UNIVERSITY OF CALIFORNIA, SAN DIEGO

The Combinatorics of nabla p_n and connections to the Rational Shuffle Conjecture

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by

Emily Sergel

Committee in charge:

Professor Adriano Garsia, Chair
Professor Ronald Graham
Professor Russell Impagliazzo
Professor Jeffrey Remmel
Professor Nolan Wallach

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The dissertation of Emily Sergel is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

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VITA

2011	B. S. in Mathematics and Computer Science *summa cum laude*, Rutgers University, New Brunswick
2011-2016	Graduate Teaching Assistant, Research Assistant, and Associate Instructor, University of California, San Diego
2016	Ph. D. in Mathematics, University of California, San Diego

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E. Sergel Leven, "A proof of the Square Paths Conjecture", *arXiv*:1601.06249.

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ABSTRACT OF THE DISSERTATION

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Emily Sergel

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Professor Adriano Garsia, Chair

The symmetric function operator, ∇, introduced by Bergeron and Garsia (1999), has many astounding combinatorial properties. The (recently proven) Shuffle Conjecture of Haglund, Haiman, Loehr, Remmel, and Ulyanov (2005) relates ∇e_n to parking functions. The rational Compositional Shuffle Conjecture of the author, Bergeron, Garsia, and Xin (2015) relates a whole family of operators (closely linked to ∇) to rational parking functions. Loehr and Warrington (2007) conjectured a relationship between ∇p_n and preference functions. We prove this conjecture and provide another combinatorial interpretation in terms of parking functions. This new formula reveals a connection between ∇p_n and an operator appearing in the rational Compositional Shuffle Conjecture at $t = 1/q$.

Chapter 1

Background and History

This work explores the relationship between combinatorics and symmetric function theory. A function of n variables $f(x_1, x_2, \ldots, x_n)$ is symmetric if $f(x_1, x_2, \ldots, x_n) = f(x_{\sigma_1}, x_{\sigma_2}, \ldots, x_{\sigma_n})$ for every permutation $\sigma \in S_n$. Extending this to infinitely many variables gives the ring Λ of symmetric functions. For an introduction to symmetric function theory, see Macdonald [Mac95]. Many properties of symmetric functions can be beautifully expressed using combinatorics. For example, the Pieri rule gives a way to multiply Schur functions and complete symmetric functions by counting tableaux.

These two subjects - combinatorics and symmetric function theory - are also closely related to representation theory. The Frobenius map gives a correspondence between Schur functions and irreducible representations of S_n. So for any S_n-module (or bi-module) one can define the Frobenius characteristic - a weighted sum of Schur functions which gives the multiplicity of each irreducible representation according to degree (or bi-degree). Any Schur-positive symmetric function is the Frobenius characteristic of some module. Hence finding combinatorial formulas for Schur-positive symmetric functions gives information about the representation theory of the corresponding module.

We can express many useful transformations of symmetric functions with plethystic notation. Suppose $E(t_1, t_2, \ldots,)$ is a Laurent polynomial in indeterminates t_1, t_2, etc. We define the plethystic substitution of E into the power symmetric function p_k by

$$p_k[E(t_1, t_2, \dots)] = E(t_1^k, t_2^k, \dots).$$

Then if f is any symmetric function, we can express it as a polynomial in the power symmetric functions $Q(p_1, p_2, \dots)$. Let

$$f[E] = Q(p_1[E], p_2[E], \dots).$$

That is, plethystic substitution of a particular expression E is a ring homomorphism on Λ.

Note that if $X = x_1 + x_2 + \dots$, then $f[X] = f$ for all $f \in \Lambda$. Furthermore, if A and B are any Laurent polynomials and $n \geq 1$, then $h_n[A + B] = \sum_{k=0}^{n} h_{n-k}[A]h_k[B]$ and $e_n[A + B] = \sum_{k=0}^{n} e_{n-k}[A]e_k[B]$. An introduction to plethysm can be found in [Mac95]. For further details, see Loehr and Remmel [LR11].

In 1988, Macdonald [Mac88] introduced a new basis for the ring of symmetric functions. This basis was later modified by Garsia and Haiman [GH96b] to form the modified Macdonald polynomial basis $\{\widetilde{H}_\mu[X; q, t]\}$. They were also interested in finding the Frobenius characteristic for a certain S_n bi-module, called the module of Diagonal Harmonics. They conjectured [GH96a] a formula for the modified Macdonald expansion of this Frobenius characteristic and Haiman [Hai01] later proved their conjecture using algebraic geometry. However, this formula is not obviously Schur positive or even polynomial.

More specifically, let λ be a partition and λ' the conjugate. Consider the French Ferrers diagram of λ. For each cell c, let the number of cells North, East, South, or West of c in λ be denoted $\mathrm{leg}(c)$, $\mathrm{arm}(c)$, $\mathrm{coleg}(c)$, and $\mathrm{coarm}(c)$, respectively. Then we have the following expressions, which appear in the theory of

Macdonald polynomials.

$$n(\lambda) = \sum_{c \in \lambda} \mathrm{leg}(c) = \sum_{i=1}^{l(\lambda)} \lambda_i (i-1) \qquad \tilde{h}_\lambda(q,t) = \prod_{c \in \lambda} q^{\mathrm{arm}(c)} - t^{\mathrm{leg}(c)+1}$$

$$T_\lambda = t^{n(\lambda)} q^{n(\lambda')} \qquad \tilde{h}'_\lambda(q,t) = \prod_{c \in \lambda} q^{\mathrm{leg}(c)} - t^{\mathrm{arm}(c)+1}$$

$$B_\lambda(q,t) = \sum_{c \in \lambda} q^{\mathrm{coarm}(c)} t^{\mathrm{coleg}(c)} \qquad w_\lambda(q,t) = \tilde{h}_\lambda(q,t) \tilde{h}'_\lambda(q,t)$$

$$\Pi_\lambda(q,t) = \prod_{\substack{c \in \lambda \\ c \neq (0,0)}} \left(1 - q^{\mathrm{coarm}(c)} t^{\mathrm{coleg}(c)}\right)$$

Garsia and Haiman's formula for the Frobenius characteristic of the module of Diagonal Harmonics can be written as

$$DH_n[X;q,t] = \sum_{\mu \vdash n} \frac{T_\mu \widetilde{H}_\mu[X;q,t](1-t)(1-q) B_\mu(q,t) \Pi_\mu(q,t)}{w_\mu(q,t)}.$$

Bergeron and Garsia [BG99] noted that this is very close to the modified Macdonald expansion of e_n, which is

$$e_n = \sum_{\mu \vdash n} \frac{\widetilde{H}_\mu[X;q,t](1-t)(1-q) B_\mu(q,t) \Pi_\mu(q,t)}{w_\mu(q,t)}.$$

Inspired by this similarity, they defined the linear symmetric function operator ∇, which acts by $\nabla \widetilde{H}_\mu = T_\mu \widetilde{H}_\mu$. In this language, $DH_n[X;q,t] = \nabla e_n$. They then explored ∇'s numerous remarkable properties, some of which we will see in later chapters.

One way to view problems involving ∇ is as a basis expansion problem. To find the multiplicity of a particular irreducible character in the module of Diagonal Harmonics, one may expand e_n in terms of the modified Macdonald polynomials, apply ∇, and then expand each modified Macdonald polynomial in terms of Schur functions. At present, there is no combinatorial formula for completing this last step.

Another result of Haiman [Hai01] gives a simple formula for the dimension of the same module: $(n+1)^{n-1}$. Both of Haiman's proofs used deep results from algebraic geometry. Together they inspired a search for a combinatorial interpretation - a collection of $(n+1)^{n-1}$ objects with some statistics giving a weighted enumeration of ∇e_n - that would show the polynomiality of ∇e_n. Such an interpretation was (conjecturally) found by Haglund, Haiman, Loehr, Remmel and Ulyanov [HHL$^+$05]. Their conjecture is known as the Shuffle Conjecture, and was only recently proved. It is still an open problem to find a combinatorial formula for the Schur expansion of ∇e_n.

In 1966, Konhiem and Weiss [KW66] studied a combinatorial problem involving cars trying to park on a one-way street. The resulting objects are known as parking functions. For our purposes, it is more helpful to think of parking functions as labeled paths. A Dyck path in the $n \times n$ lattice is a path $(0, 0)$ to (n, n) of North and East steps which stays weakly above the line $y = x$. A parking function is a Dyck path with labels $\{1, 2, \ldots, n\}$ on North steps which are column-increasing. We write the labels of a parking function in the cell just East of each North step. This visualization was introduced by Garsia and Haiman [GH96a]. For example, see Figure 1.1. Konheim and Weiss [KW66] proved there are $(n + 1)^{n-1}$ parking functions of size n. In reference to their origins, the labels of a parking function are known as cars.

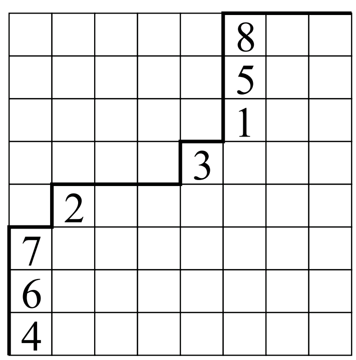

Figure 1.1: A parking function with 8 cars.

The symmetric function ∇e_n has weights with indeterminates t and q. Hence we want to express ∇e_n as an enumeration of parking functions with two integer statistics - the power of t and q - and a symmetric function. The most natural of these statistics is the area - the number of full cells between the main diagonal $y = x$ and the underlying Dyck path. In Figure 1.1, the area is 8.

The other two statistics use the notion of diagonals. Let the k-diagonal be the set of cells cut by the line $y = x + k$. In particular, the main diagonal $y = x$ is the 0-diagonal. In Figure 1.1, there are 3 cars in the 0-diagonal, 2 cars in the 1-diagonal, and 3 cars in the 2-diagonal.

The dinv of a parking function counts certain inversions in σ. If two cars are in the same diagonal and the larger occurs further right, we say they create a primary dinv. If two cars are in adjacent diagonals so that the larger car is higher and further left, they create a secondary dinv. The dinv of a parking function is the total number of primary and secondary dinvs. In Figure 1.1, there is one primary dinv between 2 and 8, and three secondary dinvs caused by the pairs (1,6), (3,6), and (5,7). Hence the dinv is 4.

Finally, the word σ of a parking function is the permutation obtained by reading cars from highest to lowest diagonal and right to left within each diagonal. In Figure 1.1, $\sigma = 8\,2\,7\,5\,6\,1\,3\,4$. This statistic will be used to give a symmetric function weight to each parking function. Recall that the ides of a permutation σ is the descent set of σ^{-1}. Alternatively, it is the set of i so that $i+1$ occurs left of i in σ. In the example, $\mathrm{ides}(\sigma) = \{1, 4, 6, 7\}$.

One can define a composition of n by looking at successive differences in an subset of $\{1, 2, \ldots, n-1\}$ along with the difference between n and the largest element. Let the composition corresponding to the ides of the word of a parking function PF be denoted $\mathrm{pides}(PF)$. Then we can weight each parking function PF with the (Compositional) Schur function $s_{\mathrm{pides}(PF)}$. Equivalently, it can be weighted by the quasi-symmetric function $F_{\mathrm{ides}(PF)}$. Here if $S \subset \{1, 2, \ldots, n-1\}$, F_S is the following degree n fundamental quasi-symmetric function defined by Gessel [Ges84].

$$F_S = \sum_{\substack{0 \leq a_1 \leq a_2 \leq \cdots \leq a_n \\ i \in S \Rightarrow a_i < a_{i+1}}} x_{a_1} x_{a_2} \ldots x_{a_n}$$

Let $\mathcal{P}F_n$ be the set of all parking functions on n cars. Then the classical Shuffle Conjecture states

$$\nabla e_n = \sum_{PF \in \mathcal{P}F_n} t^{\mathrm{area}(PF)} q^{\mathrm{dinv}(PF)} Q_{\mathrm{ides}(PF)}.$$

In [HMZ12], Haglund, Morse and Zabrocki refined the Shuffle Conjecture using the following plethystic symmetric function operators.

$$C_a\, P[X] \;=\; \left(-\frac{1}{q}\right)^{a-1} P\!\left[X - \frac{1-1/q}{z}\right] \sum_{m\geq 0} z^m h_m[X]\Big|_{z^a}$$

Their conjecture, which is stated below, was recently proved by Carlsson and Mellit in [CM15]. Here comp(PF) is the composition of n giving the distances between points (i,i) on PF's underlying path. For example, the parking function in Figure 1.1 has comp $=(4,1,3)$.

Theorem 1.0.1 (Carlsson-Mellit)**.** *For all compositions* $\rho \models n$*,*

$$\nabla C_{\rho_1}\cdots C_{\rho_k} 1 = \sum_{\substack{PF\in\mathcal{PF}_n\\ \mathrm{comp}(PF)=\rho}} t^{\mathrm{area}(PF)} q^{\mathrm{dinv}(PF)} F_{\mathrm{ides}(PF)}.$$

Let PF be a parking function. Define touch(PF) to be the number of parts of comp(PF), i.e., the number of cars in the main diagonal $y=x$. For $n\in\mathbb{N}$ and $1\leq k\leq n$, Garsia and Haglund [GH02] defined the symmetric functions $E_{n,k}$ so that

$$e_n\left[X\frac{1-z}{1-q}\right] = \sum_{k=1}^{n} \frac{(z;q)_k}{(q;q)_k} E_{n,k}[X].$$

where

$$(z;q)_n = (1-z)(1-zq)\cdots(1-zq^{n-1}).$$

Haglund, Morse and Zabrocki [HMZ12] showed

Theorem 1.0.2 (Haglund-Morse-Zabrocki)**.** *For all* $1\leq k\leq n$*,*

$$E_{n,k} = \sum_{\rho\models n,\; l(\rho)=k} C_{\rho_1}\cdots C_{\rho_k} 1.$$

Hence Theorem 1.0.1 implies

Corollary 1.0.1. *For all* $n\in\mathbb{N}$ *and all* $1\leq k\leq n$*,*

$$\nabla E_{n,k} = \sum_{\substack{PF\in\mathcal{PF}_n\\ \mathrm{touch}(PF)=k}} t^{\mathrm{area}(PF)} q^{\mathrm{dinv}(PF)} F_{\mathrm{ides}(PF)}.$$

In the next chapter, we use this intermediate refinement of the Shuffle Conjecture to give combinatorial meaning to ∇p_n, proving a conjecture of Loehr and Warrington [LW07]. The following chapter explores an algebra of symmetric function operators closely related to ∇. These operators are used for the symmetric function side of a family of extended Shuffle Conjectures known as the Rational Shuffle Conjectures. These conjectures are open at the time of writing, though there is extensive experimental evidence supporting them. The final chapter gives a second interpretation for ∇p_n which appears as a special case of the formulas in the previous chapter and connects these seemingly disparate problems.

Chapter 2

A proof of the Square Paths Conjecture

2.1 Introduction

We prove here that ∇p_n can be expressed as a weighted sum of certain labeled lattice paths (called preference functions or labeled square paths). This formula for ∇p_n was originally conjectured by Loehr and Warrington [LW07]. Preference functions are intimately related with parking functions. Both were introduced by Konheim and Weiss [KW66] in 1966. A preference function is a map $f : [n] \to [n]$. For convenience, we will also write it as the vector $(f(1), f(2), \ldots, f(n))$. A parking function is any preference function such that $|f^{-1}([k])| \geq k$ for all $1 \leq k \leq n$. Konheim and Weiss motivated this definition by describing a parking procedure in which n cars try to park in n spaces on a one-way street according to a preference function f. The cars will all succeed in parking if and only if the preference function is a parking function.

For our purposes, it is more helpful to think of the lattice-path interpretation of preference, which matches the interpretation of parking functions given in Chapter 1. Start with an empty $n \times n$ lattice. Write each car which prefers spot 1 (each $i \in f^{-1}(1)$) in column 1, starting at the bottom, from smallest to largest. Then move to the lowest empty row and write all the cars which prefer spot 2

$(f^{-1}(2))$ in column 2 from smallest to largest and bottom to top. Continue this procedure until all the cars have been recorded. Then draw in the unique smallest lattice path which consists of North and East steps and stays above each car. For example, see Figure 2.1.

This gives a bijective correspondence between the n^n preference functions and the set of North-East paths from $(0,0)$ to (n,n) which (1) have column-increasing labels adjacent to North steps and (2) end with an East step. The underlying lattice paths here are also known as square paths and the labels are known as cars. Furthermore, such a labeled path corresponds to a parking function if and only if the underlying path stays (weakly) above the line $y = x$. The underlying paths here are known as Dyck paths.

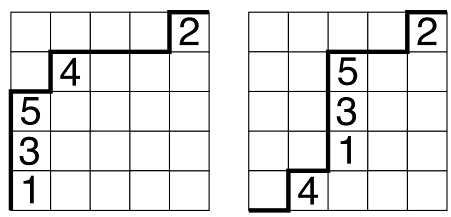

Figure 2.1: The labeled paths corresponding to $(1,5,1,2,1)$ and $(3,5,3,2,3)$.

In [LW07], Loehr and Warrington conjectured a formula for ∇p_n as an enumeration of all preference functions. Their statistics are similar to those used in the Shuffle Conjecture. The word of a preference function, for example, is calculated just as the word of a parking function is: the cars are read from highest to lowest diagonal and from right to left within each diagonal. We will again write $\mathrm{ides}(Pr)$ for the inverse descent set of the word of a preference function Pr. The preference function on the right of Figure 2.1 has word $5\,2\,3\,1\,4$ and ides $= \{1,4\}$.

The dinv of a preference function has three components: the usual primary and secondary dinvs (within any diagonals) and a new component that we will call tertiary dinv. The tertiary dinv is simply the number of cars strictly below the main diagonal $y = x$. For example, the preference function on the right of Figure 2.1 has dinv $= 3$. That is, it has no primary dinv, one secondary dinv (between cars 2 and 5), and two tertiary dinvs (contributed by cars 1 and 4).

To define the area of a preference function, we need to name diagonals. In particular, we will refer to the diagonal $y = x + k$ as the k-th diagonal. For any preference function Pr, let $l(Pr)$ be the number of negative diagonals which are nonempty. This is known as the deviation of the preference function. Note that Pr is a parking function iff $l(Pr) = 0$. Then area(Pr) is the sum over all cars of Pr to which a car in diagonal k contributes $k + l(Pr)$. That is, cars in the lowest diagonal contribute 0, cars in the next lowest diagonal contribute 1, and so on. In the left side of Figure 2.1, the deviation is 1 and area $= 4$.

It is easy to see that the two definitions given for dinv and word coincide when we view parking functions as (special) preference functions. To see the equivalence of the two definitions for the area of a parking function, note that a car in diagonal k lies in a row with k full cells between the underlying path and the main diagonal.

Let $\mathcal{P}ref_n$ be the set of all preference functions on n cars.

Conjecture 2.1.1 (Loehr-Warrington)**.**

$$(-1)^{n-1}\nabla p_n = \sum_{Pr \in \mathcal{P}ref_n} t^{\text{area}(Pr)} q^{\text{dinv}(Pr)} F_{ides(\sigma(Pr))}$$

The main result of this chapter is a proof of Conjecture 2.1.1. In Section 2.2, we extend a notion of Haglund and Loehr [HL05] and use it to enumerate, by area and dinv alone, those preference functions with a fixed set of cars in each diagonal. In Section 2.3, we will discuss the effects of shifting cars between diagonals on the enumeration we obtained in Section 2.2. This will allow us to relate the enumeration of preference functions by area and dinv to the enumeration of parking functions by area and dinv. Finally, in Section 2.4, we will show how to use the results of Section 2.3 to relate the full enumerations (using area, dinv, and ides) of preference and parking functions by extending a result of Hicks [Hic13]. This, combined with a symmetric function identity and the Compositional Shuffle Conjecture, proves the Square Paths Conjecture.

In fact, we prove something much stronger: a relationship between the full enumerations of parking and preference functions with the same "diagonal word" (which we introduce in the next section). This is analogous to Hicks' [Hic13] conjecture that relations between different incarnations of the Compositional Shuffle Conjecture may be refined to the level of parking functions with fixed sets of cars in diagonals. This suggests that there may be quasi-symmetric refinements for the symmetric functions sides of the Shuffle Conjecture and Square Paths Conjecture which correspond to these combinatorial enumerations.

2.2 Schedules for preference functions

In this section we make use of the diagonal word statistic and the schedule of a parking function. These concepts were introduced by Haglund and Loehr in [HL05] and expanded upon by Hicks in [Hic13]. We follow the latter's notation.

The *diagonal word* of a preference function Pr, denoted diagword(Pr), is a permutation whose runs give the cars in each diagonal of Pr from highest to lowest diagonal. That is, cars from a single diagonal are listed in increasing order. This should not be confused with Pr's word, σ, which lists cars from each diagonal in the order they actually appear. For example, the two preference functions in Figure 2.1 have words $\sigma = 4\,5\,3\,2\,1$ and $\sigma = 5\,2\,3\,1\,4$, respectively, but diagonal words $4\,5\,3\,1\,2$ and $5\,2\,3\,1\,4$.

This concept was first introduced to enumerate parking functions as follows. Let $\tau \in S_n$. Suppose the last run of τ has length k. Then for $1 \leq i \leq k$, let $w_i = i$. For $k < i \leq n$, let w_i be the number of elements of τ_{n+1-i}'s run which are larger than τ_{n+1-i} plus the number of cars smaller than τ_{n+1-i} in the next run. If PF is a parking function with diagonal word τ, then $W = (w_i)$ is called its schedule. We also say that W is the schedule of τ. There are $\prod_{i=1}^{n} w_i$ parking functions with diagonal word τ and they can be built by inserting the cars of τ from right to left into an empty parking function.

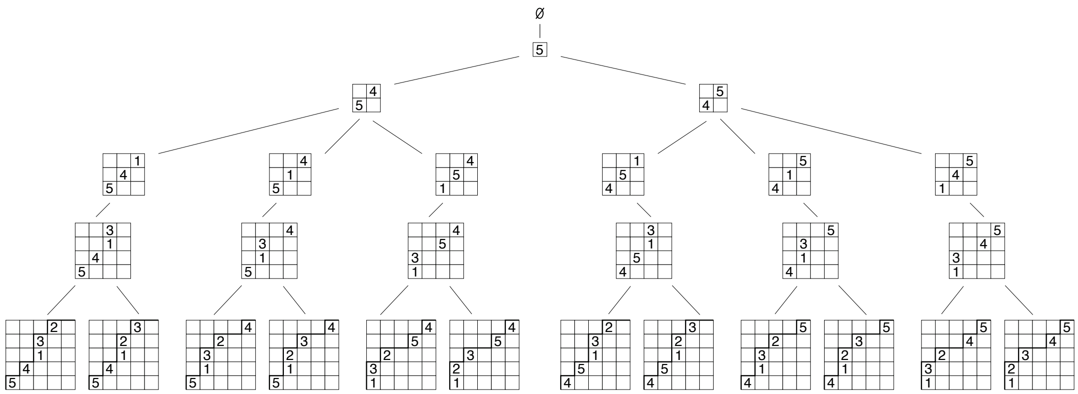

Figure 2.2: All preference functions with diagonal word 2 3 1 4 5 and deviation 0.

