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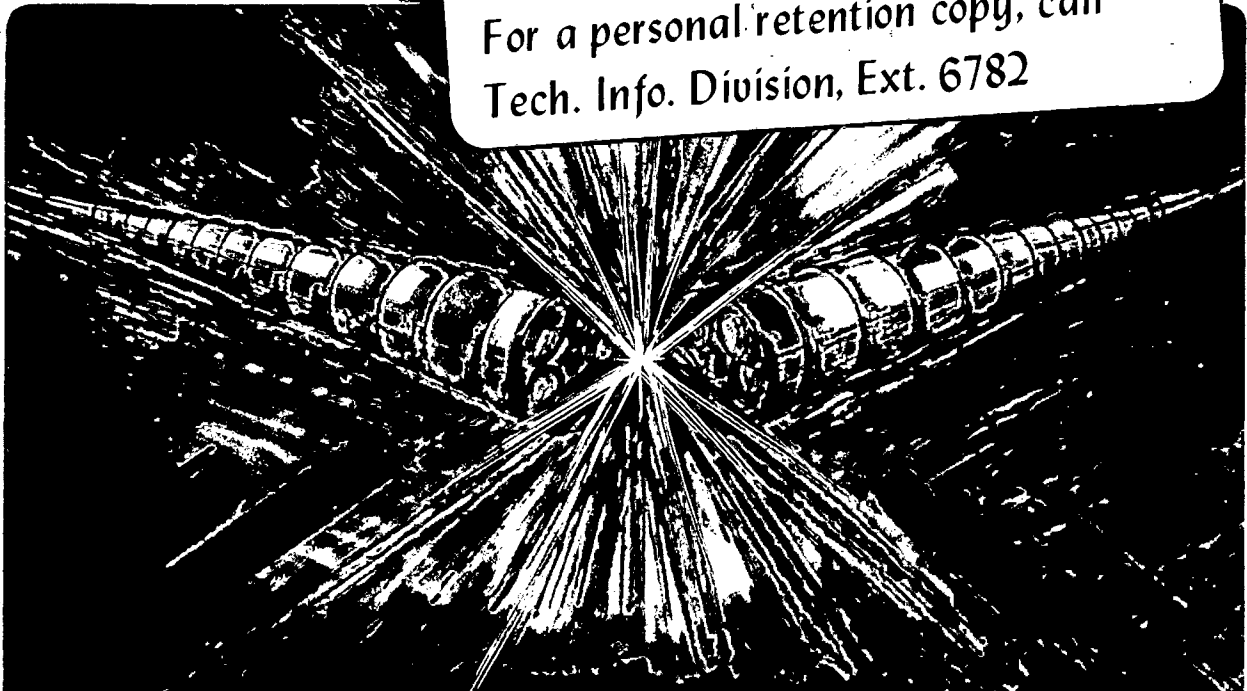
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Finite Resolution Approximations to the
Asymptotic Distribution for Dynamical Systems*

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ABSTRACT

The asymptotic distribution of points in phase space visited by orbits of a dynamical system determines how one is to evaluate averages of physical quantities over the phase space. In any numerical or laboratory experiment the details of this distribution are restricted by resolution. We present a method for calculating the asymptotic distribution commensurate with any given experimental resolution. Also we show how to evaluate the evolution from any initial distribution to the asymptotic distribution. Detailed calculations are given for the mapping $x_{n+1} = B\sin(\pi x_n)$. $0 \leq B \leq 1$, which takes the interval $0 \leq x \leq 1$ into itself.

