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### Publication Date

1986-05-01



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Presented at the 27th U.S. Symposium on Rock Mechanics,  
Tuscaloosa, AL, June 23-25, 1986, and published  
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May 1986

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## **Frictional Stability of Heterogeneous Surfaces in Contact**

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May 1986

## Frictional Stability of Heterogeneous Surfaces in Contact

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### Abstract

Utilizing the principles of elastic-brittle fracture mechanics, stress-displacement relationships are derived for an elastic solid containing a slip plane consisting of various configurations of collinear arrays of cracks. These models take into account the heterogeneous nature of slip planes as observed in the lab and in the field, and are relevant to the behavior of rock containing faults, joints, bedding planes, and fractures. All of our models exhibit slip weakening behavior, and show regimes of both stable (aseismic) and unstable (seismic) deformation. For given configurations of cracks and applied boundary conditions, our results show explicitly the energy released through unstable crack growth. For instance, it is found that two equal length cracks growing together release more seismic energy than, say, a large crack growing into a small crack. Also, by allowing cracks to coalesce, our models show multiple stick-slip events without crack healing or other geochemical means.

### Introduction

In this paper, constitutive relationships are developed for the shear response of an elastic solid containing a slip plane comprised of displacement discontinuities. This is relevant to the behavior of rock containing faults, joints, bedding planes, and fractures. Analytical models for friction between the surfaces of rock have developed along several different lines. Many of the models utilize Hertzian contact theory, see for example Archard(1957), Greenwood and Williamson (1966), and Swan(1983). Other theories involve the brittle failure of wedged shaped asperities as given by Byerlee(1967), or the plastic yielding of asperities as analysed by Bowden and Tabor(1954). Patton(1966) and Ladanyi and Archambault(1970) consider the movement along, as well as the brittle failure of, wedged shaped asperities, in order to account for the normal dilatation associated with shearing in rock. Empirical models for friction have also been successful, see Barton and Choubey(1977).

In all of the above models for friction, slip is assumed to occur simultaneously over the whole slip plane. Recent experimental work (e.g., Okubo and Dieterich, 1984, and Mogi, 1985) indicates that slip does not usually occur in this fashion; instead, the slip plane at a given instant of time under an applied shear load can consist of patches of slipped material surrounded by locked material (barrier model, see Aki, 1984), or patches of locked material surrounded by slipped material (asperity model, see Aki, 1984). A slip event consists of the propagation of the slipped zones into the locked zones, and can be stable or unstable depending on the imposed boundary conditions and the stress-displacement relationship for the given geometry of slipped and locked regions. Observations of friction in stiff testing machines show that often the stress necessary to initiate slip degrades with ongoing slip, termed slip weakening (Okubo and Dieterich, 1984), and also regular periodic oscillations between stable and unstable behavior occur, termed stick-slip (Cook, 1981). Neither of these observations can be modelled with the homogeneous friction models described above. The above mentioned phenomena are very relevant to the mechanics of earthquake rupture and global plate motions, as reviewed in the papers by Rudnicki (1981), Rice(1983), and Kanamori(1986). The occurrences of foreshocks and aftershocks, and the use of gap theory to predict large earthquakes, all support the view that the surfaces of faults are very heterogeneous, as discussed by Mogi (1985). Also frictional stability is important in slip motion along joints or faults caused by changes in stress produced by underground excavations and thermal changes, such as in nuclear waste repositories or mining.

In this paper, frictional slip surfaces are assumed to be comprised of heterogeneous distributions of displacement discontinuities, so that under a given applied shear stress, some portions of the surface will have slipped while other portions remain locked. Along the slipped portions we assume Amonton's law,

