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Multilingual Students’ Arguments in Early Algebra: Writing, Refining, and Revising Through Conferencing

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MULTILINGUAL STUDENTS' ARGUMENTS IN EARLY ALGEBRA: WRITING, REFINING, AND REVISING THROUGH CONFERENCING

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Abstract

MULTILINGUAL STUDENTS' ARGUMENTS IN EARLY ALGEBRA:
WRITING, REFINING, AND REVISING THROUGH CONFERENCING

Salvador Huitzilopochtli

This qualitative study sought to understand middle school students' developing mathematical arguments in a linguistically and culturally supportive classroom that featured mathematical writing and oral conferencing. Writing tasks and conferencing focused on developing the core algebraic practice of justifying by emphasizing audience and revision.

Inequitable learning opportunities in mathematics education continue to precipitate academic failure and under-achievement among underrepresented and minoritized (URM) students. Classrooms that make use of mathematical writing and discussions and focus on student reasoning can enhance learning opportunities for URM students (Moschkovich, 2013). This study examined how the arguments of middle school students changed in a classroom where mathematical writing and conferences, conducted during remote instruction caused by the global COVID-19 pandemic, provided opportunities for reflection and potential revision. The study was guided by the following questions:

1. When asked to do mathematical writing and supported with conferencing in a remote context, what kinds of arguments did students make?

2. How did the mathematical arguments of individuals change over the course of a unit of instruction on generating, selecting, and justifying claims?

3. In what ways did students revise their mathematical arguments during conferencing?

Examination of student work revealed the ways that their efforts to justify changed. Upon examination of the Convince Forms, I found that students expanded arguments from describing procedures to making arguments and using examples in mathematically sound ways, and from making no claims to selecting claims and even generating claims of their own. After examination of the mathematical conferences, I found that students expanded their efforts to justify, employing additional *proof schemes* (Healy & Hoyles, 1998), and revised conjecture-testing procedures (i.e., exemplifying) and meanings for formal words.

The findings highlight how students who are multilingual, low-achieving, or designated as special education engage in mathematical argument with support. Moreover, this study illustrates how mathematical argument can be conceptualized as a constellation of approaches that include refining how different parts of an argument can be used in dialectic with the others, i.e., how the use of examples, the further generalization of claims, and further exploration of how to justify can support each other.

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Chapter 1: Statement of Problem

Despite decades of reforms, inequitable learning opportunities in mathematics education continue to precipitate academic failure and under-achievement among underrepresented and minoritized (URM) students. This is largely due to the limited access URM students have to the study of rigorous or advanced mathematics and deficit views of URM students that undergird and presumably warrant the need for limiting such access. Exclusion from mathematics has a long history in professional mathematics and the research community (Inniss, Lewis, Mitrea, Okoudjou, Salerno, Su, & Thurston, 2021), as well as in education (González, 2017; NCSM & TODOS, 2019). Limiting access to mathematics can have negative impacts on the lives of individuals as they subsequently have limited access to more lucrative work in the STEM fields, but it also has negative impacts on the communities they represent as community needs go unaddressed in favor of projects that serve United States economic and military domination. Moreover, individuals are denied the possibility of studying mathematics as a way to live fuller lives (Su, 2020) and communities are denied the innovation that URM students of mathematics might offer (McGee, 2020a & 2020b).

The issue of access manifests at different scales and along different criteria. ‘Access’ refers to the availability of resources to students, for example, quality mathematics teachers, rigorous curriculum, and a classroom environment that honors and invites participation (Gutiérrez, 2009). Due to issues such as disinvestment, tracking, and re-segregation, URM students have less access to high quality mathematics instruction (Haas Institute, 2017; NCSM & TODOS, 2019). Re-segregation and the persistence of tracking undermine educational opportunity and achievement (Haas Institute, 2017) and despite long-standing calls to de-track mathematics education (National Council of Teachers of Mathematics, 2018), tracking remains a persistent practice of structural racism and impedes access to robust teaching and curricula for URM students (Haas Institute, 2017; Oakes, Rogers & Lipton, 2006). Multilingual students also suffer the effects of tracking, but often for different reasons than one might expect. Current systems for identifying and classifying multilingual students do not take the heterogeneity of this group of students into account (NASEM, 2018). In most cases, students' prior mathematics knowledge is not assessed and placement into mathematics courses is based on *English proficiency* (NASEM, 2018). Since many of the English support courses are tracked together with support for mathematics, multilingual students are often relegated to lower tracks of mathematics without regard to their actual mathematics proficiency.

Algebra courses, above all, continue to function as gatekeepers that regulate access to higher mathematical and educational opportunities (Stein, Kaufman, Sherman, & Hillen, 2011). Along racial lines, this exclusion has been documented among Black students (Morton & Riegle-Crumb, 2019) and Mexican American students (Donato & Hanson, 2012; González, 2017). Even when URM students can get into an algebra course, they are less likely to have access to a rigorous course that examines advanced topics or has a highly qualified teacher (Morton & Riegle-Crumb, 2020).

Deficit views of students undergird rationales for the separation of those students whose educability is perceived as suspect. Deficit views of students focus narrowly on what students cannot do, rather than what they can do (Valencia, 2010). In doing so, these views of students can obscure the social, structural, and institutional ways that both access and resources have been attenuated or withheld from URM students. Moreover, the study of mathematics is fraught with stereotypes about intelligence and has become a unique battle ground for asserting and contesting deficit frames of students (McGee & Martin, 2011).

Deficit views of students encompass perceived failings that reside in individuals (vis-à-vis race and gender) or the communities and cultures they represent (including culturally specific values or linguistic practices). Some of the forms that deficit framings of students have taken include genetic arguments about girls' mathematics abilities (Boaler & Sengupta-Irving, 2006; Fennema, 1974; Leder, 2019), intelligence arguments about Latinx and Black students (González, 2017; Martin, 2009; McGee & Martin, 2011), and arguments that the language of a student (or their community) lacks the required sophistication to engage in the study of mathematics (Hudicourt-Barnes, 2003). Addressing deficit views that are encoded into policy or present in the beliefs of educators is crucial to increase access to mathematics education for URM students.

This qualitative study sought to understand students' developing mathematical arguments in a linguistically and culturally supportive classroom that featured mathematical writing and conferencing to learn the foundations to mathematical proof and proving by developing the practice of justifying. The study adhered to recommendations from research in mathematics educational equity and mathematical writing to support students' efforts to develop claims and justify them as they learn about mathematical argument. Moreover, the study responded to deficit framings of students by incorporating a research design for a strengths-based approach to teaching and learning that assumes students bring a wealth of cognitive, cultural, and linguistic resources from home and community (Wang, Lang, Bunch, Basch, McHugh, Huitzilopochtli & Callanan, 2021; Yosso, 2005).

I assumed that students bring cognitive, linguistic, and cultural strengths to the classroom. The mathematical brilliance of students was taken as "axiomatic" (Gholson, Bullock, & Alexander, 2012). Rather than engage in debates about the educability of URM students, I assumed that students' mathematical brilliance could be taken as fact. I further assumed that, given the appropriate conditions, multilingual students could achieve on par with their monolingual peers (Barwell, Moschkovich, & Setati Phakeng, 2018; Goodrich, Thayer, & Leiva, 2021; NASEM, 2018). I also assumed that language proficiency was not prerequisite to but an outcome of effective content instruction (NASEM, 2018; Walqui & Bunch, 2019).

Dissertation overview

The study sought to understand students' developing mathematical arguments in a middle school, early algebra classroom that featured mathematical writing and oral conferencing. This study connected three areas of research. First, a body of research that shows how an early algebra approach that emphasizes reasoning, justifying, representing, and generalizing can help prepare students for success in algebra and the study of more advanced mathematics. Justifying is a core algebraic practice and prepares students for the larger mathematical process standard of proof and proving, (Leinwand, Brahier, & Huinker, 2014). Second, research in mathematical writing has shown that writing supports struggling students' mathematical understanding (Baxter, Woodward, & Olson, 2005), metacognition (Pugalee, 2004), and conceptual understanding (Countryman, 1992; Pugalee, 2005). Third, research in mathematics educational equity has shown that focusing on students' reasoning as they engage mathematical practices and leveraging students' cultural and linguistic strengths can cultivate the kind of learning environment where all students can thrive. This study connected these areas of research by choosing a setting where the cultivation of such a learning environment is possible and writing practices that emphasize audience and revision could be used to support students as they engage the core algebraic practice of justifying.

This study used a sociocultural view of mathematics learning and writing. I assumed that mathematics learning and writing activities are discursive and framed by knowledge, practices, and discourse, and that meaning is both situated and negotiated (Moschkovich, 2015). In this framework, language functions as a tool that mediates learning, organizing thinking and giving structure to meaning (Vygotsky, 1978).

I conducted the study at “Esperanza Elementary School” for a variety of reasons. First, Esperanza features a bilingual (Spanish-English) curriculum and purports to have strengths-based, positive views of multilingual students. Second, Esperanza primarily serves multilingual students (96% designated as Latinx), many of whom qualified for free or reduced lunch. These students are typically viewed as “not ready” to engage rigorous mathematics classes that emphasize discussion and writing, so the setting was ideal for this study.

The study focused on enacting lessons on mathematical writing and mathematical argument, and then conducting mathematical conferences with a sample of participants to examine how they revised their work. The lessons were adapted from existing curricular sources: Core Connections 3 (Dietiker, Baldinger, Kassarjian, & Shreve, 2013), published by CPM Educational Program and the Mathematics Assessment Project’s *Evaluating Statements: Consecutive Sums* lesson, published by the Mathematics Assessment Resource Service (MARS, 2015). In addition, I supplemented the lessons with tasks from Knuth, Choppin, & Bieda’s (2009) article *Middle school students’ production of mathematical justifications.* These tasks focused on number property and took the form of “number tricks”.

The study was guided by the following questions:

1. When asked to do mathematical writing and supported with conferencing in a remote context, what kinds of arguments do students make?
2. How did the mathematical arguments of individuals change over the course of a unit of instruction on generating, selecting, and justifying claims?
3. In what ways did students revise their mathematical arguments during conferencing?

In the chapters that follow, I elaborate a conceptual framework (Chapter 2) and design for the study (Chapter 3), present findings (Chapters 4 and 5), and discuss conclusions and implications (Chapter 6). In Chapter 2, I elaborate the conceptual framework and review three areas of research that inform the study: early algebra, equity in mathematics education, and mathematical writing. In Chapter 3, I describe the methodology, data, analysis, and other relevant study materials. In Chapter 4, “Burgeoning arguments: Shifts in generality and the use of examples”, I share findings relevant to the first research question. In that chapter, I summarize the kinds of responses students generated for different task types. I also describe how students expanded from writing descriptions to generating claims, how they expanded on the ways they used examples, and how many students went from writing no claims to generating claims in Lessons 1 and 2 and selecting and justifying them in Lesson 3. In Chapter 5, “Conferencing as a place for oral revision”, I present findings relevant to the second research question. I show how students revised arguments, procedures to test conjectures, and use of formal terms during the conferences. In Chapter 6, I summarize conclusions and describe some implications for theory and teaching practice.

Chapter 2: Conceptual Framework and Review of Literature

This study engages three bodies of literature: early algebra, equity in mathematics education, and mathematical writing. I summarize findings from each of these bodies of literature to frame this study and provide backing for some of the methodological and analytic choices made during the course of the study.

These three fields are related in ways that are mutually supportive. The early algebra approach began over concerns related to the "algebra problem" (Kaput, 2008), which refers to the historical approach of mathematics education that requires most students to master "shopkeeper arithmetic" and reserves the study of algebra for the "elite" (Kaput, 2008). The algebra problem has two aspects: 1) the entrenchment of an arithmetic-then-algebra curricular approach that overly emphasizes computation in the early grades and 2) the inequitable distribution of access to advanced mathematics, which disproportionately leads to failure and dropout among students from underrepresented and minoritized[1] (URM) communities.

Research on mathematics educational equity is relevant to each of these aspects. First, equity in mathematics education addresses *access* to rigorous curricula as a central concern (Gutiérrez, 2009 & 2012). Secondly, overemphasis on procedures does not support deep student learning (Hiebert & Grouws, 2007) but for students from URM communities, this is often the only kind of instruction and curricula to which they have access (Leinwand, Brahier, & Huinker, 2014); especially,

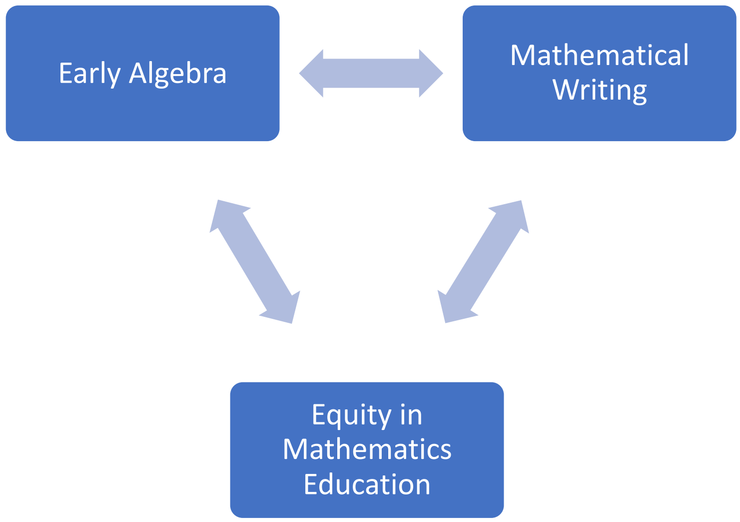
[1] Following McGee (2020b), I use the term "underrepresented and minoritized" (URM) to describe communities and their members that have historically been minoritized or marginalized through the exercise of social, political, or economic power. In mathematics education, this includes Black, Indigenous, Latinx, female, gender non-conforming, and cognitively diverse students.

multilingual students whose teachers frequently hold deficit-oriented beliefs (Moschkovich, 2007; Faltis & Valdés, 2016). While I more fully describe research on equity in mathematics education in a later section, I note the importance of creating equitable learning environments by enacting pedagogical approaches that include teaching for conceptual understanding, using language (both oral and written) to support mathematical reasoning, and embracing students' full linguistic repertoires as resources rather than obstacles (Moschkovich, 2013a; NASEM, 2018; TODOS & NCSM, 2018).

Research on mathematical writing is well positioned to provide pedagogical recommendations and practices that support deeper mathematical understandings and align with the goals of equity in mathematics classrooms. Mathematical writing is assumed to support metacognition as students organize, reflect, and revise mathematical understandings (Pugalee, 2004 & 2005). As such, mathematical writing can facilitate the construction of meaning through the use of language, which supports conceptual understanding and aligns with the goals and practices of equitable instruction. The mutually supportive relationship between the early algebra approach, research on mathematical equity, and the use of mathematical writing are illustrated in Figure 1, below.

Figure 2.1

Three Areas of Research as Mutually Implicative and Supportive.



In the first section of the chapter, I define ‘language’ and briefly describe its role in mathematics education. This description is important as language can be conceptualized in a variety of ways, each with a concomitant set of theoretical assumptions (Solano-Flores, 2010). I focus on sociocultural views of language and describe what is meant by *mathematical language* (Halliday, 1978; O’Halloran, 2015; Pimm, 1987; Schleppegrell, 2007).

In the second section, I describe a framework for equity (Gutiérrez, 2009 & 2012). Then, I summarize current research and describe some major issues, such as deficit views of students and how they can be used to pathologize URM students of mathematics. I focus on reasoning and justifying as practices that promote equity and understanding of mathematical concepts. I also focus on the use of language and writing to support the attainment of educational equity in mathematics classrooms, especially with regard to the mathematical practice of justifying.

In the third section, I define and describe early algebra and early algebraic thinking as engaging four core practices: generalizing, representing, justifying, and reasoning (Blanton et al., 2011; Kaput, 2008). In particular, I focus on reasoning and justifying. I also discuss research in *justification and proof* (Ferrini-Mundy & Martin, 2000, Leinwand, Brahier, & Huinker, 2014). This section describes the terrain for the content of the study and the instructional approaches enacted.

In the fourth section, I summarize research on mathematical writing. I describe major perspectives on research on mathematical writing and then focus on four *purposes* (Casa et al., 2016) for mathematical writing. I conclude this section by describing a variety of ways to support students' mathematical writing, including the use and consideration of students' linguistic resources, the rhetorical situation, conferencing, and revision.

A Note on the Role of *Language*

Language is a concept central to this study. In general, a researcher's view of language influences the research that they conduct explicitly and implicitly (Solano-Flores, 2010). As such, it is important to define 'language' and describe the ways that it has been conceptualized, particularly within mathematics education research. Below, I review some ideas from mathematics education literature and then describe how I conceptualize language in this study.

In their review article on language and communication in mathematics education, Morgan and colleagues (2014) discuss a definition of language that includes several aspects: first, language as *words*, verbal and non-verbal modes, or mathematical symbol systems; second, *national languages* (e.g., English, Spanish, Vietnamese, etc.); and third, phraseology, vocabulary, and syntax of particular groups—i.e., *registers* (Halliday, 1978). *Register* refers to language correlated with "occasions of use" (Ferguson, 1994). In mathematics, the unique registers are comprised of vocabulary, syntax, intonation, etc. that might vary from other uses of language (Schleppegrell, 2007; Sfard, 2007), as well as multi-modal forms of expression such as symbols and visuals (O'Halloran, 2015), and even their unique positioning when recorded on paper (Pimm, 1987). Each modality has the potential to enhance mathematical communication in unique ways (O'Halloran, 2015). Barwell and colleagues (2018) define *mathematical language* as distinct from language writ large. The term 'language' is described as including everyday language, generally, while the term *mathematical language* is described as including technical words and vocabulary, discursive practices necessary for participation in mathematical activities (including mathematical symbols systems and ways of using them), and lexico-grammatical features of mathematical texts (Barwell, Moschkovich, & Setati Phakeng, 2018).

Radford & Barwell (2016) critically reviewed research articles on language in mathematics education paying special attention to the role of power and the political nature of language. They considered everyday language in contrast to mathematical language, as they examined the major themes and theoretical approaches of the research reviewed. Two major pedagogical approaches, "transmissive" and "child-centered" models, focus Radford & Barwell's (2016) discussion of research on language. An important aspect of this review is the treatment and discussion of power. The discussion of power describes the different research articles in the review as "react[ing]" to the transmissive teaching model (Radford & Barwell, 2016, p.292). Child-centered models are also critiqued as foreclosing on student freedom because these pedagogies fail to question some of the more clandestine forces that shape schooling and learning, for example, curricula and their "political and economic orientation" (Radford & Barwell, 2016, p.305). This discussion is largely beyond the scope of my study; however, it is relevant to the use and role of curricula, as well as the learning environment. Curricula can position students in ways that affects their learning (Herbel-Eisenmann & Wagner, 2007). As such, Radford & Barwell (2016) argue for a new critical language in mathematics education research to better apprehend the tensions between (forms of) language as political "through and through" (p.305). The political nature of these tensions can be traced, in part, back to legacies of colonialism, which still exist and affect interactions among teachers, students, and curricula (Hsu, 2015 & 2017; Wang, Lang, Bunch, Basch, McHugh, Huitzilopochtli & Callanan, 2021). Consequently, mathematics education researchers bear responsibility to know their students (Moschkovich, 2013a) and be sensitive to linguistic histories and hierarchies (Faltis & Valdés, 2016) so that they can contextualize their work as *political acts* (Aguirre et al, 2017; Gutiérrez, 2013) and make informed choices that support the liberatory potential of a good mathematics education.

Following previous research, I define language to include everyday language (including national languages) and mathematical language (Barwell, Moschkovich, & Setati Phakeng, 2018). Aligned with Moschkovich (2015), this study assumes a sociocultural view of language where language is viewed as a *tool* that mediates learning (Vygotsky, 1978). Moreover, students' use of language to make meaning and shape knowledge is regarded as essential to reveal students' reasoning and communication. Language is taken to be "socioculturally situated in mathematical practices and the classroom setting" such that students can draw on multiple modes (e.g., oral, written, or others), multiple representations (such as objects, drawings, tables, or symbols), or multiple registers (including everyday language) to make meaning and communicate mathematically (Moschkovich, 2015). From this view of language, this study seeks to use multiple modes of language (focusing on writing) to support students as they develop capacities for mathematical communication (Aguirre & Bunch, 2012).

I further clarify that the inclusion of everyday language is not conceptualized in a binary with mathematical language. Aligned with the theories of Vygotsky, I consider everyday language as a valuable tool for sense-making (Vygotsky, 1986). Everyday language facilitates student movement into and through academic tasks and can serve as a vehicle to clarify and negotiate meaning of concepts (Bunch, 2014; Moschkovich, 2002; Nemirovsky, Barros, Noble, Schnepp, & Solomon, 2005). Thus, everyday language is an important intellectual resource for all students, but critical for multilingual students. It is critical for multilingual students because multilingual students are often viewed in deficit ways (NASEM, 2018) and those views might attenuate how freely students can use their everyday language (Wang et al., 2021), especially if their everyday language might be viewed as requiring “remedy”. I view the use of everyday language as a critical sense-making resource for students and an essential part of affirming student identities, rather than a problem to be fixed (Civil, 2006; Goffney, Gutiérrez, & Boston, 2018).

Equity and Mathematics Education

For many students taking algebra for the first time, typically in the eighth or ninth grades, the results are an “unmitigated disaster” (NRC, 1998). Algebra courses continue to serve as a “gatekeeper,” fraught with inequity (Martin, 2009; Stein, Kaufman, Sherman, & Hillen, 2011; Stephens et al., 2017). Moreover, it has been argued that mathematics functions similarly to Whiteness[2] (Gutierrez, 2017b) and mathematics education represents a “white institutional space” (Battey & Leyva, 2016; Martin, 2013) such that students from URM communities are positioned as outsiders within an “inhospitable culture” (McGee, 2020a, p.634). Indeed, the historical and ongoing exclusion of URM students and scholars of mathematics is a documented and persistent problem (Inniss, Lewis, Mitrea, Okoudjou, Salerno, Su, & Thurston, 2021). Supporting URM students of mathematics involves taking a research-based approach and addressing persistent deficit views of students and communities that educators might hold. In this section I review an equity framework

[2] Mathematics is hypothesized to operate as Whiteness when 1) contributions to mathematics made by all cultures are not acknowledged and 2) it is used as a standard by which others are judged (Gutiérrez, 2017b)

for mathematics education (Gutierrez, 2009 & 2012), describe recommendations for equitable classroom instruction, describe some of the resources that students bring to bear in the classroom (including multilingual students), and describe some of the ways deficit-based views of students do harm.

An Equity Framework

Gutierrez (2009) elaborates a framework to study and enact equitable practices along two axes: a dominant axis, which considers *access* and *achievement*; and a critical axis, which considers *identity* and *power*. The approach of the dominant axis seeks to effectively prepare students for participation in the economy and privileges the status quo. In common parlance, the approach helps students "play the game" without changing structures or curricula (Gutierrez, 2009 & 2012). It proposes to examine why we see the different outcomes vis-à-vis access and achievement. *Access* relates to tangible resources that facilitate participation in mathematics, for example "quality teachers, adequate technology and supplies*, rigorous curriculum, classroom environment that invites participation,* reasonable class sizes, tutoring, etc." (Gutierrez, 2009, p.5, emphasis added). *Achievement* refers to the tangible results, including patterns of course taking, standardized test scores, and participation in math courses at different educational levels. The approach of the critical axis seeks to "change the game" (Gutierrez, 2009) by acknowledging how access and achievement are mediated by one's position vis-à-vis race, class, gender, and language (Gutierrez, 2009). *Identity* relates to how students view the utility of mathematics in their lives and the extent to which they have opportunities to draw upon the cultural and linguistic resources they bring to the classroom. Finally, *power* involves how interactional practices and positioning are mediated by the structures that are either created within the classroom or imported from without, for example, how larger social hierarchies might be enacted in the microcosm of the classroom.

TODOS & NCSM (2019) expand from an equity framework to a *social justice* orientation. TODOS & NCSM (2019) distinguish the social justice stance as interrogating and challenging the roles power, privilege, and oppression play both *in* and *out* of the classroom. The social justice stance is expansive because it considers actions beyond those of individuals in the teaching/learning environment, but also community engagement practices and institutional structures. While the expanded considerations are beyond the scope of the study, they certainly influence the experiences of teachers and students as school structures and policies mediate access to algebra (Stein et al., 2011).

Equity Recommendations for the Classroom

Research in equity in mathematics education emphasizes instruction focused on mathematical reasoning and engagement with mathematical practices where students' cultural and linguistic repertoires of practice (Gutiérrez & Rogoff, 2003; Moschkovich, 2013a, 2013b) are taken as resources for learning.

The National Council of Teachers of Mathematics (NCTM) assert that with access to high-quality curricula and instruction (where teachers appreciate students' social and cultural contexts), "Persistent and unacceptable gaps" can disappear (Leinwand, Brahier, & Huinker, 2014, p.65). Some of the recommendations regarding curriculum and instruction include a call for teachers to *draw on community resources* to support student identities and promote engagement. Additionally, teachers should foster a sense of community and promote *mathematical communication* in multiple modes (such as talk and writing). The third recommendation is to focus on *mathematical practices*, including the use of tasks that allow multiple entry points and promote students to reason mathematically, at multiple levels, and devise their own strategies. For example, the practice of justifying is a mathematical practice, but can also be viewed as an *equity practice* (Bieda & Staples, 2020). Creating opportunities for students to justify their mathematical claims provides access to mathematical reasoning and deeper learning while fostering their mathematical agency (Bieda & Staples, 2020).

In other work, Moschkovich (2013b) used a framework for cultural relevance (Brenner, 1998) to describe equitable teaching strategies. Moschkovich defined equitable teaching practices as supporting *mathematical reasoning, mathematical discourse*, as well as *broadening participation* to increase opportunities to learn (i.e., improving *access*). To these ends, Moschkovich recommends honoring and leveraging students' repertoires of practice (Gutiérrez & Rogoff, 2003) by getting to know the students and their communities and using that knowledge to support instruction. This community knowledge can animate the framework through each of its three components: *cultural content, social organization,* and *cognitive resources* (Moschkovich, 2013b). Knowledge about the cultural and linguistic backgrounds of the students (and the communities they represent) can support instruction by, for example, providing familiar contexts for problems, familiar interactional or communicational practices, or using everyday language to support understanding of mathematical ideas and concepts.

Goffney, Gutiérrez, & Boston (2018) also describe teaching practices to *rehumanize* mathematics education for Black, Indigenous, and Latinx students. The term 'rehumanize' is used for two reasons: first, these practices are meant to counter *de*-humanizing teaching practices that students often encounter in mathematics classrooms, and secondly, throughout history, humanizing mathematics practices have existed within Black, Latinx, and Indigenous communities. The recommended practices would support experiences where students don't have to "leave their cultures and identities at the door" when learning mathematics. Goffney and colleagues (2018) suggest the following practices (p.167-168, emphasis added):

1. Teachers need to have a working *knowledge of the cultures and identities of the students* in their classes.
2. Teachers need to have a *working knowledge of the impact of structural barriers and systemic racism* for creating common dehumanizing learning experiences for Black, Indigenous, and Latinx students.
3. Rehumanizing teaching practices requires a *departure from common teaching practices* that privilege lecture and direct instruction, focus on diagnostic or evaluative approaches for remediation, and employ a color-blind or neutral approach where Black, Latinx, and Indigenous students "leave their culture at the door," students are grouped based on their perceived ability, and group

membership provides expanded or constrained opportunities to learn and do mathematics.

4. Rehumanizing mathematics includes *making connections between the mathematics content and students' experiences and interests*.
5. Rehumanizing mathematics uses a *variety of teaching practices*, many of which focus on co-constructing mathematics knowledge through collaborations where authority is shared between the teacher and students, broadening types of participation, using multiple strategies for supporting students to demonstrate their mathematical competence and smarts in the public space of the classroom, and facilitating productive work on tasks and projects with multiple entry points and multiple layers that promote deep thinking.

Each of the above recommendations, from NCTM, Moschkovich (2013b), and Goffney et al. (2018), emphasizes that teachers and instruction should have knowledge of the cognitive, cultural, and linguistic resources that students bring from their communities so that they can connect it to mathematical content. Moreover, instructional practices should broaden possibilities for participation by moving away from traditional teaching methods in favor of placing mathematical authority and agency in the hands of students as they engage mathematical practices (for example, see Bieda & Staples, 2020). In addition, teachers and instruction should emphasize mathematical discourse through culturally sensitive social arrangements and focus on mathematical reasoning.

Reasoning supports conceptual understanding and is considered part of effective mathematics teaching (Leinwand, Brahier, & Huinker, 2014). In the context of equity, teachers should consider students' linguistic and cultural repertoires of practice as resources not obstacles (Gutiérrez, Morales, & Martinez, 2009; Moschkovich, 2013b). This includes their everyday, or colloquial, language practices, which can communicate mathematical meaning (Moschkovich, 2002; Nemirovsky, Barros, Noble, Schnepp, & Solomon, 2005) and serve as raw material with which students construct meaning. Engaging students' repertoires of practice contributes to the creation of a Third Space (Gutiérrez, Baquedano-Lopez, & Tejada, 1999) wherein students can use their diverse cultural and linguistic resources to support learning; therefore, contributing to an equitable learning environment (Ramirez & Celedón-Pattichis, 2012).

Strengths and Resources

All students arrive in the classroom with a variety of resources available to them, however they are not all valued the same (Civil, 2006). Even teacher noticing of student strengths can be mediated by culture (Louie, 2018). Equitable teaching practices rest on (a) the support of students' reasoning, conceptual understanding, and discourse and (b) the broadening of possibilities for participation (Moschkovich, 2013a). The problem is that discourse and participation therein are not possible if the resources available to students are not valued by the teacher or peers, or do not match narrow conceptions of mathematical proficiency (for example, correct use of vocabulary). Some of the resources that are available to students exist in the classroom, for example objects, drawings, graphs, or gestures, or are imported from outside of the classroom, for example everyday language practices and experiences (Moschkovich, 2013a). Students can effectively use everyday language to discuss and clarify technical terms (Moschkovich, 2002; Nemirovsky et al., 2005). Nemirovsky and colleagues emphasize that in the "tensions and ambiguities" that arise when using everyday and technical language, the mathematical register is brought to life for the students (Nemirovsky et al., 2005, p.199).

It is important to note that the strengths that students bring to the classroom can, and should, be assumed (Gholson, Bullock, & Alexander, 2012; Martin, 2019). Teachers and researchers, therefore, have a responsibility to notice these strengths and support students in making use of them to learn mathematics. In order to do this, three recommendations include: notice student strengths, recognize relevant mathematical practices, and expand on what counts as mathematics (Huitzilopochtli, Foxworthy Gonzalez, Moschkovich, McHugh, & Callanan, 2021). When teachers can not recognize the strengths that students bring to the classroom as such, then it becomes easy to render those strengths invisible and describe students only in terms of what they "don't know" or "can't do".

Rendering Resources Invisible

Given the historical and ongoing exclusion of URM students and scholars from the field of mathematics (Inniss et al., 2021) the gatekeeping function that algebra courses in middle and high school have served come as no surprise and can be viewed as an extension of that exclusivity. The gatekeeping function of algebra is fundamentally an issue of *access* to “rigorous curriculum” and “a classroom environment that invites participation” (Gutierrez, 2009; Gutierrez, Morales, & Martinez, 2009; Morton & Riegle-Crumb, 2019; Tate & Rousseau, 2002). Equity research that is relevant to both gender (Boaler & Sengupta-Irving, 2006 & 2016; Fennema, 1974; Leder, 2019) and race (Martin, 2009) locates achievement issues among students from URM communities in the *learning environment* rather than the students. That is, students from URM communities are less likely to have access to the kind of environments that could support effective teaching and learning (Gutierrez, Morales, & Martinez, 2009). Past reform efforts made use of “colorblind” approaches; however, mathematics education is itself a racialized experience (Martin, 2006). For example, Martin (2006) documents how race is “central” when African Americans construct meaning from their experiences with mathematics. According to one of the interviewees in his study (2006), “Raheem” recalled being placed in “regular math class” instead of algebra in the eighth grade despite scoring in the 90th percentile in seventh grade. Raheem connected his placement as an “example of the fact that Black children, even when you do well, your educational future is not planned out properly for you…” (Martin, 2006, p.211). Raheem recalled eighth grade as the first year he “hated” math. “Colorblind” approaches to policy obscure the historical and structural ways that racism is enacted, thus exacerbating the problem (Martin, 2006; Choi, 2008).

Deficit and “readiness” arguments have also been used to explain or justify why students from URM communities should not have access to algebra early on

(Tate & Rousseau, 2002). Deficit views focus narrowly on what students do not have or cannot do (Valencia, 2010). The perceived failures of individuals are attributed to presumed deficiencies of the individual students or of their families and communities. By focusing narrowly on individuals, deficit thinking obscures structural factors. Such deficit views negatively impact students' classroom experiences, course placements, and opportunities to learn STEM, and this intellectual, symbolic, and epistemological violence can have material consequences on student outcomes (Martin, 2019; NASEM, 2018). Martin's (2009) discussion of race in mathematics education discusses the normalization of Whiteness and the deeming of Black, Indigenous, and Latinx students as "changeworthy" (Martin, 2009). Martin argues that the codification of deficit frames, in both research and policy initiatives "*necessarily* positions researchers, policy makers, and practitioners to assume and accept beliefs about inferiority" of particular groups of students (Martin, 2009, p.316). Similarly, Boaler & Sengupta-Irving reviewed research on mathematics and gender and found that researchers either pathologized aspects of women's engagement with mathematics, i.e., either the women's attitudes or affect were assumed to precipitate inferior achievement or hypothesized that genetic differences accounted for perceived deficits in women's performance in mathematics. Boaler & Sengupta-Irving noted that "boys' achievements, participation and behaviors were implicitly positioned as 'normative' and the benchmark against which girls were understood" (p.11). Unproductive beliefs related to readiness extend to language considerations as well. For example, many policies and practices are based on the erroneous belief that language must be mastered before students can engage with rigorous content or that language is learned separately from content (Faltis & Valdés, 2016). The normalization of monolingual speech, maleness, and Whiteness is indicative of the ways that mathematics can operate as Whiteness (Gutierrez, 2017a) and why mathematics teachers, in particular, are called on to develop *political knowledge* (Gutierrez, 2013, 2017b). Political knowledge and explicit attention to equity is important because "equity may not be a natural by-product of reform teaching" (Tate & Rousseau, 2002, p. 290).

Early Algebra and Early Algebraic Thinking

In this section, I review definitions for early algebra and early algebraic thinking. I describe early algebra in terms of core practices (Stephens, Ellis, Blanton, & Brizuela, 2017), and pedagogical approaches (Carraher & Schliemann, 2007). I also describe justification in terms of proof and proving, a related topic.

Recall that the "algebra problem" refers in part to the high dropout and failure rates of students taking algebra for the first time, especially students from URM communities (Kaput, 2008). The *early algebra* approach to algebra education refers to efforts that seek to remedy the algebra problem by infusing early mathematics education (typically from early elementary through middle school years) with algebra related instruction to support success in high school algebra (Carraher, Schliemann, & Schwartz, 2008). In the broadest sense, early algebra represents a significant departure from traditional algebra instruction. Whereas traditional algebra instruction typically involves a year-long course with a strong symbolic orientation (Kieran, 2007), early algebra builds on the background contexts of problems, introduces formal notation gradually, and interweaves existing topics of early mathematics (Carraher, Schliemann, & Schwartz, 2008) in instructional environments that focus on the development of early algebraic *thinking* in elementary and middle grades (Stephens et al., 2017). The development of early algebraic thinking is accomplished by engaging students in four core practices of algebra: generalizing, representing, justifying, and reasoning (Blanton, Levi, Crites, & Dougherty, 2011; Kaput, 2008; Stephens et al., 2017).

I use Kaput's (2008) framework for algebraic knowledge and activity, which describes algebra in terms of two core aspects and three strands that embody them. The first aspect views algebra as stating generalizations with symbols. The second aspect of algebra involves reasoning with the generalizations (Kaput, Carraher, & Blanton, 2008). These aspects are embodied in three strands that focus on abstracting of computations, functions and relations, and the application of algebraic modeling languages (Kaput, Carraher, & Blanton, 2008). Aligned with other early algebra research, this study shifts the emphasis from "algebra" as list of topics to "algebraic thinking" as a set of thinking practices to engage indeterminate quantities in analytic ways (Radford, 2011).

Blanton and colleagues (2011) describe algebraic thinking as centering on four core practices: generalizing, representing, justifying, and reasoning. In this study, justifying features centrally. The practice of justifying involves students using informal mathematical arguments to "construct mathematical certainty" about generalizations (Stephens et al., 2017). Justifying is shaped by the mathematical content. For example, students' efforts to justify in a geometry class are very different when compared to an algebra class.

Generalizing refers to the act of creating a symbolic object that represents a multiplicity (Kaput, Blanton, & Moreno, 2008) and includes a focus on individual students as well as generalization achieved as a "collective act" (Stephens, et al., 2017). Generalizing is widely recognized as part of algebraic thinking (Stephens, et al., 2017). The use of symbols plays a role with representing, the second practice, as well. For example, consider the problem below:

> *Chair and Leg Problem*
>
> *Suppose that you have some chairs, and each chair has 4 legs. How would you describe the relationship between the number of chairs and the corresponding number of chair legs?* (Blanton et al., 2011, p.9)

The task requires students to consider a general relationship between an "unknown but varying number of chairs and the corresponding number of chair legs" (Blanton et al., 2011, p.9). The resulting generalization can be represented in several ways, including symbolically. In early algebra, symbols and symbolizing are "arguably as important" as generalizing (Stephens, et al., 2017). Early algebraic instruction defines symbol systems broadly and includes a variety of semiotic systems, for example, everyday language, graphs, tables etc. (Carraher & Schliemann, 2007; Kaput, 2008). The relationship in the *Chair and Leg Problem* could be represented as:

- "The number of chair legs is four times the number of chairs"

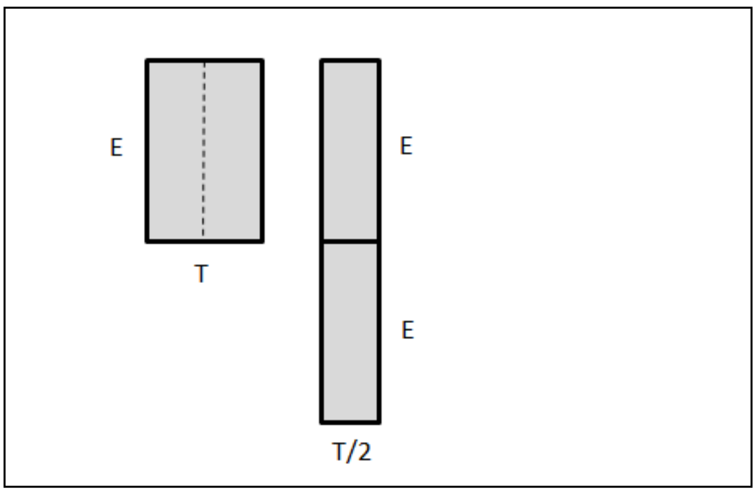
- $l = 4 \times c$
- $l = 4c$
- or using a table, graph, etc. (Blanton et al., 2011, p.9)

Representing, justifying, and reasoning are considered practices of algebraic thinking "in the service of *actions with or on generalizations*" (Stephens, et al., 2017, p. 388, original italics). That is, absent the generalizations or actions upon them, the other core practices are not necessarily algebraic. Consider the analogy that the movements of swimmers only qualify as "swimming" if they are propelling themselves through water. Tasks presented in this study support student engagement with these core practices.

Early algebraic instructional practices utilize a multiplicity of symbolic systems that are not, necessarily, conventional algebraic notation systems for representing and justifying (Brizuela & Ernest, 2008; Cooper & Warren, 2011; Russel, Schifter, & Bastable, 2011; Ellis, 2011; and Blanton & Kaput, 2011). Some researchers have capitalized on students' use of representation-based proofs in 2nd – 6th grade classrooms so that students can demonstrate algebraic thinking (Russel, Schifter, & Bastable, 2011; Schifter, 2009). It is important for younger students to develop use of drawings, models, or story contexts to prove mathematical assertions (Russel, Schifter, & Bastable, 2011). Representation-based proofs should demonstrate generalizing and how the claim follows the premise "in the structure of the representation" (Russel, Schifter, & Bastable, 2011, p.57). For example, when finding the product of 18 x 45, a student provided figures (see Figure 2.3) when asked to prove why the strategy of halving and doubling works. ‘Halving and doubling’ refers to a procedure that can simplify multiplication where one factor is halved and the other doubled to compute the product more easily. For example, 18 X 45 could take more time to compute than 9 X 90 (which has been halved/ doubled).

Figure 2.2

Student’s Representation-based Proof of Halving and Doubling (Schifter, 2009).



In the discussion of students’ use of notation with meaning, the authors stated that “Once students have considerable experience stating generalizations in words and justifying these general claims by using representations of the operations, they have images and explanations to which they can connect algebraic symbols” (Russel, Schifter, & Bastable, 2011, p.64). What is important here is that the representations (concrete and visual) and the words (spoken and written) help students bridge to the symbolic world of algebra. The authors go on to state that teachers should introduce notation at a point when “students have *already* articulated their ideas in words and images" (Russel, Schifter, & Bastable, 2011, p.66, emphasis added). This allows them to maintain meaning for the symbols and use resources they bring to the classroom.

Reasoning also plays a key role in early algebra. The National Council of Teachers of Mathematics (NCTM) described *reasoning and proof* as a process standard, i.e., a way of "acquiring and using content knowledge" (Ferrini-Mundy & Martin, 2000, p.29), that involves three activities: 1) making and investigating conjectures, 2) constructing and evaluating mathematical arguments and proofs for oneself and others, and 3) selecting and using various *types* of reasoning and methods of proof (p.57-58). 'Proof' refers to a "formal way of expressing particular kinds of *reasoning* and *justification*" (Ferrini-Mundy & Martin, 2000, p.56, emphasis added). In other work, *adaptive reasoning* has been defined as the "capacity for logical thought, reflection, explanation, and justification" (Leinwand, Brahier, & Huinker, 2014; NRC, 2001, p.116). NRC (2001) adds that adaptive reasoning involves thinking logically about relationships among concepts in a given situation (p.129). This broad definition can include informal explanation and justification, formal proof and deductive reasoning, even intuitive and inductive reasoning (NRC, 2001, p.129). Using informal reasoning to "connect new learning with prior knowledge" is part of the foundation for effective mathematics teaching (Leinwand, Brahier, & Huinker, 2014, p.9) and NCTM recognizes that class discussions support such reasoning, especially in the case of younger children (p.58). Reasoning has also been embedded in the Common Core State Standards (CCSS) in two of the Math Practice (MP) Standards[3]:

- MP2. Reason abstractly and quantitatively.
- MP3. Construct viable arguments and critique the reasoning of others.

In early algebra, reasoning has been identified as integral to the study of early mathematics (Carraher, Schliemann, & Schwartz, 2008). In typical early algebra instructional practice, students are positioned by teachers such that they do not "draw conclusions solely through logic and syntactical rules. Instead, they use a mix of intuition, beliefs, and presumed facts coupled with principled *reasoning* and *argument*" (Carraher, Schliemann, & Schwartz, 2008, p.236, emphasis added). This study centers the practice of reasoning to support students as they lay foundations to do formal proof and proving in algebra and beyond by engaging in mathematical argument.

Mathematical arguments and proof are difficult for students of mathematics to learn (Blanton, Stylianou, & David, 2009; Healy & Hoyles, 2000). Part of the difficulty is found in the need for students to learn a new *basis of belief* (ECEMS, 2011):

> Mathematical thought concerning proof is different from thought in all other domains of knowledge, including the sciences as well as everyday experience; *the concept of formal proof is completely outside mainstream thinking*. Teachers of mathematics at all levels (mathematicians, mathematics

[3] Available at: http://www.corestandards.org/Math/Practice/

educators, schoolteachers, etc.) thus require students to acquire a new, non-natural basis of belief when they ask them to prove (Fischbein, 1982). We all need to be acutely aware of this situation. (p.51, original emphasis)

In the construction of a new basis for believing, the use of students' full linguistic repertoires, when reasoning and justifying, is essential to the development of necessary understandings.

