatrix} \mathcal{O}(\sigma)^2 & \mathcal{O}(\sigma)\cdot\mathcal{O}(f) \\ \mathcal{O}(\sigma)\cdot\mathcal{O}(f) & \mathcal{O}(f)^2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix},$$

which can be obtained by a change of basis from the usual intersection matrix for a hyperbolic lattice U:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

We prefer to take $\mathcal{O}(f)$ and $\mathcal{O}(\sigma)$ as our basis because of their geometric meaning. Recall that the stack $\mathcal{W}_{\min,2}$ parametrizes families of Weierstrass fibrations resolving to minimal elliptic surfaces. We will construct a morphism

$$G : \mathcal{F}_U \to \mathcal{W}_{\min,2},$$

which is a relative version of the morphism sending an elliptic K3 surface to its associated Weierstrass fibration. Let $\pi : X \to S$ be a family of U-polarized K3 surfaces, equipped with bundles $\mathcal{O}(f)$ and $\mathcal{O}(\sigma)$ on X, up to an étale cover of S. The surjection

$$\pi^*\pi_*\mathcal{O}(f) \to \mathcal{O}(f)$$

defines a morphism

$$p : X \to \mathbb{P}(\pi_*\mathcal{O}(f)^\vee)$$

over S. The relative effective Cartier divisor associated to $\mathcal{O}(\sigma)$ allows us to define a section s of p. The surjection

$$p^*p_*\mathcal{O}(3\sigma) \to \mathcal{O}(3\sigma)$$

defines a morphism $i : X \to \mathbb{P}(p_*\mathcal{O}(3\sigma)^\vee)$. Let Y denote the image of X under i. Then Y is a family of Weierstrass fibrations over S. This construction defines the morphism

$$G : \mathcal{F}_U \to \mathcal{W}_{\min,2}.$$

Remark 4.3.2. We note that in constructing Y, we chose line bundles $\mathcal{O}(f)$ and $\mathcal{O}(\sigma)$. Technically, we could only do so étale locally. The projective bundle $\mathbb{P}(\pi_*\mathcal{O}(f)^\vee) \to S$ will only descend to a smooth proper morphism, locally of finite type, with geometric fibers isomorphic to $\mathbb{P}^1$: it will not necessarily be the projectivization of a vector bundle on S. Second, even once we pass to an étale cover, $\mathcal{O}(f)$ and $\mathcal{O}(\sigma)$ are only defined up to pullbacks from $\mathrm{Pic}(S)$. If we made different choices for $\mathcal{O}(f)$ and $\mathcal{O}(\sigma)$ the resulting Weierstrass fibration would be canonically isomorphic to the original one because for any vector bundle $\mathcal{E}$ and line bundle $\mathcal{L}$, $\mathbb{P}(\mathcal{E} \otimes \mathcal{L})$ is canonically isomorphic to $\mathbb{P}(\mathcal{E})$.

Consider the following Cartesian diagram, which defines the stack $\widetilde{\mathcal{F}}_U$.

$$\begin{tikzcd} \widetilde{\mathcal{F}}_U \arrow[r, "G'"] \arrow[d] & \mathcal{E}_2 \arrow[d] \\ \mathcal{F}_U \arrow[r, "G"] & \mathcal{W}_{\mathrm{min},2} \end{tikzcd}$$

The vertical morphisms are μ_2-banded gerbes. In fact, we can explicitly describe the functor of points for $\widetilde{\mathcal{F}}_U$. A morphism from a scheme S to $\widetilde{\mathcal{F}}_U$ is a family

$$(\pi : X \to S, \mathcal{O}(f), \mathcal{O}(\sigma), \mathcal{N})$$

where $(\pi : X \to S, \mathcal{O}(f), \mathcal{O}(\sigma))$ is a family of U-polarized K3 surfaces and $\mathcal{N}$ is a line bundle on S such that

$$\mathcal{N}^{\otimes 2} \cong \det \pi_* \mathcal{O}(f).$$

Recall that $\mathcal{E}_2$ has a universal rank 2 vector bundle with trivial first Chern class $\mathcal{V}$ and a universal line bundle $\mathcal{M}$. By construction of the map G and its base change G', we have that

$$G'^* \mathcal{V} = \pi_* \mathcal{O}(f)^\vee \otimes \mathcal{N},$$

where $\mathcal{N}$ is the universal square root of $\det \pi_* \mathcal{O}(f)$. We will abuse notation and denote the universal K3 surface on $\mathcal{F}_U$ and $\widetilde{\mathcal{F}}_U$ both by π.

Lemma 4.3.3. *The class $c_2(\pi_* \mathcal{O}(f)^\vee \otimes \mathcal{N})$ on $\widetilde{\mathcal{F}}_U$ is the pullback of a tautological class on $\mathcal{F}_U$.*

Proof. Note that

$$\begin{aligned} c_2(\pi_*\mathcal{O}(f)^\vee \otimes \mathcal{N}) &= c_1(\mathcal{N})^2 + c_1(\pi_*\mathcal{O}(f)^\vee)c_1(\mathcal{N}) + c_2(\pi_*\mathcal{O}(f)^\vee) \\ &= \frac{1}{4}c_1(\det \pi_*\mathcal{O}(f))^2 - \frac{1}{2}c_1(\pi_*\mathcal{O}(f))c_1(\det \pi_*\mathcal{O}(f)) + c_2(\pi_*\mathcal{O}(f)) \\ &= -\frac{1}{4}c_1(\pi_*\mathcal{O}(f))^2 + c_2(\pi_*\mathcal{O}(f)). \end{aligned}$$

It thus suffices to show that the Chern classes of $\pi_*\mathcal{O}(f)$ are tautological. By Grothendieck–Riemann–Roch, we have

$$\mathrm{ch}(\pi_!\mathcal{O}(f)) = \pi_*(\mathrm{ch}(\mathcal{O}(f)) \cdot \mathrm{td}(T_\pi)).$$

By definition, the classes on the right hand side are tautological. We note that

$$\pi_!\mathcal{O}(f) = \pi_*\mathcal{O}(f)$$

because π is a relative K3 surface. By comparing degree 1 parts of both sides, we see that $c_1(\pi_*\mathcal{O}(f))$ is tautological. By comparing degree 2 parts, we see that $c_2(\pi_*\mathcal{O}(f))$ is tautological. $\square$