i.e.,  $\tau^f = \mu\sigma$ , where  $\tau^f$  is the frictional stress,  $\sigma$  the applied normal stress, and  $\mu$  is the coefficient of friction. Stress concentrations will form at the boundaries of the slipped portions and, as the applied stress is increased, the slipped portions will propagate in a cracklike manner at an appropriate value of the critical energy release rate, extending the slipped zones into the locked zones. The basic assumptions for the analysis in this paper are those of elastic-brittle fracture mechanics (e.g., Rice, 1980) and, utilizing these assumptions, stress-displacement constitutive relationships for various geometries of slipped and locked regions are derived. The derivations in this paper follow the analyses of Kemeny and Cook (1986a) for strain softening due to random distributions of cracks, and extend the analyses of Kemeny and Cook (1986b) for slip weakening due to faults consisting of equidistant collinear and coplanar arrays of cracks. The fault models of Kemeny and Cook (1986b) investigate several important phenomena, such as the effects of crack geometry and crack interaction on the shape of the stress displacement curves, and the criteria for the instability of a slip weakening material. Crack interaction is found to be responsible for the stress strain curves looping back towards the origin, resulting in instability even for infinitely stiff boundary conditions. Also, in three dimensions, coplanar noninteracting cracks and asperities are found to propagate in the shape of ellipses, where the ratio of the major to minor axes of the crack or ellipse is  $1/(1-\nu)$ , where  $\nu$  is Poisson's ratio. Both of the above findings have important implications in predicting the occurrence and magnitude of major earthquakes (Kanamori, 1986, and Mogi, 1985), as well as the fundamentals of rock friction.

Here, we consider slip or fault planes consisting of collinear, but non-equidistant, arrays of cracks. These include collinear arrays of cracks with unequal lengths and unequal spacings between cracks. Owing to the added complexity in the non-equidistant case, only two dimensional arrays of cracks are considered in this analysis. Calculated stress strain curves for the non-equidistant models reveal several important results. First of all, the non-equidistant models exhibit multiple sequences of stable to unstable movement (stick-slip), whereas the previous equidistant models exhibited only one sequence of stick-slip. This has important implications for the occurrence and magnitude of foreshocks and aftershocks. Also, it is found that the largest amount of seismic energy is released when nearly equal length cracks grow together, rather than, say, a large crack growing into a smaller one.

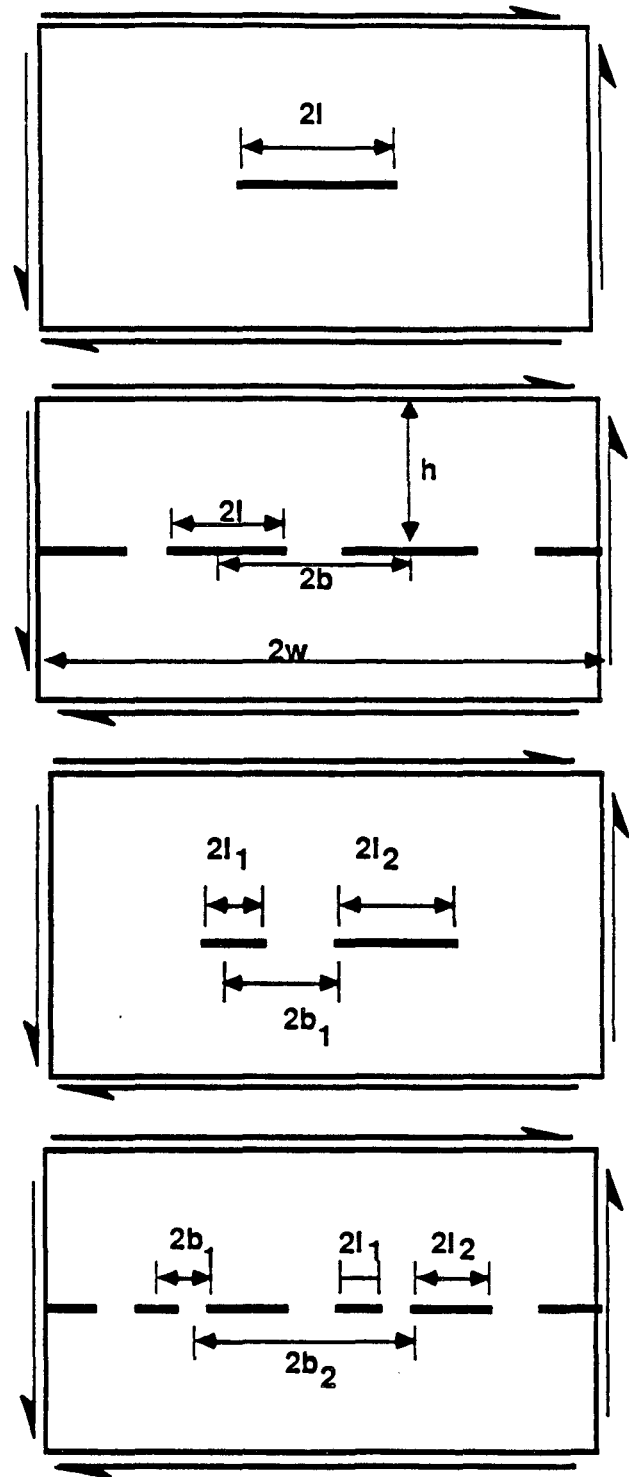


Figure 1. Two dimensional elastic solid containing a slip plane consisting of various configurations of collinear cracks: a) single crack, b) row of equal length cracks, c) two unequal length cracks, d) repetition of two unequal length cracks.