Hicks [Hic13] introduced a visualization of this as a tree. In Figure 2.2, we show how parking functions with diagonal word 2 3 1 4 5 are built by inserting. The schedule numbers of τ are $(1, 2, 3, 1, 2)$. Note that at each level of the tree, the degree of each node is the schedule number corresponding to the car being inserted. Furthermore, the children of each node are arranged so that, from left to right, the change in dinv between parent and child is 0, 1, …, $w_i - 1$. This is essentially the proof of the following theorem, which is due to Haglund and Loehr [HL05].

Theorem 2.2.1 (Haglund-Loehr)**.** *Let $\tau \in S_n$ with schedule (w_i). Then*

$$\sum_{\substack{PF \in \mathcal{PF}_n \\ \text{diagword}(PF) = \tau}} t^{\text{area}(PF)} q^{\text{dinv}(PF)} = t^{\text{maj}(\tau)} \prod_{i=1}^{n} [w_i]_q.$$

We extend the notion of schedules to preference functions as follows. Suppose $l \geq 0$ and $\tau \in S_n$ with at least $l+1$ runs. Let $1 \leq c \leq n$. If c is in one of the last l runs of τ, then define $w^{(l)}(c)$ to be the number of elements smaller than c in its own run plus the number of elements larger than c in the previous run. If c is in the $(l+1)$-*st from last* run, define $w^{(l)}(c)$ to be the number of elements to the right of c in the same run. Otherwise define $w^{(l)}(c)$ to be the number of elements larger than c in its own run plus the number of elements smaller than c in the next run.

For example, let $\tau = 2\,3\,1\,4\,5$. Then τ consists of 2 runs and we have $w^{(1)}(3) = 1$, $w^{(1)}(2) = 2$, $w^{(1)}(1) = 2$, $w^{(1)}(4) = 1$, and $w^{(1)}(5) = 2$. We say that $(w^{(l)}(c))$ are the l-schedule numbers of τ. It is easy to see that the original schedule numbers (w_i) correspond to the 0-schedule numbers of τ, but they appear in a different order. We will use the new schedule numbers $(w^{(l)}(c))$ to build preference functions with diagonal word τ and deviation l. See Figure 2.3 for the tree whose leaves are preference functions with diagonal word $2\,3\,1\,4\,5$ and deviation $l = 1$. Note that $w^{(1)}(c)$ gives degrees of the nodes when car c is inserted.

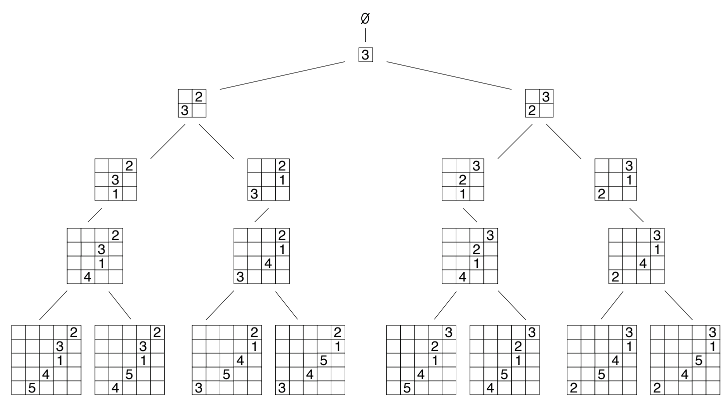

Figure 2.3: All preference functions with diagonal word $2\,3\,1\,4\,5$ and deviation 1.

Theorem 2.2.2. *Let $\tau \in S_n$ with runs of lengths $\rho_k, \ldots, \rho_1, \rho_0$. Let $0 \leq l \leq k$.*

$$\sum_{\substack{Pr \in \mathcal{P}ref_n \\ \text{diagword}(Pr)=\tau \\ l(Pr)=l}} t^{\text{area}(Pr)} q^{\text{dinv}(Pr)} = t^{\text{maj}(\tau)} q^{\rho_0 + \cdots + \rho_{l-1}} \prod_{c=1}^{n} [w^{(l)}(c)]_q.$$

Proof. Each element in the *i-th from last* run of τ will contribute $i - 1$ to area. Therefore the factor $t^{\text{maj}(\tau)}$ on the right hand side of Theorem 2.2.2 accounts for the area on the left hand side. It remains to enumerate the desired preference functions by dinv.

To do this, first insert each car c which occurs in the first $k + 1 - l$ runs of τ from right to left starting in diagonal 0 and moving up a diagonal between runs. At each step, we will have $w^{(l)}(c)$ choices which, when ordered from right to left, will contribute $0, 1, \ldots, w^{(l)}(c) - 1$ to primary and secondary dinv. Since these cars belong to nonnegative diagonals, they contribute nothing to the tertiary dinv.

Next, insert the cars of the remaining l runs from left to right starting in diagonal -1 and moving into the next lowest diagonal at the start of each new run. Such a car c can either appear directly below a larger car from the previous run (i.e., an element from the next highest diagonal of τ) or directly left of a (previously inserted, hence smaller) car in the same run (i.e., same diagonal). Therefore we have $w^{(l)}(c)$ choices. These choices, when ordered from left to right, will contribute $0, 1, \ldots, w^{(l)}(c) - 1$ to primary and secondary dinv.

Since these cars appear below diagonal 0, they also contribute to tertiary dinv. There are $\rho_0 + \cdots + \rho_{l-1}$ such cars, so the tertiary dinv "factors out," just as area did. And, as we observed above, each car contributes $[w^{(l)}(c)]_q$ to the enumeration of primary and secondary dinv. $\square$

2.3 Shifting diagonals and schedules

This section is devoted to proving the following general result about preference functions.

Theorem 2.3.1. *Let $\tau \in S_n$ with schedule (w_i). Suppose that the runs of τ have lengths $\rho_r, \ldots, \rho_1, \rho_0$. If $1 \le l \le r$, then the multi-set of l-schedule numbers of τ is equal to $\{w_i : 1 \le i \le n\} \cup \{\rho_l\} \setminus \{\rho_0\}$. Hence*

$$\sum_{\substack{Pr \in \mathcal{P}ref_n \\ \mathrm{diagword}(Pr)=\tau \\ l(Pr)=l}} t^{\mathrm{area}(Pr)} q^{\mathrm{dinv}(Pr)} = t^{\mathrm{maj}(\tau)} q^{\rho_0+\cdots+\rho_{l-1}} \frac{[\rho_l]_q}{[\rho_0]_q} \prod_{i=1}^{n} [w_i]_q.$$

Our proof of this theorem requires a surprising lemma regarding partitions. See Figure 2.4 for an illustration of the lemma applied to $\lambda = (3, 3, 2, 1, 0)$ with $a = 4$ and $b = 5$.

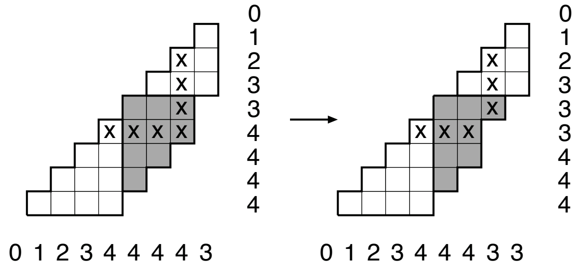

Figure 2.4: Diagrams of $((3,3,2,1,0)+\delta_5)\cup\delta_4$ and $((3,2,2,1,0)+\delta_5)\cup\delta_4$.

Lemma 2.3.1. *Let $a,b>0$ and let $\lambda=(\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_b)$ be a partition, with nonnegative parts, contained in the rectangle $a\times b$. That is $\lambda_1\leq a$ and $l(\lambda)=b$. We will write λ' for the conjugate of λ considered as a partition in the $b\times a$ rectangle. We also write δ_n for the sequence $(0,1,\ldots,n-1)$ for all $n\in\mathbb{N}$. Then the sequences*

$$(\lambda+\delta_b)\cup\delta_a \text{ and } (\lambda'+\delta_a)\cup\delta_b$$

have the same multi-set of entries. Here the sum of sequences is coordinate-wise and $\cup$ denotes concatenation.

Proof. Note that the claim holds if λ is the empty partition. So let $\emptyset\neq\lambda$ be contained in the rectangle $a\times b$ and suppose the claim holds for all partitions contained in λ (with b parts). Suppose λ_1 occurs k times in λ. Then the k-th entry of $\lambda+\delta_b$ is λ_1+k-1. Furthermore, the λ_1-st entry of $\lambda'+\delta_a$ is $k+\lambda_1-1$. For example, in Figure 2.4, $k=2$ and $\lambda_1=3$, so the marked row corresponds to the k-th entry of $\lambda+\delta_b$ and the marked column corresponds to the λ_1-st entry of $\lambda'+\delta_b$, and they have equal length.

Let μ be the partition obtained from λ by reducing its k-th entry from λ_1 to λ_1-1. E.g., if $\lambda=(3,3,2,1,0)$ then $\mu=(3,2,2,1,0)$. The entries of $\mu+\delta_b$ are identical to the entries of $\lambda+\delta_b$ *except* that the k-th entry is now $(\lambda_1-1)+k-1$. Similarly, the only entry of $\mu'+\delta_a$ which differs from $\lambda'+\delta_a$ is the λ_1-st entry, which is now $(k-1)+\lambda_1-1$.

For any sequence σ, let $\{\sigma\}$ denote the multi-set of σ's entries. Then

$$\{(\lambda+\delta_b)\cup\delta_a\}=\{(\mu+\delta_b)\cup\delta_a\}\cup\{\lambda_1+k-1\}\setminus\{(\lambda_1-1)+k-1\}$$

and

$$\{(\lambda' + \delta_a) \cup \delta_b\} = \{(\mu' + \delta_a) \cup \delta_b\} \cup \{k + \lambda_1 - 1\} \setminus \{(k-1) + \lambda_1 - 1\}.$$

Since the claim holds for μ, it also holds for λ. By induction, it holds for all partitions. $\square$

In Figure 2.4 we can see the geometric intuition behind our proof of the Lemma. Namely, the marked corner lies in a row and a column of equal length. In fact *all* removable corners of λ lie in equal rows and columns. Hence removing any one of them preserves the correspondence between row parts (i.e., $(\lambda + \delta_b) \cup \delta_a$) and column parts (i.e., $(\lambda' + \delta_a) \cup \delta_b$).

Proof of Theorem 2.3.1. We claim that

$$\{w^{(l-1)}(c) : 1 \leq c \leq n\} \setminus \{\rho_{l-1}\} = \{w^{(l)}(c) : 1 \leq c \leq n\} \setminus \{\rho_l\}. \tag{2.3.1}$$

as multi-sets for all $1 \leq l \leq r$. Note that if c is the leftmost element of the $(m+1)$-*st from last* run, then $w^{(m)}(c) = \rho_m$, hence there is no trouble with the multi-set subtractions above. Once (2.3.1) is shown, we will have

$$\{w^{(0)}(c) : 1 \leq c \leq n\} \setminus \{\rho_0\} = \{w^{(m)}(c) : 1 \leq c \leq n\} \setminus \{\rho_m\}.$$

for each $1 \leq m \leq r$, which is equivalent to the desired formula.

Let $1 \leq l \leq r$. Note that $w^{(l-1)}(c) = w^{(l)}(c)$ unless c is in the l-*th or* $(l+1)$-*st from last* run of τ. This is because the calculation of a schedule number depends only on its place τ and whether the car in question lies in a positive, zero, or negative diagonal. Shifting the deviation by one only changes the positive/zero/negative "status" of cars from two runs. For example, consider the case $\tau = 3\,7\,1\,5\,8\,2\,6\,4$ with $l = 1, 2, 3$.

$$\begin{aligned} c &= 3\,7\;\; 1\,5\,8\;\; 2\,6\;\; 4 \\ w^{(0)}(c) &= 2\,2\;\; 2\,2\,2\;\; 1\,1\;\; 1 \\ w^{(1)}(c) &= 2\,2\;\; 2\,2\,2\;\; 2\,1\;\; 1 \\ w^{(2)}(c) &= 2\,2\;\; 3\,2\,1\;\; 2\,2\;\; 1 \\ w^{(3)}(c) &= 2\,1\;\; 2\,2\,2\;\; 2\,2\;\; 1 \end{aligned}$$

We can see here that schedule numbers only change within two runs of τ whenever we shift l. Therefore it is sufficient to prove our claim for τ with a single descent and $l = 1$ (that is, for the case when the preference functions in question are contained in two diagonals).

Suppose $\tau \in S_n$ with a single descent. For a finite set A, let $A^{\uparrow}$ denote the word consisting of the elements of A in increasing order. Then $\tau = B^{\uparrow} A^{\uparrow}$ for some disjoint A, B. Let $\lambda \subseteq |A| \times |B|$ be the partition whose ith part is the number of elements of A which are smaller than the i-th largest element of B. Then λ' is the partition whose jth part is the number of elements of B which are larger than the j-th smallest element of A.

Let $w_i^{(l)} = w^{(l)}(c)$ for $c = \tau_{n+1-i}$. Then for i from 1 to $|A|$, $w_i^{(0)} = i$, and for j from 1 to $|B|$, $w_{|A|+j}^{(0)} = \lambda_j + j - 1$. Hence the 0-schedule numbers of τ form the multi-set $\{(\lambda + \delta_{|B|}) \cup \delta_{|A|}\} \cup \{|A|\} \setminus \{0\}$. On the other hand, for i from 1 to $|A|$, $w_i^{(1)} = \lambda'_{|A|-i+1} + |A| - i$, and for j from 1 to $|B|$, $w_{|A|+j}^{(1)} = j$. Then the 1-schedule numbers of τ form $\{(\lambda' + \delta_{|A|}) \cup \delta_{|B|}\} \cup \{|B|\} \setminus \{0\}$.

For example, consider $\tau = 3\,4\,5\,8\,1\,2\,6\,7\,9$. Then $A = \{1, 2, 6, 7, 9\}$ and $B = \{3, 4, 5, 8\}$. This gives $\lambda = (4, 2, 2, 2)$ and $\lambda' = (4, 4, 1, 1, 0)$. Furthermore, we have

$$\begin{aligned}
c &= 3\,4\,5\,8\;\;1\,2\,6\,7\,9 \\
w^{(0)}(c) &= 5\,4\,3\,4\;\;5\,4\,3\,2\,1 \\
&= \overset{3}{\underset{2}{+}}\,\overset{2}{\underset{2}{+}}\,\overset{1}{\underset{2}{+}}\,\overset{0}{\underset{4}{+}}\;\;5\,4\,3\,2\,1 \\
w^{(1)}(c) &= 4\,3\,2\,1\;\;4\,5\,3\,4\,4 \\
&= 4\,3\,2\,1\;\;\overset{0}{\underset{4}{+}}\,\overset{1}{\underset{4}{+}}\,\overset{2}{\underset{1}{+}}\,\overset{3}{\underset{1}{+}}\,\overset{4}{\underset{0}{+}}
\end{aligned}$$

If we remove a single copy of $\rho_0 = |A|$ from $\{w_i^{(0)}\}$ and a single copy of $\rho_1 = |B|$ from $\{w_i^{(1)}\}$ and insert the missing 0's, then Lemma 2.3.1 applies. Hence $\{w_i^{(0)}\} \setminus \{\rho_0\} = \{w_i^{(1)}\} \setminus \{\rho_1\}$ as desired. $\square$

Corollary 2.3.1. *Let $\tau \in S_n$ with schedule (w_i) and let k be the length of its last run. We have*

$$\sum_{\substack{Pr \in \mathcal{P}ref_n \\ \text{diagword}(Pr)=\tau}} t^{\text{area}(Pr)} q^{\text{dinv}(Pr)} = t^{\text{maj}(\tau)} \frac{[n]_q}{[k]_q} \prod_{i=1}^{n} [w_i]_q = \frac{[n]_q}{[k]_q} \sum_{\substack{PF \in \mathcal{P}F_n \\ \text{diagword}(PF)=\tau}} t^{\text{area}(PF)} q^{\text{dinv}(PF)}.$$

Proof. We simply note that if τ's runs are given by $\rho_r, \ldots, \rho_1, \rho_0$ (so that $\rho_0 + \cdots + \rho_r = n$ and $\rho_0 = k$), then

$$\begin{aligned}
\sum_{\text{diagword}(Pr)=\tau} t^{\text{area}(Pr)} q^{\text{dinv}(Pr)} &= \sum_{l=0}^{r} \left(\sum_{\substack{\text{diagword}(Pr)=\tau \\ l(Pr)=l}} t^{\text{area}(Pr)} q^{\text{dinv}(Pr)} \right) \\
&= t^{\text{maj}(\tau)} \frac{1}{[\rho_0]_q} \left(\sum_{l=0}^{r} q^{\rho_0 + \cdots + \rho_{l-1}} [\rho_l]_q \right) \prod_{i=1}^{n} [w_i]_q \\
&= t^{\text{maj}(\tau)} \frac{[n]_q}{[k]_q} \prod_{i=1}^{n} [w_i]_q.
\end{aligned}$$

This gives the first equality. To obtain the second, apply Theorem 2.2.1. □

2.4 Dealing with Inverse Descents

In order to address the Square Paths Conjecture, we need to enumerate preference functions by area, dinv *and* ides. In her thesis, Hicks [Hic13] shows that the ides "factors out" of the desired enumeration for parking functions. We follow her notation here and prove the corresponding result for preference functions.

For any permutation τ, we can partition the set $\{1, 2, \ldots, n\}$ according to whether i appears directly left of $i+1$ in τ. Call each such part a *consecutive block* of τ. E.g., the consecutive blocks of $\tau = 8\,9\,5\,4\,6\,7\,1\,2\,3$ are $\{8, 9\}$, $\{5\}$, $\{4\}$, $\{6, 7\}$, $\{1, 2, 3\}$. Let $\text{Yconsec}(\tau)$ be the Young subgroup of S_n which permutes elements in the same consecutive block of τ. In the example, $\text{Yconsec}(\tau) = S_{\{1,2,3\}} \times S_{\{4\}} \times S_{\{5\}} \times S_{\{6,7\}} \times S_{\{8,9\}}$.

Lemma 2.4.1. *Let $l \geq 0$. Suppose $\tau \in S_n$ has at least $l+1$ runs. Then*

$$\begin{aligned}
&\sum_{\substack{Pr \in \mathcal{P}ref_n \\ \text{diagword}(Pr)=\tau \\ l(Pr)=l}} t^{\text{area}(Pr)} q^{\text{dinv}(Pr)} F_{\text{ides}(Pr)} \\
&\quad = \left(\sum_{\substack{Pr \in \mathcal{P}ref_n \\ \text{diagword}(Pr)=\tau \\ l(Pr)=l}} t^{\text{area}(Pr)} q^{\text{dinv}(Pr)} \right) \cdot \left(\frac{\sum_{\pi \in \text{Yconsec}(\tau)} q^{\text{inv}(\pi)} F_{\text{ides}(\tau) \cup \text{ides}(\pi)}}{\sum_{\pi \in \text{Yconsec}(\tau)} q^{\text{inv}(\pi)}} \right).
\end{aligned}$$

The case $l = 0$ of this lemma is equivalent to Corollary 74 of [Hic13]. Its proof extends without issue to this more general setting. However, for the sake of completeness, we provide a sketch of this proof below.

Proof Sketch. Let $\mathcal{P}ref_{\tau,l}$ be the set of preference functions with diagonal word τ and deviation l. Note that $\mathrm{ides}(\tau) \subseteq \mathrm{ides}(Pr)$. This is because $i \in \mathrm{ides}(\tau)$ iff $i+1$ occurs in a higher diagonal of Pr than i, which means that $i+1$ will precede i in $\sigma(Pr)$. Any other element of $\mathrm{ides}(Pr)$ corresponds to some i and $i+1$ in the same consecutive block of τ. Hence, each $Pr \in \mathcal{P}ref_{\tau,l}$ can be uniquely decomposed into a pair consisting of another preference function $Pr' \in \mathcal{P}ref_{\tau,l}$ with $\mathrm{ides}(Pr') = \mathrm{ides}(\tau)$ and a permutation $\pi \in \mathrm{Yconsec}(\tau)$ so that if we permute the cars of Pr' according to π, we obtain Pr.

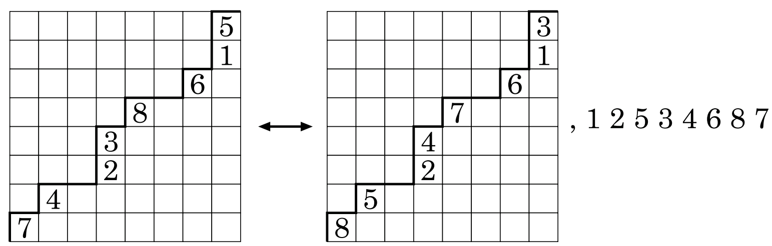

Figure 2.5: A decomposition of a preference function by consecutive blocks.

For example, consider Figure 2.5. On the left side of the figure, we have a preference function Pr with diagonal word $\tau = 3\,4\,5\,7\,8\,1\,2\,6$ and deviation $l = 1$. Furthermore $\mathrm{ides}(Pr) = \{2, 4, 6, 7\}$ and $\mathrm{ides}(\tau) = \{2, 6\}$. On the right we have a preference function Pr' with $\mathrm{ides}(Pr') = \mathrm{ides}(\tau)$ and a permutation π consisting of a cycle on $\{3, 4, 5\}$ and a transposition on $\{7, 8\}$. The consecutive blocks of τ are $\{1, 2\}, \{3, 4, 5\}, \{6\}, \{7, 8\}$, so $\pi \in Yconsec(\tau)$.

In general, we have that $\mathrm{ides}(Pr) = \mathrm{ides}(\tau) \cup \mathrm{ides}(\pi)$ and $\mathrm{dinv}(Pr) = \mathrm{dinv}(Pr') + \mathrm{inv}(\pi)$. Note that Pr and Pr' have identical dinv pairs and cars below the diagonal with one exception. Pr contains primary dinv between consecutive cars and Pr' does not. But π encodes the way that consecutive cars within a diagonal (within a single consecutive block of τ) are interleaved and hence how many primary dinvs occur between them. Similarly, Pr and Pr' share ides except those caused by pairs i and $i+1$ in the same diagonal, which are recorded by π.

Let $\mathcal{P}ref^{id}_{\tau,l}$ be the set of preference functions $Pr \in \mathcal{P}ref_{\tau,l}$ which corresponds to itself and the identity permutation under this decomposition. Then we have

$$\sum_{Pr \in \mathcal{P}ref_{\tau,l}} t^{\mathrm{area}(Pr)} q^{\mathrm{dinv}(Pr)} = \left(\sum_{Pr' \in \mathcal{P}ref^{id}_{\tau,l}} t^{\mathrm{area}(Pr')} q^{\mathrm{dinv}(Pr')} \right) \cdot \left(\sum_{\pi \in \mathrm{Yconsec}(\tau)} q^{\mathrm{inv}(\pi)} \right)$$

and

$$\sum_{Pr \in \mathcal{P}ref_{\tau,l}} t^{\mathrm{area}(Pr)} q^{\mathrm{dinv}(Pr)} F_{\mathrm{ides}(Pr)}$$
$$= \left(\sum_{Pr' \in \mathcal{P}ref^{id}_{\tau,l}} t^{\mathrm{area}(Pr')} q^{\mathrm{dinv}(Pr')} \right) \cdot \left(\sum_{\pi \in \mathrm{Yconsec}(\tau)} q^{\mathrm{inv}(\pi)} F_{\mathrm{ides}(\tau) \cup \mathrm{ides}(\pi)} \right).$$

Combining these equations gives the desired result. □

Fixing τ, if we sum Lemma 2.4.1 over l and compare with the case $l = 0$, we see that the ides-less enumerations of preference functions and parking functions differ from the full enumeration by the same factor. This fact, combined with Corollary 2.3.1 gives the following.