In this study, I use Harel & Sowder's (1998) notion of *proof schemes* to frame analysis of the arguments. Typically, students make use of three classes of proof schemes: external conviction, empirical, and analytical (Harel & Sowder, 1998). Absent formal instruction, empirical proof schemes are likely to be the only ones that students possess (Harel & Sowder, 1998). External conviction proof schemes include ritual, authoritarian, and symbolic arguments. In the ritual external proof scheme, students' doubts are removed by the *appearance* of a valid proof or doubts are not removed because the proof did not appear "proof-*y*" enough, for example if a proof does not make use of symbols. The authoritarian external proof scheme typically relies on a teacher or textbook as a source for mathematical assertions. In this proof scheme, *what* is true is considered important, i.e., not *why* it is true. The symbolic external proof scheme involves the manipulation of symbols without considering the meaning of such manipulation.

An empirical proof scheme is one where conjectures are "validated, impugned, or subverted by appeals to physical facts or sensory experiences" (Harel & Sowder, 1998, p.252). There are two classes of empirical proof scheme: perceptual and inductive empirical proof schemes. Perceptual empirical arguments rely on "rudimentary mental images". Inductive empirical proof schemes can use calculated examples, without any warrants or backing, and are taken as demonstrations of a conjecture's validity or counterexamples to demonstrate invalidity.

Analytical proof schemes seek to settle a conjecture in general. Analytic proof schemes can be transformational or axiomatic. Transformational proof schemes are ones where "justifications are concerned with general aspects of a situation and involve reasoning oriented toward settling the conjecture in general" while axiomatic proof schemes are reflected in students who are comfortable working with mathematics as an organized body of knowledge "so that subsequent results are logical consequences of preceding ones" (Harel & Sowder, 1998).

While some researchers consider proof schemes in a hierarchical trajectory where empirical proof schemes are considered inferior to analytic ones (e.g., see Knuth, Choppin, & Bieda, 2009, or Küchemann & Hoyles, 2011), I consider proof schemes as a constellation where competencies in each proof scheme can be developed independently, as well as in relation to one another. This perspective aligns with Harel & Sowder's (1998) position on proof schemes and the importance of setting, context, and content:

> It is important *not* to regard the taxonomy in a hierarchical, single-niche sense. A given person may exhibit various proof schemes during one short time span, perhaps reflecting her or his familiarity for, and relative expertise in, the

> contexts, along with her or his sense of what sort of justification is appropriate in the setting of the work. (p.277, emphasis added)

The proof scheme framework supports a strengths-based approach because I examine what students did and said and then I describe the mathematical merit in the context of a particular mathematical task. I do not compare this knowledge to what "experts" do or in a hierarchical learning trajectory that places student knowledge as inferior. As Harel & Sowder (1998) noted: "…it is the individual's scheme of doubts, truths, and convictions, in a given social context, that underlies our characterization of proof schemes" (p.244). This is important because *social context* is acknowledged as playing a role in how a student's "doubts, truths, and convictions" are expressed.

Mathematical Writing

Mathematical writing is central to this study and is defined as "any representation (Ferrini-Mundy & Martin, 2000) that students record on paper (including prose, symbols, and other visuals) resulting from prompts that press to include one's mathematical reasoning" (Cohen, Miller, Casa, & Firmender, 2015). Due to the COVID-19 pandemic, study participants were not able to attend in-person schooling. Consequently, I expanded the definition of mathematical writing to include other inscriptions, such as visuals and text that students record and generate in digital formats (e.g., type-written, or other responses recorded in online forms, "chat," or "whiteboard" features of online meeting platforms). Each of these modalities (prose, symbols, and visuals) has potential to contribute unique meaning or nuance to mathematical communication (O'Halloran, 2015). Consequently, mathematical writing also includes algebraic symbolic writing, such as “2x + 3 = 15”. This kind of symbolic writing is important to include as it uniquely communicates the kind of generality that reveals mathematical structure (Grabiner, 2012; Sfard, 1995). For example, consider how “A + B = B + C” captures a multiplicity of examples while “4 + 7 = 7 + 4” describes a single example of the commutative property of addition.

This study integrates different approaches to mathematical writing and seeks to support both conceptual understanding of mathematical content, as well as the act of writing as a social practice (Cope & Kalantzis, 2009; Lea & Street, 2006). Lampert & Cobb (2003) noted that mathematical writing activities and instruction have historically been approached from two perspectives: *writing to learn* and *learning to write. Writing to learn* is hypothesized to support metacognition (Clarke, Waywood, & Stephens, 1993; Pugalee, 2004) and serve as a *tool* to organize, clarify, and revise thinking (Gillespie, Graham, Kiuhara, & Hebert, 2014). For example, teachers might assign transactive writing, which informs, explains, describes, or persuades a reader (Pugalee, 2005) and provides opportunities for students to reflect on understandings and processes related to mathematics. Process writing in support of “writing to learn” can include writing assignments such as “freewrites”, learning logs, auto-biographies, journals, stories, poems, etc. (Countryman, 1992). On the other hand, l*earning to write* perspectives consider how vocabulary, syntax, genres, and multi-modal symbol systems unique to mathematics require instruction for students to use them to make meaning mathematically (Morgan, 1998; O’Halloran, 2015; Schleppegrell, 2007).

Casa and colleagues (2016) elaborated a framework for mathematical writing that appears to integrate different approaches and focused on four purposes for writing in mathematics: Exploratory, Informative/ Explanatory, Argumentative, and Mathematically Creative. This framework reflected aspects from both *writing to learn* and *learning to write* approaches. For example, *exploratory* writing was characterized as sense-making and includes brainstorming as well as efforts to resolve confusion. *Informative/ explanatory* writing included descriptions, representations (and connections between them), and definitions. *Argumentative* writing involved construction and critique of arguments, as well as conjectures and justifications. *Creative* writing documented original ideas, problems, and solutions, demonstrating flexibility in thinking and making connections within and outside of mathematics. It did not involve the kind of creative writing that is generally not typical in mathematics contexts, such as poems or stories.

In this study, I introduced pedagogical activities that leveraged the different purposes for mathematical writing in ways that support the process of generating a mathematical argument. Students had opportunities to use exploratory writing, for example, as they completed the 'Convince Yourself' writing activity. This mirrors what Harel & Sowder (1998) might refer to as the "ascertainment phase of a proof scheme" (p.279). Likewise, students used explanatory/informative writing in the 'Convince a Friend' writing activity and argumentative writing in the 'Convince a Skeptic' writing activity. This mirrors what Harel & Sowder (1998) refer to as the "persuasion phase" (p.279). Students also had opportunities to use creative mathematical writing (Casa et al., 2016) as they reflected upon their arguments, made connections, and documented original ideas in their journal writing. In order to explore the promise of writing to support conceptual understanding, I looked for changes in the students' arguments and writing in terms of process and object views of algebra, generalization, use of symbols, and other considerations. For example, in a pilot study conducted previously, I found that one group of students reasoned with an object view of algebra while another group reasoned with a process view of algebra (Moschkovich, Schoenfeld, & Arcavi, 1993).

Richness of Context

A central tenet of early algebra education is to support the development of meaning and understanding by grounding tasks in a rich context (Carraher & Schliemann, 2007). Richer contextualization of problems also supports multilingual students to engage in mathematical writing tasks (Banes, 2019); however, the richness of the context can be developed in different ways. One way is to ground problems with direct experience with quantities, which could facilitate quantitative reasoning (Ellis, 2011). Representing problems in everyday experience adds richness that can serve as common ground for multilingual students to engage problems (Dominguez, 2011). Dominguez found that multilingual students were "more predisposed" to use one language over another, given a different purpose. For example, bilingual students were more likely to use Spanish during *exploratory* or joint activity and English for tasks requiring more individual approaches and "traditional" schoolwork (Dominguez, 2011, p.324-5). Dominguez (2011) also found that students were “more involved” with others’ ideas in Spanish (p.325). This finding corroborates what others have found, albeit in different disciplinary contexts—like history (Bunch, 2014). When problem contexts are based in students’ everyday experiences and they have opportunities to discuss, argue, and negotiate (i.e., explore solutions), then they could have increased opportunities for meaningful learning to take place.

The richness of a problem context can also come from the features of tasks as they are presented. In Banes’ (2019) dissertation, several features facilitated the kind of mathematical arguments that were conceptually rich and complete. First, tasks that ask students to justify a given claim or qualify a claim and justify are easier for students to engage. Examples and details of each type of task will be provided in Chapter 3 (Methodology) and discussed in detail in the chapters that discuss findings (Chapters 4 and 5). Briefly, these tasks invite students to justify a claim that is either:

- *Given* to the student.
- *Selected* by the student using a non-neutral qualifier, such as:
    - Agree/ Disagree
    - Always/ Sometimes/ Never True
- *Open*. To be generated by the student.

Banes (2019) suggests a progression of tasks to scaffold students to make claims with increasing independence: first, given claim tasks, then selected-claim tasks, and then open-claim tasks. These categorizations are presumed to scaffold the process to generate a claim by modeling precise claims and providing opportunities to discuss the claim statement. This support is important because students' capacities for justifying are not the same as their capacities for generating claims. Data from a pilot study to this dissertation suggest that students' efforts to generate claims and justify them should be treated separately.

The second feature of a good task is to include substantial details for the rhetorical situation. The rhetorical situation should include a clear audience, purpose, and the requirements for a "quality" response (Banes, 2019, p.40). Notably, Banes suggests that the purpose and audience can be fictional. If the purpose, audience, etc. are clear, then the students "might be more likely to produce a strong mathematical argument" (Banes, 2019, p.40). I interpret this to mean that 'realia' is not as important to students as 'being real', i.e., being transparent with the situational demands and the expectations for performance. The third feature is to clearly identify the audience. The fourth feature is to include richer contextualization and "teaching" situations with many avenues to explain problem situations, which is reminiscent of approaches that seek to exploit "entry points" for the study of algebra (Carraher & Schliemann, 2007). Banes (2019) suggests that the context should be meaningful to students, similar to Dominguez's (2011) findings, and be situationally complex.

Per Banes' (2019) recommendation and to the extent possible, the cooperating teacher and I organized a progression of tasks to scaffold students' increasing independence. The first two weeks of the intervention emphasized tasks that ask students to justify a given claim, for example, "Prove the following statement: The sum of four consecutive numbers is always even" (my example). The next two weeks broadened the emphasis to include tasks that ask students to qualify a claim and then justify, and from that point forward I incorporated tasks that ask students to generate a claim and then justify. The richness of the tasks focused on a rhetorical richness. Elaboration of the rhetorical situation included a context meaningful to the students and clearly defined audience (Banes, 2019; Casa et al., 2015), which is hypothesized to support students as they engage writing as a social practice (Cope & Kalantzis, 2009; Lea & Street, 2006). The “Convince” form was used to suggest oneself, a friend, and a skeptic as possible audiences, each with a different set of social and rhetorical expectations that might stimulate different kinds of mathematical writing and communication.

Supporting Student Writing

In this section I describe the ways that the cooperating teacher and I drew on a number of pedagogical principles related to research on mathematical writing. For example, the cooperating teacher and I provided opportunities for students to use their full linguistic repertoires as resources (Moschkovich, 2013a, 2013b) and provided a linguistically sensitive social environment (Ramirez & Celedon-Pattichis, 2012), which included using the *resources* form as a source for potentially familiar mathematical contexts. The cooperating teacher and I used an instructional framework for “amplifying” curriculum (Walqui, 2019). The framework suggests organizing units of instruction and individual lessons with coherent instructional goals and considering three “moments” in each lesson: Preparing Learners, Interacting with Text (or Concept), and Extending Understanding (Walqui, 2019).

The cooperating teacher and I made use of resources (such as the Convince Form) to scaffold participation (Walqui & Bunch, 2019). The structure of the scaffold is presumed to routinize student participation and "create predictability" when students consider audience and engage with argument (Walqui & Bunch, 2019).

We provided additional support by providing opportunities for conferencing, revision, and writing in math journals. Conferencing is a productive practice to support students' writing (Hillocks, 2006; Newkirk, 1989), but it is especially so for multilingual students, who might have difficulty with feedback offered in only one mode (Aguirre & Bunch, 2012). Conferencing and revision have been shown to enhance student understanding (Swain, 2009) and mathematics education research has called for additional opportunities for students from URM communities (including multilingual students) to review and revise their arguments (Moschkovich, 2018; Ramirez & Celedon-Pattichis, 2012). Using mathematical journals serves two purposes. First, the use of journals supports metacognition and learning (Bangert-Drowns, Hurley, & Wilkinson, 2004; Pugalee, 2004) by soliciting explanations and other writing about the learning process itself from students. The use of journals is specifically recommended to support multilingual learners as an exit task (Aguirre & Bunch, 2012) or as part of larger writing to learn strategies (Ortmeier-Hooper, 2013). Secondly, the journals provided ethnographic data and student feedback regarding the utility of the writing activities. This data was used to inform instructional decisions from day-to-day and lesson-to-lesson.

Conclusion

In this chapter, I described three areas of research, mathematics equity, early algebra, and mathematical writing, as mutually supportive. Mathematics educational equity is central to the study, but an often-overused term. As such, I would like to clarify that I chose the site and teacher with a specific notion of equity in mind. Equity must also account for a socially just educational opportunity. Mathematics education researchers and practitioners, by moving beyond "diversity-equity-inclusion" rhetoric, can create possibilities for mathematics education that are not circumscribed by economic or political projects, such as American economic or military dominance. Asking "What mathematics? For whom? And for what purpose?" can help mathematics education to engender socially just possibilities (Aguirre, Mayfield-Ingram, & Martin, 2013). In short, how can mathematics education practitioners and researchers support the kind of educational practices that place mathematics as a tool in the hands of students (and the communities they represent) for their own purposes? I selected Esperanza Elementary because it embodied many of these ideals.

Chapter 3: Methodology

The purpose of this qualitative study was to understand students' developing mathematical arguments in a middle school, early algebra classroom that featured mathematical writing and oral conferencing. Mathematical writing is a practice that rarely occurs in situ (Applebee & Langer, 2011; Mastroianni, 2013; Powell et al., 2017) and the practice remains ill-defined (Casa et al., 2016). Moreover, the focus on language and writing is supported by research on language diversity that suggests that the conditions for the success of students from non-dominant communities are "a matter of equity, intimately bound up with deep-running social divisions in society" (Barwell, Moschkovich, & Setati Phakeng, 2017). Under the right conditions, bilingual, multilingual, and second language learners can learn *at least* as well as their monolingual counterparts (Barwell, Moschkovich, & Setati Phakeng, 2017; Goodrich, Thayer, & Leiva, 2021). This study followed this recommendation and considered not only three semiotic systems of mathematical discourse (i.e., everyday language, mathematics symbols systems, and visual displays), but multiple modes (such as oral and written text), multiple representations, and different types of written text (Barwell, Moschkovich, & Setati Phakeng, 2017). This study examined students' explanations and arguments and described the development of their mathematical reasoning when supported by mathematical writing activities. The research questions that guided the study include:

1. When asked to do mathematical writing and supported with conferencing in a remote context, what kinds of arguments do students make?

2. How did the mathematical arguments of individuals change over the course of a unit of instruction on generating, selecting, and justifying claims?

3. In what ways did students revise their mathematical arguments during conferencing?

This qualitative study focused on supporting students' construction and refinement of mathematical arguments with mathematical writing practices. The writing practices were part of the instructional design and iteratively modified as tools to support learning. I assumed a socio-constructivist view of learning, which drew on the theories of Vygotsky and Piaget. Students are assumed to play an active role in the construction of knowledge and language (alongside other tools) is assumed to mediate learning and interactions (Vygotsky, 1978). Moreover, the study considered students' full linguistic and mathematical repertoires of practice as resources for meaning making and learning, not obstacles to overcome (Gutierrez & Rogoff, 2003; Moschkovich, 2013a). This study focused on the classroom-level and the teaching unit (Van den Akker, 2013; as cited in Prediger, Gravemeijer, & Confrey, 2015) and theory generation. The aim of the study was to focus on the learning processes associated with the use of mathematical writing to support mathematical arguments with middle schoolers studying early algebra and, specifically, the practice of justifying.

I assumed an ethnographic stance, framed by a naturalistic paradigm (Moschkovich, 2019; Moschkovich & Brenner, 2000). The ethnographic stance can be summarized as embodying three principles: 1) consider multiple points of view, 2) study cognitive activity in context, and 3) connect theory verification and theory generation (Moschkovich, 2019; Moschkovich & Brenner, 2000). To consider multiple points of view means, for example, considering student learning "in their own terms", and not comparing student knowledge to expert knowledge (Moschkovich & Brenner, 2000). Studying the cognitive activity in context means discussing setting and context. In accordance with the naturalistic paradigm, which frames the ethnographic stance, I refer to the *setting* as "the physical and social environment" and the *context* as the "relationship between the setting and how participants interpret the setting" (Moschkovich, 2019, p.64). Because the "algebra problem" reflects inequitable educational practices that have historically limited access to rigorous curricula for students from underrepresented or minoritized (URM) communities, the ethnographic stance is important to illuminate how the context might be unique for the study participants. The mathematics classroom can be "inhospitable" to URM students (McGee, 2020a).

In order to understand aspects of the context, I observed the classroom for a unit of instruction, conducted open-ended interviews with students (i.e., student conferences), and I used data gathered from the students' journal writing and "quick writes" to ascertain students' views of the writing activities and inform instructional decisions as iterations of the intervention proceeded. I also taught the unit on mathematical argument as a "guest teacher". I taught four lessons over a total of eight class sessions. Other data that informed instructional decisions came from research team meetings (which included the cooperating teacher and researcher) and other data sources, such as the "Mathematical Resources Inventory" (Appendix A) that students completed as part of the study and video data of whole-class interactions.

Setting and Participants

The data were collected at "Esperanza Elementary School"[4], a rural charter school in Northern California's Central Valley. I selected the site and teacher based on some promising features. First, the site heterogeneously grouped students, i.e., without regard to "English language fluency" (personal communication with cooperating teacher). This is important because it reflects a move away from segregation and sorting practices that are known to erode educational equity. Secondly, the cooperating teacher stated a firm belief in the utility and benefits of using mathematical discussions and writing. Third, the cooperating teacher (and school ethos—as a bilingual school) supported a multilingual approach to instruction and embraced the linguistic strengths that students bring to the classroom. Students were unabashedly themselves and I hope to render their mathematical brilliance visible.

Esperanza Elementary is a K-8, dual-immersion school whose mission includes five core values: academic achievement, parents as partners, student pride and citizenship, *being bilingual and biliterate*, and *celebrating culture* (from school website, emphasis added). The average teacher at Esperanza has 10 years of experience, which reflects the district average. Esperanza's teaching staff has a higher proportion of Latinx teachers when compared to the district. Esperanza's staff is

[4] To protect privacy, all names of people and places are pseudonyms.

80.8% Hispanic or Latino and 19.2% White while the district rates are 24.1% and 71.3%, respectively. In comparison, the state rates for California are 21.1% and 61.2% when last reported[5]. The major student populations at Esperanza identify as Hispanic or Latino (96.2%) and White (2.9%). At the district level, these demographic sub-groups represent 83.2% and 14% of the students, respectively. At Esperanza, 80.7% of the students qualify for free or reduced lunch. Since 58.8% of the students qualify as 'English Learners' and 7% of the students are designated as 'Redesignated Fluent English Proficient', roughly two out of three students could be considered "ever ELs"[6] (NASEM, 2018). Some of these descriptive data are summarized in Table 3.1. In Table 3.1, the first two columns show percentages of students and teachers (by ethnicity) for the district and the second two columns show percentages of students and teachers (by ethnicity) for Esperanza Elementary School. Henceforth, I will refer to these students as "multilingual" to reflect a multilingual perspective on learning and eschew the normalization of the monolingual English speaker[7]. Labels, such as "ELL", "EL", "LEP", etc., have been critiqued as

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[5] http://www.ed-data.org/state/CA

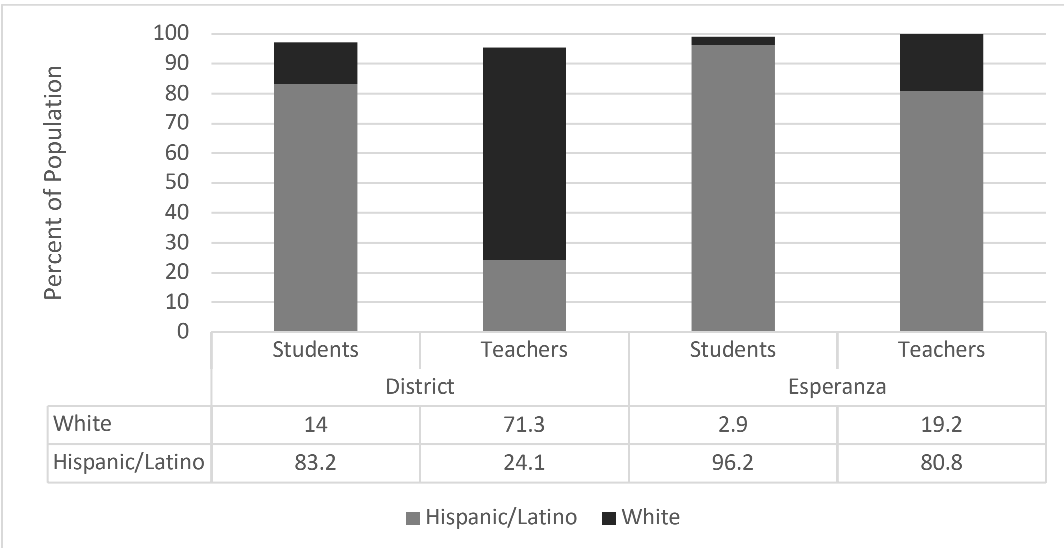
[6] "Ever EL" is a term that can be used to enhance statistical analyses and claims by aggregating any students that have *ever* had the designation of "English Learner" (NASEM, 2018).

[7] In California, these designations are made based on the results of a Home Language Survey (HLS) (CDE, 2020) and subsequent examination of English fluency that are legally required by the education code. For example, upon registration for school, if a parent or guardian answers 'No' to all survey questions, then the student can be designated as English-Only. If a parent or guardian answers 'Yes' to any survey question, then the child is tested for English fluency. If they pass, then the student can be designated 'Initial Fluent English Proficient' (IFEP). If the child does not pass, then they can be designated as an 'English Learner' (EL), 'English Language Learner' (ELL), or 'Limited English Proficient' (LEP). The child will have annual opportunities for assessment of English fluency. If the child passes in a later year, and meets any additional criteria the district might have, then they can be designated as 'Redesignated Fluent English Proficient' (RFEP). HLS Questions include: 1) "Which language did your child learn when they first began to talk?", 2)" Which language does your child most frequently speak at home?", 3) "Which language do you (the parents and guardians [sic] most frequently use when speaking with your child?", and 4) "Which language is most often spoken by

"inherently deficit oriented" (Barwell, Moschkovich, & Setati Phakeng, 2017; de Araujo, Roberts, Willey & Zahner, 2018; Faltis & Valdés, 2016).

Table 3.1

Summary of District and Site Demographics as Percents



I chose this site and teacher because I wanted to study mathematics learning in a place that was well-positioned to have the kind of exposure and experience with multilingual students and communities that could foster positive views of multilingualism. In the United States, 20% of the population over five years of age lives in a home where a language other than English is spoken (Crouch, 2012; as cited in Faltis & Valdés, 2016). In contrast, less than 5% of teacher workforce grew up in such homes (Faltis & Valdés, 2016). Moreover, many teachers report feeling unprepared to teach multilingual students and hold deficit views of these students

adults in the home? (parents, guardians, grandparents, or any other adults)" (California Department of Education, 2020).

(Faltis & Valdés, 2016) so I chose a site and teacher that might hold strengths-based views of students and have positive views of and experiences with multilingualism. "Ms. García" is a bilingual educator with 15 years of teaching experience who is empathetic to students' struggles with language. She believes that students have a right to their "home language" and should not be required to "assimilate" in ways that strip them of heritage languages (interview, February 2020). Moreover, Ms. García used positive behavioral practices (i.e., not punitive) and participated in a three-year professional development project for discussion-based, reform curriculum implementation. After observing several teachers who expressed interest in participating in the study, I chose to work with this teacher because of her strong alignment with the precepts engendered in this study.

My role in the study was to work closely with the classroom teacher and then enact the unit on mathematical argument. I supported daily instruction by providing feedback on some of the writing tasks that were already present in the curriculum and making recommendations for additional writing tasks that might support student learning. During the unit of instruction on mathematical argument, I enacted the lessons. Additional information on the lessons is available in a subsequent section ("Overview of Lessons").

My language background and positioning are also relevant to the study. I was a student who was designated as 'English Only' during my K-12 schooling. I attended public schools in California for all but three years of my life. I spent third, fourth, and fifth grades in a parochial school. Although my parents are both Mexican and arrived at the United States as adolescents, they did not teach their children (my siblings and I) Spanish to ensure that we would not be placed in lower academic tracks. They were mistrustful of the education system's potential treatment of their children as Spanish speakers. Consequently, they told us tell school officials that none of us spoke Spanish; thus, requiring us to lie to school staff. Eventually, I took Spanish courses in high school and gained additional fluency (beyond reading and writing) throughout my twenties. Learning to speak Spanish was important for me because of my involvement in community projects. In particular, my work with immigrant communities, in educational settings, and in my personal life as a *Danzante Azteca*[8], where Spanish serves as a *lingua franca*. For me, it has been important to preserve this heritage language for political, educational, and personal reasons. Reflecting on the process of reclaiming a heritage language, I have gained insights into the role that home languages can play in a variety of educational settings and the difficulties associated with learning a second language. As a multilingual student, teacher, and researcher, I have first-hand experience that informs my work.

The study included 44 student participants (17 female and 27 male students) and one collaborating teacher. The students represented all the district's categories for mathematics proficiency and "language fluency": "English Learner" (N=19), "Reclassified Fluent English Proficient" or "Initially Fluent English Proficient" (N=20), and students designated as "English Only" (N=5). All students participated in

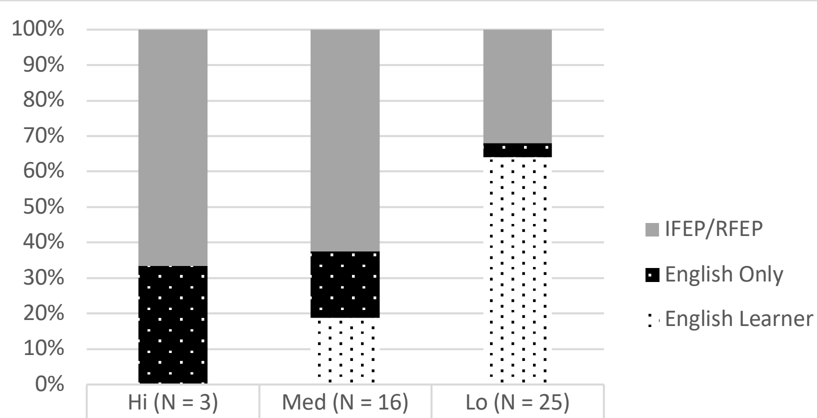
[8] Aztec Dance (*Danza Azteca*) is a folkloric dance tradition from Mexico that serves as cultural maintenance and epistemology.

writing activities and whole-class discussions, but to the extent possible, I strategically sampled focal students to represent a cross-section of the classroom population by mathematical achievement (low, middle, or high, based on district norming exams).

I consulted with the classroom teacher to find students that represented various levels of mathematics achievement. Focal students were chosen for two reasons: first, previous studies on argumentation have typically involved only high-achieving students (for example, see Healy & Hoyles, 2000) and, secondly, focal students participated in conferences with the researcher. The conferences will be described further in the section on analysis. The different student categories are summarized in Table 3.2, below.

Table 3.2

Distribution of Participants by "Language Fluency" Category. Columns Represent Mathematics Achievement Categories Based on District-wide Exam.



	Hi (N = 3)	Med (N = 16)	Lo (N = 25)
IFEP/RFEP	2	10	8
English Only	1	3	1
English Learner		3	16

The focal students were selected based on several criteria. First, I selected students representing high, medium, and low mathematics achievement (as determined by the teacher). Much of the research on proof and justifying has involved high-achieving students (for example, see Healy & Hoyles, 2000), and I was interested in examining how students at all levels constructed arguments. In terms of mathematics proficiency, focal students included one designated “high”, three designated “medium”, and three designated “low”. This sample is roughly representative of the larger group of 44 participants (see Table 3.2). I assume that high-needs students are capable learners and doers of mathematics and might have unique learning characteristics and strengths that might otherwise go unnoticed.

Second, I selected students representing a spectrum of language fluency designations. Focal students included three designated “English Only” and three students “Redesignated” as fluent in English. Only one focal student was designated as an English Learner. These numbers did not reflect the larger population of participants, where 16 (of 44) were “English Learners”. Likewise, students designated as “English Only” were overrepresented in the focal students, in comparison to the whole set of participants. In terms of gender, three of the focal students were female and four were male, per district demographic data.

Third, I prioritized students who completed most Convince Forms and represented what other students did. So, focal students were selected from those students who submitted a Convince Form for most lessons (i.e., at least two out of three). Of the 14 students who completed at least two Convince Forms, seven were selected as focal students to represent the greatest diversity in terms of gender, mathematics proficiency, and language fluency, per district demographic data. Then, I selected students who submitted the best example of similar types of responses when compared to their peers. For example, three students had difficulty with the term "consecutive" and reasoned empirically (Yolanda, Marta, and Maggie), but I chose Yolanda as a focal student because of her clear responses during the conference. Characteristics of the focal students are summarized in Table 3.4.

Table 3.4

Focal Students with Demographic Information

Focal Student	Gender	Ethnicity	Math Proficiency[9]	Language Designation	Conference
Antonio	M	Hispanic[10]	Med	English Only	1, 2
Ricardo	M	Hispanic	Med	English Only	1
Nathan	M	White	Med	English Only	N/A
Renato	M	Hispanic	Low	English Learner	N/A
Yolanda	F	Hispanic	Low	Redesignated	2
Amaya	F	Hispanic	Low	Redesignated	2
Nina	F	Hispanic	High	Redesignated	1

A Note on the Impact of COVID-19

When the COVID-19 pandemic precipitated a shelter-in-place order in California (March 19, 2020), data collection at the school site was already in progress. I had completed the observation of a full unit of instruction, administered a pre-test for algebraic thinking, had just started the writing activities, and was on track

[9] As described by the teacher.
[10] The school district uses "Hispanic". I will use Latinx.

to conclude data collection in May of 2020. During this time, I was able to observe Esperanza Elementary's typical activity (see the vignette on a typical day later in this chapter). The shelter-in-place order required I cease data collection and I was no longer able to continue as planned. That was the last time I saw the eighth graders with whom I had spent the school year. The academic year was completed as public schools unceremoniously transitioned to distance-learning models, research projects were deprioritized by the local school district, and I used what little data I had gathered to further refine my analytic tools during the Spring Quarter of 2020.

During the summer of 2020, I resumed communication with the classroom teacher and in September we started anew. Rebooting the study meant recruiting more study participants. The reboot also required that I coordinate with the UC Santa Cruz Human Subjects Committee and the local school district to ensure the study could move forward in the online setting with sufficient participant protections. I converted relevant paper forms (for example, the "Convince" form, pre and post assessments, and consent forms) into electronic forms with the capacity to collect a variety of response formats, e.g., type-written, photos (for hand-written work), and audio/video files. Due to concerns over privacy, the collection of video data was minimized regarding the number of participants and total time.

The move to online settings introduced new issues. For example, I used Google Forms (GForms) and Google Docs (GDocs) to collect data, rather than collecting physical documents. GForms have capacity to survey students with multiple-choice items and check boxes, as well as gather text and digital files (such as photos, voice recordings, and video recordings). Unfortunately, there is no equation editor available in GForms or in GDocs, which might have affected how students responded to open-ended questions involving mathematical reasoning. That is, the use of a QWERTY keyboard, unlike a blank sheet of paper, might have constrained how students responded to questions involving mathematics. In response, many students completed written work as they normally would, on paper, and then submitted photos of the work. In this way, they were able to submit any representations that were either visual or symbolic (i.e., representations that are not easily made on a QWERTY keyboard).

On the other hand, submitting photos was not taken for granted and introduced a different set of issues. For example, as students were working with a wide variety of devices (e.g., Chromebooks, laptops, iPads, smartphones, etc.), they struggled to resolve issues related to compatibility or platform-use on their own. Also, connectivity was a major issue in the geographic region where the study took place. This was brought to national attention by a viral photo of two students using the free WiFi available at a local restaurant (see Figure 3.1). Reliable connectivity was a formidable obstacle to participation in the study and likely contributed to the low participation rates. This might have contributed to low participation rates for Lessons 1 and 2.

Figure 3.1

Elementary School Students Using Free WiFi at a Local Restaurant (Ebrahimji, 2020)

Another set of issues arose with the use of online meeting platforms. First, it was not possible to record small-group discussions. As such, I was not able to gather data related to classroom discussions as I had originally planned. Whole-class discussions could be recorded but all recordings were made by the cooperating teacher and then the files were shared with me later. The classroom discussions were shaped by the Zoom and GMeets platforms in ways that differ from typical classroom settings. For example, only one person could speak at a time, students could turn cameras off (and most often did), and “break-out” groups could not be recorded or accessed by anyone other than the host (i.e., the teacher). Secondly, students were reluctant or unable to participate in class discussions. Student disengagement was an obstacle that required constant attention.

Finally, I would like to acknowledge the emotional burden that students bore during that time. It was a time of great uncertainty, and the COVID-19 virus ran rampant through many such agricultural communities. One study found that from mid-July to November of 2020, when the study took place, 13% of the participants, who were all farmworkers, tested positive for SARS-CoV-2 (the virus that causes COVID-19), compared to 5% of California as a whole (Manke, 2020). Most of the farmworkers (58%) continued to go to work while infected or symptomatic due to fear of job loss. Consequently, student attendance suffered (from COVID-19 illness) and morale was extremely low, likely due to the stress being felt in the larger community.

Preparation

As part of the preparation for the study, the cooperating teacher and I crafted and reviewed writing prompts, I observed a unit of instruction, and I familiarized students with the different purposes for mathematical writing. During the summer, we met three times to discuss writing prompts, finalize the timeline for the study, and finalize lessons for the unit of instruction. I observed a unit of instruction to document the typical instructional practices and use of writing. Documentation of the "instructional starting point" (Cobb, Jackson, & Dunlap, 2017) was important to familiarize myself with the kinds of writing practices available in the curriculum and typical instruction, as well as to ensure the ecological validity of introduced tasks (Moschkovich, 2019). The curriculum is a common, reform-based textbook (Dietiker, Baldinger, Shreve, Kassarjian, & Nikula, 2013) that regularly uses discussion and writing as a feature of daily activities.

A typical morning at Esperanza Elementary School (Pre-COVID-19)

"I arrive at the school 15 minutes before the start of the first period. I park on the side of the school because the front parking lot is crowded with a line of vehicles dropping off students. I pass a small school garden and enter the front of the school. The main hallway is covered with examples of students' academic and other work. There are people walking through the hallway and I am greeted with the sounds of talk in Spanish and English. I enter the office and exchange greetings with the office staff. We recognize each other at this point and signing-in is a brief routine.

I walk across the playground toward the classroom, which is a 'temporary' classroom trailer that has been there for years. Children are playing on the jungle gym while others are on the blacktop or basketball courts. Some children are playing what seems to be a low-stakes basketball game while others are playing what looks like 'tag'. I greet a few teachers along the way. They also recognize me, and we greet each other with, "Buenos dias, maestra[11]" or "maestro". I arrive at the entrance of the classroom early for class to find a locked door and empty classroom. More students are playing on the field. A group of girls is playing what looks like 'tag' with a ball. They laugh and chase, even dragging each other to the ground on occasion. I rarely see such physical play in a school setting. Another group of students is playing softball.

[11] Translation: "Good morning, teacher."

A teacher opens the adjacent classroom and I ask, ‘Are these students in PE?’ She says that they are not, and that school has not started. I follow up asking where they got the equipment. The teacher says that some students bring their own equipment and teachers let them use the rest. A bell rings, to mark a passing period to get to class, and the students either pack up or return equipment and line up at the doors of their classes. ‘Ms. Garcia’[12] arrives, greets the crowd, and enters the class alone. After a few moments, she opens the door and greets each student with either a unique or default handshake, quickly checks in on personal matters with some students, and collects homework as they enter the classroom and take their assigned seats.” (fieldnotes, March 12, 2021)

Data collection

This study proceeded through three phases: *preparing, enacting,* and *reflecting* (Gravemeijer & Prediger, 2019). As a reminder, all data was collected virtually. I made use of online forms, online meeting platforms, and shared documents to organize activities and collect data. I included a variety of data sources to allow analysis of student learning in the classroom that is summarized in Table 2, below. During the preparation phase, I collected relevant de-identified ethnographic data about the site and its students, including historical performance data disaggregated by grade and demographic sub-groups to establish a historical context

[12] Pseudonym.

for the site and students. I began by observing the classroom practices[13] for one unit of instruction[14] (approximately four weeks) to document the "instructional starting point" (Cobb, Jackson, & Dunlap, 2017, p.212). Observations focused on identifying and describing different kinds of mathematical writing and discussions that occurred in the classroom using the Purposes framework (Casa et al., 2015) when writing was involved (see Appendix B). The Purposes framework describes mathematical writing as exploratory, explanatory, argumentative, or creative (Casa et al., 2015). During this phase I interviewed the cooperating teacher to probe her beliefs about education and the role of language and gain additional details about her professional preparation.

I administered and collected a "Mathematical Resources Inventory" (see Appendix A—adapted from Ortmeier-Hooper's, 2013, "Mapping our literacies and resources" activity for multilingual students) from the students. The inventory asked students to elaborate upon people, places, and events that influence their schoolwork, as well as hobbies that, although not necessarily related to mathematics, could be used to provide a familiar context (for example, see Dominguez's, 2008, work) and supply potential raw material to enhance the 'semiotic budget' available to students (Walqui, 2019). I administered and collected a questionnaire about other linguistic and mathematical background information (Appendix C). The questionnaire sought

[13] Due to the ongoing COVID-19 pandemic, the term "classroom" includes virtual spaces that the teacher and students occupied as part of the regular school day. For example, class sessions held in online meeting platforms are considered classroom activity.

[14] I observed a complete unit of instruction because of its potential to include both a variety of lesson types (e.g., introducing a new topic, reviewing for assessment, etc.) and typical instructional practices (e.g., whole group and small group discussions, writing tasks, etc.) as a coherent whole. In the online environment, small groups were typically managed as "break-out" groups.

basic information about students' previous mathematics instruction to complement historical mathematics data (from the district and site), demographic information, and explore mathematics attitudes.

During the enactment phase of the study, I gathered data from pre- and post-assessments and enacted lessons: one lesson on mathematical writing and three lessons on mathematical argument. The pre/ post early algebraic thinking assessment was administered to determine how students' thinking changed over time using an adapted version of an algebraic thinking test used in previous research in early algebra[15] (Appendix D; Adapted from Chimoni, Pitta-Pantazi, & Christou, 2018). Items in each of the administrations were slightly different to minimize the possibility of item recall. I enacted lessons on mathematical writing and mathematical argument. I collected samples of student work, took observational notes, and video recorded class sessions that took place on virtual meeting platforms.

[15] The pre and post tests were not part of the analysis for the dissertation.

Table 3.3

Summary of Study Activities

<table>
<tr><th></th><th>Activity</th><th>Description</th><th>Data/Artifacts Collected</th></tr>
<tr><td rowspan="3">Prepare (6)</td><td>Coordinate with teacher</td><td>Coordinate with teacher to clarify learning goals.</td><td>• Observational data
• Teacher Interview
• Gather Consent/Assent</td></tr>
<tr><td>Observation</td><td>Observe writing practices to establish an "instructional starting point".</td><td>• Observational data
• Examine student writing
• Observe typical writing practices</td></tr>
<tr><td>Research Team Meetings</td><td>Meet to review and plan the argument lessons.</td><td>• Observation data</td></tr>
<tr><td rowspan="5">Enact (12 weeks)</td><td>Pre-Assessment</td><td>Gather background data</td><td>• Algebraic Thinking Pre-Test
• 'Resources' and student questionnaire</td></tr>
<tr><td>Math Writing lesson</td><td>Introduce purposes for mathematical writing to students.</td><td>• Observational data
• Journal and Quick Writes</td></tr>
<tr><td>Argument Lessons & Conferences</td><td>Enact three lessons on mathematical argumentation. Meet with teacher to review and plan next lesson. Conference with focal students.</td><td>• Observational data
• Video recording and samples of student writing:
    o From conferences</td></tr>
<tr><td>Post-Assessment</td><td>Post-test for Algebraic Thinking.</td><td>Algebraic Thinking Assessment (post-test)</td></tr>
<tr><td colspan="3">18 Weeks total</td></tr>
<tr><td>Reflect</td><td colspan="2">Winter, Spring, Summer 2021</td><td>Review all data and conduct retrospective analysis.</td></tr>
</table>

The duration, frequency, and length of writing tasks reflect recommendations from a meta-study of mathematical writing studies (Bangert-Drowns, Hurley, & Wilkinson, 2004) to the extent possible. As such, mathematical writing tasks were implemented two times per week for a minimum of one semester (approximately 18 instructional weeks), for 5-10 minutes per class period in a lesson (2-3 class periods in a lesson). The lessons integrated writing activities, whole-class discussions, small group work, and a 10-minute (maximum) individual writing session for students to complete and submit an argument.

Throughout the study, student writing was used to formatively assess mathematical understanding, inform instructional decisions, and as the shared focus for mathematical conferences. As a formative assessment, the cooperating teacher used student writing to make day-to-day adjustments when lesson planning. As the lessons on mathematical argument were completed, I used students' mathematical writing (i.e., responses to the Convince Form) for assessment and to inform my feedback during conferencing. In this manner, the mathematical conferences provided opportunity to discuss, reflect upon, and revise the students' own mathematical writing. I discuss the Convince Form and the use of 'audience' at the end of the next section, 'Overview of Lessons'.

Overview of Lessons

In total, there were four lessons enacted with the students: one preliminary lesson on mathematical writing and then three lessons on mathematical argument. The mathematical writing lesson focused on the Four Purposes for mathematical writing (Casa et al., 2016). The purpose of the lesson is to provide instruction on the purposes and characteristics of mathematical writing. In addition, students continued to engage in mathematical writing throughout the observation period and subsequent lessons on mathematical argument. Since longer exposures to writing practices could be related to increased beneficial effects (Bangert-Drowns, Hurley, & Wilkinson, 2004), mathematical writing prompts were used throughout the semester. To be clear, the purpose of the mathematical writing lesson was not to “introduce” students to mathematical writing. Both the teacher and curriculum made use of mathematical writing even prior to the researcher’s involvement. The lesson was intended to *clarify* characteristics and criteria for good mathematical writing in each of the purposes and then provide opportunities to write.

The three argument lessons were designed to support students as they developed understandings for the need and demands of mathematical argument. Moreover, the lessons included a focus on learning processes, which is an aspect of topic-specific research (Prediger, Gravemeijer, & Confrey, 2015). The materials and activities were hypothesized to stimulate and leverage various forms of learning by eliciting student thinking. In addition, I assumed that students are “epistemic agents of their own" who bring to bear their own experience and resources. Students are not “incomplete adults (Kaput,1999)” (Prediger et al., 2015, p.881).

The lessons centered on the topic of ‘number property’ and were organized around a progression of tasks that supported students’ increasing capacities for composing claims and justifications (summarized in Appendix E). The progression was shaped by a variety of tasks that invited students to make claims and justify them with varying levels of support for generating a claim. Justifying-a-given-claim tasks provided students with a claim and asked them to justify it. For example, in Lesson 1, students were asked to "Show that when you add any two even numbers, your answer is always even". The claim that the sum of two even numbers was "always even" was given to the students and they were invited to justify it. Tasks that asked students to qualify a claim provided students with a claim but asked them to choose a non-neutral qualifier for the claim (e.g., '*always*, *sometimes*, or *never'*) and then justify their choice. For example, in Lesson 3, students were asked to decide whether *the sum of five consecutive whole numbers is divisible by 5* is always, sometimes, or never true, and then justify their choice. Tasks that invited students to generate a claim invited students to generate and justify a general claim, given a mathematical situation. For example, in Lesson 3, students considered all possible sums generated by adding *three consecutive whole numbers*, and then conjectured about *which kinds* of numbers can be made (see Figure 3.3).