### Crack Models

In this section, stress displacement relationships are calculated for a two dimensional (plane strain) 'shear box' type of configuration, where, along the slip plane, various configurations of cracks are placed. Figure 1 shows the boundary conditions, and the specific geometries for slip planes considered in this paper. A fixed shear displacement is applied at a distance  $h$  from the discontinuity, which would, in the absence of the discontinuity, result in a uniform shear stress,  $\tau^\infty$ , throughout the sample. Also, by Amonton's law, the crack faces subjected to a normal stress  $\sigma$  can sustain

a frictional stress,  $\tau^f = \mu\sigma$ . Under these conditions, the crack faces are subjected to only mode II loading, and the crack tip stress intensity factors are proportional to  $\tau$ , where  $\tau = \tau^\infty - \tau^f$ . Four different crack configurations for slip planes are considered, as shown in Figure 1. The first two, a single crack of length  $2l$ , and a row of cracks each of length  $2l$  and spacing  $2b$ , are taken from Kemeny and Cook (1986b). Newly considered geometries are two unequal cracks of lengths  $2l_1$  and  $2l_2$  and spacing  $2b_1$ , and a periodic repetition of the two unequal length cracks with period  $2b_2$ . As material containing one of these cracks configurations is sheared, at some point the cracks will begin to propagate, producing non-linear stress displacement curves. It is assumed that the cracks will propagate, in the plane of the existing cracks, when the energy release rate,  $G$ , given by:

$$G = \frac{(1 - \nu^2)}{E} K_{II}^2$$

attains a critical value,  $G_c$ , where  $E$  is Young's modulus and  $K_{II}$  is the mode II stress intensity factor. The assumption of straight crack growth seems reasonable for shear along pre-existing planes, and results in crack coalescence as the cracks grow together. In this analysis, only the displacements due to the cracks are considered, i.e., the elastic displacements due to the surrounding material are removed. Stress-displacement relationships for the configurations in Figure 1 are derived by utilizing a relationship between the average shear displacement of the body due to the discontinuities,  $\bar{\delta}$ , and the crack tip stress intensity factors for the configurations, as derived in Kemeny and Cook(1986b) using Betti's reciprocal theorem. For all the configurations, analytical solutions for the stress intensity factors are used (e.g., Rooke and Cartwright, 1976); however, for the last two configurations, numerical integration is required to calculate the average crack displacements.

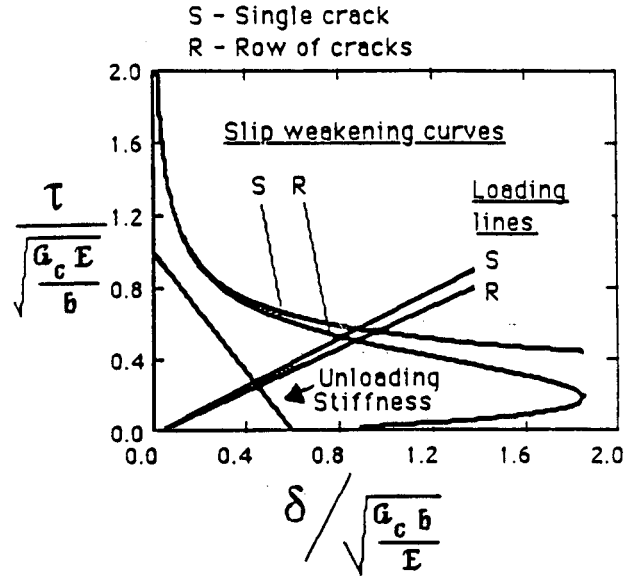


Figure 2. Linear loading lines and slip weakening curves for slip planes consisting of a single crack and a row of equal length cracks. (assumed  $\nu = 0.2$ )