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In the analysis of nonlinear dynamical systems an object of some interest is the asymptotic distribution function, $D_A(\underline{x})$, on the N dimensional state space $\underline{x} = (x_1, \dots, x_N)$ of the system.¹ This distribution describes the disposition of almost all orbits after the system has evolved long enough in time from an initial state x_0 . Averages of phase space functions $G(\underline{x})$ are given by

$$\langle G \rangle = \frac{\int d\underline{x} D_A(\underline{x}) G(\underline{x})}{\int d\underline{x} D_A(\underline{x})} \quad (1)$$

when the asymptotic distribution is reached. For nonintegrable Hamiltonian systems $D_A(\underline{x})$ is the microcanonical distribution--namely, uniform population of the constant energy surface. We consider here more general systems than Hamiltonian.

This note describes a method for approximating $D_A(\underline{x})$. It is motivated by the observation that in any laboratory or numerical experiment on a dynamical system, the observed distribution will be a "coarse grained" representation of $D_A(\underline{x})$ reflecting the experimental resolution. $D_A(\underline{x})$ consists of a series of delta functions arising from the deterministic aspect of the dynamics. Experimental resolution smooths out the true $D_A(\underline{x})$ by presenting the answer in bins of an approximate size. If present, external noise will also smooth out D_A ², such noise can be easily incorporated in our method when its characteristic function is known. Henceforth we assume external noise to be absent.

We choose to evaluate not $D_A(x)$ but its Fourier transform

$$\tilde{D}_A(\underline{m}) = \int \frac{d\underline{x}}{(2\pi)^N} e^{i\underline{m}\cdot\underline{x}} D_A(\underline{x}) \quad (2)$$

which is smooth although $D_A(\underline{x})$ is singular. Truncating the number of modes in the equation determining $D_A(\underline{m})$, we are able, by operations of a finite order, to calculate $D_A(\underline{x})$ with a specified, improvable resolution.

Our focus in this note is on discrete dynamical systems whose evolution is given by

$$\underline{x}_{n+1} = F(\underline{x}_n) \quad (3)$$

This yields the state of the system at "time" $n+1$, given the state at n . For dynamical flows governed by a differential equation we can discretize it in time to reach (3) or study the Poincare' or return map for the flow. With small changes, the analysis here applies to flows as well.

We take the dynamical system to be ergodic³, which means

$$\langle G \rangle = \int d\underline{x} D_A(\underline{x}) G(\underline{x}) = \lim_{L \rightarrow \infty} \frac{1}{L} \sum_{k=0}^{L-1} G(F^k(\underline{x}_0)) \quad (4)$$

where we have normalized $D_A(\underline{x})$ to one and $F^k(\underline{x}_0)$ is the k^{th} iterate of the map beginning at \underline{x}_0 . Denoting

$$D_L(\underline{x}, \underline{x}_0) = \frac{1}{L} \sum_{k=0}^{L-1} \delta^N(\underline{x} - F^k(\underline{x}_0)), \quad (5)$$

it follows that

$$D_A(\underline{x}) = \lim_{L \rightarrow \infty} D_L(\underline{x}, \underline{x}_0). \quad (6)$$

The interpretation of (5) is quite natural: the asymptotic distribution is the set of points which the orbit beginning at \underline{x}_0 visits as the orbit length becomes infinite.

Equation (6) contains the implication that the asymptotic distribution is independent of \underline{x}_0 . For almost all points in a given basin of attraction that is what we should expect. If there are many basins of attraction for the dynamical systems, each may have a different asymptotic distribution. Even within a given basin of attraction, there may be a set of points leading to a different $D_A(\underline{x})$ which is singular with respect to the distribution we expect to be calculating here. A heuristic way to specify what $D_A(\underline{x})$ we are evaluating is to appeal to a result of Kifer⁴ which says, for our purposes, that adding a small external noise source leads to an asymptotic distribution as close to our $D_A(\underline{x})$ as we wish. Since the noise eliminates singular distributions by providing a "spread" in phase space, we can say that the distribution we are evaluating is the same as would result from introducing an infinitesimal noise source, calculating D_A , and then removing the noise.