Corollary 2.4.1. *Let $\tau \in S_n$ and let k be the length of its last run. Then*

$$\sum_{\substack{Pr \in \mathcal{P}ref_n \\ \mathrm{diagword}(Pr)=\tau}} t^{\mathrm{area}(Pr)} q^{\mathrm{dinv}(Pr)} F_{\mathrm{ides}(Pr)} = \frac{[n]_q}{[k]_q} \sum_{\substack{PF \in \mathcal{PF}_n \\ \mathrm{diagword}(PF)=\tau}} t^{\mathrm{area}(PF)} q^{\mathrm{dinv}(PF)} F_{\mathrm{ides}(PF)}.$$

Now we can relate the right hand side of this equation to ∇ using a corollary of the Compositional Shuffle Conjecture. Summing Corollary 2.4.1 over all τ whose last run has length k and applying Corollary 1.0.1 gives

Theorem 2.4.1. *For all $1 \le k \le n$,*

$$\sum_{\substack{Pr \in \mathcal{P}ref_n \\ \mathrm{touch}(Pr)=k}} t^{\mathrm{area}(Pr)} q^{\mathrm{dinv}(Pr)} F_{\mathrm{ides}(Pr)} = \frac{[n]_q}{[k]_q} \nabla E_{n,k}$$

where $\mathrm{touch}(Pr)$ *is the number of cars in diagonal $-l(Pr)$ for any preference function Pr. (It is also the length of the last run of* $\mathrm{diagword}(Pr)$*.)*

Finally, we need a symmetric function identity relating p_n to the polynomials $\{E_{n,k}\}$. The following identity was proved by Can and Loehr [CL06] in their proof of a special case of the Square Paths Conjecture. It seems this was known earlier to Garsia and Haglund [GH02].

Theorem 2.4.2 (Garsia-Haglund)**.** *For all $n \geq 1$,*

$$(-1)^{n-1} p_n = \sum_{k=1}^{n} \frac{[n]_q}{[k]_q} E_{n,k}.$$

Hence summing Theorem 2.4.1 over k and applying Theorem 2.4.2 gives the Square Paths Conjecture.

Theorem 2.4.3. *For all $n \geq 1$,*

$$\sum_{Pr \in \mathcal{P}ref_n} t^{\mathrm{area}(Pr)} q^{\mathrm{dinv}(Pr)} F_{\mathrm{ides}(Pr)} = (-1)^{n-1} \nabla p_n.$$

Acknowledgement. *This chapter is a reproduction of a paper with the same name.*

Chapter 3

A new plethystic symmetric function operator and the Rational Compositional Shuffle Conjecture at $t = 1/q$

3.1 Introduction

The specializations at $t = 1/q$ of all the Shuffle conjectures (including the classical cases) are still open to this date. What makes this specialization particularly fascinating is that both sides of the stated identities have combinatorial interpretations. Nevertheless proving these identities is quite challenging even in the simplest cases. For instance from the Rational Shuffle Conjecture we can easily derive the following identity, for any coprime pair (m, n).

$$\sum_{D \in \mathcal{D}_{m,n}} q^{\text{coarea}(D)+\text{dinv}(D)} = \frac{1}{[m]_q} \begin{bmatrix} m+n-1 \\ n \end{bmatrix}_q \tag{3.1.1}$$

Here the sum is over Dyck paths in the $m \times n$ lattice rectangle, coarea(D) gives the number of lattice squares above the path and dinv(D) is a Dyck path statistic that can also be given a relatively simple geometric construction. The identity obtained by setting $q = 1$ in (3.1.1) is an immediate consequence of the Cyclic

Lemma, which suggests that this classical result may have a natural q-analogue. The investigations that yielded the present results have been directed towards giving a concrete setting to a variety of identities stated or implied in recent work by the Algebraic Geometers, particularly in [BS12] and [SV13]. Unfortunately most of this work appears in language that requires considerable algebraic geometrical background. We have been privileged to have had some of these results translated into a language that we could understand by Eugene Gorsky and Andrei Negut. Many of the theorems we prove here have their origin in this algebraic geometrical literature. Our contribution is to provide proofs that are accessible to the algebraic combinatorial audience. We hope that in doing so, the new results we obtain may be conducive to progress in this challenging area of Algebraic Combinatorics.

We will be dealing here with an algebra $\mathcal{A}$ of linear operators acting on the space Λ of symmetric functions in an infinite alphabet $X = \{x_1, x_2, x_3, \ldots\}$ with coefficients in the field $\mathbb{Q}(q,t)$ of Rational functions in the two indeterminates q and t. Given a symmetric function $F[X] \in \Lambda$, it will be convenient to denote by "$\underline{F}$" the operator "*multiplication by* $F[X]$." As is customary, we will denote by "$F^\perp$" the operator dual of $\underline{F}$ with respect to the classical Hall scalar product of symmetric functions.

For a coprime pair (m,n) the $Q_{m,n}$ operators have an elementary definition which, as far as we understand, is due to Burban-Schiffmann in [BS12]. By taking the lattice point (a,b) in the $m\times n$ rectangle that is closest to and below the segment $(0,0) \to (m,n)$ and setting $(c,d) = (m,n) - (a,b)$ we obtain a decomposition

$$(m,n) = (a,b) + (c,d) \tag{3.1.2}$$

which here and after will be referred to as "Split(m,n)." The fact that (a,b) is the closest point forces both pairs (a,b) and (c,d) to be coprime. Therefore we can recursively set

$$Q_{m,n} \; = \; \frac{1}{M} \left[Q_{c,d}, Q_{a,b} \right] \, , \quad (M = (1-t)(1-q)) \tag{3.1.3}$$

with base cases

$$Q_{0,1} = -\underline{e}_1 \quad \text{and} \quad Q_{1,0} = D_0. \tag{3.1.4}$$

Here e_1 is the customary elementary symmetric function and D_0 belongs to a family of operators $\{D_k\}_{k\in\mathbb{Z}}$ introduced in [GHT99] and defined by setting for $F[X]\in\Lambda$

$$D_k F[X] \;=\; F\left[X+\frac{M}{z}\right]\sum_{r\geq 0}(-z)^r e_r[X]\Big|_{z^k}. \tag{3.1.5}$$

In dealing with the present subject, plethystic notation is absolutely indispensable. Readers that are not familiar with this device are referred to [GXZ11] for an introduction to its use. The readers will also find in [BGLX15] and [BGLX14] elementary proofs of all the auxiliary results that we will need in this writing. Those papers were written precisely to render this subject accessible to the algebraic combinatorial audience in a completely self contained manner. In particular it is shown in [BGLX14] that to compute the action of an operator $Q_{m,n}$ we do not need to recurse to the base cases in (3.1.4), but rather use as a shortcut the identities

$$Q_{1,k} \;=\; D_k. \tag{3.1.6}$$

It should be mentioned that the original identities justifying the use of this shortcut were first given in [GHT99].

The definition of the operator $Q_{u,v}$ for a non coprime pair (u,v) relies on a truly amazing property of the algebra generated by the operators D_k, certainly noticed in [BS12] and possibly in other algebraic geometrical literature. To state it, it will be convenient to write a non coprime pair in the form $(u,v)=(km,kn)$ with (m,n) coprime and $k>1$. This given, we can recursively define the operator $Q_{km,kn}$ by choosing any of the lattice points (a,b) in the rectangle $km\times kn$ that are strictly below and closest to the segment $(0,0)\leftarrow(km,kn)$ and then setting

$$Q_{km,kn} \;=\; \frac{1}{M}\left[Q_{km-a,kn-b},Q_{a,b}\right]. \tag{3.1.7}$$

This definition is made possible because the choice of (a,b) forces both (a,b) and $(km-a,kn-b)$ to be coprime. Moreover, all the operators resulting from such a choice of (a,b) can be shown to act identically on symmetric functions.

Another fundamental fact discovered by the Algebraic Geometers is that the Q operators indexed by collinear vectors do commute. More precisely for any coprime pair m,n and any two integers k,h we have

$$[Q_{km,kn},Q_{hm,hn}] \;=\; 0. \tag{3.1.8}$$

Elementary, but by no means simple, proofs of all these properties are given in [BGLX14]. The complexity of these proofs is due to the recursive nature of the definition in (3.1.3). Our discovery here is that with the specialization $t = 1/q$ we can obtain several explicit identities from which these fundamental properties are immediate.

More precisely let $D_{u,v}$ denote the operator whose action on the symmetric function $F[X] \in \Lambda$ is defined by setting

$$D_{u,v}F[X] \;=\; F\left[X + \frac{M\,[u]_q}{z}\right] \sum_{r\geq 0}(-z)^r e_r\left[[u]_t\,X\right]\Big|_{z^v}, \tag{3.1.9}$$

where we must set here $t = 1/q$.

Theorem 3.1.1. *If a, b, c, d, u, v are any integers related by the vector identity $(a, b) + (c, d) = (u, v)$, we have for non vanishing a, c, u*

$$\frac{1}{M}\left[D_{c,d}, D_{a,b}\right]\Big|_{t=1/q} \;=\; q^{1+bc}\frac{[a]_q\,[c]_q}{[u]_q}\frac{(1-q^{da-bc})}{(1-q)}D_{u,v}\Big|_{t=1/q}. \tag{3.1.10}$$

This identity has the following immediate corollary.

Theorem 3.1.2. *For any coprime pair (m, n) and $k \geq 1$ we have*

$$q^{(km-1)(kn-1)/2+(k-1)/2}Q_{km,kn}\Big|_{t=1/q} = q^{(km-1)kn}\frac{[k]_q}{[km]_q}D_{km,kn}. \tag{3.1.11}$$

The two identities in (3.1.10) and (3.1.11) have a variety of consequences. For instance we can immediately see from (3.1.10) that the collinearity of (a, b) and (c, d) implies that $D_{a,b}$ and $D_{c,d}$ commute. We thus obtain a much simpler proof of this commutativity result for the $Q_{u,v}$ operators when $t = 1/q$. Another immediate consequence of (3.1.10) is that the algebra generated by the D_k operators at $t = 1/q$ is spanned by the *convex* monomials in the $D_{u,v}$ operators. Here a monomial

$$D_{u_1,v_1}D_{u_2,v_2}D_{u_3,v_3}\cdots D_{u_l,v_l},$$

is called *convex* if and only if we have

$$\frac{v_1}{u_1} \geq \frac{v_2}{u_2} \geq \cdots \geq \frac{v_l}{u_l}.$$

To state an important consequence of (3.1.11), we need some background. Let us recall that the classical Shuffle conjecture of Haglund, Haiman, Loehr, Remmel and Ulyanov in [HHL$^+$05] may be stated as the identity

$$Q_{n+1,n}(-1)^n = \sum_{PF\in\mathcal{PF}} t^{\mathrm{area}(PF)} q^{\mathrm{dinv}(PF)} s_{\mathrm{pides}(PF)}[X] \tag{3.1.12}$$

where the sum is over parking functions in the $n \times n$ lattice square, area(PF) and dinv(PF) are parking function statistics we will define later in a much more general context, and $s_{\mathrm{pides}(PF)}[X]$ denotes the Schur function indexed by the composition which gives the inverse descent set of a permutation naturally associated to a parking function. In a recent paper [GN15], E. Gorsky and A. Negut formulated an infinite variety of Shuffle conjectures, one for each coprime pair (m, n). They may be stated in a form similar to (3.1.12), namely

$$Q_{m,n}(-1)^n = \sum_{PF\in\mathcal{PF}_{m,n}} t^{\mathrm{area}(PF)} q^{\mathrm{dinv}(PF)} s_{\mathrm{pides}(PF)}[X] \tag{3.1.13}$$

where the sum is over parking functions in the $m \times n$ lattice rectangle, and the parking function statistics occurring in (3.1.13) are highly non trivial modifications of the statistics involved in (3.1.11). Now, Theorem 3.1.2 has the following immediate corollary.

Theorem 3.1.3. *For any coprime pair (m, n) and $k \geq 1$ we have*

$$q^{(km-1)(kn-1)/2+(k-1)/2} Q_{km,kn}(-1)^{kn}\Big|_{t=1/q} = \frac{[k]_q}{[km]_q} e_{kn}\left[X\,[km]_q\right]. \tag{3.1.14}$$

In particular, by combining (3.1.14) with the Gorsky-Negut conjectures at $t = 1/q$ we obtain the identity

$$\frac{1}{[m]_q} e_n\left[X\,[m]_q\right] = \sum_{PF\in\mathcal{PF}_{m,n}} q^{\mathrm{coarea}(PF)+\mathrm{dinv}(PF)} s_{\mathrm{pides}(PF)}[X] \tag{3.1.15}$$

with coarea$(PF) = (m-1)(n-1)/2 - \mathrm{area}(PF)$.

This given, we may ask if the right hand side of (3.1.14), can also be given a parking function interpretation. It turns out that this is indeed the case. More precisely we will prove the following theorem.

Theorem 3.1.4. *Upon the validity of the extended Compositional Shuffle Conjecture in [BGLX15] it follows that*

$$\frac{[k]_q}{[km]_q} e_{kn}\left[X\,[km]_q\right] = \sum_{PF\in\mathcal{P}F_{km,kn}} q^{\mathrm{coarea}(PF)+\mathrm{dinv}(PF)}\,[\mathrm{ret}(PF)]_q\, s_{\mathrm{pides}(PF)}[X] \tag{3.1.16}$$

where $\mathrm{ret}(PF)$ *is a statistic which indicates the height of the first return to the diagonal by the Dyck path of* PF *in the* $km\times kn$ *lattice rectangle.*

The precise definitions of all the parking function statistics occurring in (3.1.16) will be given in the sequel.

We must mention that it would follow from (3.1.16) combined with the theory of LLT polynomials that the left hand side is a Schur positive symmetric polynomial. However, we will show that this particular result can be given a much more elementary proof.

It is important to notice that operators $D_{u,v}$ can be used for any integral values of u and v. Now it follows from (3.1.11) for $m=1$ and $n=0$ that

$$Q_{k,0}\Big|_{t=1/q} = D_{k,0}. \tag{3.1.17}$$

It was known to the Algebraic Geometers that the family of operators $\{Q_{k,0}\}_{k\geq 1}$ have the modified Macdonald basis $\{\widetilde{H}_\mu[X;q,t]\}_\mu$, introduced in [GH96b], as a complete set of eigenfunctions. More precisely, we have

$$Q_{k,0}\widetilde{H}_\mu[X;q,t] = \Big(1-(1-t^k)(1-q^k)B_\mu(q^k,t^k)\Big)\widetilde{H}_\mu[X;q,t], \tag{3.1.18}$$

where for a partition $\mu=(\mu_1,\mu_2,\ldots,\mu_l)$ we set $B_\mu(q,t)=\sum_{i=1}^{l} t^{i-1}\sum_{j=1}^{\mu_i} q^{j-1}$. Since it can be shown that the polynomial $\widetilde{H}_\mu[X;q,t]$ specializes, at $t=1/q$ to a scalar multiple of $s_\mu\left[\frac{X}{1-q}\right]$, it follows from (3.1.18) and (3.1.17) that we must also have the following.

Theorem 3.1.5.

$$D_{k,0}s_\mu\left[\frac{X}{1-q}\right] = \Big(1-(1-q^{-k})(1-q^k)B_\mu(q^k,q^{-k})\Big)s_\mu\left[\frac{X}{1-q}\right] \tag{3.1.19}$$

Proving this identity directly from the definition in (3.1.9) leads to some highly non trivial combinatorial problems. However, with some effort, as we shall see, a less direct but still entirely elementary path to (3.1.19) can actually be found. In fact this particular effort led to the discovery of the following formula for the action of $D_{u,v}$ on the basis $\left\{s_\mu\left[\frac{X}{1-q}\right]\right\}_\mu$.

Theorem 3.1.6. *For any $u, v > 0$ and any partition μ we have*

$$D_{u,v}s_\mu\left[\frac{X}{1-q}\right] = (q^u-1)\sum_{i=1}^{|\mu|+v} q^{up(\mu)_i+v-ui}s_{p(\mu)+ve_i}\left[\frac{X}{1-q}\right] \tag{3.1.20}$$

where $p(\mu)$ is the weak composition of length $|\mu|+v$ obtained by adjoining zeros to the parts of μ and e_i is the i^{th} coordinate vector of length $|\mu|+v$.

Remark 3.1.1. *We should mention that there is another interesting by-product of our introduction of the operator $D_{u,v}$. We learned from Eugene Gorsky (see also Section 6.10 of [Gor13]) that in [Ste10] it is shown that for a suitable constant factor $c_{m,n}(q)$ we have, for (m,n) a coprime pair*

$$Q_{m,n}\Big|_{t=1/q} = c_{m,n}(q)\nabla^{\frac{m}{n}}\underline{p}_n\nabla^{-\frac{m}{n}}\Big|_{t=1/q} \tag{3.1.21}$$

with ∇ the operator introduced in [BG99]. Now it turns out that one can easily derive (3.1.21) from (3.1.20) for $(u,v) = (m,n)$, directly from the definition of ∇ given in [BG99].

This chapter is divided into five further sections. In Section 3.2 we give an elementary proof of Theorem 3.1.1. This type of proof has been successfully used in various similar situations where we needed a straightforward proof of an identity that was discovered by another path. In this section we also give proofs of Theorems 3.1.2 and 3.1.3.

In Section 3.3 we give the elementary argument that proves the polynomiality and Schur positivity of the symmetric function in (3.1.14).

In Section 3.4 we give our parking function setting for the symmetric function in (3.1.14). We also give there a simplified version of the parking function statistics that occur in the formulation of the Rational Compositional Shuffle Conjecture that take account of the most recent developments in this subject.

In Section 3.5 we prove Theorems 3.1.5 and 3.1.6 and explore some of their consequences. What is interesting is that the path to these proofs uses an argument that may be conducive to the discovery of a variety other identities of similar type.

In Section 3.6 we give an outline of the theoretical steps that led to the discovery of the operators $D_{u,v}$ and Theorem 3.1.1.

3.2 Commutator properties of our new operators

Our main goal in this section is an elementary proof of Theorem 3.1.1 and its immediate corollaries. In a later section we will try to give a glimpse of the machinery that led to the discovery of the operators $D_{u,v}$ and yielded the original proof of this identity. To give the reader an idea of the basic difference between these two approaches, we need to recall a device which was extensively used in all previous work in the theory of Macdonald polynomials. We will refer to it as the "Ω" notation. The point of departure of plethystic substitutions is the operation of evaluating the power symmetric function p_k at a formal power series $E = E(t_1, t_2, t_3, \ldots)$ containing an unlimited number of indeterminates. We simply set

$$p_k[E] \;=\; E(t_1^k, t_2^k, t_3^k, \ldots) \tag{3.2.1}$$

Since every symmetric function can be expressed as a polynomial in the power functions, this definition allows to evaluate $F[E]$ for any given symmetric function F. This is what we refer to as the plethystic substitution of E in F. In this vein we set

$$\Omega[E] \;=\; \exp\left(\sum_{k\geq 1} \frac{p_k[E]}{k}\right). \tag{3.2.2}$$

Clearly, this definition implies that for any two expressions A and B we have

$$\Omega\,[A+B] = \Omega[A] \times \Omega[B] \quad \text{and} \quad \Omega\,[A-B] = \Omega[A]/\Omega[B] \tag{3.2.3}$$

In particular it also follows from (3.2.1) and (3.2.2), that if $X = x_1 + x_2 + x_3 + \cdots$ then

$$\sum_{r\geq 0} (-z)^r e_r[X] \;=\; \Omega\,[-zX]\,. \tag{3.2.4}$$

Using this device the definition of the operators D_k in (3.1.5) can be rewritten in the form

$$D_k F[X] \;=\; F\left[X+\frac{M}{z}\right]\Omega\left[-zX\right]\Big|_{z^k} \tag{3.2.5}$$

Successive applications of two operators D_a and D_b to a symmetric function $F[X]$, in this notation, leads to the identities

$$\begin{aligned}
D_bD_aF[X] \;&=\; D_bF\left[X+\frac{M}{z_1}\right]\Omega\left[-z_1X\right]\Big|_{z_1^a}\\
&=\; F\left[X+\frac{M}{z_1}+\frac{M}{z_2}\right]\Omega\left[-z_1(X+\frac{M}{z_2})\right]\Omega\left[-z_2X\right]\Big|_{z_1^a z_2^b}\\
&=\; F\left[X+\frac{M}{z_1}+\frac{M}{z_2}\right]\Omega\left[-(z_1+z_2)X\right]\Omega\left[-\frac{Mz_1}{z_2}\right]\Big|_{z_1^a z_2^b}\\
&=\; F\left[X+\frac{M}{z_1}+\frac{M}{z_2}\right]\Omega\left[-(z_1+z_2)X\right]\\
&\qquad\qquad \times\frac{(1-z_1/z_2)(1-qtz_1/z_2)}{(1-tz_1/z_2)(1-qz_1/z_2)}\frac{1}{z_1^a z_2^b}\Big|_{z_1^0 z_2^0}
\end{aligned} \tag{3.2.6}$$

where the last equality results from multiple applications of the identities in (3.2.2). By contrast if we carry out this calculation, the way a computer would do it, we would end up with the following sequence of identities.

$$\begin{aligned}
D_bD_aF[X] \;&=\; D_b\sum_{r_1\geq 0}F^{(r_1)}[X]\frac{1}{z_1^{r_1}}\Omega\left[-z_1X\right]\Big|_{z_1^a}\\
&=\; D_b\sum_{r_1\geq 0}F^{(r_1)}[X](-1)^{r_1+a}e_{r_1+a}[X]\\
&=\; \sum_{r_1\geq 0}F^{(r_1)}\left[X+\frac{M}{z_2}\right](-1)^{r_1+a}e_{r_1+a}\left[X+\frac{M}{z_2}\right]\Omega\left[-z_2X\right]\Big|_{z_2^b}\\
&=\sum_{r_1,r_2\geq 0}F^{(r_1,r_2)}[X]\frac{1}{z_2^{r_2}}(-1)^{r_1+a}\sum_{s=0}^{r_1+a}e_{r_1+a-s}[X]\frac{1}{z_2^s}e_s\left[M\right]\Omega\left[-z_2X\right]\Big|_{z_2^b}\\
&=\sum_{r_1,r_2\geq 0}F^{(r_1,r_2)}[X](-1)^{r_1+a}\sum_{s=0}^{r_1+a}e_{r_1+a-s}[X]e_s\left[M\right](-1)^{r_2+s+b}e_{r_2+s+b}[X]
\end{aligned} \tag{3.2.7}$$

where for convenience we have set

$$F^{(r_1)}[X]=F\left[X+Mu_1\right]\Big|_{u_1^{r_1}} \quad\text{and}\quad F^{(r_1,r_2)}[X]=F\left[X+Mu_1+Mu_2\right]\Big|_{u_1^{r_1}u_2^{r_2}}.$$

We can see from this example that the second calculation of the action of the operator D_bD_a is completely elementary and straight forward. But, in more complex situations, this approach is not conducive to discovery but only to delivering the verification of an identity discovered by other means. On the other hand the calculation of this action carried out in (3.2.6), in several significant instances, has led to discovery and proof of surprising identities. Nevertheless, we must add that due care must be taken in expressing the Rational function, argument of the constant term in (3.2.6), as an appropriate Laurent series in z_1, z_2. A systematic way of carrying this out in greater generality has been developed in [Xin04] and [Xin05].