### Previous Results

For slip planes consisting of 1) a single crack and 2) a row of equal length cracks, the following closed form stress-displacement relationships have been derived by Kemeny and Cook(1986b), respectively:

$$\bar{\delta} = \frac{G_c^2 E}{w \tau^3 \pi (1 - \nu^2)} \quad (1)$$

$$\bar{\delta} = \frac{8 \tau b (1 - \nu^2)}{\pi E} \ln \cos \left[ \tan^{-1} \frac{G_c E}{\tau^2 2b (1 - \nu^2)} \right] \quad (2)$$

The dimensionless form of equations (1) and (2) is plotted in figure 2. The single crack and row of crack results are plotted together by letting  $w=b$  in the single crack equation. Thus the single crack results are actually for a row of cracks with no interaction effects. The curves in Figure 2 represent the loci of points in stress displacement space where crack extension will start to occur, similar to a yield surface in plasticity theory. The curves are referred to as slip weakening curves, since they show strength degradation with increasing slip. Assuming some initial crack density, the theory predicts that linear behavior will occur up to the point where the initial cracks start to propagate, and as the cracks propagate, the deformation follows the slip weakening curve. The linear loading lines are also drawn in Figure 2, and are given by the following

formula, taken from Kemeny and Cook (1986b):

$$\bar{\delta} = \frac{l^2 \tau \pi (1 - \nu^2)}{w E} \quad (3)$$

$$\bar{\delta} = \frac{8 \tau b (1 - \nu^2)}{\pi E} \ln \cos \left( \frac{\pi l}{2b} \right) \quad (4)$$

for the single crack and the row of cracks models, respectively.

Finally, the stability of a slip weakening material is governed by the unloading stiffness of the material surrounding the fault plane. The unloading stiffness for the boundary conditions assumed in this analysis, i.e., a fixed shear displacement at a distance  $h$  from the fault plane, is plotted in Figure 2 and is given by the following formula (Stewart, 1981):

$$\bar{\delta} = \frac{-h(1 + \nu) \tau}{E} \quad (5)$$

The unloading stiffness is independent of the details of the system producing motion across the plane, and instability in this system occurs when the slope of the slip weakening curve for the fault exceeds the unloading stiffness. In general, then, the deformation sequence for these models consists of an initial linear loading up to the slip weakening curve, deformation along the slip weakening curve as the cracks propagate, and instability when the slope of the slip weakening curve exceeds the slope of the unloading stiffness of the surrounding material. The normalized results in figure 2 show that the linear loading lines are proportional to the initial crack density,  $l_0/b$ , where  $l_0$  is the initial crack length, and the unloading stiffness is proportional to  $h/b$ . The main points in figure 2 are that 1) slip weakening behavior is exhibited by both the single crack and the row of cracks models, 2) the row of cracks model possesses a vertical tangent, which defines an instability point even for a very stiff (nearly vertical) unloading stiffness, and, 3) the slope of the slip weakening curve goes to zero past the vertical tangent, as the stiffness of the joint is reduced to zero. Further results are given in Kemeny and Cook (1986b).

### New Results

The above behavior is now contrasted with the behavior of a fault plane consisting of non-equidistant arrays of cracks. Unlike the results shown in equations (1) and (2), closed form solutions for the slip weakening curves cannot be derived easily for the case of non-equidistant cracks. Rather, solutions are in the form of two simultaneous equations, as given below for the case of a fault plane consisting of two unequal length cracks:

$$\tau = \sqrt{\frac{G_c E}{(1 - \nu^2)}} F_1(l_1, l_2, b) \quad (6)$$

$$\bar{\delta} = \frac{2 \tau (1 - \nu^2)}{E b} F_2(l_1, l_2, b) \quad (7)$$

The functions  $F_1$  and  $F_2$  are given in the appendix and are easily derived from the crack tip stress intensity factors for this configuration (Rooke and Cartwright, 1976) and formulas in Kemeny and Cook (1986b). Equation (6) represents, for given values of  $l_1$ ,  $l_2$ , and  $b$ , the critical stress at which point one of the crack tips will begin to propagate (the inner tip of the largest crack in this case). Equation (7) gives the average crack displacements for this value of critical stress and crack geometry. Together they give the stress displacement behavior for the fault plane consisting of two unequal length cracks.