To derive an equation for D_A we note the distribution $D_L(\underline{x}, \underline{x}_0)$ is connected with the conditional probability $P(\underline{x}, k | \underline{x}_0, 0)$ ⁵ that the system is at \underline{x} after k steps, given it originated at \underline{x}_0 . Since

$$P(\underline{x}, k | \underline{x}_0, 0) = \delta^N(\underline{x} - \underline{F}^k(\underline{x}_0)), \quad (7)$$

one sees

$$D_L(\underline{x}, \underline{x}_0) = \frac{1}{L} \sum_{k=0}^{L-1} P(\underline{x}, k | \underline{x}_0, 0). \quad (8)$$

The conditional probability satisfies

$$P(\underline{x}, k | \underline{x}_0, 0) = \int d\underline{w} P(\underline{x}, k | \underline{w}, k-1) P(\underline{w}, k, -1 | \underline{x}_0, 0). \quad (9)$$

Introducing the Fourier amplitudes

$$A_k(\underline{m}, \underline{x}_0) = \int d\underline{x} e^{-i\underline{m} \cdot \underline{x}} P(\underline{x}, k | \underline{x}_0, 0), \quad (10)$$

(9) yields the recursion relation

$$A_k(\underline{m}, \underline{x}_0) = \sum_{\underline{n}} T_{\underline{m}, \underline{n}} A_{k-1}(\underline{n}, \underline{x}_0) \quad (11)$$

in which

$$T_{\underline{m}, \underline{n}} = \int \frac{d\underline{z}}{(2\pi)^N} e^{i\underline{n} \cdot \underline{z} - i\underline{m} \cdot \underline{F}(\underline{z})}. \quad (12)$$

Fourier transforming $D_L(\underline{x}, \underline{x}_0)$ and using the recursion relation, we find, in a matrix notation,

$$(1-T) \tilde{D}_L = \frac{1}{L} (1-T^L) A_0. \quad (13)$$

From this we see that $D_A(m)$ is the eigenvector of T with eigenvalue 1. With regard to the remarks above, we see that it is independent of \underline{x}_0 .

This last relation for $\tilde{D}_L(\underline{m}, \underline{x}_0)$ allows us to determine the approach to D_A as well as D_A itself. Introduce the eigenvalues $\lambda_0 = 1, \lambda_2, \dots$ and eigenvectors $V_a(\underline{m})$ $a = 0, 1, 2, \dots$ of T . The λ_a have magnitude less than or equal to one if the transformation matrix T does not take orbits off to infinity. Equation (13) only makes sense in the limit of large L when this is the case, so we assume it in our discussion. (Our statements above about the uniqueness of the asymptotic distribution are here iterated by the requirement that the eigenvalue unity be nondegenerate.) From (13) we see

$$\tilde{D}_L(\underline{m}, \underline{x}_0) = \tilde{D}_A(\underline{m}) + \frac{1}{L} \sum_{a=1} [(1 - (\lambda_a)^L)/(1 - \lambda_a)]^x \quad (14)$$

$$\times V_a(\underline{m}) \left[\sum_{\underline{n}} \bar{V}_a(\underline{n}) A_0(\underline{n}, \underline{x}_0) \right].$$

This shows how the distribution $D_A(\underline{x})$ is reached.

Our observation is now that a truncation of the modes m in $T_{\underline{m}, n}$ provides a controllable and improvable resolution in the $D_A(\underline{x})$ predicted by that truncation. Restricting ourselves to $m_j = 0, \pm 1, \pm 2, \dots, \pm N_D$ for each component of m , results in a resolution N_D^{-1} in each component of x . Precisely this resolution comes from placing one's experimental data in \underline{x} space into N_D bins for each component of x . At any finite truncation we have a finite dimensional T matrix whose eigenvector with eigenvalue unity determines $\tilde{D}_A(\underline{m})$.