The proof of Theorem 3.1.1 in this section will use the approach illustrated in (3.2.7). The proof that follows the approach in (3.2.6) led to the discovery of the operators $D_{u,v}$ and their commutator identities. This second proof will be given in Section 3.6.

Recalling that the action of the operator $D_{u,v}$ is defined by setting

$$D_{m,n}F[X] \;=\; F\left[X + [m]_q \frac{M}{z}\right] \Omega\left[-z\,[m]_t\,X\right]\Big|_{z^n} \tag{3.2.8}$$

with the convention that $t = 1/q$, we have the following.

Theorem 3.2.1. *If a, b, c, d, m, n are any integers related by the vector identity $(a, b) + (c, d) = (m, n)$ we have for non vanishing a, c, m, then*

$$\frac{1}{M}\left[D_{c,d}, D_{a,b}\right]\Big|_{t=1/q} \;=\; q^{1+bc}\frac{[a]_q\,[c]_q}{[m]_q}\frac{1-q^{da-bc}}{1-q}D_{m,n}\Big|_{t=1/q}. \tag{3.2.9}$$

Proof. Using the notation and the sequence of steps outlined in (3.2.7) we get

$$\begin{aligned}
D_{a,b}F[X] &= F\left[X + \frac{M\,[a]_q}{z_1}\right]\Omega\left[-z_1\,[a]_t\,X\right]\Big|_{z_1^b} \\
&= \sum_{r_1 \geq 0} F^{(r_1)}[X]\frac{1}{z_1^{r_1}}\Omega\left[-z_1\,[a]_t\,X\right]\Big|_{z_1^b} \\
&= \sum_{r_1 \geq 0} F^{(r_1)}[X](-1)^{r_1+b}e_{r_1+b}\left[[a]_t\,X\right]
\end{aligned}$$

and consequently

$$D_{c,d}D_{a,b}F[X] = \sum_{r_1,r_2\geq 0} F^{(r_1,r_2)}[X]\frac{(-1)^{r_1+b}}{z_2^{r_2}} e_{r_1+b}\left[[a]_t\left(X+\frac{M\,[c]_q}{z_2}\right)\right] \times \Omega\left[-z_2\,[c]_t\,X\right]\Big|_{z_2^d}$$

$$= \sum_{r_1,r_2\geq 0} F^{(r_1,r_2)}[X]\frac{(-1)^{r_1+b}}{z_2^{r_2}} \sum_{s=0}^{r_1+b} e_{r_1+b-s}\left[[a]_t\,X\right] e_s\left[[a]_t\,M\,[c]_q\right] \times \frac{1}{z_2^s}\Omega\left[-z_2\,[c]_t\,X\right]\Big|_{z_2^d}$$

$$= \sum_{r_1,r_2\geq 0} F^{(r_1,r_2)}[X](-1)^{r_1+b} \sum_{s=0}^{r_1+b} e_{r_1+b-s}\left[[a]_t\,X\right] e_s\left[M\,[a]_t\,[c]_q\right] \times (-1)^{r_2+d+s} e_{r_2+d+s}\left[[c]_t\,X\right]. \tag{3.2.10}$$

Now we can easily see that

$$M\,[a]_t\,[c]_q \;=\; \frac{(1-q)(1-1/q)(1-q^{-a})(1-q^c)}{(1-1/q)(1-q)} \;=\; -q^{-a}(1-q^a)(1-q^c).$$

Thus

$$e_s\left[M\,[a]_t\,[c]_q\right] \;=\; \frac{(-1)^s}{q^{as}} h_s\left[(1-q^a)(1-q^c)\right] \;=\; \frac{(-1)^s}{q^{as}}(1-q^a)(1-q^c)\frac{1-q^{sm}}{1-q^m},$$

and (3.2.10) becomes

$$\frac{D_{c,d}D_{a,b}}{(1-q^a)(1-q^c)}F[X] = \sum_{r_1,r_2\geq 0} F^{(r_1,r_2)}[X](-1)^{r_1+b} \sum_{s=0}^{r_1+b} e_{r_1+b-s}\left[[a]_t\,X\right] \times \frac{(-1)^s}{q^{as}}\frac{1-q^{sm}}{1-q^m}(-1)^{r_2+d+s} e_{r_2+d+s}\left[[c]_t\,X\right]$$

$$= \sum_{r_1,r_2\geq 0} F^{(r_1,r_2)}[X](-1)^{r_1+r_2+n} \sum_{s=0}^{r_1+b} e_{r_1+b-s}\left[[a]_t\,X\right] \times \frac{q^{-as}-q^{cs}}{1-q^m} e_{r_2+d+s}\left[[c]_t\,X\right].$$

Or better

$$\frac{(1-q^m)D_{c,d}D_{a,b}}{(1-q^a)(1-q^c)}F[X] \;=\; A \,-\, B \tag{3.2.11}$$

with

$$A \;=\; \sum_{r_1,r_2\geq 0} F^{(r_1,r_2)}[X](-1)^{r_1+r_2+n} \sum_{s=0}^{r_1+b} e_{r_1+b-s}\left[[a]_t\,X\right] q^{-as} e_{r_2+d+s}\left[[c]_t\,X\right]$$

and

$$B \,=\, \sum_{r_1,r_2\geq 0} F^{(r_1,r_2)}[X](-1)^{r_1+r_2+n}\sum_{s=0}^{r_1+b} e_{r_1+b-s}\left[[a]_t\,X\right] q^{cs} e_{r_2+d+s}\left[[c]_t\,X\right].$$

Simple manipulation allows us to rewrite A and B in the more convenient forms

$$A \,=\, q^{ad}\sum_{r_1,r_2\geq 0} F^{(r_1,r_2)}[X]q^{ar_2}(-1)^{r_1+r_2+n}\sum_{s=0}^{r_1+b} e_{r_1+b-s}\left[[a]_t\,X\right] e_{r_2+d+s}\left[q^{-a}\,[c]_t\,X\right]$$

and

$$B \,=\, q^{cb}\sum_{r_1,r_2\geq 0} F^{(r_1,r_2)}[X]q^{cr_1}(-1)^{r_1+r_2+n}\sum_{s=0}^{r_1+b} e_{r_1+b-s}\left[q^{-c}\,[a]_t\,X\right] e_{r_2+d+s}\left[[c]_t\,X\right].$$

Carrying out the interchanges $a \leftrightarrow c$ and $b \leftrightarrow d$ gives

$$\widetilde{A} \,=\, q^{cb}\sum_{r_1,r_2\geq 0} F^{(r_2,r_1)}[X]q^{cr_2}(-1)^{r_1+r_2+n}\sum_{s=0}^{r_1+d} e_{r_1+d-s}\left[[c]_t\,X\right] e_{r_2+b+s}\left[q^{-c}\,[a]_t\,X\right]$$

and

$$\widetilde{B} \,=\, q^{ad}\sum_{r_1,r_2\geq 0} F^{(r_2,r_1)}[X]q^{ar_1}(-1)^{r_1+r_2+n}\sum_{s=0}^{r_1+d} e_{r_1+d-s}\left[q^{-a}\,[c]_t\,X\right] e_{r_2+b+s}\left[[a]_t\,X\right].$$

Thus from (3.2.11) we derive that

$$\frac{(1-q^m)D_{a,b}D_{c,d}}{(1-q^a)(1-q^c)}F[X] \,=\, \widetilde{A} \,-\, \widetilde{B}$$

and consequently

$$\frac{(1-q^m)\,[D_{c,d},D_{a,b}]}{(1-q^a)(1-q^c)}F[X] \,=\, A \,-\, B \,-\, (\widetilde{A} \,-\, \widetilde{B}) \,=\, (A+\widetilde{B}) \,-\, (B+\widetilde{A}). \quad (3.2.12)$$

Now we may rewrite A by setting $u = r_1+b-s$ so $0 \leq u \leq r_1+b$ and $r_2+d+s = r_1+r_2+n-u$ obtaining

$$A \,=\, q^{ad}\sum_{r_1,r_2\geq 0} F^{(r_1,r_2)}[X]q^{ar_2}(-1)^{r_1+r_2+n}\sum_{u=0}^{r_1+b} e_u\left[[a]_t\,X\right] e_{r_1+r_2+n-u}\left[q^{-a}\,[c]_t\,X\right].$$

For $\widetilde{B}$ we set $u = r_2+b+s$ so $r_2+b \leq u \leq r_1+r_2+n$ and then make the switch $r_1 \leftrightarrow r_2$ to obtain

$$\begin{aligned}\widetilde{B} \,&=\, q^{ad}\sum_{r_1,r_2\geq 0} F^{(r_2,r_1)}[X]q^{ar_1}(-1)^{r_1+r_2+n}\sum_{u=r_2+b}^{r_1+r_2+n} e_u\left[[a]_t\,X\right] e_{r_1+r_2+n-u}\left[q^{-a}\,[c]_t\,X\right]\\ &=\, q^{ad}\sum_{r_1,r_2\geq 0} F^{(r_1,r_2)}[X]q^{ar_2}(-1)^{r_1+r_2+n}\sum_{u=r_1+b}^{r_1+r_2+n} e_u\left[[a]_t\,X\right] e_{r_1+r_2+n-u}\left[q^{-a}\,[c]_t\,X\right].\end{aligned}$$

This gives

$$\begin{aligned}
A+\widetilde{B} \;=\; & q^{ad}\sum_{r_1,r_2\geq 0} F^{(r_1,r_2)}[X]q^{ar_2}(-1)^{r_1+r_2+n} \\
& \qquad\qquad \times \sum_{u=0}^{r_1+r_2+n} e_u\left[[a]_t X\right] e_{r_1+r_2+n-u}\left[q^{-a}[c]_t X\right] \\
& + \sum_{r_1,r_2\geq 0} F^{(r_1,r_2)}[X](-1)^{r_1+r_2+n} e_{r_1+b}\left[[a]_t X\right] e_{r_2+d}\left[[c]_t X\right] \\
= \; & q^{ad}\sum_{r_1,r_2\geq 0} F^{(r_1,r_2)}[X]q^{ar_2}(-1)^{r_1+r_2+n} e_{r_1+r_2+n}\left[[m]_t X\right] \\
& + \sum_{r_1,r_2\geq 0} F^{(r_1,r_2)}[X](-1)^{r_1+r_2+n} e_{r_1+b}\left[[a]_t X\right] e_{r_2+d}\left[[c]_t X\right] \\
= \; & q^{ad}\sum_{r_1,r_2\geq 0} F^{(r_1,r_2)}[X]\frac{q^{ar_2}}{z^{r_1+r_2}}\Omega\left[-[m]_t X\right]\Big|_{z^n} \\
& + \sum_{r_1,r_2\geq 0} F^{(r_1,r_2)}[X](-1)^{r_1+r_2+n} e_{r_1+b}\left[[a]_t X\right] e_{r_2+d}\left[[c]_t X\right] \\
= \; & q^{ad}F\left[X+\frac{M[m]_q}{z}\right]\Omega\left[-[m]_t X\right]\Big|_{z^n} \\
& + \sum_{r_1,r_2\geq 0} F^{(r_1,r_2)}[X](-1)^{r_1+r_2+n} e_{r_1+b}\left[[a]_t X\right] e_{r_2+d}\left[[c]_t X\right].
\end{aligned}$$

With an entirely analogous sequence of steps we obtain the identity

$$\begin{aligned}
B+\widetilde{A} \;=\; & q^{cb}F\left[X+\frac{M[m]_q}{z}\right]\Omega\left[-[m]_t X\right]\Big|_{z^n} \\
& + \sum_{r_1,r_2\geq 0} F^{(r_1,r_2)}[X](-1)^{r_1+r_2+n} e_{r_2+d}\left[[c]_t X\right] e_{r_1+b}\left[[a]_t X\right].
\end{aligned}$$

Thus (3.2.12) finally yields

$$\begin{aligned}
& \frac{(1-q^m)\left[D_{c,d},D_{a,b}\right]}{(1-q^a)(1-q^c)}\left[D_{c,d},D_{a,b}\right]F[X] \\
& \qquad\qquad = (q^{ad}-q^{cb})F\left[X+\frac{M[m]_q}{z}\right]\Omega\left[-[m]_t X\right]\Big|_{z^n}
\end{aligned}$$

proving the identity

$$\frac{1}{M}\left[D_{c,d},D_{a,b}\right]F[X] \;=\; \frac{q[a]_q[c]_q}{[m]_q}\frac{q^{cb}-q^{ad}}{1-q}F\left[X+\frac{M[m]_q}{z}\right]\Omega\left[-[m]_t X\right]\Big|_{z^n},$$

which is just another way of writing (3.2.9). $\square$

In the remainder of this section we will derive a number of immediate consequences of the identity in (3.2.9). We will state and prove them as a succession of Corollaries, the last two of which are simply restatements of Theorems 3.1.1 and 3.1.2. For convenience we will here and after use the symbol "$Q^s_{u,v}$" as a short hand for "$Q_{u,v}\Big|_{t=1/q}$." We will start with an auxiliary identity that we will use on various occasions.

Lemma 3.2.1. *For integers a, b, c, d, we have*

$$\frac{(a-1)(b+1)}{2} + \frac{(c-1)(d+1)}{2} + bc + 1 = \frac{(a+c-1)(b+d+1)}{2} + \frac{bc-ad+1}{2}.$$

Corollary 3.2.1. *If m and n are relatively prime, then we have*

$$Q^s_{m,n} = q^{(n+1)(m-1)/2} D_{m,n} / \left[m\right]_q.$$

Proof. We will proceed by induction on m. The base case $m = 1$ is easy: we have $Q^s_{1,n} = D_n\Big|_{t=1/q} = D_{1,n}$, so the corollary holds in this case.

It is best to start by an example. Let us consider $Q^s_{2,n} = \frac{1}{M}\left[Q^s_{1,d}, Q^s_{1,b}\right]$, where $\mathrm{Split}(2,n) = (1,b) + (1,d)$, with $d - b = 1$. Then by (3.2.9)

$$\begin{aligned} Q^s_{2,n} &= \frac{1}{M}\left[D_{1,d}, D_{1,b}\right] = q^{b+1}\frac{[1]_q\,[1]_q}{[2]_q}\frac{(1-q^{d-b})}{1-q}D_{2,n} \\ &= q^{(m-1)(n+1)/2}\frac{1}{[2]_q}D_{2,n}. \end{aligned}$$

One more example: Let us say $\mathrm{Split}(3,n) = (1,b) + (2,d)$ with $d - 2b = 1$. Then again by (3.2.9)

$$\begin{aligned} Q^s_{3,n} &= \frac{1}{M}\left[Q^s_{2,d}, Q^s_{1,b}\right] = q^{(d+1)/2}\frac{1}{[2]_q}\left[D_{2,d}, D_{1,b}\right] = q^{(d+1)/2}q^{2b+1}\frac{1}{[3]_q}D_{3,n} \\ &= q^{(n+1)}\frac{1}{[3]_q}D_{3,n}. \end{aligned}$$

Assume the corollary holds for smaller m. Now suppose

$$\mathrm{Split}(m,n) = (a,b) + (c,d).$$

That is we have $a + c = m, b + d = n, ad - bc = 1$. Then

$$\begin{aligned}
Q^s_{m,n} &= \frac{1}{M}\left[Q^s_{c,d}, Q^s_{a,b}\right] \\
&= q^{(c-1)(d+1)/2+(a-1)(b+1)/2}\frac{1}{[a]_q\,[c]_q}\frac{1}{M}\left[D_{c,d}, D_{a,b}\right] \\
&= q^{(c-1)(d+1)/2+(a-1)(b+1)/2}\cdot q^{bc+1}\frac{1}{[a+c]_q}\frac{1-q^{ad-bc}}{1-q}D_{a+c,b+d} \\
&= q^{(c-1)(d+1)/2+(a-1)(b+1)/2+bc+1}\frac{1}{[m]_q}D_{m,n} \\
&= q^{(m-1)(n+1)/2}\frac{1}{[m]_q}D_{m,n}.
\end{aligned}$$

This completes the induction and the proof. $\square$

Corollary 3.2.2. *For any coprime pair (m, n) we have*

$$q^{(m-1)(n-1)/2}Q^s_{m,n}(-1)^n = \frac{1}{[m]_q}\,e_n\left[X\,[m]_q\right].$$

Proof.

$$\begin{aligned}
Q^s_{m,n}(-1)^n &= q^{(m-1)(n+1)/2}\frac{1}{[m]_q}D_{m,n}(-1)^n \\
&= q^{(m-1)(n+1)/2}\frac{1}{[m]_q}(-1)^n\Omega{-zX}\,[m]_t z^{-n}\Big|_{z^0} \\
&= q^{(m-1)(n+1)/2}\frac{1}{[m]_q}e_n\left[X\,[m]_t\right] \\
&= q^{(m-1)(n+1)/2}q^{-(m-1)n}\frac{1}{[m]_q}e_n\left[X\,[m]_q\right] \\
&= q^{-(m-1)(n-1)/2}\frac{1}{[m]_q}e_n\left[X\,[m]_q\right].
\end{aligned}$$

$\square$

Corollary 3.2.3. *For any coprime pair (m, n) and $k \geq 1$ we have*

$$Q^s_{km,kn} = q^{(km-1)(kn+1)/2-(k-1)/2}\frac{[k]_q}{[km]_q}D_{km,kn}.$$

Proof. Suppose $\mathrm{Split}(m, n) = (a, b)+(c, d)$. That is $a+c = m, b+d = n, ad-bc = 1$. It should be clear that since a and b are also relatively prime, we can choose $Split(km, kn) = (a, b) + ((k - 1)a + kc, (k - 1)b + kd)$ so that it is easily obtained by linear algebra that

$$\det\begin{bmatrix} a & (k-1)a+kc \\ b & (k-1)b+kd \end{bmatrix} = k \text{ and } \det\begin{bmatrix} a+c & (k-1)a+kc \\ b+d & (k-1)b+kd \end{bmatrix} = 1.$$

Thus $c' = (k-1)a+kc$ and $d' = (k-1)b+kd$ are relatively prime. We have

$$\begin{aligned} Q^s_{km,kn} &= \frac{1}{M}\left[Q^s_{(k-1)a+kc,(k-1)b+kd}, Q^s_{a,b}\right] \\ &= q^{(c'-1)(d'+1)/2+(a-1)(b+1)/2}\frac{1}{[a]_q\,[c']_q}\frac{1}{M}\left[D_{c',d'}, D_{a,b}\right] \\ &= q^{(c'-1)(d'+1)/2+(a-1)(b+1)/2}\cdot q^{bc'+1}\frac{1}{[a+c']_q}\frac{1-q^{ad'-bc'}}{1-q}D_{a+c',b+d'} \\ &= q^{(c'-1)(d'+1)/2+(a-1)(b+1)/2+bc'+1}\frac{[k]_q}{[km]_q}D_{km,kn} \\ &= q^{(km-1)(kn+1)/2-(k-1)/2}\frac{[k]_q}{[km]_q}D_{km,kn}. \end{aligned}$$

□

Corollary 3.2.4. *For any coprime pair (m,n) and $k \geq 1$ we have*

$$Q^s_{km,kn}(-1)^{kn} = q^{-(km-1)(kn-1)/2-(k-1)/2}\frac{[k]_q}{[km]_q}e_n\left[X\,[km]\right].$$

Proof. We have

$$\begin{aligned} Q^s_{km,kn}(-1)^{kn} &= q^{(km-1)(kn+1)/2-(k-1)/2}\frac{[k]_q}{[km]_q}D_{km,kn}(-1)^{kn} \\ &= q^{(km-1)(kn+1)/2-(k-1)/2}\frac{[k]_q}{[km]_q}D_{km,kn}(-1)^{kn}(-1)^{kn}\Omega{-zX\,[km]_t}\Big|_{z^{kn}} \\ &= q^{(km-1)(kn+1)/2-(k-1)/2}\frac{[k]_q}{[km]_q}e_n\left[X\,[km]_t\right] \\ &= q^{(km-1)(kn+1)/2-(k-1)/2}q^{-(km-1)n}\frac{[k]_q}{[km]_q}e_n\left[X\,[km]_q\right] \\ &= q^{-(km-1)(kn-1)/2-(k-1)/2}\frac{[k]_q}{[km]_q}e_n\left[X\,[km]_q\right]. \end{aligned}$$

□

3.3 Polynomiality and positivity.

The main goal of this section is to prove that the quotient

$$\frac{[k]_q}{[km]_q} e_{kn}\left[X\,[km]_q\right] \tag{3.3.1}$$

is a Schur positive symmetric polynomial. This will be obtained by combining the next four auxiliary propositions.

The first fact on which the proof is based is the following classical result.

Proposition 3.3.1. *For any $n \geq 1$ we have to matrices $\|c_{\lambda,\rho}\|_{\lambda,\rho\vdash n}$ and $\|d_{\lambda,\rho}\|_{\lambda,\rho\vdash n}$ such that*

$$a)\ s_\lambda = \sum_{\rho\vdash n} c_{\lambda,\rho} p_\rho \quad \textit{and} \quad b)\ p_\rho = \sum_{\lambda\vdash n} d_{\lambda,\rho} s_\lambda. \tag{3.3.2}$$

Proof. Frobenius proved that (3.3.2) a) and b) hold with

$$a)\ c_{\lambda,\rho} = \chi^\lambda_\rho / z_\rho \quad \text{and} \quad d)\ d_{\lambda,\rho} = \chi^\lambda_\rho \tag{3.3.3}$$

where χ^λ_ρ is the value of Young's irreducible S_n character indexed by λ at the conjugacy class indexed by ρ and where, for $\rho = 1^{\alpha_1}2^{\alpha_2}\dots n^{\alpha_n}$, we have

$$z_\rho = 1^{\alpha_1}2^{\alpha_2}\cdots n^{\alpha_n}\alpha_1!\alpha_2!\cdots\alpha_n!. \qquad \square$$

This given, our polynomiality result can be stated as follows.

Proposition 3.3.2. *If $(m,n) = 1$ and $k \geq 1$ then for all $\lambda \vdash kn$ we have*

$$\frac{[k]_q}{[km]_q} s_\lambda\left[[km]_q\right] \in \mathbb{Q}[q] \tag{3.3.4}$$

where for any integer $s \geq 0$ we set $[s]_q = 1 + q + \cdots + q^{s-1}$.