Proof of Theorem 2.2.4. Each of the stacks $\mathcal{E}_2$, $\mathcal{W}_{\min,2}$, $\mathcal{F}_U$, and $\widetilde{\mathcal{F}}_U$ has the same coarse moduli space E_2. They thus all have isomorphic Chow rings, and proper push forward $A_*(Z) \to A_*(E_2)$ is an isomorphism of Chow groups, where Z is any of the four stacks above [Vis89, Proposition 6.1]. By Theorem 2.2.2, $A^1(E_2)$ is generated by the push forward of a_1. By [Pet19, Theorem 2.1] or the proof of [vdGK05, Corollary 4.2], the tautological class λ is nonvanishing on $\mathcal{F}_U$. It follows that $A^1(\mathcal{F}_U)$ is generated by λ, so $A^1(\mathcal{F}_U) = R^1(\mathcal{F}_U)$. By Theorem 2.2.2, $A^2(E_2)$ is generated by the push forwards of a_1^2 and c_2. By Lemma 4.3.3, the class c_2 pulls back to a class in $A^2(\widetilde{\mathcal{F}}_U)$ that is the pullback of a tautological class from $A^2(\mathcal{F}_U)$. It follows that $A^2(\mathcal{F}_U) = R^2(\mathcal{F}_U)$, as the images of a_1^2

and c_2 in $A^2(E_2)$ can both be obtained by pushing forward tautological classes from $\mathcal{F}_U$. Therefore, $A^*(\mathcal{F}_U) = R^*(\mathcal{F}_U)$. The fact that $A^*(\mathcal{F}_U) = R^*(\mathcal{F}_U)$ is Gorenstein with socle in codimension 16 follows from Theorem 2.2.2. □

4.3.4 Codimension one classes

By Theorems 2.2.2 and 2.2.4, $A^1(\mathcal{F}_U)$ is of rank one and the Hodge class λ is a generator. It is natural to ask how to represent κ-classes explicitly in terms of the Hodge class λ.

Proposition 4.3.4. *The following four linear combinations of κ-classes are independent of the choice of universal line bundles. Moreover, they are all multiples of the Hodge class λ.*

$$\kappa_{3,0,0} + \frac{1}{4}\kappa_{1,0,1} = \frac{7}{2}\lambda, \quad 3\kappa_{2,1,0} - \frac{1}{4}\kappa_{1,0,1} + \frac{1}{4}\kappa_{0,1,1} = \frac{1}{2}\lambda,$$
$$3\kappa_{1,2,0} - \frac{1}{4}\kappa_{0,1,1} = -3\lambda, \quad \kappa_{0,3,0} = 0.$$

where $\kappa_{i,j,k} := \pi_*\left(c_1(\mathcal{O}(\sigma))^i \cdot c_1(\mathcal{O}(f))^j \cdot c_2(\mathbb{T}_\pi)^k\right)$.

Proof. A direct computation shows the above four κ combinations are invariant under $f \mapsto f + \pi^*(l)$ and $\sigma \mapsto \sigma + \pi^*(l')$ for any $l, l' \in A^1(\mathcal{F}_U)$.

By Theorem 2.2.2, we know $A^1(\mathcal{F}_U)$ is of rank one, so it is sufficient to check the identities by computing their intersection numbers with a suitable test curve:

$$\iota : C \to \mathcal{F}_U.$$

To construct the curve, we use the resolved version of the STU model in [KMPS10]. The STU model is a smooth Calabi-Yau 3-fold, endowed with a map:

$$\pi^{STU} : X^{STU} \to \mathbb{P}^1.$$

It arises as an anti-canonical section of a toric 4-fold Y. The fan datum for Y can be found in [KMPS10, Section 1.3]. We use their notation. There are 10 primitive rays $\{\rho_i; 1 \leq i \leq 10\}$, and the corresponding divisors are denoted as $D_i \in \mathrm{Pic}(Y)$. The anti-canonical class is:

$$-K_Y = \sum_{i=1}^{10} D_i.$$

The general fiber of π^{STU} is a smooth elliptic K3 surface, but there are 528 singular fibers [KMPS10, Proposition 1], each of which has exactly one ordinary double point singularity. Let $\epsilon : C \to \mathbb{P}^1$ be a double cover branched along the 528 points corresponding to the singular fibers. The pullback of X^{STU} by ϵ has double point singularities, and by resolving them we obtain the resolved STU model:

$$\widetilde{\pi}^{STU} : \widetilde{X}^{STU} \to C.$$

All fibers of $\widetilde{\pi}^{STU}$ are smooth elliptic K3 surfaces. Moreover the toric divisors $D_5, D_3 \in \mathrm{Pic}(Y)$ restrict to the universal section and fiber for $\widetilde{\pi}^{STU}$. The family $\widetilde{\pi}^{STU}$ defines a curve in the moduli space $\mathcal{F}_U$:

$$\iota : C \to \mathcal{F}_U.$$

The intersection number $\iota^*(\lambda)$ is computed in [KMPS10, Proposition 2]:

$$\iota^*(\lambda) = 4E_4(q)E_6(q)[0] = 4,$$

where E_4 and E_6 are Eisenstein series, and we take the coefficient of q^0.

For the κ-classes, it suffices to perform the computation over the non-resolved STU model. Since the tautological classes we consider are all invariant, we may assume the universal line bundles on $\mathcal{F}_U$ pull back to the toric divisors D_5, D_3. For $\kappa_{3,0,0}$, we have:

$$\iota^*(\kappa_{3,0,0}) = 2 \cdot \pi_*^{STU}\left(D_5^3 \cdot \sum_{i=1}^{10} D_i\right),$$

where the factor of 2 comes from the double cover ϵ. Using toric geometry, all monomials of the form $D_i \cdot D_j \cdot D_k \cdot D_l$ can be explicitly determined. We obtain:

$$\iota^*(\kappa_{3,0,0}) = 16.$$

Other intersection numbers can be computed analogously. We record the final answers:

$$\iota^*(\kappa_{3,0,0}) = 16 \quad \iota^*(\kappa_{1,0,1}) = -8 \quad \iota^*(\kappa_{2,1,0}) = -4$$

$$\iota^*(\kappa_{0,1,1}) = 48 \quad \iota^*(\kappa_{1,2,0}) = 0 \quad \iota^*(\kappa_{0,3,0}) = 0.$$

The four identities in the proposition then follow immediately. □

This chapter is, in full, adapted from the material as it appears in

- Samir Canning and Bochao Kong, *"The Chow rings of moduli spaces of elliptic surfaces over $\mathbb{P}^1$"*, Algebraic Geometry 10.4 (2023).

The dissertation author was the co-primary investigator and author of this paper.