To illustrate our discussion we have studied the one dimensional map⁶

$$x_{n+1} = B \sin(\pi x_n), \quad 0 \leq B \leq 1, \quad (15)$$

which maps the interval $0 \leq x \leq 1$ into itself. In Fig. 1 we display the asymptotic attractor for $0.82 \leq B \leq 0.997$. For $B \leq 1/\pi$ the attractor is the stable fixed point $x = 0$. For $1/\pi < B \leq 0.72$ there is another stable fixed point. At $B \sim 0.72$ this fixed point bifurcates into the stable two cycle still visible at $B = 0.82$ in Fig. 1.

Our "data" are collected by dividing the x axis into N_x bins, and, at a fixed B , counting the number of points of the attractor in each bin. Properly normalized, this is precisely the coarse grained version of $D_A(x)$ described above.

For the map above T_{mn} is

$$T_{mn} = \frac{1}{\pi} \int_0^\pi d\theta e^{i(2n\theta - 2\pi m B \sin\theta)} = J_{2n}(2\pi m B) + iE_{2n}(2\pi m B), \quad (16)$$

where J and E are the ordinary Bessel and Lommel functions. Choosing $|m|, |n| \leq N_b$ we have determined the eigenvector of (16) with eigenvalue unity. Returning to x space we display $D_A(x)$ and the result of binning data from the direct numerical evaluation of the attractor. In Fig. 2 we have taken $N_b = 25$, $B = 0.847$, and put the numerical data into 50 bins. In this region the attractor is a period four orbit and our method does a credible job in reproducing the distribution. In Fig. 3 we have taken $B = 0.955$ with the binning the same as before. For this value of B we are in a regime of chaotic

motion just past the bifurcations of a stable 3 cycle. In each display the orbit was iterated 10,050 times and the last 10^4 points were put into the bins. This leads to a statistical error of about 7 in the numerical data. All distributions are normalized to one.

The information in the other eigenvalues and eigenfunctions of T allows us to follow the evolution of any initial density of points $f(x_0)$. We choose $f(x_0)$ to be normalized to one. From (14) the Fourier component of the average of this density with $D_L(m, x_0)$ behaves as

$$\begin{aligned} \tilde{f}_L(\underline{m}) = & \tilde{D}_A(\underline{m}) + \frac{1}{L} \sum_a [(1 - (\lambda_a)^L)/(1 - \lambda_a)]^X \\ & \times v_a(\underline{m}) \left[\sum_{\underline{n}} v_a(\underline{n}) \tilde{f}(\underline{n}) \right], \end{aligned} \quad (17)$$

since $A_0(\underline{n}, \underline{x}_0) = \exp(i\underline{n}\underline{x}_0)$. This shows that any initial distribution approaches $D_A(x)$. The same information permits us to find the behavior of correlation functions on phase space.

$$C_{AB}(k) = \int d\underline{x} A(\underline{x}) B(F^k(\underline{x})) D_A(\underline{x}) - \langle A \rangle \langle B \rangle. \quad (18)$$