Proof. Note that to show (3.3.4) we need only show that every root of

$$1 + q + q^2 + \cdots + q^{km-1} \;=\; 0 \tag{3.3.5}$$

is a root of the polynomial

$$[k]_q\, s_\lambda\left[[km]_q\right].$$

To show this we use (3.3.2) (for $n \to kn$) and write for ζ a root of (3.3.5).

$$[k]_\zeta \, s_\lambda \left[[km]_\zeta\right] = \sum_{\rho \vdash kn} c_{\lambda,\rho} \, [k]_\zeta \, p_\rho \left[[km]_\zeta\right] \tag{3.3.6}$$

Thus if the left hand side of (3.3.6) does not vanish we will necessarily have a $\rho \vdash kn$ such that

$$[k]_\zeta \, p_\rho \left[[km]_\zeta\right] \neq 0. \tag{3.3.7}$$

In particular if for some r we have $\rho_r = i$ then

$$1 + \zeta^i + \zeta^{2i} + \cdots \zeta^{(km-1)i} \; = \; p_i \left[[km]_\zeta\right] \neq 0. \tag{3.3.8}$$

Since

$$(1 - q^i) \left(1 + q^i + q^{2i} + \cdots q^{(km-1)i}\right) \; = \; 1 - q^{(km)i}$$

from (3.3.8) and ζ is a root of (3.3.5), we derive that

$$a) \;\; \zeta^i = 1 \quad \text{and} \quad b) \;\; \zeta \neq 1. \tag{3.3.9}$$

Now note that since the ρ in (3.3.7) may be written in the form $\rho = \prod_{i=1}^{kn} i^{\alpha_i}$ with $\sum_{i=1}^{kn} i\alpha_i = kn$ then we must also have

$$\zeta^{kn} = \prod_{i=1}^{kn} (\zeta^i)^{\alpha_i} \; = \; 1.$$

Now the assumed coprimality of the pair (m, n) gives that $k = gcd(km, kn)$ and this combined with the fact that ζ is a root of (3.3.5) forces

$$\zeta^k = 1.$$

But then b) of (3.3.9) yields

$$1 + \zeta + \zeta^2 + \cdots + \zeta^{k-1} = 0$$

which is in plain contradiction with (3.3.7). This contradiction forces every root of (3.3.5) to be a root of

$$[k]_q \, s_\lambda \left[[km]_q\right] \; = \; 0$$

as desired. $\square$

The next device we use is the following well known fact

Proposition 3.3.3. *Any principal evaluation of a Schur Function is unimodal. More precisely for any $\lambda \vdash n$ and any $m > 1$ the polynomial*

$$s_\lambda \left[[m]_q \right] \tag{3.3.10}$$

is unimodal.

Proof. This is exercise 4 page 137 of Macdonald's book [Mac95]. Since the solution in [Mac95] is only sketched, for the sake of completeness we carry out Macdonald's exercise in full detail. Macdonald considers the evaluation

$$s_\lambda \left[\frac{x_1^m - x_2^m}{x_1 - x_2} \right] \;=\; s_\lambda \left[s_{m-1} \left[x_1 + x_2 \right] \right] \tag{3.3.11}$$

as a character of $GL_2 [\mathbb{C}]$. Using this he derives that for some weakly positive integer constants c_{r_1,r_2} we must have the expansion

$$s_\lambda \left[s_{m-1} \left[x_1 + x_2 \right] \right] = \sum_{r_1+r_2=d,\, r_1 \geq r_2} c_{r_1,r_2} s_{[r_1,r_2]} \left[x_1 + x_2 \right] \tag{3.3.12}$$

where for convenience we have set

$$d = (m-1)n. \tag{3.3.13}$$

Notice that we have

$$\begin{aligned} s_{[r_1,r_2]} \left[x_1 + x_2 \right] \;&=\; x_1^{r_1} x_2^{r_2} + x_1^{r_1-1} x_2^{r_2+1} + \cdots + x_1^{r_2+1} x_2^{r_1-1} + x_1^{r_2} x_2^{r_1} \\ &=\; x_2^d \left((\frac{x_1}{x_2})^{r_1} + (\frac{x_1}{x_2})^{r_1-1} + \cdots + (\frac{x_1}{x_2})^{r_2} \right) \;=\; x_2^d \sum_{s=r_2}^{r_1} q^s \end{aligned}$$

where for convenience we have set $\frac{x_1}{x_2} = q$. Thus with a slight change of notation we may rewrite (3.3.12) in the form

$$\begin{aligned} s_\lambda \left[s_{m-1} \left[x_1 + x_2 \right] \right] &= x_2^d \sum_{r_2=0}^{\lfloor d/2 \rfloor} c_{r_2} \sum_{s=0}^{d} q^s \chi \left(r_2 \leq s \leq d - r_2 \right) \\ &= x_2^d \sum_{s=0}^{d} q^s \sum_{r_2=0}^{s \wedge (d-s)} c_{r_2} \end{aligned} \tag{3.3.14}$$

from which the unimodality assertion immediately follows by setting $x_2 = 1$ and $x_1 = q$. $\square$

Our positivity result is a consequence of the following simple but powerful fact.

Proposition 3.3.4. *Let $g(q) = b_0 + b_1 q + \cdots + b_r q^r$ and assume that, for $d = r + s$, the polynomial*

$$f(q) = (1 + q + \cdots + q^s)g(q) = \sum_{l=0}^{d} c_l\, q^l \tag{3.3.15}$$

is unimodal with peak at p and non-negative coefficients. Then $g(q)$ also has non-negative coefficients.

Proof. We proceed by contradiction. Suppose some of the coefficients of $g(b)$ are negative. Since $c_0 = c_d \geq 0$ let b_i and b_j (with $0 < i \leq j < d$) be the leftmost and rightmost negative coefficients of $g(q)$ respectively. Now if $i \leq p$ then

$$c_i = \sum_{u=0\vee(i-s)}^{i} b_u \quad \text{and} \quad c_{i-1} = \sum_{u=0\vee(i-1-s)}^{i-1} b_u.$$

This gives

$$c_i - c_{i-1} \;=\; b_i \;-\; \chi\,(i - 1 - s > 0)\; b_{i-1-s} < 0,$$

a contradiction!

If $r - j \leq d - p$ then do the same argument for $\tilde{f}(q) = q^d f(1/q)$ and $\tilde{g}(q) = q^r g(1/q)$. So we are left with $i > p$ and $r - j > d - p$. But that cannot happen since it implies that $i > s + j > j$. □

As a corollary we obtain our desired goal.

Theorem 3.3.1. *For any coprime pair (m, n) and any $k \geq 1$ we have that the symmetric function*

$$\frac{[k]_q}{[km]_q} e_{kn}\left[X\,[km]_q\right] \tag{3.3.16}$$

is a Schur positive symmetric polynomial

Proof. The Cauchy formula gives

$$\frac{[k]_q}{[km]_q} e_{kn}\left[X\,[km]_q\right] \;=\; \sum_{\lambda\vdash kn} s_{\lambda'}[X]\,\frac{[k]_q\, s_\lambda\left[[km]_q\right]}{[km]_q}.$$

Now we have proved that $[k]_q\, s_\lambda \left[[km]_q\right]$ is divisible by $[km]_q$, we have also proved that $s_\lambda \left[[km]_q\right]$ is palindromic unimodal. Thus it follows from this that also $[k]_q\, s_\lambda \left[[km]_q\right]$ is palindromic unimodal. We can then apply Proposition 3.3.4 with $f(q) = [k]_q\, s_\lambda \left[[km]_q\right]$ and

$$g(q) \;=\; \frac{[k]_q\, s_\lambda \left[[km]_q\right]}{[km]_q}$$

and conclude that $g(q) \in \mathbb{N}[q]$, proving the Schur positivity of the polynomial in (3.3.16). $\square$

Remark 3.3.1. *The polynomiality of the symmetric function in (3.3.16) for the special case $k = 1$ was first proved by Mark Haiman in [Hai94]. The question arose from the discovery in [GH96a] that in the case $m = n + 1$ (3.3.16) (for $k = 1$) is the Frobenius characteristic of an appropriate single grading of the Diagonal Harmonic Module of S_n. This also prompted Haiman to look for some reason that justified this polynomial to be Schur positive. With remarkable foresight Haiman investigates the more general (m, n) case and provided a mechanism for proving that the symmetric function*

$$\frac{e_n \left[X\, [m]_q\right]}{[m]_q} \tag{3.3.17}$$

is a polynomial and Schur positive if and only if (m, n) is a coprime pair. However, the Schur positivity was shown in [Hai94] by constructing a quotient

$$\mathbb{Q}\left[x_1, x_2, \ldots, x_n\right] / (e_1, f_2, \ldots, f_n)$$

with Frobenius characteristic the polynomial in (3.3.17), where e_1 is the ordinary elementary symmetric function and $f_1, f_2, \ldots, f_n$ is a sequence of polynomials satisfying the following properties.

1. *Each f_i is homogeneous of degree m,*

2. *they satisfy the identities $\sigma f_i = f_{\sigma_i}$ (for $1 \le i \le n$ and all $\sigma \in S_n$),*

3. *$f_1 + f_2 + \cdots + f_n = 0$ and*

4. $e_1, f_2, \ldots . f_n$ *are a regular sequence.*

A sequence f_i *satisfying (1), (2), (3), and (4) was constructed by Hanspeter Kraft a few years later but never published. More recently a very natural example of such a sequence was discovered by Dunkl in [Dun98] and used later by Gorsky in his work [Gor13] on torus knots invariants. The Schur positivity of the polynomial in (3.3.17) in the coprime case follows from this and Mark Haiman's result. The challenge now is to construct an equally natural quotient with Frobenius characteristic equal to the polynomial in (3.3.16).*

3.4 A parking function setting for our Frobenius characteristics

The main goal of this section is a proof of Theorem 3.1.4. To carry this out we need to briefly review the statement of the Rational Compositional Shuffle Conjecture. We will start by introducing the symmetric function tools that are used in its formulation.

The basic ingredient here is the identity

$$Q_{0,k} \;=\; \frac{qt}{qt-1}\, \underline{h}_k\, [X(1/qt-1)] \quad \text{(for all integers } k \geq 1). \tag{3.4.1}$$

From this it follows that a basis for the subspace of operators of bi-degrees $(0, n)$ for $n \geq 1$ is given by the collection

$$\{Q_{0,\lambda}\}_\lambda \;=\; \left\{ \prod_{i=1}^{l(\lambda)} Q_{0,\lambda_i} \right\}_\lambda . \tag{3.4.2}$$

This fact (see [BGLX15] for an elementary treatment) can be used to construct an operator of bi-degree (km, kn) for any coprime pair (m, n), any integer $k \geq 1$ and any given homogeneous symmetric function $F[X]$ of degree k, by the following two steps.

1. *Compute the expansion*

$$F \;=\; \sum_{\lambda \vdash k} c_\lambda \prod_{i=1}^{l(\lambda)} Q_{0,\lambda_i}, \tag{3.4.3}$$

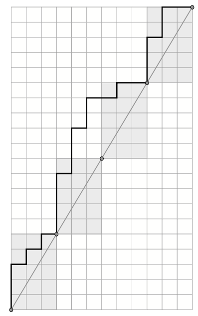

Figure 3.1: A path in the 12×20 lattice rectangle.

2. *and then set*

$$\mathbf{F}_{km,kn} = \sum_{\lambda \vdash k} c_\lambda \prod_{i=1}^{l(\lambda)} Q_{\lambda_i m, \lambda_i n}. \tag{3.4.4}$$

The commutativity of the operators Q_{u_1,v_1} and Q_{u_2,v_2} with (u_1, v_1) and (u_2, v_2) collinear vectors assures that (3.4.4) well defines the operator $\mathbf{F}_{km,kn}$. In [BGLX15] a variety of Shuffle conjectures were formulated based on the above construction and various choices of the symmetric function $F[x]$. The simplest one corresponds to choosing $F = e_k$ (the k^{th} elementary symmetric function). The corresponding operator which we denote $\mathbf{e}_{km,kn}$ has a truly remarkable connection to the Theory of parking functions.

A single example will suffice to get across this connection. We have displayed in Figure 3.1 a 12×20 lattice rectangle with a path that proceeds by North and East unit steps, always remaining weakly above the diagonal $(0, 0) \to (12, 20)$. A parking function in the 12×20 lattice rectangle is obtained by labeling the cells immediately to the right of the north steps of such a path by the integers $1, 2, \ldots, 20$ (referred to as *cars*) in a column increasing manner. This given, one of the conjectures formulated in [BGLX15] may expressed as the identity

$$\mathbf{e}_{km,kn}(-1)^{k(n+1)} = \sum_{PF \in \mathcal{P}F_{km,kn}} t^{\text{area}(PF)} q^{\text{dinv}(PF)} F_{\text{pides}(PF)} \tag{3.4.5}$$

where the sum is over all parking functions in the $km \times kn$ lattice rectangle, and the parking function statistics occurring in (3.4.5) are as will be defined shortly. Here the symbol "$F_{\text{pides}(PF)}$" stands for the Gessel's [Ges84] fundamental quasi-symmetric function indexed by the composition "pides(PF)".

Now the Extended Compositional Shuffle Conjecture states a refined version of (3.4.5) where the symmetric function F in the above construction of the operator $\mathbf{F}_{km,kn}$, is chosen to assure that the sum in (3.4.5) is restricted to be over an appropriately selected subfamily of parking functions in the $km \times kn$ lattice rectangle. These special choices of F are obtained by means of the modified Hall-Littlewood operators C_a whose action on a symmetric function $F[x]$ is defined by setting

$$C_a P[X] = (-\frac{1}{q})^{a-1} P\left[X - \frac{1-1/q}{z}\right] \sum_{m\geq 0} z^m h_m[X] \Big|_{z^a} \tag{3.4.6}$$

Again, a single example will suffice to illustrate our choices. Suppose that we want to restrict the sum in (3.4.5) to be carried out only over the parking functions whose supporting Dyck path hits the diagonal precisely at the first and third and fourth possible places, as the path depicted in the above display. To achieve this we simply choose the symmetric function $F = C_1 C_2 C_1\, 1$. More generally, given a composition $p = (p_1, p_2, \ldots, p_l)$ of the integer k, let us denote by $\mathbb{C}^{(p)}_{km,kn}$ the operator obtained by choosing $F = C_{p_1} C_{p_2} \cdots C_{p_l}\, 1$ in the above construction. This given, the Extended Compositional Shuffle Conjecture states that

$$\mathbb{C}^{(p)}_{km,kn}(-1)^{k(n+1)} \;=\; \sum_{PF\in\mathcal{P}F_{km,kn}(p)} t^{\mathrm{area}(PF)} q^{\mathrm{area}(PF)} F_{\mathrm{pides}(PF)} \tag{3.4.7}$$

where the sum is over parking functions whose path hits the diagonal precisely in l of the k possible places, as prescribed by the parts of the composition $p = (p_1, p_2, \ldots, p_l)$. The reason (3.4.7) refines (3.4.6) is due to the remarkable identity

$$e_k \;=\; \sum_{p\models k} C_{p_1} C_{p_1} \cdots C_{p_{l(p)}}\, 1.$$

Here the sum is over all compositions of k

Keeping all this in mind, we are now in a position to show that the identity in (3.1.16) is one of the many consequences of the identity in (3.4.7). More precisely we will show that Theorem 3.1.4 is a corollary of the following stronger result.

Theorem 3.4.1. *The validity of (3.4.7) for any coprime pair (m, n), $k \geq 1$ and any composition $p \models k$ implies the identity*

$$Q_{km,kn}(-1)^{kn} = \sum_{PF\in\mathcal{P}F_{km,kn}} t^{\mathrm{area}(PF)-\mathrm{ret}(PF)+1} \left[\mathrm{ret}(PF)\right]_t q^{\mathrm{dinv}(PF)} F_{\mathrm{pides}(PF)} \tag{3.4.8}$$

where all the parking function statistics are as in the Extended Shuffle Conjectures. The "$\mathrm{ret}(PF)$*" statistic gives the smallest positive i such that the supporting path of PF goes through the point (im, in).*

Proof. For brevity we will start with the following identity, valid for any integer $1 \le d \le k$ and $p \models k-d$

$$\sum_{p\models k-d;} C_d C_{p_1} C_{p_2} \cdots C_{p_{l(p)}} \mathbf{1} \;=\; (-\frac{1}{q})^{d-1} s_{d,1^{k-d}} - (-\frac{1}{q})^{d}\, s_{1+d,1^{k-d-1}} \tag{3.4.9}$$

(see [HMZ12] for a proof). Note next that it follows from (3.4.9) that

$$\sum_{d=a}^{k} \sum_{p\models k-d;} C_d C_p\ 1 = \sum_{d=a}^{k} (-\frac{1}{q})^{d-1} s_{d,1^{k-d}} - \sum_{d=a+1}^{k} (-\frac{1}{q})^{d-1}\, s_{d,1^{k-d}} = (-\frac{1}{q})^{a-1} s_{a,1^{k-a}} \tag{3.4.10}$$

Our next step is to rewrite (3.4.1) in a more suitable form. Now a use of the Cauchy identity gives

$$\begin{aligned} Q_{0,k} &= \frac{(qt)^{1-k}}{qt-1}\, \underline{h}_k\,[X(1-qt)] = \frac{(qt)^{1-k}}{qt-1} \sum_{\mu\vdash k} \underline{s}_\mu[X]\underline{s}_\mu\,[1-qt] \\ &= \frac{(qt)^{1-k}}{qt-1} \sum_{r=0}^{k-1} \underline{s}_{k-r,1^r}[X](-qt)^r(1-qt) = -(qt)^{1-k} \sum_{r=0}^{k-1} \underline{s}_{k-r,1^r}[X](-qt)^r \\ &= (-1)^k \sum_{a=1}^{k} \underline{s}_{a,1^{k-a}}[X](-qt)^{1-a}, \end{aligned} \tag{3.4.11}$$

where for the third equality we use the following, easily verified, special evaluation of a Schur function

$$s_\mu\,[1-m] \;=\; \begin{cases} (-m)^a(1-m) & \text{if } \mu = k-r, 1^r \text{ for } 0 \le r \le k-1 \\ 0 & \text{otherwise,} \end{cases} \tag{3.4.12}$$

valid for any monomial m.

Hence by combining (3.4.10) and (3.4.11) we derive that

$$\begin{aligned} (-1)^k Q_{0,k} &= \sum_{a=1}^{k} (-qt)^{1-a} s_{a,1^{k-a}} = \sum_{a=1}^{k} t^{1-a} \sum_{d=1}^{k} \chi(d \ge a) \sum_{p\models k-d} C_d C_p 1 \\ &= \sum_{d=1}^{k} \sum_{p\models k-d} C_d C_p 1 \sum_{a=1}^{d} (1/t)^{a-1} = \sum_{d=1}^{k} \sum_{p\models k-d} [d]_{1/t}\, C_d C_p 1. \end{aligned} \tag{3.4.13}$$

This given, the particular case $F = Q_{0,k}$ of the above construction gives that for any coprime pair (m, n) we have

$$\mathbf{Q_{0,k}}_{km,kn} = Q_{km,kn}. \tag{3.4.14}$$

Likewise by choosing $F = C_d C_p 1$ for $p \models k-d$ we obtain the operator $\mathbf{C_{d,p}1}_{km,kn}$ which, by the Rational Shuffle Compositional conjecture, satisfies

$$\mathbf{C_{d,p}1}_{km,kn}(-1)^{k(n+1)} = \sum_{PF \in \mathcal{PF}_{km,kn}(d)} t^{\mathrm{area}(PF)} q^{\mathrm{dinv}(PF)} F_{\mathrm{pides}(PF)} \tag{3.4.15}$$

where the sum is over parking functions in the $km \times kn$ rectangle whose supporting path returns to the diagonal for the first time in row dn. Thus combining (3.4.15) with (3.4.14) and (3.4.13) we obtain that

$$Q_{km,kn}(-1)^{kn} = \sum_{d=1}^{k} \sum_{p \models k-d} [d]_{1/t} \sum_{PF \in \mathcal{PF}_{km,kn}(d)} t^{\mathrm{area}(PF)} q^{\mathrm{dinv}(PF)} F_{\mathrm{pides}(PF)}$$

which is only another way of writing (3.4.8). $\square$

We can now finally give our

Proof of Theorem 3.1.4. We will show here the identity in (3.1.16), written in the form

$$\frac{[k]_q}{[km]_q} e_{kn}\left[X\, [km]_q\right] = \sum_{PF \in \mathcal{PF}_{km,kn}} q^{\mathrm{coarea}(PF)+\mathrm{dinv}(PF)} \left[\mathrm{ret}(PF)\right]_q F_{\mathrm{pides}(PF)}[X] \tag{3.4.16}$$

To this end, notice that setting $t = 1/q$ in (3.4.8) gives

$$\begin{aligned} &Q_{km,kn}(-1)^{kn}\Big|_{t=1/q} \\ &= \sum_{PF \in \mathcal{PF}_{km,kn}} q^{\mathrm{ret}(PF)-\mathrm{area}(PF)-1} \left[\mathrm{ret}(PF)\right]_q q^{-(\mathrm{ret}(PF)-1)} q^{\mathrm{dinv}(PF)} F_{\mathrm{pides}(PF)} \end{aligned} \tag{3.4.17}$$

But we can now use (3.1.14) which can be rewritten in the form

$$\frac{[k]_q}{[km]_q} e_{kn}\left[X\, [km]_q\right] = q^{(km-1)(kn-1)/2+(k-1)/2} Q_{km,kn}(-1)^{kn}\Big|_{t=1/q} \tag{3.4.18}$$

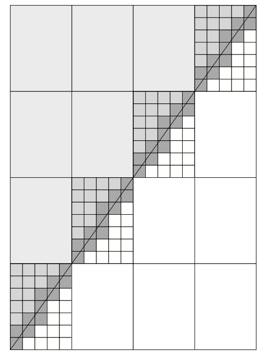

Figure 3.2: The 20×28 lattice rectangle and main diagonal.

This given, a comparison of the combination of (3.4.18) and (3.4.17) with (3.4.16) shows that we need only to show the equality

$$\operatorname{coarea}(PF) + \operatorname{area}(PF) = (km-1)(kn-1)/2 + (k-1)/2 \tag{3.4.19}$$

This can be easily justified by the following geometric argument. We have depicted in Figure 3.2 the $km \times kn$ lattice rectangle for $k = 4$ and $(m,n) = (5,7)$. Now by the definition $\operatorname{area}(PF)$ gives the number of lattice cells below the supporting path of PF and weakly above the diagonal $(0,0) \to (km,kn)$ and $\operatorname{coarea}(PF)$ gives the number of lattice cells above the path. Thus to show (3.4.19) we need only verify that the right hand side gives the number of lattice cells weakly above the diagonal $(0,0) \to (km,kn)$. Now it is easy to see from the display that the number of lattice cells cut by the diagonal in any one of the diagonal 4×7 blocks is by one short of $4+7$. This implies that the number of uncut lattice cells above the diagonal within each diagonal block is none other than $(mn - m - n + 1)/2 = (m-1)(n-1)/2$. Moreover, the total number of lattice cells within the upper non-diagonal blocks is $mn \times \binom{k}{2}$. Thus (3.4.19) is none other than a consequence of the equality

$$k\frac{(m-1)(n-1)}{2} + mn\binom{k}{2} = \frac{(km-1)(kn-1)}{2} + \frac{k-1}{2}.$$

This completes our proof of (3.4.16). $\square$

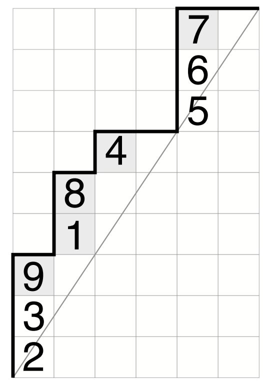

Figure 3.3: A $6,9$-parking function.

In the remainder of this section we will present the latest version of the parking function statistics that occur in the various formulations of the Rational Compositional Shuffle Conjecture.

Let (m,n) be coprime pair of positive integers and let $k \geq 1$. Recall that a km,kn-Dyck path is sequence of north and east steps in the $km \times kn$ lattice rectangle which starts in the southwest corner, ends in the northeast corner, and stays weakly above the main diagonal $y = \frac{n}{m}x$. A km,kn-parking function is a km,kn-Dyck path with labels $\{1,2,\ldots,kn\}$, known as cars, adjacent to north steps and increasing within each column. For example, see Figure 3.3. Let $\mathcal{P}F_{km,kn}$ denote the set of all km,kn-parking functions.