Each lesson was completed over two sessions (55 minutes each), made use of discussion boards and online forms, and concluded with a task inviting the students to write a mathematical argument (for a summary, see Appendix J). The first lesson introduced mathematical argumentation and focused on given-claim tasks to familiarize students with basic concepts, vocabulary, and the concept of mathematical argumentation (as different from everyday arguments). The final task for lesson 1 is called "Fibonacci Rectangles" (see Figure 3.1). Note that Parts D and E of the task support students to look for patterns in a data table, predict a distant value, and then make a general prediction.

Figure 3.1

"Fibonacci Rectangles" Task (Dietiker et al., 2013 Problem PI-4, p.527).

Students were asked:

A. How is the sequence growing? [1, 1, 2, 3, 5, 8, 13, …]

B. If the measures of the sides of a rectangle are consecutive Fibonacci numbers, it is called a "Fibonacci Rectangle." Here are the first four [illustration]. What is the area of each rectangle? What is the sum of the first two? The first three?

C. Copy/Complete the table (up to 8).

Number of Rectangles	Sum of Areas
2	3
3	9
4	
5	
6	
7	
8	

D. Use your table. Look for any special numbers. Can you find any patterns?

E. How can you predict the sum of the areas of Fibonacci Rectangles? Use your patterns to predict the sum of the areas of the first 15 Fibonacci Rectangles.

When uploading the work, students were asked to upload a photo of their written work and type an explanation of the prediction and how it works.

The second mathematical argument lesson continued to explore the topic of ‘number property’ through “Number tricks” (Knuth, Choppin, & Bieda, 2009) and the “Happy Numbers[16]” task (see Figure 3.2). The Happy Numbers task prompted students to find two- or three-digit happy numbers (without calculating) and to justify why their method works. Happy numbers are found by separating the digits of a number, squaring each digit, and then adding the resulting products. If the resulting sum is ‘1’, then the number is ‘happy’. If not, then the process repeats until it results in a ‘1’ or a sum repeats (in which case the number is not ‘happy’). This process is illustrated in Figure 3.2.

Figure 3.2

“Happy Numbers” Task (above) and Determination Procedure (below) (Dietiker et al., 2013, p.534)

Students were asked:

A. There are 17 Happy Numbers (from 1 – 99). Find as many as you can.

B. Find five 3-digit happy numbers.

C. Eva found out that 478 is a happy number.

    i. What other numbers *must* be happy numbers?

[16] “Happy Numbers” are numbers whose individual digits are squared and then added. This process is repeated until the sum is either “1” (in which case the number is “happy”) or a sum is repeated (in which case the number is not “happy”).

ii. How do you know?

iii. Find at least 10 new happy numbers.

Determination of Happy Numbers in Lesson 2:

Example 1: 23

$2^2 + 3^2 = 4 + 9 = 13$

$1^2 + 3^2 = 1 + 9 = 10$

$1^2 + 0^2 = \mathbf{1}$

Happy

Example 2: 34

$3^2 + 4^2 = 9 + 16 = 25$

$2^2 + 5^2 = 4 + 25 = 29$

$2^2 + 9^2 = 4 + 81 = 85$

$8^2 + 5^2 = 64 + 25 = 89$

…

$5^2 + 8^2 = 25 + 64 = 89$ (repeat)

Not Happy

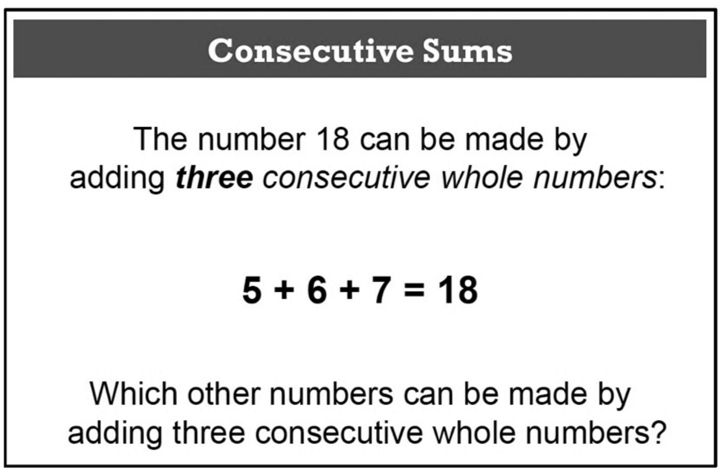
During the second lesson, I invited students to comment to each other in the discussion board. For example, I asked students about the "pros" and "cons" of each justification, why they did or did not find it convincing, and how examples and calculations were used. Like the first lesson, the students were asked to upload photos and text to an online form to justify any conjectures made about Happy Numbers.

At this point, some focal students had an opportunity to conference with the researcher and revise one of their arguments. Additionally, students completed a math journal (see Appendix G for prompts). These additional activities were intended to support student learning by providing an opportunity to integrate the feedback with an additional opportunity to compose and justify the claim, and then reflect on the overall learning process.

The third lesson involved enacting the "Evaluating Statements: *Consecutive Sums"* lesson (Appendix A; MARS, 2015). The mathematical learning goals of this lesson were for students to 1) state and test mathematical conjectures, and 2) understand and use *alternative methods* of proof (MARS, 2015; emphasis added). Note that the lesson does not seek rigorous proof as a learning goal. Typical of early algebraic instructional practice, the goal is to introduce students to the conceptual and procedural rigors of proof and proving so that they are positioned for success in future mathematics coursework. The lesson features tasks that asked students to justify a given claim, qualify a claim and justify, and then generate a claim and justify.

Figure 3.3

Task 1 Consecutive Sums (MARS, 2015, p. T-5).



The final task for lesson 3 was to determine whether some mathematical claims about different sums of consecutive numbers were *always, sometimes,* or *never* true. Here are some sample claims students classified:

A. The sum of four consecutive whole numbers is divisible by 4.

B. The sum of five consecutive whole numbers is divisible by 5.

C. To find the sum of consecutive whole numbers, find the middle number and then multiply it by how many numbers there are. (MARS, 2015, p.S-6)

All tasks reflected grade-level standards and required students to justify mathematical conjectures (i.e., create mathematical arguments that reasoned and generalized) that were given to them as well as those that they made themselves.

An important difference between Lesson 3 and the previous lessons is in the presentation of the Convince Form. Due to low rates of participation in the first two lessons, I revised the Convince Form into editable slides that each student could access and submit through their classroom's online platform (Google Classroom; see Appendix I). My hope was to place the form in a familiar platform, alongside their classwork, so that students could access it more readily. The mathematical writing that students submitted was used as formative assessment to inform future instructional decisions.

In each of the mathematical argument lessons, students were asked to respond to the final tasks by filling out a Convince Form (see Appendices H and I). The Convince Form focused on supporting students by describing and scaffolding the process of argumentation as convincing oneself, a friend, and a skeptic (adapted from Mason, Burton, & Stacey, 1982 and Schoenfeld, 2009). The Convince Form enriched the rhetorical situation for students by suggesting different kinds of audiences. The form also invited students to engage in a variety of writing *purposes* (Casa et al., 2015), such as explanatory and argumentative writing. Presenting the argument process in terms of audience has been lauded as "one of the best descriptions of the proof process" (Schoenfeld, 2009). It described the varying aspects of argumentation by prompting students to consider audiences with differing needs and affordances, which is a recommended instructional practice to support students when teaching language and mathematics (Wilkinson, 2018). Secondly, the form invited students to go beyond what is typically required for everyday arguments. One implication noted in Banes' (2019) work is that mathematical argumentative writing should be presented as "over explaining", for example to a 'skeptic', so that evidence, counter-examples, and rebuttals are viewed as appropriate. Such rigor can be an obstacle for students because it violates other social norms around conversation and arguing (Grice, Cole, & Morgan, 1975; as cited in Banes, 2019). A second implication of Banes' (2019) work is that instruction should include tasks with greater rhetorical specification. This would also include the use of rubrics and even the strategic use of a "less mathematical" audience to create a need for so much explanation, for example "Use words and pictures to explain why the conjecture is false to a 3rd grader" (my own example). The Convince Form was designed to support students' argumentation by framing phases of students' work with particular kinds of writing purposes (Casa et al., 2015), i.e., by promoting exploratory writing during the "convince yourself" phase, informative/ explanatory writing during the "convince a friend" phase, and argumentative writing (including consideration of counterarguments and sufficiency of evidence) during the "convince a skeptic" phase. In the first two lessons, the Convince Form was a fillable form students accessed by clicking on a link (see Appendix H). Due to low participation rates in the first two mathematical argument lessons, the Convince Form was revised into editable slides that each student could access and submit through their classroom website (see Appendix I).

In addition to typical writing activities, I conducted student writing conferences to provide an opportunity for revision of their mathematical reasoning. Conferencing is a productive practice to support students' writing (Hillocks, 2006) and enhance understanding (Swain, 2009), but it is especially productive for multilingual students, who might have difficulty with feedback offered in only one mode (Aguirre & Bunch, 2012). I conducted one round of conferences with the focal students that took place between the second and third lessons and a second round with the remaining students after the third lesson (see *Summary of Unit of Instruction*, Appendix B). In the first round of conferencing, students were opportunistically sampled (Patton, 2002) because they were among the few students that responded to the prompt. In the second round, students were sampled to ensure maximum variation (Patton, 2002).

Conferences were video recorded via Zoom and followed the description provided by Newkirk (1989). In addition to video recording, I took notes and worked to avoid 'common mistakes', such as opening with Yes/No questions, asking multiple questions at once, or praising particular responses (Brenner, 2006). The conferences focused on one or two important items and lasted approximately 10-15 minutes each. My role as the interviewer was to balance evaluations and responses with encouragement for students to take initiative and evaluate their own writing and argumentation. The student's job was to answer the question, "What did you think" of your written argument (Newkirk, 1989, p.318). Ideally, the student and I mutually constructed an agenda as I allowed time and space for students to vocalize their own reflections and revisions. I engaged the conferences like an interview so that the student had time and “space to express meaning in his or her own words and give direction to the interview process” (Brenner, 2006, p.357).

Summary of Data Collection and Protocols

After each daily session, the collaborating teacher and I met to analyze the activity and consider instruction for subsequent sessions. We reviewed data as part of the iterative cycle (to inform change) and in the retrospective analysis. Data from the MRI was used to inform problem contexts. For example, if students indicated that they play dominoes at home, then dominoes could serve as a familiar context for a mathematics problem (example from Nasir, 2005).

Analysis

Video data was collected in two different settings. The first was the online meeting platforms where classes were held. Regular classroom sessions included the collaborating teacher, student participants, and the researcher (as participant observer) in the GoogleMeets platform. The second setting I videorecorded was the conferences. Conferences were conducted on Zoom and included the researcher and an individual student, with the collaborating teacher present as an observer. Both settings were designed to produce written work (including in the chat features of each online platform), oral mathematical discussion for transcription, and video data.

The first round of conferences involved five students and focused on their Convince Form responses for Lesson 2 (Happy Numbers task). These five students were selected from the seven students who filled out the Convince Form in Lesson 2. The second round of conferences involved seven students and focused on their Convince Form responses for Lesson 3 (Consecutive Sums task). These seven students reflected the students who agreed to participate and appeared for the conference. I invited 24 students for a conference (20 conferences with four on stand-by).

A round of conferences followed Lessons 2 and 3 with mostly different students in each (one student volunteered for both rounds). The purpose of the conference was to support students to revise or extend the Convince Form responses they provided to the final tasks of Lessons 2 and 3 through questioning. The approach of the conference was to present students with the work they submitted and then ask a series of eliciting and probing questions to further their mathematical reasoning (Franke et al., 2009). The structure of the conference resembled an interview (Brenner, 2006). I reviewed each student's work, prepared one to four questions, and limited each conference to 10 – 12 minutes. The questions focused on one or two questions to review and assess the students' initial work, and then one or two questions to elicit or press for further student reasoning and encourage attention toward mathematical structure. The starting point was what work each student submitted in the most recent Convince Form. In the first round of conferences, we discussed the Happy Numbers task (5 students) and in the second round of conferences we discussed the Consecutive Sums task (7 students—including one who volunteered for a second conference) for a total of 12 conferences and 11 unique students (see Appendix J for complete tasks).

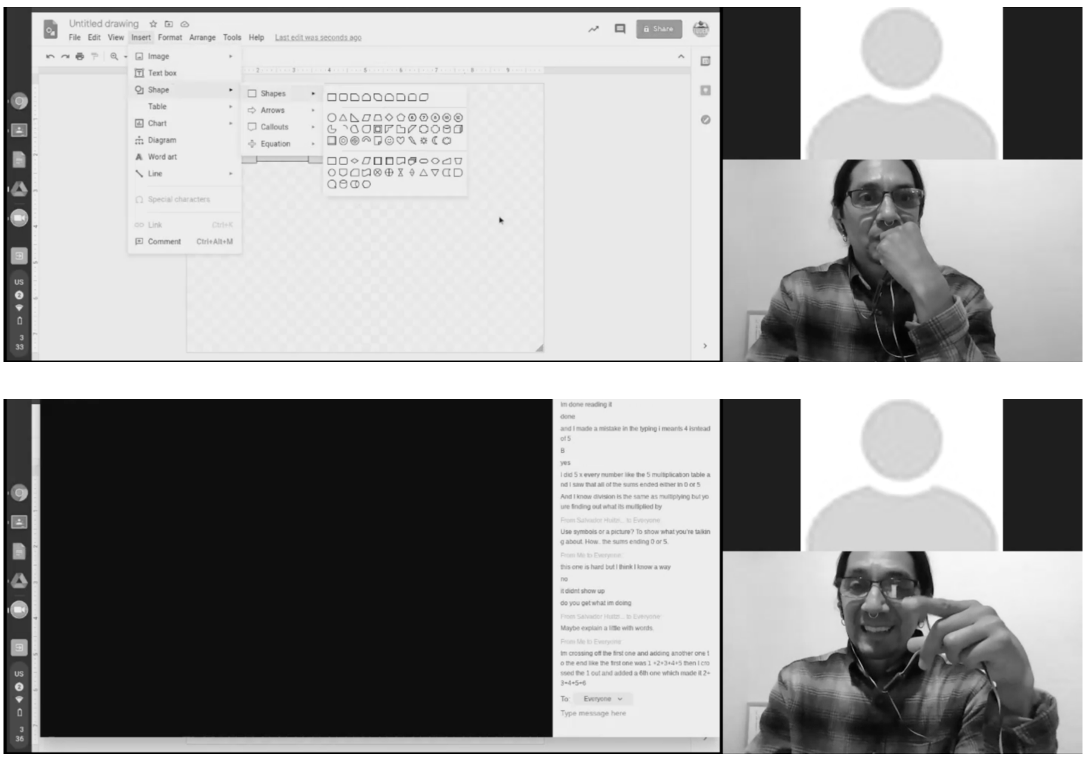
I conducted and recorded conferences with online meeting platforms (i.e., Zoom). Data sources included the video recording (and transcripts), chat files, related work that is referenced throughout the conference, contemporaneous notes, and any writing generated during the conference (including, for example, students' hand-written work shown on camera and visual representations). I examined these data to determine how the students revised their arguments or other aspects of their work.

I used the *proof schemes* (external, empirical, or analytical) framework (Harel & Sowder, 1998) to categorize the kinds of arguments students used in the previously submitted work and then examined whether the students used a new scheme during the conference. I used the proof scheme framework because it reflected a non-hierarchical way to categorize how students were arguing in each context. Moreover, students might use a different scheme depending on familiarity with mathematical content or their sense of what was "appropriate in the setting of the work" (Harel & Sowder, 1998, p.277), which is what happened with four participants in this study. I used the parts of an argument elaborated by Toulmin (2003) to describe details of student work. The criteria for each classification are described in the Conceptual Framework chapter (p.41).

I considered multiple data sources, at times in coordination, to review student arguments and trace whether and how the students revised their initial argument. See Figure 3.5 for a screenshot of what the conferences looked like in the online setting.

Figure 3.5

Screenshots of Video Recording. Data Sources Included Talk, Images Shared On-screen (top) and Comments in the 'Chat' Box (bottom).



Participants in each round of conferencing met differing criteria. In the first round of conferencing, every student who submitted the original Convince Form was invited (N = 6; see Appendix H). Five of these students assented and participated in a conference. In the second round of conferencing, I invited 16 students (40 possible students). Seven of them assented and participated. To varying degrees, these students completed the modified Convince Form available for the Consecutive Sums task (see Appendix I). For each of the focal students, I am including students' mathematical proficiency because most empirical studies of mathematical argument have focused on high-achieving students (for example, see Healy & Hoyles, 2000). I sampled from a broader population of student participants so that findings can be relevant to all students and to demonstrate how students of all proficiency levels can and do make arguments.

Video data from the whole-class discussions and conferences were analyzed in two ways. First, global and local arguments in the whole-class discussions were modeled using the method outlined by Knipping & Reid (2019). Knipping & Reid (2019) suggest a method to describe classroom arguments (in whole class and small group settings) that builds on Toulmin's (2003) theory and has three stages: reconstructing the sequence and meaning of classroom talk, analyzing local (first) and global (second) argument structures, and then comparing the structures to reveal rationales. This method used data, claims (or 'conclusions'), warrants, and backing (Knipping & Reid, 2019; Toulmin, 2003) as parts of the argument. The result was a schematized map of arguments elaborated in classroom discussion.

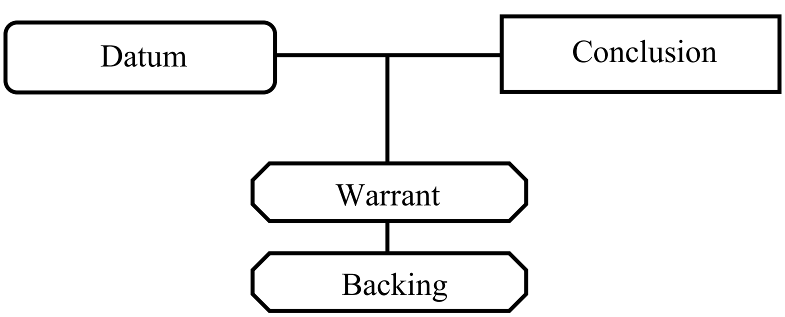
The second analysis involved transcripts of mathematical discussions. Transcription is a process laden with theory (Moschkovich, 2008; Ochs, 1979). As such, the emphasis was on illuminating students' reasoning and transcripts follow a particular set of conventions to account for all modalities available (see Appendix F). For example, transcripts included the use of parentheses ("( )") when there was doubt about an utterance and double parentheses (“(( ))”) to indicate additional annotations. Since the use of gesture has been documented as a resource for ELs (Barwell, Moschkovich, & Setati Phakeng, 2017), I described gestures in the transcripts without bracketing or parentheses. Gestures were presented as integrated with speech, i.e., not interpreted as subordinate to oral speech.

I analyzed student writing using a rubric that considers the *form* of the argument, the *rationale* (Banes, 2019), and the *generalizing* of the argument. The form of the argument was described in terms of the presentation and was related to representations. For example, do the students make use of examples, tables, symbols, words, or visual representations? I used the term ‘rationale’ to analyze students’ writing because it emphasized conceptual understanding, reasoning, and justifying (following Banes, 2019; and Kosko & Zimmerman, 2015). The rationale included claims, the accompanying efforts to justify, and included any reasoning in which the students engaged. Generalizing can be thought of as the process by which students “identify structure and relationships in mathematical situations” (Blanton et al., 2011, p.9). As students generalized, they might find structure in a variety of representations. My analysis emphasized how they engaged generalizing, even if they were not using symbols. For example, students might have engaged generalizing using different kinds of and purposes for examples (Ellis et al., 2012). To detail exactly how the students were reasoning and the extent to which they were generalizing, I used the *proof schemes* (Healy & Hoyles, 2000) framework to analyze arguments. For a full description, see the “Conceptual Framework” chapter (p.29).

To the extent possible, descriptions of student work included photos of the original work and schematic representations of the arguments using the Toulmin model (Knipping & Reid, 2019; Toulmin, 2003). This model described the constitutive parts of an argument and included claims, data, warrants, and backing. While mathematical assertions can be referred to in a variety of ways (e.g., *claims, conclusions,* or *conjectures*), I primarily used the word 'claims' to refer to mathematical assertions made by the students. Data referred to the facts and calculations that served as a foundation for the claim. Warrants were the "rules, principles, [and] inference licenses" that generally ratified the step from data to claim (Toulmin, 2003, p.91). Backing included the "assurances" that lend credence to the warrants (Toulmin, 2003, p.96). A diagram of a generic argument is shown in Figure 3.5. Whenever necessary, I used this schematic to clarify how students structured arguments.

Figure 3.5

Toulmin Model of an Argument (Knipping & Reid, 2019).



Chapter 4: Burgeoning arguments: Shifts in generality and the use of examples

In this chapter I present my analysis of student written arguments submitted in each of the three lessons in the unit. I use a strengths-based approach that utilizes multiple analytic tools to address Research Questions 1 and 2:

1. When asked to do mathematical writing and supported with conferencing in a remote context, what kinds of arguments do students make?
2. How did the mathematical arguments of individuals change over the course of a unit of instruction on generating, selecting, and justifying claims?

Here I describe the organization of the chapter and review overall findings before proceeding. The first section will catalog the types of responses that students provided for tasks that asked them to generate a claim (Lessons 1 and 2) and tasks that ask them to qualify a claim (Lesson 3). Due to the distinct nature of these tasks, first I describe student responses for tasks that asked them to generate a claim in broad terms. And then, I characterize the ways students justified how they qualified claims. The second section elaborates evidence to show how students changed from writing descriptions of procedures to making arguments. I examine the work of focal students Nathan and Nina to illustrate the change. The third section presents evidence to show how students' use of examples changed in ways that are logically sound (rather than succumbing to common logical fallacies). I present evidence from the written work of focal students Renato and Amaya. The fourth section examines how students began to generate claims (Lessons 1 and 2) and qualify claims (Lesson 3), depending on the task. I examine data from each of the lessons and a variety of students to illustrate the ways that the students' written responses changed.

The above questions seek to understand how students' argumentations showed improvement and how they changed over time. I present three major findings related to the first two research questions. First, students' responses changed from writing descriptions of procedures to making arguments. My analysis shows that students' written responses changed such that they used examples in-lieu-of-arguments less and began to include increasingly general arguments. Secondly, students used examples in more logically sound ways. Students changed from, for example, using only specific examples that might support a claim toward using examples as evidence. Third, students went from writing no claims at all (either when prompted to generate a claim or to qualify a claim) to qualifying claims when offered a choice of non-neutral qualifier or generating claims when invited.

The findings listed above show that, by the end of the study, some students made mathematical arguments (general, partial, or empirical) when justifying their claims while most continued to wrestle with developing and generating claims. The first two lessons featured tasks that ask students to generate a claim ("Fibonacci Rectangles" and "Happy Numbers"; see Appendix J). Lessons 1 and 2 were enacted over two class sessions, each, and separated by one week. The third lesson also took two class sessions but took place two weeks after the end of Lesson 2. Types of responses for Lessons 1 and 2 are summarized in Table 4.1. It is important to note that even in Lesson 1, some students generated claims and attempted to justify them (three out of 14). More importantly, while all three attempts to justify arguments in Lesson 1 included a logical fallacy, no such fallacies presented themselves in student work from Lesson 2. Alas, the majority of the 44 participants did not submit any response for Lessons 1 and 2.

Table 4.1

Types of Student Responses from Convince Forms Collected in Lessons 1 and 2.

Open Claim Task	***Lesson 1***	***Lesson 2***
No response	*30*	*35*
Generated no claim	*2*	*2*
Description	*9*	*5*
Generated a claim	*3**	*2*
Total participants	***44***	***44***

*Claims were made but arguments included logical fallacies.

Lesson 3 culminated with a task asking students to qualify a claim and saw marked improvement in participation, as well as in the types and elaboration of mathematical arguments. Forty of the 44 participants (90%) qualified claims. These 40 students qualified an average of 5.3 claims (out of six possible) at a rate of 77.5% correct (i.e., nearly four out of the five qualifications were correct). Students were then asked to justify two of the claims they qualified. Eleven of the 40 participants that qualified claims made at least one argument (either empirical, partial, or general) to justify their qualifications. These 11 students qualified an average of 5.9 claims (out of six possible) at a rate of 73.9% correct. Note that the rate of correct responses for those who attempted to justify was slightly lower than the group that simply qualified claims. See Table 4.2 for a summary of these data.

Table 4.2

Types of Responses from Convince Forms Collected in Lesson 3.

Qualifying a claim		*Item 1*	*Item 2*	*Percent Correct*
No response		*4*	*7*	
Qualified claim, only		*29*	*28*	*78.4%*
Qualified and Justified Claims	*Empirical*	*6*	*6*	
	Partial	*2*	*0*	
	General	*3*	*3*	*73.9%*
	Total	***44***	***44***	***77.5%***

The 29 students that did not justify chose *always, sometimes,* or *never* with 78.4% accuracy. Twenty-seven of these 29 students were multilingual (see Table 4.3).

Table 4.3

Summary of Responses for Items 1 and 2 of Lesson 3 (Qualifying a Claim) by Language Classification.

Item 1 Responses	*English Only*	*Multilingual*	*Row Total*
Claim only	*2*	*27*	*29*
No response		*4*	*4*
Empirical	*1*	*5*	*6*
Partial		*2*	*2*
General	*2*	*1*	*3*
Total	***5***	***39***	***44***
Item 2 Responses	*English Only*	*Multilingual*	*Row Total*
Claim only	2	26	28
No response		7	7
Empirical	1	5	6
General	2	1	3
Total	**5**	**39**	**44**

Methodology

The analyses presented in this chapter have two foci: in the first I examined students' mathematical writing (i.e., their Convince Form submissions) to characterize the ways that they responded to the tasks, and then I examined how those responses might have changed over the course of the study. The mathematical writing that students generated served as a written record of the kinds of responses they made (analysis #1) and how those responses changed over duration of the study (analysis #2). Ultimately, the students' mathematical writing was used to describe the ways that they might have argued mathematically vis-à-vis the proof schemes frame (Harel & Sowder, 1998).

Analysis of the written documents began by considering what the task asked students to do (e.g., justifying claims they are either asked to generate or qualify). Depending on what the task asked them to do, responses were categorized accordingly. In alignment with the naturalistic paradigm, I consider student learning "in their own terms" (Moschkovich, 2019; Moschkovich & Brenner, 2000). This paradigm also supports a strengths-based approach to analysis of student work. I present students as active agents in their own learning by emphasizing what they *do* (rather than what they "cannot do") and characterizing their actions with verbs (rather than nominalizations or passive voice). This approach seeks to document how students are learning to use a variety of proof schemes (Harel & Sowder, 1998). While some researchers consider proof schemes in a hierarchical trajectory where, for example, empirical proof schemes are considered inferior to analytic ones (e.g., see

Knuth, Choppin, & Bieda, 2009, or Küchemann & Hoyles, 2011). I align with Harel & Sowder's (1998) position on proof schemes and the importance of setting, context, and content:

> It is important *not* to regard the taxonomy in a hierarchical, single-niche sense. A given person may exhibit various proof schemes during one short time span, perhaps reflecting her or his familiarity for, and relative expertise in, the contexts, along with her or his sense of what sort of justification is appropriate in the setting of the work. (p.277, emphasis added)

The second part of the analysis examined the written arguments of students from lesson to lesson. I examined how the work of individual students changed over time. The concept of "cherry-picking" emerged as relevant to this analysis. Cherry picking is "to select the most desirable"[17] data from what is available to support preconceived ideas. It is a common form of logical fallacy that can be done intentionally or unintentionally (Klass, 2008). When it arose in the students' written work and discussions, we brought attention back to examples (i.e., data) that did not support the claims to occasion further reflection. As a reminder, a description of the Toulmin method, including parts and use, is provided in the Methodology chapter (Chapter 3).

Because the analyses examined the quality of students' written work over the duration of the study, I selected focal students by prioritizing consistent participation alongside demographic diversity, distribution across achievement levels, and distribution across "English proficiency" levels. Fourteen of the 44 participants completed at least two of the Convince Forms for Lessons 1, 2, and 3. I selected Nina,

[17] https://www.merriam-webster.com/dictionary/cherry-pick

Nathan, Renato because they completed Convince Forms for all lessons. Amaya completed Convince Forms for the Lessons 1 and 3, only.

In the next sections, I describe the types of responses that students submitted, show how students' responses became more general (using written work from focal students Nina and Nathan), show how some students' use of examples became more sophisticated (using written work from focal students, Renato and Amaya), and show how students' written responses went from including no claims at all (either when prompted to generate a claim or to qualify a claim) to qualifying claims when offered a choice of non-neutral qualifier or generating claims when prompted. As a reminder, I provide a description of the process for focal student selection in the Methodology chapter (Chapter 3).

Response Types

In this section, I describe the different types of written responses that students made throughout the unit. Since the final task for each lesson involved students either generating a claim (Lessons 1 and 2) or a qualifying a claim (Lesson 3), I consider the first two lessons together and then the third. Also, I consider *types of claims* separately from *types of justifying.* Early data[18] suggested that the kinds of claims that students made were not necessarily congruous with how they justified them. For example, some students could make a general claim and justified it with a general argument, but others might use an empirical argument or no argument at all. As

[18] These data were gathered before the shelter-in-place order was given (March 2020). Subsequent to the order, I analyzed the data and saw that the generality and validity of a claim was not necessarily related to the generality and validity of how they justified it.

students moved between generating a claim (or qualifying a claim) and justifying it, this allowed them to make choices about how to justify their claims. Moreover, since the processes involved in generating a claim (which could include abductive reasoning—see Minnameier, 2004, or Rivera & Becker, 2007a) are likely different from those involved in justifying the claim (which could include inductive and deductive reasoning), I separate them.

Responses for Generating-a-claim Tasks

Next, I review the types of responses for the first two lessons, which culminated with generating-a-claim tasks. Complete descriptions of each type of response will include a brief characterization and an example of student work (see Table 4.1 for a list).

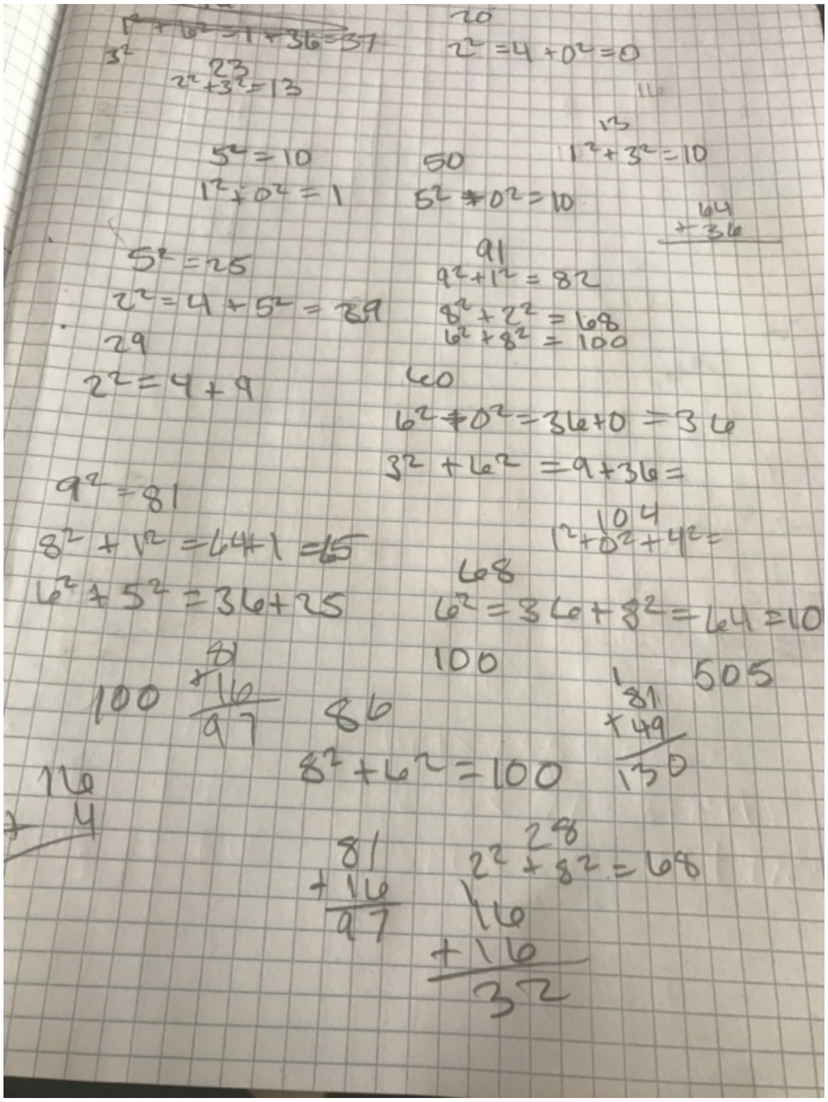
Overview. Table 4.1 is organized so that differences in response type over the two lessons can be highlighted. Lessons 1 and 2 are separated in Table 4.1 to highlight how the types changed over time. Some key observations include that the number of descriptions decreased from 9 in Lesson 1 to 5 in Lesson 2, while the number of students that generated no claim remained the same. It is also important to note that three students generated a claim in Lesson 1 but did not successfully justify them. In Lesson 2, different students generated a claim, but these students successfully justified them. Regarding language designations, I note that most students who did not respond to the Convince Form were designated as either EL, IFEP or RFEP. I discuss this in a later section (See *Conclusions and implications for curriculum and instruction,* this chapter).

No Response. This category includes all students who did not submit any photos or type-written responses in the online forms for Lessons 1 or 2.

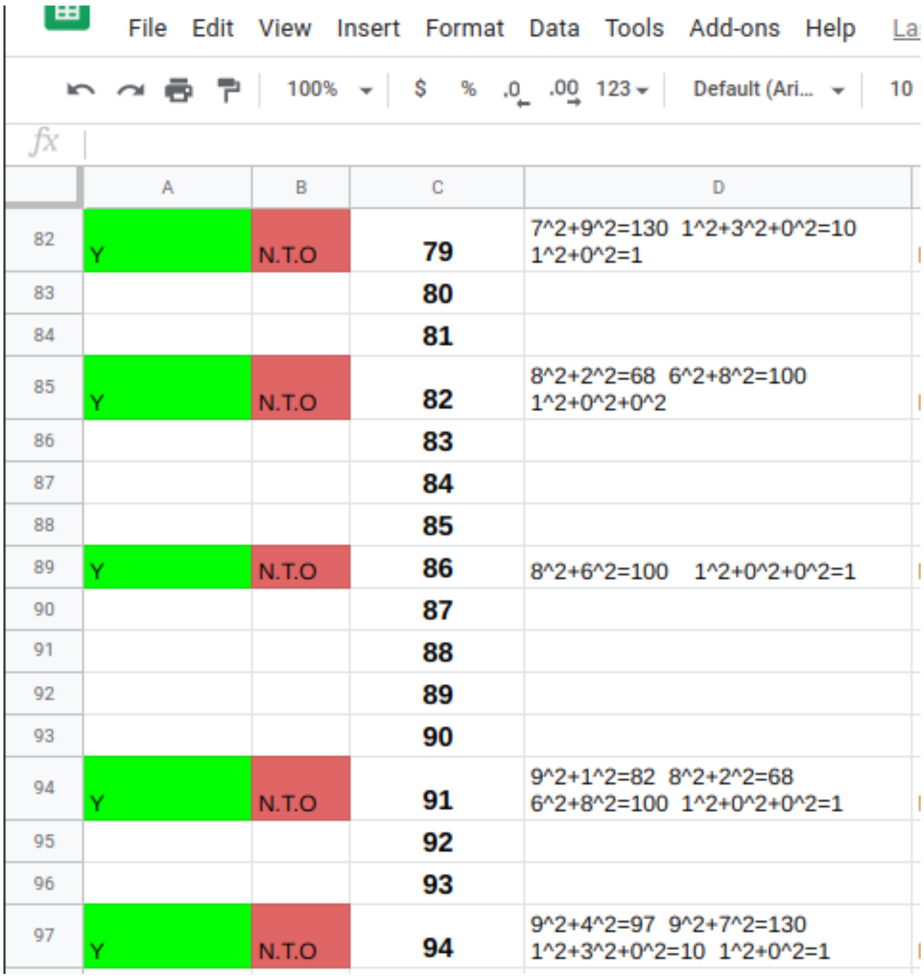
Generated No Claim. This category included responses that contained calculations, tables, or other visuals but no claim or warrant to constitute an argument. For example, Nina provided a picture of her work from Lesson 2 (see Figure 4.2). It features calculation and a statement that leaves the reader to interpret meaning from the photo.

Figure 4.2

Nina's Response from Lesson 2, Including Written Work (top), a List of Happy Numbers (bottom), and a Type-written Statement (middle).



In the picture above, I showed my work and how I explored the problem. You can see that some of the numbers I worked on were happy numbers and some were not.



	A	B	C	D
82	Y	N.T.O	79	7^2+9^2=130 1^2+3^2+0^2=10 1^2+0^2=1
83			80	
84			81	
85	Y	N.T.O	82	8^2+2^2=68 6^2+8^2=100 1^2+0^2+0^2
86			83	
87			84	
88			85	
89	Y	N.T.O	86	8^2+6^2=100 1^2+0^2+0^2=1
90			87	
91			88	
92			89	
93			90	
94	Y	N.T.O	91	9^2+1^2=82 8^2+2^2=68 6^2+8^2=100 1^2+0^2+0^2=1
95			92	
96			93	
97	Y	N.T.O	94	9^2+4^2=97 9^2+7^2=130 1^2+3^2+0^2=10 1^2+0^2=1

Description. Descriptions were typically explanations of the arithmetic procedures that a student used to complete a given part of the task. For example, if Part B of Task 1 asked students to calculate the areas of the first four Fibonacci Rectangles[19], then a student might put a description of how they did so (see Figure 4.3). Descriptions such as these generally foregrounded how students generated the examples on their pages. In the first lesson, nine of the fourteen submissions included descriptions of procedures. In the second lesson, before any conferencing took place,

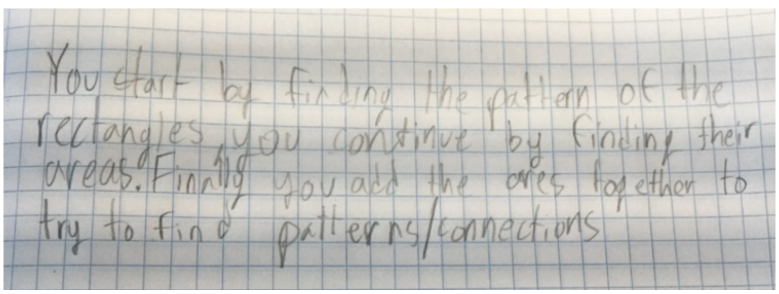
[19] See Appendix A for complete tasks.

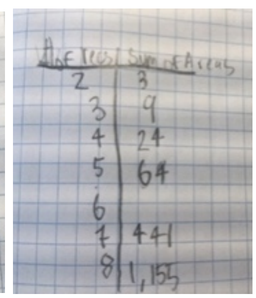
descriptions of procedures were also the predominant response. Individual student responses varied in form.

Figure 4.3

Nathan's Description of How to Calculate the Areas of Fibonacci Rectangles and Find Their Sums, Including a Typed Explanation to a "Friend" (top) and Submitted Photos (bottom left and right).

"I started by finding the size of the rectangles then I continued on by putting them onto a chart and adding them all together. The first + the second + the third etc."





Generated a Claim. This category refers to the student responses that included a mathematical claim. In the Happy Numbers task, a task that prompts students to generate a claim from Lesson 2, students were asked to make an argument about 2-digit or 3-digit happy numbers, or "something else". Nathan generated the following claim: "10 to any exponent is a happy number…". In order to generate a claim, students had to state a claim beyond simply restating the problem or describing a procedure.

Responses for Qualifying-a-claim Tasks

Below I describe the types of responses from the third lesson, which culminated with students qualifying a claim and then justifying their choice of qualification. Complete descriptions of the types of responses summarized in Table 4.2 (above) will include a brief characterization and example of student work.

Overview.

Table 4.2 is organized so that differences between the two opportunities to justify how they qualified a claim (Item #1 and Item #2) can be highlighted. In Lesson 3, students were given the opportunity to make two arguments by justifying two of the claims they qualified as either 'always', 'sometimes', or 'never' true. I note that empirical arguments were predictably the more common form of argumentation. Six empirical arguments were submitted for each of the items (not, necessarily, by the same students). Descriptions of responses follow.

No Response.

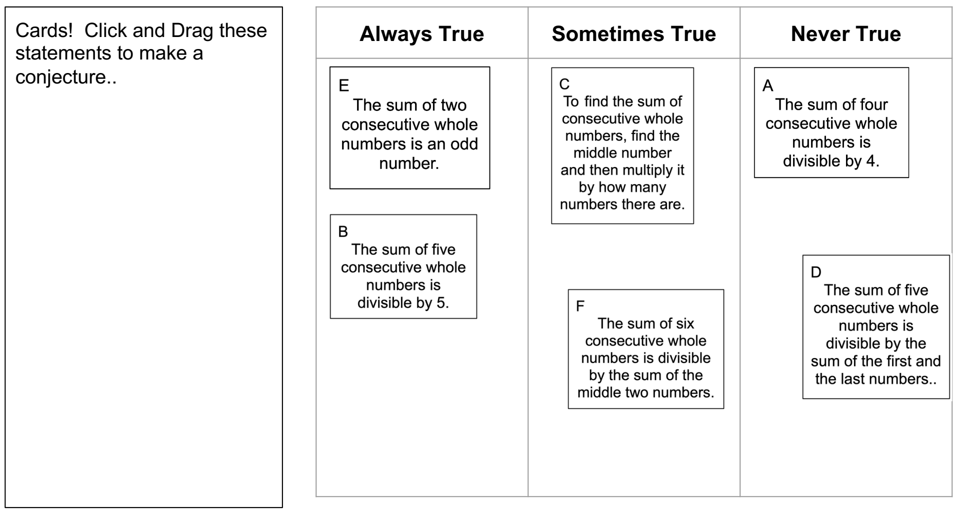
This category includes all students who did not submit any photos or type-written responses in the online forms for Items #1 or #2 of Lesson 3.

Qualified only.

These students qualified claims but did not justify. Students would qualify a claim as "always", "sometimes", or "never" true by clicking on a conjecture (A – F) and dragging it to a column of their choosing (See Figure 4.4 for an example). In this response type, no other written calculations, explanations, or attempts to justify were included.

Figure 4.4

Sample of a Student who Qualified-only. Alisha's Cards Shown With Her Own Placements.



Qualified and justified.

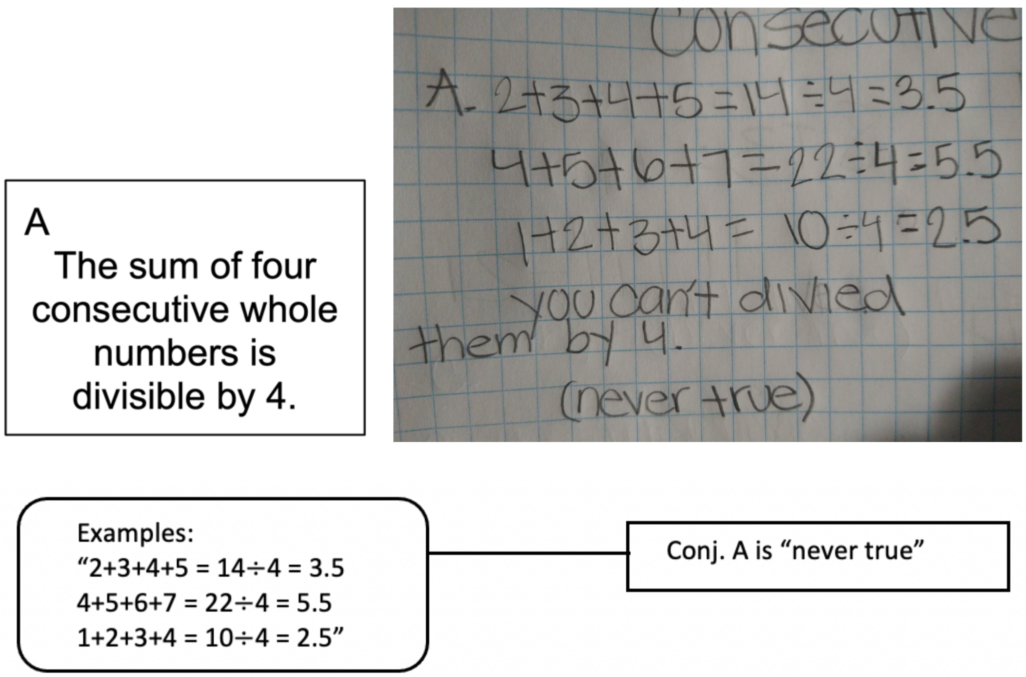
This category includes all students who qualified claims and justified them, i.e., they made an argument. There are three kinds of argument that emerged from the data: empirical arguments, partial arguments, and general arguments. Each of these arguments is described further here.