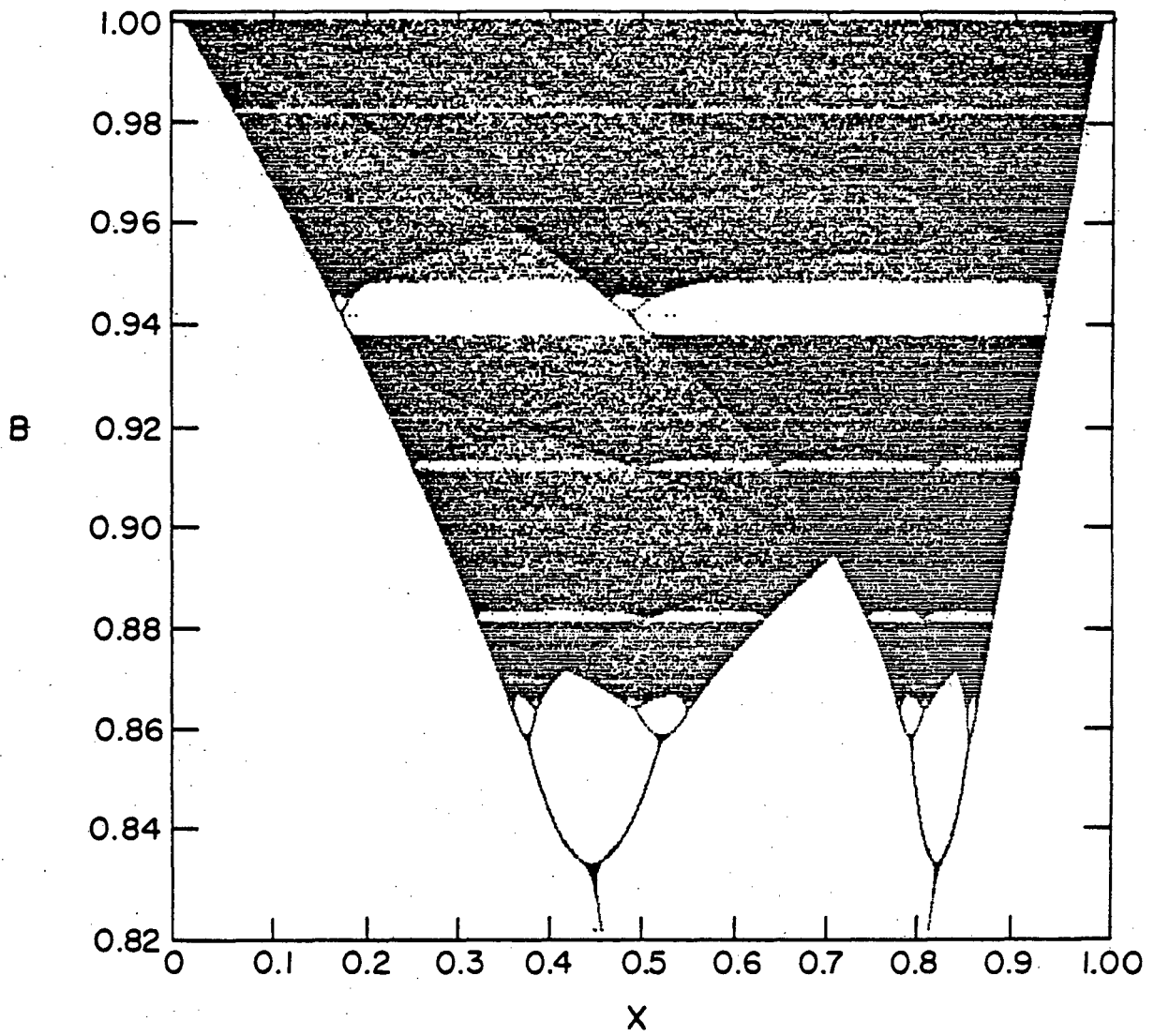
For large k , $C_{AB}(k)$ falls as $\exp(-rk)$ with r determined by the eigenvalue of T whose magnitude is closest to 1. These matters and the application of the present method to both flows and models for physical systems will appear in our forthcoming paper.⁷