The original Rational Shuffle Conjecture of [GN15] states that for coprime m,n, we can express $Q_{m,n}(-1)^n$ as a weighted sum of m,n-parking functions. This enumeration involves the statistics $\text{area}(PF)$, $\text{dinv}(PF)$, and the word $\sigma(PF)$. The simplest of these is $\text{area}(PF)$, which is the number of full cells between the path and the main diagonal $y = \frac{n}{m}x$. In the adjacent example, the area is 5 and the corresponding cells are shaded.

The dinv and word statistics both make use of a rank function defined by $\text{rank}(x,y) = kmy - knx + \lfloor \frac{x}{m} \rfloor$. This causes the points weakly above the diagonal to have distinct nonnegative ranks, with points further from the main diagonal having higher rank. This way, the rank function generalizes the notion of diagonals from classical parking function theory [HHL$^+$05]. Let the rank of a car be the rank of the southwest corner of that car's cell. Then the word, $\sigma(PF)$, is just the permutation of $\{1,2,\ldots,kn\}$ obtained by listing the cars from highest to lowest rank. In the example above, the ranks of cars 1 through 9 are 9, 0, 6, 13, 2, 8, 14, 15, and 12, respectively. Hence the word of that parking function is $\sigma(PF) = 8\,7\,4\,9\,1\,6\,3\,5\,2$.

We set

$$\mathrm{tdinv}(PF) = \sum_{\text{cars } i<j} \chi(\mathrm{rank}(i) < \mathrm{rank}(j) < \mathrm{rank}(i) + km). \tag{3.4.20}$$

Here *tdinv* is short for "temporary dinv" because we will modify this statistic to obtain $\mathrm{dinv}(PF)$. In the example above, $\mathrm{tdinv}(PF) = 9$ because the inequalities in (3.4.20) are satisfied for the pairs $(1,4)$, $(1,7)$, $(1,9)$, $(2,5)$, $(3,6)$, $(4,7)$, $(4,8)$, $(6,9)$ and $(7,8)$.

The original formulation of Hikita [Hik14] as modified by Gorsky-Mazin [GM13], [GM14] expressed the dinv statistic as a combination of tdinv and two other statistics. However, Hicks and Leven [HL15] showed that this can be simplified as follows. Let $\lambda(PF)$ be partitions whose english Ferrers diagram is formed by the cells above PF. In the example above, $\lambda(PF) = (4,4,4,2,1,1)$. This given, for an m,n-Parking Function we set

$$\begin{aligned}\mathrm{dinv}(PF) &= \mathrm{tdinv}(PF) - \#\left\{c \in \lambda(PF) : \frac{arm(c)}{leg(c)} \le \frac{m}{n} < \frac{arm(c)+1}{leg(c)+1}\right\} \\ &\qquad \text{if } m < n \text{ and} \\ &= \mathrm{tdinv}(PF) + \#\left\{c \in \lambda(PF) : \frac{arm(c)}{leg(c)} > \frac{m}{n} \ge \frac{arm(c)+1}{leg(c)+1}\right\} \\ &\qquad \text{if } m > n.\end{aligned} \tag{3.4.21}$$

Here we must use the conventions $\frac{0}{0} = 0$ and $\frac{x}{0} = \infty$ when $x \neq 0$.
In the example above, we have $m < n$, thus $\mathrm{dinv}(PF) = \mathrm{tdinv}(PF) - 4 = 5$, since there are 3 cells with $arm = 0$ and $leg = 0$ and one cell with $arm = 2$ and $leg = 3$.

We now have all the ingredients that occur in any of the Rational Shuffle Conjectures including the Compositional ones in [BGLX15]. In particular, (3.4.5) conjectures the equality

$$\mathbf{e}_{km,kn}(-1)^{k(n+1)} = \sum_{PF \in \mathcal{PF}_{km,kn}} t^{\mathrm{area}(PF)} q^{\mathrm{dinv}(PF)} F_{\mathrm{pides}(PF)} \tag{3.4.22}$$

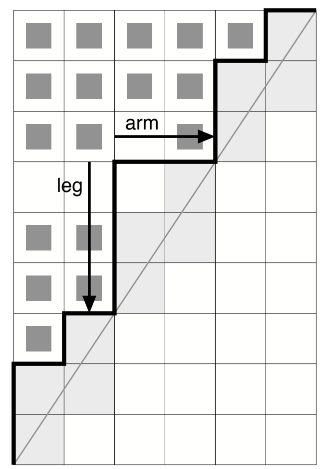

Figure 3.4: The dinv of a Dyck path.

where pides(PF) is the composition that gives the descent set of the inverse of the permutation $\sigma(PF)$ defined above, and $F_{\text{pides}(PF)}$ is the Gessel [Ges84] fundamental quasi-symmetric function indexed by the composition pides(PF). Here we use the convention that for a composition $p \models u$ if $S(p)$, is the subset of $\{1, 2, \ldots, u-1\}$ that corresponds to p, then we set

$$F_p(x_1, x_2, \ldots, x_v) \; = \sum_{1 \le a_1 \le a_2 \le \cdots \le a_v \le v,\, i \in S(p) \to a_i < a_{i+1}} x_{a_1} x_{a_2} \cdots x_{a_v}.$$

The same conventions apply to the statistics occurring in (3.4.8), namely our conjecture that

$$Q_{km,kn}(-1)^{kn} = \sum_{PF \in \mathcal{PF}_{km,kn}} t^{\text{area}(PF) - \text{ret}(PF) + 1} \left[\text{ret}(PF)\right]_t q^{\text{dinv}(PF)} F_{\text{pides}(PF)}. \tag{3.4.23}$$

Remark 3.4.1. *We must mention that the conjectured equality in (3.4.16) has a specialization that extends the equality in (3.1.1) to the non coprime case. More precisely, we have*

$$\sum_{D \in \mathcal{D}_{km,kn}} \left[\text{ret}(PF)\right]_q q^{\text{coarea}(D) + \text{dinv}(D)} \; = \frac{[k]_q}{[km]_q} \begin{bmatrix} kn + km - 1 \\ kn \end{bmatrix}_q. \tag{3.4.24}$$

In fact, scalar multiplication of both sides of (3.4.16) by $e_{kn}[X]$ *gives*

$$\sum_{D\in\mathcal{D}_{km,kn}} [\mathrm{ret}(PF)]_q\, q^{\mathrm{coarea}(D)+\mathrm{dinv}(D)} = \frac{[k]_q}{[km]_q}\langle e_{kn}\left[X\,[km]_q\right],\ e_{kn}[X]\rangle$$
$$= \frac{[k]_q}{[km]_q}\langle h_{kn}\left[X\,[km]_q\right],\ h_{kn}[X]\rangle$$

and (3.4.24) then follows from the identities

$$\langle h_{kn}\left[X\,[km]_q\right],\ h_{kn}[X]\rangle = h_{kn}\left[[km]_q\right] = \begin{bmatrix} kn+km-1 \\ kn \end{bmatrix}_q.$$

We should mention that, for Dyck paths, the dinv statistic is obtained by counting the number of cells c of the english partition above the path (see Figure 3.4) whose *arm* and *leg* satisfy the inequalities

$$\frac{arm(c)}{leg(c)+1} \le \frac{m}{n} < \frac{arm(c)+1}{leg(c)}.$$

In the figure we have placed a green square in each of the cells that contribute to the dinv.

Remark 3.4.2. *This should complete our presentation of the combinatorial side of the Rational Shuffle Conjectures except for two important observations. Firstly we should notice that (3.4.16) and (3.1.16) differ in that (3.1.15) has "*$s_{\mathrm{pides}(PF)}$*" replacing "*$F_{\mathrm{pides}(PF)}$*" in (3.4.16). We stated (3.1.15) and various analogous identities in the introduction in this manner, since this makes them easier to verify on a computer. In fact, the validity of this replacement, is one of the surprising consequences of a result of Egge, Loehr and Warrington [ELW10] concerning Gessel fundamental expansions of symmetric functions.*

The second observation results from a direct comparison of (3.4.22) and (3.4.23). Notice that we have $1 \le \mathrm{ret}(PF) \le k$ *since the path must end at the point* (km, kn)*. Furthermore,* $\mathrm{area}(PF) \ge \mathrm{ret}(PF) - 1$*. This is because each time* PF *fails to touch the point* (im, in)*, a cell must fall between the path and the main diagonal. Therefore all the powers of* t *appearing in (3.4.23) when* $\mathrm{ret}(PF) > 1$ *are non-negative. It follows from this that the difference of the right and sides* *of (3.4.23) and (3.4.22) can be shown to be positive linear combination of LLT polynomials which have in turn been shown to be Schur positive. So another strong evidence supporting the validity of these conjectures is that computer data confirms the Schur positivity of the difference of the left hand sides of (3.4.23) and (3.4.22).*

3.5 The action of the operators $\mathbf{D_{u,v}}$ on the basis $\left\{\mathbf{s}_\mu\left[\frac{\mathbf{X}}{\mathbf{1-q}}\right]\right\}_\mu$

Our main goals in this section are the proofs of Theorems 3.1.5 and 3.1.6. To carry this out we need auxiliary notation and some preliminary identities which may initially appear only remotely connected with these goals.

Our basic tool here is a new variant of the Macdonald operator "D_n^1" of [Mac95]. We will denote it "R_v." Its action on a symmetric polynomial $P\left[X_n\right]$ is obtained by setting

$$R_v P\left[X_n\right] \;=\; \frac{1}{\Delta\left[X_n\right]}\sum_{i=1}^{n} T_{x_i}^{q} x_i^{v} \Delta\left[X_n\right] P\left[X_n\right], \tag{3.5.1}$$

where "$T_{x_i}^{q}$" is the linear operator which carries out the substitution $x_i \to q x_i$. Here $\Delta\left[X_n\right]$ is the Vandermonde determinant

$$\Delta\left[X_n\right] = \prod_{1\le i<j\le n}(x_i - x_j) = \sum_{\sigma\in S_n}\epsilon(\sigma)\sigma x^{\delta}, \tag{3.5.2}$$

with $\epsilon(\sigma)$ the sign of σ and $\delta = (n-1, n-2, \ldots, 1, 0)$. The actual value of n in both (3.5.1) and (3.5.2) is immaterial provided that we choose it greater than v plus the degree of P. The following identity, which in particular shows that R_v preserves symmetry, will play a crucial role.

Proposition 3.5.1. *For any integral vector $\mu = (\mu_1 \ge \mu_2 \ge \cdots \ge \mu_n \ge 0)$ we have*

$$R_v s_\mu\left[X_n\right] \;=\; \sum_{i=1}^{n} q^{v+\mu_i+n-i} s_{\mu+ve_i}\left[X_n\right] \tag{3.5.3}$$

with e_i the n dimensional coordinate vector with i^{th} component equal to 1.

Proof. Since by definition

$$s_\mu[X_n] \;=\; \frac{1}{\Delta[X_n]} \sum_{\sigma \in S_n} \epsilon(\sigma)\sigma x^{\mu+\delta}$$

we may write

$$\begin{aligned} R_v s_\mu[X_n] &= \frac{1}{\Delta[X_n]} \sum_{i=1}^{n} T^q_{x_i} x_i^v \sum_{\sigma \in S_n} \epsilon(\sigma)\sigma x^{\mu+\delta} \\ &= \frac{1}{\Delta[X_n]} \sum_{i=1}^{n} \sum_{\sigma \in S_n} \epsilon(\sigma) T^q_{x_{\sigma_i}} x^v_{\sigma_i} \sigma x^{\mu+\delta} \\ &= \frac{1}{\Delta[X_n]} \sum_{i=1}^{n} q^{v+\mu_i+n-i} \sum_{\sigma \in S_n} \epsilon(\sigma)\sigma x^{\mu+ve_i+\delta}. \end{aligned}$$

This proves (3.5.3). □

The next identity shows that R_v may be given an expression that is similar to the one obtained by Macdonald for his D_n^1 operator.

Proposition 3.5.2. *We have*

$$R_v \;=\; q^v \sum_{i=1}^{n} A_i(x;q) x_i^v T^q_{x_i} \tag{3.5.4}$$

with

$$A_i(x;q) = \prod_{\substack{j=1 \\ j\neq i}}^{n} \frac{qx_i - x_j}{x_i - x_j}. \tag{3.5.5}$$

Proof. The definition in (3.5.1) may also be rewritten as

$$R_v P[X_n] \;=\; q^v \sum_{i=1}^{n} \Big(\frac{1}{\Delta[X_n]} T^q_{x_i} \Delta[X_n] \Big) x_i^v T^q_{x_i} P[X_n].$$

This shows 4.4 with

$$A_i(x,q) \;=\; \frac{1}{\Delta[X_n]} T^q_{x_i} \Delta[X_n].$$

However, we see that

$$\frac{1}{\Delta[X_n]} T^q_{x_i} \Delta[X_n] = \prod_{r,s\neq i} \frac{x_r - x_s}{x_r - x_s} \prod_{i<s} \frac{qx_i - x_s}{x_i - x_s} \prod_{r<i} \frac{x_r - qx_i}{x_r - x_i} \;=\; \prod_{\substack{j=1 \\ j\neq i}}^{n} \frac{qx_i - x_j}{x_i - x_j},$$

as desired. □

One of the difficulties in using Macdonald's original definition of the operators D_n^k stems from the fact that formulas expressing a symmetric polynomial $P[X_n]$ in terms of the variables themselves are quite impractical for significant values of n. For this reason, neither the definition in (3.5.1) nor its alternate form in (3.5.4) are much help in computing the action the operators D_n^k when it matters. However, for any of the operators R_v, we do have a plethystic formula which is computationally as well as theoretically very convenient. To state and prove this result we need some auxiliary identities.

Proposition 3.5.3. *For any formal series* $F(x) = \sum_{k\geq 0} c_k x^k$ *and for all integers* $v \geq 0$ *we have*

$$\begin{aligned} &F(1/z)\Omega\left[(q-1)zX_n\right]\Big|_{z^v} \\ &\quad = \frac{\chi(v=0)}{q^n}F(0) + \frac{q-1}{q^n}\sum_{i=1}^n\Big(\prod_{\substack{j=1\\ j\neq i}}^n \frac{qx_i - x_j}{x_i - x_j}\Big)(qx_i)^v F(qx_i). \end{aligned} \tag{3.5.6}$$

Proof. Starting from the partial fraction expansion

$$\Omega\left[(q-1)zX_n\right] = \prod_{i=1}^n \frac{1-zx_i}{1-zqx_i} = \frac{1}{q^n} + \frac{q-1}{q^n}\sum_{i=1}^n\Big(\prod_{\substack{j=1\\ j\neq i}}^n \frac{qx_i - x_j}{x_i - x_j}\Big)\frac{1}{1-zqx_i},$$

we get

$$\begin{aligned} &F(1/z)\Omega\left[(q-1)zX_n\right]\Big|_{z^v} \\ &= \frac{F(1/z)}{q^n}\Big|_{z^v} + \frac{q-1}{q^n}\sum_{i=1}^n\Big(\prod_{\substack{j=1\\ j\neq i}}^n \frac{qx_i - x_j}{x_i - x_j}\Big)\frac{1}{1-zqx_i}F(1/z)\Big|_{z^v} \\ &= \frac{\chi(v=0)}{q^n}F(0) + \frac{q-1}{q^n}\sum_{i=1}^n\Big(\prod_{\substack{j=1\\ j\neq i}}^n \frac{qx_i - x_j}{x_i - x_j}\Big)\Big(\sum_{r\geq 0}(zqx_i)^r\Big)\Big(\sum_{s\geq 0}F_s/z^s\Big)\Big|_{z^v} \\ &= \frac{\chi(v=0)}{q^n}F(0) + \frac{q-1}{q^n}\sum_{i=1}^n\Big(\prod_{\substack{j=1\\ j\neq i}}^n \frac{qx_i - x_j}{x_i - x_j}\Big)\Big(\sum_{\substack{r-s=v,\\ r,s\geq 0}}(qx_i)^r F_s\Big) \\ &= \frac{\chi(v=0)}{q^n}F(0) + \frac{q-1}{q^n}\sum_{i=1}^n\Big(\prod_{\substack{j=1\\ j\neq i}}^n \frac{qx_i - x_j}{x_i - x_j}\Big)\Big(\sum_{s\geq 0}(qx_i)^{s+v}F_s\Big). \end{aligned}$$

This proves our proposition. $\square$

As a corollary we obtain the following basic identity.

Theorem 3.5.1. *For any symmetric polynomial $P[X_n]$ we have*

$$R_v P[X_n] \;=\; \frac{\chi(v=0)}{1-q} P[X_n] \;+\; \frac{q^{n+v}}{q-1} P[X_n - (1-q)/z]\,\Omega[(1-1/q)zX_n]\Big|_{z^v}. \tag{3.5.7}$$

Proof. Notice that in view of (3.5.4) we can rewrite (3.5.1) in the form

$$\begin{aligned} R_v P[X_n] \;&=\; q^v \sum_{i=1}^{n} A_i(x;q) x_i^v P[X_n - (1-q)x_i] \\ &=\; q^v \sum_{i=1}^{n} \Big(\prod_{\substack{j=1\\ j\neq i}}^{n} \frac{qx_i - x_j}{x_i - x_j} \Big) x_i^v P[X_n - (1-q)x_i]\,. \end{aligned} \tag{3.5.8}$$

On the other hand Proposition 3.5.3 for $F(z) \to F(z/q)$ yields

$$\begin{aligned} &F(1/qz)\Omega[(q-1)zX_n]\Big|_{z^v} \\ &\qquad = \frac{\chi(v=0)}{q^n} F(0) \;+\; \frac{q-1}{q^n} \sum_{i=1}^{n} \Big(\prod_{\substack{j=1\\ j\neq i}}^{n} \frac{qx_i - x_j}{x_i - x_j} \Big) (qx_i)^v F(x_i). \end{aligned}$$

Or better

$$\begin{aligned} &q^v \sum_{i=1}^{n} \Big(\prod_{\substack{j=1\\ j\neq i}}^{n} \frac{qx_i - x_j}{x_i - x_j} \Big) x_i^v F(x_i) \\ &\qquad = \frac{\chi(v=0)}{1-q} F(0) \;+\; \frac{q^n}{q-1} F(1/qz)\Omega[(q-1)zX_n]\Big|_{z^v}. \end{aligned}$$

Using this with $F(z) = P[X_n - (1-q)z]$ and using (3.5.8) gives

$$\begin{aligned} R_v P[X_n] \;&=\; q^v \sum_{i=1}^{n} \Big(\prod_{\substack{j=1\\ j\neq i}}^{n} \frac{qx_i - x_j}{x_i - x_j} \Big) x_i^v P[X_n - (1-q)x_i] \\ &=\; \frac{\chi(v=0)}{1-q} P[X_n] \;+\; \frac{q^n}{q-1} P[X_n - (1-q)/qz]\,\Omega[(1-1/q)qzX_n]\Big|_{z^v}. \end{aligned} \tag{3.5.9}$$

Notice next that for any two formal power series $A(z), B(z)$ we have the identity

$$\begin{aligned} A\left[1/qz\right]B\left[qz\right]\Big|_{z^v} &= \sum_{r,s} A_r B_s (\frac{1}{qz})^r (qz)^s\Big|_{z^v} = \sum_{r,s} (zq)^{s-r} A_r B_s\Big|_{z^v} \\ &= \sum_{s-r=v} q^v A_r B_s = q^v \sum_r A_r B_{r+v} = q^v A\left[1/z\right]B\left[z\right]\Big|_{z^v} \end{aligned}$$

and thus (3.5.9) becomes

$$R_v P\left[X_n\right] = \frac{\chi(v=0)}{1-q} P\left[X_n\right] + \frac{q^{n+v}}{q-1} P\left[X_n - (1-q)/z\right]\Omega\left[(1-1/q)zX_n\right]\big|_{z^v}.$$

This proves (3.5.7). □

In particular, setting $P\left[X_n\right] = s_\mu\left[X_n\right]$ and using (3.5.3) we obtain

$$\begin{aligned} &\sum_{i=1}^{n} q^{v+\mu_i+n-i} s_{\mu+ve_i}\left[X_n\right] \\ &\quad = \frac{\chi(v=0)}{1-q} s_\mu\left[X_n\right] + \frac{q^{n+v}}{q-1} s_\mu\left[X_n - (1-q)/z\right]\Omega\left[(1-1/q)zX_n\right]\big|_{z^v}. \end{aligned}$$

or better yet

$$\begin{aligned} &q^{n+v} s_\mu\left[X_n - (1-q)/z\right]\Omega\left[(1-1/q)zX_n\right]\big|_{z^v} \\ &\quad = \chi(v=0) s_\mu\left[X_n\right] + (q-1)\sum_{i=1}^{n} q^{v+\mu_i+n-i} s_{\mu+ve_i}\left[X_n\right] \end{aligned} \tag{3.5.10}$$

At this point it is more convenient to separate the cases $v > 0$ and $v = 0$. We will begin with the following immediate corollary of Theorem 3.5.1.

Proposition 3.5.4. *For any $u, v > 0$ and any partition μ we have*

$$q^{uv} s_\mu\left[X - (1-q^u)/z\right]\Omega\left[(1-q^{-u})zX\right]\Big|_{z^v} = (q^u - 1)\sum_{i=1}^{|\mu|+v} q^{u(p(\mu)_i+v-i)} s_{p(\mu)+ve_i}[X] \tag{3.5.11}$$

where $p(\mu)$ is the weak composition of length $|\mu| + v$ obtained by adjoining zeros to the parts of μ and e_i is the i^{th} coordinate vector of length $|\mu| + v$.

Proof. For $v > 0$ (3.5.10) can be rewritten in the form

$$s_\mu\left[X_n - (1-q)/z\right]\Omega\left[1-1/q)zX_n\right]\big|_{z^v} = (q-1)\sum_{i=1}^{n} q^{\mu_i+v-i} s_{\mu+ve_i}\left[X_n\right]$$

and the replacement $q \to q^u$ gives

$$q^{uv} s_\mu \left[X_n - (1-q^u)/z\right] \Omega\left[(1-q^{-u})zX_n\right]\Big|_{z^v} \;=\; (q^u - 1)\sum_{i=1}^{n} q^{u(\mu_i+v-i)} s_{p(\mu)+ve_i}\left[X_n\right]$$

This given, (3.5.11) follows since the Schur functions involved in this expression stabilize after $n \geq |\mu| + v$. □

Keeping this in mind let us recall that our goal here is to work out the action of the operator $D_{u,v}$ on the basis $\left\{s_\mu\left[\frac{X}{1-q}\right]\right\}_\mu$. The following identity provides the link that ties this goal with the identities in (3.5.7) and (3.5.11).