Empirical.

An empirical argument is one where conjectures are "validated, impugned, or subverted by appeals to physical facts or sensory experiences" (Harel & Sowder, 1998, p.252). In the case of middle school students, this can take the form of an inductive empirical argument, where calculated examples, without any warrants or backing, are taken as demonstrations of a conjecture's validity or counterexamples to demonstrate invalidity. There are also perceptual empirical arguments that rely on "rudimentary mental images" (Harel & Sowder, 1998). An example of an empirical argument is shown in Figure 4.5. Amaya argued that the sum of four consecutive whole numbers is never divisible by four by citing three examples.

Figure 4.5

Amaya's Empirical Argument that Conjecture A is Never True (diagram below).



Partial.

This category includes responses where students made a claim and included a warrant, but absent any other backing, data, or examples, the argument is incomplete. It is similar to the *Proof Production Level* 2 category presented in other work (Knuth, Choppin, & Bieda, 2011). For example, Evelyn asserted that the sum of four consecutive whole numbers is *never* divisible by four "…because the number always is equal to a decimal number and not a whole number" (from type-written response). Absent any examples or backing for the warrant, the claim remained unproven.

General.

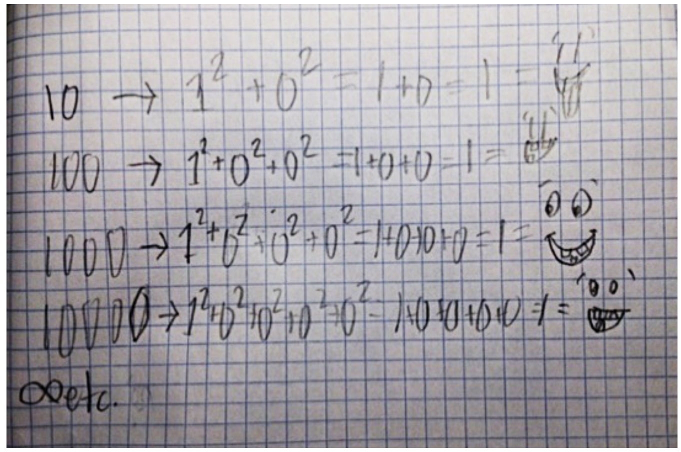
General arguments are those that reflect analytic proof schemes (Harel & Sowder, 1998). Analytic proof schemes can be *transformational* or *axiomatic.* Transformational proof schemes are ones where "justifications are concerned with *general* aspects of a situation and involve reasoning oriented toward settling the conjecture in *general*" (Harel & Sowder, 1998, p.673, emphasis added) while axiomatic proof schemes are reflected in students who are comfortable working with mathematics as an organized body of knowledge "so that subsequent results are logical consequences of preceding ones" (p.674). The example below (see Figure 4.6) comes from Nathan's work in Lesson 2. Nathan's argument contains a claim ("10 to any exponent is a happy number"), data (4 examples), and a warrant ("1 to any power is one and 0 to any power is zero so 1 + any number of 0's is still one"). I also include a diagram using the Toulmin model as an organizing scheme to highlight the parts of the argument.

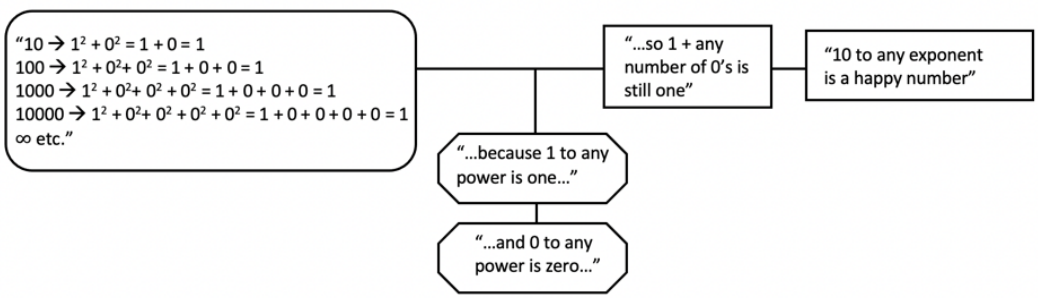
In summary, this section characterized student responses to tasks that ask students to either generate a claim (Lessons 1 and 2) or qualify a claim (Lesson 3). These characterizations included images of student work, descriptions of distinguishing characteristics, and where possible references to previous research.

Figure 4.6

Nathan's General Argument. Typed Argument (left), Photo of Examples as Data (right), and Diagram (bottom).

"10 to any exponent is a happy number because 1 to any power is one and 0 to any power is zero so 1 + any number of 0's is still one"





Shifts Toward Generalizing

In this section, I show how students' work changed over time, going beyond describing how they generated examples toward justifying with general arguments. I review the written work of focal students Nathan and Nina, as students who exemplify the generalizing that emerged, to illustrate how their written work changed.

Focal Student 1: Nathan

In response to the Fibonacci's Rectangles task (Lesson 1), Nathan described a procedure (see Figure 4.3, above). Nathan is a middle-achieving student whose language fluency is recorded as "English Only". When asked to generate and justify a claim about predicting sums of areas of Fibonacci Rectangles (Lesson 1), he described the procedure to generate a data table (by calculating the areas and then adding them). In response to the Happy Numbers task (Lesson 2), Nathan made a general argument. As a class, students spent time finding as many of the two-digit happy numbers (17 possible) as time permitted. Then, students were asked to make an argument about happy numbers. Nathan argued that "10 to any exponent is a happy number…" (see Figure 4.6, above). His claim is general, i.e., relevant to *any* power of ten. Moreover, Nathan justified the argument with implicit references to two mathematical properties. Nathan's statements that, "1 to any power is one" implicitly refers to the identity property of multiplication while "0 to any power is zero" implicitly refers to the multiplication property of 0, which states that zero times any number is zero. Albeit a simple argument, it seems to exemplify an axiomatic proof scheme because Nathan is settling the conjecture in general and working with mathematics as an "organized body of knowledge" (Harel & Sowder, 1998).

In response to the Consecutive Sums task (Lesson 3), Nathan similarly argued, primarily with words and examples, that the sums of four consecutive whole numbers are *never* divisible by four ('Conjecture A'). Nathan's argument contained the example, "1+2+3+4 = 10" (see Figure 4.7). From here, Nathan noted that "it keeps adding 4 to 10". At this point, I highlight a key aspect of Nathan's argument: the representation of the multiples of four with words. To conclude, he stated that "10 + any multiple of 4 will not be divisible by four." While Nathan made the statement as a matter of fact, the argument is general without the use of variables or a visual.

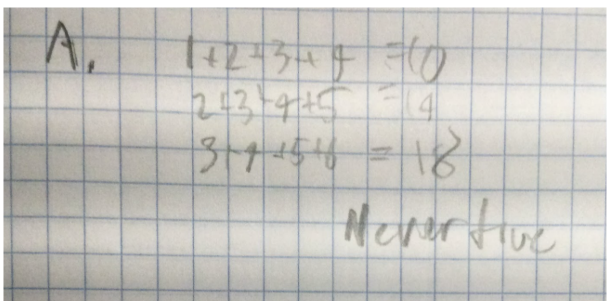
The absence of variables does not detract from the fact that Nathan represented with words what Usiskin (1988) referred to as an equation, or open mathematical sentence, where the variable is unknown: “10 + any multiple of 4” can be understood as $10 + 4n$. Nathan demonstrated a conceptual understanding of one purpose for variables.

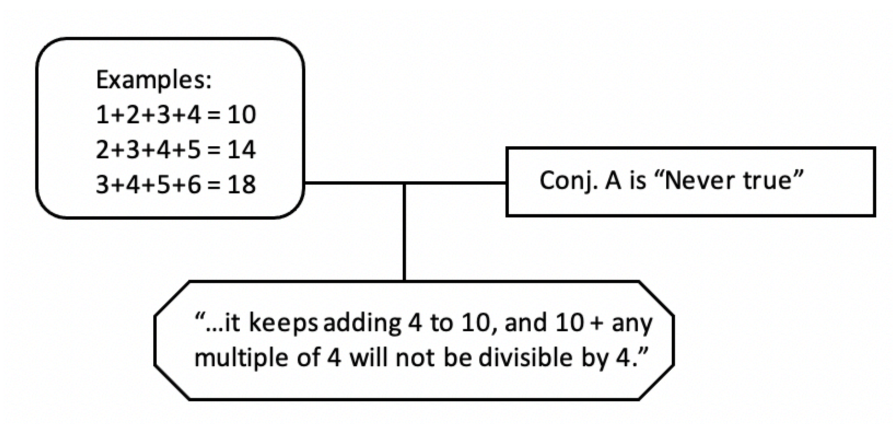
Nathan’s written work changed from describing how to generate a data table to making a general argument and using the concept of variable. In Lesson 1, he described the procedure for calculating values to place in a table. In Lesson 2, he implicitly referred to two mathematical properties to make a general argument. In Lesson 3, he used the concept of variable when he justified his claim. After Lesson 1, Nathan described more than calculations but mathematical properties and concepts to support his arguments. He used language to represent and leverage mathematical properties (e.g., “0 to any power is zero”) as warrants and then invoked the mathematical concept of variable (“10 + any multiple of 4” as $10 + 4n$) using mostly words.

Figure 4.7

Nathan's General Argument from Lesson 3, Including Claim and Rationale (top), Examples (middle), and Diagram (bottom).

I think that A is never true because it keeps adding 4 to 10, and 10 + any multiple of 4 will not be divisible by 4. For example, 1+2+3+4=10 and 10, is NOT divisible by 4.





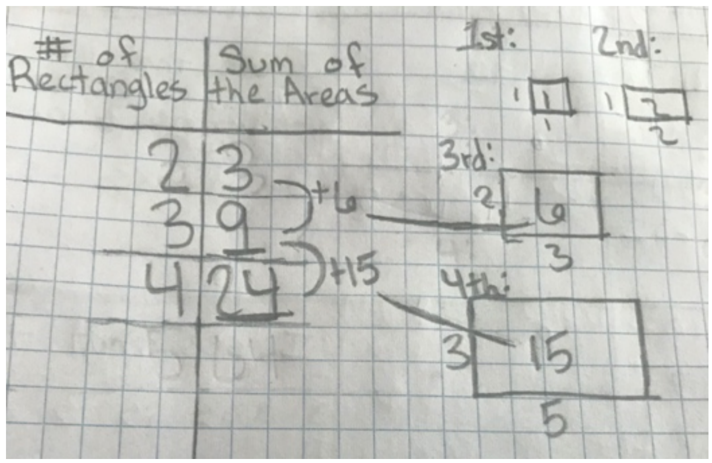
Focal Student 2: Nina

Nina is a high-achieving student whose language fluency is recorded as RFEP. In Lesson 1, Nina provided a description for a procedure. In Lesson 2, Nina was asked to predict sums of areas of Fibonacci Rectangles and to generate an argument about why the prediction works. She filled in the table of values and described the procedure to calculate values in the table. The description of the procedure is shown in Figure 4.8.

Figure 4.8

Nina's Description from Lesson 1, Including Her Narrative (left) and Annotations (right).

The Sum of the Areas will always be the area, of the rectangle you're on, plus the previous Sum of the Areas.



Nina's response was categorized as generating no claim (see Figure 4.2, above). The response was a complete listing of each happy number that she found. The submission was a complete response to Part A of the task, but not a response to the final prompt, which invited students to find more Happy Numbers (without calculating), containing either two or three digits, and justify why their method works. After Lesson 2, Nina and I conferenced about her response. In Lesson 3, Nina submitted two general arguments (see Figure 4.9 for diagrams and Figures 4.10 and 4.11 for original work).

Figure 4.9

Diagrams of Nina's General Arguments for Conjectures E (top) and F (bottom).

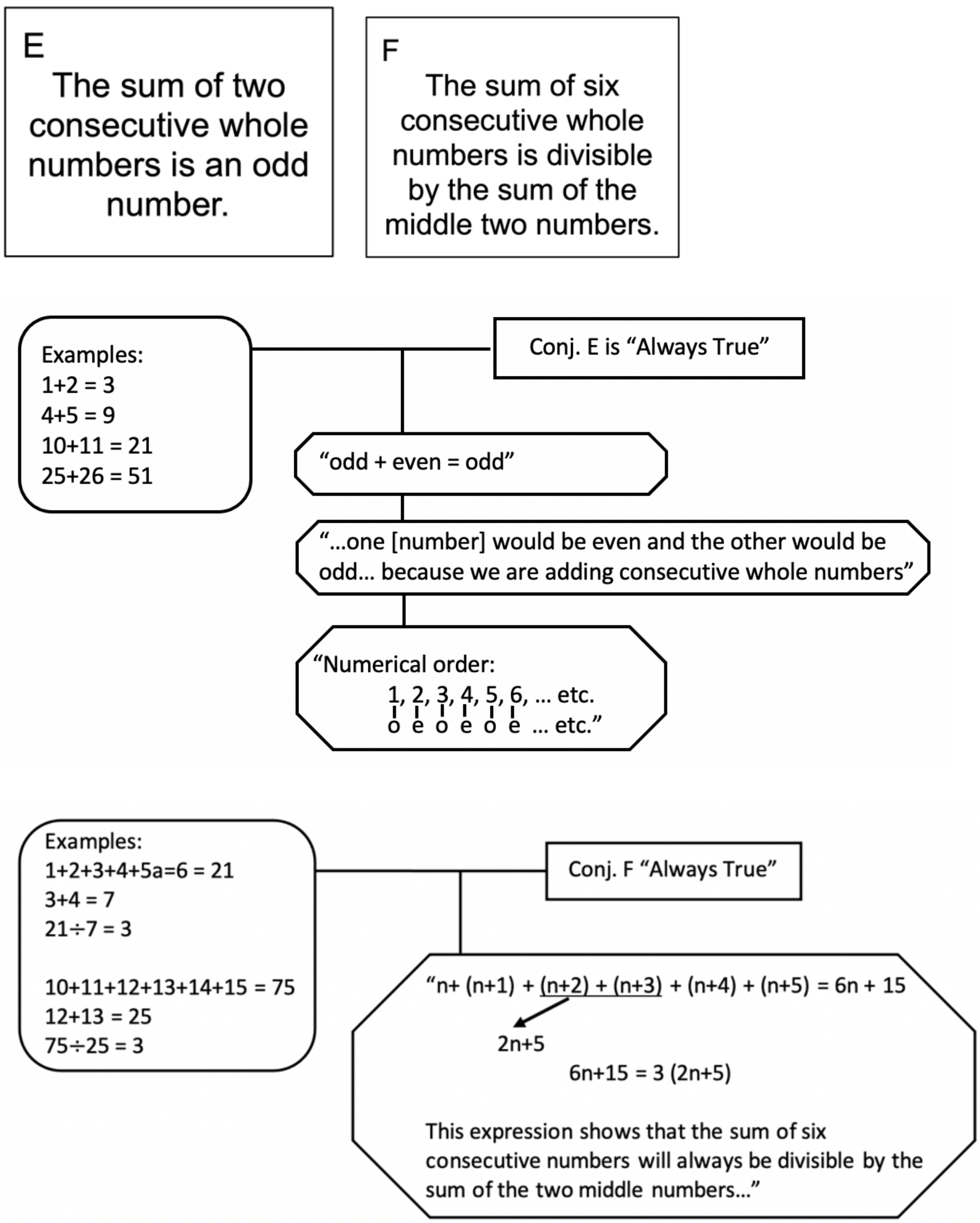
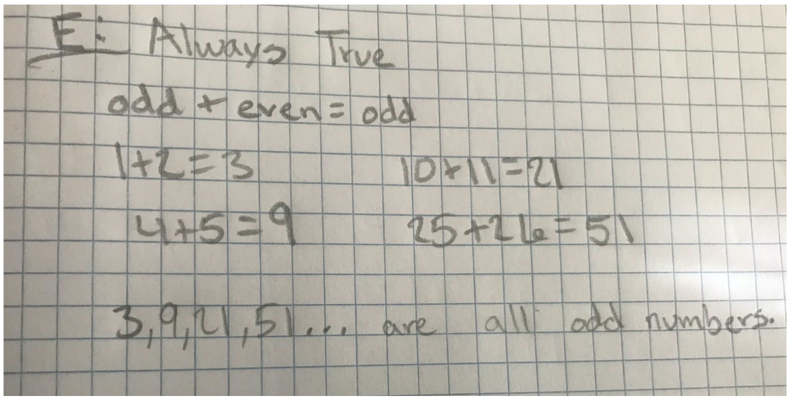


Figure 4.10

Nina's Written Argument for Conjecture E. 'Convince Yourself' (top), 'Convince a Friend' (middle), and 'Convince a Skeptic' (bottom).



When I was exploring conjecture E, I remembered that, out of the two consecutive whole numbers I needed to add, one would be even and the other would be odd. This comes in handy because, when you add an odd number and an even number together, you get an odd number. Therefore, when you add two consecutive whole numbers, you will always get and odd number as the answer.

E:

o: odd number
e: even number

Numerical order:

1, 2, 3, 4, 5, 6...etc.
o e o e o e...etc.

When you add an even number and an odd number you will always get an odd number.

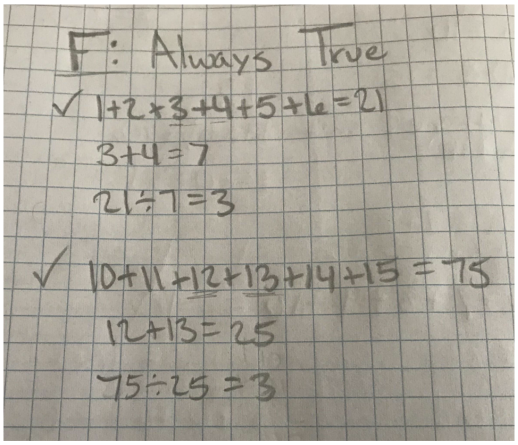
$3 + 10 = 13$ (o e o) $\quad 12 + 21 = 33$ (e o o)

One of the numbers I would need to add in this conjecture would have to be odd and the other number would have to be an even number. This is because we are adding consecutive whole numbers. Therefore, the sum of two consecutive numbers will always be an odd number.

$3+4=7$ (o e o) $\quad 6+7=13$ (e o o)

Figure 4.11

Nina's Written Arguments for Conjecture F. 'Convince Yourself' Response (top), 'Convince a Friend' Response (middle), and 'Convince a Skeptic' Response (bottom).



F: Always True

✓ $1+2+3+4+5+6=21$

$3+4=7$

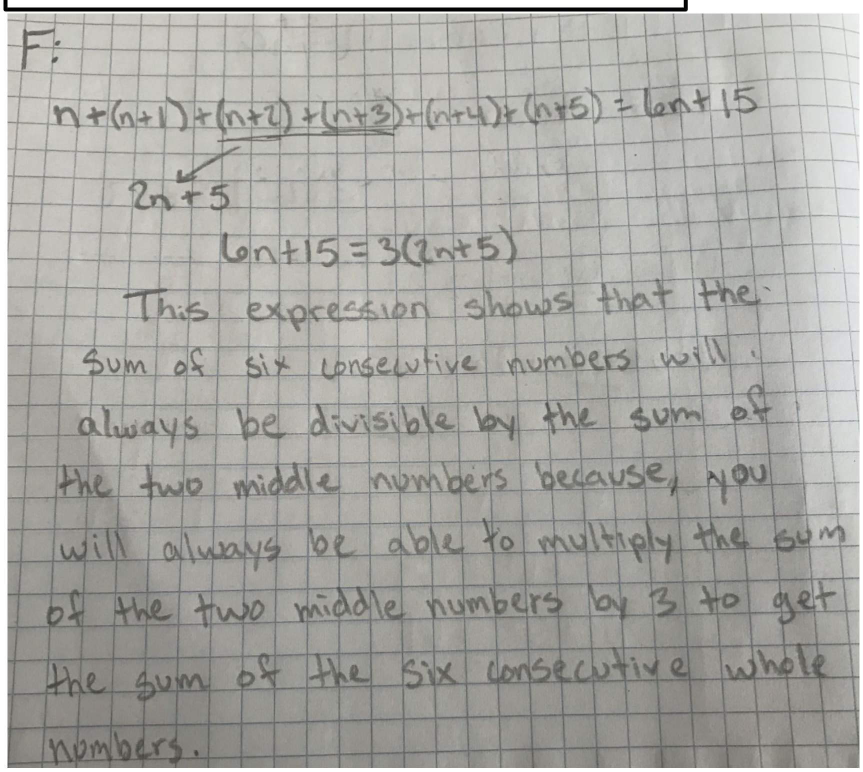
$21\div 7=3$

✓ $10+11+12+13+14+15=75$

$12+13=25$

$75\div 25=3$

When I was exploring conjecture F I noticed that every time I tried it out, the conjecture was correct.



F:

$n+(n+1)+(n+2)+(n+3)+(n+4)+(n+5)=6n+15$

$2n+5$

$6n+15=3(2n+5)$

This expression shows that the sum of six consecutive numbers will always be divisible by the sum of the two middle numbers because, you will always be able to multiply the sum of the two middle numbers by 3 to get the sum of the six consecutive whole numbers.

Nina's arguments used algebraic concepts, such as modeling a situation with variables, and seemed to use axiomatic proof schemes. In her argument for Conjecture E (which stated that "The sum of two consecutive whole numbers is an odd number"), Nina used warrants, backing, and data to support her selection that the conjecture was "Always" true. Nina's warrant stated that the conjecture must be true because when adding two consecutive whole numbers "one of the numbers … would have to be odd and the other number would have to be an even number." As backing, she included a "numerical order" list of whole numbers that is annotated with the letters "o" and "e", presumably to designate odd and even numbers and illustrate them alternating (see Figure 4.10). Nina suggested that the pattern could be extended indefinitely using ellipses and the abbreviated phrase, "etc." Her data included what appeared to be four strategically chosen examples. Two of the examples included sequences of numbers that begin with even numbers, "4+5=9" and "10+11=21," and the other two included sequences of numbers that begin with odd numbers, "1+2=3" and "25+26=51". This argument is general and shows Nina's capacity for creating and using her own resources, such as the "numerical order" list with annotations, and strategic and thoughtful use of examples (Ellis et al., 2012).

Nina's second argument was closer to an algebraic proof because it relied on the algebraic properties that make transformations of equations possible, such as the distributive property. Nina qualified Conjecture F, "The sum of six consecutive whole numbers is divisible by the sum of the middle two numbers," as *always* true. Like her argument for Conjecture E, Nina recorded some examples in the "Convince Yourself" portion of the Convince Form, essentially an empirical argument. Then, she "convinced a friend" by stating that she "noticed" the conjecture was true for every example. Then, when "convinc[ing] a skeptic", she used variables to model the situation. Nina modeled the sum of six consecutive numbers as "n + (n+1) + (n+2) + (n+3) + (n+4) + (n+5)" or "6n+15" (see Figure 4.11). This equation embodies Usiskin's (1988) use of variable for the purpose of 'equations' where the variable is unknown. Nina highlighted the middle two terms as having a sum of "2n+5" and implicitly used the distributive property to assert that "…you will always be able to multiply the sum of the two middle numbers by 3 to get the sum of the six consecutive whole numbers." Nina left implicit that if the sum of the two middle numbers is a factor of the sum of the six consecutive whole numbers, then the sum of the six consecutive whole numbers is divisible by the sum of the two middle numbers (by the definition of 'factor'). Nina changed from writing descriptions and lists to making generalized arguments that provided warrants, backing, and examples to justify the ways she qualified claims.

Nina's written work changed from describing procedures and listing examples (in Lessons 1 and 2) to making general arguments that included mathematical resources she created, and modeling with variables. Nina used writing to communicate the kind of generality that is required of a mathematical proof. Specifically, her use of ellipses, the phrase "etc.", and the word "always" point toward settling her qualification of Conjecture E *in general* (see Figure 4.10). When justifying her qualification of Conjecture F as always true, Nina modeled the situation with variables and then used language (rather than transforming the equations) to indicate that the product of three and the "middle two numbers" (as '2n + 5') will always yield the sum of six consecutive integers (as '6n + 15').

Using Examples in Logically Sound Ways

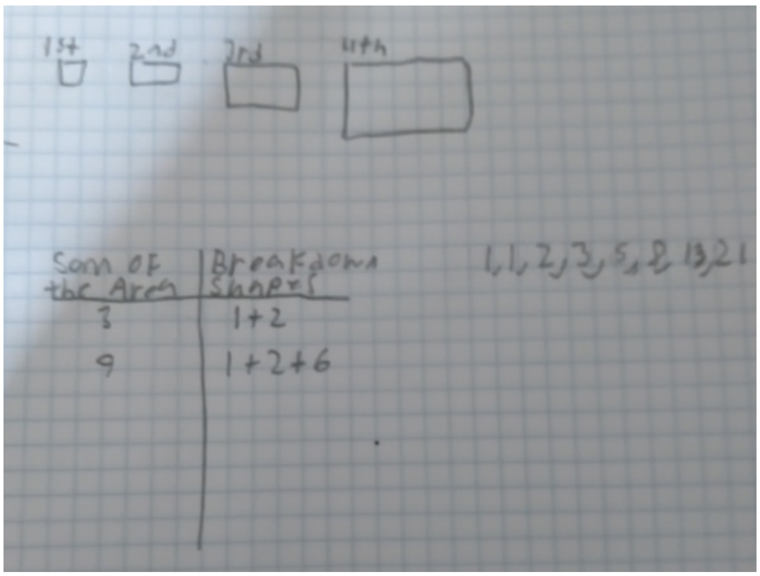
In this section, I show how students' responses changed from using examples to support their claims (without considering examples that do not) to using examples as data or evidence in a mathematical argument that is consistent with all available examples. I review the written work of focal students Renato and Amaya to illustrate the change. While empirical arguments are not considered general (as they do not consider the need for a general justification), the change reflects a use of examples that is logically sound and part of a larger process for understanding a problem situation.

Focal Student 3: Renato

Renato is classified as a "low-achieving student", but he learned to justify the ways he qualified claims in the final task. For Lesson 1, students worked on the Fibonacci Rectangles task. They were asked to conjecture about any patterns present in the tables they generated. Renato submitted a photo that shows a table with the first two sums of rectangle areas for the Fibonacci Rectangle task coded as no-claim (see Figure 4.12). The photo contains the first four rectangles that are generated using the numbers in the Fibonacci sequence. The sequence that Renato lists includes "1, 1, 2, 3, 5, 8, 13, 21" and his rectangles have dimensions 1 X 1, 1 X 2, 2 X 3, and 3 X 5. The table has columns for the "sum of the area" and "breakdown shapes". The two listed sums of 3 and 9 are correct for the sums of the first two and three rectangles, respectively. The student's response did not include a pattern or a claim, so this response was coded as 'generated no claim'.

Figure 4.12

Renato's Photo Submission for Lesson 1.

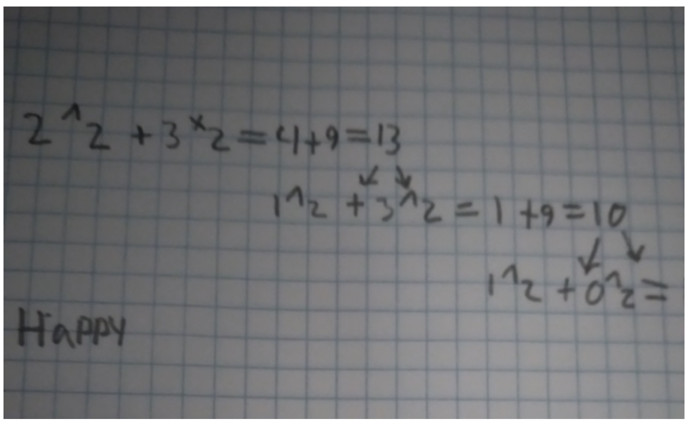


In Lesson 2, the Happy Numbers task asked students to find two- or three-digit happy numbers (without calculating) and justify why their method works. Recall that Happy numbers are found by separating the digits of a number, squaring each digit, and then adding the resulting products. If the result is 1, then the number is 'happy'; however, if not, then the process repeats until it results in a 1 or a number is repeated (in which case the number is not 'happy').

In his response for Lesson 2, Renato described the procedure specified above (see Figure 4.13). Strictly speaking, the description is not correct because Renato indicated "multiply[ing]" by two rather than squaring, but the example in the photo correctly illustrates the arithmetic procedure and shows 23 as a “happy” number. Renato performed the procedure correctly, although he described it without canonical mathematical language.

Figure 4.13

Renato’s Photo Submission (left) and Accompanying Description (right): Lesson 2.



you have to get a number separate the number and then multiply each number by two.and if it ends at one then its happy and if it doesnt then its not happy

In response to the Consecutive Sums task (Lesson 3), Renato used examples to support the ways he qualified claims. To begin, Renato qualified four of the six conjectures correctly (see Figure 4.14). Of those, he selected two to justify (Conjectures A and B). When prompted to ‘convince a friend’ that the qualifications are correct, Renato listed one example for each (see Figure 4.15), generating empirical arguments. He did not post any additional work in response to the prompt for ‘convincing a skeptic’.

Figure 4.14

How Renato Qualified Claims

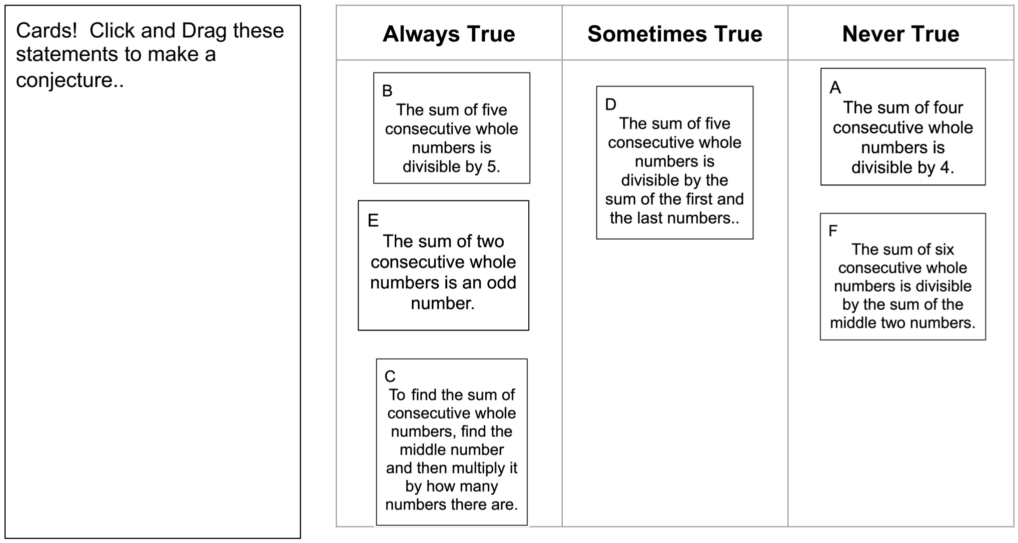
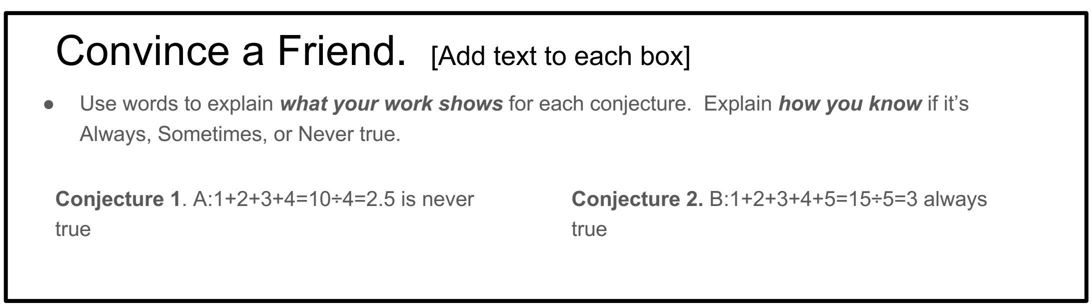


Figure 4.15

Renato's Empirical Argument for Conjectures A and B



As with all empirical arguments, Renato did not address the need for a more general argument. Based on the strength of a single example, the claims are assumed to be justified as *always true* (as in the case of Conjecture B) or *never true* (as in the case of Conjecture A). Generality notwithstanding, the way Renato qualified the claims and their respective examples were correct. At this point, Renato was poised to consider how to represent this mathematical situation (e.g., with variables or other visual representation). In comparison to the first two lessons, Renato's response changed to reflect an empirical proof scheme, thus showing progress. In Lesson 1, he was just beginning to engage the task (showing some calculations without having generated a claim) and in Lesson 2, his response was to describe a procedure.

Renato's written work shows a student whose engagement with examples changed over the lessons. The first two lessons culminated in tasks that prompted students to generate a claim. In Lesson 1, Renato submitted two examples. He made no claim about patterns in the table. In Lesson 2, Renato provided an example and described the process for generating examples to fill a table. Thus, he used language to describe how he *would* fill the table. Lesson 3 culminated in a task prompting students to qualify claims for which Renato correctly qualified four of the six claim cards and justified the selections of two (Conjectures A and B) with empirical arguments. This illustrates how Renato's use of examples changed as they are no longer the main substance of Renato's responses, but evidence to support a larger argument about how the claims were qualified.

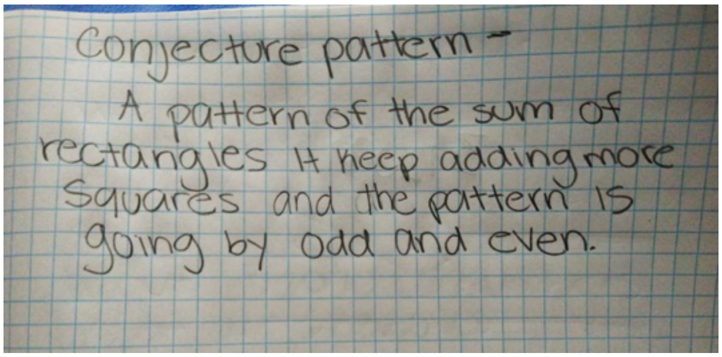
Focal Student 4: Amaya

Amaya is classified as a mathematically "low-achieving student" and learned to avoid logical fallacies and argue empirically. In Lesson 1, she attempted to generate a claim and justify it. The Fibonacci Rectangles task asks students to find patterns in the sums of rectangles whose dimensions are derived from the Fibonacci sequence. Amaya generated the claim that "A pattern of the sum of rectangles it [sic] keep adding more squares and *the pattern is going by odd and even*" (see Figure 4.16, emphasis added). It appears that Amaya was suggesting that one possible pattern in the sums is the alternation of the sums as odd and even numbers. Amaya also included a photo of her data table. Note that she did not support the claim with the data available in the table (see Figure 4.16). There are both even and odd sums present in the "sum of the areas" column, but they do not go "by odd and even".

Eventually, Amaya changed to using her examples as data in an empirical argument that considered all available examples. In Lesson 2, Amaya did not submit a response. In Lesson 3, Amaya submitted her card sort for the Consecutive Sums task, where claims were qualified by placing conjecture cards in columns labeled "Always True", "Sometimes True", and "Never True". Amaya correctly placed three cards (Conjectures A, B, and E) and possibly a fourth (Conjecture C; see Figure 4.17). Amaya submitted empirical arguments to justify how she qualified Conjectures A and B. She argued that the sum of four consecutive whole numbers (Conjecture A) is *never* divisible by four, citing three examples (see Figure 4.5, above). Similarly, she argued that the sum of five consecutive whole numbers (Conjecture B) is *always* divisible by five (see Figure 4.18), citing the same number of examples.

Figure 4.16

Amaya's Claim (top) and Data Table (bottom) from Lesson 1



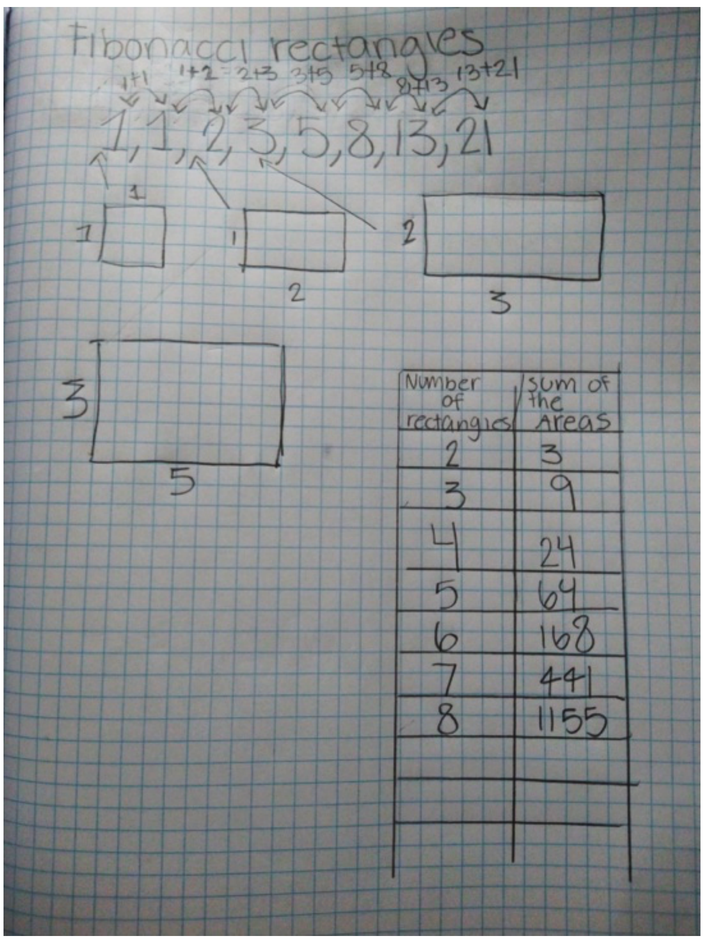


Figure 4.17

Amaya's Qualified Claims for Lesson 3

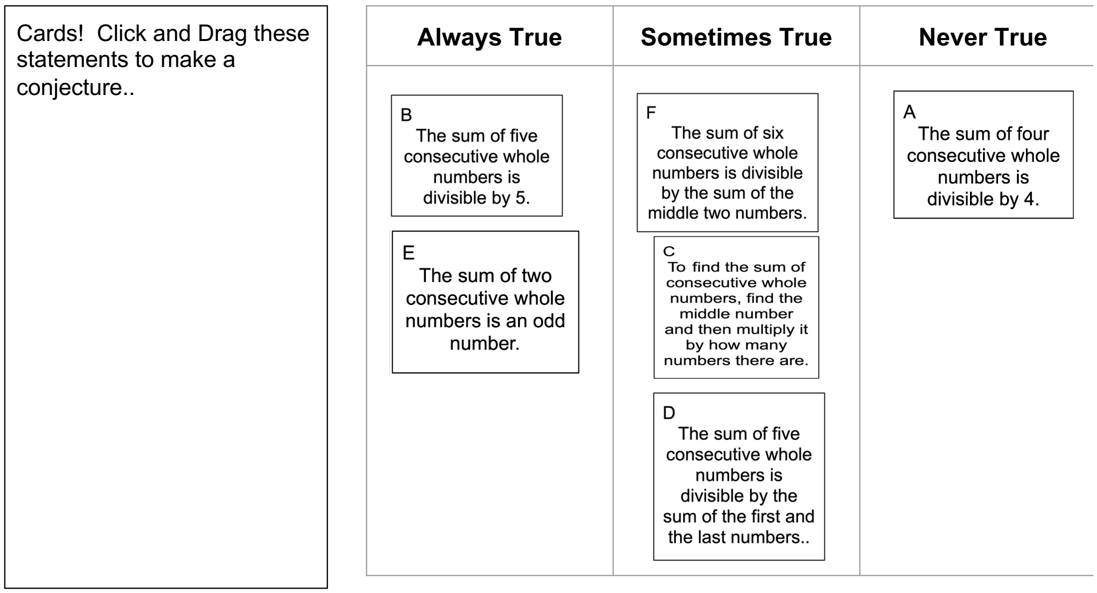
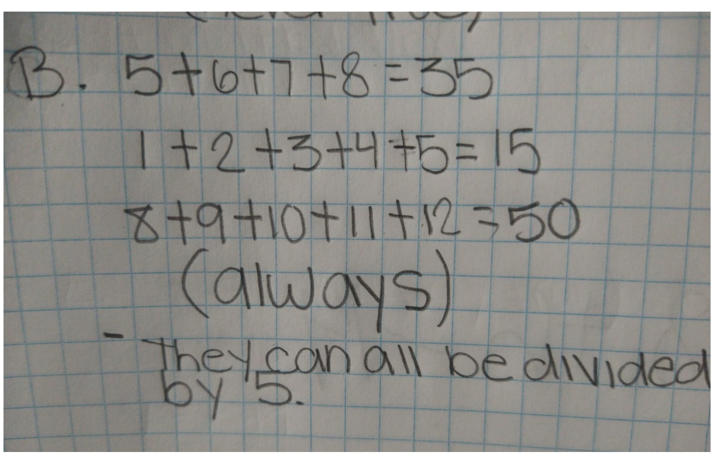


Figure 4.18

Amaya's Empirical Argument for Conjecture B, Lesson 3



From Lesson 1 to Lesson 3, Amaya changed how she used the available examples. In Lesson 1, she generated a claim that the Fibonacci rectangle pattern was going “by odd and even” (Figure 4.16). This preconceived idea was not supported by all the data available in her table, which indicated values of 3, 9, 24, 64, 168, 441, and 1155. So, this argument represents cherry picking because only some of the data support the claim: a logical fallacy. By Lesson 3, Amaya was selecting examples as empirical data to justify the ways she qualified claims, but all the available examples support the way the claims are qualified. Amaya’s written work highlights the need to draw students’ attention to a collection of examples as an example space (Watson & Mason, 2005) rather than focusing narrowly on only those examples that support a priori beliefs.

Generating Claims

In this section, I show how, lesson-to-lesson, students shifted from not generating any claims to generating claims in one of two forms: generating claims or qualifying claims. I review descriptive data from Lessons 1, 2, and 3 and show how some students initially had difficulty generating claims and justifying them (Lesson 1), shifted toward generating claims and justifying (in Lesson 2) and then qualifying and justifying claims (in Lesson 3). In this section, I review overall responses to the tasks and illustrate changes in how students made claims with the written work of focal students Oscar and Filomeno.

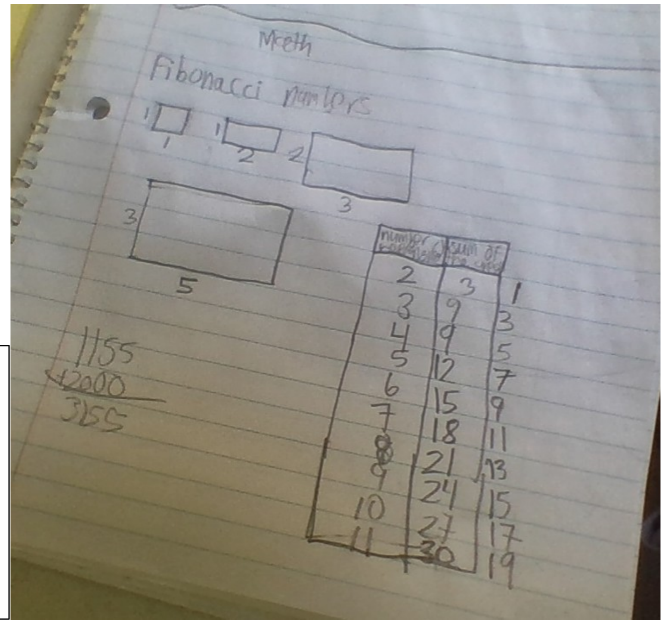
In Lesson 1, only three of the 14 respondents (there were 44 total participants) generated a claim in response to the Fibonacci Rectangles task, which asked them to generate a claim (see Table 4.1). Nine participants submitted descriptions of procedures, two participants submitted work without generating a claim, and 30 participants submitted no response. None of the responses that included claims were justified (or even justifiable). One example was discussed in Amaya's written work, above (see Figure 4.16). The claim she generated was countered by her own presented data (examples). Likewise, Oscar and Filomeno generated claims that were not supported by the data and examples they provided. Both appear to be attempts at empirical arguments of the second type: perceptual (Harel & Sowder, 1998).