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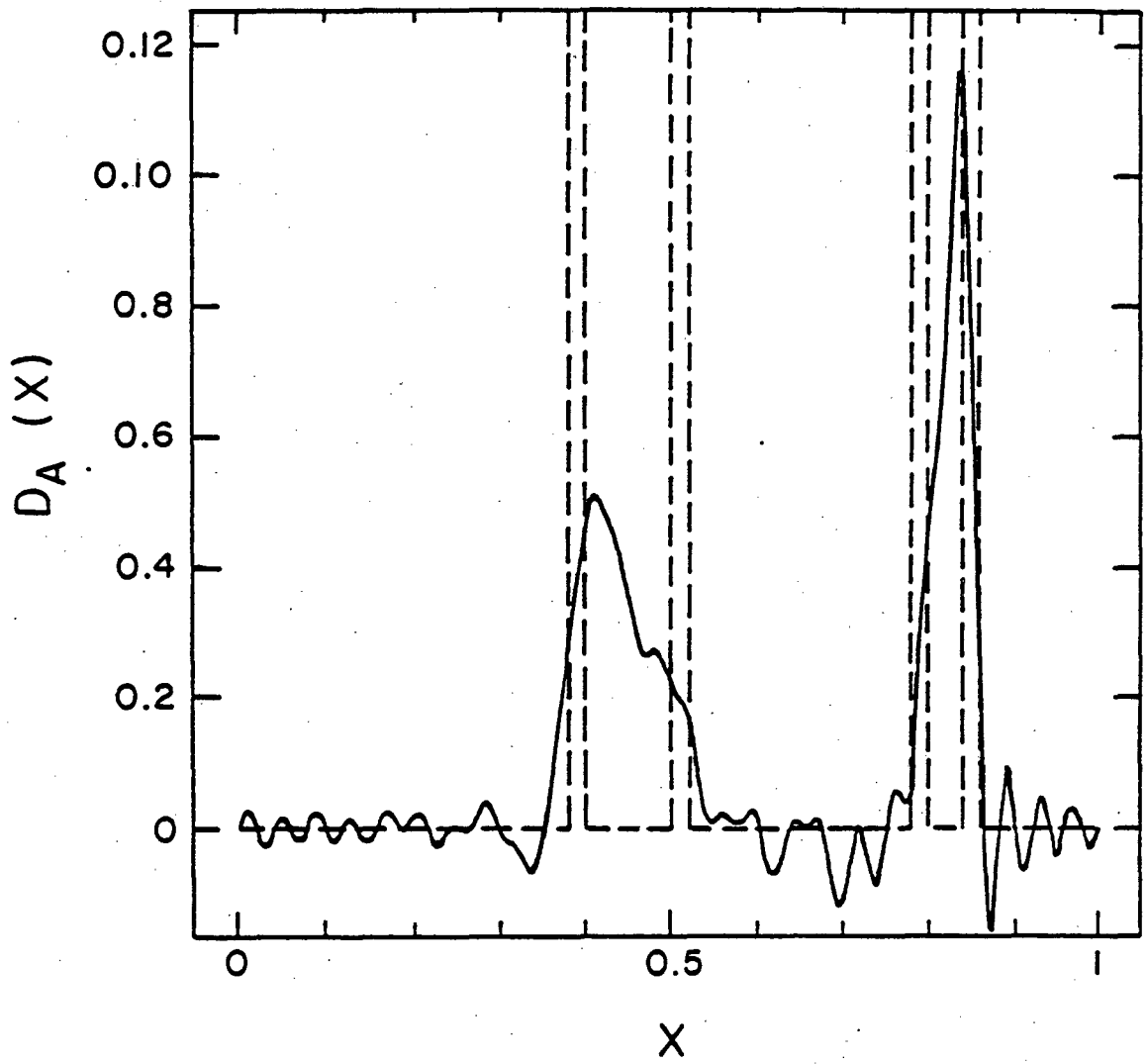
Figure Captions

- Figure 1. The asymptotic attractor for the mapping $x_{n+1}=B\sin(\pi x_n)$. The map has been iterated 550 times for 450 values of B in the range $0.8 < B < 0.997$. For each B the first 50 values of x_n were discarded and the subsequent values displayed.
- Figure 2. For $B=0.847$ we show the result of placing 10^4 values of x on the asymptotic attractor of Fig. 1 into 50 bins. Also displayed is the asymptotic distribution $D_A(x)$ calculated by our method for $N_b=25$.
- Figure 3. Same as Fig. 2 except now $B=0.955$. The attractor here is quite complex.