Proposition 3.5.5. *Suppose that for some $v \geq 0$ we have*

$$s_\mu\left[X - (1-q^u)/z\right] \Omega\left[(1-q^{-u})zX\right]\Big|_{z^v} \;=\; G\left[X; q\right] \tag{3.5.12}$$

then

$$D_{u,v} s_\mu\left[\frac{X}{1-q}\right] \;=\; q^v G\left[\frac{X}{1-q}; q\right] \tag{3.5.13}$$

Proof. Notice first the following sequence of equalities.

$$\begin{aligned} q^v G\left[X;q\right] \;&=\; q^v s_\mu\left[X - (1-q^u)/z\right]\Omega\left[z(1-q^{-u})X\right]\Big|_{z^v} \\ &=\; s_\mu\left[X - (1-q^u)/qz\right]\Omega\left[qz(1-q^{-u})X\right]\Big|_{z^v} \\ &=\; s_\mu\left[X + \frac{(q-1)}{1-q}(1-q^u)/qz\right]\Omega\left[-qz\frac{1-q^{-u}}{q-1}X(1-q)\right]\Big|_{z^v} \\ &=\; s_\mu\left[X + \frac{(1-1/q)}{1-q}(1-q^u)/z\right]\Omega\left[-z\frac{1-q^{-u}}{1-q^{-1}}X(1-q)\right]\Big|_{z^v} \end{aligned}$$

Next the replacement $X \to \frac{X}{1-q}$ gives

$$\begin{aligned} q^v G\left[\frac{X}{1-q};q\right] \;&=\; s_\mu\left[\frac{X}{1-q} + \frac{(1-1/q)}{1-q}(1-q^u)/z\right]\Omega\left[-z\frac{1-q^{-u}}{1-q^{-1}}X\right]\Big|_{z^v} \\ &=\; s_\mu\left[\frac{X + (1-1/q)(1-q^u)/z}{1-q}\right]\Omega\left[-z\frac{1-q^{-u}}{1-q^{-1}}X\right]\Big|_{z^v} \\ &=\; s_\mu\left[\frac{X + (1-q)(1-1/q)\left[u\right]_q/z}{1-q}\right]\Omega\left[-z\frac{1-q^{-u}}{1-q^{-1}}X\right]\Big|_{z^v}. \end{aligned} \tag{3.5.14}$$

Now recalling that by definition we have (for $t = 1/q$)

$$D_{u,v}F[X] = F\left[X + M\left[u\right]_q/z\right]\Omega\left[-z\left[u\right]_t X\right]\Big|_{z^v},$$

we see that (3.5.14) proves (3.5.13). □

We are thus able to obtain our

Proof of Theorem 3.1.6. By combining Propositions 3.5.4 and 3.5.5 with

$$G\,[X;q] \;=\; q^{-uv}(q^u-1)\sum_{i=1}^{|\mu|+v} q^{u(p(\mu)_i+v-i)} s_{p(\mu)+ve_i}[X]$$

we obtain

$$D_{u,v}s_\mu\left[\frac{X}{1-q}\right] \;=\; (q^u-1)\sum_{i=1}^{|\mu|+v} q^{up(\mu)_i+v-ui} s_{p(\mu)+ve_i}\left[\frac{X}{1-q}\right]$$

as desired. □

Our next task is to take care of the case $v = 0$ of (3.5.7). This may be rewritten as

$$q^n P\,[X_n-(1-q)/z]\,\Omega\,[(1-1/q)zX_n]\,\big|_{z^0} = P\,[X_n] \,-\, (1-q)R_0 P\,[X_n]\,.$$

Choosing $P\,[X_n] = s_\mu\,[X_n]$ and using (3.5.3) for $v = 0$ we get

$$q^n s_\mu\,[X_n-(1-q)/z]\,\Omega\,[1-1/q)zX_n]\,\big|_{z^0} = \Big(1 \,-\, (1-q)\sum_{i=1}^{n} q^{\mu_i+n-i}\Big) s_\mu\,[X_n]\,. \tag{3.5.15}$$

Our next step is to transform (3.5.15) into a relation which contains no explicit dependence on n. To this end, recalling that we write a partition of n in the form $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n \geq 0)$ we set

$$B_\mu(q,t) \;=\; \sum_{\mu_i>0} t^{i-1}(1+q+\cdots+q^{\mu_i-1}) \;=\; \sum_{i=1}^{n} t^{i-1}\frac{1-q^{\mu_i}}{1-q}. \tag{3.5.16}$$

This polynomial, usually called the *biexponent generator of* μ, plays an essential role in the Theory of Macdonald polynomials, in our case this results in the following basic identity.

Proposition 3.5.6.

$$s_\mu\left[X_n-\frac{1-q}{z}\right]\Omega\left[(1-q^{-1})zX_n\right]\Big|_{z^0} \;=\; \Big(1 \,-\, (1-q^{-1})(1-q)B_\lambda(q,q^{-1})\Big)s_\mu\,[X_n] \tag{3.5.17}$$

Proof. Changing t into $1/q$ in (3.5.16) and multiplying both sides by q^n we can rewrite it in the form

$$q^n\Big(1-(1-q^{-1})(1-q)B_\mu(q,t)\Big) \;=\; 1 \;-\; (1-q)\sum_{i=1}^{n} q^{\mu_i+n-i}. \tag{3.5.18}$$

Using this in (3.5.15) gives

$$q^n s_\mu\left[X_n-\frac{1-q}{z}\right]\Omega\left[(1-q^{-1})zX_n\right]\Big|_{z^0} = q^n\Big(1-(1-q^{-1})(1-q)B_\mu(q,t)\Big)s_\mu\left[X_n\right].$$

Canceling the factor q^n proves (3.5.17) as desired. □

Remark 3.5.1. *By setting $v=0$ in (3.5.3) and (3.5.4) we derive that*

$$\sum_{i=1}^{n} A_i(x;q)T^q_{x_i}s_\mu\left[X_n\right] \;=\; \Big(\sum_{i=1}^{n} q^{\mu_i+n-i}\Big)s_\mu\left[X_n\right]$$

This identity was used by Macdonald in [Mac95] to prove that his polynomial $P_\mu(X_n,q,t)$ reduces to $s_\mu\left[X_n\right]$ at $t=q$. Now the original definition of the modified Macdonald polynomial $\widetilde{H}_\mu\left[X;q,t\right]$ was obtained by setting

$$\widetilde{H}_\mu\left[X;q,t\right] \;=\; c_\nu(q,t)P_\mu\left[X/(1-1/t);q,1/t\right] \tag{3.5.19}$$

where $P_\mu\left[X;q,t\right]$ is none other than $P_\mu\left[X_n;q,t\right]$ (for $\mu\vdash n$) with X_n replaced by the infinite alphabet $X=x_1+x_2+\cdots$. and $c_\nu(q,t)$ is a polynomial in q,t whose nature is immaterial here. Thus it follows from (3.5.19) that

$$\widetilde{H}_\mu\left[X;q,1/q\right]=c_\nu(q,q^{-1})P_\mu\left[X/(1-q);q,q\right]=c_\nu(q,q^{-1})s_\mu\left[\frac{X}{1-q}\right].$$

Thus for all practical purposes, in the present context, which arises from our setting $t=1/q$ in all our $Q_{u,v}$ and their identities, the basis $\{\widetilde{H}_\mu\left[X;q,t\right]\}_\mu$ need only be replaced by the basis $\left\{s_\mu\left[\frac{X}{1-q}\right]\right\}_\mu$. In this vein we can easily obtain an alternate way of interpreting the identity in (3.5.17).

Theorem 3.1.5. *For all integers $u\geq 1$ we have*

$$D_{u,0}s_\mu\left[\frac{X}{1-q}\right] \;=\; \Big(1\;-\;(1-q^{-u})(1-q^u)B_\mu(q^u,q^{-u})\Big)s_\mu\left[\frac{X}{1-q}\right]. \tag{3.5.20}$$

Proof. Making the replacements $X_n \to X$ and $q \to q^u$ in (3.5.16) gives

$$s_\mu \left[X - \frac{1-q^u}{z} \right] \Omega \left[(1-q^{-u})zX \right] \Big|_{z^0} = \Big(1 - (1-q^{-u})(1-q^u) B_\mu(q^u, q^{-u}) \Big) s_\mu \left[X \right].$$

Next we do $X \to \frac{X}{1-q}$ and get

$$s_\mu \left[\frac{X}{1-q} - \frac{1-q^u}{z} \right] \Omega \left[-\frac{1-q^{-u}}{1-1/q}(z/q)X \right] \Big|_{z^0} \\ = \Big(1 - (1-q^{-u})(1-q^u) B_\mu(q^u, q^{-u}) \Big) s_\mu \left[\frac{X}{1-q} \right]$$

and this (with $t = 1/q$) may be rewritten as

$$s_\mu \left[\frac{X + (1-1/q)(1-q)\left[m\right]_q/(z/q)}{1-q} \right] \Omega \left[-\left[m\right]_t (z/q)X \right] \Big|_{z^0} \\ = \Big(1 - (1-q^{-u})(1-q^u) B_\mu(q^u, q^{-u}) \Big) s_\mu \left[\frac{X}{1-q} \right]$$

or better

$$s_\mu \left[\frac{X + M\left[m\right]_q/z}{1-q} \right] \Omega \left[-\left[m\right]_t zX \right] \Big|_{z^0} \\ = \Big(1 - (1-q^{-u})(1-q^u) B_\mu(q^u, q^{-u}) \Big) s_\mu \left[\frac{X}{1-q} \right]. \qquad (3.5.21)$$

Recalling that by definition we have

$$D_{u,v} F[X] = F \left[X + M\left[u\right]_q/z \right] \Omega \left[-\left[u\right]_t zX \right] \Big|_{z^v},$$

we see that (3.5.20) is simply another way of writing (3.5.21). This completes our proof. $\square$

Remark 3.5.2. *If we follow the sequence of steps that yielded the identity in (3.5.20) we will notice that this identity is but a direct consequence of the identity in (3.5.17) with the replacement $X_n \to X$, that is*

$$s_\mu \left[X - \frac{1-q}{z} \right] \Omega \left[(1-q^{-1})zX \right] \Big|_{z^0} = \Big(1 - (1-q^{-1})(1-q) B_\mu(q, q^{-1}) \Big) s_\mu \left[X \right]. \qquad (3.5.22)$$

Thus in principle, a shortcut in the proof of Theorem 3.1.5, may appear to be the verification of (3.5.22).

To better appreciate the power of the path we followed in the proof of (3.5.20), it will be instructive to see what kind of combinatorial identities we are led to in trying to carry this out. Now working first on the left hand side of (3.5.22) gives

$$\begin{aligned} LHS \;&=\; s_\mu[X] + \sum_{k\geq 1} \sum_{\nu=(k-a,1^a)} s_{\mu/\nu'}[X](-1/z)^k s_\nu\left[1-q\right]\Omega\left[(1-q^{-1})zX\right]\Big|_{z^0} \\ &=\; s_\mu[X] + (1-q)\sum_{k\geq 1}(-1)^k \sum_{a=0}^{k-1} s_{\mu/(a+1,1^{k-a-1})}[X](-q)^a h_k\left[1-q^{-1})X\right] \end{aligned} \tag{3.5.23}$$

and since

$$h_k\left[1-q^{-1})X\right] \;=\; (1-q^{-1})\sum_{b=0}^{k-1}(-q)^b s_{k-b,1^b}[X]$$

(3.5.23) becomes

$$LHS = s_\mu[X] + (1-q)(1-q^{-1})\sum_{k\geq 1}(-1)^k \sum_{a,b=0}^{k-1}(-q)^{q+b} s_{\mu/(a+1,1^{k-a-1})}[X] s_{k-b,1^b}[X]. \tag{3.5.24}$$

Taking the scalar product of both sides of (3.5.22) by $s_\lambda[X]$ and using (3.5.24), routine manipulations reduce (3.5.22) to the equivalent identity

$$\sum_{k\geq 1}(-1)^k \sum_{a,b=0}^{k-1}(-q)^{a+b}\langle s_{\mu/(a+1,1^{k-a-1})}\,,\, s_{\lambda/(k-b,1^b)}\rangle \;=\; -\chi(\lambda=\mu)B_\mu(q,q^{-1}). \tag{3.5.25}$$

A standard result on scalar products of skew Schur functions (see [GR85]), asserts that the scalar product summand is none other than the number of permutations that fit the shape $\mu/(a+1,1^{k-a-1})$ whose inverse fits the shape $\lambda/(k-b,1^b)$. This given, the only conclusion we can draw from this calculation is that (3.5.25) is one truly remarkable combinatorial consequence of Theorem 3.1.5.

3.6 The original proof of Theorem 3.1.1 by the partial fraction method

In this section we explain how we discovered Theorem 3.1.1. Indeed Theorem 3.1.1 involves different series expansions of a single Rational function, which is best understood by using the partial fraction method of the fourth named author. To this end we need to work in the field $K = \mathbb{Q}((z_N))((z_{N-1}))\cdots((z_1))$ of iterated Laurent series to obtain series expansion of Rational functions. The readers are referred to [Xin04] for the original development of the field of iterated Laurent series. Here we only recall that K defines a total group order on its monomials given by

$$z_1^{a_1}\cdots z_N^{a_N}\begin{cases} <_K 1, & \text{if } a_1=\cdots=a_{i-1}=0\,\&\,a_i>0 \\ =1, & \text{if } a_1=\cdots=a_N=0 \\ >_K 1, & \text{if } a_1=\cdots=a_{i-1}=0\,\&\,a_i<0. \end{cases}$$

We shall simply write this order by $z_1<z_2<\cdots<z_N<1$. The series expansion of $(1-w)^{-1}$ for a monomial $w\neq 1$ (called small or large) is thus given by

$$\frac{1}{1-w}=\begin{cases} \sum_{n\geq 0} w^n, & \text{if } w<_K 1 \\ \frac{1}{-w(1-1/w)} = -\sum_{n\geq 0} w^{-n-1}, & \text{if } w>_K 1. \end{cases}$$

To start with, let us recall that for any Laurent polynomials $L(z_1,z_2)\in\mathbb{Q}\left[z_1^{\pm 1},z_2^{\pm 1}\right]$, we always have that

$$L(z_1,z_2)\Big|_{z_1^0z_2^0} = L(z_2,z_1)\Big|_{z_1^0z_2^0} \qquad \text{holds in } \mathbb{Q}\left[z_1^{\pm 1},z_2^{\pm 1}\right].$$

In particular, antisymmetric Laurent polynomials have constant term 0. Such properties no longer hold for Rational functions in $K_1=\mathbb{Q}((z_1))((z_2))$. For example, $f(z_1,z_2)=\frac{z_1+z_2}{z_1-z_2}$ is clearly antisymmetric, but in K_1 we have

$$\frac{z_1+z_2}{z_1-z_2}\Big|_{z_1^0z_2^0} = \frac{1+z_2/z_1}{1-z_2/z_1}\Big|_{z_1^0z_2^0} = (1+z_2/z_1)\sum_{n\geq 0}(z_2/z_1)^n\Big|_{z_1^0z_2^0} = 1.$$

Indeed, the exchanging of the two variables z_1,z_2 transforms the constant term in K_1 to a constant term in $K_2=\mathbb{Q}((z_2))((z_1))$. We shall have

$$f(z_1,z_2)\Big|_{z_1^0z_2^0}^{K_1} = f(z_2,z_1)\Big|_{z_1^0z_2^0}^{K_2}$$

where the left hand side is a constant term in K_1, but the right hand side is a constant term in K_2. This type of exchanging of variables would be sufficient for us to prove Theorem 3.1.1. See [Xin05] for general formulation on the change of variables in a field of Malcev-Neumann series.

Proof of Theorem 3.1.1. With $t = 1/q$, we have

$$\begin{aligned}
D_{c,d}D_{a,b}\,F[X] &= D_{c,d}\,F\Big[X + [a]_q \frac{M}{z_1}\Big]\Omega\Big[-z_1 X[a]_t\Big] z_1^{-b}\Big|_{z_1^0} \\
&= F\Big[X + [c]_q \frac{M}{z_2} + [a]_q \frac{M}{z_1}\Big]\Omega\Big[-z_1[a]_t\big(X + [c]_q \frac{M}{z_2}\big)\Big]\Omega\Big[-z_2 X[c]_t\Big] z_1^{-b} z_2^{-d}\Big|_{z_1^0 z_2^0} \\
&= F\Big[X + [c]_q \frac{M}{z_2} + [a]_q \frac{M}{z_1}\Big]\Omega\Big[-X\big(z_1[a]_t + z_2[c]_t\big)\Big]\Omega\Big[-M[a]_t[c]_q \frac{z_1}{z_2}\Big] z_1^{-b} z_2^{-d}\Big|_{z_1^0 z_2^0} \\
&= F\Big[X + [c]_q \frac{M}{z_2} + [a]_q \frac{M}{z_1}\Big]\Omega\Big[-X\big(z_1[a]_t + z_2[c]_t\big)\Big]\frac{(1-\frac{z_1}{z_2})(1-q^c t^a \frac{z_1}{z_2})}{(1-q^c\frac{z_1}{z_2})(1-t^a\frac{z_1}{z_2})} z_1^{-b} z_2^{-d}\Big|_{z_1^0 z_2^0}
\end{aligned}$$

where in the last step we have used the fact that $M\,[a]_t\,[c]_q = (1-t^a)(1-q^c) = 1 - t^a - q^c + q^c t^a$. This constant term has to be understood as in a field of iterated Laurent series where $q^c z_1/z_2$ and $t^a z_1/z_2$ are small. In the general q, t case, we can set $q < t < z_1 < z_2 < 1$. But here we can not set q to be small, since that will force $t = 1/q$ to be large. We choose to work in the field of iterated Laurent series K_1 defined by the order $z_1 < z_2 < q < 1$ (one can take q as a constant). More precisely, we can set, for example, $K_1 = \mathbb{Q}(q)((z_2))((z_1))\,[[x_1, x_2, \ldots]]$.

Let us write

$$\begin{aligned}
D_{c,d}D_{a,b}F[X] = F\Big[X + [c]_q\,\frac{M}{z_2} + [a]_q\,\frac{M}{z_1}\Big]& \\
\times\,\Omega\Big[-X\big(z_1[a]_t + z_2[c]_t\big)\Big] G(z_1, z_2; c, d, a, b)\Big|_{z_1^0 z_2^0}^{K_1},& \qquad (3.6.1)
\end{aligned}$$

where

$$G(z_1, z_2; c, d, a, b) = \frac{(1-\frac{z_1}{z_2})(1-q^c t^a \frac{z_1}{z_2})}{(1-q^c\frac{z_1}{z_2})(1-t^a\frac{z_1}{z_2})} z_1^{-b} z_2^{-d}.$$

Now switching (c, d) and (a, b) gives

$$\begin{aligned}
D_{a,b}D_{c,d}F[X] = F\Big[X + [a]_q \frac{M}{z_2} + [c]_q \frac{M}{z_1}\Big]\Omega\Big[-X\big(z_1[c]_t + z_2[a]_t\big)\Big]& \\
\times\, G(z_1, z_2; a, b, c, d)\Big|_{z_1^0 z_2^0}^{K_1}.&
\end{aligned}$$

Observe that when $t = 1/q$, we have

$$G(z_2, z_1; a, b, c, d) = \frac{(1 - \frac{z_2}{z_1})(1 - t^c q^a \frac{z_2}{z_1})}{(1 - t^c \frac{z_2}{z_1})(1 - q^a \frac{z_2}{z_1})} = \frac{(1 - \frac{z_1}{z_2})(1 - q^c t^a \frac{z_1}{z_2})}{(1 - q^c \frac{z_1}{z_2})(1 - t^a \frac{z_1}{z_2})}$$
$$= G(z_1, z_2; c, d, a, b).$$

By exchanging z_1 and z_2, we obtain that $D_{a,b}D_{c,d}F[X]$ is the same constant term as in (3.6.1), but working in the field of iterated Laurent series K_2 defined by the order $z_2 < z_1 < q < 1$. So we are indeed computing the difference of the constant terms of a single "Rational function" in two different working fields.

By partial fraction decomposition in z_1, applied to the coefficients in the x's and then sum, we have

$$F\Big[X + [c]_q \frac{M}{z_2} + [a]_q \frac{M}{z_1}\Big] \Omega\Big[-X\big(z_1[a]_t + z_2[c]_t\big)\Big] G(z_1, z_2; a, b, c, d) =$$
$$= p^{\geq 0}(z_1) + p^{<0}(z_1) + \frac{A_1}{1 - q^c \frac{z_1}{z_2}} + \frac{A_2}{1 - t^a \frac{z_1}{z_2}},$$

where when we restrict to each coefficient of the x's, $p^{\geq 0}(z_1)$ is a polynomial in z_1, $p^{<0}(z_1)$ is a Laurent polynomial that only contains negative powers in z_1, and A_1 and A_2 are free of z_1 given by

$$A_1 = F\Big[X + [c]_q \frac{M}{z_2} + q^c [a]_q \frac{M}{z_2}\Big] \Omega\Big[-X\big(z_2 t^c [a]_t + z_2 [c]_t\big)\Big] \frac{(1-t^c)(1-t^a)}{(1-t^a t^c)} z_2^{-b} t^{-bc} z_2^{-d},$$
$$= -q^{bc} \frac{(1 - q^c)(1 - q^a)}{(1 - q^{a+c})} F\Big[X + [a + c]_q \frac{M}{z_2}\Big] \Omega\Big[-X z_2 [a + c]_t\Big] z_2^{-b-d}; \tag{3.6.2}$$

$$A_2 = F\Big[X + [c]_q \frac{M}{z_2} + t^a [a]_q \frac{M}{z_2}\Big] \Omega\Big[-X\big(z_2 q^a [a]_t + z_2 [c]_t\big)\Big] \frac{(1-q^a)(1-q^c)}{(1-q^a q^c)} z_2^{-b} q^{-ba} z_2^{-d}$$
$$= \frac{(1 - q^a)(1 - q^c)}{(1 - q^a q^c)} F\Big[X + [a + c]_q \frac{M}{z_2 q^a}\Big] \Omega\Big[-X z_2 q^a [a + c]_t\Big] z_2^{-d-b} q^{-ba}.$$

We need the following formula (obtained by the change of variables $z_2 \to z_2 t^a$).

$$A_2\Big|_{z_2^0} = q^{da} \frac{(1 - q^a)(1 - q^c)}{(1 - q^{a+c})} F\Big[X + [a + c]_q \frac{M}{z_2}\Big] \Omega\Big[-X z_2 [a + c]_t\Big] z_2^{-d-b}\Big|_{z_2^0} \tag{3.6.3}$$

Now we take the constant term in z_1 first, working in K_1 and K_2 separately. Since the two monomials $q^c z_1/z_2$ and $t^a z_1/z_2$ are both small in K_1 but large in K_2, we have

$$\begin{aligned} D_{c,d}D_{a,b}F[X] &= p^{\geq 0}(0)\Big|_{z_2^0} + A_1\Big|_{z_2^0} + A_2\Big|_{z_2^0}, \\ D_{a,b}D_{c,d}F[X] &= p^{\geq 0}(0)\Big|_{z_2^0}. \end{aligned}$$

It follows that

$$\frac{1}{M}\left[D_{c,d}, D_{a,b}\right]F[X] = \frac{1}{M}\Big(A_1\Big|_{z_2^0} + A_2\Big|_{z_2^0}\Big).$$

Applying formulas (3.6.2) and (3.6.3) gives the desired result. $\square$

The idea of the proof of Theorem 3.1.1 can be generalized as follows, which is easy to prove but turns out to be very useful.

Proposition 3.6.1. *Let K_1 and K_2 be two different field of iterated Laurent series. Suppose that we have the following partial fraction expansion.*

$$F(z) = p_0(z) + \frac{p_{-1}(z)}{z^m} + \frac{p_1(z)}{(1-u_1 z)^{k_1}} + \cdots + \frac{p_N(z))}{(1-u_N z)^{k_N}}$$

Then we have

$$F(z)\Big|_{z^0}^{K_1} - F(z)\Big|_{z^0}^{K_2} = \sum_{u_i z <_{K_1} 1 \ \& \ u_i z >_{K_2} 1} p_i(0) - \sum_{u_j z >_{K_1} 1 \ \& \ u_j z <_{K_2} 1} p_j(0).$$

In words, the difference of the two constant terms only came from those denominators that are contributing (i.e., with $u_i z < 1$) in one field but dually contributing (i.e., with $u_i z > 1$) in the other field.