Oscar is characterized as a "low-achieving student" but is a participant in the Special Education program. A full-time Instructional Assistant supports his activity in the classroom tasks because of the nature of his cognitive diversity. Oscar's claim was that the sums of areas for Fibonacci Rectangles "…were going up by 3 every time" (see Figure 4.19). While his table of values seems to corroborate that claim, the calculations that produced the "sum of the areas" column were not correct.

Figure 4.19

Oscar's Generated Claim (left) and Data (right) for Lesson 1.

I convinced a friend by explaining him that the fibonnaci numbers in the squares and rectangles were going up by 3 every time.

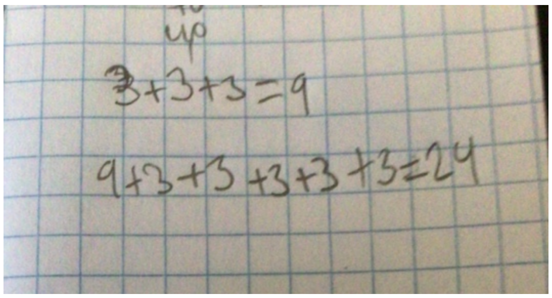


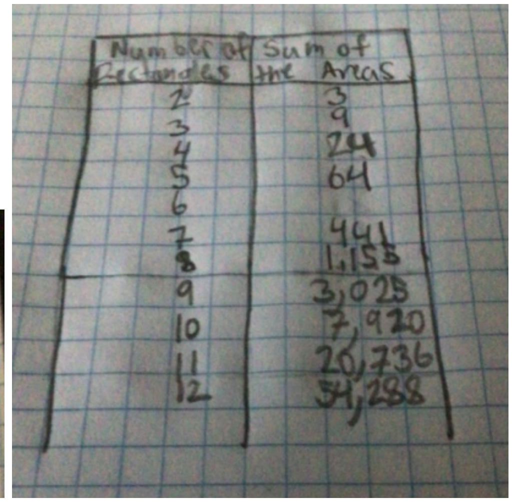
Filomeno also made a claim that was not supported by his own data. He asserted that each subsequent term in the table could be reached by adding some number of threes (see Figure 4.20). Only some of the data in the table supported this claim. This appears to also be a case of cherry-picking.

Figure 4.20

Filomeno's Generated Claim with Two Examples (upper and lower left) with Data Table (right)

I kept on adding 3 to match up with the numbers in the sheet.





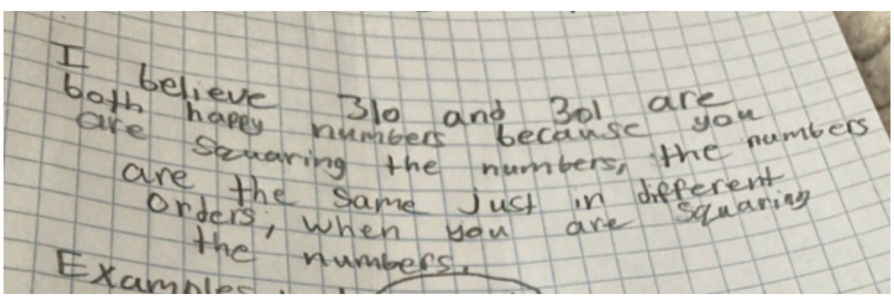
The table of values that Filomeno supplied includes values that support the claim and others that do not. For example, 9, 24, 441, and others, can be represented as a sum of some number of threes. However, other values cannot be represented as the sum of some number of threes (i.e., 64 and 3,025). Consequently, it is possible that Filomeno generated the conjecture without verifying each datum in the table. Alternatively, Filomeno may have been aware of the discrepancy but ignored it. Either way, it could also be considered a form of cherry picking.

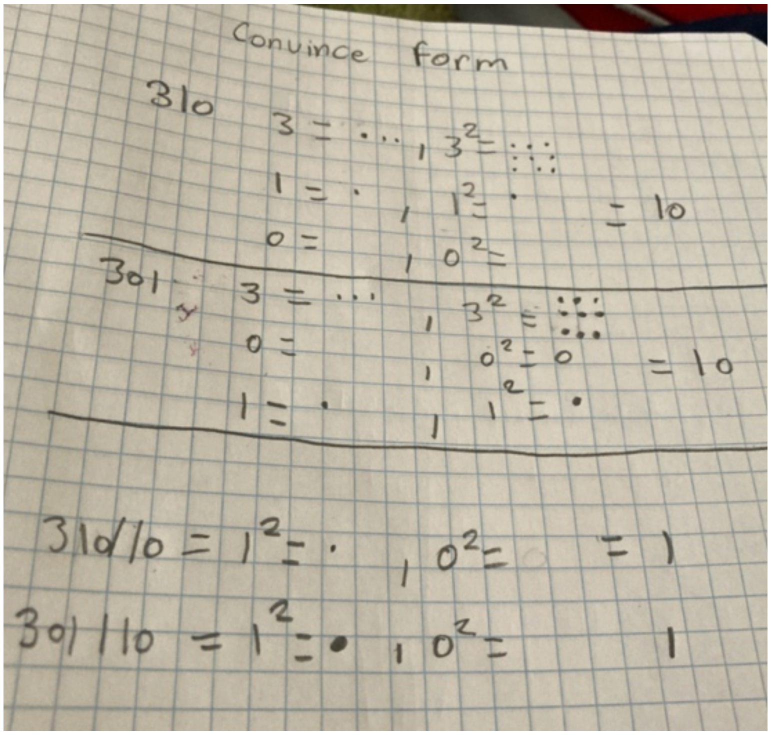
For both focal students, Oscar's and Filomeno's arguments appear to represent attempts at *perceptual* empirical arguments. These are arguments based on "rudimentary mental images" (Harel & Sowder, 1998). For example, for Filomeno, the data in the table appear to follow a pattern where each sum contains a multiple of three (written as a sum of some number of threes). However, two values do not satisfy the claim. The written work for both students appears to illustrate a *habit of* *mind* called “pattern sniffing” (Cuoco, Goldenberg, & Mark, 1996) and suggests that they are on their way to generating a viable claim.

In Lesson 2, two of the nine participants generated and submitted claims for the Happy Numbers task, which prompts students to generate a claim. Most students did not submit responses, but five students submitted descriptions and two submitted work with no claim (see Table 4.1). Antonio and Nathan submitted claims that they generated and justified. Antonio asserted that two numbers were “the same”, i.e., both happy numbers, because the order of the digits did not matter (see Figure 4.21). Although this is not a general claim, it does demonstrate that two numbers, 301 and 310, both result in being happy numbers because the digits are “…just in different orders [sic]”.

Figure 4.21

Antonio’s Generated Claim (top) and Data (bottom) Submitted in Lesson 2.





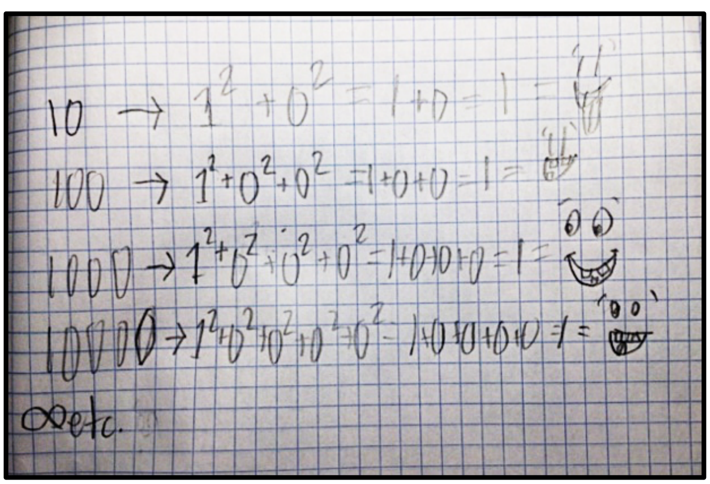
Antonio’s calculations showed how squaring the digits of 301 and 310 will result in the same sum of 1. Thus, they are both happy numbers. Nathan, on the other hand, provided a far more general claim because it applies to any power of ten. Nathan claimed that “10 to any exponent is a happy number…” (see Figure 4.22). His claim is general because it is relevant to *any* power of ten. Moreover, Nathan justified the argument implicitly using two mathematical properties. Nathan’s statements that, "1 to any power is one" implicitly uses the identity property of multiplication while "0 to any power is zero" implicitly uses the multiplication property of 0. Albeit a simple argument, it was the first argument generated by a student that represented an axiomatic proof scheme (Harel & Sowder, 1998).

While Antonio’s and Nathan’s arguments differ in terms of generality, they appear to represent significant mathematical progress. Focal student Nathan was the first student to generate and justify a general claim: applicable to a *class* of numbers (i.e., the powers of ten). Antonio's written work illustrates a critical inflection point in learning to construct arguments. Antonio's focus, the 'sameness' of two specific examples, relies on the mathematical property of commutativity, i.e., not simply on the calculations that show that both numbers are Happy. How much longer before Antonio comes to the realization that the digits of *any happy number* can be rearranged to create more happy numbers? The kind of exploration in which Antonio engaged could be a gateway to generating more general claims and justifying them. Antonio's argument is discussed at length in a future section (Chapter 5) and in the student-researcher conference which took place after Lesson 2.

Figure 4.22

Nathan's Argument (left) and Examples (right) from Lesson 2.

"10 to any exponent is a happy number because 1 to any power is one and 0 to any power is zero so 1 + any number of 0's is still one"



The Consecutive Sums task (Lesson 3) was different from the other two major tasks that students completed. First, students qualified claims by sorting "conjecture cards" as *always, sometimes,* or *never* true (6 total cards available). So, claims (or conjectures) did not need to be generated by the students. Secondly, due to low participation rates (Lessons 1 and 2), I changed the presentation of the Convince Form. Rather than the online form, which was accessible through external links, I made editable slides available for each student in the Google Classroom platform (which is where their classwork is typically housed).

Overall, Lesson 3 saw a dramatic increase in participation. Most participants qualified claims (40 of 44). These students qualified an average of 5.3 claims (out of six possible) at a rate of 77.5% correct (i.e., nearly four out of the five qualifications were correct). Eleven of the 40 participants made at least one argument (either empirical, partial, or general) to justify the ways they qualified the claims while the rest qualified claims without justifying or did not respond. See Table 4.4 for a summary.

Table 4.4

Summary of Responses for Items 1 and 2 of Lesson 3 (Qualifying-a-claim Tasks).

Qualifying a claim		*Item 1 Responses*	*Item 2 Responses*
No response		4	7
Qualified Claim, only		29	28
Qualified and justified Claim	*Empirical*	6	6
	Partial	2	0
	General	3	3
	Total	***44***	***44***

In summary, the changes in the students' participation reflect a move from not generating claims (or not making viable claims) in Lesson 1 to generating and justifying mathematical claims in Lesson 2, and then selecting and justifying claims in Lesson 3. In Lesson 1, most students (30 of the 44 participants) did not submit work. Most of the submissions included descriptions of procedures (9 students) and some generated non-viable claims that appear to be attempts at perceptual empirical arguments. No viable claims were generated in Lesson 1. In Lesson 2, much like Lesson 1, most students did not submit work (N = 35) and descriptions of procedures were the most common response (N = 5). A critical difference is that the claims generated in Lesson 2 were both viable, one of which represented an analytic proof scheme. Whereas Lessons 1 and 2 culminated in tasks prompting students to generate a claim, Lesson 3 culminated in a task prompting students to qualify claims. The majority of students (40 of the 44 participants) qualified claims and 11 of those students attempted to justify how they qualified claims with either empirical, partial, or general arguments. The focal students were typical of the latter group of 11 students who went beyond simply qualifying claims and included an argument.

This represented a significant improvement in participation and the kinds of arguments students made. Moreover, students who qualified claims (only) had a slightly *higher* percent correct than those who qualified claims and then attempted to justify how they qualified the claims (78.4% versus 74.2%). The high level of participation and increased presence and variety of arguments supports the notion that students should be scaffolded to make better arguments by first giving them task that ask them to qualify claims and then justify them before moving to tasks that ask students to develop and generate their own claims to justify. Tasks that ask students to qualify claims (rather than generate them) model the kinds of claims that are possible, as well as the kind of generality that is valued in mathematical argument and proof.

Summary of Findings

As students learned to generate claims and justify them through their written work, the data suggest several considerations. Students' early efforts were susceptible to logical fallacies, specifically in the form of cherry picking. As students worked to generate a claim, they were either willing to ignore some of the available data or did not (correctly) account for all of the available data. Consequently, there appears to be a need to focus students on the entire corpus of examples as related to whatever mathematical claims are at play.

Discussion

Student descriptions of procedures resemble the *procedural explanations* found in other research (for example, see Hebert & Powell, 2016). Procedural explanations use mathematical vocabulary to describe the procedures employed by students rather than mathematical concepts. The difference is critical because only the latter engenders writing for the purposes of mathematical *reasoning* (Casa et al., 2015). For the participants of this study, a procedural explanation related to, for example, the steps taken to determine whether a number is happy is only a first step in a larger process of inquiry that ultimately seeks to generate more generalized claims applicable to *classes of numbers* that might be 'happy'.

A next step in learning to generate claims, students would need to consider their examples as interconnected (Watson & Mason, 2005). A shift from generating examples (or describing how to generate them) to using them as evidence to support a claim is also significant. For example, in the case of Antonio. Antonio implicitly used the commutative property of addition as a warrant for the "sameness" of two examples. Teachers should notice this kind of reasoning and support it. That shift could signal a transition from the task of generating examples, where calculating each example is the focus, toward searching for and noticing patterns in the examples, with examples serving a variety of purposes (Ellis et al., 2012).

The analysis also showed that some students shifted from not generating claims to generating and justifying claims in Lesson 2 and qualifying and justifying claims in Lesson 3. In Lesson 1, only three of the 44 participants attempted to make a claim, but none of the claims were valid. In Lesson 2, two students made valid claims and justified them. By Lesson 3, 40 of the 44 participants qualified claims from a given set, 10 of whom even made an argument (either empirical, partial, or general). Initially, I was discouraged by the students who qualified claims, only because this seemed a minimal level of participation. However, students who qualified claims (only) in Lesson 3 averaged 78.4% accuracy (i.e., roughly 4 out of 5 cards were correctly placed as 'always,' 'sometimes,' or 'never' true). The 'qualified claims (only)' category represented the largest group of students (29 students for Item 1 and 28 students for Item 2).

One explanation for these responses could be the relative ease of *qualifying* a claim when compared to *generating* a claim. Since qualifying a claim required less effort than generating one, more students participated. Indeed, Banes (2019) found stronger student responses, i.e., students' written efforts to justify, associated with tasks that asked students to qualify claims (what she referred to as "selected-claim" tasks) and did not require students to "develop and articulate" their own claim (p.25). Alternatively, the increase in participation could have been a consequence of a change in presentation of the Convince Form. In Lessons 1 and 2, I supplied students with an outside link to a fillable form where they could upload photo files and type responses to prompts (see Appendix H). In contrast, Lesson 3's form was placed as an embedded activity (editable slides) among the other classroom assignments on the Google Classroom platform (see Appendix I). They clicked and dragged the claims to whichever column seemed an appropriate way to qualify them (either as "always true", "sometimes true", or "never true") and then had the options to justify their choice by typing text and/or uploading photos to additional slides. It is possible that the inclusion of the form with their other classroom assignments provided a familiar platform and greater motivation to complete it.

While the "absence" of justifying by three out of four students is concerning, the students qualified claims with great accuracy. So, why did students who took time to qualify so many claims so accurately *not* take the time to justify their qualifications? It is possible that many students were simply overly concerned with getting the “right answer” quickly (Lampert, 1990). Having submitted their answers, perhaps they did not feel a need to continue.

The students who generated their own claims, Nathan and Antonio, represent an important shift when compared to the beginning of the study. Like their peers, neither of these students generated a claim in Lesson 1. However, in Lesson 2, Nathan generated a claim about a class of numbers (“10 to any exponent”, i.e., the powers of 10). This shift toward generating claims about a *class* of numbers, rather than individual examples, represents the kind of generalization that is emblematic of mathematical arguments.

Conclusions and Implications for curriculum and instruction

We need to expand how we analyze arguments so that we include an analysis of the claims. Numerous students communicated difficulty when asked to generate their own claim (i.e., with open-claim tasks). However, fewer had difficulty with tasks that prompted them to qualify a claim, so this might be a good starting place for all students. Using tasks that prompt students to qualify a claim could be a good place to practice building an example space and looking for mathematical structure. In turn, students could use their newly acquired capacities for navigating example spaces to refine their capacities for developing, refining, and ultimately generating claims. This kind of dialectic approach needs to be included in curriculum and instruction in order to support students ongoing efforts to learn the proof process.

By the end of the study, 90% (40 out of 44) participants (most of whom had not previously submitted work) were selecting claims, although only 11 of them justified their selections with arguments. Among those 29 students that did not justify, they chose *always, sometimes,* or *never* with 78.4% accuracy. Twenty-seven of these 29 students were multilingual (see Table 4.3). This could suggest that language is an issue for justifying, as well as generating claims. However, many of the students who put claims, only, were also lower achieving students. Of the 29 that made claims-only in Lesson 3, 19 were from the low-achieving group of students and nine were from the middle-achieving group. Only one was from the high-achieving group. It could be that for lower-achieving students, the intensive demands on mathematical calculation and communication are creating a barrier to focusing their efforts on justifying.

Additional research needs to provide further analyses of students' empirical arguments to see how those change. Drawing distinctions between different kinds of empirical arguments might be a useful step toward understanding how students engage examples and, eventually, guiding students to engage the underlying structures that an example space might offer. Expanding how we analyze empirical arguments could be a productive way to guide students toward more general arguments that utilize analytic proof schemes. Before introducing tasks that prompt students to generate a claim, curricula should introduce tasks that prompt students to justify a given claim or qualify a claim and justify their choices (Banes, 2019). Students need time to see the connections and relationships between generating claims, justifying them, and the data (often as examples) upon which they base their reasoning for themselves. That means that ample time must be allotted for students to explore and construct an example space and a variety of claims. Moreover, students should be encouraged to use different types of examples for different purposes (Ellis et al., 2012). For example, teachers could support students to consider not only ‘easy’ or ‘first-thought-of’ examples but expand to include ‘conjecture-breaking’ or ‘generic’ types of examples for the purposes of ‘understanding’, ‘checking’, or ‘supporting a general argument’ (Ellis et al., 2012).

Chapter 5: Conferencing as a place for oral revision

In this chapter I present analyses of the Convince Form and the mathematical conferences. In my analysis of the conferences, I examine students' writing, other inscriptions, and their classroom discussions (as a mathematical text) to document how they revised or extended their submitted work. This analysis addresses Research Question 3:

3. In what ways did students revise their mathematical arguments during conferencing?

Here I describe the organization of the chapter in two sections. The first section describes the methodology relevant to the analyses in this chapter. The second section examines two focal students (Nina and Antonio) to illustrate how some students revised their responses to make an argument or revised their argument. Nina's written work illustrates how she changed from a list of calculations to generate novel examples without a need for calculating. Antonio's written work illustrates how he revised his empirical argument, expanding to include a visual representation, and moved toward an analytic argument. In the third section, I examine the written work of Yolanda and Ricardo to illustrate how some students revised their procedures for testing claims and usage of formal terms. The chapter concludes with a discussion of the results and then implications for research and practice.

Research question #3 sought to further understand the kinds of arguments, if any, students generated in writing. In the previous chapter, I analyzed student responses in the Convince Forms. In this chapter, I analyze how students revised or extended their work during oral conferencing. The analysis shows that during conferencing, some students revised procedures and their use of formal terms while others revised arguments. I illustrate the findings through the work of four focal students that show how during conferencing 1) two students revised their responses to make an argument or revised their argument (Nina and Antonio) and 2) two students revised the procedures related to testing claims and their use of formal terms (Yolanda and Ricardo).

During the conferencing, 10 of the 11 students orally revised the work they submitted in the Convince Forms, which were submitted at the end of each lesson. Table 5.1 summarizes the students' names by conference round and the revisions they made. As a reminder, the first round of conferences involved five students and focused on their Convince Form responses for Lesson 2 (Happy Numbers task). The second round of conferences involved seven students and focused on their Convince Form responses for Lesson 3 (Consecutive Sums task). Additional information on the conferences can be found in the Methodology section, below, and the Methodology chapter (Chapter 3).

Table 5.1

Summary of Conferences by Round, Includes Revisions and Pre/post Conference Codes.

Task	Student	Pre-Conference Codes	Conference Revisions	Post-Conference codes
Round 1- Happy Numbers (HN)	Nina*	Generated No Claim	Nina revised a list of calculated examples to Included novel examples generated without calculating.	Qualified and Justified (empirical)
	Noah	Generated No Claim	Clarified how to calculate exponents and test for a Happy Number.	Generated No Claim
	Norbert	Description	Revised description of the Happy Number procedure and related argument.	Qualified and Justified (empirical)
	Ricardo*	Description	Ricardo revised the Happy Number procedure when we clarified words and phrases.	Description
	Antonio*	Qualified and Justified (partial)	Revised claim about 301 and 310 being the same (i.e., Happy Numbers), and generated novel examples without calculating.	Qualified and Justified (general)
Round 2- Consecutive Sums (CS)	Jose	Qualified Claim, Only	Jose revised his empirical argument by providing definitions for odd/even numbers that informed a new warrant.	Qualified and Justified (partial)
	Marta	Qualified Claim, Only	Revised procedures for Conjecture B and generated an empirical argument. Clarified the term “consecutive”.	Qualified and Justified (empirical)
	Yolanda*	Qualified Claim, Only	Revised procedures for Conjecture C and generated an empirical argument. We clarified the term “consecutive”.	Qualified and Justified (empirical)
	Maggie	Qualified Claim, Only	Revised response for Conjecture B to generate an empirical argument. She needed support with calculation and formal terms.	Qualified and Justified (empirical)
	Josefina	Qualified and Justified (empirical)	Revised empirical argument on Conjecture B. She needed support with formal terms (“average”).	Qualified and Justified (empirical)
	Amaya	Qualified and Justified (empirical)	No revision.	Qualified and Justified (empirical)
	Antonio*	Qualified and Justified (empirical)	Antonio revised his empirical argument on Conjecture B to include a warrant based on a visual he generated.	Qualified and Justified (general)

*Focal Student in this chapter.

In the first round, most students revised their work to reflect a new proof scheme or other revisions, such as revision of a procedure or use of vocabulary. In the second round of conferences, which focused on the Consecutive Sums task, four of the seven students entered having qualified claims, only (see Figure 4.4 for description). These students explored definitions, relevant vocabulary, and procedures, ultimately leading to them expanding into empirical and even somewhat general arguments. Only one student was committed to the ideas they started with, not revising any part of their response.

Methodology

A fuller description of the methodology is available in the Methodology chapter (Chapter 3). Briefly, I will summarize the approach here. I conducted two rounds of conferences. The conferences were conducted like student interviews (Brenner, 2006). First, I prepared for the conference by examining student responses to the Convince Form, in which students were prompted to convince themselves, a friend, and a skeptic of the certainty of their mathematical claim(s). Then, I prepared 1-3 follow-up questions to elicit further reflection. Then, I met individually with students for 10 – 12 minutes. I use proof schemes (Harel & Sowder, 1998) and the Toulmin model of argument (Toulmin, 2003) as frames to describe student arguments and how they changed.

As described earlier, the role of the mathematical writing that was submitted in the Convince Forms was to assess student engagement with mathematical structure and provide a *shared focus of attention* for the conference (Moschkovich, 2004). The lesson called for teachers to assess student work and provide written feedback. The written feedback could involve (MARS, p.T-3):

1. Writing one or two questions on the student's work and returning that to the student, or
2. Selecting a few questions that would help the "majority" of the students, writing them on the board, and discussing them with the class.

After assessing the student work, I presented the questions in the conference setting to ensure that the feedback was understood (Aguirre & Bunch, 2012) and elicited further engagement with mathematical structure. Thus, the Convince Form provided a shared focus of attention, and the oral conference provided a time and place where we could further organize, clarify, reflect upon, or revise the ideas in the written responses to support learning. That is, we used writing as a tool for mathematical learning (Gillespie et al., 2014).

Because the analysis examined the ways students revised written work while conferencing, I selected focal students from both rounds of conferencing, alongside demographic diversity, achievement levels and English proficiency. Focal students Nina, Antonio, and Ricardo all responded to all three Convince Forms when given the opportunity. Yolanda only responded to the final Convince Form but agreed to participate in the conference. I invited six of the nine students who filled out Convince Forms for the first round of conferencing (after Lesson 2) and five attended. I invited 20 of the 40 students who filled out Convince Forms (after Lesson 3; with an additional four students on stand-by) and seven attended. The four focal students were selected from the 11 unique students that attended conferences.

Supporting Creation and Development of Arguments

In this section I review the evidence that students made or revised arguments, by using a new proof scheme. Below, I review the written work of Nina (conferencing on Happy Numbers task) and Antonio (conferencing on Happy Numbers task and Consecutive Sums task).

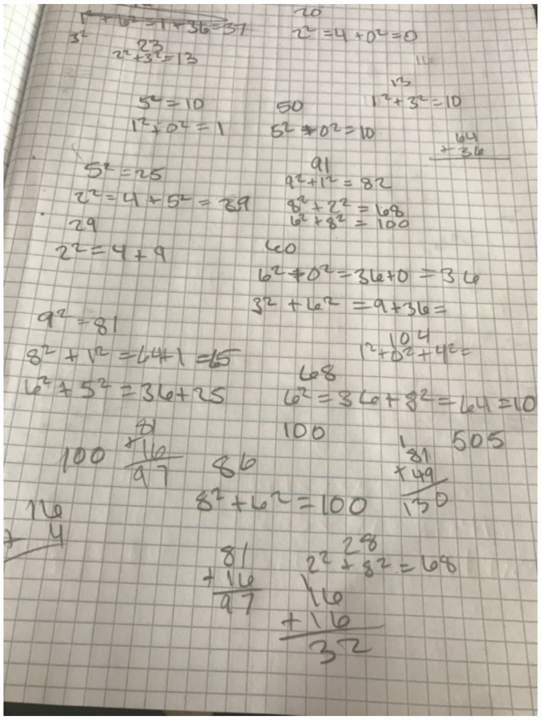
Focal Student 1: Nina

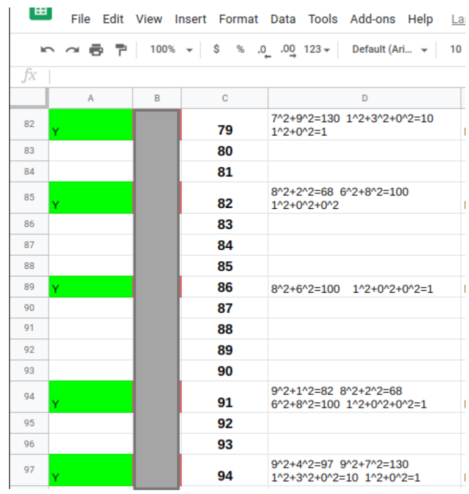
During the conference, Nina described two phenomena that moved past listing examples to deducing some novel happy numbers.

Prior to the conference, I reviewed her response to the Happy Numbers task, which was coded as ‘generated no claim’. As noted in the previous chapter, Nina’s response included no claims or arguments (see Figure 4.2). Her work is again presented in Figure 5.2. Nina submitted a page of calculations, six screen shots to show additional calculations (only one of which is shown in Figure 5.2), a list of numbers she designated as happy (with a “Y”), and a brief statement.

Figure 5.2

Nina's 'No Claim' Response from Lesson 2, Prior to Conference: Written Work (left), a List of Happy Numbers (upper right), and a Type-written Statement.





In the picture above, I showed my work and how I explored the problem. You can see that some of the numbers I worked on were happy numbers and some were not.

We began the conference by reviewing Nina's list of examples available in the screenshots. She said that she "noticed" that, in addition to 68, 86 is a happy number (turn 8), elaborating that they have the same "digits" (turn 10). I asked, "what pattern are you noticing? Can you name the pattern?" and she responded, "I'm not sure" (turns 11-12). Following her observation, I asked her if other numbers have the "same numbers" and she correctly provided additional examples of "…94 and 49… and 97, 79" (turns 13 – 16). I asked Nina why it was happening. Eventually, she explained that squaring the digits and adding them would give the same "sequence" (turns 17 – 22).

7 Salvador What kinds of patterns are you seeing? Here's 68.

8	Nina	I notice that 86 also... Is also a happy number.
9	Salvador	Yeah. 86 IS also a happy number.
10	Nina	They have the same, like, um numbers. The digits.
11	Salvador	Yeah. They do. Ok. Well… can you… What do you think about that? What pattern are you noticing? Can you name the pattern?
12	Nina	I'm not sure.
13	Salvador	Well, you said they have the same digits. Do any of the other numbers have the same digits?
14	Nina	Yes.
15	Salvador	which ones?
16	Nina	Like, 94 and 49. and 97, 79.
17	Salvador	Good. Good. Good. So. So… Now if I ask you… How about this then, now if I ask you, why is that? Do have any idea why that's happening? Like if 68 is a happy number why does that mean 86 will be a happy number? How is that working?
18	Nina	Because you're.. Uh.. Square the individual numbers and … yeah.
19	Salvador	Cause you square the individual numbers? And then what?
20	Nina	And then you add them?
21	Salvador	uh huh.
22	Nina	So, you're gonna get the same, like, sequence for the … yeah.

Nina had difficulty articulating a claim at this point, so I asked her if she could use her idea to find three-digit happy numbers. Nina asserted that “you can get the same digits but put a zero in between or at the end” (turn 28). I asked her why it works, and Nina said, “Because when you square zero, you ‘re gonna get zero, so it doesn’t make any change in the final answer” (turn 34). At that point, I asked Nina to use her ideas to determine which numbers *must* be happy numbers if 478 is a happy number (Part C of the Happy Numbers task- Turn 37). Nina correctly determined 4,780, 4,078, and 8,740 were all happy without the need for calculation (turns 40 – 42). Thus, her argument deepened and expanded beyond a need to calculate each novel happy number.

37	Salvador	…and use your pattern. I’m asking you to use your pattern. So, maybe not to calculate too much but if your pattern… if your ideas are right, then what other numbers could you make with that 4-7-8?
38	Nina	You can add a zero in between.
39	Salvador	Ok. So, like.. And get what? What number would work, then?
40	Nina	Like, 4,780?
41	Salvador	ok. That's good. What other numbers, maybe?
42	Nina	Like 4,078 or eight hun-- eight thousand seven hundred and forty… and forty. Yeah.
43	Salvador	Oh, you changed the order, too, that time. 8,740?

44	Nina	yeah.
45	Salvador	Would it still work if you just changed the order of the 478?
46	Nina	Yeah. Because they're the same numbers… you use the same numbers.

By generating three novel happy numbers, Nina is demonstrating some implicit knowledge of the mathematical structure at play. She used zeros as an additional digit to create more happy numbers because they do not change the value of the sums. This change in her argument suggests the implicit use of the identity property of addition. Her statement that adding zero "doesn't make any change in the final answer" implicitly refers to the additive identity property, which states that any number added to zero sums to the number (i.e., $a + 0 = a$).

Nina also changed the order of the digits of happy numbers to generate novel happy numbers because rearranging the addends will not change the value of the sum. This move reflects the implicit use of the commutative property of addition because she is transposing the digits of Happy Numbers to account for why previously found Happy Numbers, like 94 and 49, were both happy and also to generate new happy numbers without a need for calculation. Nina states that the digits can be rearranged because they use the "same numbers" (Turn 46) but does not explicitly name or describe the commutative property (which is typically stated as: $a + b = b + a$).

These shifts represent a significant change: from listing examples of happy numbers to generating novel happy numbers *without the need for calculating*. While Nina encountered difficulty when prompted to articulate the patterns she was describing, she was generating examples for the purpose of "support[ing] a general argument" (Ellis et al., 2012). This type of example can be used to support deductive proofs whereas the original list she provided was more aligned with a different purpose, i.e., "check[ing]" to see if a conjecture holds (Ellis et al., 2012). Nina's work so far is best classified as making an empirical argument because she has not explicitly stated her claim(s) or warrant(s). However, her work is laying the foundation for a more deductive proof. As evidence amasses, Nina would need to explicitly articulate her claim and use of mathematical properties as warrants. It is also moving towards an analytic proof scheme by seeking a general solution to the implicit claim that the digits of any Happy Number can be rearranged to generate new ones. Our mathematical discussion (in the form of a conference) provided new "text" that Nina could use to revise her argument.

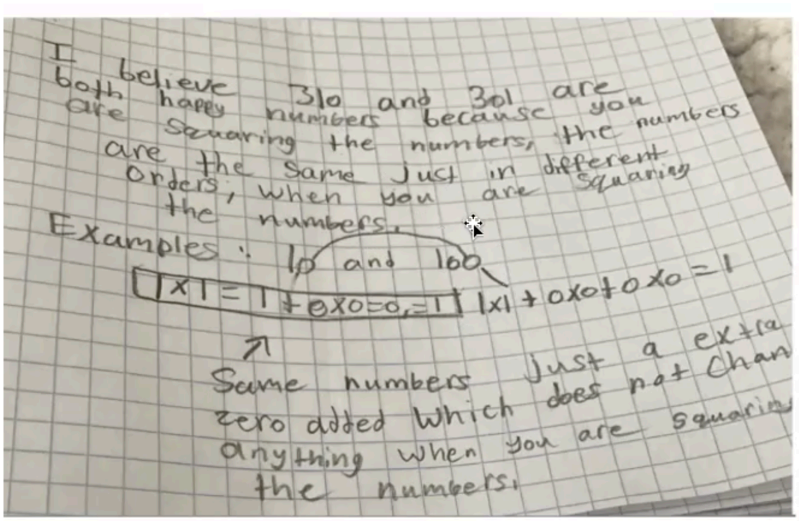
Focal Student 2: Antonio

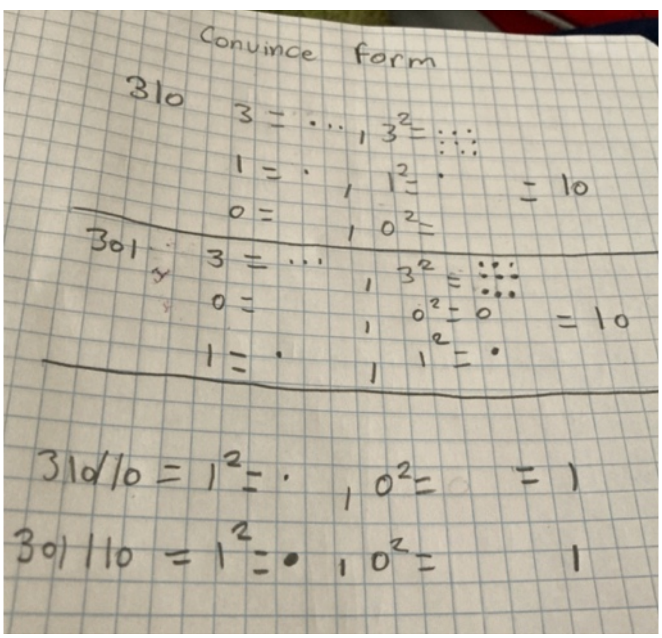
In this section I review both of Antonio's conferences to illustrate changes in his arguments. During Antonio's first conference, he, like Nina, described how the order of happy numbers could be changed and that adding a digit of zero could generate novel happy numbers. During his second conference, he shifted from an empirical argument to an analytic argument with the use of a representation he improvised.

Antonio's first conference was very similar to Nina's in two ways. First, he indicated that the commutative property and the identity property of addition were useful for determining novel happy numbers. Secondly, he had difficulty explicitly articulating his assertions about how he was generating novel happy numbers. Prior to the conference, I reviewed Antonio's Convince Form response to Lesson 2 and coded it as generating a claim in response to a task that prompts students to generate a claim (see Figure 4.21 and related discussion). In summary, Antonio asserted that 301 and 310 were "the same", i.e., both happy numbers, because the order of the digits did not matter (see Figure 4.21). While this is not a general claim, he did demonstrate that two numbers, 301 and 310, both result in being happy numbers because the digits are "…just in different orders [sic]". This work, which served as the starting point for the conference, is presented again in Figure 5.3.

Figure 5.3

Antonio's Claim (top) and Evidence (bottom) Submitted in Lesson 2.





Antonio's initial work includes two important phrases. The first is that "the numbers are the same just in different orders" and "same numbers just a [sic] extra zero added which does not change anything…" (Figure 5.3). Once the conference was under way, I asked Antonio if these ideas would be true for other happy numbers (turn 27, emphasis added). He responded that he would try with 28 and 280 (turn 32).

27	Salvador	**Do you think that'll be true for other happy numbers?**
28	Antonio	Maybe. Like, um, like 30 and 300. **Like, it's the same thing because the zero doesn't change the effect of the-- wait, wait.** Let's see. Naw, naw. 30 and 300 are like kinda the same. That's what I think. Wait. Did I put 30 and 300 as an example? No. I did that one.
29	Salvador	You put 310. Yeah, and 301.
30	Antonio	Yeah. **Cause I saw that the 1 and the zero were-- like, it's the same thing**. It's just the 1 and zero are switched in 301 and 310. In 301, the zero's first and in 301 the one is first.
31	Salvador	Right. So, what I'm saying-- yeah. What I'm asking you is, um, maybe look at the list of other happy numbers and-- **is your conjecture true for, maybe, another happy number?** Wait, pick another happy number and see if it still works with another one.
32	Antonio	**I'll try it with 28 and 280.**
33	Salvador	ok. [wait] Let me know when you're ready, ok?

Eventually, Antonio found that 28 and 280 are both happy, providing some confirming data that adding a zero digit would produce a novel happy number (turn 60). At that point, I asked Antonio if he could say something "general" about the pattern he found (turn 61). In response, he commented that "…every single one of these… examples—the zero is nothing" (turn 62). He used more general language in the phrase "every single one". While the argument is empirical, the use of generalized language might suggest Antonio's awareness of a need for a more general argument: one that is applicable to a class of numbers rather than individual examples. His use of language seems to suggest an eye toward settling the conjecture in general, which would be a foot in the realm of analytic arguments.

At that point, we discussed a fictitious student, Eva, who found that 478 was happy (Part C of the Happy Numbers task). I asked Antonio to use the ideas we discussed to predict other happy numbers. Antonio generated three more novel happy numbers: 4,780, 748, and 874.

60	Antonio	…equals 1. Zero squared equals zero and then another zero squared and that equals one. Now, I have to do 280. 2 squared. 64 plus zero. 68. yeah. **I think I… 280 and 28 are like the same**.
61	Salvador	ok. Good. So**, what can you say in general about-- like, what pattern did you find**? Like, what's your conjecture? So, now you have two examples, but what's like your conjecture?
62	Antonio	So, for if you go to 70.. No. 70 is not-- 68. I feel like 68 and 680 are like the same thing because when I-- and **every single one of these, like, examples-- the zero is nothing**. So, it equals to the-- it would be like the same as 6 squared plus 8 squared because the zero does not equal anything.

At that point, Antonio said, “I get it now” (turn 76). I asked him what his conjecture was now, and he replied:

> …you could switch the numbers around, as long as the numbers stay the same. Like, if there's three digits and you switch ‘em around, it won't change the sums because you're squaring all three of the numbers. You're not, like, what's it called? How do I say this? You're not.. You're not uh.. like, I don't know how to explain it, but like, if it were anything else beside like what we're doing, it probably wouldn't be. (turn 80).

Antonio articulated that “switching” the digits of a happy number could generate a new happy number and eventually described the commutative property of addition as a warrant. I asked him what “we” know about addition, and he responded “…that one thing where you switch the numbers it doesn’t change. I forgot. Isn’t it, like, 4 plus 8 and you switch 8 plus 4 it’s, like, the same thing. I forgot what it’s called.” (turn 84). I reminded him that the name of the property he is describing is the commutative property and he responded, “there you go!” (turn 86).

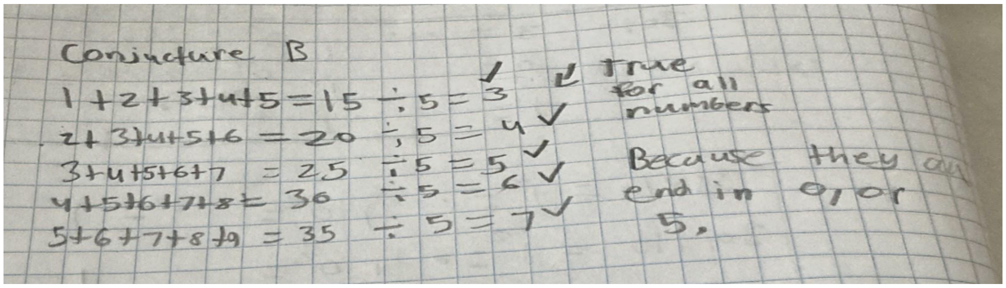
During Antonio’s first conference his argument changed from justifying why two specific numbers (301 and 310) were both happy (an empirical argument), to justifying why he could generate any number of happy numbers by rearranging the digits or adding a zero digit (an analytic argument).

During the second conference, we discussed his response to the Consecutive Sums task. In this task, students were given six claims (Conjectures A – F), asked to qualify each claim as *always, sometimes,* or *never* true, and then justify how they qualified each claim. Per Antonio's choice, we focused on Conjecture B, which made the claim that "The sum of five consecutive whole numbers is divisible by 5." While Antonio's initial response engendered an empirical proof scheme that leveraged the examples in a table of values as data, by the end of the conference he used an analytic proof scheme that used a visual representation for the sums of five consecutive integers to justify how he qualified the claim. At some point prior to the conference, Antonio's Zoom microphone broke, so he responded to my oral prompts using the 'chat' and 'screen share' features of the meeting platform, only.

Antonio's initial work stated that the sum of five consecutive whole numbers is always divisible by five because the sums end in zero or five (see Figure 5.4). This appeared to be a *perceptual* empirical argument, which rely on "rudimentary mental images" (Harel & Sowder, 1998). We reviewed the response and I said, "Alright, so, this is what you put for Conjecture B. And, um, maybe in the chat real quick maybe just put what it is that we're seeing" (turn 13). In response, Antonio typed "There [sic] adding 1 to each number like in the first one it was 1 +2+3+4+5=15 and then on the seceond [sic] one it was 2 +3+4+5+6 which equaled 20" (turn 14). This statement did not seem to match his written work, which emphasized sums ending in zero or five. So, I responded "…you said the sum would end in zero or five and I wanted to know-- how do you know that?" (turn 23). Antonio said, "I did 5 x every number like the 5 multiplication table and I saw that all of the sums ended either in 0 or 5" (turn 24). Then, I asked him to use "symbols or a picture" to illustrate what he was describing.

Figure 5.4

Antonio's Argument for Conjecture B, Submitted Prior to Conference. Convince Yourself (top) and Convince a Skeptic (bottom).