Chapter 5

Tautological relations via localization

We have seen in Definition 2.3.1 and Section 4.3.1 that the moduli spaces $\mathcal{F}_{2\ell}$ and $\mathcal{F}_\Lambda$ carry rich cycles/classes originating from tautological constructions. Using Hodge classes, Noether-Lefschetz classes and kappa classes, [MOP17] proposed the following definition of the tautological ring:

Definition 5.0.1. The *tautological ring* $R^\star(\mathcal{F}_\Lambda) \subset A^\star(\mathcal{F}_\Lambda)$ is the $\mathbb{Q}$-subalgebra generated by all pushforwards of all monomial combinations of kappa classes and Hodge classes from all Noether-Lefschetz loci. The *Noether-Lefschetz ring* $NL^\star(\mathcal{F}_\Lambda)$ is the $\mathbb{Q}$-subalgebra of $A^\star(\mathcal{F}_\Lambda)$ generated by the Noether-Lefschetz classes.

We have computed the divisorial tautological relations in Proposition 4.3.4, where κ-classes are expressed in terms of the Hodge class λ. In this chapter, we will use localization to derive the tautological relations in $A^1(\mathcal{F}_U)$. The program was initially proposed in [MOP17], we apply the techniques to the moduli space of elliptic K3 surfaces $\mathcal{F}_U$. All three types of tautological classes will appear naturally in the process. Furthermore, this method does not require knowing that the dimension of $A^1(\mathcal{F}_U)$ is one.

5.1 Quot scheme localization

Let $\pi : \mathcal{X}_U \to \mathcal{F}_U$ be the universal K3 surface map, and $\mathcal{O}(\sigma), \mathcal{O}(f)$ be the universal line bundles with the following intersection matrix:

$$\begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}.$$

Take $H = \sigma + df$ with $d \geq 2$. We consider the π-relative Quot scheme $\mathcal{Q}^{\pi}_{H,\chi}(\mathbb{C}^2)$. Which parametrizes quotients:

$$0 \to E \to \mathbb{C}^2 \otimes \mathcal{O}_X \to F \to 0$$

over a surface $(X, \mathcal{O}(f), \mathcal{O}(\sigma))$, where F is of rank 0, $c_1(F) = H$ and $\chi(F) = \chi$. The Quot scheme is not smooth, but it admits a virtual fundamental class $[\mathcal{Q}^{\pi}_{H,\chi}(\mathbb{C}^2)]^{\mathrm{vir}}$, which allows us to do intersection theory.

Lemma 5.1.1. *[MOP17, Lemma 1] The Quot scheme $\mathcal{Q}^{\pi}_{H,\chi}(\mathbb{C}^r)$ admits a virtual fundamental class $[\mathcal{Q}^{\pi}_{H,\chi}(\mathbb{C}^r)]^{\mathrm{vir}}$. The π-relative virtual dimension is $r\chi + H^2$. The obstruction theory is governed by the sheaves:* $\mathrm{Ext}^1(E, F)$ *and* $\mathrm{Ext}^2(E, F)$.

However, the naive virtual fundamental class $[\mathcal{Q}^{\pi}_{H,\chi}(\mathbb{C}^r)]^{\mathrm{vir}}$ actually vanishes, as we have a trivial factor in the obstruction theory. To resolve this issue, we need to consider the refined virtual fundamental class $[\mathcal{Q}^{\pi}_{H,\chi}(\mathbb{C}^r)]^{\mathrm{red}}$.

Proposition 5.1.2. *[MOP17, Lemma 2] For a K3 surface (X, H) as above, there exists a natural surjective map:*

$$\mathrm{Ext}^1(E, F) \to H^2(\mathcal{O}_X) = \mathbb{C},$$

and a reduced virtual fundamental class $[\mathcal{Q}^{\pi}_{H,\chi}(\mathbb{C}^r)]^{\mathrm{red}}$ of π-relative dimension $r\chi + H^2 + 1$.

We will only consider $r = 2$, and the π-relative reduced virtual dimension will be

$2\chi + 2d - 1$. We will derive relations from the vanishing pushforward:

$$(-1)^{\chi+\ell} p_\star \left(0^{2\chi+2d} \cap [\mathcal{Q}^\pi_{H,\chi}(\mathbb{C}^2)]^{\mathrm{red}}\right) \in A^1(\mathcal{F}_U, \mathbb{Q}), \tag{5.1.1}$$

where the class ζ is the pullback of the hyperplane class via the morphism:

$$\mathcal{Q}^\pi_{H,\chi}(\mathbb{C}^2) \to \mathbb{P}(\pi_\star \mathcal{H}).$$

To obtain non-trivial tautological terms from (5.1.1), we will use the virtual localization formula. Consider the torus action $\mathbb{C}^\star$ on $\mathbb{C}^2$ with weight 0 and 1:

$$\mathbb{C}^2 = \mathbb{C}[0] + \mathbb{C}[1].$$

This induces the action of $\mathbb{C}^\star$ on $\mathcal{Q}^\pi_{H,\chi}(\mathbb{C}^2)$. The virtual localization formula [GP99] states that the virtual fundamental integrals can be computed at the fixed points of the torus action:

$$\int_{\left[\mathcal{Q}_{H,\chi}(\mathbb{C}^2)\right]^{\mathrm{vir}}} \alpha = \sum_{\mathrm{F}} \int_{[\mathrm{F}]^{\mathrm{vir}}} \frac{\widetilde{\alpha}|_{\mathrm{F}}}{e_{\mathbb{C}^\star}\left(\mathrm{N}^{\mathrm{vir}}_{\mathrm{F}}\right)}$$

where F runs over the fixed points of the torus action, $\mathrm{N}^{\mathrm{vir}}_{\mathrm{F}}$ is the virtual normal bundle of F, and $\widetilde{\alpha}$ is any lift of α to the $\mathbb{C}^\star$-equivariant Chow. The equation holds in the equivariant Chow ring of $\mathcal{Q}^\pi_{H,\chi}(\mathbb{C}^2)$, which is a $\mathbb{Q}(t)$-algebra. The formula holds true for the reduced virtual fundamental class as well. Furthermore, the pushforward (5.1.1) can be computed fiberwise as reduced virtual class integrals. In our case, we will need to pick the equivariant lift $\widetilde{0}$, and we will lift to the equivariant parameter t. If no confusion arises, we will set the equivariant parameter t to 1 during the computation.