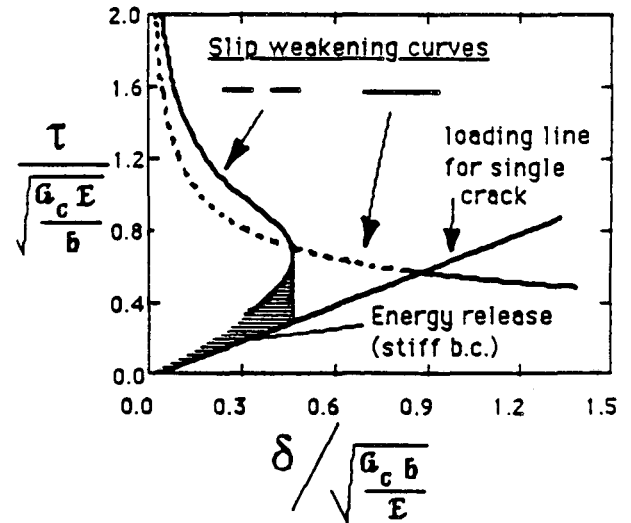


Figure 3. Normalized slip weakening curves for slip planes consisting of two equal length cracks and for a single larger crack derived from the two cracks. (assumed  $\nu = 0.2$ )



### Case of Two Equal Length Cracks

We first consider the case of a fault plane consisting of two equal length cracks ( $l_1=l_2$  in equations 6 and 7), which represents a case where the crack spacing is non-equidistant. It is found that two equal length cracks will grow from the inside, rather than from the outside or from both sides, due to the higher stress intensity factor on the inside cracks, and also because the strength of the system degrades with increasing slip, so that the strength is continually exceeded first at the inner tips. Normalized results for the case of two equal length cracks from equations (6) and (7) are plotted in Figure 3. Eventually the two cracks will grow together to form a single crack, and so also plotted in Figure 3 are the loading line and the slip weakening curve for the single crack that results from this coalescence (from equations 1 and 3). Several important features can be noted in Figure 3. The stress displacement curve for the two growing cracks exhibits slip weakening behavior, and possesses a vertical tangent, as in the model with a row of equal length cracks. However, now the slope of the curve past the vertical tangent approaches the stiffness of the single large crack made from the two cracks, rather than zero as in the row of cracks model. If we assume a stiff system where the unloading stiffness is nearly vertical, then instability will occur for the two crack system near the point of vertical tangency. During instability, the stress will drop, with little increase in displacement, until it intersects the linear loading line for the single larger crack system. The energy released by the instability is shown by the shaded region in Figure 3. The stress on the resulting single crack may be well below that necessary to cause this crack to propagate. Upon further loading, the system will follow the linear loading for the single large crack until it intersects the slip weakening curve for the single crack, at which point the outer tips will now start to propagate.

The above solution is now extended to a fault plane consisting of a periodic repetition of two equal length cracks with period  $b_2$ , as in the fourth configuration in figure 1 (with  $l_1=l_2$ ). An exact solution for this configuration is not available, however, an approximate solution can be constructed if the pairs of cracks are initially well separated ( $b_1 \ll b_2$ , see figure 1). For this assumption, use is made initially of equations (6) and (7) for a single pair of cracks, and, when each pair of cracks have grown into single cracks, the solution for a row of equal length cracks is used (equation 2), rather than the solution for a single crack (equation 1).

Results for the above mentioned case are presented in figure 4, and show several interesting features. The curves in Figure 4 are similar to those in Figure 3, except now two instabilities occur, the first instability

due to each of the pairs of cracks joining to form a row of equal length cracks, and the second instability as the row of equal length cracks interact with each other, as in Figure 2. Thus, multiple stick-slip events are produced, not by crack healing, stress corrosion cracking, or other geochemical methods, but by the heterogeneity of the fault surface alone. Also important to note are the relative magnitudes of the energy releases between the first and second instabilities. If the first instability releases more energy, then it can be considered a main event, with the second instability an aftershock. If the second instability produces more energy, then the first instability is a foreshock, followed by the main event. For the curves shown in Figure 4, the two instabilities have energy releases of roughly equal orders of magnitude. However, as will be shown in the next section, for two unequal length cracks, rather than pairs of equal length cracks, far less energy may be released. Also, it was assumed in the shaded areas of Figure 4 that the surrounding ground is very stiff, so that the unloading stiffness is nearly vertical, which may not be the case.