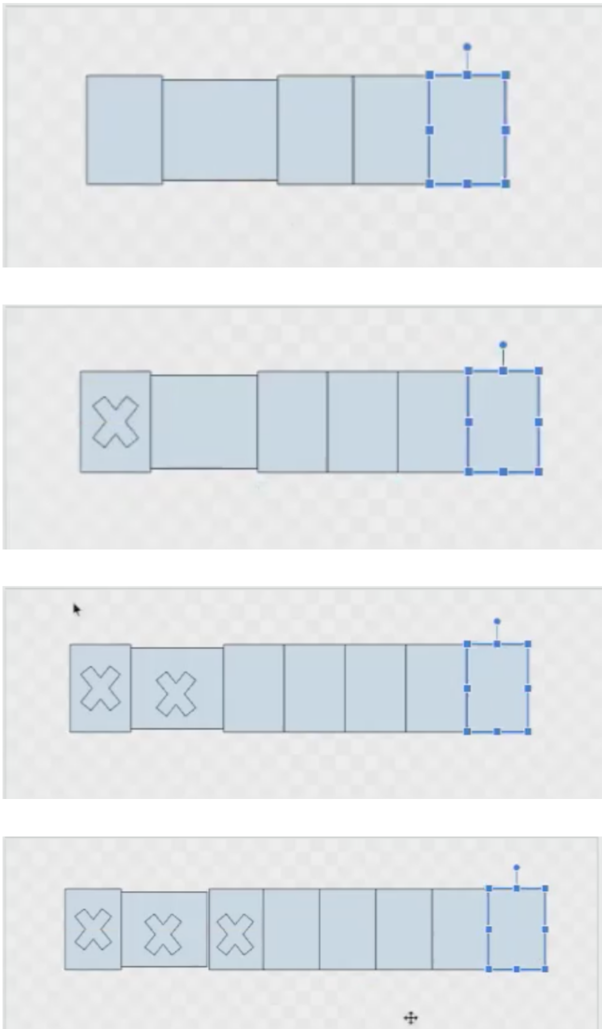


Conjecture B is works for any sum because for example if it is adding 5 consecutive numbers the sum would end with a 0 or a 5 , and if it ends with 0 and 5 it could be divided by 5 , proof : 1+2+3+4+5=15, 15 divided by 5 is 3, and then 2+3+4+5+6=20 , 20 divided by 5 is 4 , the next one is 3+4+5+6+7=25 , 25 divided by 5 is 5 and then you keep going.

At that point, Antonio represented generating examples like placing and removing cards. Antonio generated the images in Figure 5.5 by using the "Insert drawing" feature of a Google Doc. When it was complete, he chatted me, "do you get what im [sic] doing" and I responded, "Maybe explain a little with words" (from chat file, 12/10/2020). He explained, "Im [sic] crossing off the first one and adding another one to the end like the first one was 1 +2+3+4+5 then I crossed the 1 out and added a 6th one which made it 2+3+4+5+6" (from chat file, 12/10/2020). I pressed for elaboration asking aloud, "So, what happens when you trade that 1 for a 6?" and he responded in chat, "it adds one to the next number it increases the sum youre [sic] gonna get". I pressed again asking, "how much does it increase by?" and he responded in chat, "by 5." (chat file, 12/10/2021).

Figure 5.5

Antonio's Representations for the Sum of Five Consecutive Whole Numbers



The series of visuals shown in Figure 5.5 represents sums of five consecutive whole numbers. Per Antonio's description, the left-top image represents the sum, 1+2+3+4+5 = 15. The second image shows the first rectangle crossed out and a 6^{th} rectangle added, representing 2+3+4+5+6 = 20. In effect, "trading" the one for a six adds five to the sum. Likewise, the images suggest that trading the 2 for a 7 and the 3 for an 8 would also results in net gains of 5 to the sum. In this manner, the sums always add five to the original sum and are, therefore, always divisible by five. This representation is recursive, but nevertheless could represent a step toward an analytic proof scheme because it seeks to settle the conjecture in general.

In this section I reviewed evidence that two students either created or changed their arguments (vis-à-vis proof schemes) during the conferences. In Nina's written work, she considered her list of examples more deeply and generated novel happy numbers without the need for calculation and implicitly referenced the commutative and identity properties of addition as warrant for doing so. In his first conference, Antonio also expanded from his focus on two examples to leverage the commutative and identity properties of addition to settle his Happy Numbers conjecture in general. In Antonio's second conference, he shifted from observing a pattern in a table to generating a representation to illustrate why each new sum of five consecutive numbers will add five to the previous and, consequently, remain divisible by 5. In sum, the students' arguments changed from a focus on calculating examples to a focus on generating novel examples, implicitly using mathematical properties as warrants for their claims, and striving to explicitly articulate a general mathematical argument.

Revising Procedures and Revealing Areas for Support

In this section I review the evidence that two students revised procedures and their use of formal terms. Overall, four of the 11 students who participated in conferences used part of the time to clarify and revise procedures related to the tasks. I began the conferences by reviewing what they had submitted in the Convince Form. And then, I used eliciting and probing questions to understand and extend their reasoning. Our conferences often included my suggesting a new example or I would guide them through an example and then support them to generate additional examples on their own. When they used examples to test claims on their own, I took that as evidence of a revision.

Additionally, five[20] of the 11 students who completed conferences clarified vocabulary. One of us would ask a question about a term which might lead to a revision of how to use formal vocabulary. For example, when considering the Consecutive Sums task, I asked Maggie, "Do you remember what consecutive means?" (Turn 13, Maggie's transcript). She responded, "No" so we reviewed the meaning and moved on. When students used formal vocabulary in ways that aligned with the teacher's usage, when they previously had not, I took that as evidence of a revision. I review the written work of Yolanda and Ricardo to illustrate how students revised procedures or formal terms.

Focal Student 3: Yolanda

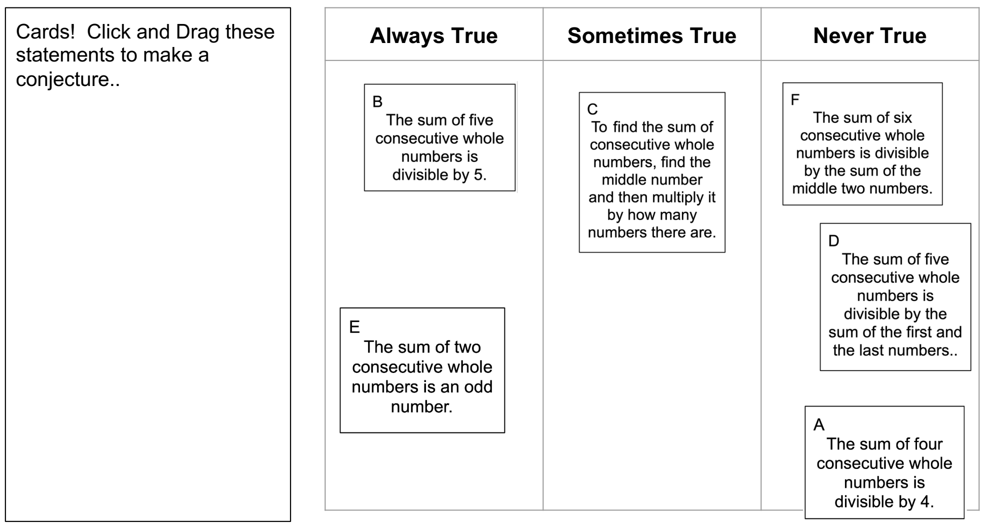
Up to the point of the conference, Yolanda had participated minimally in the study activities. For example, although she participated in classroom discussion and activities, she did not submit arguments for Lessons 1 or 2. In Lesson 3, which culminated in the Consecutive Sums task, Yolanda qualified claims by completing the card sort (see Figure 5.6). Yolanda's card sort was mostly correct (83%), but she did not justify any of the choices (i.e., did not complete the Convince Form). During the conference, we discussed the claim that "To find the sum of consecutive whole

[20] Three of the students were the same as those who clarified procedures.

numbers, find the middle number and then multiply it by how many numbers there are" (Conjecture C): the only conjecture that Yolanda indicated as *sometimes* true.

Figure 5.6

Yolanda's Qualified Claims for Consecutive Sums Task.



Afterwards, Yolanda decided to discuss Conjecture C. I then asked her if she could test the claim, to which she responded, "I don't know" (turn 42). I picked an example for her to test, 3 + 4 + 5, and she responded by simply adding them to get 12 (turn 44). At this point, I supported her to test the conjecture by walking her through the steps to determine whether multiplying the middle number by the number of addends equals the sum (turns 49 – 58). Then I asked her to come up with her own example to test and she suggested "2 + 4 + 6" (turn 64). Because she used consecutive *even* numbers, I asked if those were consecutive numbers and she responded, "No. How do you know what a consecutive number is?" (turns 65 – 66).

45	Salvador	Ok. It equals 12. But does it work? Is it sometimes true for the other one? Like, how you said, like, uh.. I'll read it again. To find the sum of consecutive whole numbers, find the middle number and then multiply by how many numbers there are? Does this work?
46	Yolanda	Yes?
47	Salvador	Why does it work?
48	Yolanda	Because.. Well, actually, I don't know.
49	Salvador	Well, let's take it in pieces. What's the middle number?
50	Yolanda	The middle number? Like from the problem?
51	Salvador	Yeah.
52	Yolanda	Four?
53	Salvador	ok. And how many numbers are there?
54	Yolanda	Three.
55	Salvador	Ok. So, if you multiply the middle number by how many numbers, what do you get?
56	Yolanda	Twelve.
57	Salvador	So, did it work?
58	Yolanda	Yeah.
59	Salvador	Ok. So, it worked. On that one. Um, let's think of more examples. Can you think of another one on your own?
60	Yolanda	No. [laughs]

61	Salvador	[laughs]. Just try. Try. Try another one with three numbers.
62	Yolanda	Um..
63	Salvador	Or try one with four numbers or try one with five numbers. Just, kinda-- You're gonna need to try some examples, maybe. Let’s just focus on examples for a minute.
64	Yolanda	**Two plus four plus six?**
65	Salvador	Well, I'll put it in the chat. The only thing about this, I'm gonna say, is-- Ima ask you: are these consecutive numbers?
66	Yolanda	No. How do you know what a consecutive number is?
67	Salvador	Oh. Ok. Good. Good question…

This interaction suggested that Yolanda was not confident with the procedure for testing the claim in Conjecture C. When I asked if she could test it, she said “I don’t know”. When I provided three consecutive numbers to test, she responded with the sum, which did not constitute a complete test of the conjecture. The interaction also suggests that she was not using the term “consecutive” in the way intended by the conjecture, since Yolanda suggested three consecutive *even* numbers rather than whole numbers (turn 64). In addition to Yolanda, four of the 11 students who conferenced used key terms with a meaning different than intended. This included terms such as ‘consecutive’ (three students), ‘squaring’, ‘average’, ‘pull apart’, and the circumflex (“^”), which will be discussed later. I continued the conference by clarifying what “consecutive” means and reviewing a procedure to test the conjecture with another two examples (turns 83 – 116).

At this point, I asked Yolanda to make an example, “maybe with more numbers” (turn 119). She suggested “9 + 10 + 11 + 12” (turn 120). Yolanda ran into difficulty when she realized that there was no middle number per se. Yolanda described her conundrum in the following way:

> Because it's four and if you-- you could put four in a group so then it would be two and two. But then, **there wouldn't be a middle number**. Or, at least, there can't be. And the middle numbers could be 10 and 11.. but it would have to be one. (turn 138, emphasis added)

In her comment, Yolanda indicated that there is no middle number in a set of four consecutive whole numbers. At that point, I suggested trying a sum of five consecutive numbers and she agreed. I asked her if the conjecture works for 1 + 2 + 3 + 4 + 5 (turn 143) and Yolanda repeated a difficulty with the testing procedure she encountered earlier—adding rather than multiplying.

143	Salvador	What if we had this. Does it work now? [types “1 + 2 + 3 + 4 + 5 =” in chat]
144	Yolanda	Yeah.
145	Salvador	Same thing. So, like, what's the sum? And the, does it work?
146	Yolanda	The middle number would be three.
147	Salvador	Ok. Keep going.
148	Yolanda	Three and.. And there’s five. Three and five, right?
149	Salvador	mm. hmm.
150	Yolanda	Eight. Right? [inaudible]

151 Salvador Oh. Remember the original thing.

152 Yolanda Yeah. Eight.

153 Salvador It's ‘times’. It's ‘find the middle number and then multiply by how many numbers’. So, it's..

154 Yolanda Ohhhhhhh.

155 Salvador Times five, not plus five.

156 Yolanda oh. Fifteen.

157 Salvador Yeah. Good. Ok. So, does that work? Is that the sum?

158 Yolanda Yeah.

Yolanda went through each step to test the conjecture but made a calculation error. Yolanda verified that the middle number was a 3 (turn 146), and that there were five numbers (turn 148). Then, rather than multiply the 3 by 5 to confirm that the product and sum of consecutive whole numbers is equal, she added 3 and 5 to get 8 (turn 152). I reminded her that the test procedure calls for multiplying and she made the adjustment.

At this point, I turned my focus back to Yolanda’s qualifying Conjecture C as *sometimes* true. I invited Yolanda to “say more” about when the conjecture does and does not work. Yolanda responded, “Because if you do 4, I don’t think it’ll be able to work but if you do more, it could possibly work. Or if you do less” (turn 160). I asked if “3 + 4” worked and Yolanda said, “That can’t work” (turn 162) because “it’s only two and there would at least have to be three” (turn 164). To conclude, we reviewed what we had done, and I asked her to summarize when she thought the conjecture worked. In the chat, she responded, “it works when there is [sic] 3 and 5+ numbers but it won’t work if it’s 4 or 2 numbers” (chat file 12/7/2020).

During the conference, Yolanda benefitted from support with the procedures for testing Conjecture C using examples and using the term “consecutive” with the intended meaning. Yolanda refined her working definition of what “consecutive” means, as we can see in Turn 80 when she completed a sequence of three numbers and in Turn 112 when she suggested testing 6, 7, and 8. Her initial suggestion to test “2 + 4 + 6” (turn 64) used consecutive even numbers (not consecutive whole numbers). My clarification supported her work on the task, this time using the intended meaning for consecutive. Then, she articulated that the conjecture worked for 3 and 5 addends but not for 2 or 4 addends (chat file 12/7/2020).

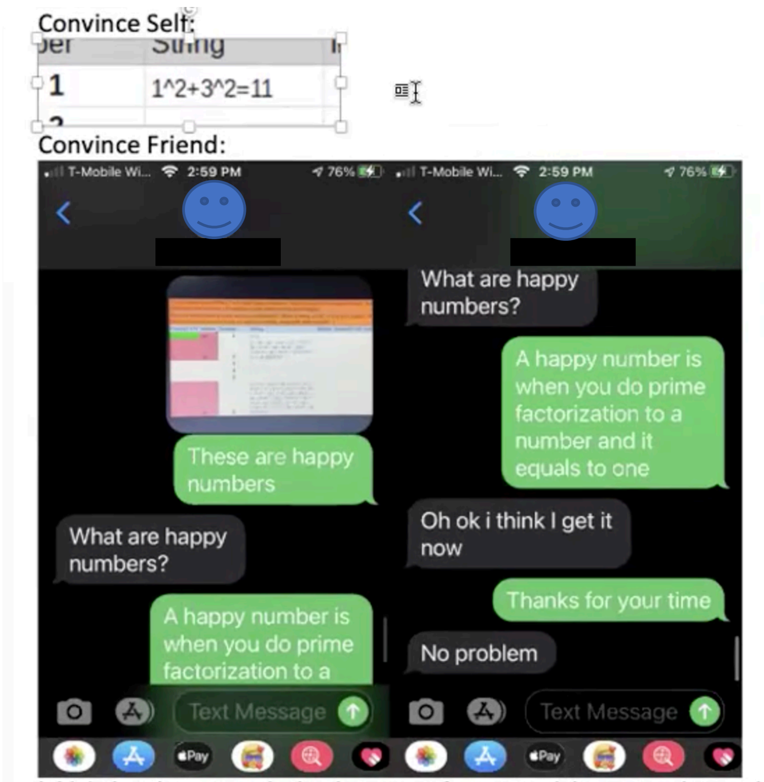
Focal Student 4: Ricardo

Prior to the conference, Ricardo had two opportunities to turn in arguments. For Lessons 1 and 2, he submitted a description of a procedure. Before beginning the conference, I reviewed Ricardo’s response for Lesson 2 (the Happy Numbers task), which we would discuss. The Happy Numbers task prompts students to find two- or three-digit happy numbers without calculation and justify their method (for a complete description see Figure 3.2 and related discussion). Prior to the conference, Ricardo submitted a description of the procedure to find happy numbers where part of the procedure to test an example is illustrated with a calculation and described in a text thread with a friend using a non-canonical reference to ‘prime factorization’ (Figure 5.7).

During the conference, I worked to clarify vocabulary and the testing procedure with Ricardo. Just as we got started, Ricardo asked a clarifying question about the "little arrow thing". He asked if the circumflex ("^") represented multiplication (for example in the expression "3 ^ 2"). I clarified that it is used to represent exponentiation when the use of superscript is not available. Then, we turned to the work he submitted. I asked him to explain to me what he was doing in the example (Figure 5.6, top). He said that he did not remember (turns 31 – 32). I reviewed the procedure with him and asked him what "squaring" something means, as a check for understanding (turn 35). Ricardo responded, "Isn't it like, um, double it or…" (turn 36). We reviewed exponents and then we went on to review some of the happy numbers found in class.

Figure 5.7

Ricardo's Description of Happy Numbers for Discussion in the Conference.



I invited Ricardo to focus on exploring the first nine whole numbers and he recognized an error another student made testing the number "1". I asked Ricardo if the student was right (see Figure 5.8). He responded, "I think… well, it would be 1 plus 1, so… but shouldn't there be a zero?" We discussed that it could be "$0^2 + 1^2$", so the sum is 1 and 1 is happy. Ricardo moved on to test other numbers and asked a clarifying question about the testing procedure: "For eight, does zero to the fourth power plus four to the second power work? Or does it have to be second power?" (Figure 5.9 and Turn 60, below). This question revealed that he was not yet confident in using the testing. Rather than squaring 8, he wrote "8 = 0^4 + 4^2" (Figure 5.9).

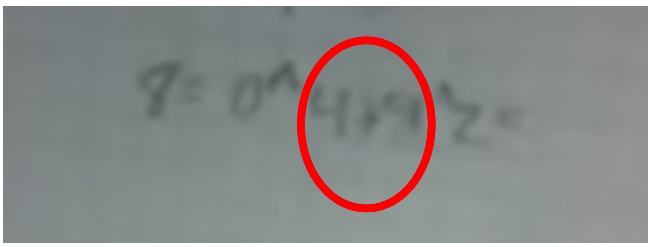
Figure 5.8

Student Error Caught by Ricardo.

Number	String
1	1^1+1^1= 2
2	

Figure 5.9

Ricardo's Notes for Testing '8'.



This kind of error, placing "4+4" at the center of an arrangement of other symbols (see highlighted portion of Figure 5.9 and Turn 62 below), could also indicate that he was using his own meaning, rather than the intended meaning, for "pulling apart" digits of a number. In prior classwork, examples used only two-digit numbers to describe the testing procedure for Happy Numbers (see Figure 3.2). As a reminder, the digits of the number 23, for example, would be "split", "pulled apart", or otherwise separated and then squared like this: $2^2 + 3^2$. The results would be added, and the process repeated until a sum is 1 or a value repeated. When confronted with a single-digit number, 8, Ricardo appears to have split the *value* of 8 and then appended whatever additional information seemed to make sense.

60	Ricardo	um. For eight, does zero to the fourth power plus four to the second power work? Or does it have to be second power?
61	Salvador	Um.. They all have to be to the second power, so, like,.. For eight, I don't know. Walk me through it. Show me-- show me what you mean. Is it-- Maybe hold your paper up. Can you hold your paper up and show me?
62	Ricardo	[student shows paper: “8 = 0^4 + 4^2”]
63	Salvador	ok. Hold on. Let me see. [looks]. How come you-- oh. Ok. You got zero… no. Don't do to the fourth power. It would always be squared. Like it would be zero squared and then it would be eight squared.
64	Ricardo	Like this? [student shows revised work: "8 = 0^2 + 8^2"]
65	Salvador	Yeah. Zero squared plus eight squared. You could use, like, regular exponents on your paper. I'm just doing it with the little arrows on the computer cause, um, I don't know how to make the small 2. You know how they make the little, tiny 2. But it's ok. Either way is ok.
66	Ricardo	um. [works silently]. Um. For six I did zero to the second power plus six to the second power.
67	Salvador	ok
68	Ricardo	And for 5 I did the same, except five to the second power.

Ricardo revised the first part of the testing procedure and attempted two more numbers, each time beginning, but not completing, the procedure (Turns 66 and 68). For example, he tested 5 and 6 and suggested that 6 could be happy (turn 72). When I asked him if the sum was 1, he said, “No” (turn 74). We reviewed the testing procedure again and the conference time expired before he could successfully revise the entire testing procedure.

During conferencing, Ricardo used support for the testing procedure and in using formal terms with the intended meaning. The procedure to test whether a number is happy or not can require substantial calculation (see Figure 5.1 for examples), thus presenting a formidable obstacle for entry into the heart of the task for some students. Ricardo caught a mistake another student when finding the happy number involved a single step, which is the case with the number 1. Ricardo did not generate additional examples without help. By the end of the conference, Ricardo squared digits and added the products, successfully completing the first step in the procedure to test the claim. Difficulties with foundational concepts and interpreting formal vocabulary (e.g., “squaring” and calculating exponents) and novel symbols (i.e., the circumflex as “^”) presented obstacles to Ricardo’s progress on the task. Moreover, the open-ended nature of testing a number for being ‘happy’ could have presented another obstacle. During the conference, I provided support when I could identify these difficulties and supported the student’s access to the task.

Summary of Findings

In this chapter, I presented evidence to show how students generated or expanded their arguments (Nina and Antonio). Nina began with a list of calculations, indicating which numbers were Happy. The list was provided because she was asked to find each of the 17 Happy Numbers between 1 and 99. In the end, Nina was generating novel Happy Numbers with four digits (4,780, 4,078, and 8,740) without the need for calculation. She implicitly used the commutative and identity properties of addition as warrants for her implicit claim that rearranging the digits of or adding a zero-digit to a Happy Number will generate a novel Happy Number. Antonio's first conference was very similar in that he also generated novel Happy Numbers (4,780, 748, and 874) based on an implicit claim that one could take the digits of that number and "switch them around" to generate more. Likewise, he described and implicitly used the commutative and identity properties of addition as warrants. These findings are significant because they show how students' persistence with examples led to a change in purpose. Initial examples were chosen out of convenience or even by prescription of the task, then later examples were generated through deduction.

I also presented evidence that students revised the testing procedure and use of formal terms (Yolanda and Ricardo). Yolanda revised the testing procedure when we clarified what "consecutive" means. She revised her usage for 'consecutive' to go on and test additional examples. Yolanda needed sustained support when completing the test procedure, but she discerned that her claim—that "To find the sum of consecutive whole numbers, find the middle number and then multiply it by how many numbers there are" is *sometimes* true—holds for 3 and 5 addends, but not 2 or 4 addends. This is an empirical argument, as it focuses on these particular cases, but she is making progress toward something more general. How long before she recognizes that *odd* numbers of addends have a middle number whereas *even* numbers of addends do not? Ricardo revised his usage for the term "squaring" as well as use of the circumflex ("^"). While this is not a formal term, its usage in mathematics is common when superscript is not available. During conferencing, Ricardo's also revealed how much support he needed, only having revised a part of the testing procedure.

Discussion

In this chapter I documented how students' arguments changed and provided evidence that students from a variety of mathematical proficiencies created or revised an argument, as well as how students revised testing procedures for claims and usage of formal terms. I presented data from two focal students, Nina and Antonio, to illustrate how students either created or revised their arguments using a *proof schemes* framework (Harel & Sowder, 1998). I also presented data from two additional focal students, Yolanda and Ricardo, to show how we clarified testing procedures and formal terms.

Nina and Antonio made or improved their arguments during the conferences. Nina began her conference having submitted a list of examples. During the conference, she noticed that the digits of 68 could be transposed to create another happy number, 86. She also noticed that adding a zero digit could generate additional happy numbers. These kinds of noticings can occur when students have access to an *example space* and time to (re)organize the available examples (Watson & Mason, 2005). Nina and Antonio noticed some similarities or differences among the examples allowed them to anticipate what *should* be a happy number. That is, they both generated novel happy numbers without the need for calculating based on what they noticed about the available examples.

The shift in focus from examples to noticing how the examples are the same or different is not trivial. The example space can be a critical tool for enabling the noticing but the noticing itself is part of a larger process of inquiry. Rivera & Becker (2007a, 2007b) describe an account of the process of inquiry as consisting of three kinds of reasoning: abduction, induction, and deduction. In this trivium, the process of inquiry begins with what "may be" (abduction), followed by testing what is "actually operative" (induction), and then stating what "must be" (deduction) (Rivera & Becker, 2007a). Abduction foregrounds inference and is creative (Rivera & Becker, 2007a). In the conferences, I supported abduction, as well as the other stages of inquiry, so that students were able to revise their arguments in relation to important aspects of the mathematical structure embedded in the examples. In doing so, students were positioned to infer generalities, test their inferences with novel examples, and reason inductively.

Yolanda and Ricardo revised two procedures and refined their meanings for formal vocabulary words, increasing their access to the task. Academic language can function as a "symbolic border" that prevents students who speak stigmatized varieties of English and multilingual students from full participation in learning activities (Valdés, 2017). At the same time, Yolanda made significant progress toward defining cases for which her claim is "sometimes" true. During the conferences, I supported students' reasoning by directly clarifying the intended meaning for important formal terms and symbols (such as "consecutive" and the circumflex, " ^ "). During the conferences, students also revised relevant testing procedures. The conferences served as a safe space where their difficulties were revealed and revising procedures or refining meanings was a central goal.

Students benefitted from support for expanding the types and purposes of examples. For example, (re)arranging the available examples led to Nina and Antonio to conjecture about how they examples were related (i.e., they remarked on the digits being the same for some examples). In turn, they generated novel examples, thus expanding to use novel examples to check a conjecture and possibly support a general argument. Testing novel examples did not always lead to the confirmation of a conjecture. For example, Norbert conjectured that numbers ending with 7, 0, or 3 would be happy (based on the fact that 7, 10, and 13 were all happy, and then 97, 100, and 103 were all happy) (Turn 44, Norbert transcript). When he tested 17, 20, and 23, it failed. This failure is a triumph in that it is a demonstration of the student's engagement with empirical reasoning and expanding by using his examples for the purpose of *conjecture busting* (Ellis et al., 2011).

Finally, I present a note about "funneling". Funneling "involves a series of closed questions that progressively move students toward saying something desired by the teacher" that is ultimately not productive for learning (Munson, 2018, p.83). I reviewed one interview (with Amaya) and found that I unintentionally used questioning to funnel the student a particular direction. Amaya was discussing Conjecture A ("The sum of four consecutive whole numbers is divisible by 4."). She said that it is never true because, "[inaudible] Oh. They didn't work because they give you a odd number for the-- or they give you… They probably won't work because it might give you, like, *a decimal number*." (Turn 36, emphasis added). Rather than explore her comment about decimals, I guided the conversation toward another topic. After a while, and she responded, "I don't know" to my questions (Turns 82 and 120) and I got a sense that the conference was not productive. She is the only student who made no revision in the conference. I reflected that when we follow students' reasoning, there is less mystery for the student because they are not guessing at what the teacher might be thinking. In following the students' reasoning, we validate their thinking and then provide their own thinking back to them for further consideration. Rather than seek to understand and support the student's reasoning, funneling seeks to push students toward a predetermined answer the teacher is looking for (Munson, 2018; Wood, 1998). Funneling can be a serious pitfall and engenders deficit views of students by privileging the teacher's ideas.

Chapter 6: Discussion, Conclusions, and Implications

The purpose of this qualitative study was to understand students' developing mathematical arguments in a middle school, early algebra classroom that featured mathematical writing and oral conferencing. This included examining how students used the Convince Form to develop and justify claims, and then how conferencing with students about their written arguments might have supported them to further revise. The unit provided students opportunities to write mathematical arguments as a regular part of classroom instruction and introduced a Convince Form and mathematical conferencing to support their efforts to justify mathematical claims submitted as part of the instructional unit on mathematical argumentation. The unit consisted of three lessons: two of which (Lessons 1 and 2) culminated in tasks that asked students to generate a claim and justify it, and a third lesson that asked students to qualify up to six claims and justify how they qualified them. In addition, I conducted two rounds of mathematical conferences: one round after Lesson 2, and one round after Lesson 3. The research questions below guided this study:

1. When asked to do mathematical writing and supported with conferencing in a remote context, what kinds of arguments do students make?
2. How did the mathematical arguments of individuals change over the course of a unit of instruction on generating, selecting, and justifying claims?
3. In what ways did students revise their mathematical arguments during conferencing?

To answer the first research question, I examined the written work students submitted in the Convince Forms from each lesson. In the first analysis, I cataloged the kinds of responses students submitted for each task type. I described the responses given in Convince Form submissions for Lessons 1 and 2, which used the original form (Appendix H; available as a Google Form) and culminated in tasks asking students to generate and justify a claim. I also described the responses given in the modified Convince Form (Appendix I; available in their Google Classroom as editable slides) used in Lesson 3, which culminated in a task asking students to qualify and justify claims. In Lessons 1 and 2 students were prompted to generate and justify their own claim about mathematical situations described in the tasks. In Lesson 3, students were asked to qualify up to six claims using a non-neutral qualifier (e.g., always, sometimes, or never true) and justify the ways they qualified the claims.

Research Question #1 and #2 sought to understand how students developed claims and justified them in a unit that included mathematical writing and conferencing. The mathematical writing samples, which were gathered as responses to the Convince Form, provided a written record of the kinds of responses that students submitted and how those responses changed over the duration of the study. The findings described in Chapter 4 highlight how students' responses changed in three major ways. First, they moved away from writing descriptions of procedures to writing arguments using empirical and even analytic proof schemes. Second, students increasingly used examples in mathematically sound ways. Third, students went from writing responses that contained no claims to qualifying claims (when offered a list of non-neutral qualifiers) or generating claims (when invited).

To answer the third research question, I examined the written work and video recordings of the conferences to see how the opportunities for discussion and revision supported students to revise their arguments. I considered the *proof schemes* (external, empirical, or analytical; Harel & Sowder, 1998) students used in the previously submitted work and then examined how students revised their responses. I found that some students revised their work by using a new proof scheme or amended the initial scheme they had used. I also found that students revised the procedures they used to test/check examples and how they used formal vocabulary. The conferences provided time and space to explore examples and students' reasoning more deeply. Students had increased opportunities to continue generating examples and I guided them to reflect on their examples. These opportunities contributed to how students used examples for new purposes and made their claims and warrants more explicit.

Research Question #3 sought to understand how students used writing practices engendered in the Convince Form and mathematical conferences to revise or improve their arguments. To answer this question, the mathematical writing samples gathered in the Convince Form were used to 1) assess the student engagement with mathematical structure, and 2) provide a shared focus of attention (Moschkovich, 2004), which was the basis for the conferences. The findings described in Chapter 5 include: 1) students revised their responses using new proof schemes or elaborating their argument and 2) students revised procedures related to testing claims and their use of formal terms. Thus, the support provided in the conferencing made a difference in terms of facilitating student revision of arguments, procedures, and meanings for vocabulary.

Relation to Literature

Regarding Research Questions #1 and #2, I found that when prompted for written arguments, some students changed from writing descriptions of procedures to generating claims in Lessons 1 and 2. Previous research has documented the kinds of procedural explanations that students generate when they describe procedures in lieu of generating mathematical arguments (Hebert & Powell, 2016). While Hebert & Powell (2016) found that procedural descriptions were "not favorable for instruction" (p.1531), others have found that when students use every day and technical language, "…the mathematical register is brought to life…" (Nemirovsky et al., 2005, p.199). There is evidence of this rousing to "life" in the students' movement from descriptions to generating claims.

Changing from writing descriptions of procedures to generating claims is significant and was difficult for students to achieve. The change demonstrates that some students were engaging the process to develop and generate claims, a crucial step in any argument. It reflects a shift in *attention* from how to calculate a single or small collection of examples to the patterns engendered in those examples. This aligns with Ellis' (2011) recommendation that grounding problems in "direct experience with quantity" can support students to generalize. The shift in attention from calculating individual examples to building sets of examples can help students to eventually “see” the mathematical structures that shape the patterns. At the beginning of the unit, students appeared to be focused on calculating examples. Then, their attention focused on describing how those examples should be calculated. This was what some students submitted as “arguments” in Lessons 1 and 2.

In Lesson 3, I found that when prompted to qualify claims and justify how they qualified them, students correctly qualified claims and at times justified them using empirical or analytic proof schemes. Like other research involving middle schoolers justifying claims, I found that empirical arguments were most common (see Knuth, Choppin, & Bieda, 2009 or Healy & Hoyles, 2000). I also found that students engaged examples in more mathematical ways.

Students’ changing responses can be understood in terms of how they used the *example space*[21] (Watson & Mason, 2005). As students moved beyond descriptions to develop claims and consider approaches to justifying, their initial attempts involved multiple instances of cherry-picking. The initial move away from describing procedures is important because such procedural writing is not considered to involve mathematical reasoning (Casa et al., 2015) and does not reflect a holistic view of the example space. The process of inquiry has been modeled as having three stages: Entry, Attack, and Review (Mason, Burton, & Stacey, 1982). Part of the entry phase is a process called *specializing* whereby students try specific cases, making

[21] An example space can be thought of as a collection of examples that is interconnected and structured by a larger mathematical generality (Watson & Mason, 2005).

calculations, to look for insights that might apply generally. The movement from generating examples, to "cherry picking" examples, and later toward using examples as data in an argument might reflect how students are using examples with an awareness of the entire example space (Watson & Mason, 2005). This awareness is required if students are to notice any patterns within the example space and then make general claims about classes of numbers.

As the awareness of patterns grows, students might begin to use examples for different purposes and refine their claims, suggesting a dialectic relationship between example use and "seeing" mathematical structure. If a dialectic relationship is presumed to exist between students' construction of example spaces and further refinement of claims about the example space, then it would behoove teachers and curricula to support *more* engagement, individually and in groups, with calculating, organizing, and reflecting upon examples. Currently, hierarchical views of proof and proving (for example, see Chimoni et al., 2018; Healy & Hoyles, 2000; Knuth, Choppin, & Bieda, 2009; and Küchemann & Hoyles, 2011), which consider the calculation and use of examples as inferior to more general arguments, provide implicit encouragement to move past the calculation and use of examples quickly. At the middle school level, students might benefit from a deeper engagement with an example space to explore mathematical structures at play.

This study provides empirical evidence that students who are multilingual, low-achieving, or designated as special education can engage in mathematical argument and illustrates how they can do so with support. This study contributes uniquely to the literature by illustrating in detail the ways that, with support, students who are multilingual, low-achieving, or designated as special education engage in mathematical argument. Many students used the support engendered in the discussions, collaboration, and writing tasks to engage the process of inquiry. Specifically, they engaged in the *specializing* process, and some moved toward later stages of the inquiry. Beyond Entry, the Attack and Review stages each involve a different way of engaging examples, justifying, and claims (Mason, Burton, & Stacey, 1982). Importantly, examples are never completely *gone* from the process.

The use of 'audience' in the mathematical writing tasks was hypothesized to support students by enhancing the writing expectations but many students ignored the hypothetical audience. While the use of audience was recommended by Banes (2019) and others (Casa et al., 2015), especially for multilingual students learning language *and* mathematics (Wilkinson, 2018), the students did not consistently generate responses appropriate to the proposed audience. It is possible that the use of an authentic audience might be more compelling for students. For example, Aguirre, Mayfield-Ingram, & Martin (2013) describe a project[22] where students presented student disciplinary data related to school suspensions to the principal of the school. In a case such as this, the use of an authentic audience could be more compelling and create a need for students to be more convincing.

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[22] See Chapter 4 entitled, "*Cultivating Mathematical Agency: "He Was Suspended for Being Mexican"*.

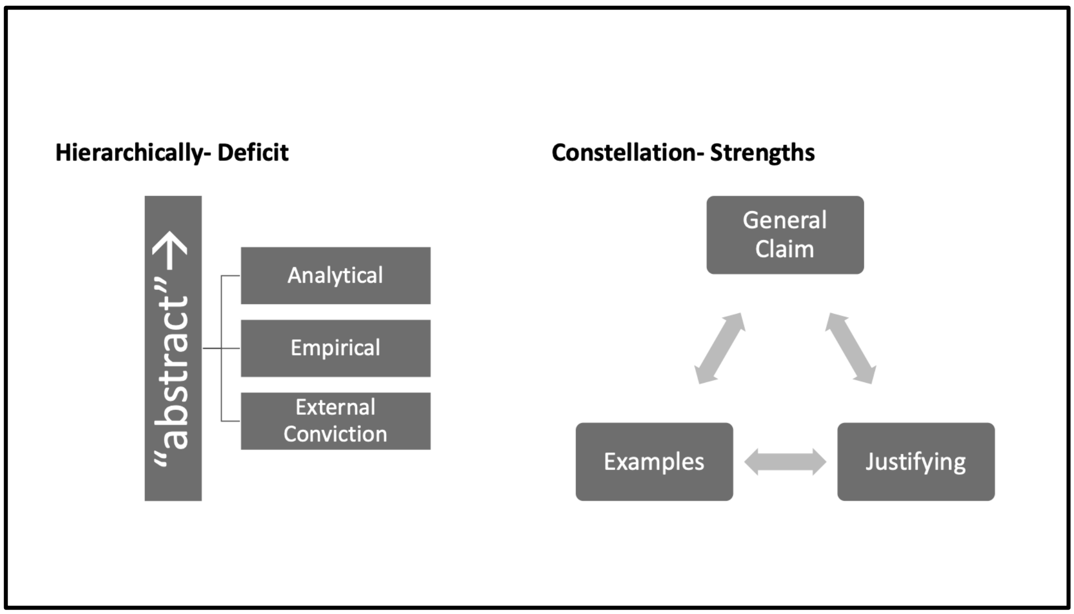
In response to the third research question, I found that students revised their responses using new proof schemes or elaborating their arguments. These changes in student work corroborate findings in previous research. For example, Ellis (2007) suggested that students benefit from iterative cycles of generalizing and justifying as one of four "mechanisms for change" to support increasingly sophisticated forms of algebraic reasoning (p.208). The other mechanisms included "focus," "generalizations that promote deductive reasoning," and "influence of deductive reasoning on generalizing" (Ellis, 2007). Each of these mechanisms is reported to work dynamically (and non-hierarchically) where focus on quantitative reasoning and supporting analytic proof schemes can help students learn to generalize. Ellis (2007) emphasizes that rich tasks are important, as well as the role of teachers. Importantly, Ellis concludes that "Although correct algebraic generalizations and deductive forms of proof remain a critical instructional goal, [the] study suggests that students' incorrect, nondeductive generalizations and proofs may serve as an important bridge toward this goal" (p.224). The conclusion that "incorrect" generalizations can be a bridge toward deductive proofs is relevant to this study because the incipient proof schemes that students communicate are important building blocks for more sophisticated understandings of proof, but only if students have opportunities to reflect on them and revise their arguments.

This work extends Ellis' (2007) work by showing how conferencing provided a productive instructional support for students writing and revising arguments. The conferences illustrate the need to support students to generate and (re)organize example spaces so that they can get to a point of inferring generalities. Once the support was in place, students responded by noticing structural aspects of the task and testing them, i.e., extending their empirical reasoning. For example, Nina and Antonio noticed how digits could be rearranged or a zero digit could be added to create novel happy numbers. They found some empirical support for a more general idea. They were subsequently positioned to begin the process of proving their general ideas deductively. The importance of the need for support in earlier stages of justifying was crucial for focal students Yolanda and Ricardo. With support, they clarified procedures and vocabulary so that they could move past the initial parts of the task.

The analysis supports a view of the learning of mathematical argument not as a series of stages but as the refinement and revision of claims, justifying schemes, and the use of examples in a spiraling, dialectic process (see Figure 6.1). A view of learning mathematical argument that is framed by stages typically positions empirical arguments as inferior to analytic ones (for example, see Healy & Hoyles, 2000; Knuth, Choppin, & Bieda, 2009; and Küchemann & Hoyles, 2011). Unfortunately, for many URM students, mathematics education engenders a racialized experience (Martin, 2006; McGee, 2017) and debates around their "readiness" to study rigorous mathematics can reflect racial inequities and disproportionately affect their access to rigorous mathematics education. Moreover, for multilingual students, fluency in English can be used as an added layer of gatekeeping (NASEM, 2018). By conceptualizing the learning of mathematical argument as the refinement and revision of claims, justifying schemes, and use of examples in a spiraling, dialectic process, students' linguistic and mathematical strengths are not cast in deficit ways but embraced and developed throughout the learning of mathematical argument.

Figure 6.1

Comparison of Proof Schemes as Hierarchically Organized and as a Constellation of Approaches to Proving



The analysis also shows that tasks prompting students to *qualify a claim* and then justify how they qualified are more accessible than tasks prompting students to *generate a claim*. Banes (2019) suggests that qualifying claims (which she describes as "selecting claims") can scaffold students' efforts to justify by removing the need to develop or generate a claim on their own. This kind of access is an important way to support students' entry into the dynamic process suggested by Ellis (2007).

This study documented increased student participation and increased variety of proof schemes when students qualified claims before justifying them and thus provides empirical support for Banes' (2019) suggestion. Generating a claim might be a formidable obstacle to the dynamic process of justifying. Consequently, the use of claims that students can qualify and then begin the process of justifying could serve as a more accessible entry point. In the future, research should consider generating a claim and justifying a claim, separately. This could guide researchers to understand each part as its own process but also understand how these two activities might support each other. Supporting students' by encouraging multiple types of and purposes for example-use can help them through such a process by creating opportunities for them to discern the mathematical structures that shape the example space. Mathematical arguments can be falsely dichotomized as "empirical" or "deductive". The analysis here shows that empirical reasoning is not the "floor" and supporting students earlier in the inquiry process can be beneficial.

During the conferences, a lack of "academic language" and "basic skills" did not seem to prevent students from learning about mathematical argument. Students were able to revise procedures and vocabulary related to the task *at the same time* that they revised their arguments. This is important because the assumption that students require fluency either with language or basic skills is a commonly used rationale for withholding rigorous mathematical content (NASEM, 2018; Martin, 2009). I also found that students were making different kinds of arguments, and in some cases the same student made different kinds of arguments. Hence, different proof schemes are not necessarily used or learned hierarchically (i.e., from concrete—or empirical—to abstract). This is significant because proof schemes are often presumed to be learned in a fixed learning trajectory where empirical proofs, which make use of examples, precede analytic ones, which are related to general aspects of a mathematical situation (Chimoni et al, 2018; Healy & Hoyles, 2000; Knuth, Choppin, & Bieda, 2009; and Küchemann & Hoyles, 2011). This analysis refutes arguments about student lack of "readiness" to study algebra if they are making empirical arguments.

This study builds on the proof scheme taxonomy by considering the proof schemes as a constellation of competencies that can be developed independently, as well as in relation to one another. This perspective is counter to views that position empirical arguments as inferior to analytic ones. In this analysis, empirical arguments and analytic arguments were related through a dialectic process that involved a deep engagement with quantitative reasoning, generating and refining claims, and opportunities to discuss, describe, and revise written arguments.

Future Research

One limitation of this study was the low participation rates, especially early on. The COVID-19 pandemic forced millions of students and teachers into remote instructional settings. Consequently, myriad issues arose that worked against full participation. For example, the lower rates and quality of internet connectivity found in the local community became a formidable obstacle to student participation. The difficulties that students encountered when navigating new platforms and submitting work via online forms sharply attenuated participation and could erode some of the conclusions drawn in this study. A study in a more traditional classroom setting (i.e., in-person) might provide additional data in support of the findings presented here.

Pedagogy of Acompañamiento (Sepúlveda, 2011) would be useful to future analyses of these data because it is relevant to the kind of conferencing conducted in this study. Sepúlveda (2011) describes acompañamiento as “a fellowship and engagement with one another without goals and objectives” (p.560). This pedagogical approach to literacy could be expanded to approach mathematical reasoning, thus framing future study of such reasoning in mathematics conferences. While I had some “goals and objectives”, they were not specific (in terms of students coming to my own pre-determined conclusion). Acompañamiento is predicated on a few guiding principles: 1) that social justice can’t be done “to” students, 2) affirmation of student ideas and experience, and 3) relationships based in empathetic love. This approach could be used alongside other approaches in mathematics research that foreground relationships between teachers and students. This can be a problem because of the deficit views teachers hold of multilingual students (Faltis & Valdés, 2016). Also, the pedagogy of acompañamiento is based in a loving ethos that requires that teachers suspend a priori notions in favor of seeking to understand *student* reasoning and expression.