5.2 The calculation for $d = 2, \chi = 1$

In this section, we will fix $d = 2$ and $\chi = 1$. We will compute the pushforward (5.1.1) for the reduced virtual class. The fixed locus will parametrize sequences of the following form:

$$0 \to \mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2 \hookrightarrow \mathcal{O}_X \oplus \mathcal{O}_X \to \mathcal{F} = \mathcal{F}_1 \oplus \mathcal{F}_2 \to 0.$$

Given that $\mathcal{F}$ has zero-dimensional and one-dimensional supports, the kernel must split as

$$\mathcal{E}_1 = \mathcal{I}_Z \otimes \mathcal{O}_X(-C_1), \quad \mathcal{E}_2 = \mathcal{I}_W \otimes \mathcal{O}_X(-C_2)$$

where $C_1 + C_2 = \sigma + 2f$ and

$$Z, W \subset X$$

are zero-dimensional subschemes of lengths z and w respectively, and in our case:

$$z + w = 2 - C_1 \cdot C_2.$$

The fixed locus generally takes the form of a fiber product of relative Hilbert schemes of points and relative projective bundles corresponding to curves varying in the linear systems:

$$\mathcal{X}^{[z]} \times_{\mathcal{F}_U} \mathbb{P}_1 \times_{\mathcal{F}_U} \mathcal{X}^{[w]} \times_{\mathcal{F}_U} \mathbb{P}_2.$$

Over a generic point of $\mathcal{F}_U$, the splitting $C_1 + C_2 = \sigma + 2f$ can only involve combinations of σ and f. On the Noether-Lefschetz loci, extra curves on the surfaces result in additional contributions. Therefore, the overall summation will include contributions from Noether-Lefschetz loci. The fixed locus involves the Hilbert scheme of points, where the computation can be reduced to a product of surfaces. This reduction process introduces

κ-classes and the Hodge class. Moreover, the trivial factor in deformation-obstruction theory will also contribute to the Hodge class.

5.2.1 Generic Contributions

In our case, the generic splitting has finitely many possibilities. We list all the possible configurations of $\mathcal{E}_1 \oplus \mathcal{E}_2$, up to a flip:

1. $\mathcal{I}_2 \ \oplus \ \mathcal{O}(-\mathcal{H})$
2. $\mathcal{I}_{\Delta_1} \ \oplus \ \mathcal{I}_{\Delta_2} \otimes \mathcal{O}(-\mathcal{H})$
3. $\mathcal{O} \ \oplus \ \mathcal{I}_2 \otimes \mathcal{O}(-\mathcal{H})$
4. $\mathcal{I}_{\Delta_1} \otimes \mathcal{O}(-\sigma - f) \ \oplus \ \mathcal{O}(-f)$
5. $\mathcal{O}(-\sigma - f) \ \oplus \ \mathcal{I}_{\Delta_2} \otimes \mathcal{O}(-f)$
6. $\mathcal{O}(-\sigma) \ \oplus \ \mathcal{O}(-2f)$

We will demonstrate the computations for case (1), (2) and (3). These three cases include all crucial computation details. For the rest of the cases, we will only list the final results.

Case (1): $\mathcal{I}_2 \ \oplus \ \mathcal{O}(-\mathcal{H})$

The fixed locus in this case is $\mathcal{X}^{[2]} \times_{\mathcal{F}_U} \mathbb{P}$, where the Hilbert scheme of points $\mathcal{X}^{[2]}$ comes from the first factor and the projective bundle $\mathbb{P}$ comes from the linear systems of H. Recall that the reduced virtual class is determined by the K-theory class:

$$\mathrm{Ext}^{\bullet}(\mathcal{E}, \mathcal{F}) + \mathbb{C}.$$

The $\mathbb{C}^\star$ action on $0 \to \mathcal{E} \to \mathbb{C}^2 \otimes \mathcal{O}_X \to \mathcal{F} \to 0$ introduces fixed and moving K-theory classes:

$$\text{Fixed} = \text{Ext}^\bullet\left(\mathcal{E}_1, \mathcal{F}_1\right) + \text{Ext}^\bullet\left(\mathcal{E}_2, \mathcal{F}_2\right) + \mathbb{C}.$$

The moving part of the theory has two components. The one with weight 1 is:

$$\text{Mov}[1] = \text{Ext}^\bullet(\mathcal{E}_1, \mathcal{F}_2),$$

and the one with weight -1 is:

$$\text{Mov}[-1] = \text{Ext}^\bullet(\mathcal{E}_2, \mathcal{F}_1).$$

We need to compute the obstruction class from the fixed theory. Notably, we have a smooth fixed locus, so the obstruction class can be computed as follows:

$$\text{Obs} = \text{Ext}^\bullet\left(\mathcal{E}_1, \mathcal{F}_1\right) + \text{Ext}^\bullet\left(\mathcal{E}_2, \mathcal{F}_2\right) + \mathbb{C} - \text{Tan(Fixed Locus)}$$

Let $\mathcal{L} = \mathcal{O}_{\mathbb{P}}(1)$. We can simplify the fixed K-theory class as:

$$\begin{aligned}
\text{Fixed} &= 2\mathbb{C} + \text{Ext}^\bullet(\mathcal{I}_2, \mathcal{O}_2) + \text{Ext}^\bullet(\mathcal{H}^{-1} \otimes \mathcal{L}^{-1}, \mathbb{C}) - \text{Ext}^\bullet(\mathbb{C}, \mathbb{C}) \\
&= \text{Ext}^\bullet(\mathcal{I}_2, \mathbb{C}) - \text{Ext}^\bullet(\mathcal{I}_2, \mathcal{I}_2) + \mathbb{C}^{\ell+2} \otimes \mathcal{L} \\
&= \text{Tan}(\mathbb{P}^{\ell+1}) + \mathbb{C} + \text{Tan}(X^{[2]}) - 2\mathbb{C} + \text{Ext}^\bullet(\mathbb{C}, \mathbb{C}) - \text{Ext}^\bullet(\mathcal{O}_2, \mathbb{C}) \\
&= \text{Tan}(\mathbb{P}^{\ell+1}) + \text{Tan}(X^{[2]}) - ((\mathcal{O}_1^{[2]})^\vee - \mathbb{C}),
\end{aligned}$$

where $H^2 = 2\ell = 2$. In this case, we conclude that the reduced obstruction bundle is:

$$\text{Obs} = (\mathcal{O}_1^{[2]})^\vee - \mathbb{C}.$$

Note that the fiber-wise trivial factor $\mathbb{C}$ will glue to the pullback of the the Hodge line bundle $\mathbb{E}^\vee$ on $\mathcal{M}_{\mathbb{L}}$. We will further reduce it but the final contribution for this case will be a multiple of the Hodge class λ. Next, we compute the moving part with weight 1:

$$\mathrm{Mov}[1] = 2\mathbb{C} - (\mathcal{O}^{[2]})^\vee - \mathbb{C}^{\ell+2} \otimes \mathcal{L}^{-1} + (H^{[2]})^\vee \otimes \mathcal{L}^{-1}.$$