Proof. With the given partial fraction decomposition, taking constant term in z under K_1 gives

$$F(z)\Big|_{z^0}^{K_1} = p_0(0) + \sum_{u_i z^{k_i} <_{K_1} 1} p_i(0).$$

A similar result holds for K_2. Subtracting gives the desired formula. $\square$

Remark 3.6.1. *The proposition applies whenever $F(z)$ includes something like $\Omega\left[X + M/z\right]$ or $\Omega\left[-zX\right]$ as factors. In that case m or $p_0(z)$ does not exist. But $F(z)$ can be first expanded as a power series in the x's, and then apply the proposition to the coefficients in the x's.*

Remark 3.6.2. *If we take* $K_1 = \mathbb{C}((z))$ *and* $K_2 = \mathbb{C}((z^{-1}))$*, then the proposition gives*

$$F(z)\Big|_{z^0}^{K_1} - F(z)\Big|_{z^0}^{K_2} = \sum_i p_i(0).$$

This can be shown to be equivalent to the well-known fact that for any given Rational function, its residues at all points (including ∞*) sum to* 0*.*

Acknowledgement. *This chapter is a reproduction of a paper with the same name and is joint work with Adriano Garsia, Guoce Xin, and Nolan Wallach. The present author's primary contributions form Section 3.4.*

Chapter 4

A new interpretation of ∇p_n

4.1 Symmetric function identities

The main result of this chapter a new combinatorial interpretation for ∇p_n. In particular, we show the following.

Theorem 4.1.1. *For all $n \geq 1$,*

$$(-1)^{n-1}\nabla p_n = \sum_{PF \in \mathcal{P}F_n} [\mathrm{ret}(PF)]_q \, t^{\mathrm{area}(PF)} q^{\mathrm{dinv}(PF)} F_{\mathrm{ides}(PF)}.$$

This is obtained by rewriting p_n as a positive sum of the functions $C_\alpha 1$ which appear in the Compositional Shuffle Conjectures. We begin with a direct consequence of the definition

$$C_a \, P[X] \; = \; \left(-\frac{1}{q}\right)^{a-1} P\left[X - \frac{1-1/q}{z}\right] \sum_{m \geq 0} z^m h_m[X] \Big|_{z^a}$$

and build up to the desired identity. Several of our intermediate equations were already known, but we prove them here both for the sake of completeness and because, in many cases, our proofs are much simpler than those previously known.

Theorem 4.1.2. *For $a, m \geq 1$,*

$$a)\ C_a e_m[X] = \left(\tfrac{-1}{q}\right)^{a-1} s_{a,1^m} - \left(\tfrac{-1}{q}\right)^{a} s_{a+1,1^{m-1}}, \quad \text{and} \quad b)\ e_n = \sum_{\rho \models n} C_\rho 1.$$

Hence

$$c)\ \sum_{\rho\models n-a} C_a C_\rho 1 = \left(\tfrac{-1}{q}\right)^{a-1} s_{a,1^{n-a}} - \left(\tfrac{-1}{q}\right)^{a} s_{a+1,1^{n-a-1}} \quad \textit{for} \ \ n > a$$

$$\textit{and} \qquad d)\ C_n 1 = \left(\tfrac{-1}{q}\right)^{n-1} s_n[X].$$

Proof. First note that d) is immediate from the definition of C_n and that c) follows from a) and b). Now we can directly compute a) by applying the addition formula (for both h and e plethystic substitutions) and Pieri's formula.

$$\begin{aligned}
(-q)^{a-1} C_a e_m[X] &= \sum_{r=0}^{m} e_{m-r}[X](-1)^r e_r\left[\tfrac{1-1/q}{z}\right]\Omega[zX]\Big|_{z^a} \\
&= e_m h_a + (1-\tfrac{1}{q})\sum_{r=1}^{m}(-1)^r e_{m-r} h_{r+a} \\
&= e_m h_a + (1-\tfrac{1}{q})\sum_{r=1}^{m}(-1)^r \left(s_{r+a,1^{m-r}} + \chi(r<m)\, s_{r+a+1,1^{m-r-1}}\right) \\
&= s_{a,1^m} + s_{a+1,1^{m-1}} + (1-\tfrac{1}{q}) s_{a+1,1^{m-1}} \\
&= s_{a,1^m} + \tfrac{1}{q} s_{a+1,1^{m-1}}.
\end{aligned}$$

Since b) is trivial for $n = 1$, we can proceed by induction on n.

$$\begin{aligned}
\sum_{\rho\models n} C_\rho 1 &= \sum_{a=1}^{n} C_a e_{n-a} \\
&= \sum_{a=1}^{n-1}\left(\left(\tfrac{-1}{q}\right)^{a-1} s_{a,1^m} - \left(\tfrac{-1}{q}\right)^{a} s_{a+1,1^{m-1}}\right) + \left(\tfrac{-1}{q}\right)^{n-1} s_n \\
&= s_{1^n} - \left(\tfrac{-1}{q}\right)^{n-1} s_n + \left(\tfrac{-1}{q}\right)^{n-1} s_n = e_n.
\end{aligned}$$

□

Theorem 4.1.3. *For all* $1 \le b \le n$,

$$s_{b,1^{n-b}} = (-q)^{b-1} \sum_{\rho\models n} \chi(\rho_1 \ge b) C_\rho 1.$$

Proof. Summing Theorem 4.1.2 d) and c) for $b \le a < n$, we get

$$\sum_{a=b}^{n} \sum_{\rho\models n-a} C_a C_\rho 1 = \left(\tfrac{-1}{q}\right)^{b-1} s_{b,1^{n-b}}.$$

But this can be rewritten as

$$\sum_{\rho\models n}\chi(\rho_1\geq b)C_\rho 1=\left(\tfrac{-1}{q}\right)^{b-1}s_{b,1^{n-b}}.$$

□

Theorem 4.1.4. *For all $n\geq 1$,*

$$(-1)^{n-1}p_n=\sum_{\rho\models n}[\rho_1]_q\,C_\rho 1.$$

Applying ∇ and the Compositional Shuffle Theorem, we obtain Theorem 4.1.1.

Proof. By the Murnaghan-Nakayama rule, we have

$$(-1)^{n-1}p_n=\sum_{b=1}^{n}(-1)^{b-1}s_{b,1^{n-b}}.$$

Then by Theorem 4.1.3,

$$(-1)^{n-1}p_n=\sum_{b=1}^{n}q^{b-1}\sum_{\rho\models n}\chi(\rho_1\geq b)\,C_\rho 1.$$

Interchanging the order of summation gives

$$(-1)^{n-1}p_n=\sum_{\rho\models n}\left(\sum_{b=1}^{n}q^{b-1}\chi(\rho_1\geq b)\right)C_\rho 1$$

which is another way of writing the desired equation. □

By taking $t=1/q$, (3.4.13) gives a direct connection between $Q_{km,kn}(-1)^{kn}$ and $(-1)^{k-1}\nabla p_k$. This is also reflected in the combinatorial similarity between Theorems 3.4.1 and 4.1.1. Unfortunately, there is no expansion of $Q_{0,k}\,1$ of a sum of $\{E_{k,l}\}_{1\leq l\leq k}$ with rational coefficients in t and q. This impossibility of such an expansion can already be seen when $k=4$. If some other analog of Theorem 4.2.1 can be found, it may give a hint as to a new combinatorial interpretation of $Q_{km,kn}(-1)^{kn}$ in terms of some sort of "rational preference functions." The main challenge would be to define an appropriate notion of dinv for these rational preference functions.

4.2 A refinement and a conjecture

In Chapter 2, we made use of the following (known) symmetric function identity. We provide a new, simplified proof below.

Theorem 4.2.1. *For $n \geq 1$,*

$$(-1)^{n-1}p_n = \sum_{k=1}^{n} \frac{[n]_q}{[k]_q} E_{n,k}.$$

Proof. Recall that the symmetric functions $E_{n,k}$ are implicitly defined by

$$\begin{aligned} e_n\left[X\frac{1-z}{1-q}\right] &= \sum_{k=1}^{n} \frac{(z;q)_k}{(q;q)_k} E_{n,k}[X] \\ &= \sum_{k=1}^{n} \frac{(1-z)(1-zq)\dots(1-zq^{k-1})}{(1-q)(1-q^2)\dots(1-q^k)} E_{n,k} \end{aligned}$$

Note that if we expand $e_n\left[X\frac{1-z}{1-q}\right]$ in terms of the power symmetric basis, the coefficient of p_λ will be divisible by $\prod_{i=1}^{l(\lambda)}(1-z^{\lambda_i})$, and hence divisible by $(1-z)^{l(\lambda)}$. Therefore, dividing by $(1-z)$ and letting $z \to 1$ gives

$$(-1)^{n-1}p_n\frac{1}{1-q^n} = \sum_{k=1}^{n} \frac{1}{1-q^k} E_{n,k}.$$

Multiplying both sides by $(1-q^n)$ gives the desired equation. $\square$

The commutativity relations between the C operators imply that every symmetric function $C_\rho 1$ for $\mu \models n$ can be uniquely "straightened" and expanded in the basis $\{C_\lambda\}_{\lambda \vdash n}$. Hence Theorem 4.1.4 and Theorem 4.2.1 give

Theorem 4.2.2. *For $1 \leq k \leq n$,*

$$\frac{[n]_q}{[k]_q} E_{n,k} = \sum_{\rho \models n,\ l(\rho)=k} C_\rho 1.$$

Then, by Theorem 2.4.1, we have

Theorem 4.2.3. *For $1 \leq k \leq n$,*

$$\sum_{\substack{Pr \in \mathcal{P}ref_n \\ \mathrm{touch}(Pr)=k}} t^{\mathrm{area}(Pr)} q^{\mathrm{dinv}(Pr)} F_{\mathrm{ides}(Pr)} = \sum_{\substack{PF \in \mathcal{PF}_n \\ \mathrm{touch}(PF)=k}} [\mathrm{ret}(PF)]_q\, t^{\mathrm{area}(PF)} q^{\mathrm{dinv}(PF)} F_{\mathrm{ides}(PF)}.$$

It would be interesting to find a bijective proof of this theorem. This is trivial when $k = n$. At the other extreme, $k = 1$, the return is always n and Corollary 2.4.1 provides a combinatorial (enumerative) proof, but not a bijective one. Even a proof of this type is not known currently to the author for other values of k.

Furthermore, experimental evidence supports the following refinement.

Conjecture 4.2.1. *For $\tau \in S_n$,*

$$\sum_{\substack{Pr \in \mathcal{P}ref_n \\ \text{diagword}(Pr)=\tau}} t^{\text{area}(Pr)} q^{\text{dinv}(Pr)} F_{\text{ides}(Pr)} = \sum_{\substack{PF \in \mathcal{PF}_n \\ \text{diagword}(PF)=\tau}} [\text{ret}(PF)]_q \, t^{\text{area}(PF)} q^{\text{dinv}(PF)} F_{\text{ides}(PF)}.$$

As noted above, if τ's last run has length 1, this corresponds to the case $k = 1$ of Corollary 2.4.1. If τ's last run has length 2, the problem is already very interesting.

4.3 A special case

We identify each unlabeled square path SQ with the preference function Pr which has underlying path SQ and word $n\, n-1 \ldots 1$. (As a special case, a Dyck path D corresponds to the parking function PF with underlying path D and word $n\, n-1 \ldots 1$.) With this in mind, we see that the square paths (resp. Dyck paths) with the same diagonal word are precisely those with the same number of cars in consecutive diagonals. E.g. the Square Paths sharing diagonal word $\tau = 3\,4\,5\,1\,2$ are those which have five cars where two in the lowest diagonal and three are in the second lowest diagonal. Let the area, dinv, and deviation l of a square path be the area, dinv, and deviation l of the corresponding preference function. Additionally, let the ret of a Dyck path be the ret of the corresponding parking function.

In this section, we prove the unlabeled version of Conjecture 4.2.1. Taking the inner product of both sides with s_{1^n} yields $0 = 0$ for most permutations τ. Each remaining case corresponds to a composition of n. For a square path SQ, let diagcomp(SQ) be the composition whose ith part gives the number of cars in the $i+1$st lowest diagonal of SQ. Let $\mathcal{SQ}_n$ and $\mathcal{D}_n$ be the sets of all square paths and Dyck paths, respectively, in the $n \times n$ lattice. Then for any composition α, Conjecture 4.2.1 implies

$$\sum_{\substack{SQ \in \mathcal{SQ}_n \\ \mathrm{diagcomp}(SQ)=\alpha}} t^{\mathrm{area}(SQ)} q^{\mathrm{dinv}(SQ)} = \sum_{\substack{D \in \mathcal{D}_n \\ \mathrm{diagcomp}(D)=\alpha}} [\mathrm{ret}(D)]_q t^{\mathrm{area}(D)} q^{\mathrm{dinv}(D)}.$$

We will see below that a much stronger refinement holds in this case, but not in general.

Given a square path SQ with deviation $l = l(SQ)$ (hence lowest diagonal $-l$), we define a composition $\mathrm{retcomp}(SQ) = \beta$ as follows. Let $\beta_0 = 1$. For $0 \le i < l$, define β_{i+1} so that there are $\beta_{i+1} - 1$ cars in the $(i+1-l)$-th diagonal which are left of the $(\beta_i + 1)$-st car in the $(i-l)$-th diagonal in SQ. For $i \ge l$, let β_{i+1} be the number of cars in the $(i+1-l)$-th diagonal which are left of the $(\beta_i + 1)$-st car in the $(i-l)$-th diagonal in SQ. In both cases, if the $(i-l)$-th diagonal doesn't have $\beta_i + 1$ cars, simply count all the cars in the $(i+1-l)$-th diagonal.

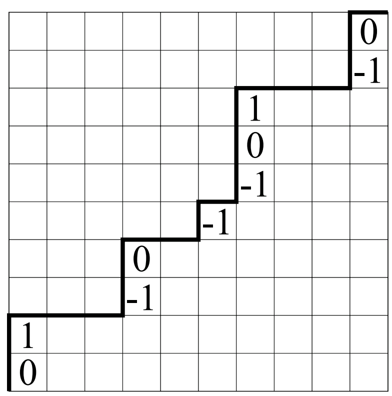

Figure 4.1: A square path with diagcomp $= (4,4,2)$, retcomp $= (1,3,2)$.

For example, see Figure 4.1. There are 0-diagonal cars left of the second car from the (-1)-diagonal. Since these are in nonpositive diagonals, we get $\beta_1 = 2+1$. There are two 1-diagonal cars left of the fourth 0-diagonal car. Since these are in nonnegative diagonals, we get $\beta_2 = 2$. Note that if the path has deviation 0, i.e. it is a Dyck path, then this gives the number of cars within each diagonal to the left of the path's first return to $y = x$. Hence $|\beta|$ generalizes the return statistic to square paths (and preference functions).

Theorem 4.3.1. *For all compositions α, β and $l \geq 0$,*

$$\sum_{\substack{SQ \in \mathcal{SQ}_n, l(SQ)=l \\ \mathrm{diagcomp}(SQ)=\alpha \\ \mathrm{retcomp}(SQ)=\beta}} t^{\mathrm{area}(SQ)} q^{\mathrm{dinv}(SQ)} = q^{(\sum_{i=0}^{l-1} \beta_i)} [\beta_l]_q \sum_{\substack{D \in \mathcal{D}_n \\ \mathrm{diagcomp}(D)=\alpha \\ \mathrm{retcomp}(D)=\beta}} t^{\mathrm{area}(D)} q^{\mathrm{dinv}(D)}.$$

Proof. First, we note that the power of t is fixed in each term of the desired equation. In particular, the area of a square path with diagcomp $= \alpha$ is $\sum_k k\alpha_k$. Furthermore, since the cars in each diagonal are in increasing order (from left to right), any two cars in the same diagonal contribute to primary dinv. This gives a total primary dinv of $\sum_i \binom{\alpha_i}{2}$ for any square path with diagcomp $= \alpha$. Finally, there is no tertiary dinv for Dyck paths, but each square path on the left hand side contributes $q^{\sum_{i=0}^{l-1} \alpha_i}$ due to tertiary dinv. This matches the additional power of q on the right hand side. So it remains to show

$$\sum_{\substack{SQ \in \mathcal{SQ}_n, l(SQ)=l \\ \mathrm{diagcomp}(SQ)=\alpha \\ \mathrm{retcomp}(SQ)=\beta}} q^{\mathrm{secdinv}(SQ)} = [\beta_l]_q \sum_{\substack{D \in \mathcal{D}_n \\ \mathrm{diagcomp}(D)=\alpha \\ \mathrm{retcomp}(D)=\beta}} q^{\mathrm{secdinv}(D)}$$

where secdinv is secondary dinv.

Recall that secondary dinv only depends on the relative positions of cars in adjacent diagonals. This is determined by the inversions in the word of i's and $(i+1)$'s which represent cars in the i-th and $(i+1)$-st diagonal as they appear from left to right. So each square path SQ with diagcomp$(SQ) = \alpha$ corresponds to a sequence of words $SQ_{i,i+1}$ consisting of α_{l+i} i's and $(i+1)$'s with $i \geq -l$. If $i < 0$, then the last element of $SQ_{i,i+1}$ must be $i+1$. If $i \geq 0$, the first element of $SQ_{i,i+1}$ must be i.

Furthermore, suppose retcomp$(SQ) = \beta$. Then for $i < 0$, then $SQ_{i,i+1}$ consists of a word with β_{l+i} i's and $\beta_{l+i+1} - 1$ $(i+1)$'s followed by a word with $\alpha_{l+i} - \beta_{l+i}$ i's and $\alpha_{l+i+1} - \beta_{l+i+1}$ $(i+1)$'s and finally an $(i+1)$. If $i \geq 0$, then $SQ_{i,i+1}$ consists of an i followed by a word with $\beta_{l+i} - 1$ i's and β_{l+i+1} $(i+1)$'s and then a word with $\alpha_{l+i} - \beta_{l+i}$ i's and $\alpha_{l+i+1} - \beta_{l+i+1}$ $(i+1)$'s.

Now let $\mathcal{W}(a, b)$ be the set of all words consisting of a small elements and

b large elements. Let $\cdot$ denote concatenation. We will use the fact that

$$\sum_{\substack{w_1\in\mathcal{W}(a,b)\\ w_2\in\mathcal{W}(c,d)}} q^{\operatorname{inv}(w_1\cdot w_2)} = q^{bc}\begin{bmatrix} a+b\\ a\end{bmatrix}_q \begin{bmatrix} c+d\\ c\end{bmatrix}_q.$$

We will also make small tweaks to this formula to account for a forced small or large element at the beginning or end.

$$\begin{aligned}
\sum_{\substack{SQ\in\mathcal{SQ}_n,\, l(SQ)=l\\ \operatorname{diagcomp}(SQ)=\alpha\\ \operatorname{retcomp}(SQ)=\beta}} q^{\operatorname{secdinv}(SQ)} &= \sum_{SQ}\left(\prod_{i\geq 0} q^{\operatorname{inv}(SQ_{i,i+1})}\right)\\
&= \prod_{i=-l}^{-1} q^{(\beta_{l+i+1}-1)(\alpha_{l+i}-\beta_{l+i})}\begin{bmatrix}\beta_{l+i}+\beta_{l+i+1}-1\\ \beta_{l+i}\end{bmatrix}_q\begin{bmatrix}\alpha_{l+i}+\alpha_{l+i+1}-\beta_{l+i}-\beta_{l+i+1}-1\\ \alpha_{l+i}-\beta_{l+i}-1\end{bmatrix}_q\\
&\quad\times\prod_{i=0}^{l(\beta)-l} q^{\beta_{l+i+1}(\alpha_{l+i}-\beta_{l+i})}\begin{bmatrix}\beta_{l+i}+\beta_{l+i+1}-1\\ \beta_{l+i}-1\end{bmatrix}_q\begin{bmatrix}\alpha_{l+i}+\alpha_{l+i+1}-\beta_{l+i}-\beta_{l+i+1}-1\\ \alpha_{l+i}+\beta_{l+i}-1\end{bmatrix}_q\\
&\quad\times\prod_{i=l(\beta)-l+1}^{l(\alpha)-l-1}\begin{bmatrix}\alpha_{l+i}+\alpha_{l+i+1}-1\\ \alpha_{l+i}-1\end{bmatrix}\\
&= \prod_{i=-l}^{-1} q^{(\beta_{l+i}-\alpha_{l+i})}\frac{[\beta_{l+i+1}]_q}{[\beta_{l+i}]_q}\\
&\quad\times\prod_{i=-l}^{l(\beta)-l} q^{\beta_{l+i+1}(\alpha_{l+i}-\beta_{l+i})}\begin{bmatrix}\beta_{l+i}+\beta_{l+i+1}-1\\ \beta_{l+i}-1\end{bmatrix}\begin{bmatrix}\alpha_{l+i}+\alpha_{l+i+1}-\beta_{l+i}-\beta_{l+i+1}-1\\ \alpha_{l+i}+\beta_{l+i}-1\end{bmatrix}_q\\
&\quad\times\prod_{i=l(\beta)-l+1}^{l(\alpha)-l-1}\begin{bmatrix}\alpha_{l+i}+\alpha_{l+i+1}-1\\ \alpha_{l+i}-1\end{bmatrix}\\
&= q^{(\sum_{i=0}^{l-1}\beta_i-\alpha_i)}\,[\beta_l]_q\sum_{\substack{D\in\mathcal{D}_n\\ \operatorname{diagcomp}(D)=\alpha\\ \operatorname{retcomp}(D)=\beta}} q^{\operatorname{secdinv}(D)}
\end{aligned}$$

The last equality follows by comparing the previous equation for general l and the case $l = 0$, which corresponds to Dyck paths. □

Corollary 4.3.1. *For all compositions α and β,*

$$\sum_{\substack{SQ\in\mathcal{SQ}_n\\ \operatorname{diagcomp}(SQ)=\alpha\\ \operatorname{retcomp}(SQ)=\beta}} t^{\operatorname{area}(SQ)}q^{\operatorname{dinv}(SQ)} = \sum_{\substack{D\in\mathcal{D}_n\\ \operatorname{diagcomp}(D)=\alpha\\ \operatorname{retcomp}(D)=\beta}} [\,|\beta|\,]_q t^{\operatorname{area}(D)}q^{\operatorname{dinv}(D)}$$

Summing over all β gives

$$\left\langle \sum_{\substack{Pr\in\mathcal{P}ref_n\\ \mathrm{diagword}(Pr)=\tau}} t^{\mathrm{area}(Pr)} q^{\mathrm{dinv}(Pr)} F_{\mathrm{ides}(Pr)},\ s_{1^n} \right\rangle$$
$$= \left\langle \sum_{\substack{PF\in\mathcal{P}F_n\\ \mathrm{diagword}(PF)=\tau}} [\mathrm{ret}(PF)]_q\, t^{\mathrm{area}(PF)} q^{\mathrm{dinv}(PF)} F_{\mathrm{ides}(PF)},\ s_{1^n} \right\rangle,$$

which is a special case of Conjecture 4.2.1.

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