Implications for Practice

Researchers and practitioners have called for instruction to include the use of rehumanizing mathematics instructional practices (Goffney et al., 2018) so that students can participate in ways that support their full access to mathematical content and activity. These types of instructional practices could include, but are not limited to, those that support co-construction of mathematical knowledge, collaboration with peers, shared mathematical authority, the use of multiple entry points for mathematical tasks, and the use of multiple forms of assessment. The analyses presented here support a focus on example spaces that could support further engagement with the refinement of mathematical claims and the available ways to justify them.

Some students revised their arguments even at the same time that *they were revising how they used* procedures and meanings for formal vocabulary. This finding directly refutes a common belief that students must master the “basic skills” of mathematics or language as a precondition to engage in rigorous mathematics. Some of the study participants did both at the same time. This study shows the details of how multilingual students constructed mathematical arguments and illustrates how multilingual students can do rigorous mathematics, while developing English. The “English Language Learner” (ELL) designation is not relevant to access to such work. Future research should explore how professional development can feature mathematical conferencing as a pedagogical strategy for supporting all students, but especially multilingual students.

More research is needed to explore how having an authentic audience might change student responses**.** The study featured an activity where students reasoned to an imagined audience (in the Convince Form). My early assumption was that an imagined audience would be sufficient for students to consider the unique rhetorical demands of, for example, convincing oneself, a friend, or a skeptic. However, many students either ignored the imagined audience or considered the imagined audience with a seemingly performative response. Future research could use classroom activities that address school or community issues in such a way that students use mathematics as a tool when working with other stakeholders and thus provide an authentic audience.

Students leveraged the work produced in shared documents when generating arguments. A student (Amaya) commented that she preferred to work collaboratively as a whole class than individually or in groups “Because, people had, like, answers. [inaudible] how they had answers and I could understand that more” (Turn 128, Amaya transcript). In her comment, Amaya referred to examples as “answers” and leveraged the crowd sourcing of examples to use in her own argument. I wonder whether the students were leveraging culturally based modes of participation in mathematical activity. For example, considering the learning-by-observation-and-pitching-in framework (LOPI; Rogoff, 2014), it might be possible to support and examine student contributions to shared documents and how they support learning. The LOPI framework represents an “informal learning” model that includes seven facets[23] that focus on students collaborating to contribute to a shared endeavor. Using shared documents to collectively develop conjectures about mathematical

[23] Seven facets: 1) community organization *incorporates* children in the range of ongoing endeavors of their families and communities, 2) learners are *eager to contribute and belong* as valued members of their families and communities, 3) the social organization of endeavors involves *collaborative engagement as an ensemble*, with *flexible leadership* as the people involved coordinate fluidly with each other, 4) the goal of learning is *transforming participation* to contribute and belong in the community, 5) learning involves *wide, keen attention,* in anticipation of or during *contribution* to the endeavor at hand, 6) communication is based on coordination among participants that builds on the *shared reference* available in their mutual endeavors, and 7) assessment includes appraisal of the success of the *support* provided for the learner as well as of the learner’s *progress toward mastery* (Rogoff, 2014).

situations might leverage a cultural practice to promote deeper engagement with the process of mathematical inquiry and support the use of culturally sustaining pedagogies (Alim, Paris, & Wong, 2020).

A final comment

Focusing on student reasoning is a long-standing recommendation in the literature on equity in mathematics education and mathematics education writ large (Goffney, Gutiérrez, & Boston, 2018; Leinwand, Brahier, & Huinker, 2014; Moschkovich, 2013a, 2013b; NCTM, 2014). However, much of mathematics instruction for URM students continues to over-emphasis the role of procedures (NCTM, 2018), especially regarding multilingual students (NASEM, 2018). Focusing on student reasoning means resisting ‘funneling’, a common pitfall for many teachers. Funneling undermines student agency and indirectly communicates deficit views of students by positioning the student’s ideas as secondary or inferior to the teacher’s ideas. Focusing on student reasoning means accompanying students’ ideas, wherever they might go. This means that teachers should suspend their judgements about “right” and “wrong”, thus communicating confidence in students’ ideas, and support students to pursue and develop their ideas. There is often a temptation to short-circuit the learning process by jumping directly to a generalized statement, but this does not serve learning. The expediency comes at the *expense* of learning.

Supporting reasoning is critical to the inquiry process, especially for URM students. Since mathematics education can be viewed as a white institutional space (Battey & Leyva, 2016; Martin, 2009) where deficit views of students are rampant vis-à-vis language (NASEM, 2018), race (Martin, 2006), gender (Fennema, 1974; Leder, 2019), and intersectionally in the case of Black and Latinx girls (Copur-Gencturk, Cimpian, Lubienski, and Thacker, 2020), then efforts to facilitate entry to mathematical inquiry must provide some kind of redress or remedy to the ubiquitous violence (symbolic, epistemological, and even physical) that is visited upon them. It is possible that focusing on student reasoning presents a kind of remedy by consciously holding the teacher's a priori goals and judgements about mathematical capacity at bay to create space for students to reveal, develop, or construct their own mathematical goals and capacities to meet them.

The gatekeeping function of mathematics, and algebra in particular, is well documented (Martin, 2009; NCSM & TODOS, 2019; NRC, 1998; Stein et al., 2011; Stephens et al., 2017). Access to rigorous mathematics can be blocked when students are seen as "unprepared" or "unable" to study more advanced mathematics. Arguments for grouping students by their perceived abilities, i.e., tracking, are typically based on an assumption that the instruction and materials will be commensurate with student "abilities" and those abilities will be accurately assessed. However, observed test scores can be attributed to the students' interactions with the *learning environment* rather than their mathematical abilities (Boaler & Sengupta-Irving, 2016; Martin, 2009). Likewise, achievement of multilingual students is also sensitive to characteristics of the learning environment (Barwell, Moschkovich, & Setati Phakeng, 2018; Goodrich, Thayer, & Leiva, 2021). Moreover, access to advanced mathematics classes for multilingual students can be further mitigated by the inappropriate usage of tests for *English* proficiency, especially when support classes for English and mathematics are tracked together (NASEM, 2018). All students in this study benefitted from access to good tasks and support in revising their arguments. By introducing support *as it was required* by these students, the conferencing mitigated the effects of gatekeeping and promoted access to advanced mathematics for these students. This study provides an example of how to support access for all students in learning to construct mathematical arguments.

References Cited

Aguirre, J. M., Bunch, G. C. (2012). What's language got to do with it?: Identifying language demands in mathematics instruction for English language learners. In S. Celedón-Pattichis & N. G. Ramirez's (Eds.) *Beyond good teaching: Advancing mathematics education for ELLs* (pp. 183-194). Reston, VA: The National Council of Teachers of Mathematics.

Aguirre, J., Herbel-Eisenmann, B., Celedón-Pattichis, S., Civil, M., Wilkerson, T., Stephan, M., Pape, S., & Clements, D. H. (2017). Equity within mathematics education research as a political act: Moving from choice to intentional collective professional responsibility. *Journal for Research in Mathematics Education*, 48(2), 124-147.

Aguirre, J., Mayfield-Ingram, K., & Martin, D. (2013). *The impact of identity in K-8 mathematics: Rethinking equity-based practices*. The National Council of Teachers of Mathematics.

Alim, H. S., Paris, D., & Wong, C. P. (2020). Culturally sustaining pedagogy: A critical framework for centering communities. In N. Nasir, C.D. Lee, R. Pea, & M. Mckinney de Royston's (Eds.) *Handbook of the cultural foundations of learning* (pp. 261-276). Routledge.

Applebee, A. N., & Langer, J. A. (2011). "EJ" Extra: A Snapshot of Writing Instruction in Middle Schools and High Schools. *The English Journal*, 100(6), 14-27.

Banes, L. C. (2019, April). Mathematical argumentative writing: towards stronger genre uptake and tasks that elicit it. Paper presented at the American Educational Research Association (AERA) annual meeting in Toronto, Canada.

Bangert-Drowns, R. L., Hurley, M. M., & Wilkinson, B. (2004). The effects of school-based writing-to-learn interventions on academic achievement: A meta-analysis. *Review of Educational Research*, 74, 29−58. doi:10.3102/00346543074001029

Barwell, R., Moschkovich, J., & Setati Phakeng, M. (2017). Language diversity and mathematics: Second language, bilingual and multilingual learners. In J. Cai's (Ed.) *First compendium for research in mathematics education*. National Council of Teachers of Mathematics: Reston, VA. 583-606.

Battey, D., & Leyva, L. A. (2016). A Framework for Understanding Whiteness in Mathematics Education. *Journal of Urban Mathematics Education,* 9(2), 49-80.

Baxter, J. A., Woodward, J., & Olson, D. (2005). Writing in mathematics: An alternative form of communication for academically low-achieving students. *Learning Disabilities Research and Practice*, 20, 119−135. doi:10.1111/j.1540-5826.2005.00127.x

Bieda, K. N., & Staples, M. (2020). Justification as an equity practice. *Mathematics Teacher: Learning and Teaching PK-12*, 113(2), 102-108.

Blanton, M. L., & Kaput, J. J. (2011). Functional thinking as a route into algebra in the elementary grades. In J. Cai & E. Knuth's (Eds.) *Early algebraization: A global dialogue from multiple perspectives* (pp. 5-23). Springer Berlin Heidelberg.

Blanton, M., Levi, L., Crites, T., Dougherty, B., & Zbiek, R. M. (2011). *Developing Essential Understanding of Algebraic Thinking for Teaching Mathematics in Grades 3-5*. Series in Essential Understandings. Reston, VA: National Council of Teachers of Mathematics.

Blanton, M.L., Stylianou, D.A., & David, M. (2009). Understanding instructional scaffolding in classroom discourse on proof. In D.A. Stylianou, M.L. Blanton, & E.J. Knuth's (Eds.) *Teaching and learning proof across the grades: A k-16 perspective* (Ch.17). New York, NY: Routledge.

Boaler, J., Sengupta-Irving, T. (2006). Nature, neglect and nuance: Changing accounts of sex, gender and mathematics. In Skelton, C., Francis, B., & Smulyan, L. (Eds.). *The Sage handbook of gender and education*. Thousand Oaks, CA: Sage Publications, pp. 207 – 220.

Boaler, J., Sengupta-Irving, T. (March, 2016). The many colors of algebra: Engaging disaffected students through collaboration and agency. *The Journal of Mathematical Behavior*. 41, 179 – 190.

Brenner, M. E. (1998). Adding cognition to the formula for culturally relevant instruction in mathematics, *Anthropology and Education Quarterly*, 29(2), 213-244.

Brenner, M. E. (2006). Interviewing in Educational Research. In J. Green, G. Camilli, & P. Elmore (Eds.) *Complementary Methods for Research in Education* (pp. 357-370). Washington, DC: American Educational Research Association/Erlbaum.

Brizuela, B. M., & Earnest, D. (2008). Multiple notational systems and algebraic understandings: The case of the “best deal” problem. In J. Kaput, D.W. Carraher, & M. Blanton (Eds.), *Algebra in the early grades*, 273-301.

Bunch, G. C. (2014). The language of ideas and the language of display: Reconceptualizing “academic language” in linguistically diverse classrooms. *International Multilingual Research Journal*, 8(1), 70-86.

California Department of Education (2020). Home Language Survey. https://www.cde.ca.gov/ta/cr/documents/hls2020.pdf

Carraher, D. W., & Schliemann, A. D. (2007). Early algebra and algebraic reasoning. In K. Lester's (Ed.) *Second handbook of research on mathematics teaching and learning*, 2, 669-705.

Carraher, D. W., Schliemann, A. D., Schwartz, J. L. (2008). Early algebra is not the same as algebra early. In J. Kaput, D.W. Carraher, & M. Blanton (Eds.), *Algebra in the early grades* (235 – 272). Mahwah, NJ: Erlbaum.

Casa, T. M., Firmender, J. M., Cahill, J., Cardetti, F., Choppin, J. M., Cohen, J., …Zawodniak, R. (2016). *Types of and purposes for elementary mathematical writing: Task force recommendations*. Retrieved from http://mathwriting.eduation.uconn.edu

Chimoni, M., Pitta-Pantazi, D. & Christou, C. (2018). Examining early algebraic thinking: insights from empirical data. *Educational Studies in Mathematics*, 98(1): 57-76. https://doi.org/10.1007/s10649-018-9803-x

Choi, J. A. (2008). Unlearning colorblind ideologies in education class. *The Journal of Educational Foundations*, 22(3/4), 53.

Civil, M. (2006, November). Working towards equity in mathematics education: A focus on learners, teachers, and parents. In S. Alatorre, J.L. Cortina, M. Sáiz, & A. Méndez's (Eds.) *Proceedings of the Twenty Eighth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 30-50).

Clarke, D. J., Waywood, A., & Stephens, M. (1993). Probing the structure of mathematical writing. *Educational Studies in Mathematics*, 25, 235−250. doi:10.1007/bf01273863

Cobb, P., Jackson, K., & Dunlap, C. (2017). Conducting design studies to investigate and support mathematics students' and teachers' learning. In J. Cai's (Ed.) *Compendium for research in mathematics education* (pp.208-233). Reston, VA: National Council of Teachers of Mathematics.

Cohen, J. A., Miller, H. C., Casa, T. M., & Firmender, J. M. (2015). Characteristics of second graders' mathematical writing. *School Science and Mathematics*, 115 (7), 344−355. doi:10.1111/ssm.12138

Cooper, T. J., & Warren, E. (2011). Years 2 to 6 students' ability to generalise: Models, representations and theory for teaching and learning. In J. Cai & E.

Knuth's (Eds.) *Early algebraization* (pp. 187-214). Springer, Berlin, Heidelberg.

Cope, B., & Kalantzis, M. (2009). “Multiliteracies”: New literacies, new learning. *Pedagogies: An international journal*, 4(3), 164-195.

Copur-Gencturk, Y., Cimpian, J. R., Lubienski, S. T., & Thacker, I. (2020). Teachers’ bias against the mathematical ability of female, Black, and Hispanic students. *Educational Researcher*, 49(1), 30-43.

Countryman, J. (1992). *Writing to learn mathematics: Strategies that work*. Portsmouth, NH: Heinemann.

Crouch, R. (with Jiandani, J.). (2012). The United States of education: The changing demographics of the United States and their schools. Retrieved from http://www.centerforpubliceducation.org/You-May-Also-Be-Interested-In-landing-page-level/Organizing-a-School-YMABI/The-United-States-of-education-The-changing-demographics-of-the-United-States-and-their-schools.html

Cuoco, A., Goldenberg, E. P., & Mark, J. (1996). Habits of mind: An organizing principle for mathematics curricula. *The Journal of Mathematical Behavior*, 15(4), 375-402.

de Araujo, Z., Roberts, S. A., Willey, C., & Zahner, W. (2018). English learners in K–12 mathematics education: A review of the literature. *Review of Educational Research*, 88(6), 879-919.

Dietiker, L., Baldinger, E., Kassarjian, M. and Shreve, B. (2013). *Core Connections 3*. V(5). Elk Grove, CA: CPM Educational Program.

Dominguez, H. (2008). *The discourse of mathematization: Bilingual students reinventing mathematics and themselves as mathematical thinkers*. Doctoral dissertation, University of Texas- Austin. ProQuest. URL: https://repositories.lib.utexas.edu/bitstream/handle/2152/3838/dominguezh97734.pdf?sequence=2

Domínguez, H. (2011). Using what matters to students in bilingual mathematics problems. *Educational Studies in Mathematics*, 76(3), 305-328.

Donato, R., Hanson, J. (2012) Legally White, Socially "Mexican": The Politics of de jure and de facto School Segregation in the American Southwest. *Harvard Educational Review*. 82 (2): 202 – 225.

Ebrahimji, A. (2020, August). School sends family a hotspot after students went to Taco Bell to use their free WiFi. *CNN.com.* https://www.cnn.com/2020/08/31/us/taco-bell-california-students-wifi-trnd/index.html

Education Committee of the European Mathematical Society. (2011). Do theorems admit exceptions? Solid findings in mathematics education on empirical proof schemes. *EMS Newsletter*, 82, 50–53.

Ellis, A. B. (2007). Connections between generalizing and justifying: Students' reasoning with linear relationships. *Journal for Research in Mathematics education*, *38*(3), 194-229.

Ellis, A. B. (2011). Algebra in the middle school: Developing functional relationships through quantitative reasoning. In J. Cai & E. Knuth's (Eds.) *Early algebraization* (pp. 215-238). Springer, Berlin, Heidelberg.

Ellis, A. E., Lockwood, E., Williams, C. C. W., Dogan, M. F., & Knuth, E. (2012). Middle school students' example use in conjecture exploration and justification. In L. R. Van Zoest, J. J. Lo, & J. L. Kratky (Eds.), *Proceedings of the 34th Annual Meeting of the North American Chapter of the Psychology of Mathematics Education* (pp. 135–142). Kalamazoo, MI: Western Michigan University.

Faltis, C. J., & Valdés, G. (2016). Preparing teachers for teaching in and advocating for linguistically diverse classrooms: A vade mecum for teacher educators. In D. Gitomer & C. Bell's (Eds.) *Handbook of research on teaching*, 549-592.

Fennema, E. (1974). Mathematics Learning and the Sexes: A Review, *Journal for Research in Mathematics Education*, 5(3), 126-139. Retrieved Jan 19, 2022, from https://pubs.nctm.org/view/journals/jrme/5/3/article-p126.xml

Ferguson, C. A. (1994). Dialect, register, and genre: Working assumptions about conventionalization. In D. Biber & E. Finegan's (Eds.), *Sociolinguistic perspectives on register* (pp. 15–30). New York: Oxford University.

Ferrini-Mundy, J., & Martin, W. G. (2000). *Principles and standards for school mathematics*. Reston: National Council of Teachers of Mathematics (NCTM).

Franke, M. L., Webb, N. M., Chan, A. G., Ing, M., Freund, D., & Battey, D. (2009). Teacher Questioning to Elicit Students' Mathematical Thinking in Elementary School Classrooms. *Journal of Teacher Education*, 60(4), 380–392.

Gholson, M., Bullock, E., & Alexander, N. (2012). On the brilliance of black children: A response to a clarion call. *Journal of Urban Mathematics Education*, 5(1), 1-7. Available at: http://ed-osprey.gsu.edu/ojs/index.php/JUME/article/view/180/106

Gillespie, A., Graham, S., Kiuhara, S., & Hebert, M. (2014). High school teachers use of writing to support students' learning: A national survey. *Reading and Writing*, 27(6), 1043-1072.

Goffney, I., Gutiérrez, R., & Boston, M. (2018). *Rehumanizing mathematics for Black, Indigenous, and Latinx students*. Reston, VA: National Council of Teachers of Mathematics.

González, G. G. (2017). Segregation and the education of Mexican children, 1900-1940. In J.F. Moreno's (Ed.) *The elusive quest for equality*, 150, 53-76.

Goodrich, J. M., Thayer, L., & Leiva, S. (2021). Evaluating Achievement Gaps Between Monolingual and Multilingual Students. *Educational Researcher*, 50 (7) 0013189X21999043.

Grabiner, J. V. (2012). Why proof? A historian's perspective. In G. Hanna & M. de Villiers' (Eds.) *Proof and proving in Mathematics Education: The 19th ICME study* (pp. 147-167). Springer, Dordrecht.

Gravemeijer, K. & Cobb, P. (2013). Design research from the learning design perspective. In: T. Plomp & N. Nieveen (Eds.), *Educational design research Part A: An introduction* (pp. 72–113), Enschede: SLO.

Gravemeijer, K., Prediger, S. (2019) Topic-specific design research: An introduction. In G. Kaiser & N. Presmeg's (Eds.) *Compendium for Early Career Researchers in Mathematics Education* (pp.33-58). ICME-13 Monograph. Cham, Switzerland: Springer.

Grice, H. P., Cole, P., & Morgan, J. L. (1975). *Syntax and semantics*.

Gutiérrez, K. D., & Rogoff, B. (2003). Cultural ways of learning: Individual traits or repertoires of practice. *Educational researcher*, 32(5), 19-25.

Gutiérrez, K. D., Baquedano-López, P., & Tejeda, C. (1999). Rethinking diversity: Hybridity and hybrid language practices in the third space. *Mind, culture, and activity*, 6(4), 286-303.

Gutiérrez, K. D., Morales, P. Z., & Martinez, D. C. (2009). Re-mediating literacy: Culture, difference, and learning for students from nondominant communities. *Review of research in education*, 33(1), 212-245.

Gutiérrez, R. (2009). Framing equity: Helping students “play the game” and “change the game.” *Teaching for Excellence and Equity in Mathematics*, 1(1), 4–8.

Gutiérrez, R. (2012). Context matters: How should we conceptualize equity in mathematics education? In B. Herbel-Eisenmann, J. Choppin, D. Wagner, & D. Pimm's (Eds.) *Equity in discourse for mathematics education* (pp. 17-33). Springer, Dordrecht.

Gutiérrez, R. (2013). Commentary: Why (urban) mathematics teachers need political knowledge. *Journal of Urban Mathematics Education*, 6(2).

Gutiérrez, R. (2017a). Living mathematx: Towards a vision for the future. *Philosophy of Mathematics Education Journal*, 32.

Gutiérrez, R. (2017b). Political conocimiento for teaching mathematics: Why teachers need it and how to develop it. In S. Kastberg, A. M. Tyminski, A. Lischka, & W. Sanchez (Eds.), *Building support for scholarly practices in mathematics methods* (pp. 11–38). Charlotte, NC: Information Age.

"Haas Institute (2017). Responding to Educational Inequality, Berkeley, CA: Haas Institute for a Fair and Inclusive Society, University of California, Berkeley, September 2016.

Halliday, M. A. (1978). Sociolinguistics aspects of mathematical education. *The social interpretation of language and meaning*, 194-204.

Harel, G., & Sowder, L. (1998). Students' proof schemes: Results from exploratory studies. *American Mathematical Society*, 7, 234-283.

Healy, L., & Hoyles, C. (2000). A study of proof conceptions in algebra. *Journal for research in mathematics education*, 396-428.

Hebert, M. A., & Powell, S. R. (2016). Examining fourth-grade mathematics writing: Features of organization, mathematics vocabulary, and mathematical representations. *Reading and Writing*, 29, 1511–1537. doi:10.1007/s11145-016-9649-5

Herbel-Eisenmann, B., & Wagner, D. (2007). A framework for uncovering the way a textbook may position the mathematics learner. *For the Learning of Mathematics*, 27(2), 8-14.

Hiebert, J., & Grouws, D. A. (2007). The effects of classroom mathematics teaching on students' learning. In F.K. Lester's (Ed.) *Second handbook of research on mathematics teaching and learning*, 1(1), 371-404.

Hillocks, G. (2006). Middle and high school composition. In P. Smagorinsky's (Ed.). *Research on composition: Multiple perspectives on two decades of change* (pp.48-77). New York, NY: Teachers College Press.

Hsu, F. (2015). The coloniality of neoliberal English: The enduring structures of American colonial English instruction in the Philippines and Puerto Rico*. L2 Journal*, 7(3).

Hsu, F. (2017). Resisting the Coloniality of English: A Research Review of Strategies. *CATESOL Journal*, 29(1), 111-132.

Hudicourt-Barnes, J. (2003). The use of argumentation in Haitian Creole science classrooms. *Harvard Educational Review*, 73(1), 73-93.

Huitzilopochtli, S. Foxworthy-Gonzalez, J., Moschkovich, J.N., McHugh, S.R., & Callanan, M.A. (2021). Noticing multilingual and non-dominant students' strengths for learning mathematics and science. In A. Essien & A. Msimanga's (Eds*.) Multilingual education yearbook 2021: Policy and practice in STEM multilingual contexts*. Springer.

Hull, G. A., & Moje, E. B. (2012). What is the development of literacy the development of? Commissioned papers on language and literacy issues in the Common Core State Standards and Next Generation Science Standards, 94, 52.

Inniss, T.R., Lewis, W.J., Mitrea, I., Okoudjou, K.A., Salerno, A., Su, F., & Thurston, D. (March 2021). Towards a fully inclusive mathematics profession. *American Mathematical Society*. http://www.ams.org/about-us/understanding-ams-history

Kaput, J., Carraher, D.W., & Blanton, M. (Eds.) (2008) *Algebra in the early grades*. Mahwah, NJ: Erlbaum.

Kaput, J. (2008). What is algebra? What is algebraic reasoning? In J. Kaput, D.W. Carraher, & M. Blanton (Eds.), *Algebra in the early grades* (5 – 19). Mahwah, NJ: Erlbaum.

Kaput, J., Blanton, M. L., Moreno, L. (2008). Algebra from a symbolization point of view. In J. Kaput, D.W. Carraher, & M. Blanton (Eds.), *Algebra in the early grades* (19 - 57). Mahwah, NJ: Erlbaum.

Kieran, C. (2007). Learning and teaching algebra at the middle school through college levels: Building meaning for symbols and their manipulation. In K. Lester's (Ed.) *Second handbook of research on mathematics teaching and learning*, 2, 707-762.

Klass, G. (2008). Just plain data analysis: Common statistical fallacies in analyses of social indicator data. Statlit.org.

Knipping, C., & Reid, D. A. (2019). Argumentation Analysis for Early Career Researchers. In G. Kaiser, N. Presmeg's (Eds.) *Compendium for Early Career Researchers in Mathematics Education* (pp. 3-31). ICME-13 Monograph. Cham, Switzerland: Springer

Knuth, E., Choppin, J., & Bieda, K. (2009). Middle school students' production of mathematical justifications. In D.A. Stylianou, M.L. Blanton, & E.J. Knuth's (Eds.). *Teaching and learning proof across the grades: A K-16 perspective*. New York, NY: National Council of Teachers of Mathematics/ Routledge.

Kosko, K. W., & Zimmerman, B. S. (2015). Emergence of argument in children's mathematical writing. *Journal of Early Childhood Literacy*, 1468798417712065.

Küchemann, D., & Hoyles, C. (2009). From empirical to structural reasoning in mathematics: Tracking changes over time. In D.A. Stylianou, M.L. Blanton, & E.J. Knuth's (Eds.) *Teaching and learning proof across the grades: A k-16 perspective* (Ch.10). New York, NY: Routledge.

Lampert, M. (1990). When the problem is not the question and the solution is not the answer: Mathematical knowing and teaching. *American educational research journal*, 27(1), 29-63.

Lampert, M., & Cobb, P. (2003). Communication and language. In J. Kilpatrick, W.G. Martin, & D. Schifter's (Eds.) *A research companion to principles and standards for school mathematics* (p.237 – 249). Reston, VA: National Council of Teachers of Mathematics.

Lea, M. R., & Street, B. V. (2006). The "academic literacies" model: Theory and applications. *Theory into Practice*, 45(4), 368–377.

Leder, G.C. (2019). Gender and mathematics education: An overview. In G. Kaiser & N. Presmeg's (Eds.) *Compendium for Early Career Researchers in Mathematics Education* (p.289-308). ICME-13 Monograph. Cham, Switzerland: Springer.

Leinwand, S., Brahier, D.J., & Huinker, D. (2014). *Principles to actions: Ensuring mathematical success for all*. Reston, VA: National Council of Teachers of Mathematics, Incorporated.

Louie, N. L. (2018). Culture and ideology in mathematics teacher noticing. *Educational Studies in Mathematics,* 97 (1), 55-69.

Manke, K. (2020, December). California farmworkers hit hard by COVID-19, study finds. *Berkeley News.* https://news.berkeley.edu/2020/12/02/california-farmworkers-hit-hard-by-covid-19-study-finds/

Martin, D. B. (2006). Mathematics learning and participation as racialized forms of experience: African American parents speak on the struggle for mathematics literacy. *Mathematical Thinking and Learning*, 8(3), 197-229.

Martin, D. B. (2009). Researching race in mathematics education. *Teachers College Record*, 111(2), 295-338.

Martin, D. B. (2013). Race, racial projects, and mathematics education. *Journal for Research in Mathematics Education*, 44(1), 316-333.

Martin, D. B. (2019). Equity, inclusion, and antiblackness in mathematics education. *Race Ethnicity and Education*, 22(4), 459-478.

Mastroianni, M.P. (2013) Writing in mathematics. In A.N. Applebee and J.A. Langer's (Eds.) *Writing instruction that works: Proven methods for middle and high school classrooms* (pp.71-93). New York, NY: Teachers College Press.

Mason, J., Burton, L., & Stacey, K. (1982). *Thinking mathematically*. Harlow.

Mathematics Assessment Resource Service (2015) Evaluating Statements: Consecutive Sums. Mathematics Assessment Project Classroom Challenges: A formative assessment lesson. Available at: http://map.mathshell.org

McGee, E. O. (2016). Devalued Black and Latino racial identities: A by-product of STEM college culture? *American Educational Research Journal*, *53*(6), 1626-1662.

McGee, E. O. (2020a). Interrogating structural racism in STEM higher education. *Educational Researcher*, 49(9), 633-644.

McGee, E. O. (2020b). *Black, brown, bruised: How racialized STEM education stifles innovation*. Cambridge, MA: Harvard Education Press.

McGee, E. O., & Martin, D. B. (2011). "You would not believe what I have to go through to prove my intellectual value!" Stereotype management among academically successful Black mathematics and engineering students. *American Educational Research Journal*, 48(6), 1347-1389.

Minnameier, G. (2004). Peirce-suit of truth–why inference to the best explanation and abduction ought not to be confused. *Erkenntnis*, 60(1), 75-105.

Morgan, C. (1998). Writing mathematically: The discourse of investigation (Vol. 9). Psychology Press.

Morgan, C., Craig, T., Schuette, M., & Wagner, D. (2014). Language and communication in mathematics education: An overview of research in the field. *ZDM*, 46(6), 843-853.

Morton, K., & Riegle-Crumb, C. (2019). Who gets in? Examining inequality in eighth-grade algebra. *Journal for Research in Mathematics Education*, 50(5), 529-554.

Morton, K., & Riegle-Crumb, C. (2020). Is School Racial/Ethnic Composition Associated With Content Coverage in Algebra. *Educational Researcher*, 49(6), 441-447.

Moschkovich, J. (2002). A situated and sociocultural perspective on bilingual mathematics learners. *Mathematical thinking and learning*, 4(2-3), 189-212.

Moschkovich, J. N. (2004). Appropriating mathematical practices: A case study of learning to use and explore functions through interaction with a tutor. *Educational Studies in Mathematics.* 55: 49 – 80.

Moschkovich, J. (2007). Examining mathematical discourse practices. *For the learning of mathematics,* 27(1), 24-30.

Moschkovich, J. (2013a). Principles and guidelines for equitable mathematics teaching practices and materials for English language learners. *Journal of Urban Mathematics Education*, 6(1), 45-57.

Moschkovich, J. (2013b). Equitable practices in mathematics classrooms: Research-based recommendations. *Teaching for Excellence and Equity in Mathematics*, 5(1), 26-34.

Moschkovich, J. N. (2008). "I went by twos, he went by one": Multiple interpretations of inscriptions as resources for mathematical discussions. *The Journal of the Learning Sciences*, 17(4), 551-587.

Moschkovich, J. N. (2015). Academic literacy in mathematics for English Learners. *The Journal of Mathematical Behavior*, 40, 43-62.

Moschkovich, J. N. (2019). A Naturalistic Paradigm: An Introduction to Using Ethnographic Methods for Research in Mathematics Education. In G. Kaiser & N. Presmeg's (Eds.) *Compendium for Early Career Researchers in Mathematics Education* (p.59-79). ICME-13 Monograph. Cham, Switzerland: Springer

Moschkovich, J. N., & Brenner, M. E. (2000). Integrating a Naturalistic Paradigm into Research on Mathematics and Science Cognition and Learning. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of Research Design in Mathematics and Science Education* (pp. 457-486). Mahwah, NJ: Lawrence Erlbaum Associates, Publishers.

Moschkovich, J.N. (2018). Considering silence and precision: Communication practices that support equitable mathematical discussions for marginalized students. In R. Hunter, M. Civil, B. Herbel-Eisenmann, N. Planas, & D. Wagner's (Eds.) *Mathematical discourse that breaks barriers and creates space for marginalized learners* (pp.61-84). Rotterdam, The Netherlands: Sense Publishers.

Moschkovich, J., Schoenfeld, A., & Arcavi. A. (1993). Aspects of understanding: On multiple perspectives and representations of linear relations and connections among them. In T.A. Romberg, E. Fennema, and T.P. Carpenter (Eds.) *Integrating research on the graphical representations of functions* (pp.69-100). Hillsdale, NJ: Erlbaum.

Munson, J. (2018). Responding to student thinking in the moment: Examining conferring practices and teacher learning in the elementary mathematics classroom (Order No. 28115152). Available from ProQuest Dissertations & Theses A&I. (2438014321). https://www.proquest.com/dissertations-theses/responding-student-thinking-moment-examining/docview/2438014321/se-2?accountid=14523

Nasir, N. I. S. (2005). Individual cognitive structuring and the sociocultural context: Strategy shifts in the game of dominoes. *The Journal of the Learning Sciences*, *14*(1), 5-34.

National Academies of Sciences, Engineering, and Medicine (2018) *English Learners in STEM Subjects: Transforming Classrooms, Schools, and Lives*.

Washington, DC: The National Academies Press. https://doi.org/10.17226/25182.

National Council of Supervisors of Mathematics, TODOS: Mathematics for all (2020). Mathematics Education Through the Lens of Social Justice: Acknowledgment, Actions, and Accountability. Available at: https://www.todos-math.org/socialjustice

National Council of Teachers of Mathematics (NCTM). (2018). *Catalyzing change in high school mathematics: Initiating critical conversations*. Reston, VA: National Council of Teachers of Mathematics.

National Research Council, & Mathematics Learning Study Committee. (2001). *Adding it up: Helping children learn mathematics*. National Academies Press.

National Research Council. (1998). *The nature and role of algebra in the K-14 curriculum: Proceedings of a national symposium*. Washington, D. C.: National Academies Press.

Nemirovsky, R., Barros, A., Noble, T., Schnepp, M., & Solomon, J. (2005). Learning Mathematics in High School: Symbolic Places and Family Resemblances. *Understanding Mathematics and Science Matters*, 185.

Newkirk, T. (1989). The first five minutes: Setting the agenda in a writing conference. In C.M. Anson's (Ed). *Writing and Response: Theory, Practice, and Research* (pp. 317-331). National Council of Teachers of English: Urbana, IL.

O'Halloran, K. L. (2015). The language of learning mathematics: A multimodal perspective. *The Journal of Mathematical Behavior*, 40, 63-74.

Oakes, J., Rodgers, J., Lipton, M. (2006). *Learning Power: Organizing for Education and Justice*. New York, NY: Teachers College Press.

Ochs, E. (1979). Transcription as theory. In E. Ochs & B. Schieffelin (Eds.), *Developmental pragmatics* (pp. 41–72). New York, NY: Academic Press.

Ortmeier-Hooper, C. (2013). *The ELL writer: Moving beyond basics in the secondary classroom*. Teachers College Press.

Patton, M. Q. (2002). Qualitative research and evaluation methods. Thousand Oaks, CA*: Sage Publications*.

Pimm, D. (1987). *Speaking mathematically: Communication in mathematics classrooms*. London & New York: Routledge.

Powell, S.R., Hebert, M.A., Cohen, J.A., Casa, T.M. & Firmender, J.M. (2017). A synthesis of mathematics writing: Assessments, interventions, and surveys. *Journal of Writing Research*, 8(3), 493 – 526. Doi: 10.17239/jowr-2017.08.03.04

Prediger, S., Gravemeijer, K., & Confrey, J. (2015). Design research with a focus on learning processes: an overview on achievements and challenges. *ZDM*, 47(6), 877-891.

Pugalee, D. K. (2004). A comparison of verbal and writing descriptions of students' problem solving processes. *Educational Studies in Mathematics*, 55 (1), 27−47. doi:10.1023/b:educ.0000017666.11367.c7

Pugalee, D. K. (2005). Writing to develop mathematical understanding. Christopher-Gordon.

Radford, L. (2011). Grade 2 students' non-symbolic algebraic thinking. In J. Cai & E. Knuth's (Eds.) *Early algebraization* (pp. 303-322). Springer, Berlin, Heidelberg.

Radford, L., & Barwell, R. (2016). Language in mathematics education research. In F.K. Lester's (Ed.) *The second handbook of research on the psychology of mathematics education* (pp. 275-313). Brill Sense.

Ramirez, N.G., Celedon-Pattichis, S. (2012) Second language development and implications for the mathematics classroom. In S. Celedon-Pattichis & N.G. Ramirez's (Eds.), *Beyond good teaching: Advancing mathematics education for ELLs*. Reston, VA: National Council of Teachers of Mathematics.

Rivera, F. D., & Becker, J. R. (2007a). Abduction–induction (generalization) processes of elementary majors on figural patterns in algebra. *The Journal of Mathematical Behavior*, 26(2), 140-155.

Rivera, F., & Becker, J. R. (2007b). Abduction in pattern generalization. In J. Woo, H. Lew, K. Park, & D. Seo's (Eds.) *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education*. Seoul: PME (pp. 97-104).

Rogoff, B. (2014). Learning by observing and pitching in to family and community endeavors: An orientation. *Human development*, 57(2-3), 69-81.

Russell, S. J., Schifter, D., & Bastable, V. (2011). Developing algebraic thinking in the context of arithmetic. In J. Cai & E. Knuth's (Eds.) *Early algebraization: A global dialogue from multiple perspectives* (43-69). Springer: Berlin Heidelberg.

Schifter, D. (2009). Representation-based proof in the elementary grades. In D.A. Stylianou, M.L. Blanton, & E.J. Knuth's (Eds.) *Teaching and learning proof across the grades* (pp. 71-86). Routledge.

Schleppegrell, M. J. (2007). The linguistic challenges of mathematics teaching and learning: A research review. *Reading & Writing Quarterly*, 23(2), 139-159.

Schoenfeld, A. (2009). The soul of mathematics (Series editor's forward). In D.A. Stylianou, M.L. Blanton, & E.J. Knuth's (Eds.), *Teaching and learning proof across the grades: A K-16 perspective*. Routledge.

Sepúlveda III, E. (2011). Toward a pedagogy of acompañamiento: Mexican migrant youth writing from the underside of modernity. *Harvard Educational Review*, *81*(3), 550-573.

Sfard, A. (1995). The development of algebra: Confronting historical and psychological perspectives. *The Journal of Mathematical Behavior*, *14*(1), 15-39.

Sfard, A. (2007). When the rules of discourse change, but nobody tells you: Making sense of mathematics learning from a commognitive standpoint. *The Journal of the Learning Sciences*. 16 (4), 565-613.

Solano-Flores, G. (2010). Function and form in research on language and mathematics education. In J.N. Moschkovich's (Ed.) *Language and mathematics in education: Multiple perspectives and directions for research* (pp. 113-149). Charlotte, NC: Information Age Publishing, Inc.

Stein, M. K., Kaufman, J. H., Sherman, M., & Hillen, A. F. (2011). Algebra: A challenge at the crossroads of policy and practice. *Review of Educational Research*, 81(4), 453-492.

Stephens, A. C., Ellis, A. B., Blanton, M., & Brizuela, B. M. (2017). Algebraic thinking in the elementary and middle grades. In J. Cai's (Ed.) *Compendium for research in mathematics education*. National Council of Teachers of Mathematics: Reston.

Su, F. (2020). *Mathematics for Human Flourishing*. Yale University Press.

Swain, M. (2009). Languaging, agency and collaboration in advanced second language proficiency. In H. Byrnes' (Ed.) *Advanced language learning: The contribution of Halliday and Vygotsky* (pp.95-108). New York, NY: Continuum.

Tate, W. F., & Rousseau, C. (2002). Access and opportunity: The political and social context of mathematics education. In L.D. English & D. Kirshner's (Eds.) *Handbook of international research in mathematics education* (271-299). Mahwah, USA: Lawrence Erlbaum.

Toulmin, S. (2003) *The uses of argument*. Cambridge: Cambridge University Press.

Usiskin, Z. (1988). Conceptions of school algebra and uses of variables. *The ideas of algebra, K-12*, 8, 19.

Valdés, G. (2017). Entry visa denied: The construction of symbolic language borders in educational settings. *The Oxford handbook of language and society*, 321-348.

Valencia, R. R. (2010). *Dismantling contemporary deficit thinking: Educational thought and practice*. Routledge.

van den Akker, J. (2013). Curricular development research as a specimen of educational design research. In T. Plomp & N. Nieveen (Eds.), *Educational design research: illustrative cases* (pp. 52–71). Enschede: SLO, Netherlands Institute for Curriculum Development.

Vygotsky, L. S. (1978). *Mind in society: The development of higher mental process*. Cambridge, MA: Harvard University Press.

Vygotsky, L.S. (1986). *Thought and Language*. Cambridge, MA: MIT Press.

Walqui, A. (2019) Designing the amplified lesson. In A. Walqui & G. Bunch's (Eds.) *Amplifying the curriculum: Designing quality learning opportunities for English learners* (pp. 47-84). Teachers College Press: New York, NY.

Walqui, A., Bunch, G. (2019) *Amplifying the curriculum: Designing quality learning opportunities for English learners* (pp. 15-46). Teachers College Press: New York, NY.

Wang, S. H., Lang, N., Bunch, G. C., Basch, S., McHugh, S. R., Huitzilopochtli, S., & Callanan, M. (2021). Dismantling Persistent Deficit Narratives About the

Language and Literacy of Culturally and Linguistically Minoritized Children and Youth: Counter-Possibilities. *In Frontiers in Education (p. 260)*. Frontiers.

Watson, A., & Mason, J. (2005). *Mathematics as a constructive activity: Learners generating examples.* Mahwah, NJ: Lawrence Earlbaum Associates.

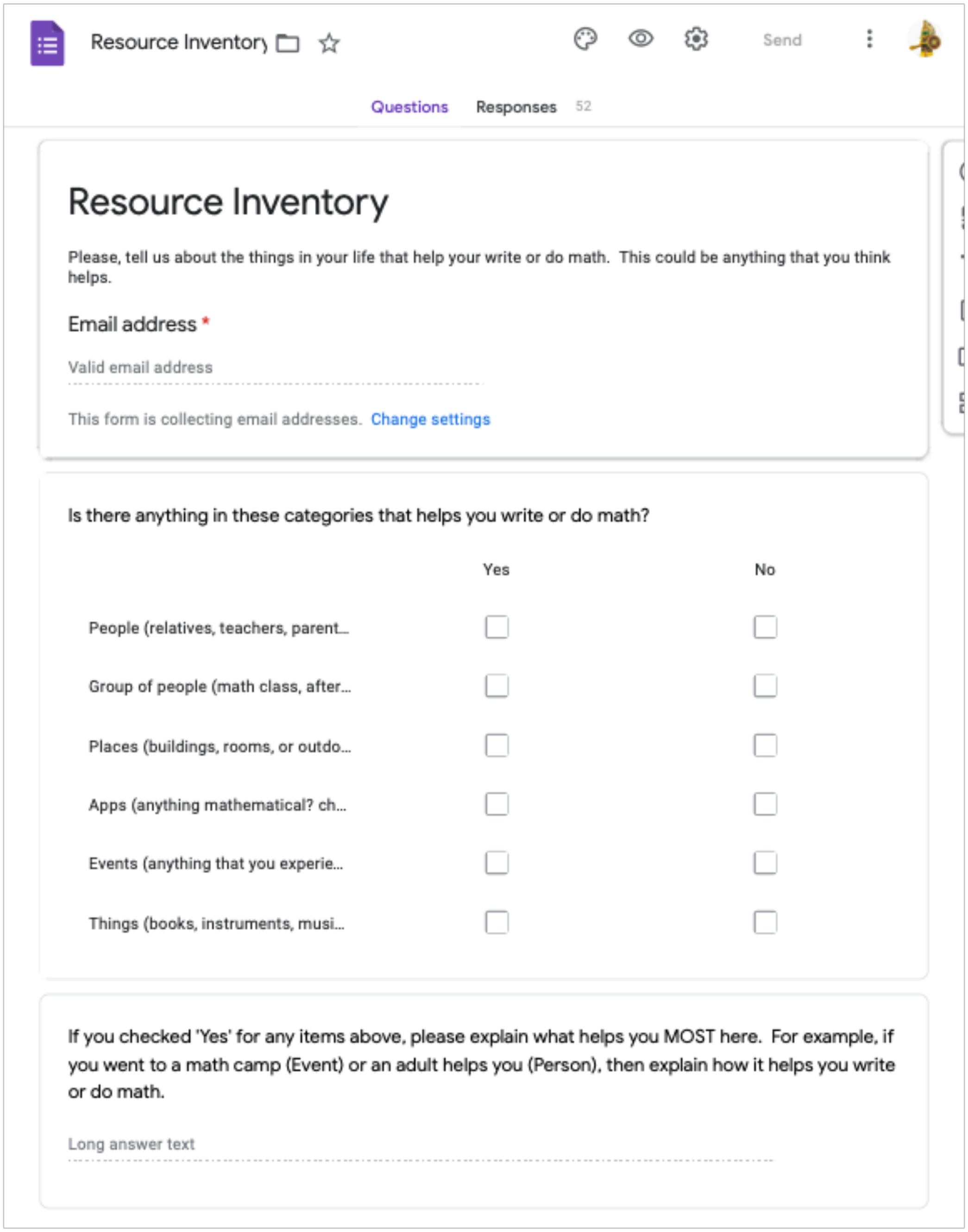
Wilkinson, L.C. (2018) Teaching the language of mathematics: What the research tells us teachers need to know and do. *Journal of mathematical behavior*, 51, 167-174.