The moving part with weight -1 is:

$$\mathrm{Mov}[-1] = \mathrm{Ext}^\bullet(H^{-1} \otimes \mathcal{L}^{-1}, \mathcal{O}_2) = H^{[2]} \otimes \mathcal{L}.$$

The localized contribution of $p_\star\left(0^6 \cap [\mathcal{Q}_{H,\chi}(\mathbb{C}^2)]^{\mathrm{red}}\right)$ on this fixed locus will be

$$\frac{e(\mathrm{Obs})}{c_+(\mathrm{Mov}[1]) \cdot c_-(\mathrm{Mov[-1]})} = \int_{X^{[2]}\times\mathbb{P}^2} \frac{c_1((\mathcal{O}^{[2]})^\vee) \cdot c_+((\mathcal{O}^{[2]})^\vee) \cdot c_+(\mathbb{C}^3 \otimes \mathcal{L}^{-1})}{c_+((H^{[2]})^\vee) \cdot c_-(H^{[2]} \otimes \mathcal{L})}$$

where the bundles $\mathcal{O}^{[2]}$ and $H^{[2]}$ are tautological line bundles on the Hilbert scheme of points $X^{[2]}$. The c_+ and c_- are total Chern polynomial evaluated at 1 and -1 respectively, essentially, we substitute the equivariant parameter t to 1. The computation can be carried out using standard techniques in [EGL01]. The final result is:

$$\int_{X^{[2]}\times\mathbb{P}^2} \frac{c_1((\mathcal{O}^{[2]})^\vee) \cdot c_+((\mathcal{O}^{[2]})^\vee) \cdot c_+(\mathbb{C}^3 \otimes \mathcal{L}^{-1})}{c_+((H^{[2]})^\vee) \cdot c_-(H^{[2]} \otimes \mathcal{L})} = -24$$

so the contribution from this fixed locus is 24λ.

Case (2): $\mathcal{I}_{\Delta_1} \oplus \mathcal{I}_{\Delta_2} \otimes \mathcal{O}(-\mathcal{H})$

We now demonstrate the computation for case (2), which is similar with Case (1), but a Grothendieck-Riemann-Roch computation is needed. In this case, the fixed locus is $\mathcal{X}^{[1]} \times_{\mathcal{F}_U} \mathcal{X}^{[1]} \times_{\mathcal{F}_U} \mathbb{P}^2 \cong \mathcal{X} \times_{\mathcal{F}_U} \mathcal{X} \times_{\mathcal{F}_U} \mathbb{P}^2$.

Let $\mathcal{L} = \mathcal{O}_{\mathbb{P}}(1)$. We can simplify the fixed K-theory class as:

$$\text{Fixed} = \text{Tan}_{X_1} + \text{Tan}_{X_2} + \text{Tan}_{\mathbb{P}} - (\mathcal{O}_1^{[1]})^\vee - \mathcal{H}_2 \otimes \mathcal{L}.$$

Note that the fiber-wise trivial factor $\mathcal{O}_1^{[1]}$ will glue to the pullback of the the Hodge line bundle $\mathbb{E}$ on $\mathcal{M}_{\mathbb{L}}$. Thus, we have:

$$\text{Obs} = \mathcal{H}_2 \otimes \mathcal{L} + \mathbb{E}^\vee.$$

The moving parts can be simplified as follows:

$$\text{Mov}[1] = 2\mathbb{C} - (\mathcal{O}_1^{[1]})^\vee - \text{Ext}^\bullet(\mathcal{I}_{\Delta_1}, \mathcal{I}_{\Delta_2} \otimes \mathcal{H}^{-1}) \otimes \mathcal{L}^{-1},$$

$$\text{Mov}[-1] = \mathbb{C}^3 \times \mathcal{L} - \mathcal{H}_2 \otimes \mathcal{L} - \text{Ext}^\bullet(\mathcal{I}_{\Delta_2} \otimes \mathcal{H}^{-1}, \mathcal{I}_{\Delta_1}) \otimes \mathcal{L}$$

The localized contribution of $p_\star\left(0^6 \cap [\mathcal{Q}_{H,X}(\mathbb{C}^2)]^{\text{red}}\right)$ on this fixed locus will be

$$\begin{aligned}
&\frac{e(\text{Obs})}{c_+(\text{Mov}[1]) \cdot c_-(\text{Mov[-1]})} \\
&= -c_1(\mathbb{E}) \int_{X \times X \times \mathbb{P}^2} \frac{e(\mathcal{H}_2 \otimes \mathcal{L}) \cdot c_-(\mathcal{H}_2 \otimes \mathcal{L}) \cdot c_-(\text{Ext}^\bullet \otimes \mathcal{L}) \cdot c_+((\text{Ext}^\bullet)^\vee \otimes \mathcal{L}^{-1})}{(1-\zeta)^3}, \\
&= -\lambda \cdot \int_{X \times X \times \mathbb{P}^2} \frac{(h_2 + \zeta)(1 - h_2 - \zeta) \cdot c_+^2((\text{Ext}^\bullet)^\vee \otimes \mathcal{L}^{-1})}{(1-\zeta)^3}.
\end{aligned}$$

where $\text{Ext}^\bullet = \text{Ext}^\bullet(\mathcal{I}_{\Delta_2} \otimes \mathcal{H}^{-1}, \mathcal{I}_{\Delta_1})$. Using Grothendieck-Riemann-Roch, we can compute:

$$\text{ch}((\text{Ext}^\bullet)^\vee) = 1 + h_1 + h_2 + \Delta_{1,2} - [(c, X)] - [(X, c)] - [(h, h)] - [(c, c)],$$

where the c is a point Chow class. In fact, the class c is a special zero cycle, satisfying the

Beauville-Voisin rule in [BV04]:

$$[(x,x)] - [(x,c)] - [(c,x)] + [(c,c)] = 0 \quad \text{in} \quad \mathrm{CH}_0(X \times X).$$

Converting $\mathrm{ch}((\mathrm{Ext}^\bullet)^\vee)$ to $c((\mathrm{Ext}^\bullet)^\vee)$ and plugging it in, the fixed locus contribution simplifies to:

$$\frac{e(\mathrm{Obs})}{c_+(\mathrm{Mov}[1]) \cdot c_-(\mathrm{Mov}[\text{-}1])} = 48\lambda.$$

Case (3): $\mathcal{O} \ \oplus \ \mathcal{I}_2 \otimes \mathcal{O}(-\mathcal{H})$

Case (3) is special among all cases, as we have a trivial factor $\mathcal{O}$ in the summand, the result will not be a multiple of the Hodge class. The obstruction theory on the fixed locus has been computed in [MOP17, Section 4.3]. We briefly summarize the result here.