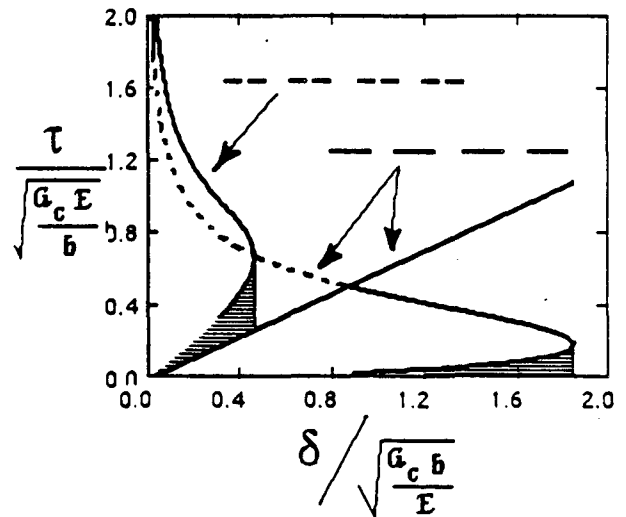


Figure 4. Normalized slip weakening curves for slip planes consisting of a periodic repetition of two equal length cracks, and a row of single cracks derived from the first configuration. (assumed  $\nu = 0.2$ )

### Case of Two Unequal Length Cracks

We now consider the case of a slip plane consisting of two unequal length cracks. It is found that for two cracks of unequal length, the stress intensity factor will always be largest on the inner tip of the largest crack, and therefore this tip will start to propagate first, and will continue to propagate until it engulfs the smaller crack. Since slipped zones on fault surfaces will never be exactly the same size, it is important to investigate the consequences, in terms of the amount

of energy released, of the above scenario. We start with two almost equal length cracks and calculate the slip weakening curve as the larger crack extends towards and coalesces with the smaller crack. The important parameter in this analysis is the ratio,  $\chi$ , of the length of the two almost equal length cracks, with the length of the final single crack after coalescence. We plot slip weakening curves for  $\chi=0.1, 0.3$ , and  $0.5$  in figure 5.  $\chi=0.5$  represents the slip weakening curve for two equal length cracks as in figure 3. The energy released due to unstable crack growth (assuming stiff boundary conditions) for each case are shown by shaded regions in Figure 5. Figure 5 shows that the highest energy is released when two equal length cracks grow together ( $\chi=0.5$ ), and very small amounts of energy are released when a large crack grows into a small crack (small  $\chi$ ).

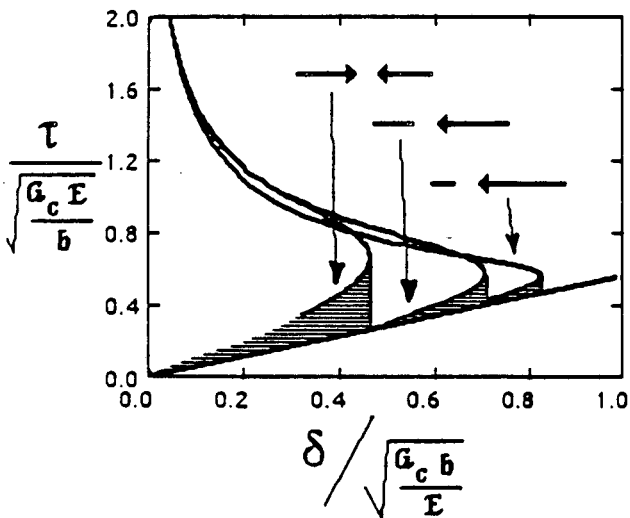


Figure 5. Normalized slip weakening curves for fault planes containing two unequal length cracks, with different ratios of  $\chi$  (ratio of length of non-propagating segment to length after coalescence). (assumed  $\nu = 0.2$ )

#### Discussion

In this paper, we try to quantify, using analytical models, some of the effects of heterogeneity in frictional response that we know exist from lab experiments and seismological data. The fact that fault heterogeneity is important is now well accepted. Its effects are often understated, however. Non-uniform response due to heterogeneity is found even in carefully ground and polished samples under shear (Mogi, 1985). Also, Kanamori (1986) shows a relationship between contact area and velocity for subducting lithospheric plates, which indicates that the velocity dependence on stress drop during stick-slip, that is observed in laboratory tests, may also be due

to changes in the geometry of surface contact.