Wood, T. (1998). Alternative patterns of communication in the mathematics classroom: Funneling or Focusing? In H. Steinbring, M. G. Bartolini-Bussi, & A. Sierpinska (Eds.), *Language and Communication in the Mathematics Classroom* (pp. 167–178). Reston, VA: National Council of the Teachers of Mathematics.

Yosso, T. J. (2014). Whose culture has capital? A critical race theory discussion of community cultural wealth. In *Critical race theory in education* (pp. 181-204). Routledge.

Appendices

Appendix A Mathematical Resources Inventory Adapted from "Mapping our literacies and resources" (Ortmeier-Hooper, 2013)



Resource Inventory

Questions Responses 52

Resource Inventory

Please, tell us about the things in your life that help your write or do math. This could be anything that you think helps.

Email address *

Valid email address

This form is collecting email addresses. Change settings

Is there anything in these categories that helps you write or do math?

	Yes	No
People (relatives, teachers, parent...	☐	☐
Group of people (math class, after...	☐	☐
Places (buildings, rooms, or outdo...	☐	☐
Apps (anything mathematical? ch...	☐	☐
Events (anything that you experie...	☐	☐
Things (books, instruments, musi...	☐	☐

If you checked 'Yes' for any items above, please explain what helps you MOST here. For example, if you went to a math camp (Event) or an adult helps you (Person), then explain how it helps you write or do math.

Long answer text

Appendix B Focus Questions for Observation.

Observation Protocol-
USING WRITING TO SUPPORT MATHEMATICAL ARGUMENTS IN EARLY ALGEBRA (Huitzilopochtli)

Site:	Date:
Period:	Phase:

Research Questions:

1. How does students' languaging about mathematical arguments (oral and written) support the development thereof?
    a. In what ways do students show improvement (e.g. in terms of process and object views of algebra, generalization, use of symbols, or others)?
    b. How does the mathematical argumentation of individuals change over the course of the study?
2. How do students use writing to improve mathematical arguments?
    a. In what ways does writing support improvement of argumentation (e.g., in terms of the purposes, writing activities, or other writing)?
    b. How do students describe the utility or helpfulness of mathematical writing?
    c. [need an additional research question about conferencing-- is it already included in RQ1? It's a form of languaging. So.. RQ1 will be addressed through writing, Conferencing, and revision]

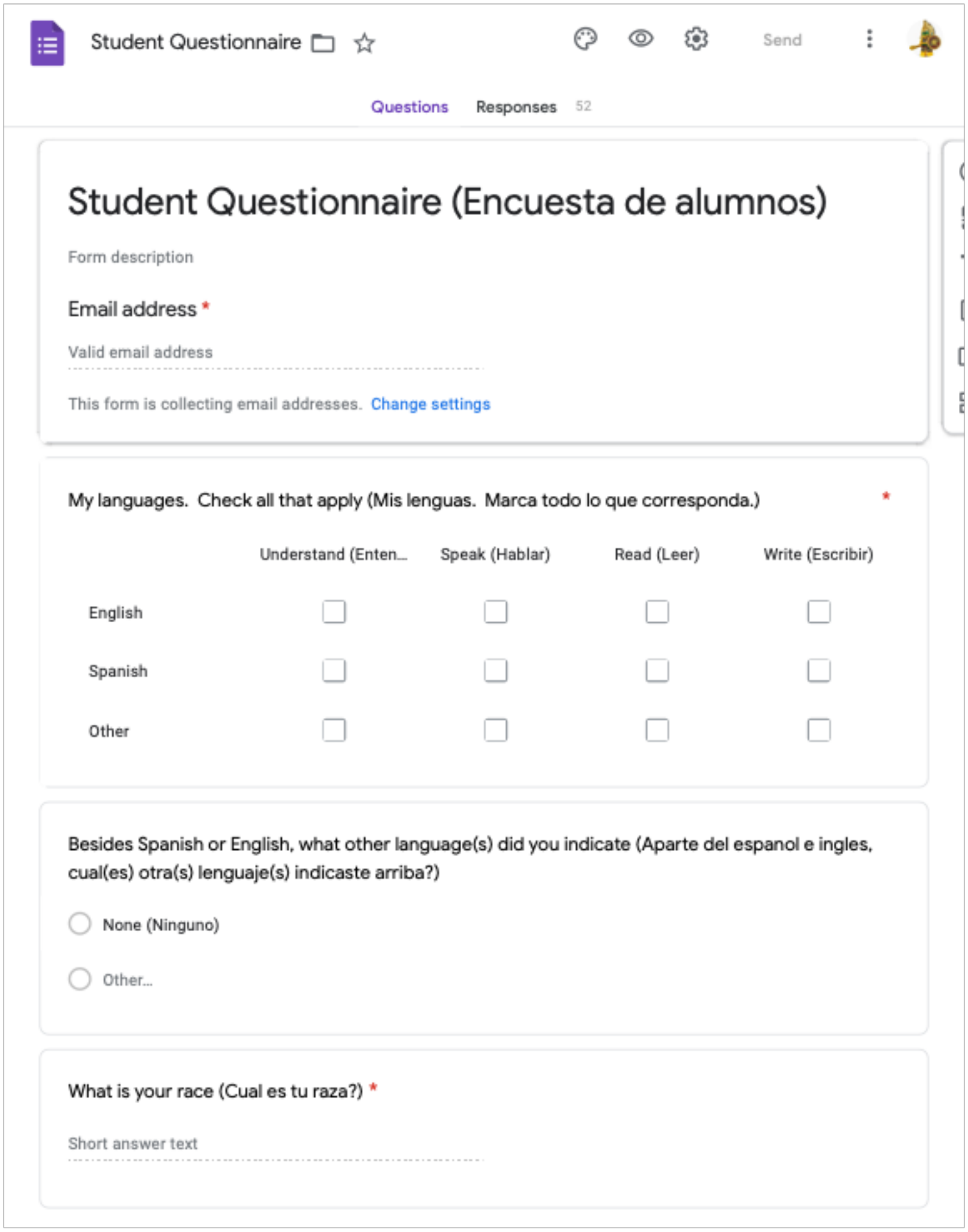
Focus Questions:

1. How does the teacher use writing during mathematics instruction?
    a. How often are students asked to write? At what point in the unit?
    b. What kinds of writing tasks are assigned (e.g., letters, journal entries, explanations, etc.)?
    c. How are the tasks supported (e.g., are rubrics provided, audience clarified, exemplars presented, etc.)?
    d. Where is the writing kept?
    e. How much time per writing session?
    f. Which purposes are used?
        i. Exploratory Writing?
        ii. Explanatory/Informational Writing?
        iii. Argumentative Writing?
        iv. Creative Writing?
    g. How does the teacher provide feedback (Feed Front, Feed Back, Feed Forward? Faltis & Valdes, 2016)?
        i. What is the focus of the feedback?
        ii. In what mode is it presented?
    h. How do the students feel about writing in mathematics class?

2. How does the teacher use discussion?
    a. Do the discussions follow the five practices (Smith & Stein, 2011)?
        i. Anticipating
        ii. Monitoring
        iii. Selecting
        iv. Sequencing
        v. Connecting
    b. Are the discussions connected to writing tasks?
    c. What kinds of participation structures are employed?

Map of Room (on reverse).

Appendix C Background Questionnaire (Mathematics Education and Linguistic)



Student Questionnaire

Send

Questions Responses 52

Student Questionnaire (Encuesta de alumnos)

Form description

Email address *

Valid email address

This form is collecting email addresses. Change settings

My languages. Check all that apply (Mis lenguas. Marca todo lo que corresponda.) *

	Understand (Enten...	Speak (Hablar)	Read (Leer)	Write (Escribir)
English	☐	☐	☐	☐
Spanish	☐	☐	☐	☐
Other	☐	☐	☐	☐

Besides Spanish or English, what other language(s) did you indicate (Aparte del espanol e ingles, cual(es) otra(s) lenguaje(s) indicaste arriba?)

- None (Ninguno)
- Other...

What is your race (Cual es tu raza?) *

Short answer text

Appendix D Early Algebraic Thinking Pre and Post Tests (Adapted from Chimoni, Pitta Patazi, & Christou, 2018)

AT Pre-Test- Fal 2020- Long Version

Algebraic Thinking (Number Property) Pre-Test-[redacted]2020. Please, use your school email to start.

The name and photo associated with your Google account will be recorded when you upload files and submit this form. Not[redacted]? Switch account

* Required

Email address *

Your email

Directions for Pre Test. Please, answer all questions on a separate piece of paper. Complete your work for each item first. When you're done, upload a photo of your written work. The pre-test is not part of your grade. You have 20 - 30 minutes.

Direcciones para el pre-examen. Favor de escribir tus respuestas en una hoja separada. Primero, acabe la respuesta. Después, sube una foto del trabajo escrito a la forma. El pre-examen no es parte de tu calificación. Tienes 20-30 minutos.

1. Do you agree with the following statement?

"The sum of two even numbers is always an odd number."

- ◯ Yes
- ◯ No

Appendix E Summary of the unit of instruction

Argument Lesson 1

Materials:

- Presentation Slides- Day 1 (URL-1)
- Discussion board- Day 1 (URL-2)
- PI-4 Resource Page- Day 2
  - Group A: (URL-3)
  - Group B: (URL-4)
- Convince Form- Day 2 (URL-5)

Session 1. Oct 20, 21

Intro/ Task 1 (10 mins)

- "Show that when you add any two even numbers, your answer is always even. Provide an explanation that would convince a classmate that the answer is *always* even"
- Show with pictures, symbols, tables, or whatever you think is convincing.
- Invite students to share screens/ ideas/ responses.

Arguments (20 mins)

- 5. Discuss *working* definitions (these are not formal definitions, but something we use to get going) for:
  - Conjecture. "A statement that you think might be true but which you are not yet absolutely sure of. It can be tested to see whether or not it is true"
  - Argument. Conjecture with a justification that includes mathematical reasons and examples as evidence.
  - Proof. "An argument showing why a statement is **certainly** true. Statements that concern infinitely many possible cases cannot be proved by listing a finite number of confirming cases"
- 10. Discuss and record justifications in the discussion board (URL-2)
  - Review discussion board layout
  - Invite students to contribute to the discussion board.

Task 2 (15 mins). 3-5 mins to work independently, 10 minutes for discussion board.

- “The sum of two consecutive numbers is always an odd number. For example, 5 + 6 = 11 and 8 + 9 = 17. Show that the sum of any two consecutive numbers is *always* an odd number.”
- Show with pictures, symbols, tables, or whatever you think is convincing.
- Discuss and record arguments
- Post in the discussion board for further comment
    - Invite students to contribute to the discussion board.
    - Discuss pros and cons of each argument. How is it convincing? How is it not? How were examples used? Did they use mathematical properties as reasons?

Homework: **Read** PI-4 (CPM p.527)-- if time permits, read problem together for understanding. No written work required yet.

Session 2. Oct. 22, 23

Intro/ PI-4 (30-40 minutes)

- Class reads task and answers clarifying questions. 5-7mins.
    - How is the pattern growing?
    - Illustrate rectangles (teacher)
        - Invite students to describe each and predict the next
    - Areas.
        - What’s area of each rectangle?
        - What’s the sum of the areas for the first two?
        - ...the first three?
    - Show the table. Teacher asks: “How do we get the next sum of areas?”
- Students work individually for 5-7 minutes on PI-4/D, E (Slide 15).. 5-7 mins.
    - Students complete **resource page**
    - Students should be prepared to share findings.
- Students return to whole-class. We use a shared resource page to fill in the table and more rectangles (teacher scribes?).
    - Share conjectures.
    - Share some arguments, examples.

Convince Form (7-10 minutes)- (URL-5)

- In notebook. Students summarize what they have found and explain it to a friend that is assumed to be their own age but not quite as good at math. Explain their conjecture, how they justified it, what the examples showed and how they used properties, definitions or other math ideas.
- In notebook. Students use pictures, symbols, tables or whatever to justify the argument to a skeptic.

- Fill/Submit online Convince Form.

Argument Lesson 2

Materials:
- Presentation Slides- Day 1 – [URL 1]
- Discussion board- Day 1
    - Group A: [URL 2]
    - Group B: [URL 3]
- PI-13 Resource Page- Day 2
    - Group A: [URL 4]
    - Group B: [URL 5]
- Convince Form- Day 2: [URL 6]
- Journal #2: [URL 7]

Session 1. Oct 27 28 (about 45 minutes)

Intro
- Closure for Fibonacci task.
-

Task 1 (10 mins)
- **"If you add any three odd numbers together, is your answer always odd?"**
- Provide an explanation that would convince your teacher that the answer is **always** odd.
- Conjectures, examples, evidence, and Comments

Discussion board (10 mins)
- 10. Discuss and record justifications in the discussion board (link)
    - Review discussion board layout
    - Invite students to contribute to the discussion board.
        - Written explanation
        - photo

Task 2 (15 mins). 3-5 mins to work independently, 10 minutes for discussion board.
- "**Zara found a cool trick. She thinks of a number between 1 and 10. She adds three, doubles the result, and then she writes the answer down. Then, she goes back to the number, doubles it, adds six, and writes the number down. Will Zara's two answers always be equal for any number between 1 and 10?**"
- Show with pictures, symbols, tables, or whatever you think is convincing.
    - More than one. Pictures? Words? symbols?

- Discuss and record arguments
- Post in the discussion board for further comment
    - Invite students to contribute to the discussion board.
    - Discuss pros and cons of each argument. How is it convincing? How is it not? How were examples used? Did they use mathematical properties as reasons?

Session 2. Oct. 22, 23 (about 45 minutes)

Intro/ **PI-13 (p.534).** Resource page:

- Class reads task and answers clarifying questions. 5-7mins.
    - Read Slide 1. Include intro up to, and including, the definition of a **happy number.**
    - Read Slide 2. Read example of "34". Illustrate why it's not a happy number.
        - Students should put each calculation in the chat.
        - Show that the sequence repeats at 89, thus, 34 is not a happy number.
- Students work individually on Question A (Slide XX).. 10 mins.
    - Students complete **resource page-**
    - Students should be prepared to share findings.
    - Students return to whole-class. We use a shared resource page to fill in the 17 2-digit happy numbers and discuss patterns.
        - Share conjectures.
        - Share some arguments, examples.
- Students return to work individually to complete the list of 17 and work on Question B. 10 mins.
    - On their own resource pages, students find 5 three-digit happy numbers.
    - Share with whole class.
- Students work on Question C and conjecture on "what other numbers must be happy numbers? How do you know? Find at least 10 more happy numbers". 10 minutes.
    - Students work on resource page
    - Students share in whole-class

Convince Form (7-10 minutes)- [URL 6]

- In notebook. Students summarize what they have found and explain it to a friend that is assumed to be their own age but not quite as good at math.

Explain their conjecture, how they justified it, what the examples showed and how they used properties, definitions or other math ideas.

- In notebook. Students use pictures, symbols, tables or whatever to justify the argument to a skeptic.
- Fill/Submit online Convince Form.

Journal #2: [URL 7]

Argument Lesson 3

Materials:

- Presentation Slides- [URL 1]
- Resource Page- Day 1. Task 1, 2 Discussions.
  - Student Form for Google Classroom: [URL 2]
  - Group Discussion:
    - Group A: [URL 3]
    - Group B: [URL 4]
- Discussion Board- Day 1. Responses to Student Samples.
  - Student Form for Google Classroom: [URL 5]
  - Group Discussion
    - Group A: [URL 6]
    - Group B: [URL 7]
- Convince ***Slides***- Day 2: (as slides!)
  - Student link for Google Classroom: [URL 8]
  - For presentation:
    - Group A: [URL 9]
    - Group B: [URL 10]

Session 1. Nov 17 18 (about 45 minutes)

Intro- START RECORDING.

- Slide 1. Hello
- Slide 2. Day 1 overview.

Task 1

- Slide 3. Task 1. Consecutive Sums. **15 minutes**. Think about the task and use the resource page to organize your thoughts and examples. Let's do some together.
  - Do 5 examples with the students. Make sure they understand and explain what "consecutive" means. The resource page should also ask the questions:
    - Experiment with some numbers
    - Try to make a conjecture about which numbers can be made.
    - Try to prove your conjecture.
    - If you have time, say and prove another conjecture about sums of consecutive whole numbers.

- **Whole Class.** In the Resource page (public), gather examples from all students, identify "common issues" [see table below] and prompt students appropriately.
- Keep track of conjectures.
- List questions for each of the conjectures on the resource page (based on "Common Issues" document).

Task 2

- Slide 4. Task 2. Consecutive Sums 2. 15 minutes. "Continue on your individual resource page and think about the conjecture. Investigate whether it's true.
    - Do a few examples with students and then allow them time to work.
    - Return to whole-class discussion. Poll students for Y/N (is the conjecture true?).
    - Ask for a few volunteers to share their work (share screens?)
- Slide 5. Questions for Sample Responses.
    - Understand what they did.
    - Q1: Explain their reasoning in your own words.
    - Q2: Why does it (not) convince you?
    - Which do you prefer? Explain your reasoning.
- Slides 6-9. Student Samples. Review 4 mins.
- Slide 10. Discussion Board.
    - What is different about the methods?
    - Which do you prefer and why?
    - What are the pluses and minuses of the different methods?
    - Which are easiest? Why?
    - Which are most convincing? Why?
    - Which are easy to use with other kinds of conjectures?

Responses to Student Samples:

- Complete Form for Q1, Q2.
    - Form should have copies of each Sample with Q1 and Q2 possible responses.

Session 2. Nov 19 20 (about 45 minutes)

Intro**.** Always, Sometimes, or Never.

- Slide 12. Review conjectures from Task 1; Day 1. 10 mins.
    - Review conjecture.
        - Review each conjecture

        - Make 2-3 examples
        - Answer questions as a group.
    - Ask questions.
- Slide 13. Review conjectures A-F and do a couple of examples (in the chat). 10-12 mins.
    - Students work on their own.
    - Use the PDF that has instructions (in Google classroom)
    - Take notes and be ready to share on public document.
- Come back as a class. Review some initial thoughts for several conjectures. 10-12 mins.
    - Describe like a card-sort.
    - Go to break-out groups and talk.
    - Make some final agreements.
    - Emphasize agreement and organized notes to submit a good photo!
- Slide 14. **11:30 am**. Describe how the Slides will work to submit work. 10 mins.
- Slide 15. Emphasize:
    - The proof is important. So, consider <u>any</u> sum.
    - Upload the photo of work.
- Slide 16. Remind students about conferences.

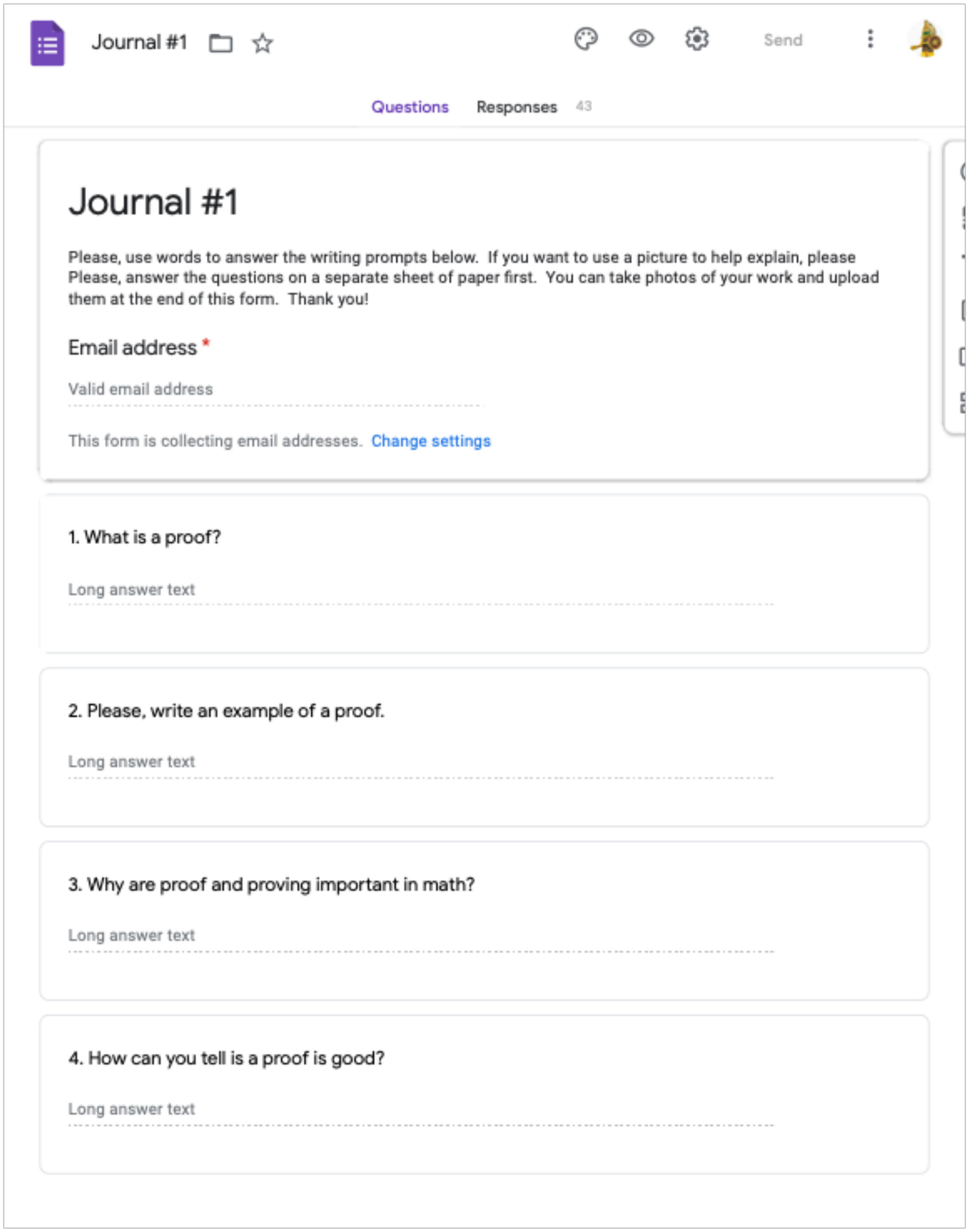
Appendix F Transcript Conventions

Transcript Conventions (adapted from Moschkovich, 2008)		
Timed pause	[3 sec]	Measured in seconds, this symbol represents an interval of silence.
Period	.	Indicates a falling pitch or intonation at the conclusion of an utterance.
Question mark	?	Indicates a rising vocal pitch or intonation at the end of an utterance.
Exclamation point	!	Indicates the end of an utterance with emphatic and animated tone.
Italics	*Italics*	Indicates emphasis on a word or phrase.
Hyphen	—	Indicates an abrupt halt between syllables or words.
Parentheses	( )	Talk for which transcriber doubt exists.
Double parentheses	(( ))	Transcript annotations

Other guidelines

- Gestural descriptions will be placed on a new line, using the different font.
- A new turn will be marked with a change of actor (evident in the numbering). So, speech and gestures will be counted in the same turn to support an integrated view of communication. If another actor interjects with a simple gesture (i.e., no speech), then it would have its own turn number as a communicative act. If it's accompanied with speech, then they would be considered part of the same communication act.
- Use phonetic spellings (e.g., 'gonna') to accurately reflect how (in) formal the speech might be.

Appendix G Math Journal Prompts



Journal #1

Questions Responses 43

Journal #1

Please, use words to answer the writing prompts below. If you want to use a picture to help explain, please Please, answer the questions on a separate sheet of paper first. You can take photos of your work and upload them at the end of this form. Thank you!

Email address *

Valid email address

This form is collecting email addresses. Change settings

1. What is a proof?

Long answer text

2. Please, write an example of a proof.

Long answer text

3. Why are proof and proving important in math?

Long answer text

4. How can you tell is a proof is good?

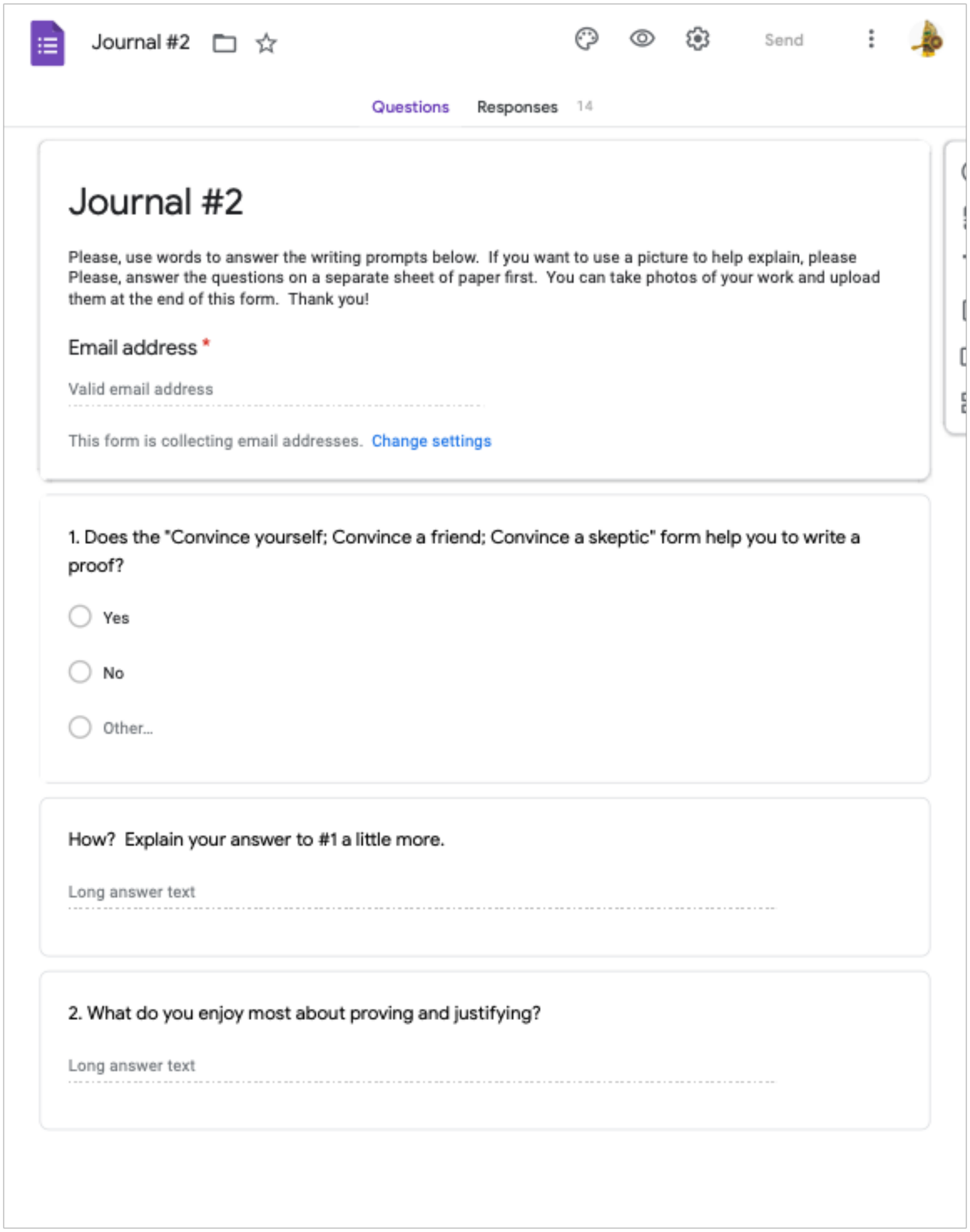
Long answer text

5. Describe your past experiences with proof and proving. This includes math classes, but also outside of math class and outside of school.

Long answer text

Upload up to five photos that help explain your answers above. Please number the drawings in the photos. In addition to photos, you can add a voice message to explain them more!

Add file



Journal #2

Questions Responses 14

Journal #2

Please, use words to answer the writing prompts below. If you want to use a picture to help explain, please Please, answer the questions on a separate sheet of paper first. You can take photos of your work and upload them at the end of this form. Thank you!

Email address *

Valid email address

This form is collecting email addresses. Change settings

1. Does the "Convince yourself; Convince a friend; Convince a skeptic" form help you to write a proof?

- Yes
- No
- Other...

How? Explain your answer to #1 a little more.

Long answer text

2. What do you enjoy most about proving and justifying?

Long answer text

3. What frustrates you the most about proof?

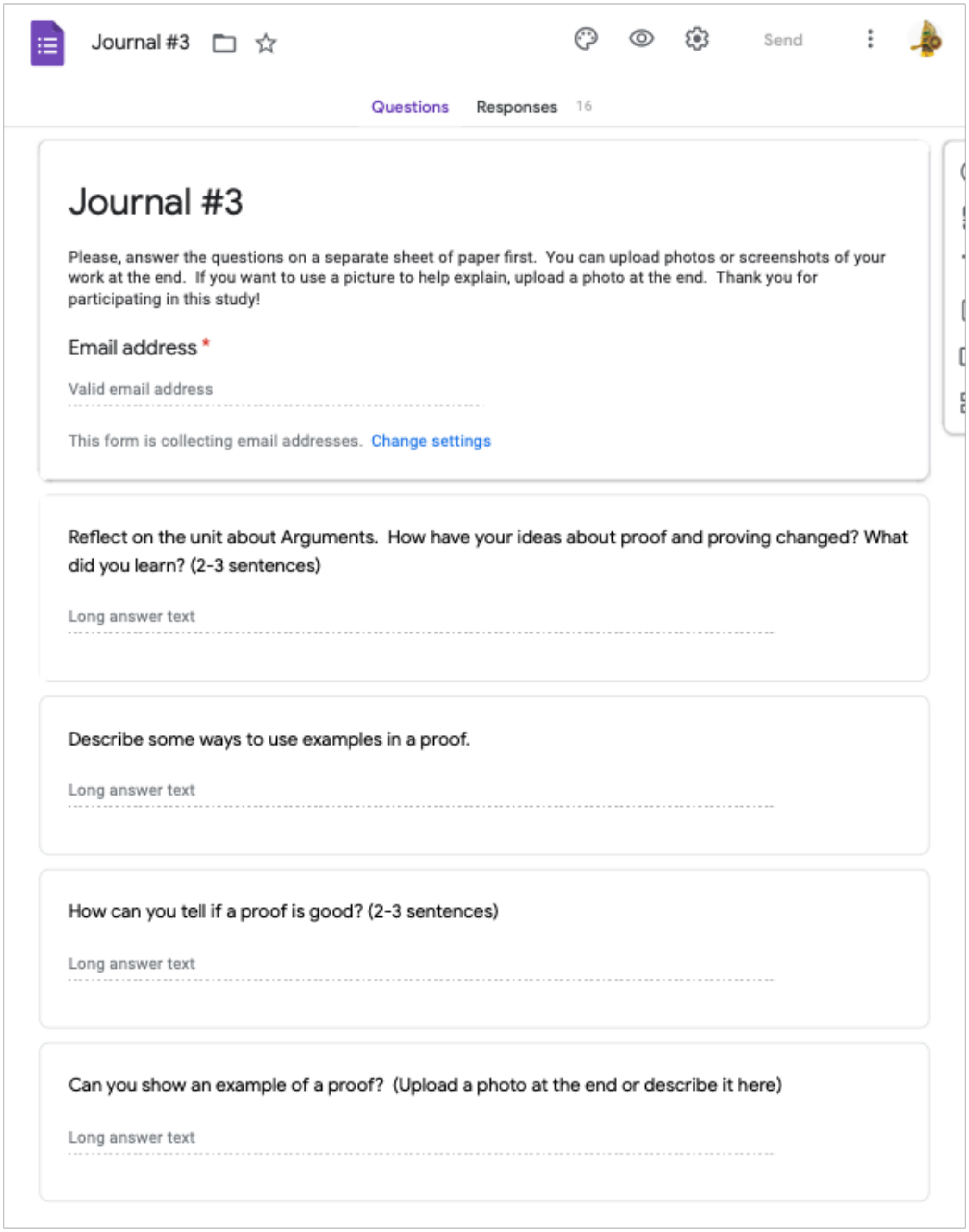
Long answer text

4. What would help you write a better conjecture or claim?

Long answer text

5. What would help you write a better proof or justification?

Long answer text



Journal #3

Questions Responses 16

Journal #3

Please, answer the questions on a separate sheet of paper first. You can upload photos or screenshots of your work at the end. If you want to use a picture to help explain, upload a photo at the end. Thank you for participating in this study!

Email address *

Valid email address

This form is collecting email addresses. Change settings

Reflect on the unit about Arguments. How have your ideas about proof and proving changed? What did you learn? (2-3 sentences)

Long answer text

Describe some ways to use examples in a proof.

Long answer text

How can you tell if a proof is good? (2-3 sentences)

Long answer text

Can you show an example of a proof? (Upload a photo at the end or describe it here)

Long answer text

How are proof and proving in a math class different from other kinds of proof (for example, in a science class, history class, or in an argument with a family member)?

Long answer text

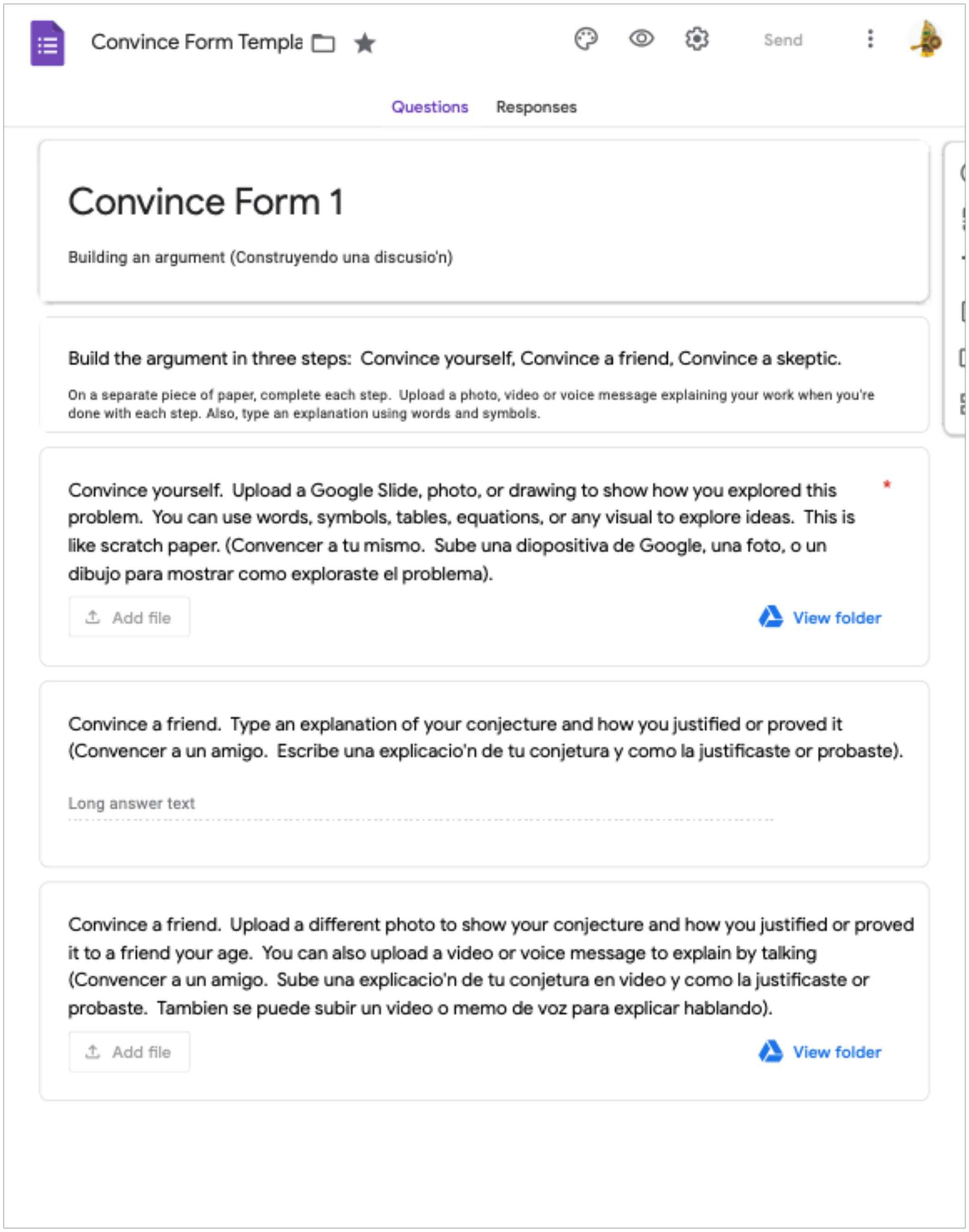
If you had a conference with Mr. Sal, describe how the math conferences were helpful or not helpful. (2 or 3 sentences)

Long answer text

Upload a photo of any writing, pictures, or examples you wrote.

Add file

Appendix H Convince Form (online)



Convince Form Templa

Send

Questions Responses

Convince Form 1

Building an argument (Construyendo una discusio'n)

Build the argument in three steps: Convince yourself, Convince a friend, Convince a skeptic.

On a separate piece of paper, complete each step. Upload a photo, video or voice message explaining your work when you're done with each step. Also, type an explanation using words and symbols.

Convince yourself. Upload a Google Slide, photo, or drawing to show how you explored this problem. You can use words, symbols, tables, equations, or any visual to explore ideas. This is like scratch paper. (Convencer a tu mismo. Sube una diopositiva de Google, una foto, o un dibujo para mostrar como exploraste el problema). *

Add file

View folder

Convince a friend. Type an explanation of your conjecture and how you justified or proved it (Convencer a un amigo. Escribe una explicacio'n de tu conjetura y como la justificaste or probaste).

Long answer text

Convince a friend. Upload a different photo to show your conjecture and how you justified or proved it to a friend your age. You can also upload a video or voice message to explain by talking (Convencer a un amigo. Sube una explicacio'n de tu conjetura en video y como la justificaste or probaste. Tambien se puede subir un video o memo de voz para explicar hablando).

Add file

View folder

Appendix I Modified Convince Form (slides)

Cards! Click and Drag these statements to make a conjecture..

A
The sum of four consecutive whole numbers is divisible by 4.

B
The sum of five consecutive whole numbers is divisible by 5.

C
To find the sum of consecutive whole numbers, find the middle number and then multiply it by how many numbers there are.

D
The sum of five consecutive whole numbers is divisible by the sum of the first and the last numbers..

E
The sum of two consecutive whole numbers is an odd number.

F
The sum of six consecutive whole numbers is divisible by the sum of the middle two numbers.

Always True	Sometimes True	Never True

1

Justify your conjecture! Convince yourself.

- Pick 2 conjectures (either A, B, C, D, E, or F).
- Upload a photo for each one. How you **convinced yourself** if it was Always, Sometimes, or Never true. Basically, how you explored the conjecture.

[Put photo 1 here using “Insert” and “Image”]

[Put photo 2 here using “Insert” and “Image”]

2

Convince a Friend. [Add text to each box]

Convince a Friend. Write an explanation of what your work shows for the first conjecture. Explain how you know if it's Always, Sometimes, or Never true.

Convince a Friend. Write an explanation of what your work shows for the second conjecture. Explain how you know if it's Always, Sometimes, or Never true.

3

Convince a Skeptic (Explain with words and/or upload a photo of your work to show symbols, visuals, examples, or other work)

A skeptic doesn't believe you. Show how your first conjecture works for **any** sum. What math ideas support your conjecture? What picture or visual? Can you demonstrate an example?

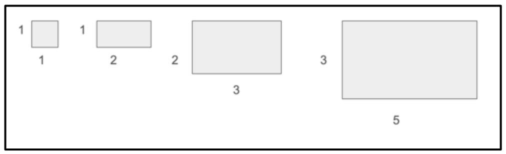
A skeptic doesn't believe you. Show how your second conjecture works for **any** sum. What math ideas support your conjecture? What picture or visual? Can you demonstrate an example?

4

Appendix J Final tasks for each lesson.

<table>
<tr>
<th>Fibonacci Rectangles (Lesson 1) (Dietiker et al., 2013 Problem PI-4, p.527)</th>
<th>Happy Numbers (Lesson 2) (Dietiker et al., 2013, Problem PI-13, p.534)</th>
<th>Consecutive Sums (Lesson 3) (MARS, 2015)</th>
</tr>
<tr>
<td>The Fibonacci Numbers are those numbers found in the following sequence:

1, 1, 2, 3, 5, 8, 13, 21, …

a. How is the sequence growing?
b. If the measures of the sides of a rectangle are consecutive Fibonacci numbers, it is called a “Fibonacci rectangle.” Here are the first four:
c.


What’s the area of each rectangle? What’s the sum of the first two? The first three?
d. Copy and complete the table (up to 8):
<table><tr><td>Number of Rectangles</td><td>Sum of the Areas</td></tr></table></td>
<td>What is a happy number?
Square each of the digits of a number and add the results.
Repeat the process until you get either:
• “1”- and the number is happy!
• Or it repeats forever. Not happy.

[Example of each shown]

a. There are 17 Happy numbers (1-99). Find as many as you can!
b. Find 5 3-digit happy numbers.
c. Eva found out that 478 is a happy number.
What other numbers must be happy numbers?
How do you know?
Find at least 10 new happy numbers.

[Added note:]</td>
<td>[In your groups] Decide whether each conjecture is Always, Sometimes or Never True. If the conjecture is Always or Never true, <u>write down how you can be sure</u>.

If the conjecture is Sometimes true, write down when it is true and when it is not.

Use ideas from the sample work you have looked at and make notes on your Math Notebooks– We will submit photos.

Agree about each conjecture before moving on. Explain your thinking on your slides (and upload the photos [t]here).

After placing the cards, pick two conjectures to prove!

For each one, use symbols, words, visuals, examples, and other ideas to prove your conjecture is Always, Sometimes, or Never true.</td>
</tr>
</table>

2	3
3	9
4	
5	
6	
7	
8	

e. Use your table. Look for any special numbers. Can you find any patterns? For example, what seems special about the sum of the areas for an odd number of rectangles?
   How are these numbers connected to the Fibonacci numbers? Extend the table if you need or want more data!

f. How can you **predict the sum** of the areas of Fibonacci rectangles?
   Use your patterns to predict the sum of the areas of the **first 15 Fibonacci rectangles**.
   Find out if your pattern worked.

Make an argument about…
...which 2-digit numbers are happy numbers and HOW you know
--or--
...how you know what 3-digit numbers are happy
--or--
...something else?

Sample Conjectures to Discuss

A The sum of four consecutive whole numbers is divisible by 4.	**B** The sum of five consecutive whole numbers is divisible by 5.
C To find the sum of consecutive whole numbers, find the middle number and then multiply it by how many numbers there are.	**D** The sum of five consecutive whole numbers is divisible by the sum of the first and last numbers.
E The sum of two consecutive whole numbers is an odd number.	**F** The sum of six consecutive whole numbers is divisible by the sum of the middle two numbers.