Let $\pi : \mathcal{X} \to \mathcal{F}_U$ be the universal K3 surface map. Let $\mathbb{V} := \pi_\star(\mathcal{H})$. In our case, it will be a vector bundle of rank $l + 2 = 4$. Let $\mathbb{P} = \mathbb{P}(\mathbb{V})$ be the projective bundle over $\mathcal{F}_U$. The fixed locus is $\mathcal{X}^{[2]} \times_{\mathcal{F}_U} \mathbb{P}$. The obstruction bundle is given by:

$$\mathrm{Obs} = \mathbb{E}^\vee \otimes \mathcal{L} \otimes \left(\left(\mathcal{H}^{-1}\right)^{[2]} \right)^\vee,$$

and the moving parts of the theory are:

$$\mathrm{Mov}[\pm 1] = \left(\mathbb{C} + \mathbb{E}^\vee + \mathcal{L}^{-1} \otimes \left(\mathcal{H}^{-1}\right)^{[2]} - \mathcal{L}^{-1} \otimes \mathbb{V}^\vee \otimes \mathbb{E}^\vee \right) [\pm 1].$$

The localized contribution of $p_\star \left(0^6 \cap [\mathcal{Q}^\pi_{H,\chi}(\mathbb{C}^2)]^{\mathrm{red}} \right)$ on this fixed locus will be:

$$q_\star \left(\mathrm{e}\left(\mathbb{E} \otimes \mathcal{L}^{-1} \otimes \left(\mathcal{H}^{-1}\right)^{[2]} \right) \cdot \frac{c_-(\mathcal{L} \otimes \mathbb{V} \otimes \mathbb{E})}{1 - \lambda} \cdot \frac{1}{c_+\left(\mathcal{L}^{-1} \otimes \left(\mathcal{H}^{-1}\right)^{[2]} \right)} \right), \tag{5.2.1}$$

where

$$q : \mathcal{X}^{[2]} \times_{\mathcal{F}_U} \mathbb{P} \to \mathcal{F}_U.$$

Furthermore, let:

$$\gamma_i = \mathrm{pr}_\star \left(s_{2+i+1} \left(\left(\mathcal{H}^{-1}\right)^{[2]} \right) \cdot c_{2-i} \left(\left(\mathcal{H}^{-1}\right)^{[2]} \right) \right), \quad 0 \le i \le 2,$$

where $\mathrm{pr} : \mathcal{X}^{[2]} \to \mathcal{F}_U$. The final contribution can be expressed as the following three parts:

$$(-1)^{\ell+1} \cdot \left(\sum_{i=0}^{n} \binom{\ell + 1 - 2n - i}{\ell + 1 - i} \cdot \gamma_i + a \cdot c_1(\mathbb{V}) + b \cdot \lambda \right),$$

where $n = d + \chi$. The constant a and b can be reduced to linear combinations of tautological integrals over $X^{[2]}$ of the following forms:

$$\alpha_i := \int_{X^{[2]}} c_{n-i} \left(\left(\mathcal{H}^{-1}\right)^{[2]} \right) \cdot s_{n+i} \left(\left(\mathcal{H}^{-1}\right)^{[2]} \right).$$

Using the recursion in [EGL01], we can compute the integrals α_i and the pushforward γ_i. We record the results:

Proposition 5.2.1. *Let* $\kappa_{i,j} := \pi_\star \left(c_1(\mathcal{O}(\mathcal{H}))^i \cdot c_2(T_\pi)^j \right)$. *Let* $2\ell = H^2$, *and we have:*

$$\begin{aligned}
\gamma_0 &= \frac{1}{2} \left((4\ell - 10)\kappa_{3,0} + \kappa_{1,1} + 10\ell \cdot \lambda \right), \\
\gamma_1 &= \frac{1}{2} \left((30 - 8\ell)\kappa_{3,0} - 7\kappa_{1,1} + (48 - 40\ell) \cdot \lambda \right), \\
\gamma_2 &= \frac{1}{2} \left((4\ell - 20)\kappa_{3,0} + 6\kappa_{1,1} + (30\ell - 48) \cdot \lambda \right), \\
\alpha_0 = 2\ell^2 - 4\ell, &\quad \alpha_1 = -4\ell^2 + 14\ell - 12, \quad \alpha_2 = 2\ell^2 - 10\ell + 12.
\end{aligned}$$

Now, by carefully expanding Equation (5.2.1), we can see in our case:

$$a = -4\alpha_0 + 6\alpha_1 - 4\alpha_2 = -20, \quad b = -8\alpha_2 + 10\alpha_1 - 8\alpha_0 = -36.$$

Using Grothendieck-Riemann-Roch, we can compute:

$$c_1(\mathbb{V}) = (-1 - \frac{\ell}{2})\lambda + \frac{1}{6}\kappa_{3,0} + \frac{1}{12}\kappa_{1,1}.$$

Putting everything together, we can compute the contribution from this fixed locus is:

$$\frac{40}{3}(\kappa_{1,1} - 4\kappa_{3,0}) - 12\lambda.$$

The process for other fixed loci will be similar. We will list the final results:

1. $\mathcal{I}_2 \ \oplus \ \mathcal{O}(-\mathcal{H})$: 24λ.

2. $\mathcal{I}_{\Delta_1} \ \oplus \ \mathcal{I}_{\Delta_2} \otimes \mathcal{O}(-\mathcal{H})$: 48λ.

3. $\mathcal{O} \ \oplus \ \mathcal{I}_2 \otimes \mathcal{O}(-\mathcal{H})$: $\frac{40}{3}(\kappa_{1,1} - 4\kappa_{3,0}) - 12\lambda$.

4. $\mathcal{I}_{\Delta_1} \otimes \mathcal{O}(-\sigma - f) \ \oplus \ \mathcal{O}(-f)$: 32λ.

5. $\mathcal{O}(-\sigma - f) \ \oplus \ \mathcal{I}_{\Delta_2} \otimes \mathcal{O}(-f)$: 32λ.

6. $\mathcal{O}(-\sigma) \ \oplus \ \mathcal{O}(-2f)$: 4λ.