All of our models exhibit slip weakening behavior, and show regimes of both stable (aseismic) and unstable (seismic) deformation. Thus all of the strain energy expended in crack growth does not necessarily contribute to seismic energy, and methods to promote stable crack propagation could aid in reducing the severity of earthquakes. For given configurations of cracks and applied boundary conditions, our results show explicitly the energy released through unstable crack growth. For instance, it is found that two equal length cracks growing together release more seismic energy than, say, a large crack growing into a small crack. Also, by allowing cracks to coalesce, our models show multiple stick-slip events without crack healing or other geochemical means.

Our results seem to shed some new light on the correlation between the energy released in the unstable propagation of slipped zones, and the seismic moment, as calculated from far field relative displacements. In particular, for stress controlled boundary conditions (horizontal unloading stiffness), our results correlate with those of Rudnicki et al. (1984) for the seismic moment due to the propagation of slipped zones that interact with each other. At the other extreme, however, for very stiff boundary conditions (vertical unloading stiffness), our results indicate that seismic energy can be released due to the unstable propagation of slipped zones with no seismic moment, and for boundary conditions in between, the magnitude of the seismic moment does not necessarily correlate with the total energy released.

In this analysis, it was assumed that the fracture strength,  $G_c$ , did not vary with position. A crack length dependent  $G_c$  was used in some of the models in Kemeny and Cook (1986b), and it was found that for strength variations less than a factor of three, the effects of a variable  $G_c$  are not significant. Other phenomena that could be incorporated in future models include time dependent behavior, and the effects of non collinear arrays of cracks. The increased understanding of earthquake events from theoretical models could improve earthquake prediction capabilities, and improve methods to reduce the severity of earthquake events.

#### Acknowledgements

Supported in part by the Office of Geologic Repositories of the Office of Civilian Waste Management of the US DOE under contract DE-AC03-76SS00098. Also, one of the authors (J.K.) wishes to acknowledge support from the Jane Lewis graduate fellowship.

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## Appendix

The functions  $F_1$  and  $F_2$  in equations (6) and (7) are given by the following formulas:

$$F_1 = \frac{-1}{2\pi b_1} \left[ \frac{x_C^2 + C_1 x_C + C_2}{\sqrt{x_C(x_A + x_C)(x_D - x_C)}} \right]^{-1}$$

$$\frac{F_2}{\sqrt{2\pi b_1}} = \left[ \frac{C_2}{\sqrt{x_D x_A x_C}} \right]^2 + \left[ \frac{x_D^2 + C_1 x_D + C_2}{\sqrt{x_D(x_A + x_D)(x_D - x_C)}} \right]^2$$

$$\left[ \frac{x_A^2 - C_1 x_A + C_2}{\sqrt{x_A(x_A + x_C)(x_A + x_D)}} \right]^2 + \left[ \frac{x_C^2 + C_1 x_C + C_2}{\sqrt{x_C(x_A + x_C)(x_D - x_C)}} \right]^2$$

where

$$x_A = \frac{2l_1}{b_1} \quad x_C = 1 - \frac{l_1}{b_1} \quad x_D = 1 - \frac{l_1}{b_1} + \frac{2l_2}{b_1}$$

$C_1 =$

$$\frac{(x_A - x_D)K(k) - 2x_A \Pi(n,k) + 2x_D \Pi(m,k) + (x_A + x_D)\Delta J}{K(k) - \Pi(n,k)}$$

$$C_2 = \frac{C_1}{K(k)} \left[ x_A K(k) - (x_A + x_D)\Pi(n,k) \right]$$

$$- \frac{1}{K(k)} \left[ x_A^2 K(k) - 2x_A(x_A + x_D)\Pi(n,k) + (x_A + x_D)^2 J(n,k) \right]$$

$$\Delta J = J(n,k) - J(m,k)$$

where  $K(k)$  is the complete elliptic integral of the 1st kind  $\Pi(m,k)$  and  $\Pi(n,k)$  are complete elliptic integrals of the third kind, and the function  $J$  is defined, for  $t = n$  or  $m$  by

$$J(t,k) = \int_0^{\pi/2} \frac{d\phi}{(1+t \sin^2 \phi)^2 \sqrt{1-k^2 \sin^2 \phi}}$$

$$\text{also, } n = \frac{(x_D - x_C)}{(x_A + x_C)} \quad m = \frac{x_A}{x_D} \quad k^2 = mn$$

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