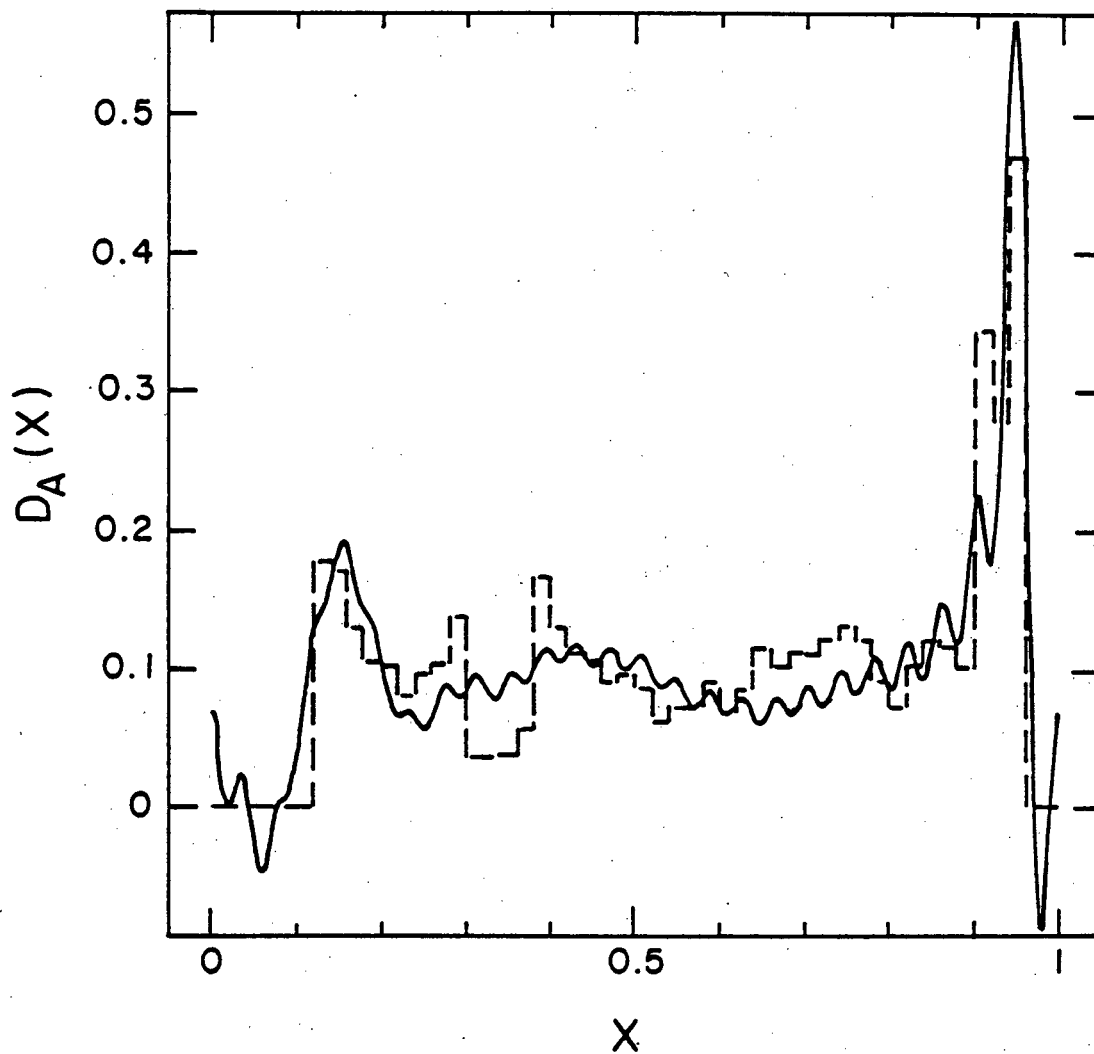
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Figure 1



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Figure 2



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Figure 3

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