5.2.2 Noether-Lefschetz Loci Contributions

The numerical constraints arising from $z + w = 2 - C_1 \cdot C_2$ dictate that only one Noether Lefschetz locus will occur in the computation, which is the reduced divisor class S^{red} corresponds to the following lattice:

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

Let σ, f and β be the basis of the lattice S. We have an additional possible configuration of $\mathcal{E}_1 \oplus \mathcal{E}_2$ (up to a flip):

1. $\mathcal{O}(-\beta) \ \oplus \ \mathcal{O}(-\sigma - 2f + \beta)$

This splitting condition is supported on $S \subset \mathcal{M}_{\mathbb{L}}$, the contribution of $p_\star\left(0^6 \cap [\mathcal{Q}_{H,\chi}(\mathbb{C}^2)]^{\text{red}}\right)$ will be a multiple of S. To determine the multiplicity, it suffices to perform the computation over a fiber. On a fixed surface X, the fixed locus will be $pt \times \mathbb{P}^1$. We follow the same recipe and have:

$$\text{Fixed} = \text{Tan}_{\mathbb{P}^1} \implies \text{Obs} = 0,$$

$$\text{Mov}[1] = \mathcal{L}_2^{-1}, \quad \text{Mov}[-1] = \mathbb{C}^2 \otimes \mathcal{L}_2 + \mathcal{L}_2.$$

Thus, the multiplicity of S can be computed as:

$$\int_{\mathbb{P}^1} \frac{e(\text{Obs})}{c_+(\text{Mov[1]}) \cdot c_-(\text{Mov[-1]})} = \frac{1}{(1-\zeta_2)(-1+\zeta_2)^2(-1+\zeta_2)} = -4.$$

Finally, we put everything together, for $d = 2, \chi = 1$, the relation we obtained by $p_\star\left(0^6 \cap [\mathcal{Q}_{H,\chi}(\mathbb{C}^2)]^{\text{red}}\right)$ is:

$$\begin{aligned} &24\lambda + 48\lambda + \frac{40}{3}(\kappa_{1,1} - 4\kappa_{3,0}) - 12\lambda + 32\lambda + 32\lambda + 4\lambda - 4S^{\text{red}} \\ &= \frac{40}{3}(\kappa_{1,1} - 4\kappa_{3,0}) + 128\lambda - 4S^{\text{red}} = 0. \end{aligned}$$

5.3 Divisorial relations on $\mathcal{F}_U$

The method in Section 5.2 can be applied repeatedly to $\mathcal{H} = \sigma + df$. If $\chi = 3 - d$, we will have Hilbert schemes of two points involved in the computation. These computations will follow the exact same procedure as in Section 5.2. We have computed the results for $2 \leq d \leq 5$, and we will list the final results.

5.3.1 Fixed loci contributions

We have given the fixed loci contributions for the case $d = 2, \chi = 1$. For completeness, we will give the fixed loci contributions for all the cases in Proposition 5.3.1. The reader can use the methodology in Section 5.2 to verify the results.

Fixed loci contributions for $\mathcal{H} = \sigma + 2f, \chi = 1$.

1. $\mathcal{I}_2 \ \oplus\ \mathcal{O}(-\mathcal{H})$: 24λ.
2. $\mathcal{I}_{\Delta_1} \ \oplus\ \mathcal{I}_{\Delta_2} \otimes \mathcal{O}(-\mathcal{H})$: 48λ.
3. $\mathcal{O} \ \oplus\ \mathcal{I}_2 \otimes \mathcal{O}(-\mathcal{H})$: $\frac{40}{3}(\kappa_{1,1} - 4\kappa_{3,0}) - 12\lambda$.
4. $\mathcal{I}_{\Delta_1} \otimes \mathcal{O}(-\sigma - f) \ \oplus\ \mathcal{O}(-f)$: 32λ.
5. $\mathcal{O}(-\sigma - f) \ \oplus\ \mathcal{I}_{\Delta_2} \otimes \mathcal{O}(-f)$: 32λ.
6. $\mathcal{O}(-\sigma) \ \oplus\ \mathcal{O}(-2f)$: 4λ.
7. Noether-Lefschetz loci: $-4S^{\mathrm{red}}$.

Fixed loci contributions for $\mathcal{H} = \sigma + 3f, \chi = 0$.

1. $\mathcal{I}_2 \ \oplus\ \mathcal{O}(-\mathcal{H})$: -184λ.
2. $\mathcal{I}_{\Delta_1} \ \oplus\ \mathcal{I}_{\Delta_2} \otimes \mathcal{O}(-\mathcal{H})$: 32λ.
3. $\mathcal{O} \ \oplus\ \mathcal{I}_2 \otimes \mathcal{O}(-\mathcal{H})$: $-40\kappa_{3,0} + 20\kappa_{1,1} + 76\lambda$.
4. $\mathcal{I}_{\Delta_1} \otimes \mathcal{O}(-\sigma - 2f) \ \oplus\ \mathcal{O}(-f)$: -4λ.
5. $\mathcal{O}(-\sigma - 2f) \ \oplus\ \mathcal{I}_{\Delta_2} \otimes \mathcal{O}(-f)$: -12λ.
6. $\mathcal{O}(-\sigma - f) \ \oplus\ \mathcal{O}(-2f)$: 4λ.
7. Noether-Lefschetz loci: $-6S^{\mathrm{red}}$.

Fixed loci contributions for $\mathcal{H} = \sigma + 4f, \chi = -1$.

1. $\mathcal{I}_2 \oplus \mathcal{O}(-\mathcal{H})$: -216λ.

2. $\mathcal{I}_{\Delta_1} \oplus \mathcal{I}_{\Delta_2} \otimes \mathcal{O}(-\mathcal{H})$: 48λ.

3. $\mathcal{O} \oplus \mathcal{I}_2 \otimes \mathcal{O}(-\mathcal{H})$: $-16\kappa_{3,0} + 12\kappa_{1,1} + 84\lambda$.

4. $\mathcal{I}_{\Delta_1} \otimes \mathcal{O}(-\sigma - 3f) \oplus \mathcal{O}(-f)$: -16λ.

5. $\mathcal{O}(-\sigma - 3f) \oplus \mathcal{I}_{\Delta_2} \otimes \mathcal{O}(-f)$: -48λ.

6. $\mathcal{O}(-\sigma - 2f) \oplus \mathcal{O}(-2f)$: -4λ.

7. Noether-Lefschetz loci: $-4S^{\mathrm{red}}$.

Fixed loci contributions for $\mathcal{H} = \sigma + 5f, \chi = -2$.

1. $\mathcal{I}_2 \oplus \mathcal{O}(-\mathcal{H})$: -72λ.

2. $\mathcal{I}_{\Delta_1} \oplus \mathcal{I}_{\Delta_2} \otimes \mathcal{O}(-\mathcal{H})$: 24λ.

3. $\mathcal{O} \oplus \mathcal{I}_2 \otimes \mathcal{O}(-\mathcal{H})$: $\frac{8}{3}(\kappa_{1,1} - \kappa_{3,0}) + 24\lambda$.

4. $\mathcal{I}_{\Delta_1} \otimes \mathcal{O}(-\sigma - 4f) \oplus \mathcal{O}(-f)$: -2λ.

5. $\mathcal{O}(-\sigma - 4f) \oplus \mathcal{I}_{\Delta_2} \otimes \mathcal{O}(-f)$: -22λ.

6. $\mathcal{O}(-\sigma - 3f) \oplus \mathcal{O}(-2f)$: -4λ.

7. Noether-Lefschetz loci: $-S^{\mathrm{red}}$.

5.3.2 Divisorial relations

We have listed all the fixed loci contributions, and we know all of them should sum to zero. We have the following result:

Proposition 5.3.1. *Let the $\kappa_{i,j} = \pi_\star \left(c_1(\mathcal{O}(\mathcal{H}))^i \cdot c_2(T_\pi)^j\right)$, we have the following divisorial relations on $\mathcal{F}_U$.*

1. *For $\mathcal{H} = \sigma + 2f, \chi = 0$, we have: $\kappa_{1,1} - 4\kappa_{3,0} - 30\lambda = 0$.*

2. *For $\mathcal{H} = \sigma + 2f, \chi = 1$, we have: $\frac{40}{3}(\kappa_{1,1} - 4\kappa_{3,0}) + 128\lambda - 4S^{\mathrm{red}} = 0$.*

3. *For $\mathcal{H} = \sigma + 3f, \chi = 0$, we have: $10(\kappa_{1,1} - 2\kappa_{3,0}) - 44\lambda - 3S^{\mathrm{red}} = 0$.*

4. *For $\mathcal{H} = \sigma + 4f, \chi = -1$, we have: $3\kappa_{1,1} - 4\kappa_{3,0} - 38\lambda - S^{\mathrm{red}} = 0$.*

5. *For $\mathcal{H} = \sigma + 5f, \chi = -2$, we have: $\frac{8}{3}(\kappa_{1,1} - \kappa_{3,0}) - 52\lambda - S^{\mathrm{red}} = 0$.*

Proof. Except for the first relation, all the other relations are obtained by summing up the fixed loci contributions listed in Section 5.3.1. The first relation can be computed from localization which only involves Hilbert schemes of one point. The computation is contained in [MOP17, Proposition 1]. In fact, [MOP17, Proposition 1] contains the following identity in $A^1(\mathcal{F}_2)$:

$$\kappa_{1,1} - 4\kappa_{3,0} - 18\lambda + 12[\mathcal{F}_U] = 0.$$

We pullback this identity to $\mathcal{F}_U$ and obtain the first relation, where we use the fact that the normal bundle of $\mathcal{F}_U$ in $\mathcal{F}_2$ will contribute $-\lambda$, see [O'G86, Lemma 1.2]. □

Remark 5.3.2. The other relations we presented here should also be compared with the ones in [MOP17, Proposition 1-4]. Although the Noether-Lefschetz loci behave more intricately under the pullback, every relation we obtained should be a pullback of the corresponding relation in $A^1(\mathcal{F}_{2\ell})$ via the morphism $\iota : \mathcal{F}_U \to \mathcal{F}_{2\ell}$.

Note that the Noether-Lefschetz classes can be defined with respect to invariants of the lattice embedding.

Definition 5.3.3. For $\Delta \in \mathbb{Z}$, and $\delta \in G/\pm 1$ where $G = \Lambda^*/\Lambda$. We define the Noether-Lefschetz divisor

$$P_{\Delta,\delta} \subset \mathcal{F}_\Lambda$$

to be the closure of the locus of Λ-quasi-polarized K3 surfaces S for which $\mathrm{Pic}(S)$ has rank $\mathrm{rank}(\Lambda)+1$ and $j : \Lambda \to \mathrm{Pic}(S)$ has discriminant Δ and coset δ. For more details, see [KMPS10].

The reduced Noether-Lefschetz divisor S^{red} can be described using the following definition by picking $\Delta = 2$ and $\delta = 0$.

As explained in [KMPS10], the Noether-Lefschetz divisors will behave better if we repackage them in the following way:

Definition 5.3.4. For integers $h, d_1, ..., d_r$, we define the Noether-Lefschetz divisor $D_{h,(d_1,...,d_r)}$ to be the weighted sum:

$$D_{h,(d_1,\ldots,d_r)} = \sum_{\Delta,\delta} m\left(h, d_1, \ldots, d_r \mid \Delta, \delta\right) \cdot [P_{\Delta,\delta}]$$

where the multiplicity $m\left(h, d_1, \ldots, d_r \mid \Delta, \delta\right)$ is the number of elements β in any lattice $(\mathbb{L}, \Lambda \hookrightarrow \mathbb{L})$ of type (Δ, δ) satisfying:

$$\langle \beta, \beta \rangle = 2h - 2, \quad \langle \beta, e_i \rangle = d_i \text{ for } i = 1, \ldots, r$$

where e_i is the basis of Λ.

The intersection theory of Noether-Lefschetz divisors are related with modular forms. For $\mathcal{F}_U$, we have the following result:

Theorem 5.3.5.

$$D_{h,(d_1,d_2)} = -E_4(q)E_6(q)\left[\frac{\det(\Lambda)}{2}\right]\cdot\lambda$$

where $E_4(q)$ and $E_6(q)$ are the Eisenstein series, and the matrix Λ is:

$$\begin{pmatrix} 0 & 1 & d_1 \\ 1 & 0 & d_2 \\ d_1 & d_2 & 2h-2 \end{pmatrix}.$$

Proof. This follows immediately from Proposition 2 in [KMPS10] and Theorem 2.2.2. $\square$

We can derive the Noether-Lefschetz divisor $S^{\mathrm{red}} = 132\lambda$ from the first two relations. Theorem 5.3.5 above tells us that $D_{0,(1,1)} = 264\lambda$. They differ by a factor of 2, due to the automorphism of the extra class $\beta \to -\beta$.

Plugging in $S^{\mathrm{red}} = 132\lambda$ into all the relations and solving a linear system, we can compute all the invariant κ-classes:

$$\kappa_{3,0,0} + \frac{1}{4}\kappa_{1,0,1} = \frac{7}{2}\lambda, \quad 3\kappa_{2,1,0} - \frac{1}{4}\kappa_{1,0,1} + \frac{1}{4}\kappa_{0,1,1} = \frac{1}{2}\lambda,$$

$$3\kappa_{1,2,0} - \frac{1}{4}\kappa_{0,1,1} = -3\lambda, \quad \kappa_{0,3,0} = 0.$$

where $\kappa_{i,j,k} := \pi_*\left(c_1(\mathcal{O}(\sigma))^i \cdot c_1(\mathcal{O}(f))^j \cdot c_2\left(\mathcal{T}_\pi\right)^k\right)$. The results are consistent with Proposition 4.3.4.

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